ON THE INJECTIVITY RADIUS IN HOFER'S GEOMETRY

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Abstract. *In this note we consider the following conjecture: given any closed symplectic manifold M, there is a sufficiently small real positive number ρ such that the open ball of radius ρ in the Hofer metric centered at the identity on the group of Hamiltonian diffeomorphisms of M is contractible, where the retraction takes place in that ball – this is the strong version of the conjecture – or inside the ambient group of Hamiltonian diffeomorphisms of M – this is the weak version of the conjecture. We prove several results that support that weak form of the conjecture.* [1] [2]

1. General facts and results

Consider a closed symplectic manifold (M,ω) of any dimension. We recall that the Hofer norm on the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of M assigns to each diffeomorphism $\phi \in \mathrm{Ham}(M)$ the infimum, over all Hamiltonians $H : M \times [0,1] \to \mathbb{R}$ whose time-one flow equals ϕ, of the mean total variation of H defined by

$$\int_0^1 (\max_M H_t - \min_M H_t)dt.$$

Given a real number $\rho \geq 0$, let us now denote by $B_H(\rho)$ the subspace of $\mathrm{Ham}(M)$ of all diffeomorphisms of Hofer norm smaller or equal to ρ.

Since the results of this paper concerns two different topologies on the group of Hamiltonian diffeomorphisms, let us make the following definition:

Definition 1. *Let X be a Fréchet manifold equiped with a C^∞ structure $\mathcal{T}$ (we will refer to this topology as the smooth or C^∞ topology of X). Endow X with a Finsler metric $\mathcal{F}$, that is to say a norm defined on each tangent space T_xX, $x \in X$, that varies smoothly with the point x. Such an object is a Finsler manifold. It induces naturally a pseudo-metric d on X by defining the distance between two points x, y as the infimum, over all smooth paths γ joining x to y, of the integral of the Finsler norm of the differential γ'. If, moreover, d is a genuine metric, we will say that the triple $(X, \mathcal{T}, d)$ is a metric Finsler manifold.*

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[2] Hofer's geometry, Lagrangian submanifolds, Quantum characteristic classes.

We have in mind here of course the example of the group of Hamiltonian diffeomorphisms of a given closed symplectic manifold endowed with the obvious smooth structure and with the metric coming from the Hofer metric.

In this paper, all homology, homotopy groups or homotopy equivalences will be defined with respect to the smooth C^∞ structure of $\mathrm{Ham}(M)$.

With this in mind, what is the topology of $B_H(\rho)$ when ρ goes to zero, or when ρ goes to ∞? When the manifold is a surface of genus larger than 0, it has been proved by Lalonde and McDuff [5] that the group $\mathrm{Ham}(M)$ has infinite diameter, i.e that $B_H(\rho)$ does not contain $\mathrm{Ham}(M)$ whatever value of ρ chosen. This was extended by Lalonde and Pestieau [6] to manifolds of the form $\Sigma \times M$ for M weakly exact and Σ the same kind of surface. The proof of the unboundedness of the group of Hamiltonian diffeomorphisms of the 2-sphere was given by Polterovich [7]. A natural conjecture is that, for ρ small enough, $B_H(\rho)$ is contractible (with respect to the smooth C^∞ topology). Note that the corresponding statement is always true for finite dimensional Finsler manifolds X, as the exponential map at x is always defined and is a diffeomorphism on a sufficiently small neighborhood of $0 \in T_xX$, see for example [12, Chapter 11]. Moreover the analogous statement holds for the group of volume preserving diffeomorphisms of a Riemannian manifold X, with its natural L^2-metric, see Ebin-Marsden [1]. In this latter case, however, it is a deep and difficult fact, although it is again essentially a statement about the existence of an exponential map in a neighborhood of the tangent bundle at a point. In the finite dimensional Finsler setting, the size of the largest ball in T_xX on which the exponential map is defined and is a diffeomorphism is called the injectivity radius at x. For a general metric space X, we may call the supremum of ρ's for which the ρ-ball around $x \in X$ is contractible, the *injectivity radius* at x, although in the finite dimensional Finsler setting, the classical injectivity radius is only a lower bound for the above generalization. Note here that in finite dimensions, the topology induced by the Finsler metric coincides with the underlying topology of the manifold; however, this need not be the case in infinite dimensions, for instance for the Hofer metric on $\mathrm{Ham}(M)$. Summarizing, if the conjecture holds, we have an interesting numerical invariant of a symplectic manifold (M, ω): the injectivity radius of $\mathrm{Ham}(M, \omega)$.

We may also ask if for ρ small enough, the inclusion map of $B_H(\rho)$ into $\mathrm{Ham}(M, \omega)$ is null-homotopic. We will refer to this as the weak conjecture since the contraction to a point of the ball $B_H(\rho)$ may then take place in the full ambient space $\mathrm{Ham}(M, \omega)$, instead of the ball itself. It is perhaps worth noting that the terminology *ball* for $B_H(\rho)$ might be misleading: $B_H(\rho)$ is not parametrized by a ball, it is a *subset* of $\mathrm{Ham}(M, \omega)$ whose topology might be a priori complicated. Note that for surfaces of genus $g \geq 1$, this weak conjecture is obvious because the whole group $\mathrm{Ham}(M, \omega)$ is contractible in that case.

Another related question, and indeed a possible way to approach the above conjecture, is to show that there is a $\rho > 0$ such that the space of smooth paths from the identity to $x \in B_H(\rho)$, minimizing the Hofer length up to δ, is contractible (with respect to the smooth topology) for some δ. From now on, such paths will be referred to as δ-minimizing. Denote the latter path space $P_\delta(id, x)$. We may in general ask for which ρ and which (M, ω) the inclusion map $P_\delta(id, x) \to P(id, x)$ is null homotopic, for all $x \in B_H(\rho)$. Interestingly, while this question may seem harder than the original conjecture, the theory of Gromov-Witten invariants, in particular quantum characteristic classes [10], gives a partial answer, which we now describe. For the moment, let us merely consider quantum classes as certain invariants of homotopy groups:

$$a \mapsto qc_{k-1}(a) \in QH(M),$$

for

$$a \in \pi_k \mathrm{Ham}(M, \omega) \simeq \pi_{k-1} \Omega \mathrm{Ham}(M, \omega),$$

(the shift by -1 is for consistency with [10]). When $k = 1$ we just get the Seidel invariant:

$$qc_0(a) = S(a),$$

see [11].

Definition 2. *We say that* **quantum classes detect rational homotopy groups** *of $Ham(M, \omega)$ if whatever k and an element $a \in \pi_k(Ham(M, \omega), \mathbb{Q})$ given, it vanishes as soon as $qc_{k-1}(a)$ vanishes.*

This is known to hold for example for $M = S^2$, and $M = \mathbb{CP}^2$ as in this case the Hamiltonian group retracts onto the compact subgroups $PSU(2)$, respectively $PSU(3)$ by a classical theorem of Smale, respectively classical theorem of Gromov, [2]. The non-zero rational homotopy groups are in degrees 3, respectively $3, 5$. These degrees are in the so called stable range: $0 \leq k - 1 \leq 2n - 2$, in the sense of [9], of rational homotopy groups of $PSU(n)$. The assertion that quantum classes detect homotopy groups of $PSU(2), PSU(3)$ then immediately follows from the main theorem of [9].

Before going further on, we mention that we will follow the following conventions: the homology is always over $\mathbb{Q}$ unless specified otherwise, and the quantum homology of a monotone symplectic manifold is also taken with $\mathbb{Q}$ coefficients and with $\mathbb{Z}_2$ grading.

Let (M, ω) be a monotone symplectic manifold, that is: $\omega = c \cdot c_1(TM)$, with monotonicty constant $c > 0$. Set $\hbar = min(c \cdot N, D(M))$ where N is the minimal positive Chern number $\langle c_1(TM), [u] \rangle$ over all u of u^*TM for $u : S^2 \to M$, and where $D(M)$ is the infinum over the positive Hofer length of non-contractible loops in $\mathrm{Ham}(M, \omega)$. If the above Chern numbers all vanish, set $\hbar = D(M)$. If $\pi_1 \mathrm{Ham}(M, \omega) = 0$, set $D(M) = \infty$.

Theorem 1. *Suppose we are given a monotone symplectic manifold (M, ω), for which quantum classes detect rational homotopy groups, then $\hbar > 0$ and the inclusion $i : P_{\delta/3,+}(id, x) \to P(id, x)$ vanishes on rational homotopy groups for $x \in B(\hbar/2 - \delta)$, for all $\delta > 0$.*

Proof. Here $P_{\delta/3,+}(id,x)$ denotes the space of paths minimizing the positive Hofer length functional:

$$L^+(\gamma) = \int_0^1 \max_M H^\gamma_t dt,$$

up to $\delta/3$, where H^γ is the generating function for γ normalized to have zero mean at each moment. Fix a $\delta/3$-minimizing $p_0 \in P(id,x)$. Given $f : S^k \to P_{\delta/3,+}(id,x)$, we get a map $\widetilde{f} : S^k \to \Omega \mathrm{Ham}(M,\omega)$, $\widetilde{f}(s) = f(s) \cdot p_0^{-1}$, for $\cdot$ the concatenation product. Clearly the length of each loop $\widetilde{f}(s)$ is less than $\hbar$. But then by the proof of [8, Lemma 3.2], $qc_k([\widetilde{f}])$ vanishes. Let us explain this. The invariant $qc_k([\widetilde{f}])$ is defined by counting pairs (u,s) for u a J_s-holomorphic section with some constraints, of the bundle $M \hookrightarrow X_s \to \mathbb{CP}^1$, obtained by using $\widetilde{f}_s$ as a clutching loop:

$$X_s = M \times D^2 \sqcup_{\widetilde{f}_s} M \times D^2,$$

where J_s is tamed by a symplectic form Ω_s on X_s, with both of these smoothly varying. Now (u,s) can contribute to the invariant only if

$$\langle c_1(T^{vert} X_s), [u] \rangle < 0,$$

for dimensional reasons that one can check easily. By assumption, each $\widetilde{f}_s$ is contractible and so as a smooth bundle $X_s \simeq M \times S^2$ this means that (u,s) can contribute only if $\langle c_1(T^{vert} X_s), [u] \rangle < -N$, so that

$$\langle \omega, [u] \rangle < -\hbar,$$

where ω is the natural form on X_{f_s} under the identification: $X_s \simeq M \times \mathbb{CP}^1$. This is the identification induced by any chosen contraction of $\widetilde{f}_s$. The result is independent of the identification as the Hamiltonian gauge group of $M \times \mathbb{CP}^1$ acts trivially on homotopy groups, see [3]. We can also make this point more transparent by using coupling forms, but we avoid introducing extra technology at this point. Finally [8, Lemma 3.2] tells us that the length of the loop $\widetilde{f}(s)$ must then be at least $\hbar$.

So we conclude that $\widetilde{f}$ is vanishing on rational homotopy groups, but then clearly the same must hold for $i \circ f$. □

The question of the injectivity radius can also be developed as follows. Given any class $\alpha \in H_*(\mathrm{Ham}(M), \mathbb{Z})$, define the *higher α-capacity* of (M,ω) as the infimum of ρ such that there exists a class $\xi \in H_*(B_H(\rho), \mathbb{Z})$ with $\iota_*(\xi) = \alpha$. Here ι is the injection of $B_H(\rho)$ inside $\mathrm{Ham}(M)$. Given any such non-zero α, the conjecture, if true, shows that the α-capacity is not zero. On the other hand, the following proposition shows that it is bounded above.

Proposition 1.1. *Given any closed symplectic manifold (M,ω) and any class $\alpha \in H_*(\mathrm{Ham}(M), \mathbb{Z})$, there is $\rho \geq 0$ such that α is realized inside $B_H(\rho)$.*

Proof. The image of a cycle α is a compact subset K of $\mathrm{Ham}(M)$. Suppose that ϕ belongs to K, with Hofer norm of ϕ denoted $E(\phi)$. Consider the ball of radius ε in Hofer's norm centered at ϕ. It contains an open set $U(\phi)$ centered at ϕ in the C^∞-topology because the C^∞ topology is finer than the Hofer topology. By the triangle inequality, the elements of $U(\phi)$ have Hofer norm at most $E(\phi)+\varepsilon$. Because K is compact, there is a finite collection of these open sets. □

Thus higher capacities belong to $(0,\infty)$ if the conjecture is true. For $M=S^2$, we show that:

Proposition 1.2. *There is ρ small enough so that the injection $B_H(\rho)\to \mathrm{Ham}(S^2)$ does not catch any of the $\mathbb{Z}$-generators of the $\mathbb{Z}$-homology of* $\mathrm{Ham}(S^2)$.

Proof. By remarks following Theorem 3 there is a $\rho_0>0$ such that all loops in $B_H(\rho_0)$ are contractible inside $\mathrm{Ham}(S^2)$. Thus the generator γ of the fundamental group of $\mathrm{Ham}(S^2)$ is not homologous inside $\mathrm{Ham}(S^2)$ to a 1-cycle lying in $B_H(\rho_0)$ since otherwise, the concatenation of the chain realizing this homology with the disc realizing the homotopy to a point would give a chain in $\mathrm{Ham}(S^2)\simeq SO(3)$ realizing a homology between γ and a point, a contradiction. This proves our statement for the generator of the first homology group of $\mathrm{Ham}(S^2)$.

Now suppose by contradiction that there is a singular 4-chain c over the integers in $\mathrm{Ham}(S^2)$ such that $\partial c=[SO(3)]-d$ where $d\in Z_3(B_H(\rho_0))$, Z_3 being the 3-cycles over the integers. Let (K,f) be the realization of c, i.e let K be a compact simplicial complex and $f:K\to \mathrm{Ham}(S^2)$ be a continuous map that realizes the homology c. Thus K is collection of simplices with integral coefficients, and with the incidence relations dictated by c. One may realize K as a piecewise linear complex with integral coefficients in some $\mathbb{R}^N$, for N large enough. The boundary of K is by definition the sum over all simplices of top dimension of the boundary of each such simplex with the coefficient coming from the simplex. By definition of K, this boundary is equal to K_0-K_1 where the subcomplex K_0 of K is an abstract singular triangulation of $[SO(3)]$ and the restriction f_0 of the map f to K_0 identifies K_0 with a singular triangulation of $[SO(3)]\subset \mathrm{Ham}(S^2)$, while the restriction f_1 of f identifies K_1 with the cycle d. Now let's compose f with the retraction $r:\mathrm{Ham}(S^2)\to SO(3)$. This gives a map $g:K\to SO(3)\subset \mathrm{Ham}(S^2)$. Now, $SO(3)$ is a smooth manifold and hence there is a continuous map $g':K\to SO(3)$ C^0-close to g and homotopic to g such that the restriction of g' to each simplex is a smooth map. Thus, if γ is a smooth curve of $SO(3)$ representing the generator of $\pi_1(SO(3))$ and transverse to all simplices of (K,g'), the inverse image $g'^{-1}(\gamma)$ of γ by g' is a 2-dimensional subspace of K that can be represented, up to smooth subdivision of K, by a smooth subcomplex $L\subset K$ with weights (coming from each simplex) with boundary equal to $\Gamma-\Gamma'$ where Γ is a 1-cycle inside K_0 mapped to γ by g' and where Γ' is a subset of K_1.

Finally, note that f maps the cycle Γ to the 1-cycle γ of $SO(3)$ and the cycle Γ' inside $B_H(\rho)$. But Γ and Γ' are homologous inside K, thus their images by f are homologous inside $\mathrm{Ham}(S^2)$. This means that there is a 1-cycle inside $B_H(\rho)$ that is homologous to the generator γ of $\mathrm{Ham}(S^2)$, a contradiction.

□

Theorem 2. *Let $(X, \mathcal{T}, d)$ be a metric Finsler manifold. Let D be the infimum of the lengths of all non contractible, smooth loops based at x_0 in X. Here the topology used for the definition of a loop and for the (non)-retraction of a loop to a point, is the smooth $\mathcal{T}$ topology, while the length is of course measured with respect to the Finsler metric. Let $B_{x_0,\varepsilon}$ be the ball of radius ε centered at $x_0 \in X$, with $\varepsilon = D/2 - \delta$. Here the distance is measured with respect to the Finsler metric d. Then the inclusion of $B_{x_0,\varepsilon}$ inside X vanishes on π_1 for all $\delta > 0$ where loops and retractions are defined with the smooth $\mathcal{T}$ topology.*

Proof. Let E be the space of all pairs (b, γ) where $b \in B_{x_0,\varepsilon}$ and γ is a smooth path in X starting at x_0 and ending at $b \in B_{x_0,\varepsilon}$ in the minimizing homotopy class relative to endpoints. By our assumptions and because the shortest length of a non contractible path is at least D, this class is uniquely determined. For a loop $\gamma : S^1 \to B := B_{x_0,\varepsilon}$, consider the pullback $E' = \gamma^* E$ of $E \to B$. We are going to show that E' has a smooth section over $S^1 = [0, 1]/\sim$. Set $b_0 = \gamma(0)$ and let p_0 be any smooth path from id to b_0 minimizing length up to $\delta/3$; we will just say $\delta/3$-minimizing from now on. In particular p_0 is in the fiber of E over b_0 – if it were in the wrong homotopy class its length would be at least $D/2$. Partition S^1 into segments s_i so that the length of each segment $\gamma|s_i$ be at most $\delta/3$. For simplicity say there are 2 segments. Then we have a canonical section of E' over s_0, which is p_0 over 0, and over $t \in s_0$ it is $p(t)$ defined by concatenating p_0 with $\gamma|_{[0,t]}$. Since each $p(t)$ has length less than $D/2$, it is in the right class, since otherwise its length would be at least $D/2$. But because each $p(t)$ is in the right class, we can change this section, using the homotopy extension property, so that over the right end point $t = 1/2$, $p(1/2)$ again minimize length up to $\delta/3$. Repeating this we get a section of E' which is 2 valued only over 0, but since the fiber is connected we can adjust it to be an actual smooth section. Given this section, we can contract γ by the associated family of paths. □

As a corollary we have:

Theorem 3. *Let (M, ω) be a symplectic manifold such that there is a lower bound D for the Hofer length of a non-contractible loop in $\mathrm{Ham}(M)$. Then the inclusion of $B_H(\rho)$ into $\mathrm{Ham}(M)$ vanishes on π_1 for $\rho = D/2 - \varepsilon$ for all $\varepsilon > 0$.*

By Lalonde-McDuff results in [5], any ruled symplectic 4-manifold or any surface satisfies this hypothesis.

Let us now consider the conjecture stating that the space $B_H(\rho)$ of all Hamiltonian diffeomorphisms of S^2 of Hofer norm less or equal to ρ is contractible inside $\mathrm{Ham}(S^2)$, for ρ small enough (weak conjecture). Let us denote by $L(\rho)$ the topological space of all images of the standard oriented equator $L \subset S^2$ by Hamiltonian diffeomorphisms of Hofer's norm less or equal to ρ. So $L(\rho)$ is included in the space $L(\infty)$ of smooth oriented embedded loops that divide the sphere into two regions of equal areas.

Proposition 1.3. *If $L(\rho)$ is contractible in $L(\infty)$, then the space $B_H(\rho)$ is also contractible in* $\mathrm{Ham}(S^2)$.

Proof. Consider the map $B_H(\rho) \to L(\rho)$ that assigns to each diffeomorphism ϕ the image under ϕ of the standard oriented equator in S^2. It is not hard to verify that this is a Serre fibration. This is a sub fibration of the Serre fibration $\mathrm{Ham}(S^2) \to L(\infty)$. The fiber F of this sub fibration is the space of all Hamiltonian diffeomorphisms of Hofer norm less or equal to ρ that preserve, not necessarily pointwise, the equator. The same space, but with no restriction on norm, is homotopy equivalent to S^1 since it retracts to the space of rotations of the closed disk. Let's denote it by F'.

Now assume that ρ is small enough so that the hypothesis of Theorem 3 be satisfied. Then the fiber F of $B_H(\rho) \to L(\rho)$ is a subspace of $F' \subset \mathrm{Ham}(S^2)$ that does not contain its S^1 generator, which implies that it is contractible in $\mathrm{Ham}(S^2)$. To see this in detail, consider the retraction of $\mathrm{Ham}(S^2)$ onto $SO(3)$; it takes F' to the space R of rotations round a fixed axis, that is to say to some $S^1 \subset SO(3)$. In this retraction, the fiber F is carried to a subspace of R. By Theorem 3, the fundamental group of F vanishes inside $\mathrm{Ham}(S^2)$ and therefore inside $SO(3)$ (after retraction). Thus F is homotopy equivalent to a subspace of R whose fundamental group vanishes in $SO(3)$. It must therefore be contractible in $SO(3)$, and consequently F is contractible in $\mathrm{Ham}(S^2)$.

Now use the fact that the base $L(\rho)$ is contractible inside $L(\infty)$ to retract each of the fibers of $B_H(\rho) \to L(\rho)$ onto the fiber at a point $L' \in L(\rho) \subset L(\infty)$. This retraction takes place in $\mathrm{Ham}(S^2)$. Then compose with the retraction of that fiber to a point inside $\mathrm{Ham}(S^2)$. $\square$

So the problem of the contractibility of $B_H(\rho)$ inside $\mathrm{Ham}(S^2)$ reduces to the problem of the contractibility of $L(\rho)$ inside $L(\infty)$.

2. The space $L(\rho)$ and the double octopus

The main question is: is it possible to find a 2-cycle inside $L(\rho)$, for arbitrarily small ρ, homologous in $L(\infty)$ to the S^2-cycle made of linear Lagrangians?

One way of approching that question is to look for the obstructions in designing an algorithm that would retract all exact Lagrangians sufficiently close to the standard oriented equator L to L. This is what we do in this section.

Each $L' \in L(\rho)$ comes with an orientation that defines two sides $H_+(L')$ and $H_-(L')$ (H for "hemisphere"). Here $H_+(L)$ and $H_-(L)$ are the standard upper and lower hemispheres. Set:

$$R_+ = H_-(L') \cap H_+(L)$$

$$R_- = H_+(L') \cap H_-(L)$$

$$G_+ = H_+(L') \cap H_+(L)$$

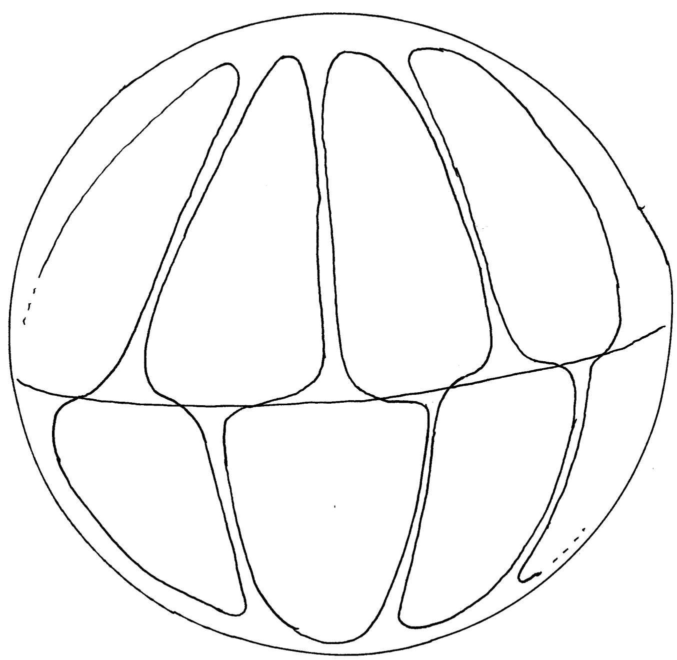
$$G_- = H_-(L') \cap H_-(L)$$

Lemma 2.1. *Each connected component R of R_+ or R_- has area bounded above by ρ.*

Proof. Lift L' to some Hamiltonian diffeomorphism ϕ of energy less or equal to ρ. Then ϕ^{-1} sends each connected component R of R_+ to the lower standard hemisphere. But such a connected component lives in the upper standard hemisphere, and so ϕ^{-1} displaces it. Thus, by the energy-capacity inequality, ρ is greater or equal to the area of R. The same argument applies as well if R is a component of R_-. □

In order to prove that $L(\rho)$ is contractible in $L(\infty)$, one would like to define an algorithm that retracts L' to L inside $L(\infty)$ in a canonical way, i.e in a way that depends continuously on L'. Note that $L(\infty)$ contains the space LL of linear Lagrangians, i.e the one consisting of all oriented great circles. This space is identified to S^2 in the obvious way, with the south pole of LL being the standard equator L and the north pole being the same equator with the opposite orientation L^{opp}. On $LL - \{L^{opp}\}$, there is a retraction to L given by reducing simultaneously the areas of the two 2-gons R_+ and R_-. Of course, the continuity of this argument breaks down at L^{opp} since the direction of the retraction depends on the slight perturbation of L^{opp} inside LL that one chooses. Note of course that, near L^{opp}, the regions R_+ and R_- are big and such configurations cannot appear in $L(\rho)$ for small enough ρ.

However one can first slightly perturb L^{opp} so that it be the graph L' of a small sinusoidal function of L. So, if the number of points in $L \cap L'$ is $2k$, the 2-sphere decomposes into a large $2k$-gon R_+ in the upper hemisphere, a large $2k$-gone R_- in the lower hemisphere, and a sequence of small alternating 2-gones of G_+ and G_-. Note that L' is in general position with respect to L. Now any reasonable algorithm would retract L' to L^{opp} and would break there. However, by the energy-capacity inequality, such a configuration L' cannot belong to $L(\rho)$. The last step is to start with L' and inflate each 2-gone of $G_+ \cup G_-$ in its own hemisphere in such a way that, at the end of this inflation, we get an element L'' of $L(\rho)$ which has both a $(\mathbb{Z}/k\mathbb{Z})$-symmetry and a $\mathbb{Z}_2$-symmetry and is made of: a $2k$-gone in R_+ whose center is at the north pole, which is an arbitrarily small thickening of a star with k branches, the end of each branch being on the equator; a $2k$-gone in R_- whose center is at the south pole, which is an arbitrarily small thickening of a star with k branches and such that the ends of the branches in R_- meet the equator at mid-points between the ends of branches of R_+; a sequence of large 2-gones alternating between G_+ and G_-. Each 2-gone in G_- can be viewed as the prolongation of a branch of R_+ as the branch crosses the equator, and similarly for G_+. This is what we call the *double octopus.* It is made of two octopuses, one based at the north pole and the other at the south pole, with very thin body and legs, both with large feet (a foot is the part of a leg that crosses the equator).



Denote by $\mathcal{O}_{k,a}$ this oriented exact Lagrangian of S^2: here k is the number of legs of each of the two octopuses, and a is the area of the intersection of the standard upper hemisphere with the upper octopus (that is to say the one based at the North pole). Thus k may be as large as we wish and a as small as we wish. Hence $\mathcal{O}_{k,a} \in L(\infty)$ has all the properties of an element of $L(\rho)$ for small ρ: both of its R_+ and R_- are arbitrarily small. If $\mathcal{O}_{k,a}$ belongs to $L(\rho)$ for small ρ, there is not much hope to construct a retraction of $L(\rho)$ to L inside $L(\infty)$.

Khanevsky and Zapolsky [4] observed that actually the double octopus does not constitute a counter-example to the main conjecture of this paper. Here is their observation:

Proposition 2.2. *For each small enough ρ, the double octopus configuration $\mathcal{O}_{k,a}$ (whatever the value $k \geq 2$ and for each a small enough) does not lie in $L(\rho)$.*

Proof. The distance between L and L^{opp} is equal to the area of S^2, that is say to 4π. Let ρ be given. Consider $\mathcal{O}_{k,a}$. Assume that it belongs to $L(\rho)$. The distance from $\mathcal{O}_{k,a}$ to L^{opp} is less or equal to the area of two consecutive legs, that is to say to $(4\pi/k) - \varepsilon$. Then, by the triangle inequality:

$$d(L, \mathcal{O}_{k,a}) \geq |d(L, L^{opp}) - d(L^{opp}, \mathcal{O}_{k,a})|$$

so this means

$$d(L, \mathcal{O}_{k,a}) \geq 4\pi - 4\pi/k + \varepsilon = \frac{4\pi(k-1)}{k} + \varepsilon$$

which shows that octopuses with $k \geq 2$ and a sufficienttly small cannot lie in arbitrarily small balls around the standard equator L.

□

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