Finite element spaces for tensor fields with specified interelement continuity conditions

Yakov Berchenko-Kogan, joint with Evan Gawlik

Florida Institute of Technology
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Tangential and normal continuity of vector fields

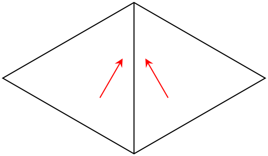
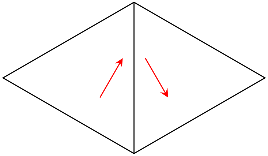



Figure: Tangential continuity (left) vs. normal continuity (right)

Tangential continuity

- Well-defined line integrals.
- In $H(\mathrm{curl})$.

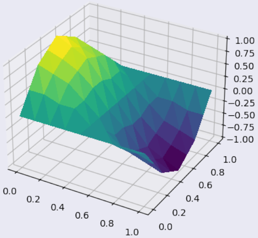
Normal continuity

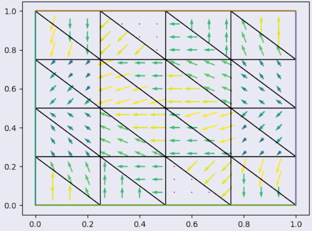
- Well-defined fluxes.
- In $H(\mathrm{div})$.

What's wrong with full continuity?

Finite element exterior calculus (FEEC) perspective: differential complexes

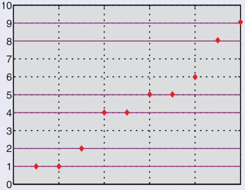
Gradients of scalar fields only have tangential continuity





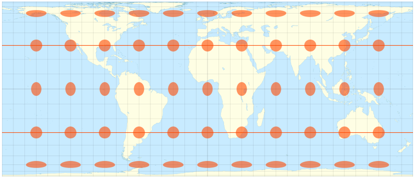
Spurious eigenvalues of the curl curl operator (AFW, 2010)

- Solve $\operatorname{curl}\operatorname{curl} u = \lambda u$, where u is a vector field on a square domain with appropriate boundary conditions.
- Using vector fields with full continuity yields false eigenvalue $\lambda = 6$.



Extrinsic

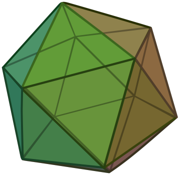
Intrinsic



Four images from Wikipedia

What's wrong with full continuity?
Geometric perspective

Extrinsic



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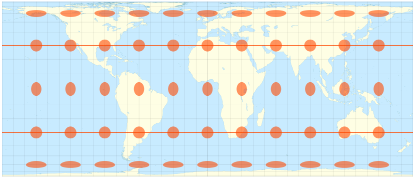

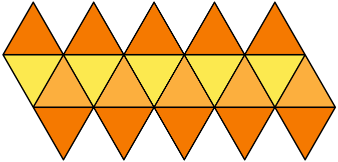

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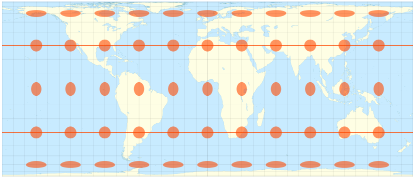

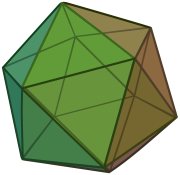

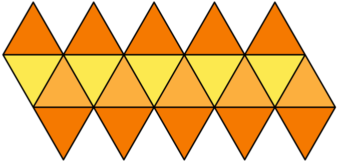

Why compute intrinsically?

- Intrinsic problems, e.g. numerical relativity, Ricci flow.
- Structure preservation: independence of embedding.

Four images from Wikipedia

What's wrong with full continuity?

Geometric perspective: Angle defect obstruction to continuous elements

- Try to construct a tangent vector field on the icosahedron.

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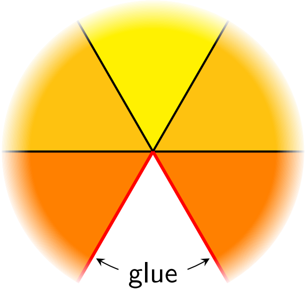

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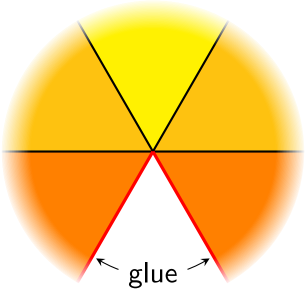

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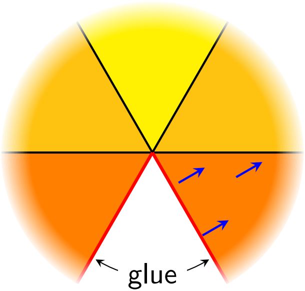

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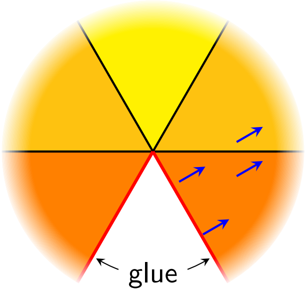

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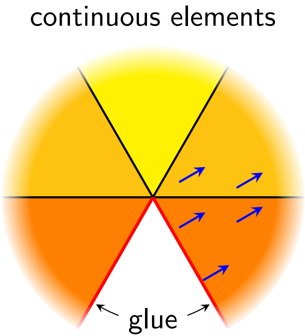

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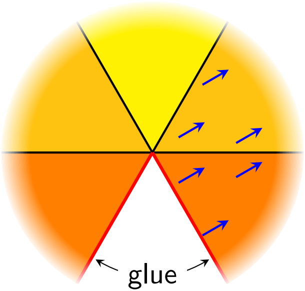

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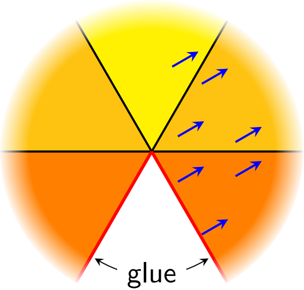

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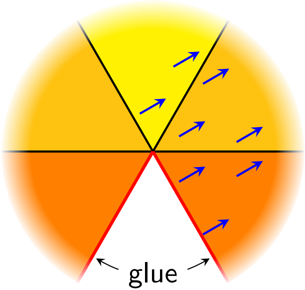

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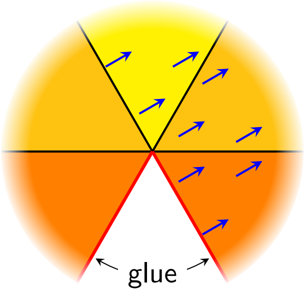

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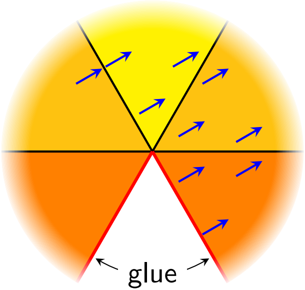

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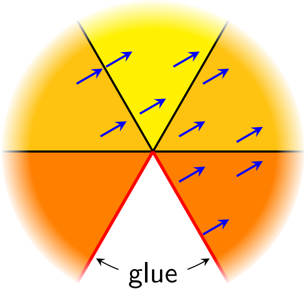

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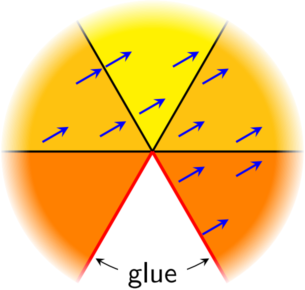

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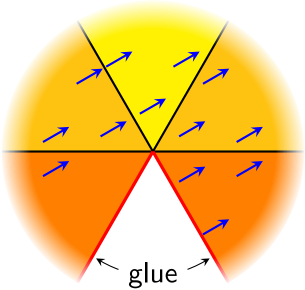

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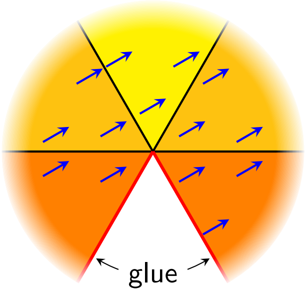

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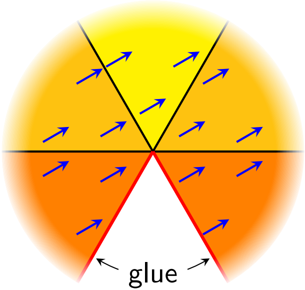

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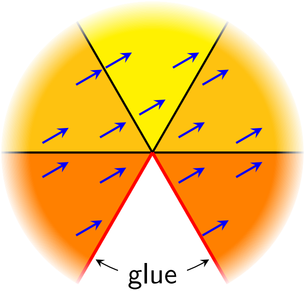

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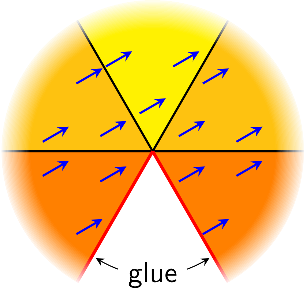

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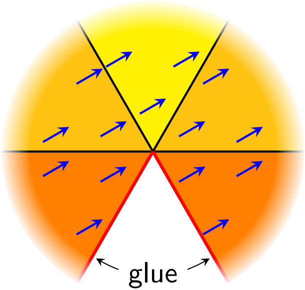

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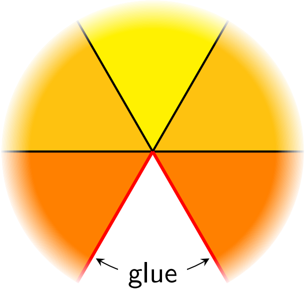

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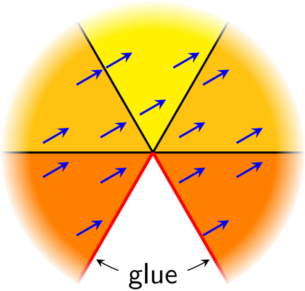

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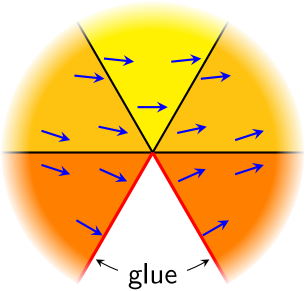

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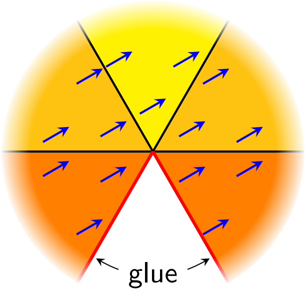

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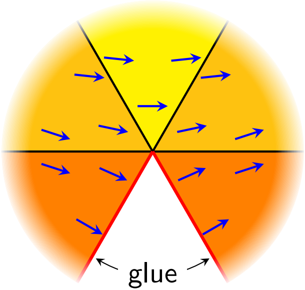

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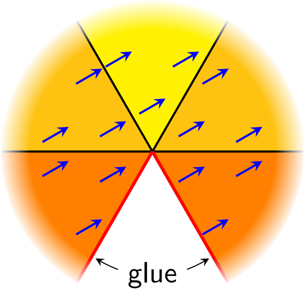

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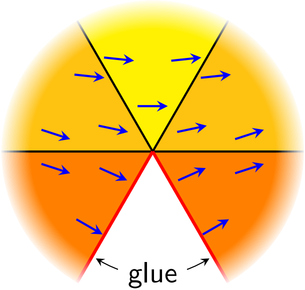

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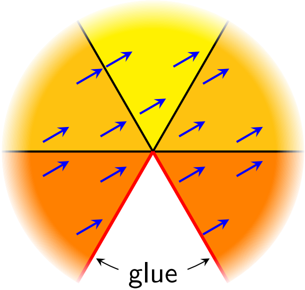

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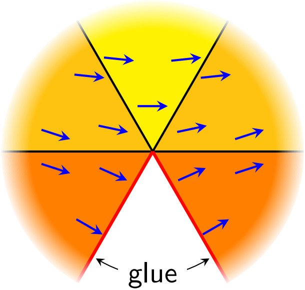

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- See also later today: Alan Demlow (10am).

Differential forms corresponding to vector field $\langle M, N, P\rangle$

One-forms Λ^1

- $M\,dx + N\,dy + P\,dz$
- Restricted to the xy-plane $z = 0$:
  - $M\,dx + N\,dy$.
  - Tangential components.

Two-forms Λ^2

- $M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$.
- Restricted to the xy-plane $z = 0$:
  - $P\,dx \wedge dy$.
  - Normal component.

Continuity conditions

- Vector fields with tangential continuity are one-forms.
- Vector fields with normal continuity are $(n-1)$-forms.

Extending FEEC to matrices and tensors

Continuity conditions for 2-tensors (matrix fields)

- tangential–tangential
- normal–normal
- normal–tangential

Applications

- Strain/stress tensors
  - Elasticity (objects deforming under stress)
  - Fluid mechanics (Stokes equations)
- Numerical geometry/relativity
  - Riemannian/Minkowski metric
  - Curvature tensor
- See also later today: Francis Aznaran (9:30am), Alan Demlow (10am), Qingguo Hong (2pm), Bowen Shi (2:30pm).

Double forms

Vector fields ($\mathbb{R}^3$)

- Vector fields with tangential continuity are one-forms Λ^1.
- Vector fields with normal continuity are two-forms Λ^2.

Matrix fields ($\mathbb{R}^3 \otimes \mathbb{R}^3$)

- Matrix fields with tangential–tangential continuity are $(1,1)$-forms $\Lambda^{1,1} := \Lambda^1 \otimes \Lambda^1$.
- Matrix fields with normal–tangential continuity are $(2,1)$-forms $\Lambda^{2,1} := \Lambda^2 \otimes \Lambda^1$.
- Matrix fields with normal–normal continuity are $(2,2)$-forms $\Lambda^{2,2} := \Lambda^2 \otimes \Lambda^2$.

More on double forms later today

- Evan Gawlik (8:30am).
- Anil Hirani (9am).

Affine-invariance vs. metric-dependence

Affine-invariant (metric-independent) finite element spaces

- FEEC differential forms Λ^k and their continuity conditions are defined without reference to a Riemannian metric.
- Same for double forms $\Lambda^{p,q}$.
- Angle defect cannot pose a problem since angle defect is not even defined without a Riemannian metric.
- In particular, for vector fields with tangential or normal continuity, FEEC works just as well on surface meshes as it does on the plane.

Metric-dependent finite element spaces

- Defining finite element spaces of vector fields with full continuity requires a Riemannian metric (even via differential form proxies).
- Behavior depends on whether angle defect is zero or not.

Affine-invariant subspaces of double forms

Theorem (Eigendecomposition of s^*s)

$$\Lambda^{p,q} = \bigoplus_m \Lambda^{p,q}_m, \qquad \max\{0, q-p\} \le m \le \min\{q, n-p\}.$$

Example

- $\Lambda^{1,1}_0$: Symmetric bilinear forms, $\varphi(X;Y) = \varphi(Y;X)$.
- $\Lambda^{1,1}_1$: Λ^2, antisymmetric bilinear forms, $\varphi(X;Y) = -\varphi(Y;X)$.

- $\Lambda^{2,1}_0$: spanned by $\alpha \otimes \beta$ such that $\alpha \wedge \beta = 0$.
  - Matrix proxy in 3D: trace-free matrices.
- $\Lambda^{2,1}_1$: Λ^3.
  - Matrix proxy in 3D: multiples of the identity matrix.

- $\Lambda^{2,2}_0$: Symmetric, satisfying the algebraic Bianchi identity.
  - Riemann curvature tensor.
- $\Lambda^{2,2}_1$: Antisymmetric, $\varphi(X,Y;Z,W) = -\varphi(Z,W;X,Y)$.
- $\Lambda^{2,2}_2$: Λ^4.

Finite element spaces

Theorem (—, Gawlik)

Apart from $\Lambda^{p,q}_q \cong \Lambda^{p+q}$ with constant coefficients, there is a finite element space for every natural space of double forms $\Lambda^{p,q}_m$ with polynomial coefficients of any degree (including zero).

Example (Constant coefficient spaces)

- $\Lambda^{1,1}_0$: symmetric matrices with tangential–tangential continuity (Regge, 1961).
  - Higher order: (Li, 2018).
- $\Lambda^{2,1}_0$ in 3D: trace-free matrices with normal–tangential continuity (Gopalakrishnan, Lederer, and Schöberl, 2019).
- $\Lambda^{2,2}_0$ in 3D: symmetric matrices with normal–normal continuity (Pechstein and Schöberl, 2011).
- $\Lambda^{2,2}_0$ (or $\Lambda^{n-2,n-2}_0$) in any dimension: finite elements for the Riemann curvature tensor.

Degrees of freedom for constant coefficient spaces

	d = 0	1	2	3	4	5	6
$\Lambda_0^{1,1}$	0	**1**	0	0	0	0	0
$\Lambda_0^{2,1}$	0	0	**2**	0	0	0	0
$\Lambda_0^{2,2}$	0	0	**1**	**2**	0	0	0
$\Lambda_1^{2,2} \cong \Lambda_0^{3,1}$	0	0	0	**3**	0	0	0
$\Lambda_0^{3,2}$	0	0	0	**3**	**5**	0	0
$\Lambda_1^{3,2} \cong \Lambda_0^{4,1}$	0	0	0	0	**4**	0	0
$\Lambda_0^{3,3}$	0	0	0	**1**	**5**	**5**	0
$\Lambda_1^{3,3} \cong \Lambda_0^{4,2}$	0	0	0	0	**6**	**9**	0
$\Lambda_2^{3,3} \cong \Lambda_1^{4,2} \cong \Lambda_0^{5,1}$	0	0	0	0	0	**5**	0

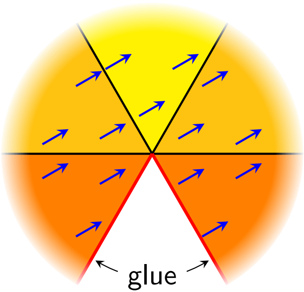
Table: Number of degrees of freedom for $\Lambda_m^{p,q}$ associated to a face of the triangulation of dimension d is $\frac{p-q+2m+1}{p+m+1}\binom{d+1}{q-m}\binom{q-m-1}{d-p-m}$.

Metric-dependent finite element spaces

Motivating problem

- Goal: construct intrinsic discretizations of tangent vector fields on smooth surfaces that are continuous across edges.
- Obstruction to using classical Lagrange $\mathcal{P}_1$ elements: angle defect.

continuous elements



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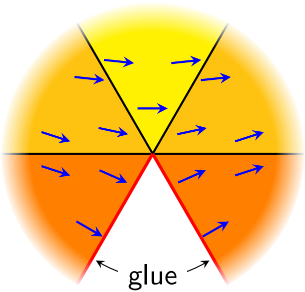

continuous across all edges
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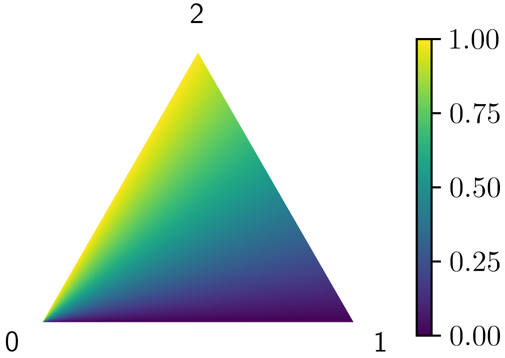
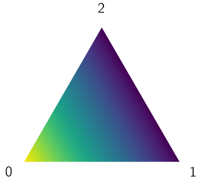
A simplicial analogue of the angular coordinate

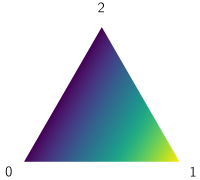


$$\frac{\lambda_2}{\lambda_1 + \lambda_2}$$

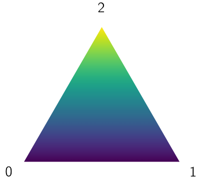
Lagrange $\mathcal{P}_1$ shape functions



λ_0

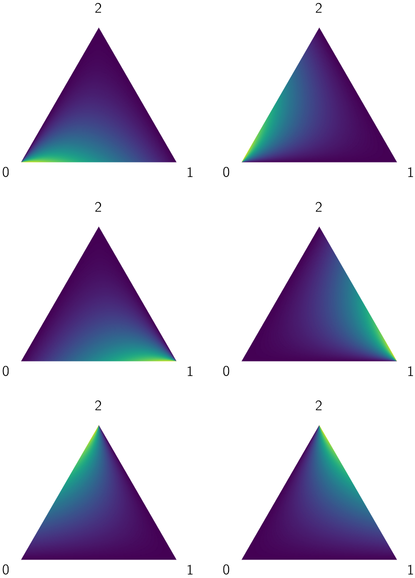


λ_1



λ_2

Blow-up $b\mathcal{P}_1$ shape functions



$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

New?

Shape function

$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_0}{\lambda_0 + \lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_2}{\lambda_2}.$$

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Earlier appearances

- Geometric invariants (Chen, 1957).

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- Geometric invariants (Chen, 1957).
- Horse betting (Harville, 1973).

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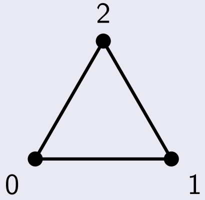
Earlier appearances

- Geometric invariants (Chen, 1957).
- Horse betting (Harville, 1973).
- Intersection homology (Brasselet, Goresky, MacPherson, 1991; Bendiffalah, 1995).

Degrees of freedom

Classical Lagrange $\mathcal{P}_1$



Barycentric coordinates: $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

- $0 : \lambda_0 = 1 \Leftrightarrow \lambda_1 = \lambda_2 = 0$
- $1 : \lambda_1 = 1 \Leftrightarrow \lambda_2 = \lambda_0 = 0$
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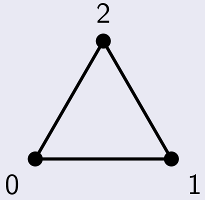

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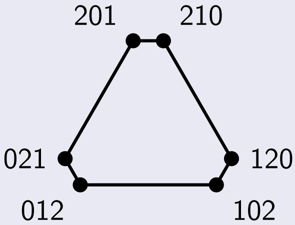
Blow-up $b\mathcal{P}_1$



- $012 : \lim_{\lambda_1\to 0} \lim_{\lambda_2\to 0}$
- $120 : \lim_{\lambda_2\to 0} \lim_{\lambda_0\to 0}$
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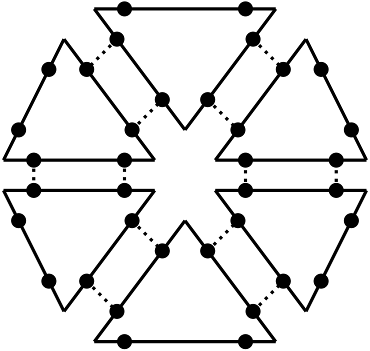
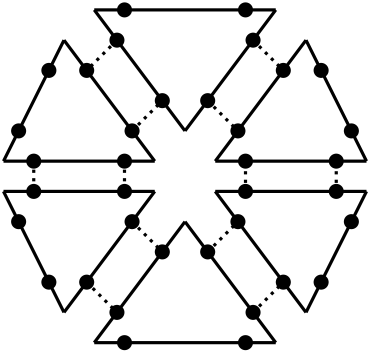
Global spaces



Blow-up finite elements

- Scalar fields: we place a number at each dot.

Global spaces



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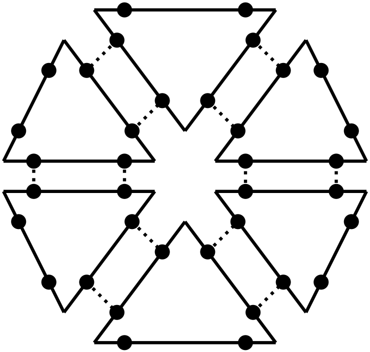

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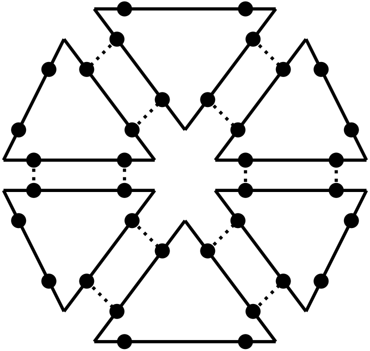

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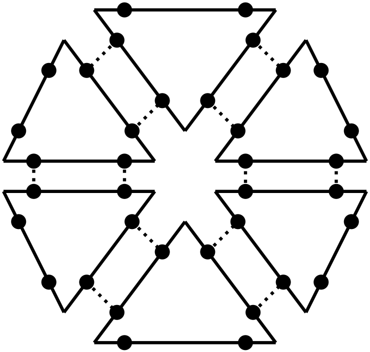

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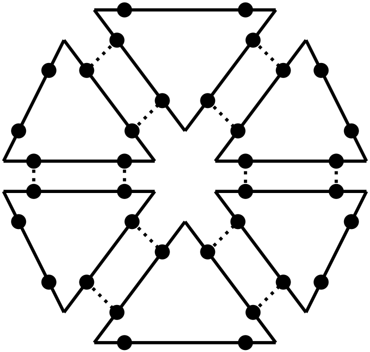

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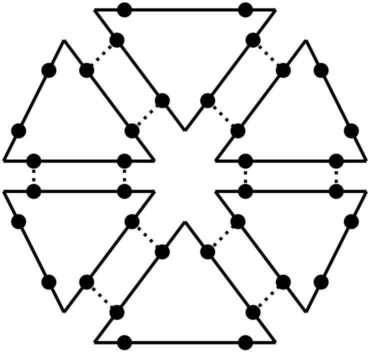

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- General tensor fields are analogous.

Vector Laplacian eigenvalue problems on surfaces

Hodge Laplacian

$$(dd^* + d^*d)v^\flat = \lambda v^\flat.$$

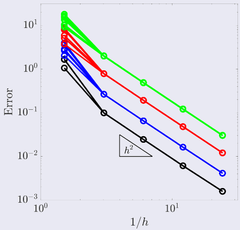
- Tangential continuity across edges suffices.
- Standard FEEC works.
- L^2 pairing suffices.

Bochner Laplacian

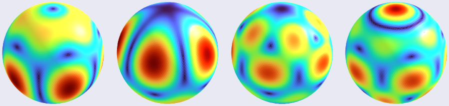
$$\nabla^*\nabla v = \lambda v.$$

- Must have full continuity across edges.
- Can't use standard FEEC.
- Needs Riemannian metric.

Bochner Laplacian on sphere using blow-up elements



Eigenvalue error



Eigenfield magnitude ($\lambda = 11, 11, 19, 19$)

There's more

So far in this talk

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  - Three radiation sources with rates λ_0, λ_1, and λ_2, sum 1.

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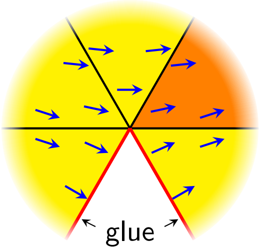
- Degrees of freedom in terms of blow-up simplex.

Blowing up



- Even on an individual triangle, the vector field is not continuous at the origin.
- But it is "continuous in polar coordinates," i.e. in r and θ.

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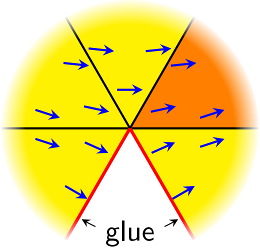

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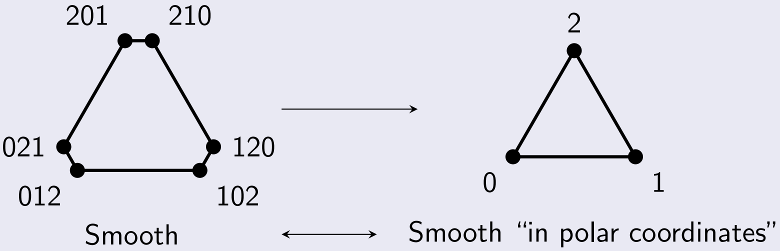
Blowing up manifolds with corners (Melrose, 1996)

- formalizes continuity/smoothness "in polar coordinates"



Thank you

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