

# Finite Element Spaces for Double Forms

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# Double forms

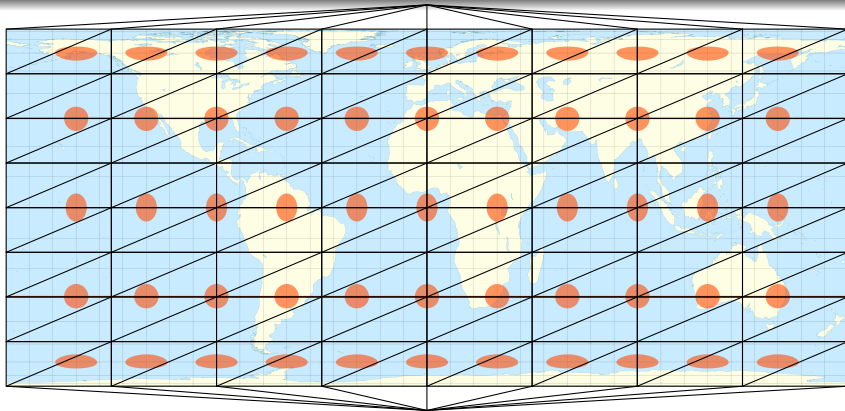
## Matrix fields $\mathbb{R}^3 \otimes \mathbb{R}^3$

- Matrix fields with tangential–tangential continuity are  $(1, 1)$ -forms  $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$ .
- Matrix fields with normal–tangential continuity are  $(2, 1)$ -forms  $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$ .
- Matrix fields with normal–normal continuity are  $(2, 2)$ -forms  $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$ .

## Applications

- Strain/stress tensors in elasticity or fluid mechanics (Stokes equations).
- Curvature tensor in numerical geometry and numerical relativity.

# Intrinsic geometry with Regge metrics



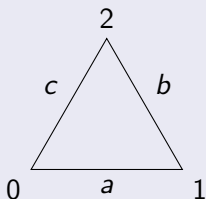
## Regge finite elements

- Record the length of each edge.
- For each triangle, use the corresponding Euclidean metric.
- Get piecewise constant metric with tang.-tang. continuity.

Map credit: Wikipedia, Gaba

# Regge metric on a reference triangle

Barycentric coordinates  $\lambda_0 + \lambda_1 + \lambda_2 = 1$



Regge metric:

$$\begin{aligned} g = & -\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0) \\ & -\frac{1}{2}b^2(d\lambda_1 \otimes d\lambda_2 + d\lambda_2 \otimes d\lambda_1) \\ & -\frac{1}{2}c^2(d\lambda_2 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_2) \end{aligned}$$

## Observations

- If  $\mathbf{v}$  is the vector from vertex 0 to vertex 1, then

$$d\lambda_0(\mathbf{v}) = -1, \quad d\lambda_1(\mathbf{v}) = 1, \quad d\lambda_2(\mathbf{v}) = 0.$$

As desired:

$$g(\mathbf{v}, \mathbf{v}) = -\frac{1}{2}a^2(-1 - 1) - \frac{1}{2}b^2(0 + 0) - \frac{1}{2}c^2(0 + 0) = a^2.$$

- Crucial:  $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$  is zero on other edges.

# Constant coefficient finite elements for bilinear forms

## Local bases for finite element spaces

- Each basis element  $\varphi$  must be associated to a face  $F$  of the triangulation, such that, for any other face  $G$ ,

$$\varphi \text{ is nonzero on } G \Leftrightarrow G \geq F.$$

## Constant coefficient symmetric bilinear forms $\Lambda_{\text{sym}}^{1,1}$

- Regge's construction works in any dimension. To each edge  $ij$ , associate

$$d\lambda_i \otimes d\lambda_j + d\lambda_j \otimes d\lambda_i.$$

## Constant coefficient antisymmetric bilinear forms $\Lambda_{\text{asym}}^{1,1}$

- Finite element spaces **do not exist** in dimension  $\geq 3$ .
- In 3D, antisymmetric bilinear forms  $\leftrightarrow$  vector fields with normal continuity.
- A nonzero constant vector field can't be tangent to three faces of a tetrahedron.

# Natural subspaces of double forms

## Theorem (Eigendecomposition of $s^*s$ )

$$\Lambda^{p,q} = \bigoplus_m \Lambda_m^{p,q}, \quad \max\{0, q - p\} \leq m \leq \min\{q, n - p\}.$$

## Example

- $\Lambda_0^{1,1}$ : Symmetric bilinear forms,  $\varphi(X; Y) = \varphi(Y; X)$ .
- $\Lambda_1^{1,1}$ :  $\Lambda^2$ , antisymmetric bilinear forms,  $\varphi(X; Y) = -\varphi(Y; X)$ .

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- $\Lambda_0^{2,1}$ : spanned by  $\alpha \otimes \beta$  such that  $\alpha \wedge \beta = 0$ .
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda_1^{2,1}$ :  $\Lambda^3$ .
  - Matrix proxy in 3D: multiples of the identity matrix.

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- $\Lambda_0^{2,2}$ : Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda_1^{2,2}$ : Antisymmetric,  $\varphi(X, Y; Z, W) = -\varphi(Z, W; X, Y)$ .
- $\Lambda_2^{2,2}$ :  $\Lambda^4$ .

## Theorem

*Apart from  $\Lambda_q^{p,q} \cong \Lambda^{p+q}$  with constant coefficients, there is a finite element space for every natural space of double forms  $\Lambda_m^{p,q}$  with polynomial coefficients of any degree (including zero).*

## Example (Constant coefficient spaces)

- $\Lambda_0^{1,1}$ : symmetric matrices with tangential–tangential continuity (Regge, 1961).
  - Higher order: (Li, 2018).
- $\Lambda_0^{2,1}$  in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda_0^{2,2}$  in 3D: symmetric matrices with normal–normal continuity (Sinwel, 2009).
- $\Lambda_0^{2,2}$  (or  $\Lambda_0^{n-2,n-2}$ ) in higher dimensions: finite elements for the Riemann curvature tensor.

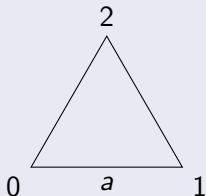
# Degrees of freedom for constant coefficient spaces

	$d$						
	0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	<b>1</b>	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	<b>2</b>	0	0	0	0
$\Lambda_0^{2,2}$	0	0	<b>1</b>	<b>2</b>	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	<b>3</b>	0	0	0
$\Lambda_0^{3,2}$	0	0	0	<b>3</b>	<b>5</b>	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	<b>4</b>	0	0
$\Lambda_0^{3,3}$	0	0	0	<b>1</b>	<b>5</b>	<b>5</b>	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	<b>6</b>	<b>9</b>	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	<b>5</b>	0

**Table:** Number of degrees of freedom for  $\Lambda^{p,q}$  associated to a face of the triangulation of dimension  $d$  is  $\frac{p-q+2m+1}{p+m+1} \binom{d+1}{q-m} \binom{q-m-1}{d-p-m}$ .



## Recall



- It was crucial that  $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$  vanishes on the other edges.

## Extension operators

- We need to be able to take a form on edge 01, and extend it so that it vanishes on the other edges.
- The metric on edge 01 is  $a^2 d\lambda_1 \otimes d\lambda_1$ .
- However, if we extend to the triangle using the formula  $a^2 d\lambda_1 \otimes d\lambda_1$ , it won't vanish on edge 12.
- We first need to use  $d\lambda_0 + d\lambda_1 = 0$  to rewrite  $a^2 d\lambda_1 \otimes d\lambda_1$  as  $-\frac{1}{2}a^2(d\lambda_0 \otimes d\lambda_1 + d\lambda_1 \otimes d\lambda_0)$  on edge 01.

# Constructing extensions

Example  $(\mathcal{P}_r \Lambda_m^{p,q} = \mathcal{P}_0 \Lambda_0^{1,1})$

- ❶ Start with a form on edge 01 with vanishing trace:  $d\lambda_1 \otimes d\lambda_1$
- ❷  $\lambda_i = u_i^2, \quad d\lambda_i = 2u_i du_i: \quad 4u_1^2 du_1 \otimes du_1.$
- ❸  $u_0 du_0 + u_1 du_1$  wedge with each factor:  
 $4u_0^2 u_1^2 (du_0 \wedge du_1) \otimes (du_0 \wedge du_1).$
- ❹ Hodge star both factors (as forms on  $\mathbb{R}^2$ ):  $4u_0^2 u_1^2.$
- ❺ Divide by  $u_0 u_1$ :  $4u_0 u_1.$
- ❻ Divide by  $(2r + p + m + 1)(2r + q - m) = 2$ :  $2u_0 u_1.$
- ❼ Exterior derivative on both factors:  $2(du_0 \otimes du_1 + du_1 \otimes du_0).$
- ❽ Apply  $(-1)^{p+q}$  times the inverse Hodge star:  
 $-2(du_1 \otimes du_0 + du_0 \otimes du_1).$
- ❾ Multiply by  $u_0 u_1$ :  $-2u_0 u_1 (du_1 \otimes du_0 + du_0 \otimes du_1).$
- ❿ Convert back to  $\lambda_i$ :  $-\frac{1}{2}(d\lambda_1 \otimes d\lambda_0 + d\lambda_0 \otimes d\lambda_1).$

# Thank you



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# Tangential and normal continuity of vector fields

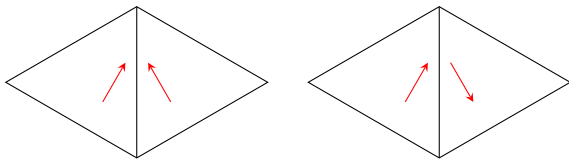


Figure: Tangential continuity (left) vs. normal continuity (right)

## Tangential continuity

- Well-defined line integrals.
- In  $H(\text{curl})$ .

## Normal continuity

- Well-defined fluxes.
- In  $H(\text{div})$ .

# Differential forms corresponding to vector field $\langle M, N, P \rangle$

## One-forms $\Lambda^1$

- $M dx + N dy + P dz$
- Restricted to the  $xy$ -plane  $z = 0$ :
  - $M dx + N dy$ .
  - Tangential components.

## Two-forms $\Lambda^2$

- $M dy \wedge dz + N dz \wedge dx + P dx \wedge dy$ .
- Restricted to the  $xy$ -plane  $z = 0$ :
  - $P dx \wedge dy$ .
  - Normal component.

## Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are  $(n - 1)$ -forms.