Probability Part 1 STAT-S520

Arturo Valdivia

01-17-23



Probability

Different Interpretations:

- ► A long run frequency of occurrence
- Expression of degrees of belief
- ► Other interpretations

Probability Model

Think of performing an experiment that results in one of possibly many outcomes. A probability model (based on Kolmogorov's axioms) is defined by:

- ▶ The Sample Space, *S*, is the collection of all outcomes
- Events are subsets of outcomes
 - ▶ We use uppercase letters to denote events, e.g., A, D, E, etc.
 - lacktriangle The set of all relevant events is denoted by Ω
- ► A probability measure is a function that assigns probabilities to events

$$P:\Omega \rightarrow [0,1]$$

Some Conventions

- P(S) = 1
- $ightharpoonup \Omega$ is a sigma-algebra, a mathematical structure with some properties beyond the scope of this course.
 - lackbox For simplicity, we can think of Ω as the set of all relevant events

Example 1

Experiment: Toss a fair coin twice. Let's represent an outcome by the results in order of occurrence.

- ► *S* = {*HH*, *HT*, *TH*, *TT*}
- ► Some events:
 - $ightharpoonup A = \{HH\}$
 - ▶ $B = \{ \text{ first coin is heads } \} = \{HH, HT\}$
 - ▶ C = { at least one tail}
- Some probabilities
 - P(A) = 1/4
 - P(B) = 1/2
 - P(C) = 3/4

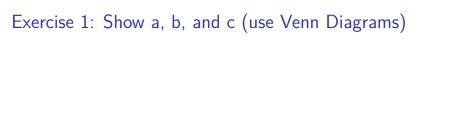
Key Property: Finite Additivity

If events A and B are disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

Using finite additivity, we can derive the following results

- a. For any event A, $P(A^c) = 1 P(A)$
 - ▶ In particular, since P(S) = 1 then $P(\emptyset) = 0$
- b. If $A \subset B$, then $P(A) \leq P(B)$
- c. $P(A \cup B) = P(A) + P(B) P(A \cap B)$



Exercise 2

Use experiment in example 1 (toss coin twice) and answer the following $\ensuremath{\mathsf{I}}$

- a. $P(A^c)$
- b. $P(A \cup B)$
- c. $P((A \cap C)^c)$

ISI CH3 Question 7 a-c

Finite Sample Spaces

- ▶ S is a finite sample space if there is $N \in \mathbb{N}$ such that $S = \{s_1, s_2, \dots, s_N\}$.
 - We do require s_1, \ldots, s_N to be different from each other, i.e., $s_i \neq s_i$ if $i \neq j$
 - A direct consequence is that the event $\{s_i\}$ is disjoint of $\{s_j\}$ for $i \neq j$
- The outcomes are equally likely if

$$P({s_1}) = P({s_2}) = \cdots = P({s_N})$$

Since $\{s_i\}$ is disjoint of $\{s_j\}$ for $i \neq j$, then using finite additivity

$$\sum_{i=1}^{N} P(\{s_i\}) = P(\bigcup_{i=1}^{N} \{s_i\}) = P(S) = 1$$

Finite Sample Spaces (continued)

▶ It follows that

$$P(\{s_i\}) = \frac{1}{N}$$

▶ For any event $A \subset S$, if S is finite with equally likely outcomes then

$$P(A) = \frac{\#A}{\#S}$$

where #A is the total number of outcomes in A

Exercise 3

Julia is among 30 students the may be randomly selected to form a committee of 3 people

- a. If the committee needs 1 president, 1 vice president, and 1 secretary, what is the probability that Julia is the president?
- b. If the committee needs 1 president, 1 vice president, and 1 secretary, what is the probability that Julia is part of this committee?
- c. If the committee needs 3 members without any given roles, what is the probability that Julia is part of this committee?

Conditional Probability

- ▶ If *B* is an event, we say that *B* has ocurred if one of the outcomes in *B* is the result of the experiment
 - In the two-coin example,
 B = { first coin is heads } = {HH, HT}. If B has occurred the resulting outcome was either HH or HT.
- The conditional probability of A given B, written P(A|B) is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1}$$

- One way to think about this is, if B has occurred, only the outcomes in B are possible (the sample space has been restricted). Now, what is the probability that A occurs?
- ▶ In the two-coin example $A = \{HH\}$ and P(A|B) = 1/2

Conditional Probability

▶ If we multiply both sides of (1) by P(B) we get

$$P(A \cap B) = P(B)P(A|B) \tag{2}$$

▶ This is the second formulation of conditional probability

Independence

Two events, A and B are independent if the probability of A does not change by the occurrence of B (or viceversa). When this happens, the following equations hold:

- $ightharpoonup P(A|B) = P(A|B^c)$
- ightharpoonup P(A|B) = P(A)
- $P(A \cap B) = P(A) \cdot P(B)$

ISI CH3 Question 7 d, e

Tree Diagrams: A and B are two events

Bayes' Rule

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(A^c) \cdot P(B|A^c)}$$

ISI CH3 Question 8 a - c