Finite Element Spaces for Double Forms

Yakov Berchenko-Kogan, joint with Evan Gawlik

Florida Institute of Technology

March 7, 2025

Double forms

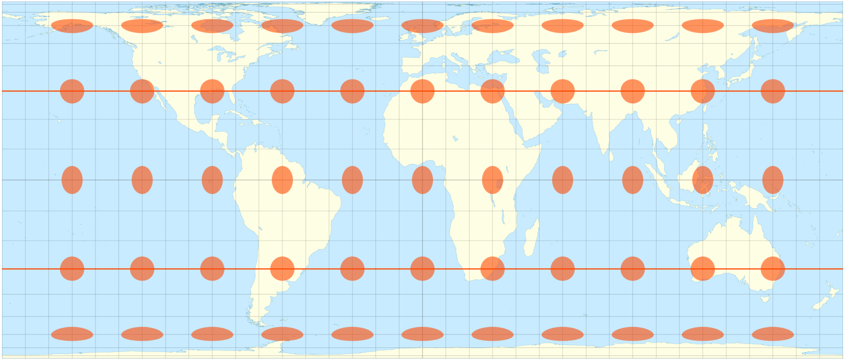
Matrix fields $\mathbb{R}^3 \otimes \mathbb{R}^3$

- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

Applications

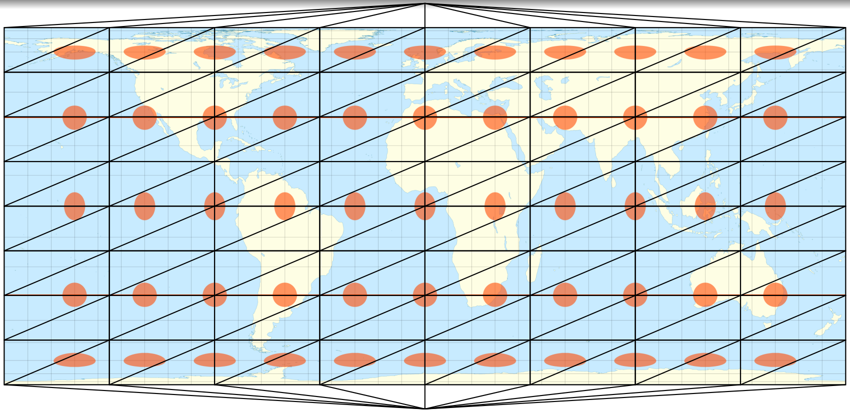
- Strain/stress tensors in elasticity or fluid mechanics (Stokes equations).
- Curvature tensor in numerical geometry and numerical relativity.

Intrinsic geometry with Regge metrics



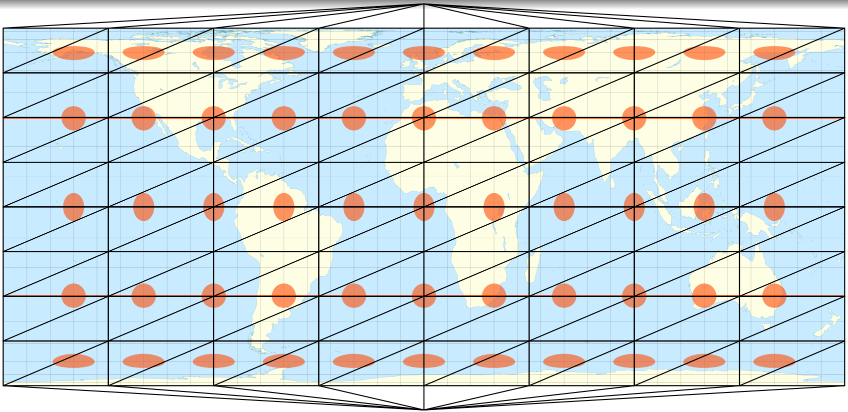
Map credit: Wikipedia, Gaba

Intrinsic geometry with Regge metrics



Map credit: Wikipedia, Gaba

Intrinsic geometry with Regge metrics



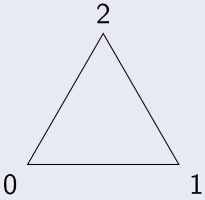
Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.–tang. continuity.

Map credit: Wikipedia, Gaba

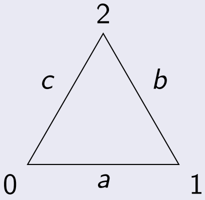
Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



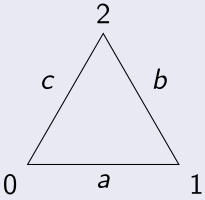
Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$

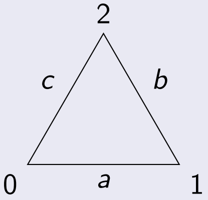


Regge metric:

$$
\begin{aligned}
g = &-\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) \\
&-\tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\
&-\tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2)
\end{aligned}
$$

Regge metric on a reference triangle

Barycentric coordinates $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric:

$$
\begin{aligned}
g = &-\tfrac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) \\
&-\tfrac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\
&-\tfrac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2)
\end{aligned}
$$

Observations

- If $\mathbf{v}$ is the vector from vertex 0 to vertex 1, then

  $$d\lambda_0(\mathbf{v}) = -1, \qquad d\lambda_1(\mathbf{v}) = 1, \qquad d\lambda_2(\mathbf{v}) = 0.$$

  As desired:

  $$g(\mathbf{v}, \mathbf{v}) = -\tfrac{1}{2}a^2(-1-1) - \tfrac{1}{2}b^2(0+0) - \tfrac{1}{2}c^2(0+0) = a^2.$$

- Crucial: $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ is zero on other edges.

Constant coefficient finite elements for bilinear forms

Local bases for finite element spaces

- Each basis element φ must be associated to a face F of the triangulation, such that, for any other face G,

$$\varphi \text{ is nonzero on } G \Leftrightarrow G \geq F.$$

Constant coefficient symmetric bilinear forms $\Lambda^{1,1}_{\mathrm{sym}}$

- Regge's construction works in any dimension. To each edge ij, associate

$$d\lambda_i \otimes d\lambda_j + d\lambda_j \otimes d\lambda_i.$$

Constant coefficient antisymmetric bilinear forms $\Lambda^{1,1}_{\mathrm{asym}}$

- Finite element spaces do not exist in dimension ≥ 3.
- In 3D, antisymmetric bilinear forms $\leftrightarrow$ vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

Natural subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \le m \le \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X;Y) = \varphi(Y;X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X;Y) = -\varphi(Y;X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

Finite element spaces

Theorem

Apart from $\Lambda^{p,q}_q \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda^{p,q}_m$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda^{1,1}_0$: symmetric matrices with tangential–tangential continuity (Regge, 1961).
  - Higher order: (Li, 2018).
- $\Lambda^{2,1}_0$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda^{2,2}_0$ in 3D: symmetric matrices with normal–normal continuity (Sinwel, 2009).
- $\Lambda^{2,2}_0$ (or $\Lambda^{n-2,n-2}_0$) in higher dimensions: finite elements for the Riemann curvature tensor.

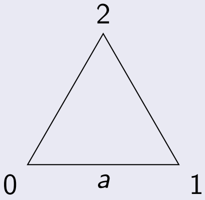
Degrees of freedom for constant coefficient spaces

	d						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

Extension

Recall



- It was crucial that $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ vanishes on the other edges.

Extension operators

- We need to be able to take a form on edge 01, and extend it so that it vanishes on the other edges.
- The metric on edge 01 is $a^2\, d\lambda_1 \otimes d\lambda_1$.
- However, if we extend to the triangle using the formula $a^2\, d\lambda_1 \otimes d\lambda_1$, it won't vanish on edge 12.
- We first need to use $d\lambda_0 + d\lambda_1 = 0$ to rewrite $a^2\, d\lambda_1 \otimes d\lambda_1$ as $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$ on edge 01.

Example ($\mathcal{P}_r\Lambda^{p,q}_m = \mathcal{P}_0\Lambda^{1,1}_0$)

1. Start with a form on edge 01 with vanishing trace: $d\lambda_1 \otimes d\lambda_1$
2. $\lambda_i = u_i^2, \quad d\lambda_i = 2u_i\, du_i$: $\quad 4u_1^2\, du_1 \otimes du_1.$
3. $u_0\, du_0 + u_1\, du_1$ wedge with each factor: $$4u_0^2u_1^2(du_0 \wedge du_1) \otimes (du_0 \wedge du_1).$$
4. Hodge star both factors (as forms on $\mathbb{R}^2$): $\quad 4u_0^2u_1^2.$
5. Divide by u_0u_1: $\quad 4u_0u_1.$
6. Divide by $(2r+p+m+1)(2r+q-m) = 2$: $\quad 2u_0u_1.$
7. Exterior derivative on both factors: $2(du_0 \otimes du_1 + du_1 \otimes du_0).$
8. Apply $(-1)^{p+q}$ times the inverse Hodge star: $$-2(du_1 \otimes du_0 + du_0 \otimes du_1).$$
9. Multiply by u_0u_1: $\quad -2u_0u_1(du_1 \otimes du_0 + du_0 \otimes du_1).$
10. Convert back to λ_i: $\quad -\frac{1}{2}(d\lambda_1 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_1).$

Thank you

J. Gopalakrishnan, P. L. Lederer, and J. Schöberl
A mass conserving mixed stress formulation for the Stokes equations.
IMA Journal of Numerical Analysis, 40 (2020), no. 3, pp. 1838–1874

L. Li
Regge finite elements with applications in solid mechanics and relativity.
PhD thesis. University of Minnesota, 2018.

A. Sinwel
A new family of mixed finite elements for elasticity.
PhD thesis. Johannes Kepler Universität Linz, 2009.

Supported by NSF DMS-2411209.

Tangential and normal continuity of vector fields

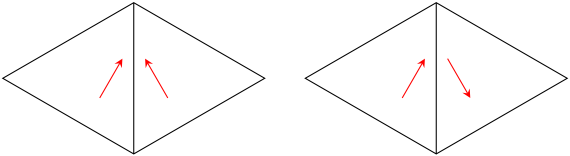


Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\mathrm{curl})$.

Normal continuity

- Well-defined fluxes.
- In $H(\mathrm{div})$.

Differential forms corresponding to vector field $\langle M, N, P\rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.