Probability Part 1

STAT-S520

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01-17-23

These slides complement ISI Ch3

Probability

Different Interpretations:

- A long run frequency of occurrence
- Expression of degrees of belief
- Other interpretations

Probability Model

Think of performing an experiment that results in one of possibly many outcomes. A probability model (based on Kolmogorov's axioms) is defined by:

- The Sample Space, S, is the collection of all outcomes
- Events are subsets of outcomes
  - We use uppercase letters to denote events, e.g., A, D, E, etc.
  - The set of all relevant events is denoted by Ω
- A probability measure is a function that assigns probabilities to events

$$P : \Omega \to [0, 1]$$

Some Conventions

- $P(S) = 1$
- Ω is a sigma-algebra, a mathematical structure with some properties beyond the scope of this course.
  - For simplicity, we can think of Ω as the set of all relevant events

Example 1

Experiment: Toss a fair coin twice. Let's represent an outcome by the results in order of occurrence.

- $S = \{HH, HT, TH, TT\}$
- Some events:
  - $A = \{HH\}$
  - $B = \{$ first coin is heads $\} = \{HH, HT\}$
  - $C = \{$ at least one tail$\}$
- Some probabilities
  - $P(A) = 1/4$
  - $P(B) = 1/2$
  - $P(C) = 3/4$

Key Property: Finite Additivity

If events A and B are disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

Using finite additivity, we can derive the following results

a. For any event A, $P(A^c) = 1 - P(A)$
   - In particular, since $P(S) = 1$ then $P(\emptyset) = 0$

b. If $A \subset B$, then $P(A) \leq P(B)$

c. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Exercise 1: Show a, b, and c (use Venn Diagrams)

Exercise 2

Use experiment in example 1 (toss coin twice) and answer the following

a. $P(A^c)$
b. $P(A \cup B)$
c. $P((A \cap C)^c)$

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Finite Sample Spaces

- S is a finite sample space if there is $N \in \mathbb{N}$ such that $S = \{s_1, s_2, \ldots, s_N\}$.
  - We do require $s_1, \ldots, s_N$ to be different from each other, i.e., $s_i \neq s_j$ if $i \neq j$
  - A direct consequence is that the event $\{s_i\}$ is disjoint of $\{s_j\}$ for $i \neq j$
- The outcomes are equally likely if

$$P(\{s_1\}) = P(\{s_2\}) = \cdots = P(\{s_N\})$$

- Since $\{s_i\}$ is disjoint of $\{s_j\}$ for $i \neq j$, then using finite additivity

$$\sum_{i=1}^{N} P(\{s_i\}) = P(\cup_{i=1}^{N}\{s_i\}) = P(S) = 1$$

Finite Sample Spaces (continued)

- It follows that
$$P(\{s_i\}) = \frac{1}{N}$$
- For any event $A \subset S$, if S is finite with equally likely outcomes then
$$P(A) = \frac{\#A}{\#S}$$
where $\#A$ is the total number of outcomes in A

Exercise 3

Julia is among 30 students the may be randomly selected to form a committee of 3 people

a. If the committee needs 1 president, 1 vice president, and 1 secretary, what is the probability that Julia is the president?
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b. If the committee needs 1 president, 1 vice president, and 1 secretary, what is the probability that Julia is part of this committee?
    -
c. If the committee needs 3 members without any given roles, what is the probability that Julia is part of this committee?
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Conditional Probability

- If B is an event, we say that B has ocurred if one of the outcomes in B is the result of the experiment
  - In the two-coin example, $B = \{$ first coin is heads $\} = \{HH, HT\}$. If B has occurred the resulting outcome was either HH or HT.
- The conditional probability of A given B, written $P(A|B)$ is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1}$$

  - One way to think about this is, if B has occurred, only the outcomes in B are possible (the sample space has been restricted). Now, what is the probability that A occurs?
  - In the two-coin example $A = \{HH\}$ and $P(A|B) = 1/2$

Conditional Probability

- If we multiply both sides of (1) by $P(B)$ we get

$$P(A \cap B) = P(B)P(A|B) \tag{2}$$

- This is the second formulation of conditional probability

Independence

Two events, A and B are independent if the probability of A does not change by the occurence of B (or viceversa). When this happens, the following equations hold:

- $P(A|B) = P(A|B^c)$
- $P(A|B) = P(A)$
- $P(A \cap B) = P(A) \cdot P(B)$

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Tree Diagrams: A and B are two events

Bayes' Rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(A^c) \cdot P(B|A^c)}$$

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