Notes for the Master Thesis

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Abstract

This is a collection of notes pertaining to concepts I needed to learn for my master's thesis.

Contents

1 Mathematical Prior

The Residue Theorem The residue theorem states that for a function $f(z)$ with a pole of order n at z_0, the integral of f about a positively oriented contour around z_0 satisfies

$$\oint \frac{\mathrm{d}z}{2\pi i} f(z) = \mathrm{Res}(f, z_0),$$

with

$$\mathrm{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left((z - z_0)^n f(z)\right).$$

The Lie Derivative For a tensor field T we define the Lie derivative in the X-direction as

$$\mathcal{L}_X T = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\gamma^{\star}_{\varepsilon X} T - T\right).$$

As the definition is similar to that of the usual derivative, it follows that it is linear in its argument and satisfies the product rule, where the product in question is now the tensor product. We have

$$(\mathcal{L}_X T)^{a_1 \ldots a_n}_{b_1 \ldots b_m} = X^a \partial_a T^{a_1 \ldots a_n}_{b_1 \ldots b_m} - \sum_{i=1}^{n} T^{a_1 \ldots a_{i-1} a a_{i+1} \ldots a_n}_{b_1 \ldots b_m} \partial_a X^{a_i} + \sum_{j=1}^{m} T^{a_1 \ldots a_n}_{b_1 \ldots b_{i-1} a b_{i+1} \ldots b_m} \partial_{b_j} X^a.$$

Pushforwards and Pullbacks Consider some function f which maps a manifold M_1 to another manifold M_2, as well as a function $g : M_2 \to \mathbb{R}$. We then define the pullback of g to M_1 by f as $f^{\star} g = g \circ f$. We also define the pushforward of a vector $V \in T_p M_1$ by f as the map $T_p M_1 \to T_p M_2$ such that $(f_{\star} V)\phi = V(f^{\star}\phi)$.

We can also define the pullback of a $(0, m)$ tensor on M_2 by f according to

$$f^{\star}\omega(V_1, \ldots, V_m) = \omega(f_{\star} V_1, \ldots, f_{\star} V_m).$$

Its components are

$$(f^{\star}\omega)_{a_1 \ldots a_m} = \omega_{\mu_1 \ldots \mu_m} \prod_{i=1}^{m} \partial_{a_i} f^{\mu_i}.$$

Finally, if f is a bijection we may also define the more general pullback of a (n, m) tensor on M_2 as

$$f^{\star} T(V_1, \ldots, V_m, \omega_1, \ldots, \omega_n) = T(f_{\star} V_1, \ldots, f_{\star} V_m, (f^{-1})^{\star}\omega_1, \ldots, (f^{-1})^{\star}\omega_n).$$

Its components are

$$(f^{\star} T)^{b_1 \ldots b_n}_{a_1 \ldots a_m} = T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m} \prod_{i=1}^{m} \partial_{a_i} f^{\nu_i} \prod_{j=1}^{n} \partial_{\mu_j} (f^{-1})^{b_j}.$$

Properties of the Pullback One important property of the pullback is its symmetry preservation. To study it, it will be sufficient to study the pullback of a $(0, 2)$ tensor, as the proof generalizes to an arbitrary number of indices.

We have

$$\begin{aligned}(f^{\star} T)_{ab} \pm (f^{\star} T)_{ba} &= T_{\mu\nu} \partial_a f^{\mu} \partial_b f^{\nu} \pm T_{\mu\nu} \partial_b f^{\mu} \partial_a f^{\nu} \\ &= (T_{\mu\nu} \pm T_{\nu\mu}) \partial_b f^{\mu} \partial_a f^{\nu},\end{aligned}$$

which implies that (anti)symmetry is preserved by the pullback. The important property is that the pullback is linear in the tensor that is pulled back, and as such the argument generalizes to (anti)symmetry with respect to an arbitrary number of indices.

Embeddings and Immersions Consider two manifolds M and N connected by a map f. f induces both the pullback and pushforward, at least when restricted to tangent and dual vectors, and it turns out that if one of these is invertible, so is the other. If f induces an invertible pushforward and pullback is called an immersion of M into N, and if f is itself invertible, it is called an embedding of M into N. The image $f(M)$ is then called a submanifold of N.

One noteworthy thing about embeddings is the induction of tensors through them, the metric being a great example. We have

$$\tilde{g}_{ab} = (f^\star g)_{ab} = g_{\mu\nu}\partial_a f^\mu \partial_b f^\nu,$$

and so the length element is

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = g_{\mu\nu}\partial_a f^\mu \partial_b f^\nu\,\mathrm{d}x^a\,\mathrm{d}x^b = g_{\mu\nu}\,\mathrm{d}y^\mu\,\mathrm{d}y^\nu = \mathrm{d}s^2\,.$$

That is, this induced metric preserves the notion of length under the mapping.

Differential Forms The set of p-forms, or differential forms, is the set of $(0,p)$ tensors that are completely antisymmetric. They are constructed using the wedge product, defined as

$$\bigwedge_{k=1}^{p} d\chi^{\mu_k} = \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) \bigotimes_{k=1}^{p} d\chi^{\mu_{\sigma(k)}}.$$

Here S_p is the set of permutations of p elements. There exists

$$n_p^N = \binom{N}{k}$$

basis elements. We note that the wedge product is antisymmetric under the exchange of two basis elements. Hence, once an ordering of indices has been chosen, any permutation will simply create a linearly dependent map.

Consider now some antisymmetric tensor ω. Introducing the antisymmetrizer

$$\bigotimes_{k=1}^{p} d\chi^{[\mu_{\sigma(k)}]} = \frac{1}{p!}\sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) \bigotimes_{k=1}^{p} d\chi^{\mu_{\sigma(k)}},$$

the symmetry yields

$$\omega = \omega_{\mu_1\ldots\mu_p} \bigotimes_{k=1}^{p} d\chi^{\mu_{\sigma(k)}} = \omega_{\mu_1\ldots\mu_p} \bigotimes_{k=1}^{p} d\chi^{[\mu_{\sigma(k)}]} = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} \bigwedge_{k=1}^{p} d\chi^{\mu_k}.$$

In other words, we can antisymmetrize the components of ω to write it as a differential form.

Two important classes of differential forms are exact and closed forms. An exact form is the exterior derivative of another, while a closed form has zero external derivative.

The Exterior Derivative We define the exterior derivative of a differential form according to

$$d\omega = \frac{1}{p!}\partial_{\mu_1}\omega_{\mu_2\ldots\mu_{p+1}} \bigwedge_{k=1}^{p+1} d\chi^{\mu_k},$$

which is a $p+1$-form. By its nature this is an antiderivative - that is,

$$d(a\wedge b) = da\wedge b + (-1)^p a\wedge db,$$

where p is the rank of a.

Differential Forms and Pullbacks We will now briefly show some properties of differential forms related to the pullback.

The first is, for two maps $f : M_1 \to M_2$ and $g : M_2 \to M_3$, that

$$((g \circ f)^\star \omega)(X_1, \dots, X_n) = \omega((g \circ f)_\star X_1, \dots, (g \circ f)_\star X_n),$$

with ω being an n-form on M_3 and X_i vectors on M_1. For a vector we have

$$((g \circ f)_\star X)(\phi) = X((g \circ f)^\star \phi) = X(\phi \circ (g \circ f)).$$

As function compositions are associative, this implies

$$(g \circ f)_\star X = g_\star f_\star X, \tag{1}$$

and thus

$$\begin{aligned}((g \circ f)^\star \omega)(X_1, \dots, X_n) =& \omega(g_\star f_\star X_1, \dots, g_\star f_\star X_n) \\ =& g^\star \omega(f_\star X_1, \dots, f_\star X_n) \\ =& f^\star g^\star \omega(X_1, \dots, X_n),\end{aligned}$$

and thus $(g \circ f)^\star = f^\star \circ g^\star$.

Next,

$$\begin{aligned}f^\star(\omega \wedge \eta)(X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}) =& \omega(f_\star X_1, \dots, f_\star X_n) \wedge \eta(f_\star X_{n+1}, \dots, f_\star X_{n+m}) \\ =& f^\star \omega(X_1, \dots, X_n) \wedge f^\star \eta(X_{n+1}, \dots, X_{n+m}),\end{aligned}$$

and thus $f^\star(\omega \wedge \eta) = f^\star \omega \wedge f^\star \eta$.

This can be used to study

$$f^\star \left(\phi d\psi_1 \wedge \dots \wedge d\psi_n\right) = f^\star \phi \cdot f^\star(d\psi_1 \wedge \dots \wedge d\psi_n).$$

We have

$$f^\star d\psi(X) = d\psi(f_\star X) = (f_\star X)(\psi) = X(f_\star \psi) = X(\psi \circ f) = d(\psi \circ f)(X),$$

and thus $f^\star d\psi = d(\psi \circ f)$, implying

$$f^\star \left(\phi d\psi_1 \wedge \dots \wedge d\psi_n\right) = (\phi \circ f) d(\psi_1 \circ f) \wedge \dots \wedge d(\psi_n \circ f).$$

A final important property comes from studying

$$\begin{aligned}f^\star(d\omega) =& f^\star \left(\frac{1}{n!} \partial_{\mu_1} \omega_{\mu_2 \dots \mu_{n+1}} d\chi^{\mu_2} \wedge \dots \wedge d\chi^{\mu_{n+1}}\right) \\ =& d\left(\frac{1}{n!} \omega_{\mu_2 \dots \mu_{n+1}} \circ f\right) d(\chi^{\mu_2} \circ f) \wedge \dots \wedge d(\chi^{\mu_{n+1}} \circ f).\end{aligned}$$

The pullback of the 1-forms nets you basis vectors in the manifold to which you pull back, and thus you have

$$f^\star(d\omega) = d(f^\star \omega).$$

Interior Multiplication The interior multiplication of a differential form with a vector is defined as

$$i_X \omega(\dots) = \omega(X, \dots).$$

Its action on a 0-form is defined to yield zero. In index notation it is the contration of the first index of ω by X. It satisfies

$$i_X(\eta \wedge \omega) = i_X(\eta) \wedge \omega + (-1)^p \eta \wedge (i_X \omega),$$

where p is the rank of η.

The Infinitesimal Homotopy Relation The infinitesimal homotopy relation relates the Lie derivative and the interior product. We will prove it by induction. First, for a 1-form, we have

$$\begin{aligned}
d(i_X\omega)+i_Xd\omega =&\partial_a(X^b\omega_b)d\chi^a+\partial_a\omega_b d\chi^a(X)\wedge d\chi^b\\
=&X^b\partial_a\omega_b d\chi^a+\partial_a(X^b)\omega_b d\chi^a+\partial_b\omega_a d\chi^b(X)\wedge d\chi^a\\
=&\left(X^b\partial_a\omega_b+\partial_a(X^b)\omega_b+X^b\partial_b\omega_a-X^b\partial_a\omega_b\right)d\chi^a\\
=&\left(\partial_a(X^b)\omega_b+X^b\partial_b\omega_a\right)d\chi^a\\
=&\mathcal{L}_X\omega.
\end{aligned}$$

We can see that this applies for 0-forms as well, as the first term will not be present in that case due to $i_xf=0$ by definition. Finally, using the product rule we have

$$\begin{aligned}
d(i_X(\omega\wedge\eta))+i_Xd(\omega\wedge\eta) =&d(i_X\omega\wedge\eta+(-1)^p\omega\wedge i_X\eta)+i_X(d\omega\wedge\eta+(-1)^p\omega\wedge d\eta)\\
=&d(i_X\omega)\wedge\eta+(-1)^{p-1}i_X\omega\wedge d\eta+(-1)^p(d\omega\wedge i_X\eta+(-1)^p\omega\wedge d(i_X\eta))\\
&+i_X(d\omega)\wedge\eta+(-1)^{p+1}d\omega\wedge i_X\eta+(-1)^p(i_X\omega\wedge d\eta+(-1)^p\omega\wedge i_X(d\eta))\\
=&(d(i_X\omega)+i_X(d\omega))\wedge\eta+(-1)^p\omega\wedge(d(i_X\eta)+i_X(d\eta)).
\end{aligned}$$

What we have shown here is that this operator follows the product rule under the wedge product. As it applies for 1-forms and 0-forms, from which higher forms can be constructed, it follows that

$$\mathcal{L}_X\omega=d(i_X\omega)+i_Xd\omega$$

for any differential form.

The Volume Form The metric gives a notion of distance, and therefore a notion of volume. This is what we will discuss now. That volume is described by a differential form follows from our notion of orientation of volumes and surfaces in two and three dimensions. As such, the volume form η should be of the form

$$\eta=f(\chi)d\chi^1\wedge\cdots\wedge d\chi^n.$$

The form of f is as of yet unknown.

Let us consider real space first. In three dimensions we have

$$\mathrm{d}V=\varepsilon_{ijk}\partial_1r^i\partial_2r^j\partial_3r^k\,\mathrm{d}\chi^1\,\mathrm{d}\chi^2\,\mathrm{d}\chi^3\,,$$

with the Cartesian position vector $\mathbf{r}(\chi)$ acting as the embedding of coordinate space in $\mathbb{R}^3$. The prefactor for the differential must be proportional to ε_{123} by matching with the chase where $\chi^i=r^i$, and because the metric in $\mathbb{R}^3$ is diagonal with elements 1, the rest must be equal to $\sqrt{\det(g)}$. We thus infer

$$\eta=\phi(\chi)\sqrt{\det(g)}d\chi^1\wedge\cdots\wedge d\chi^n.$$

To identify ϕ, we consider orthogonal coordinates. In this case the metric is diagonal with elements h_i^2. On the other hand we expect a small cube spanned by vectors $X_a=\mathrm{d}t\,\partial_a$ to have volume equal to $\mathrm{d}t^n\,h_1\dots h_n$, and thus we must have $\phi=1$. Thus the volume form is

$$\eta=\sqrt{\det(g)}d\chi^1\wedge\cdots\wedge d\chi^n.$$

The volume of a manifold is slightly less trivial to define, but we can do it if the manifold is embedded by f in some manifold with a volume form (the typical case is a manifold defined by a surface in real space). The idea we start with is flux. For a surface in real space, assuming the surface to be parametrized by s^i and have tangent vectors $\dot\gamma_i\,\mathrm{d}s^i$ (no sum) starting at a point and extending to some region limited by variations $\mathrm{d}s^i$, the flux of a field F is known to be equal to the volume spanned by F and the tangents. This suggests that

$$\mathrm{d}\Phi=\eta(F,\dot\gamma_1\,\mathrm{d}s^1\,,\dots,\dot\gamma_{n-1}\,\mathrm{d}s^{n-1})=i_J\eta(\dot\gamma_1\,\mathrm{d}s^1\,,\dots,\dot\gamma_{n-1}\,\mathrm{d}s^{n-1}).$$

In particular, the volume can be computed using the flux of a field normal to the surface with unit length. Dubbing this field ν the volume form of the manifold is then

$$\tilde\eta=f^\star(i_\nu\eta).$$

Let us now consider the exactness and closedness of the volume form. First of all, the volume form is not exact globally on a compact manifold (one without a boundary). Assuming the contrary, i.e. $d\alpha = \eta$, we find

$$\int\limits_M \eta = \int\limits_{\partial M} \alpha.$$

The left-hand side ought to be non-zero by construction, but the right-hand side clearly is, making the contradiction clear. A way to make this more flexible is to split M into two parts, and we then have

$$\int\limits_M \eta = \int\limits_{M_1} \eta + \int\limits_{M_2} \eta.$$

Assuming exactness on each half we have

$$\int\limits_M \eta = \int\limits_{\partial M_1} \alpha_1 + \int\limits_{\partial M_2} \alpha_2 = \int\limits_{\partial M_1} \alpha_1 - \alpha_2.$$

Matrix-Valued Differential Forms A matrix-valued differential form is a differential form whose components are matrices. For these we need to define a slightly different wedge product according to

$$(A \wedge B)^a{}_b = A^a{}_c \wedge B^c{}_b.$$

In words, its components are found by computing the matrix product of the corresponding components of A and B, but using the wedge product instead of the normal multiplication. The output of this is then a new matrix-valued differential form. Their exterior derivatives are defined as for normal differential forms. We will use greek indices for the differential form structure and latin indices for the matrix structure.

Non-Abelian Gauge Theory - an Example We define the field strength 2-form

$$F = dA + A^2,$$

where we now suppress wedge products, as tensor products will not appear. By definition we have

$$A^2 = (A_\mu d\chi^\mu) \wedge (A_\nu d\chi^\nu) = \frac{1}{2}(A_\mu A_\nu - A_\nu A_\mu) d\chi^\mu d\chi^\nu = \frac{1}{2}[A_\mu, A_\nu] d\chi^\mu d\chi^\nu.$$

Now, F is a differential form, meaning we can write $F = \frac{1}{2} F_{\mu\nu} d\chi^\mu d\chi^\nu$. As for dA we have

$$dA = \partial_\mu A_\nu d\chi^\mu d\chi^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) d\chi^\mu d\chi^\nu,$$

where we in the last step explicitly antisymmetrized the result. Thus we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Next, defining the gauge covariant derivative

$$\vec{\boldsymbol{\nabla}}_\mu = \partial_\mu + A_\mu.$$

We then have

$$\left[\vec{\boldsymbol{\nabla}}_\mu, \vec{\boldsymbol{\nabla}}_\nu\right] = [\partial_\mu, \partial_\nu] + [A_\mu, A_\nu] + \left[\partial_\mu, A_\nu\right] - [\partial_\nu, A_\mu].$$

For the first commutator, all components have the same matrix structure, so they commute. For the last two terms we will need to use the product rule to find

$$\left[\partial_\mu, A_\nu\right] = \partial_\mu A_\nu - A_\nu \partial_\mu = (\partial_\mu A_\nu) + A_\nu \partial_\mu - A_\nu \partial_\mu = (\partial_\mu A_\nu),$$

with the brackets highlighting the terms that are self-contained and do not act as operators. Thus we have

$$\left[\vec{\boldsymbol{\nabla}}_\mu, \vec{\boldsymbol{\nabla}}_\nu\right] = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + [A_\mu, A_\nu] = F_{\mu\nu}.$$

Next, for two vector fields we have

$$F(X,Y) = \frac{1}{2}\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right] d\chi^\mu(X) d\chi^\nu(Y) = \frac{1}{2}\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right](X^\mu Y^\nu - X^\nu Y^\mu).$$

We have

$$\vec{\nabla}_\rho X^\mu Y^\nu = \partial_\rho X^\mu Y^\nu + A_\rho(X^\mu Y^\nu) = Y^\nu \partial_\rho(X^\mu) + X^\mu \partial_\rho(Y^\nu) + X^\mu Y^\nu \partial_\rho + A_\rho X^\mu Y^\nu,$$

and in particular

$$\begin{aligned}
\vec{\nabla}_\nu X^\mu Y^\nu =& Y^\nu \partial_\nu(X^\mu) + X^\mu \partial_\nu(Y^\nu) + X^\mu Y^\nu \partial_\nu + A_\nu X^\mu Y^\nu = Y^\nu \partial_\nu(X^\mu) + X^\mu \partial_\nu(Y^\nu) + X^\mu \vec{\nabla}_Y, \\
\vec{\nabla}_\mu X^\mu Y^\nu =& Y^\nu \partial_\mu(X^\mu) + X^\mu \partial_\mu(Y^\nu) + Y^\nu \vec{\nabla}_X.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\vec{\nabla}_\mu \vec{\nabla}_\nu X^\mu Y^\nu =& \partial_\mu Y^\nu \partial_\nu(X^\mu) + \partial_\mu X^\mu \partial_\nu(Y^\nu) + \partial_\mu X^\mu \vec{\nabla}_Y + A_\mu X^\mu \vec{\nabla}_Y \\
=& \partial_\mu Y^\nu \partial_\nu(X^\mu) + \partial_\mu X^\mu \partial_\nu(Y^\nu) + (\partial_\mu X^\mu)\vec{\nabla}_Y + \vec{\nabla}_X \vec{\nabla}_Y, \\
\vec{\nabla}_\nu \vec{\nabla}_\mu X^\mu Y^\nu =& \partial_\nu Y^\nu \partial_\mu(X^\mu) + \partial_\nu X^\mu \partial_\mu(Y^\nu) + (\partial_\nu Y^\nu)\vec{\nabla}_X + \vec{\nabla}_Y \vec{\nabla}_X
\end{aligned}$$

The final result is found by first computing the difference of the above. One then notes down the result of swapping X and Y in that difference and subtracting that from what you have. First, the two connections net their commutator. The lone connections are found twice after the subtaction. Next, for the other terms the derivative can act on either factor or move to the right. The two former have terms with opposite sign cancelling them, and

$$\begin{aligned}
\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right](X^\mu Y^\nu - X^\nu Y^\mu) =& 2\left[\vec{\nabla}_X, \vec{\nabla}_Y\right] + 2\left((\partial_\mu X^\mu)\vec{\nabla}_Y - (\partial_\nu Y^\nu)\vec{\nabla}_X\right) \\
&+ 2(Y^\nu(\partial_\nu X^\mu) + X^\mu(\partial_\nu Y^\nu))\partial_\mu - 2(Y^\nu(\partial_\mu X^\mu) + X^\mu(\partial_\mu Y^\nu))\partial_\nu \\
=& 2\left[\vec{\nabla}_X, \vec{\nabla}_Y\right] + 2\left((\partial_\mu X^\mu)\vec{\nabla}_Y - (\partial_\nu Y^\nu)\vec{\nabla}_X\right) \\
&+ 2[Y,X]^\mu \partial_\mu + 2(X^\mu(\partial_\nu Y^\nu)\partial_\mu - Y^\nu(\partial_\mu X^\mu)\partial_\nu) \\
=& 2\left[\vec{\nabla}_X, \vec{\nabla}_Y\right] + 2\left((\partial_\mu X^\mu) Y^\nu A_\nu - (\partial_\nu Y^\nu) X^\mu A_\mu\right) + 2[Y,X]^\mu \partial_\mu \\
=& 2\left(\left[\vec{\nabla}_X, \vec{\nabla}_Y\right] + [Y,X]^\mu A_\mu + [Y,X]^\mu \partial_\mu\right),
\end{aligned}$$

and thus

$$F(X,Y) = \left[\vec{\nabla}_X, \vec{\nabla}_Y\right] - \vec{\nabla}_{[X,Y]}.$$

As for the exterior derivative of the field strength, we have

$$\begin{aligned}
dF =& \frac{1}{2}\partial_\mu F_{\nu\rho} d\chi^\mu d\chi^\nu d\chi^\rho \\
=& \frac{1}{2}\left(\partial_\mu(A_\nu A_\rho) - \partial_\mu(A_\rho A_\nu)\right) d\chi^\mu d\chi^\nu d\chi^\rho \\
=& \frac{1}{2}\left(\partial_\mu(A_\nu)A_\rho + A_\nu \partial_\mu(A_\rho) - \partial_\mu(A_\rho)A_\nu - A_\rho \partial_\mu(A_\nu)\right) d\chi^\mu d\chi^\nu d\chi^\rho,
\end{aligned}$$

as we have for any differential form F that $ddF = \partial_\mu \partial_\nu F_I d\chi^\mu d\chi^\nu d\chi^I = 0$, using the notation I to refer to some set of indices. Now, by comparison we have

$$FA - AF = \frac{1}{2}[F_{\mu\nu}, A_\rho] d\chi^\mu d\chi^\nu d\chi^\rho = \frac{1}{2}\left(F_{\mu\nu}A_\rho - A_\mu F_{\nu\rho}\right) d\chi^\mu d\chi^\nu d\chi^\rho,$$

using the properties of the indices under cyclic permutation. We find

$$\begin{aligned}
F_{\mu\nu}A_\rho - A_\mu F_{\nu\rho} =& \left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right) A_\rho - A_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu + [A_\nu, A_\rho]\right) \\
=& \partial_\mu(A_\nu)A_\rho - \partial_\nu(A_\mu)A_\rho - A_\mu \partial_\nu A_\rho + A_\mu \partial_\rho A_\nu,
\end{aligned}$$

and permuting indices cyclically we find

$$dF + AF - FA = 0,$$

which is a so-called Bianchi identity.

Next we study the form F^2. We have

$$dF^2 = d(F)F + FdF = (FA - AF)F + F(FA - AF).$$

Computing the trace of this involves computing the trace of objects of the form

$$F_{\mu_1\mu_2}A_{\mu_3}F_{\mu_4\mu_5}\bigwedge_{i=1}^{5} d\chi^{\mu_i}.$$

We note that if the first factor of F can be moved to the right in the trace, we will have shown that it is zero. Traces are invariant under cyclic permutation of matrices, so we can certainly move the components themselves. As for the differential forms, moving the first and second ones to the right requires either passing through 3 others, providing no net sign and yielding

$$\mathrm{tr}\big(dF^2\big) = d\mathrm{tr}\big(F^2\big) = 0.$$

Similarly we may consider the form F^n. Its trace is

$$d\mathrm{tr}(F^n) = n\,\mathrm{tr}\big(F^{n-1}dF\big) = n\,\mathrm{tr}\big(F^{n-1}(FA - AF)\big).$$

By the exact same argument we must have

$$d\mathrm{tr}(F^n) = 0.$$

Alternatively, let us prove that

$$dF^n = F^nA - AF^n.$$

Evidently it holds for $n = 1$. Next, assuming it to hold for some particular n, it follows that

$$\begin{aligned} dF^{n+1} =& d(F^n)F + F^n dF \\ =& (F^nA - AF^n)F + F^n(FA - AF) \\ =& F^{n+1}A - AF^{n+1}, \end{aligned}$$

completing the proof. Using the properties of the trace and the cyclic permutivity of the indices, we reobtain the same result.

The form F^2 is of some more interest. We have

$$F^2 = (dA)^2 + A^2dA + d(A)A^2 + A^4.$$

Let us compute its trace. The last term is the easiest to handle as it is a contraction of a symmetric matrix product with an antisymmetric differential form, and is thus zero. Antisymmetrizing the components does not help due to the trace, which allows you to cyclically permute the matrices without changing the sign. Antisymmetrizing they start with the opposite sign of another term related to the first one by cyclic permutation, and it must be zero. As for the others we have

$$\begin{aligned} \mathrm{tr}\big((dA)^2 + A^2dA + d(A)A^2\big) =& \mathrm{tr}\left(d(AdA) + \frac{2}{3}(A^2dA + d(A)A^2 + Ad(A)A)\right) \\ =& \mathrm{tr}\left(d(AdA) + \frac{2}{3}dA^3\right) \\ =& d\mathrm{tr}\left((AdA) + \frac{2}{3}A^3\right). \end{aligned}$$

Let us finally investigate the topological invariance of integrals of F^2. Under a small deformation of A, we have

$$\begin{aligned} \delta\,\mathrm{tr}(F^n) =& n\,\mathrm{tr}\big(F^{n-1}\,\delta F\big) \\ =& n\left(\mathrm{tr}\big(F^{n-1}d\delta A\big) + \mathrm{tr}\big(F^{n-1}\,\delta A\,A\big) + \mathrm{tr}\big(F^{n-1}A\,\delta A\big)\right) \\ =& n\left(\mathrm{tr}\big(F^{n-1}d\delta A\big) - \mathrm{tr}\big(AF^{n-1}\,\delta A\big) + \mathrm{tr}\big(F^{n-1}A\,\delta A\big)\right) \\ =& n\,\mathrm{tr}\big(dF^{n-1}\,\delta A\big). \end{aligned}$$

Integration of Differential Forms Consider a set of p tangent vectors X_i. The corresponding coordinate displacements are $\mathrm{d}\chi_i^a = X_i^a \,\mathrm{d}t_i$, with no sum over i. We would now like to compute the p-dimensional volume defined by the X_i and $\mathrm{d}t_i$. We expect that if any of the X_i are linearly dependent the volume should be zero. We also expect that the volume be linear in the X_i. This implies

$$\mathrm{d}V_p = \omega(X_1, \dots, X_p)\,\mathrm{d}t_1 \dots \mathrm{d}t_p$$

for some differential form ω. We now define the integral over the p-volume S over the p-form ω as

$$\int\limits_S \omega = \int \mathrm{d}t_1 \dots \int \mathrm{d}t_p\, \omega(\dot{\gamma}_1, \dots, \dot{\gamma}_p).$$

Here the γ_i are the set of curves that span S, the dot symbolizes the derivative with respect to the individual curve parameters and the right-hand integration is performed over the appropriate set of parameter values.

Stokes' Theorem Stokes' theorem relates the integral of a differential form $d\omega$ over some subset V of a manifold to an integral over ∂V of another differential form. It states that

$$\int\limits_V d\omega = \oint\limits_{\partial V} \omega.$$

2 Topology

Topological Spaces Let X be a set and $T = \{U_i | i \in I\}$ be a collection of subsets of X (I is some set of indices). The pair (X, T) (sometimes we only explicitly write X) is defined as a topological space if

- $\emptyset,\ X \in T$.
- If J is any subcollection of I, the family $\{U_j | j \in J\}$ satisfies

$$\bigcup_{j\in J} U_j \in T.$$

- If J is any finite subcollection of I, the family $\{U_j | j \in J\}$ satisfies

$$\bigcap_{j\in J} U_j \in T.$$

If the two satisfy the definition, we say that T gives a topology to X. The U_i are called its open sets.

Two cases of little interest are $T = \{\emptyset,\ X\}$ and T being the collection of all subsets of X. The two are called the trivial and discrete topologies respectively.

Metrics A metric is a map $d : X \times X \to \mathbb{R}$ that satisfies

- $d(x, y) = d(y, x)$.
- $d(x, x) \geq 0$, with equality applying if and only if $x = y$.
- $d(x, y) + d(y, z) \geq d(x,)z$.

Metric Spaces Suppose X is endowed with a metric. The collection of open disks

$$U_\varepsilon = \{x \in X | d(x, x_0) < \varepsilon\}$$

then gives a topology to X called the metric topology. The pair forms a metric space.

Continuous Maps A map between two topological spaces X and Y is continuous if its inverse maps an open set in Y to an open set in X.

Neighborhoods N is a neighborhood of x if it is a subset of X and x belongs to at least one open set contained within N.

Hausdorff Spaces A topological space is a Hausdorff space if, for any two points x, y, there exists neighborhoods U_x, U_y of the two points that do not intersect. This is an important type of topological space, as examples in physics are practically always within this category.

Homeomorphisms A homeomorphism (not to be confused with a homomorphism, however hard it may be) is a continuous map between two topological spaces with a continuous inverse. Two topological spaces are homeomorphic if there exists a homeomorphism between them.

A specific case if homeomorphicity is diffeomorphicity, which applies between two manifolds between which there is a smooth map.

Homotopy Types Two topological spaces belong to the same homotopy type if there exists a continuous map from one to the other. This is a more relaxed version of homeomorphicity, as we no longer require the map to be invertible.

Donuts and Coffee Mugs Homeomorphicity defines an equivalence relation between topological spaces. This means that we can define topological spaces into categories based on homeomorphicity.

We are now in a position to introduce the poor man's notion of topology, which considers two bodies as equivalent if one can be deformed into the other without touching two parts of the surface or tearing a part of the body. These continuous deformations correspond to homeomorphisms, but we will try to keep the discussions more to the abstract.

Topological Invariants An important question pertaining to this division of topological spaces is what separates the different categories. One possible answer is so-called topological invariants, quantities which are invariant under homeomorphism. The issue with this answer is that the full set of topological invariants has not been identified, hence they can only be used to verify that two topological spaces belong to different categories.

Homotopy and Homotopy Classes Consider maps between two manifolds. Maps that can be continuously deformed into each other are said to be homotopic. This concept is related to that of homeomorphicity, and as such there exists topological invariants for such maps too. These divide maps into homotopy classes. A particular case is $\pi_n(M)$, which is the set of homotopy classes of maps from S^n to M.

Chain Complices A chain complex is a doubly infinite sequence of mathematical objects related in a chain by a structure-preserving map ∂. The structure is thus

$$\dots C_{p-1} \xrightarrow{\partial_{p+1}} C_{p1} \xrightarrow{\partial_{p-1}} C_{p+1} \dots$$

The maps satisfy $\partial_{p-1}\partial_p = 0$.

Homology Restricting ourselves to manifolds, consider the boundary operator as a link between manifolds of different dimension, creating a chain complex. The nth homology group is then defined as

$$H_n = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

Alternatively, by defining $\ker(\partial_n) = Z_n$ and $\mathrm{im}(\partial_{n+1}) = B_n$, we can write this as

$$H_n = Z_n/B_n.$$

Its elements are homology classes, which are isomorphic to manifolds which themselves are boundaries of different manifolds.

Cohomology To define (de Rham) cohomology, we first introduce the set of closed r-forms $Z^r(M)$ and the set of exact r-forms $B^r(M)$. The cohomology group is then

$$H^r(M) = \frac{Z^r(M)}{B^r(M)}.$$

This divides up the forms in $Z^r(M)$ into classes which differ by an exact form. Two forms belonging to the same class are cohomologous.

Relating Homology and Cohomology What is so co about cohomology? Note that differential forms and manifolds fit very naturally into the language of chain complices - namely, the exterior derivative relates forms of different rank in a chain (forms with too many or a negative number of indices are identically zero), and the boundary operator relates manifolds of different dimension in a chain. The integral of a form over a manifold then induces the product

$$\langle \omega | M \rangle = \int\limits_M \omega.$$

Stokes' theorem then reads $\langle d\omega | M \rangle = \langle \omega | \partial M \rangle$. By defining the pairing with a sum of manifolds to be a sum of the integrals over the two manifolds, the pairing is bilinear, and we then have for $\omega \in Z^n$ and $M \in Z_n$ that

$$\begin{aligned}
\langle \omega + d\chi | M + \partial N \rangle &= \langle \omega + d\chi | M \rangle + \langle \omega + d\chi | \partial N \rangle \\
&= \langle \omega | M \rangle + \langle d\chi | M \rangle + \langle d(\omega + d\chi) | N \rangle \\
&= \langle \omega | M \rangle + \langle \chi | \partial M \rangle \\
&= \langle \omega | M \rangle .
\end{aligned}$$

This pairing then splits up its result depending on what representative of the homology and cohomology classes are fed into it. This also means, in some sense, that H^n belongs to the dual space of H_n. There are some subtleties to completely identifying the two as each other, however.

Retractability A domain Ω is retractable to O if there exists a smooth map ϕ_t on Ω parametrized by t such that $\phi_1(x) = x$ and $\phi_0(x) = O$.

Inverting the Exterior Derivative It holds that $d^2 = 0$, and thus one might believe that for every form ω such that $d\omega = 0$ there exists a form χ such that $\omega = d\chi$. This will, however, turn out to depend on the topological properties of the underlying space.

Assuming the underlying space to be retractable to O, we have $\phi_1^\star \omega = \omega$ and $\phi_0^\star \omega = 0$ for some closed form ω. Define η to be the vector tangential to the coordinate flow as t is varied. The fact that $d\omega = 0$ implies that $d\phi_t^\star \omega = 0$. By the definition of the Lie derivative we have $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^\star \omega = \mathcal{L}_\eta(\phi_t^\star \omega)$. We then have

$$\mathcal{L}_\eta(\phi_t^\star \omega) = (i_\eta d + d i_\eta)(\phi_t^\star \omega) = d(i_\eta(\phi_t^\star \omega)).$$

Integrating with respect to t we finally find

$$\omega = d \int\limits_0^1 \mathrm{d}t \, i_\eta(\phi_t^\star \omega),$$

and we have thus solved the problem. In the case of spaces which are not retractible, there are simple counterexamples to be found. One example would be $\sin(\theta) d\theta d\phi$ on S^2.

In the language of cohomology we can phrase this very concisely. If some form ω is the exterior derivative of another, it must follow that it is cohomologous to the zero form. This implies that if the exterior derivative is invertible, then $H^r(M) = \{0\}$.

Deligne-Beilinson Cohomology To define Deligne-Beilinson cohomology, we first need some other concepts. First, a fiber bundle is a structure (E, B, π, F), with E, B and F being topological spaces and $\pi : E \to B$ a surjection. B is called the base and F the fiber. A principal bundle is a special case of a fiber bundle where F is a (Lie) group.

Consider now some manifold M, an open cover $\{U_\alpha\}_{\alpha\in I}$ and a principal bundle $P(M, \mathrm{U}(1))$. The bundle at different points is related by a transfer function, which generally is a homomorphism of the group onto itself, and in this case is given by

$$g_{\alpha\beta} = e^{2\pi i \Lambda_{\alpha\beta}},$$

with $\Lambda_{\alpha\beta}$ being some smooth function on $U_\alpha \cap U_\beta$. For consistency we require $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$, translating to

$$\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = n_{\alpha\beta\gamma} \in \mathbb{Z}.$$

These numbers tautologically satisfy

$$n_{\beta\gamma\delta} - n_{\alpha\gamma\delta} + n_{\alpha\beta\delta} - n_{\alpha\beta\gamma} = 0.$$

Built into this is an ambiguity found by the replacement

$$g_{\alpha\beta} \to h_\alpha g_{\alpha\beta} h_\beta^{-1} \implies \Lambda_{\alpha\beta} \to \Lambda_{\alpha\beta} + \xi_\alpha - \xi_\beta.$$

We can now introduce a U(1) connection on M as a set of local 1-forms A_α such that

$$A_\beta - A_\alpha = d\Lambda_{\alpha\beta}.$$

Combining the connection with the other objects to one tuple $\mathbf{A} = (A_\alpha, \Lambda_{\alpha\beta}, n_{\alpha\beta\gamma})$, called a DB cocycle, we can also define operators δ mapping its components to objects defined on higher-order overlaps according to

$$\begin{aligned}
(\delta_0 A)_{\alpha\beta} &= A_\beta - A_\alpha = d_0\Lambda_{\alpha\beta}, \\
(\delta_1 \Lambda)_{\alpha\beta\gamma} &= \Lambda_{\beta\gamma} - \Lambda_{\alpha\gamma} + \Lambda_{\alpha\beta} = d_{-1} n_{\alpha\beta\gamma}, \\
(\delta_2 n)_{\alpha\beta\gamma\delta} &= n_{\beta\gamma\delta} - n_{\alpha\gamma\delta} + n_{\alpha\beta\delta} - n_{\alpha\beta\gamma} = 0.
\end{aligned}$$

Note the extension of the exterior derivative to be an extension of numbers to functions. The generalization of the operator δ to quantities defined on intersections of higher degree is

$$(\delta_n A)_{i_0\ldots i_n} = A_{i_1\ldots i_n} + \sum_{j=1}^{n-1} (-1)^j A_{i_0\ldots i_{j-1} i_{j+1}\ldots i_n} + (-1)^n A_{i_0\ldots i_{n-1}}.$$

It holds that $\delta_{n+1}\delta_n = 0$. To prove that this is the case, consider some index sequence of length $n-1$. In the sum appearing after δ_{n+1}, this sequence appears exactly twice, each time split by some other index. Denote the number of steps by which this split moves between the terms as k. If k is odd, the terms appear with the same sign. To find the terms containing only the desired sequence, we now evaulate the underlying sum. Because k is odd and one index has already been removed, these two contributions must therefore cancel. If k is instead odd, the argument is identical upon permuting a few words.

The ambiguities in the components of $\mathbf{A}$ can be summarized as an invariance under

$$\begin{aligned}
A_\alpha &\to A_\alpha + d_0\xi_\alpha, \\
\Lambda_{\alpha\beta} &\to \Lambda_{\alpha\beta} + \xi_\beta - \xi_\gamma - dm_{\alpha\beta} = \Lambda_{\alpha\beta} + (\delta_0\xi)_{\alpha\beta} - d_{-1}m_{\alpha\beta}, \\
n_{\alpha\beta\gamma} &\to n_{\alpha\beta\gamma} - m_{\beta\gamma} + m_{\alpha\gamma} - m_{\alpha\beta} = n_{\alpha\beta\gamma} - (\delta_1 m)_{\alpha\beta\gamma}.
\end{aligned}$$

This is very similar to a gauge transformation, which is a key motivation for developing this framework. By introducing operators δ_n and d_n acting on the appropriate components of $\mathbf{A}$, the ambiguity and connection between the components can be written as

$$D_{1,1}\mathbf{A} = 0, \ \mathbf{A} \to \mathbf{A} + D_{0,1}\Xi,$$

with

$$D_{1,1} = (\delta_0 + 0) - (\delta_1 + d_0) + (\delta_2 + d_{-1}), \ D_{0,1} = (\delta_0 + d_0) - (\delta_1 + d_{-1}),$$

and Ξ containing a number and a 0-form. $D_{0,1}\Xi$ is called a DB coboundary. The indexation is defined as such: the second index specifies the order of the exterior derivative at which one truncates to 0 and the first index specifies the degree of the overlap on which the corresponding object lives. In our case, $D_{1,1}$ means that it acts on objects with 1-forms defined on a degree-1 overlap and $D_{1,1}$ means that it acts on objects with functions defined on a degree-1 overlap.

Considering the composition of these operators, we have

$$\begin{aligned} D_{1,1}D_{0,1}\Xi =& D_{1,1}D_{0,1}\begin{bmatrix} \xi_\alpha \\ m_{\alpha\beta} \end{bmatrix} = D_{1,1}\begin{bmatrix} d_0\xi_\alpha \\ (\delta_0\xi)_{\alpha\beta} - d_{-1}m_{\alpha\beta} \\ -(\delta_1 m)_{\alpha\beta\gamma} \end{bmatrix} \\ =& \begin{bmatrix} 0 \\ (\delta_0 d_0\xi)_{\alpha\beta} - d_0\left(\delta_0\xi - d_{-1}m\right)_{\alpha\beta} \\ -(\delta_1(\delta_0\xi - d_{-1}m))_{\alpha\beta\gamma} - (d_{-1}\delta_1 m)_{\alpha\beta\gamma} \\ -(\delta_2\delta_1 m)_{\alpha\beta\gamma} \end{bmatrix} \\ =& \begin{bmatrix} 0 \\ (\delta_0 d_0\xi)_{\alpha\beta} - d_0\left(\delta_0\xi\right)_{\alpha\beta} \\ (\delta_1 d_{-1}m)_{\alpha\beta\gamma} - (d_{-1}\delta_1 m)_{\alpha\beta\gamma} \\ 0 \end{bmatrix}, \end{aligned}$$

As the exterior derivative commutes with δ, this must identically vanish. The cohomology group, which gives a gauge slice, is then given by

$$H_D^{1,1} = \ker(D_{1,1})/\mathrm{im}(D_{0,1}).$$

We are interested in the integral of field strengths corresponding to this construction. To that end, decompose some p-cycle as $C^p = \sum_{i_0} l^p_{i_0}$, where each $l^p_i \subset U_i$. There is a boundary operator on each of these given by

$$\partial l^p_{i_0} = \sum_{i_1} l^{p-1}_{i_1 i_0} - l^{p-1}_{i_0 i_1}.$$

To this belongs the idea that the decomposition is ordered - that is, $l_{ij} = 0$ if $i > j$. The generalization of this is

$$\partial l^{p-k}_{i_0\ldots i_k} = \sum_{i_{k+1}}\left(l^{p-k-1}_{i_{k+1}i_0\ldots i_k} + (-1)^{k+1} l^{p-k-1}_{i_0\ldots i_k i_{k+1}} + \sum_{j=1}^{k}(-1)^j l^{p-k-1}_{i_0\ldots i_{j-1}i_{k+1}i_j\ldots i_k}\right).$$

The minus sign describes the orientation. We then have

$$\int_{C^2} dA = \sum_{i_0}\int_{l^2_{i_0}} dA_{i_0} = \sum_{i_0}\int_{\partial l^2_{i_0}} A_{i_0} = \sum_{i_0,i_1}\int_{l^1_{i_1 i_0}} A_{i_0} - \int_{l^1_{i_0 i_1}} A_{i_0}.$$

To see how this works out, consider a specific pair i_0, i_1. They will each contribute such an integral exactly once and with opposite sign, and thus

$$\int_{C^2} dA = \sum_{i_0,i_1}\int_{l^1_{i_0 i_1}} A_{i_1} - A_{i_0} = \sum_{i_0,i_1}\int_{l^1_{i_0 i_1}} d\Lambda_{i_0 i_1} = \sum_{i_0,i_1}\int_{\partial l^1_{i_0 i_1}} \Lambda_{i_0 i_1} = \sum_{i_0,i_1,i_2}\left(\int_{l^0_{i_2 i_0 i_1}} - \int_{l^0_{i_0 i_2 i_1}} + \int_{l^0_{i_0 i_1 i_2}}\right)\Lambda_{i_0 i_1}.$$

Handling this is slightly more involved. While all indices are freely summed over, the only contributions are found when one of the indices in the integration domains are an ordered sequence. We see that this corresponds to the exact same process of index removal with alternating signs as found in the definition of δ, and thus

$$\int_{C^2} dA = \sum_{i_0,i_1,i_2}\int_{l^0_{i_0 i_1 i_2}} dn_{i_0 i_1 i_2} = \sum_{i_0,i_1,i_2} n_{i_0 i_1 i_2},$$

and so the total field strength is quantized.

To introduce so-called higher order gauge fields, one generalizes the above procedure. While this can be done to obtain $H_D^{p,q}$ for any p and q, we will only need to consider the case of $p = q$. We then introduce DB cocycles as tuples

$$\omega = \left(\omega_{\alpha_0}^{(0,p)}, \ldots, \omega_{\alpha_0 \ldots \alpha_p}^{(p,0)}, n_{\alpha_0 \ldots \alpha_{p+1}}^{p+1,-1}\right).$$

Each $\omega_{\alpha_0 \ldots \alpha_k}^{(k,p-k)}$ is a $p-k$-form defined on overlaps of degree $k+1$ related to the next element by

$$(\delta_k \omega^{(k,p-k)})_{\alpha_0 \ldots \alpha_k} = d_{p-k-1} \omega_{\alpha_0 \ldots \alpha_k}^{(k+1,p-k-1)},$$

and are thus annihilated by

$$D_{p,p} = (\delta_0 + 0) + \sum_{k=1}^{p+1} (-1)^k (\delta_k + d_{p-k}).$$

The coboundaries form the image of

$$D_{p-1,p} = \sum_{k=0}^{p} (-1)^k (\delta_k + d_{p-k-1}).$$

The quotient of the kernel of the first by the image of the second yields the cohomology group H_D^p.

A particular case to consider is 2-form gauge fields. The gauge transformation manifests as a shift by something in the image of

$$D_{1,2} = \delta_0 + d_1 - (\delta_1 + d_0) + \delta_2 + d_{-1}.$$

We have

$$D_{1,2} \begin{bmatrix} \xi_\alpha \\ \phi_{\alpha\beta} \\ m_{\alpha\beta\gamma} \end{bmatrix} = \begin{bmatrix} d_1 \xi_\alpha \\ (\delta_0 \xi)_{\alpha\beta} - d_0 \phi_{\alpha\beta} \\ -(\delta_1 \phi)_{\alpha\beta\gamma} + d_{-1} m_{\alpha\beta\gamma} \\ (\delta_2 m)_{\alpha\beta\gamma\delta} \end{bmatrix},$$

and so the generalized gauge transformation for a 2-form gauge field (ω, A, Λ, n) is

$$\begin{aligned}
\omega_\alpha \to& \omega_\alpha^{(2)} + d_1 \xi_\alpha, \\
A_{\alpha\beta} \to& A_{\alpha\beta} - \xi_\beta + \xi_\alpha - d_0 \phi_{\alpha\beta}, \\
\Lambda_{\alpha\beta\gamma} \to& \Lambda_{\alpha\beta\gamma} - \phi_{\beta\gamma} + \phi_{\alpha\gamma} - \phi_{\alpha\beta} + d_{-1} m_{\alpha\beta\gamma}, \\
n_{\alpha\beta\gamma\delta} \to& n_{\alpha\beta\gamma\delta} + m_{\beta\gamma\delta} - m_{\alpha\gamma\delta} + m_{\alpha\beta\delta} - m_{\alpha\beta\gamma}.
\end{aligned}$$

Let us also consider the corresponding integral of the field strength. We have

$$\int_{C^3} d\omega = \sum_{i_0} \int_{l_{i_0}^2} d\omega_{i_0} = \sum_{i_0} \int_{\partial l_{i_0}^2} A_{i_0} = \sum_{i_0, i_1} \int_{l_{i_1 i_0}^1} A_{i_0} - \int_{l_{i_0 i_1}^1} A_{i_0}.$$

The proof above generalizes without changes, and so

$$\int_{C^3} d\omega = \sum_{i_0, i_1, i_2} \int_{l_{i_0 i_1 i_2}^1} d\Lambda_{i_0 i_1 i_2} = \sum_{i_0, i_1, i_2, i_3} \left(\int_{l_{i_3 i_0 i_1 i_2}^0} - \int_{l_{i_0 i_3 i_1 i_2}^0} + \int_{l_{i_0 i_1 i_3 i_2}^0} - \int_{l_{i_0 i_1 i_2 i_3}^0} \right) \Lambda_{i_0 i_1 i_2}.$$

The argument previously used is identical, and so

$$\int_{C^3} d\omega = \sum_{i_0, i_1, i_2, i_3} \int_{l_{i_0 i_1 i_2 i_3}^0} (\delta_2 \Lambda)_{i_0 i_1 i_2 i_3} = \sum_{i_0, i_1, i_2, i_3} \int_{l_{i_0 i_1 i_2 i_3}^0} d_{-1} n_{i_0 i_1 i_2 i_3} = \sum_{i_0, i_1, i_2, i_3} n_{i_0 i_1 i_2 i_3},$$

which is valued in integers.

3 Quantum Pumps

Quantum Phases Quantum phases are low-temperature phases of matter. For parametrized system, the existence of quantum fluctuations entails that these systems exhibit phase transitions, realized by varying parameters of the Hamiltonian. The phase transition describes an abrupt change in the ground state.

Berry Phase, Connection and Curvature Consider a system with a Hamiltonian and eigenstates parametrized by some set of parameters χ - that is, we have for each value of χ a set of eigenstates

$$\mathcal{H}(\chi)\left|n(\chi)\right\rangle = E_n(\chi)\left|n(\chi)\right\rangle .$$

The adiabatic theorem tells us that if R is varied such that the Hamiltonian changes sufficiently slowly, a state which is initialized to an eigenstate at $t = 0$ will evolve to a corresponding eigenstate at a later time. In the general case we have

$$\left|\psi_n(t)\right\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t \mathrm{d}\tau E_n(\tau)} \left|n(\chi(t))\right\rangle .$$

The former factor is the complex exponential of the so-called Berry phase. Inserting this into the Schrödinger equation we find

$$\gamma_n = i\int_0^t \mathrm{d}\tau \left\langle n(\chi(\tau))\right|\frac{\partial}{\partial\tau}\left|n(\chi(\tau))\right\rangle .$$

Noting that

$$\frac{\partial}{\partial\tau}\left|n(\chi(\tau))\right\rangle = \frac{\mathrm{d}R}{\mathrm{d}\tau}\cdot\vec{\boldsymbol{\nabla}}_R\left|\chi(R)\right\rangle ,$$

we can define the Berry connection

$$A_n = i\left\langle n(\chi)\right|\vec{\boldsymbol{\nabla}}_\chi\left|n(\chi)\right\rangle$$

and find

$$\gamma_n = i\int_C \mathrm{d}\chi\cdot A_n.$$

C is the orbit in parameter space traversed during the time evolution.

In a slightly more sophisticated manner, the Berry connection may be taken to be a 1-form

$$A_n = i\left\langle n(R)\right|\partial_\mu\left|n(R)\right\rangle d\chi^\mu.$$

Due to Stokes' theorem, the line integral of the Berry connection about some closed path is related to the surface integral of its exterior derivative, termed the Berry curvature. Its components are

$$\Omega^{(2)}_{n,\mu\nu} = \partial_\mu A_{n,\nu} - \partial_\nu A_{n,\mu},$$

and we have

$$\int_{\partial S} A_n = \frac{1}{2}\int_S \mathrm{d}\chi^\mu\wedge\mathrm{d}\chi^\nu\,\Omega^{(2)}_{n,\mu\nu}.$$

A More Sophisticated Definition From this point on we switch to the more compact notation

$$\partial_\mu\left|n\right\rangle = \left|\partial_\mu n\right\rangle$$

and suppress the parameter dependence. The Berry curvature is given by

$$\Omega^{(2)} = dA_n = \frac{1}{2}(\partial_\mu A_{n,\nu} - \partial_\nu A_{n,\mu})d\chi^\mu\wedge d\chi^\nu,$$

and we find

$$\Omega^{(2)}_{\mu\nu} = i\left(\langle\partial_\mu n|\partial_\nu n\rangle + \langle n|\partial_\mu\partial_\nu n\rangle - \langle\partial_\nu n|\partial_\mu n\rangle - \langle n|\partial_\nu\partial_\mu n\rangle\right) = i\left(\langle\partial_\mu n|\partial_\nu n\rangle - \langle\partial_\nu n|\partial_\mu n\rangle\right).$$

This can be expressed without derivatives of the state. To do that we differentiate the eigenvalue expression to yield

$$\partial_\mu\mathcal{H}\,|n\rangle + \mathcal{H}\,|\partial_\mu n\rangle = \partial_\mu E_n\,|n\rangle + E_n\,|\partial_\mu n\rangle.$$

Using the orthogonality of the eigenstates, we have for some $n \neq m$ that

$$\langle m|\partial_\mu\mathcal{H}|n\rangle = (E_n - E_m)\,\langle m|\partial_\mu n\rangle.$$

We can now solve for the inner product on the left-hand side and its complex conjugate, as well as sum over m, to find

$$\Omega^{(2)}_{\mu\nu} = i\sum_{m\neq n}\frac{\langle n|\partial_\mu\mathcal{H}|m\rangle\,\langle m|\partial_\nu\mathcal{H}|n\rangle - \text{c.c}}{(E_n - E_m)^2}.$$

Finally we introduce a third definition

$$\Omega^{(2)} = \frac{i}{2}\oint\frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H}G^2d\mathcal{H}\big).$$

G is given by $(z-\mathcal{H})^{-1}$ and the integral is a counter-clockwise contour integral around the energy of the state in consideration. There is also the appearance of the exterior derivative of the Hamiltonian.

Does this correspond to our previous notion of the Berry curvature? To investigate, let us rewrite the above operators as

$$\mathcal{H} = \sum_n E_n\,|n\rangle\!\langle n|\,,\ G = \sum_n\frac{1}{z-E_n}\,|n\rangle\!\langle n|\,.$$

Next we note that

$$GdG^{-1} = -dGG^{-1} = -Gd\mathcal{H},$$

hence

$$GdG^{-1}GdG^{-1}G = Gd\mathcal{H}Gd\mathcal{H}G,$$

and by cyclic permutation we have

$$\begin{aligned}\mathrm{tr}\big(GdG^{-1}GdG^{-1}G\big) &= \mathrm{tr}(Gd\mathcal{H}Gd\mathcal{H}G)\\ &= \mathrm{tr}\big(G\partial_\mu\mathcal{H}G\partial_\nu\mathcal{H}G\big)d\chi^\mu d\chi^\nu\\ &= \mathrm{tr}\big(G\partial_\nu\mathcal{H}G^2\partial_\mu\mathcal{H}\big)d\chi^\mu d\chi^\nu\\ &= -\,\mathrm{tr}\big(Gd\mathcal{H}G^2d\mathcal{H}\big).\end{aligned}$$

As a warmup to the final computation, consider a case where the spectrum is parameter-independent. In the eigenbasis of the Hamiltonian we generally have

$$\begin{aligned}dG^{-1} &= \sum_n\left(-\partial_\mu E_n\,|n\rangle\!\langle n| + (z-E_n)\left(|\partial_\mu n\rangle\!\langle n| + |n\rangle\!\langle\partial_\mu n|\right)\right)d\chi^\mu,\\ dG &= \sum_n\left(\frac{\partial_\mu E_n}{(z-E_n)^2}\,|n\rangle\!\langle n| + \frac{1}{z-E_n}\left(|\partial_\mu n\rangle\!\langle n| + |n\rangle\!\langle\partial_\mu n|\right)\right)d\chi^\mu,\end{aligned}$$

and thus in this case

$$GdG^{-1}GdG^{-1}G = \sum\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\,|1\rangle\!\langle 1|\left(|\partial_\mu 2\rangle\!\langle 2| + |2\rangle\!\langle\partial_\mu 2|\right)|3\rangle\!\langle 3|\left(|\partial_\nu 4\rangle\!\langle 4| + |4\rangle\!\langle\partial_\nu 4|\right)|5\rangle\!\langle 5|\,e^{\mu\nu},$$

where the natural numbers are summed over and we abbreviate the differential form basis vector. Multiplying this out we have

$$\begin{aligned}
GdG^{-1}GdG^{-1}G &= \sum \frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} |1\rangle \left(\langle 1|\partial_\mu 2\rangle \delta_{23} + \delta_{12} \langle \partial_\mu 2|3\rangle\right) \left(\langle 3|\partial_\nu 4\rangle \delta_{45} + \delta_{34} \langle \partial_\nu 4|5\rangle\right) \langle 5| e^{\mu\nu} \\
&= \sum |1\rangle \Bigg(\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} \langle 1|\partial_\mu 2\rangle \delta_{23} \langle 3|\partial_\nu 4\rangle \delta_{45} \\
&\quad + \frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} \langle 1|\partial_\mu 2\rangle \delta_{23}\delta_{34} \langle \partial_\nu 4|5\rangle \\
&\quad + \frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} \delta_{12} \langle \partial_\mu 2|3\rangle \langle 3|\partial_\nu 4\rangle \delta_{45} \\
&\quad + \frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} \delta_{12} \langle \partial_\mu 2|3\rangle \delta_{34} \langle \partial_\nu 4|5\rangle \Bigg) \langle 5| e^{\mu\nu} \\
&= \sum |1\rangle \Bigg(\frac{1}{z-E_1} \langle 1|\partial_\mu 2\rangle \langle 2|\partial_\nu 4\rangle \langle 4| + \frac{z-E_2}{(z-E_1)(z-E_5)} \langle 1|\partial_\mu 2\rangle \langle \partial_\nu 2|5\rangle \langle 5| \\
&\quad + \frac{1}{z-E_3} \langle \partial_\mu 1|3\rangle \langle 3|\partial_\nu 4\rangle \langle 4| + \frac{1}{z-E_5} \langle \partial_\mu 1|3\rangle \langle \partial_\nu 3|5\rangle \langle 5| \Bigg) e^{\mu\nu},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}\left(GdG^{-1}GdG^{-1}G\right) &= \sum \Bigg(\frac{1}{z-E_1} \langle 1|\partial_\mu 2\rangle \langle 2|\partial_\nu 4\rangle \langle 4|1\rangle + \frac{z-E_2}{(z-E_1)(z-E_5)} \langle 1|\partial_\mu 2\rangle \langle \partial_\nu 2|5\rangle \langle 5|1\rangle \\
&\quad + \frac{1}{z-E_3} \langle \partial_\mu 1|3\rangle \langle 3|\partial_\nu 4\rangle \langle 4|1\rangle + \frac{1}{z-E_5} \langle \partial_\mu 1|3\rangle \langle \partial_\nu 3|5\rangle \langle 5|1\rangle \Bigg) e^{\mu\nu} \\
1 &= \sum \Bigg(\frac{1}{z-E_1} \langle 1|\partial_\mu 2\rangle \langle 2|\partial_\nu 4\rangle \delta_{41} + \frac{z-E_2}{(z-E_1)(z-E_5)} \langle 1|\partial_\mu 2\rangle \langle \partial_\nu 2|5\rangle \delta_{51} \\
&\quad + \frac{1}{z-E_3} \langle \partial_\mu 1|3\rangle \langle 3|\partial_\nu 4\rangle \delta_{41} + \frac{1}{z-E_5} \langle \partial_\mu 1|3\rangle \langle \partial_\nu 3|5\rangle \delta_{51} \Bigg) e^{\mu\nu} \\
&= \sum \Bigg(\frac{1}{z-E_1} \langle 1|\partial_\mu 2\rangle \langle 2|\partial_\nu 1\rangle + \frac{z-E_2}{(z-E_1)^2} \langle 1|\partial_\mu 2\rangle \langle \partial_\nu 2|1\rangle \\
&\quad + \frac{1}{z-E_3} \langle \partial_\mu 1|3\rangle \langle 3|\partial_\nu 1\rangle + \frac{1}{z-E_1} \langle \partial_\mu 1|3\rangle \langle \partial_\nu 3|1\rangle \Bigg) e^{\mu\nu}.
\end{aligned}$$

Let us now perform the contour integral about a particular energy E_n. All of them are equal to 1 if and only if n is equal to the index that appears in the denominator, hence

$$\begin{aligned}
\Omega^{(2)} &= -\frac{i}{2} \sum \left(\langle n|\partial_\mu 1\rangle \langle 1|\partial_\nu n\rangle + \langle n|\partial_\mu 1\rangle \langle \partial_\nu 1|n\rangle + \langle \partial_\mu 1|n\rangle \langle n|\partial_\nu 1\rangle + \langle \partial_\mu n|1\rangle \langle \partial_\nu 1|n\rangle \right) e^{\mu\nu} \\
&= -\frac{i}{2} \sum \left(-\langle \partial_\mu n|1\rangle \langle 1|\partial_\nu n\rangle + \langle \partial_\mu n|1\rangle \langle 1|\partial_\nu n\rangle + \langle 1|\partial_\mu n\rangle \langle \partial_\nu n|1\rangle - \langle \partial_\mu n|1\rangle \langle 1|\partial_\nu n\rangle \right) e^{\mu\nu} \\
&= \frac{i}{2} \left(\langle \partial_\mu n|\partial_\nu n\rangle - \langle \partial_\nu n|\partial_\mu n\rangle \right) e^{\mu\nu},
\end{aligned}$$

and thus

$$\Omega^{(2)}_{\mu\nu} = i \left(\langle \partial_\mu n|\partial_\nu n\rangle - \langle \partial_\nu n|\partial_\mu n\rangle \right).$$

Let us now go to the general case. It will contain an operator product

$$\begin{aligned}
&|1\rangle\langle 1| \left(-\partial_\mu E_2 |2\rangle\langle 2| + (z-E_2)\left(|\partial_\mu 2\rangle\langle 2| + |2\rangle\langle \partial_\mu 2|\right)\right) |3\rangle\langle 3| \left(-\partial_\nu E_4 |4\rangle\langle 4| + (z-E_4)\left(|\partial_\nu 4\rangle\langle 4| + |4\rangle\langle \partial_\nu 4|\right)\right) |5\rangle\langle 5| \\
&= |1\rangle \left(-\partial_\mu E_2 \delta_{12}\delta_{23} + (z-E_2)\left(\langle 1|\partial_\mu 2\rangle \delta_{23} + \delta_{12} \langle \partial_\mu 2|3\rangle\right)\right) \left(-\partial_\nu E_4 \delta_{34}\delta_{45} + (z-E_4)\left(\langle 3|\partial_\nu 4\rangle \delta_{45} + \delta_{34} \langle \partial_\nu 4|5\rangle\right)\right) \langle 5|,
\end{aligned}$$

and the trace will turn this to

$$\left(-\partial_\mu E_2 \delta_{12}\delta_{23} + (z-E_2)\left(\langle 1|\partial_\mu 2\rangle \delta_{23} + \delta_{12} \langle \partial_\mu 2|3\rangle\right)\right) \left(-\partial_\nu E_4 \delta_{34}\delta_{45} + (z-E_4)\left(\langle 3|\partial_\nu 4\rangle \delta_{45} + \delta_{34} \langle \partial_\nu 4|5\rangle\right)\right) \delta_{15}.$$

Each bracket has three terms, so let us denote their products (after adding the extra factors) as a_{ij}, with i and j denoting which terms from each of the brackets are multiplied. We know that when tracing $a_{22}+a_{23}+a_{32}+a_{33}$, we get the result. We will thus have completed the proof if we can show that the others yield no net contribution. First we have

$$\begin{aligned}\sum a_{11} &= \sum \frac{1}{(z-E_1)(z-E_3)(z-E_5)} \left(-\partial_\mu E_2 \delta_{12}\delta_{23}\right)\left(-\partial_\nu E_4 \delta_{34}\delta_{45}\right)\delta_{15} e^{\mu\nu} \\ &= \sum \frac{\partial_\mu(E_2)\partial_\nu(E_4)}{(z-E_1)(z-E_3)(z-E_5)} \delta_{12}\delta_{23}\delta_{34}\delta_{45}\delta_{15} e^{\mu\nu} \\ &= \sum \frac{\partial_\mu(E_1)\partial_\nu(E_1)}{(z-E_1)^3} e^{\mu\nu}.\end{aligned}$$

This is identically zero as it contains a contraction of symmetric components with the antisymmetric differential form basis. As for the others we have

$$\begin{aligned}\sum a_{12} &= -\sum \frac{\partial_\mu E_2 (z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} \delta_{12}\delta_{23} \langle 3|\partial_\nu 4\rangle \delta_{45}\delta_{15} e^{\mu\nu} = -\sum \frac{\partial_\mu E_1}{(z-E_1)^2} \langle 1|\partial_\nu 1\rangle e^{\mu\nu}, \\ \sum a_{13} &= -\sum \frac{\partial_\mu E_2 (z-E_4)}{(z-E_1)(z-E_3)(z-E_5)} \delta_{12}\delta_{23}\delta_{34} \langle \partial_\nu 4|5\rangle \delta_{15} e^{\mu\nu} = -\sum \frac{\partial_\mu E_1}{(z-E_1)^2} \langle \partial_\nu 1|1\rangle e^{\mu\nu}, \\ \sum a_{21} &= -\sum \frac{\partial_\nu E_4 (z-E_2)}{(z-E_1)(z-E_3)(z-E_5)} \langle 1|\partial_\mu 2\rangle \delta_{23}\delta_{34}\delta_{45}\delta_{15} e^{\mu\nu} = -\sum \frac{\partial_\nu E_1}{(z-E_1)^2} \langle 1|\partial_\mu 1\rangle e^{\mu\nu}, \\ \sum a_{31} &= -\sum \frac{\partial_\nu E_4 (z-E_2)}{(z-E_1)(z-E_3)(z-E_5)} \delta_{12} \langle \partial_\mu 2|3\rangle \delta_{34}\delta_{45}\delta_{15} e^{\mu\nu} = -\sum \frac{\partial_\nu E_1}{(z-E_1)^2} \langle \partial_\mu 1|1\rangle e^{\mu\nu},\end{aligned}$$

and these all cancel each other exactly, completing the proof.

Properties of Parametrized States We will now derive some useful properties of derivatives of states of parametrized systems. Because orthogonality is preserved we have

$$\partial_\mu \langle m|n\rangle = \langle \partial_\mu m|n\rangle + \langle m|\partial_\mu n\rangle = 0.$$

Because the identity is also preserved we have

$$\sum |\partial_\mu n\rangle\langle n| + |n\rangle\langle \partial_\mu n| = 0.$$

Applications of the Above What is the use of the Berry connection and curvature? From the point of view of computing responses in quantum systems, the answer might seem to be not much. More broadly, however, it turns out that the Berry curvature in particular is useful in characterizing systems. The idea is as follows: An important assumption of the above arguments is the existence of a non-degenerate and gapped ground state for all parameter values. If the integral $\Omega^{(2)}$ over some closed surface is zero, it follows by Stokes' theorem that there are no degeneracy points enclosed by the surface, as $\Omega^{(2)}$ is closed everywhere. As such, the Berry curvature can be used to detect degeneracy points. It also turns out that the degeneracy points are stable under deformations of the Hamiltonian, as integrals of the Berry curvature are quantized.

The Single Spin - an Example Consider a single spin-$\frac{1}{2}$ in an external field. The Hamiltonian is

$$\mathcal{H} = h_x\sigma_x + h_y\sigma_y + h_z\sigma_z.$$

With respect to the σ_z eigenstates at $\theta = \phi = 0$, which are of course angle-independent, we have

$$|{\downarrow}\rangle_{\theta,\phi} = \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right)\end{bmatrix}, \ |{\uparrow}\rangle_{\theta,\phi} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right)\end{bmatrix}, \tag{2}$$

and thus

$$A_{-,\theta} = 0, \ A_{-,\phi} = \sin^2\left(\frac{\theta}{2}\right), \ A_{+,\theta} = 0, \ A_{+,\phi} = \cos^2\left(\frac{\theta}{2}\right).$$

The Berry curvature is then

$$\Omega^{(2)}_{\pm,\theta\phi} = \mp\frac{1}{2}\sin(\theta).$$

This implies that the Berry phase induced after an adiabatic cycle is equal to half the subtended solid angle.

This example is brought up time after time in this context, but rarely elaborated upon. It was the first example considered in the field, as Berry used it in his first paper describing the Berry phase, lending it some credibility, but the addition of the Berry curvature, and in particular its integral over all of S^2, is a later invention. Its significance is never stated, but still high, as the Berry curvature is, in a sense, produced by the degeneracy point at the origin. This shows both the value of the Berry curvature as a way to probe unknown parts of the phase diagram of a system and its use for creating topological invariants to classify systems.

The Quantum Hall Effect Consider a two-dimensional electron gas in an external electromagnetic field. If the magnetic field is constant, there exists a generalized translation operator which commutes with the Hamiltonian, and we can thus introduce Bloch functions according to

$$\psi_{n,\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,\mathbf{k}}(\mathbf{r}).$$

These are eigenfunctions of the Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} + \hbar\mathbf{k} + \mathbf{A}(\mathbf{r})\right)^2 + V(\mathbf{r}),$$

with energies $E_n(\mathbf{k})$. The current carried in this system is proportional to the expectation value

$$\frac{1}{m}\left\langle \psi_{n,\mathbf{k}} \middle| \mathbf{v} \middle| \psi_{n,\mathbf{k}} \right\rangle = \frac{1}{\hbar}\left\langle u_{n,\mathbf{k}} \middle| \vec{\boldsymbol{\nabla}}_{\mathbf{k}} H \middle| u_{n,\mathbf{k}} \right\rangle.$$

Let us study this by bringing it into the framework of parametrized systems. More specifically, let us study the expectation values of generalized velocity operators

$$V_\mu = \partial_\mu \mathcal{H}.$$

To that end, we consider states $|\phi_n\rangle$ that remain close to, but not identical to, the instantaneous eigenstate $|n\rangle$, and write it as

$$|\phi_n(t)\rangle = e^{-i\int\limits_0^t \mathrm{d}\tau E_n(\tau)} \left|\tilde{\phi}_n\right\rangle.$$

We then find

$$\begin{aligned}\mathcal{H}\,|\phi_n\rangle =& e^{-i\int\limits_0^t \mathrm{d}\tau E_n(\tau)} \mathcal{H}\left|\tilde{\phi}_n\right\rangle \\ =& e^{-i\int\limits_0^t \mathrm{d}\tau E_n(\tau)} \left(E_n + i\partial_t\right)\left|\tilde{\phi}_n\right\rangle,\end{aligned}$$

and thus

$$\partial_\mu \mathcal{H}\left|\tilde{\phi}_n\right\rangle = \partial_\mu\left(E_n + i\partial_t\right)\left|\tilde{\phi}_n\right\rangle + \left(E_n - \mathcal{H}\right)\partial_\mu\left|\tilde{\phi}_n\right\rangle.$$

Because the Hamiltonian contains no time derivatives, it may equally well be computed with $|\phi_n(t)\rangle$ and $\left|\tilde{\phi}_n\right\rangle$, and we then have

$$\begin{aligned}\left\langle V_\mu\right\rangle_n =& \partial_\mu E_n + \left\langle \tilde{\phi}_n \middle| i\partial_\mu\partial_t \middle| \tilde{\phi}_n \right\rangle + \left\langle \tilde{\phi}_n \middle| \left(E_n - \mathcal{H}\right)\partial_\mu \middle| \tilde{\phi}_n \right\rangle \\ =& \partial_\mu E_n + i\left(\left\langle \tilde{\phi}_n \middle| \partial_\mu\partial_t \middle| \tilde{\phi}_n \right\rangle + \partial_t\left(\left\langle \tilde{\phi}_n \right|\right)\partial_\mu\left|\tilde{\phi}_n\right\rangle\right).\end{aligned}$$

The first term will correspond to the usual Bloch drift velocity, and the remaining terms are corrections due to the parametrized nature of the system. To work the corrections out, we use the fact that

$$\left\langle \tilde{\phi}_n \middle| i\partial_t \middle| \tilde{\phi}_n \right\rangle = \left\langle \tilde{\phi}_n \middle| \left(\mathcal{H} - E_n\right) \middle| \tilde{\phi}_n \right\rangle$$

is negligible in the adiabatic limit, yielding

$$\left\langle V_\mu\right\rangle_n = \partial_\mu E_n + i\left(\partial_t\left(\left\langle \tilde{\phi}_n \right|\right)\partial_\mu\left|\tilde{\phi}_n\right\rangle - \partial_\mu\left(\left\langle \tilde{\phi}_n \right|\right)\partial_t\left|\tilde{\phi}_n\right\rangle\right).$$

Furthermore, as the time dependence only manifests through the time-dependent parameters, one can expand the time derivative and obtain in the adiabatic limit

$$\begin{aligned}\langle V_\mu\rangle_n =&\partial_\mu E_n + i\frac{\mathrm{d}R^\nu}{\mathrm{d}t}\left(\partial_\nu\left(\left\langle\tilde{\phi}_n\right|\right)\partial_\mu\left|\tilde{\phi}_n\right\rangle - \partial_\mu\left(\left\langle\tilde{\phi}_n\right|\right)\partial_\nu\left|\tilde{\phi}_n\right\rangle\right)\\ =&\partial_\mu E_n - \Omega^{(2)}_{\mu\nu}\frac{\mathrm{d}R^\nu}{\mathrm{d}t}.\end{aligned}$$

In particular, for Bloch electrons we have $\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -e\mathbf{E}$, and the total correction to the conductivity is

$$\mathbf{J} \propto \mathbf{E}\times\int \mathbf{\Omega}^{(2)}.$$

This is a Hall-style contribution, and the quantization of the total Berry curvature produces the integer quantum Hall effect.

Higher Berry Curvature and the KS Invariant For an infinite $1d$ system, $\Omega^{(2)}$ might diverge. A convergent quantity might instead be found by splitting the Hamiltonian into a sum of local terms working at a finite range, i.e.

$$\mathcal{H} = \sum_{p\in\mathbb{Z}} \mathcal{H}_p.$$

The quantity

$$F^{(2)}_{pq} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H}_p G^2 d\mathcal{H}_q\big)$$

then decays exponentially with respect to $|p-q|$ if the Hamiltonian is gapped, and is thus well-defined. Next we can construct the two-form

$$F^{(2)}_q = \sum_{p\in\mathbb{Z}} F^{(2)}_{pq} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H} G^2 d\mathcal{H}_q\big).$$

Its exterior derivative is given by

$$dF^{(2)}_q = \sum_{p\in\mathbb{Z}} F^{(3)}_{pq}.$$

We have

$$\begin{aligned}&\partial_\rho(G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q)e^{\rho\mu\nu}\\ =&\left(\partial_\rho G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\rho\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p\partial_\rho G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p G^2\partial_\rho\partial_\nu\mathcal{H}_q\right)e^{\rho\mu\nu}\\ =&\left(\partial_\rho G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p(\partial_\rho GG + G\partial_\rho G)\partial_\nu\mathcal{H}_q\right)e^{\rho\mu\nu}\\ =&\left(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p(G\partial_\rho\mathcal{H}G^2 + G^2\partial_\rho\mathcal{H}G)\partial_\nu\mathcal{H}_q\right)e^{\rho\mu\nu}\\ =&\left(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p(G\partial_\rho\mathcal{H}G^2 + G^2\partial_\rho\mathcal{H}G)\partial_\nu\mathcal{H}_q\right)e^{\rho\mu\nu}\\ =&\left(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p G\partial_\rho\mathcal{H}G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p G^2\partial_\rho\mathcal{H}G\partial_\nu\mathcal{H}_q\right)e^{\rho\mu\nu},\end{aligned}$$

and thus

$$\begin{aligned}&\mathrm{tr}\big(\partial_\rho(G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q)\big)e^{\rho\mu\nu}\\ =&\,\mathrm{tr}\big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p G\partial_\rho\mathcal{H}G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p G^2\partial_\rho\mathcal{H}G\partial_\nu\mathcal{H}_q\big)e^{\rho\mu\nu}\\ =&\,\mathrm{tr}\big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_p G^2\partial_\nu\mathcal{H}_q - G\partial_\rho\mathcal{H}G^2\partial_\mu\mathcal{H}_q G\partial_\nu\mathcal{H}_p + G\partial_\mu\mathcal{H}_p G^2\partial_\rho\mathcal{H}G\partial_\nu\mathcal{H}_q\big)e^{\rho\mu\nu}.\end{aligned}$$

Somehow we are to find

$$F^{(3)}_{pq} = \frac{i}{6}\oint\frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(G^2 d\mathcal{H}Gd\mathcal{H}_p G d\mathcal{H}_q - Gd\mathcal{H}G^2 d\mathcal{H}_p G d\mathcal{H}_q\big) - (p\leftrightarrow q).$$

To compute this we expand in eigenstates of the Hamiltonian according to

$$\begin{aligned}G^2 d\mathcal{H} G d\mathcal{H}_p G d\mathcal{H}_q &= \sum \frac{|1\rangle\langle 1|\,|2\rangle\langle 2|\,d\mathcal{H}\,|3\rangle\langle 3|\,d\mathcal{H}_p\,|4\rangle\langle 4|\,d\mathcal{H}_q}{(z-E_1)(z-E_2)(z-E_3)(z-E_4)}\\ &= \sum \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(z-E_1)^2(z-E_2)(z-E_3)}.\end{aligned}$$

Let us now compute the contour integral around the ground state. The contributions from where only one number is zero is

$$\begin{aligned}&\sum -|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q \left(\frac{1}{(E_0-E_2)^2(E_0-E_3)} + \frac{1}{(E_0-E_2)(E_0-E_3)^2}\right)\\ &\quad + \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_1)^2(E_0-E_3)} + \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)^2(E_0-E_2)}.\end{aligned}$$

Introducing

$$G_0 = \sum_{n\neq 0} \frac{1}{E_0-E_n}\,|n\rangle\langle n|\,,$$

this can be written as

$$-|0\rangle\langle 0|\left(d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q + d\mathcal{H}G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_q\right) + G_0^2 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0 d\mathcal{H}_q + G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q.$$

Similarly, when two of the numbers are zero we get the contribution

$$\begin{aligned}&\sum \frac{1}{2}\left(2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_3)^3} + 2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_2)^3}\right) - 2\frac{|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)^3}\\ &= |0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0^3 d\mathcal{H}_q + |0\rangle\langle 0|\,d\mathcal{H}G_0^3 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q - 2G_0^3 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q.\end{aligned}$$

Finally, if none or all of them are zero there is no contribution. Next, we have

$$G d\mathcal{H} G^2 d\mathcal{H}_p G d\mathcal{H}_q = \sum \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(z-E_1)(z-E_2)^2(z-E_3)}.$$

The contributions after computing the contour integral are

$$\begin{aligned}&\sum -|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q \left(\frac{1}{(E_0-E_1)^2(E_0-E_3)} + \frac{1}{(E_0-E_1)(E_0-E_3)^2}\right)\\ &\quad + \frac{|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_2)^2(E_0-E_3)} + \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)(E_0-E_2)^2}\\ &= -G_0^2 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0 d\mathcal{H}_q - G_0 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0^2 d\mathcal{H}_q + |0\rangle\langle 0|\,d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q + G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q\end{aligned}$$

when one number is zero and

$$\begin{aligned}&\sum \frac{1}{2}\left(2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_3)^3} + 2\frac{|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)^3}\right) - 2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_2)^2}\\ &= |0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0^3 d\mathcal{H}_q + G_0^3 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q - 2\,|0\rangle\langle 0|\,d\mathcal{H}G_0^3 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q\end{aligned}$$

when two are. The final result is thus

$$\begin{aligned}F^{(3)}_{pq} &= \frac{i}{6}\Big(-\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q\right\rangle - \left\langle d\mathcal{H}G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_q\right\rangle + \left\langle d\mathcal{H}_p G_0 d\mathcal{H}_q G_0^2 d\mathcal{H}\right\rangle\\ &\quad + \left\langle d\mathcal{H}_q G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\right\rangle + \langle d\mathcal{H}\rangle\left\langle d\mathcal{H}_p G_0^3 d\mathcal{H}_q\right\rangle + \left\langle d\mathcal{H}G_0^3 d\mathcal{H}_p\right\rangle\langle d\mathcal{H}_q\rangle\\ &\quad - 2\left\langle d\mathcal{H}_q G_0^3 d\mathcal{H}\right\rangle\langle d\mathcal{H}_p\rangle + \left\langle d\mathcal{H}_p G_0 d\mathcal{H}_q G_0^2 d\mathcal{H}\right\rangle + \left\langle d\mathcal{H}_p G_0^2 d\mathcal{H}_q G_0 d\mathcal{H}\right\rangle\\ &\quad - \left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q\right\rangle - \left\langle d\mathcal{H}_q G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\right\rangle - \langle d\mathcal{H}\rangle\left\langle d\mathcal{H}_p G_0^3 d\mathcal{H}_q\right\rangle\\ &\quad - \langle d\mathcal{H}_p\rangle\left\langle d\mathcal{H}_q G_0^3 d\mathcal{H}\right\rangle + 2\left\langle d\mathcal{H}G_0^3 d\mathcal{H}_p\right\rangle\langle d\mathcal{H}_q\rangle\Big) - (p \leftrightarrow q)\\ &= \frac{i}{6}\Big(-2\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q\right\rangle - \left\langle d\mathcal{H}G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_q\right\rangle + 2\left\langle d\mathcal{H}_p G_0 d\mathcal{H}_q G_0^2 d\mathcal{H}\right\rangle\\ &\quad + \left\langle d\mathcal{H}_p G_0^2 d\mathcal{H}_q G_0 d\mathcal{H}\right\rangle + \left\langle d\mathcal{H}_q G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\right\rangle - \left\langle d\mathcal{H}_q G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\right\rangle\\ &\quad + 3\left\langle d\mathcal{H}G_0^3 d\mathcal{H}_p\right\rangle\langle d\mathcal{H}_q\rangle - 3\left\langle d\mathcal{H}_q G_0^3 d\mathcal{H}\right\rangle\langle d\mathcal{H}_p\rangle\Big) - (p \leftrightarrow q),\end{aligned}$$

where all the expectation values are computed in the ground state.

This quantity is somewhat difficult to manage, but one can reduce it somewhat. First, states excited outside of the support of $\mathcal{H}_p$ and $\mathcal{H}_q$ do not contribute, as they are orthogonal to the ground state and can pass through $\mathcal{H}_p$ and $\mathcal{H}_q$, as well as their exterior derivatives. By a similar token, $F^{(3)}_{pq}$ is non-zero only if $\mathcal{H}_p$ and $\mathcal{H}_q$ have overlapping support. This also implies that the only terms in the Hamiltonian that contribute are the ones with support overlapping with both $\mathcal{H}_p$ and $\mathcal{H}_q$.

Using these quantities we can construct a 3-form Berry curvature

$$\Omega^{(3)}(f) = \frac{1}{2}\sum_{p,q\in\mathbb{Z}} F^{(3)}_{pq}(f(q)-f(p)).$$

f is some sigmoid function, its particular shape turning out to be unimportant. A simple choice is $f(p) = \Theta(p-a)$ for some $a \in \mathbb{Z}+\frac{1}{2}$. For this particular choice we have

$$\begin{aligned}
\Omega^{(3)}(f) =& \frac{1}{2}\sum_{p,q\in\mathbb{Z}} F^{(3)}_{pq}(\Theta(q-a)-\Theta(p-a)) \\
=& \frac{1}{2}\sum_{p\in\mathbb{Z},\ q>a} F^{(3)}_{pq}(1-\Theta(p-a)) - \frac{1}{2}\sum_{p\in\mathbb{Z},\ q<a} F^{(3)}_{pq}\Theta(p-a) \\
=& \frac{1}{2}\sum_{p<a,\ q>a} F^{(3)}_{pq} - \frac{1}{2}\sum_{p>a,\ q<a} F^{(3)}_{pq} \\
=& \sum_{p<a,\ q>a} F^{(3)}_{pq},
\end{aligned}$$

using the antisymmetry of $F^{(3)}_{pq}$.

Finally we can define the KS invariant

$$Q_{\mathrm{KS}} = \int \Omega^{(3)}(f),$$

with integration over the full parameter space of the Hamiltonian. This is a topological invariant.

The Dimerized Spin Chain Consider an infinite spin chain with Hamiltonian

$$\mathcal{H}_{1d} = \sum_{p\in\mathbb{Z}} \mathcal{H}^1_p(w) + \sum_{p\in 2\mathbb{Z}+1} \mathcal{H}^{2,+}_{p,p+1}(w) + \sum_{p\in 2\mathbb{Z}} \mathcal{H}^{2,-}_{p,p+1}(w).$$

The parameter takes values on S^3. There are three kinds of terms here. The first is

$$\mathcal{H}^1_p(w) = (-1)^p(w_1\sigma^1_p + w_2\sigma^2_p + w_3\sigma^3_p),$$

which is some fluctuating on-site term. The two others are

$$\mathcal{H}^{2,\pm}_{p,p+1}(w) = g^{\pm}(w)\sum_{\mu=1,2,3}\sigma^\mu_p\sigma^\mu_{p+1},$$

with two functions

$$g^+(w) = \begin{cases} w_4, & 0\le w_4\le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad g^-(w) = \begin{cases} -w_4, & -1\le w_4\le 0, \\ 0, & \text{otherwise}. \end{cases}$$

This type of interaction defines five distinct regimes:

- $w_4 = 1$, where there is only odd-even bonding.
- $0 < w_4 < 1$, where there is odd-even bonding and on-site interactions.
- $w_4 = 0$, where there is only on-site interaction.
- $-1 < w_4 < 0$, where there is even-odd bonding and on-site interactions.

- $w_4 = 1$, where there is only even-odd bonding.

To compute the 3-form Berry curvature and KS invariant, we rewrite the Hamiltonian as a sum of local terms. These are

$$\mathcal{H}_p(w) = \mathcal{H}^1_p(w) + x\mathcal{H}^{2,\pm}_{p,p+1}(w) + (1-x)\mathcal{H}^{2,\mp}_{p-1,p}(w).$$

The top sign is for odd p. The new parameter x is an extra control parameter, taken to be fixed. Its introduction is an explicit representation of the ambiguity of the choice of local terms.

For the sigmoid function f we choose a Heaviside function, this time leaving us with two variants - f with $a \in 2\mathbb{Z} - \frac{1}{2}$ and f' with $a \in 2\mathbb{Z} + \frac{1}{2}$. To see how they differ, consider the regime $w_4 > 0$. In this case f splits the dimer in two and f' switches on between two dimers.

Because the local terms in the Hamiltonian only interact at range 1 in either direction, the eigenstates of the system for any parameter choice are product states over each dimer. This means

$$\Omega^{(3)}(f) = \Omega^{(3)}(f') = F^{(3)}_{a-\frac{1}{2},a+\frac{1}{2}},$$

with the particular choice of a distinguishing the two cases. $\Omega^{(3)}(f)$ is only non-trivial if the sites $a \pm \frac{1}{2}$ belong to the same dimer, hence $\Omega^{(3)}(f) = 0$ unless $w_4 > 0$ and $\Omega^{(3)}(f') = 0$ unless $w_4 < 0$.

We will need to diagonalize the dimer, so we first transform the basis from an angle-independent one into one parallel with the Zeeman field using a unitary operator U. This transforms states according to $|\psi\rangle \to |\psi\rangle_{\theta,\phi} = U|\psi\rangle$ and any operator according to $A \to a = UAU^\dagger$, the explicit angle dependence having been removed from the left-hand side of both equalities. This angle dependence is instead baked into the basis. The small-letter notation will be useful for clarification when a matrix representation is invoked. Having applied this transformation we choose simultaneous eigenstates of $S'_{z,p} + S'_{z,p+1}$ and $(S'_p + S'_{p+1})^2$, which are also eigenstates of $(S'_p)^2$ and $(S'_{p+1})^2$. The vector appearing in the Zeeman term has length $\sqrt{1-w_4^2}$, meaning

$$h_p = -2\sqrt{1-w_4^2}S'_{z,p} + 4xw_4 S'_p \cdot S'_{p+1}, \ h_{p+1} = 2\sqrt{1-w_4^2}S_{z,p+1} + 4(1-x)w_4 S'_p \cdot S'_{p+1}$$

for $p = a - \frac{1}{2}$. Furthermore, as

$$S'_p \cdot S'_{p+1} = \frac{1}{2}((S'_p + S'_{p+1})^2 - (S'_p)^2 - (S'_{p+1})^2),$$

we have

$$h_p = -\sqrt{1-w_4^2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + 2xw_4 \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix},$$

$$h_{p+1} = \sqrt{1-w_4^2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + (1-x)w_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

in the eigenbasis of total spin, and the total dimer Hamiltonian is

$$h = w_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} - 2\sqrt{1-w_4^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenstates $|1,1\rangle$ and $|1,-1\rangle$ are still eigenstates of the total Hamiltonian, with energy w_4. In addition there are two eigenstates found by diagonalizing

$$\begin{bmatrix} w_4 & -2\sqrt{1-w_4^2} \\ -2\sqrt{1-w_4^2} & -3w_4 \end{bmatrix}.$$

The energies are $\pm 2 - w_4$, with eigenstates

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -\sqrt{1+w_4} & \sqrt{1-w_4} \\ \sqrt{1-w_4} & \sqrt{1+w_4} \end{bmatrix}.$$

We proceed by introducing hyperspherical coordinates

$$w_1 = \sin(\alpha)\cos(\theta),\ w_2 = \sin(\alpha)\sin(\theta)\cos(\phi),\ w_3 = \sin(\alpha)\sin(\theta)\sin(\phi),\ w_4 = \cos(\alpha),$$

for which we have

$$h_p = -\sin(\alpha)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} + x\cos(\alpha)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{bmatrix},$$

$$h_{p+1} = \sin(\alpha)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\end{bmatrix} + (1-x)\cos(\alpha)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{bmatrix},$$

$$h_p + h_{p+1} = h = \cos(\alpha)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{bmatrix} - 2\sin(\alpha)\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}.$$

The eigenstates of individual spin are given in equation 2, and we then have

$$|1,1\rangle_{\theta,\phi} = \begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right)e^{-2i\phi}\\ \frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi}\\ \sin^2\left(\frac{\theta}{2}\right)\\ 0\end{bmatrix},\ |1,0\rangle_{\theta,\phi} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\sin(\theta)e^{-2i\phi}\\ \cos(\theta)e^{-i\phi}\\ \frac{1}{\sqrt{2}}\sin(\theta)\\ 0\end{bmatrix},\ |1,-1\rangle_{\theta,\phi} = \begin{bmatrix}\sin^2\left(\frac{\theta}{2}\right)e^{-2i\phi}\\ -\frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi}\\ \cos^2\left(\frac{\theta}{2}\right)\\ 0\end{bmatrix},\ |0,0\rangle_{\theta,\phi} = \begin{bmatrix}0\\ 0\\ 0\\ e^{-i\phi}\end{bmatrix}$$

with respect to the total spin basis for $\theta = \phi = 0$. We can then explicitly write

$$U = \begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right)e^{-2i\phi} & -\frac{1}{\sqrt{2}}\sin(\theta)e^{-2i\phi} & \sin^2\left(\frac{\theta}{2}\right)e^{-2i\phi} & 0\\ \frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi} & \cos(\theta)e^{-i\phi} & -\frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi} & 0\\ \sin^2\left(\frac{\theta}{2}\right) & \frac{1}{\sqrt{2}}\sin(\theta) & \cos^2\left(\frac{\theta}{2}\right) & 0\\ 0 & 0 & 0 & e^{-i\phi}\end{bmatrix}.$$

Let us also derive an expression for g_0. The eigenstates of the Hamiltonian in the angle-dependent basis are

$$v_{-2-\cos(\alpha)} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ \sqrt{1-\cos(\alpha)}\\ 0\\ \sqrt{1+\cos(\alpha)}\end{bmatrix},\ v_{\cos(\alpha),1} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ v_{\cos(\alpha),2} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ v_{2-\cos(\alpha)} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ -\sqrt{1+\cos(\alpha)}\\ 0\\ \sqrt{1-\cos(\alpha)}\end{bmatrix}.$$

Forming these into a matrix V and computing VDV^{-1} for

$$D = \begin{bmatrix}-\frac{1}{2(1+\cos(\alpha))} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{2(1+\cos(\alpha))} & 0\\ 0 & 0 & 0 & -\frac{1}{4}\end{bmatrix}$$

nets us

$$g_0 = \begin{bmatrix}-\frac{1}{2(1+\cos(\alpha))} & 0 & 0 & 0\\ 0 & -\frac{1}{8}(1+\cos(\alpha)) & 0 & \frac{1}{8}\sin(\alpha)\\ 0 & 0 & -\frac{1}{2(1+\cos(\alpha))} & 0\\ 0 & \frac{1}{8}\sin(\alpha) & 0 & -\frac{1}{8}(1-\cos(\alpha))\end{bmatrix}.$$

The three angles are now neatly separated, as ϕ and θ only enter in U and α only enters in the combination of eigenstates after U has been applied. Using the explicit formula we then have

$$\begin{aligned}F^{(3)}_{p,p+1} =& \frac{i}{6}\Big(-2\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1}\right\rangle - \left\langle d\mathcal{H}G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_{p+1}\right\rangle + 2\left\langle d\mathcal{H}_p G_0 d\mathcal{H}_{p+1} G_0^2 d\mathcal{H}\right\rangle\\ &+ \left\langle d\mathcal{H}_p G_0^2 d\mathcal{H}_{p+1} G_0 d\mathcal{H}\right\rangle + \left\langle d\mathcal{H}_{p+1} G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\right\rangle - \left\langle d\mathcal{H}_{p+1} G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\right\rangle\\ &+ 3\left\langle d\mathcal{H}G_0^3 d\mathcal{H}_p\right\rangle\left\langle d\mathcal{H}_{p+1}\right\rangle - 3\left\langle d\mathcal{H}_{p+1}G_0^3 d\mathcal{H}\right\rangle\left\langle d\mathcal{H}_p\right\rangle\Big) - (p \leftrightarrow p+1).\end{aligned}$$

In order to get non-trivial results, we must compute these expectation values in an angle-independent basis. To that end, we note that all the operators involved only depend on α in the angle-dependent basis. We then write $A = U^\dagger a U$ and consider its expectation value in some angle-dependent state $|\psi\rangle = U^\dagger |\psi\rangle_{\theta,\phi}$. We then have

$$\langle\psi|A|\psi\rangle = \langle\psi|UU^\dagger a U U^\dagger|\psi\rangle_{\theta,\phi} = \langle\psi|a|\psi\rangle_{\theta,\phi}\,.$$

This means, for instance, that

$$\begin{aligned}\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1}\right\rangle =& \left\langle U(d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1})_{\theta,\phi}U^\dagger\right\rangle_{\theta,\phi} \\ =& \left\langle U d(U^\dagger h U)U^\dagger g_0^2 U d(U^\dagger h_p U)U^\dagger g_0 U d(U^\dagger h_{p+1}U)U^\dagger\right\rangle_{\theta,\phi}.\end{aligned}$$

This should generally be a polynomial of order up to 2 in x. To simplify the calculation we could start by eliminating terms. For example, the order-2 terms come from the contributions where h_p and h_{p+1} look identical. By antisymmetry with respect to the lattice sites these terms vanish, and the remaining contributions are of order 1 at most.

By antisymmetry we may compute any component of this form, so we choose $F^{(3)}_{p,p+1,\alpha\theta\phi}$, for which the two latter terms vanish.

This is going to take some time, so let's start. We have

$$\begin{aligned}&\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1}\right\rangle \\ =& -\frac{i}{64}\cos(\theta)\left(\cos(\alpha)-1\right)\left(2\sin(\alpha)\cos^2(\alpha) - 2\cos^3(\alpha) - 5\sin(\alpha)\cos(\alpha) + 3\cos(\alpha)^2 + \sin(\alpha) + 4\cos(\alpha) + 3\right)\end{aligned}$$

Using the explicit formula above we somehow find

$$\Omega^{(3)}_{\alpha\theta\phi} = \frac{1}{2}(2+\cos(\alpha))\tan^2\left(\frac{\alpha}{2}\right)\sin(\theta)$$

for $0 < \alpha < \frac{\pi}{2}$. We then have

$$\begin{aligned}Q_{\mathrm{KS}} =& \int_{S^3}\Omega^{(3)}_{\alpha\theta\phi} = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2\pi} \mathrm{d}\alpha\,\mathrm{d}\theta\,\mathrm{d}\phi\,(2+\cos(\alpha))\tan^2\left(\frac{\alpha}{2}\right)\sin(\theta) \\ =& 2\pi\int_0^{\frac{\pi}{2}}\mathrm{d}\alpha\,(2+\cos(\alpha))\tan^2\left(\frac{\alpha}{2}\right) = 2\pi.\end{aligned}$$

What if we were to use a different sigmoid? Swapping to f', which is zero for $w_4 > 0$. Elsewhere we find

$$\Omega^{(3)}_{\alpha\theta\phi} = \frac{1}{2}(2-\cos(\alpha))\cot^2\left(\frac{\alpha}{2}\right)\sin(\theta),$$

and thus

$$\begin{aligned}Q_{\mathrm{KS}} =& 2\pi\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}\alpha\,(2-\cos(\alpha))\cot^2\left(\frac{\alpha}{2}\right) \\ =& -2\pi\int_{\frac{\pi}{2}}^{0}\mathrm{d}\beta\,(2-\cos(\pi-\beta))\cot^2\left(\frac{\pi-\beta}{2}\right) \\ =& 2\pi\int_0^{\frac{\pi}{2}}\mathrm{d}\beta\,(2+\cos(\beta))\tan^2\left(\frac{\beta}{2}\right) \\ =& 2\pi,\end{aligned}$$

and indeed the choice of sigmoid was irrelevant.

Higher Berry Curvature in Arbitrary Dimensions We will construct higher Berry curvatures for lattice models in arbitrary dimensions by means of the language of chain complices. The links will be comprised of objects $A_{p_0 \dots p_n}$, called n-chains, which depend on $n+1$ points on a lattice Λ, are antisymmetric under permutations of the points and decay exponentially away from the diagonal $p_0 = \dots = p_n$. These belong to the space $C_n(\Lambda)$. The map linking the chains is defined as

$$(\partial A)_{p_1 \dots p_n} = \sum_{p_0 \in \Lambda} A_{p_0 \dots p_n}.$$

Dual to this we introduce n-cochains, which are functions $\alpha(p_0, \dots, p_n)$ that depend on $n+1$ lattice points, are antisymmetric under permutations and have the following property: When restricted to any δ-neighborhood of the diagonal, it vanishes when any of the points is outside some finite set. These belong to the space $C^n(\Lambda)$.

Having defined the two, we introduce the pairing

$$\langle A|\alpha\rangle = \frac{1}{(n+1)!} \sum_{p_0,\dots,p_n} A_{p_0,\dots,p_n} \alpha(p_0, \dots, p_n).$$

The cochains are linked by an operator δ which satisfies $\delta^2 = 0$ and

$$\langle A|\delta\alpha\rangle = \langle \partial A|\alpha\rangle\,,\ A \in C_n,\ \alpha \in C^{n-1}.$$

To identify it, we write explicitly

$$\langle \partial A|\alpha\rangle = \frac{1}{n!} \sum_{p_1,\dots,p_n} \left(\sum_{p_0} A_{p_0,\dots,p_n} \right) \alpha(p_1, \dots, p_n).$$

We want to factorize this as a sum containing A itself and some other object, so to do that, let us consider a simple example, namely $n = 1$. We have

$$\sum_{p_1,p_2} A_{p_1,p_2}\alpha(p_2) = \frac{1}{2} \sum_{p_1,p_2} A_{p_1,p_2}\alpha(p_2) - A_{p_2,p_1}\alpha(p_2) = \frac{1}{2} \sum_{p_1,p_2} A_{p_1,p_2}(\alpha(p_2) - \alpha(p_1)).$$

What we have achieved with this rewrite is pairing A with an element of C^1, as the cochain now has dependence on two points. Performing a similar process for higher n we find

$$\delta\alpha(p_0, \dots, p_n) = \sum_{i=0}^{n} (-1)^i \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_n).$$

The chains also have an exterior product, defined as

$$\alpha \cup \gamma(p_0, \dots, p_{n+m}) = \frac{1}{(n+m+1)!} \sum_{\sigma \in S_{n+m+1}} (-1)^{\mathrm{sgn}(\sigma)} \alpha(p_{\sigma(0)}, \dots, p_{\sigma(n)})\gamma(p_{\sigma(n)}, \dots, p_{\sigma(n+m)}).$$

It satisfies

$$\alpha \cup \gamma = (-1)^{nm} \gamma \cup \alpha,\ \ \delta(\alpha \cup \gamma) = \delta\alpha \cup \gamma + (-1)^n \alpha \cup \delta\gamma.$$

We now have the mathematical machinery to define the higher Berry curvature. Partially by inspiration from the calculation in $d = 1$, the starting point will be $n-2$-chains $F^{(n)}$, which are valued in n-forms. From some starting point, these forms are defined by the descent equation

$$dF^{(n)} = \partial F^{(n+1)}.$$

To construct a higher Berry curvature, we will need to contract these with some $n-2$-cochain - in other words, we will use some construction of the form

$$\Omega^{(d+2)} = \left\langle F^{(d+2)} \middle| \alpha \right\rangle.$$

The matching of numbers makes sense, as for $d = 0$ we want a 2-form (higher) Berry curvature with underlying dependence on at most one lattice point. We will also be interested in a higher KS invariant

$$\int \Omega^{(d+2)}.$$

The details of this construction may be of importance, so let us discuss the topological properties of the above. Under a deformation of the parameter manifold, the requirement that the KS invariant be conserved implies that $\Omega^{(d+2)}$ must be closed. For the general construction above we have

$$d\Big\langle F^{(d+2)}\Big|\alpha\Big\rangle = \Big\langle dF^{(d+2)}\Big|\alpha\Big\rangle = \Big\langle \partial F^{(d+3)}\Big|\alpha\Big\rangle = \Big\langle F^{(d+3)}\Big|\delta\alpha\Big\rangle .$$

The requirement that $\Omega^{(d+2)}$ be closed thus implies that α must be closed. On the other hand, assuming α to be exact we have

$$\Omega^{(d+2)} = \Big\langle F^{(d+2)}\Big|\delta\gamma\Big\rangle = \Big\langle \partial F^{(d+2)}\Big|\gamma\Big\rangle = \Big\langle dF^{(d+1)}\Big|\gamma\Big\rangle = d\Big\langle F^{(d+1)}\Big|\gamma\Big\rangle,$$

and as such the KS invariant is trivial. We thus see the importance of the cohomology of the cochain complex. It can be shown that the nth cohomology of the cochain complex is isomorphic to the nth cohomology of R^d. The latter is non-trivial for $n = d$ only, and is one-dimensional in this case. The significance of this result is twofold. First, it is the final nail in the coffin for $F^{(d+2)}$ being a d-chain. Second, it dictates that we may construct the higher Berry curvature by choosing any one generator of C^d.

Let us now finish the construction. To arrive at a sensible definition, the starting point we choose is

$$F_p^{(2)} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H}G^2 d\mathcal{H}_p\big).$$

Next we introduce functions f_i, some particular choice of sigmoid functions each depending the coordinate $x^i(p)$ of some lattice point p. The higher Berry curvature is then defined as

$$\Omega^{(d+2)} = \Big\langle F^{(d+2)}\Big|\delta f_1 \cup \cdots \cup \delta f_d\Big\rangle .$$

Before looking at some properties of the higher Berry curvature, let us first verify that this construction is sensible. We will do this by considering the two simplest examples. The first example is $d = 0$, for which the higher Berry curvature is

$$\Omega^{(2)} = \Big\langle F^{(2)}\Big|1\Big\rangle ,$$

as there is no lattice for $d = 0$. Evaluating this we find

$$\Omega^{(2)} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H}G^2 d\mathcal{H}\big),$$

and the definition is consistent with the usual Berry curvature. Note also that $\Omega^{(2)} = dA = \partial F^{(2)}$, and so the Berry connection is the true start.

Next, for $d = 1$ we introduce the sigmoid functions f and form a 1-cochain according to

$$\delta f(p,q) = f(p) - f(q).$$

The 1-chain for this case is $F^{(3)}$, which satisfies $dF^{(2)} = \partial F^{(3)}$. The higher Berry curvature for this case is

$$\Omega^{(3)} = \Big\langle F^{(3)}\Big|\delta f\Big\rangle = \frac{1}{2}\sum_{p,q} F_{pq}^{(3)}(f(p) - f(q)),$$

which is the same result as we previously arrived at.

Finally we investigate some properties of the higher Berry curvature. Firstly, it is closed, as

$$\begin{aligned} d\Omega^{(d+2)} =& \Big\langle dF^{(d+2)}\Big|\delta f_1 \cup \cdots \cup \delta f_d\Big\rangle \\ =& \Big\langle \partial F^{(d+3)}\Big|\delta f_1 \cup \cdots \cup \delta f_d\Big\rangle \\ =& \Big\langle F^{(d+3)}\Big|\delta(\delta f_1 \cup \cdots \cup \delta f_d)\Big\rangle \\ =& 0. \end{aligned}$$

Its cohomology class is also invariant under shifts by some compactly supported function. To show this, note that

$$\begin{aligned}\left\langle F^{(d+2)}\middle|\delta g\cup f_2\cup\cdots\cup\delta f_d\right\rangle =&\left\langle F^{(d+2)}\middle|\delta(g\cup f_2\cup\cdots\cup\delta f_d)\right\rangle\\ =&\left\langle \partial F^{(d+2)}\middle|g\cup f_2\cup\cdots\cup\delta f_d\right\rangle\\ =&\left\langle dF^{(d+2)}\middle|g\cup f_2\cup\cdots\cup\delta f_d\right\rangle\\ =&d\left\langle F^{(d+2)}\middle|g\cup f_2\cup\cdots\cup\delta f_d\right\rangle.\end{aligned}$$

As such, the exact choice of sigmoid is irrelevant.

Dimensional Reduction in Topological Insulators Topological insulators are generally described by some fermion model. Two classes of such models are A and AIII. The two are defined by not respecting time reversal and charge conjugation as symmetries. In addition, AIII has some unitary operator Γ that anticommutes with the Hamiltonian, also known as having a chiral symmetry. The existence of the chiral symmetry implies symmetry of the energy spectrum about the Fermi level.

It turns out that every model of class A in even spatial dimension can be related to a model of class AIII in one lower dimension. The process of relating the two is termed dimensional reduction. The scheme is as follows: Start with a Dirac model in $d = 2n + 2$ dimensions given by

$$\mathcal{H} = m\Gamma^{2n+3}_{(2n+3)} + \sum_{a=1}^{2n+2} k_a\Gamma^a_{(2n+3)},$$

with $\Gamma^a_{(2n+3)}$ being generators of the $2n + 3$-dimensional Clifford algebra. This is the class A model. The dimensional reduction is performed by setting $k_{2n+2} = 0$, yielding a model in $d = 2n + 2$ in class AIII, as the Hamiltonian now anticommutes with $\Gamma^{2n+2}_{(2n+3)}$.

Generally for these models we introduce Bloch states $|u^{\pm}_a(k)\rangle$, with k being confined to the first Brillouin zone, a being a band index and the sign dictating whether the state is occupied. From this we can define a non-Abelian Berry curvature for the occupied state according to

$$A_{ab,\mu} = \left\langle u^-_a(k)\middle|\partial_\mu\middle|u^-_b(k)\right\rangle.$$

The Berry curvature is then $F = dA + A^2$. Models in class A in $d = 2n+2$ can be characterized by the $n+1$th Chern character

$$\mathrm{ch}_{n+1}(F) = \frac{1}{(n+1)!}\,\mathrm{tr}\left(\left(\frac{iF}{2\pi}\right)^{n+1}\right),$$

as well as its integral, called the Chern number. The Chern character is the exterior derivative of the $2(n+1)-1$th Chern-Simons form, defined as

$$Q_{2n+1} = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1}\int_0^1 \mathrm{d}t\,\mathrm{tr}(AF^n_t),\ \ F_t = tdA + t^2A^2 = tF + (t^2-t)A^2.$$

Note that the Chern-Simons form is not gauge invariant. Instead, for a gauge transformation

$$A' = g^{-1}Ag + g^{-1}dg,\ \ F' = g^{-1}Fg,$$

we have

$$\begin{aligned}Q'_{2n+1} =&\frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1}\int_0^1 \mathrm{d}t\,\mathrm{tr}\Big((g^{-1}Ag+g^{-1}dg)\left(tg^{-1}Fg+(t^2-t)(g^{-1}Ag+g^{-1}dg)^2\right)^n\Big)\\ =&\frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1}\int_0^1 \mathrm{d}t\,\mathrm{tr}\Big((g^{-1}Ag+g^{-1}dg)\left(tg^{-1}Fg+(t^2-t)(g^{-1}A^2g+g^{-1}Adg++g^{-1}dgg^{-1}Ag+g^{-1}dgg^{-1}dg)\right.\end{aligned}$$

One term here is of course Q_{2n+1}. The other is the result of setting $A = g^{-1}dg$ and $F = 0$ in the Chern-Simons form. The remaining terms are the exterior derivative of some $2n$-form, somehow.

To introduce a similar concept for class AIII, we first introduce the projection matrix $P(k)$ onto the occupied bands and $Q = 1 - 2P$. The chiral symmetry somehow implies that the Q matrix can be written as

$$Q = \begin{bmatrix} 0 & q \\ q^\dagger & 0 \end{bmatrix},$$

with q being a unitary matrix, in the basis where Γ is diagonal. We can now introduce the winding number for a model in AIII and $d = 2n + 1$ as

$$\nu_{2n+1} = \int \omega_{2n+1}(q),\ \omega_{2n+1}(q) = \frac{(-1)^n n!}{(2n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \operatorname{tr}\big((q^{-1}dq)^{2n+1}\big).$$

The Chern-Simons forms define a characteristic class on odd-dimensional manifolds, meaning we can use them to characterize systems. We therefore introduce the Chern-Simons invariant

$$\mathrm{CS}_{2n+1} = \int Q_{2n+1},$$

with the integral being over the first Brillouin zone. Returning to the issue of gauge invariance, evaluating the Chern-Simons form at $g^{-1}dg$ gives the same result as the winding number evaluated at g. The winding number is an integer which in this case measures the winding g about the Brillouin zone. Note that $\pi_{\mathrm{BZ}}(U(N)) = \mathbb{Z}$. It therefore follows that the Wilson loop

$$W_{2n+1} = e^{2\pi i \mathrm{CS}_{2n+1}}$$

is gauge invariant.

Let us move on to a more specific example. The simplest would be starting in $d = 2$ with the Hamiltonian

$$\mathcal{H} = k_x\sigma_x + k_y\sigma_y + m\sigma_z.$$

Introducing $\lambda = \sqrt{k^2 + m^2}$, the eigenstates are given by

$$\left|u^-(k)\right\rangle = \frac{1}{\sqrt{2\lambda(\lambda+m)}}\begin{bmatrix} -k_x + ik_y \\ \lambda + m \end{bmatrix},\ \left|u^+(k)\right\rangle = \frac{1}{\sqrt{2\lambda(\lambda-m)}}\begin{bmatrix} k_x - ik_y \\ \lambda - m \end{bmatrix}.$$

Writing

$$\partial_\mu f(\sqrt{k^2+m^2}) = f' \frac{k_\mu}{\sqrt{k^2+m^2}}$$

we find

$$\begin{aligned}
A_x =& \frac{1}{\sqrt{2\lambda(\lambda+m)}}\begin{bmatrix} -k_x - ik_y & \lambda + m \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2\lambda(\lambda+m)}} - \frac{(-k_x+ik_y)(4\lambda+2m)}{2(2\lambda(\lambda+m))^{\frac{3}{2}}}\frac{k_x}{\lambda} \\ -\frac{\frac{m}{2\lambda^2}}{2\sqrt{\frac{1}{2}+\frac{m}{2\lambda}}}\frac{k_x}{\lambda} \end{bmatrix} \\
=& \frac{1}{2\lambda(\lambda+m)}\begin{bmatrix} -k_x - ik_y & \lambda + m \end{bmatrix}\begin{bmatrix} -1 - \frac{(-k_x+ik_y)(2\lambda+m)}{2\lambda(\lambda+m)}\frac{k_x}{\lambda} \\ -\frac{mk_x}{2\lambda^2} \end{bmatrix} \\
=& \frac{1}{2\lambda(\lambda+m)}\left(k_x + ik_y - \frac{k^2(2\lambda+m)k_x}{2\lambda^2(\lambda+m)} - \frac{m(\lambda+m)k_x}{2\lambda^2}\right) \\
=& \frac{ik_y}{2\lambda(\lambda+m)} + \frac{k_x}{4\lambda^3(\lambda+m)^2}\left(2\lambda^2(\lambda+m) - k^2(2\lambda+m) - m(\lambda+m)^2\right) \\
=& \frac{ik_y}{2\lambda(\lambda+m)} + \frac{k_x}{4\lambda^3(\lambda+m)^2}\left(2\lambda^3 + 2\lambda^2 m - 2k^2\lambda - k^2 m - m(\lambda^2 + m^2 + 2\lambda m)\right) \\
=& \frac{ik_y}{2\lambda(\lambda+m)}.
\end{aligned}$$

By symmetry we have

$$
\begin{aligned}
A_y =& \frac{1}{2\lambda(\lambda+m)} \begin{bmatrix} -k_x - ik_y & \lambda+m \end{bmatrix} \begin{bmatrix} i - \frac{(-k_x+ik_y)(2\lambda+m)}{2\lambda(\lambda+m)} \frac{k_y}{\lambda} \\ -\frac{mk_y}{2\lambda^2} \end{bmatrix} \\
=& -\frac{i}{2\lambda(\lambda+m)} \begin{bmatrix} -k_x - ik_y & \lambda+m \end{bmatrix} \begin{bmatrix} -1 - \frac{(-k_x+ik_y)(2\lambda+m)}{2\lambda(\lambda+m)} \frac{ik_y}{\lambda} \\ -\frac{mik_y}{2\lambda^2} \end{bmatrix} \\
=& -\frac{i}{2\lambda(\lambda+m)} \left(k_x + ik_y - \frac{k^2(2\lambda+m)ik_y}{2\lambda^2(\lambda+m)} - \frac{m(\lambda+m)ik_y}{2\lambda^2} \right) \\
=& -\frac{ik_x}{2\lambda(\lambda+m)}.
\end{aligned}
$$

The Berry curvature is then

$$
\begin{aligned}
F_{xy} =& -\frac{i}{\lambda(\lambda+m)} + \frac{i(2\lambda+m)}{2\lambda^3(\lambda+m)^2}(k_x^2+k_y^2) \\
=& -\frac{i}{\lambda(\lambda+m)} \left(1 - \frac{(2\lambda+m)}{2\lambda^2(\lambda+m)}(\lambda^2-m^2) \right) \\
=& -\frac{i}{2\lambda^3(\lambda+m)} \left(2\lambda^2 - (\lambda-m)(2\lambda+m) \right) \\
=& -\frac{im}{2\lambda^3}.
\end{aligned}
$$

Approximating the integral over the first Brillouin zone to an integral over all momenta we have

$$
\begin{aligned}
\mathrm{Ch}_1 =& \frac{i}{2\pi}\int F = \frac{m}{4\pi}\int \mathrm{d}^2k \, \frac{1}{(k^2+m^2)^{\frac{3}{2}}} \\
=& \frac{m}{4\pi|m|} \frac{4\pi^2\Gamma\left(\frac{1}{2}\right)\Gamma(1)}{4\pi\Gamma\left(\frac{3}{2}\right)\Gamma(1)} \\
=& \frac{m}{2|m|} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \\
=& \frac{m}{2|m|}.
\end{aligned}
$$

Setting $k_y = 0$ instead nets us a model in class AIII. To define a topological invariant for this model we reparametrize it and switch form such that the Hamiltonian is

$$
\mathcal{H} = k_x\sigma_x - m\sigma_y.
$$

The eigenstates are

$$
\left|u^-(k)\right\rangle = \frac{1}{\sqrt{2}\lambda} \begin{bmatrix} -k_x - im \\ \lambda \end{bmatrix}, \ \left|u^+(k)\right\rangle = \frac{1}{\sqrt{2}\lambda} \begin{bmatrix} k_x + im \\ \lambda \end{bmatrix}..
$$

The projection operator onto occupied states is therefore

$$
P = \frac{1}{2} \begin{bmatrix} 1 & \frac{-k_x - im}{\lambda} \\ \frac{-k_x+im}{\lambda} & 1 \end{bmatrix},
$$

from which we find

$$
Q = \frac{1}{\lambda} \begin{bmatrix} 0 & k_x + im \\ k_x - im & 0 \end{bmatrix}.
$$

The Hamiltonian commutes with σ_z, which is diagonal in the chosen basis, and thus we find

$$
q = \frac{k_x + im}{\lambda}.
$$

The winding number is given by the form

$$\begin{aligned}
\omega_1 =& \frac{i}{2\pi}\operatorname{tr}\bigl(q^{-1}dq\bigr)\\
=& \frac{i}{2\pi}\frac{\lambda}{k_x+im}\left(\frac{1}{\lambda}-\frac{(k_x+im)k_x}{\lambda^3}\right)dk_x\\
=& \frac{i}{2\pi}\frac{k_x-im}{\lambda}\frac{\lambda^2-(k_x+im)k_x}{\lambda^3}dk_x\\
=& \frac{im}{2\pi}\frac{k_x-im}{\lambda}\frac{m-ik_x}{\lambda^3}dk_x\\
=& \frac{m}{2\pi}\frac{(k_x-im)(k_x+im)}{\lambda^4}dk_x\\
=& \frac{m}{2\pi\lambda^2}dk_x,
\end{aligned}$$

and we have

$$\begin{aligned}
\nu_1 =& \int \omega_1\\
=& \int_{-\infty}^{\infty} \mathrm{d}k_x\, \frac{m}{2\pi\lambda^2}\\
=& m\frac{\Gamma\left(\frac{1}{2}\right)^2}{2\sqrt{\pi}\Gamma(1)\Gamma\left(\frac{1}{2}\right)}(m^2)^{-\frac{1}{2}}\\
=& \frac{m}{2|m|}.
\end{aligned}$$

To compute the Chern-Simons invariant, we need the Berry connection for the occupied states. Reusing a previous calculation we find

$$A = -\frac{im}{2\lambda^2}dk_x.$$

The Chern-Simons form is then given by

$$Q_1 = \frac{i}{2\pi}\int_0^1 \mathrm{d}t\,\operatorname{tr}(A) = \frac{i}{2\pi}\operatorname{tr}(A),$$

and the Chern-Simons invariant is

$$\begin{aligned}
\mathrm{CS}_1 =& \int Q_1\\
=& \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_x\,\frac{m}{2\lambda^2}\\
=& \frac{m}{4|m|}.
\end{aligned}$$

This is half the winding number, and this is in fact not a coincidence. The Wilson loop is then given by

$$W = e^{\pm i\frac{\pi}{2}}.$$

Summary of Articles The first article is about the Berry phase in gapped systems, studied through effective field theory. Promoting parameters of the theory to background fields nets new terms to the effective action, so-called Weiss-Zumino-Witten terms. Considering such terms yields constraints on the phase diagram of the theory, and in particular implies the existence of gapless points which are stable under (some kinds of) deformations. Such points are called diabolical points. For theories with finite degrees of freedom, the existence of such points is identified using the Berry curvature. For field theories one generalization of the Berry phase is the WZW terms appearing in an effective action.

The WZW terms can be related to the ordinary Berry phase by considering interfaces in the model and performing dimensional reduction along these. The result is that the integrals of the Berry curvature are the same for the full model and the effective model along the interface. By that argument these terms also distinguish themselves from the higher Berry curvature, which is mathematically distinct from the usual Berry curvature.

The second article is about dimensional reduction of topological insulators. It demonstrates how Hamiltonians that differ with respect to dimensionality and symmetry properties can be related via dimensional reduction.

4 Quantum Field Theory

Effective Actions To define the effective action, we first introduce

$$E[J] = -i \ln(Z[J]),$$

where Z is the generating functional of some quantum filed theory. E is essentially a measure of the vacuum energy as a function of the source J. There is also a strong analogy to statistical mechanics at play, with Z playing the role of the partition function and E the role of the Helmholtz free energy. Its functional derivatives are given by

$$\frac{\delta E}{\delta J_a(x)} = -\frac{i}{Z}\frac{\delta Z}{\delta J^a(x)} = \frac{\int [\mathrm{D}\phi]\, \phi^a(x) e^{i\left(S+\int \mathrm{d}^d y J(y)\phi(y)\right)}}{\int [\mathrm{D}\phi]\, e^{i\left(S+\int \mathrm{d}^d y J(y)\phi(y)\right)}}.$$

In analogy with statistical mechanics, this can be considered a classical vacuum expectation value in the presence of a source, hence we term it $\phi_J(x)$. Its evaluation at $J = 0$ nets us the familiar correlation function. As for two-point correlators we have

$$\frac{\delta}{\delta J_a(x)}\frac{\delta}{\delta J_b(y)} E = -\, i \left(i^2 \left\langle \phi^a(x)\phi^b(y) \right\rangle_J - i^2 \left\langle \phi^a(x) \right\rangle_J \left\langle \phi^b(y) \right\rangle_J \right).$$

This demonstrates the explicit removal of disconnected Feynman diagrams in E. The general result is

$$\left(\prod_{i=1}^{n} \frac{\delta}{\delta J_{a_i}(x_i)} \right) E = -i^{n+1} \left\langle \prod_{i=1}^{n} \phi^{a_i}(x_i) \right\rangle_{\mathrm{conn}},$$

which will be useful for computing terms in the effective action. This implies that E is a restricted sum of Feynman diagrams, containing only diagrams that are connected.

E is a functional of the currents, but a functional of the fields (or, rather, their expectation values, as that is the closest we're gonna get) might be more of interest. To do this, we first introduce the notation

$$\eta^a(x) = \frac{\delta E}{\delta J_a(x)},$$

so as to distinguish between the expectation value in the presence of the source and the actual quantum field. In analogy with statistical mechanics we achieve this with the Legendre transform

$$\Gamma[\eta] = E[J_\eta] - \int \mathrm{d}^d x\, \eta^a(x) J_{a,\eta}(x).$$

As we are used to, the above defines $J_{a,\eta}$ as a functional satisfying

$$\frac{\delta \Gamma}{\delta \eta^a(x)} = -J_a(x).$$

The quantity Γ is the effective action. Note that E and Γ coincide for $J = 0$.

We will now try to obtain a method for computing the effective action. To do this we write

$$e^{i\Gamma[\eta]} = \int [\mathrm{D}\phi]\, e^{i\left(S[\phi]+\int \mathrm{d}^d x \phi^a J_a - \eta^a J_a\right)} = \int [\mathrm{D}\phi]\, e^{i\left(S[\phi]+\int \mathrm{d}^d x (\phi^a - \eta^a) J_a\right)} = \int [\mathrm{D}\chi]\, e^{i\left(S[\chi+\eta]+\int \mathrm{d}^d x \chi^a J_a\right)}.$$

Next we perform the expansion

$$S[\eta+\chi] = S[\eta] + S_2[\eta,\chi] + S_{\mathrm{int}}[\eta,\chi] + S_{\mathrm{lin}}[\eta,\chi],$$

with

$$S_2 = \frac{1}{2}\int \mathrm{d}^d x\, \chi^a \Delta_{ab}(\eta)\chi^b,\;\; S_{\mathrm{lin}}[\eta,\chi] = \int \mathrm{d}^d x \left(\int \mathrm{d}^d y\, \frac{\delta\mathcal{L}(x)}{\delta\phi(y)}\bigg|_{\phi=\eta} + J_a \right)\chi^a.$$

This yields

$$e^{i\Gamma[\eta]} = e^{iS[\eta]}\int [\mathrm{D}\chi]\, e^{i\left(S_{\mathrm{lin}}+S_{\mathrm{int}}+\frac{1}{2}\int \mathrm{d}^d x \chi^a \Delta_{ab}(\eta)\chi^b\right)} = e^{iS[\eta]}\int [\mathrm{D}\chi]\, e^{\frac{i}{2}\int \mathrm{d}^d x \chi^a \Delta_{ab}(\eta)\chi^b} \sum_{n=0}^{\infty} \frac{i^n}{n!}\left(S_{\mathrm{lin}}+S_{\mathrm{int}}\right)^n.$$

In other words, the effective action has one term given by the usual action evaluated at the configuration in question, one from the logarithm of the determinant of Δ and a set of terms obtained from bubble diagrams with Δ^{-1} as the propagator and S_{lin} and S_{int} containing interactions. Now, the above seems to imply the existence of tadpoles in the effective action, but due to the condition on the current these cancel exactly. This means that all Feynman diagrams contributing to the effective action are 1-particle irreducible - that is, cannot be split into two by cutting a single line.

Notes on Convention We will be considering fermionic models in arbitrary dimensions, and for this we will need Dirac matrices. These satisfy

$$\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{d+1}}\gamma_5) = -i^{\frac{d-1}{2}}\,(-2)^{\frac{d+1}{2}}\,\varepsilon^{\mu_1\ldots\mu_{d+1}}$$

in odd-dimensional space and

$$\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{d+1}}) = -i^{\frac{d}{2}-1}\,(-2)^{\frac{d}{2}}\,\varepsilon^{\mu_1\ldots\mu_{d+1}}$$

in even-dimensional space. We will also need hermitian variants of these, found by adding a factor of $-i$, and we then find

$$\mathrm{tr}\left(\Gamma^{(l)}_{a_1}\ldots\Gamma^{(l)}_{a_l}\right) = i^{\frac{l-1}{2}}\,(-2)^{\frac{l-1}{2}}\,\varepsilon_{a_1\ldots a_l}.$$

We will also need to perform loop integrals. By performing a Wick rotation we find

$$\int \frac{\mathrm{d}^{d+1}k}{(2\pi)^d}\frac{(k^2)^a}{(k^2+\Delta)^b} = i\frac{\Gamma\left(b-a-\frac{d+1}{2}\right)\Gamma\left(a+\frac{d+1}{2}\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(b\right)\Gamma\left(\frac{d+1}{2}\right)}\Delta^{a+\frac{d+1}{2}-b}.$$

The Quantum Hall Effect We will obtain the quantum Hall effect as an effective theory of a fermionic model given in $2+1$ dimensions by

$$\mathcal{L} = \overline{\psi}(\not{D}+M)\psi,$$

with a slowly-varying background gauge field. The effective action is given by

$$\Gamma = -i\ln\bigl(\det(\not{D}+M)\bigr) = -i\,\mathrm{tr}\bigl(\ln(\not{D}+m)\bigr) = C_0 - i\,\mathrm{tr}\left(\ln\left(1+\frac{i\not{A}}{\not{\partial}+M}\right)\right),$$

having absorbed the coupling constant into the gauge field. The term in the expansion we are looking for is of order 2, with momentum prescription given in figure 1. Note that the photon lines in this figure are grounded to the background fields.

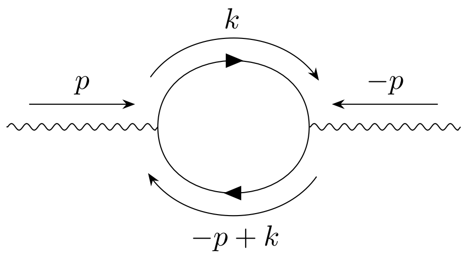

Figure 1: Feynman diagram for the second-order term in the effective action.

Thus we have

$$\begin{aligned}\Gamma \supset & \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{tr}\left(\frac{1}{i\gamma^\mu k_\mu + M}\cdot -i\gamma^\nu A_\nu(-p)\frac{1}{i\gamma^\rho(-p+k)_\rho + M}\cdot -i\gamma^\sigma A_\sigma(p)\right)\\ =& -\frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{A_\nu(-p)A_\sigma(p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\,\mathrm{tr}((i\gamma^\mu k_\mu - M)\gamma^\nu(i\gamma^\rho(-p+k)_\rho - M)\gamma^\sigma).\end{aligned}$$

In the long-wavelength limit we truncate all terms of higher order than 1 in $\frac{p}{k}$. This makes the integrand even in k, yielding

$$\Gamma = -\frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{A_\nu(-p)A_\sigma(p)}{(k^2+M^2)^2}\,\mathrm{tr}((i\gamma^\mu k_\mu - M)\gamma^\nu(i\gamma^\rho(-p+k)_\rho - M)\gamma^\sigma).$$

This produces three terms depending on how many Dirac matrices are included in the trace. Including two matrices is trivial, as the loop integral converges in this case. The case of four matrices is of interest, as it produces a divergent term. This case is given by

$$\begin{aligned}\Gamma \supset & \frac{i}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_\mu A_\nu(-p)(-p+k)_\rho A_\sigma(p)}{(k^2+M^2)^2}\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)\\ =& 2i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_\mu A_\nu(-p)(-p+k)_\rho A_\sigma(p)}{(k^2+M^2)^2}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right),\end{aligned}$$

and the middle term in particular is given by

$$\Gamma \supset -2i\int \frac{\mathrm{d}^3p}{(2\pi)^3}A^\mu(-p)A_\mu(p)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k^2}{(k^2+M^2)^2}.$$

This loop integral is divergent, and can be regularized using a Pauli-Villars regularizer. This is done by introducing a new term where the fermion propagator is replaced according to

$$\frac{1}{i\gamma^\mu k_\mu + M} \to \frac{1}{i\gamma^\mu k_\mu + M} - \frac{1}{i\gamma^\mu k_\mu + \Lambda},$$

with Λ being some regularizer to be increased to infinity at the end of the calculation. The final term in the effective action, which is the one of interest, is given by

$$\begin{aligned}\Gamma \supset & \frac{M}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{A_\nu(-p)p_\rho A_\sigma(p)}{(k^2+M^2)^2}\,\mathrm{tr}(\gamma^\nu\gamma^\rho\gamma^\sigma)\\ =& \frac{iM\Gamma\left(\frac{1}{2}\right)}{8\pi^{\frac{3}{2}}|M|}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\varepsilon^{\mu\nu\rho}A_\mu(-p)p_\nu A_\rho(p)\\ =& \frac{M}{8\pi|M|}\int \mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.\end{aligned}$$

Adding the regularizer yields

$$\Gamma = \frac{1}{8\pi}\left(\frac{M}{|M|} + \frac{\Lambda}{|\Lambda|}\right)\int \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho.$$

The corresponding current is given by

$$J^\mu = \frac{\delta\Gamma}{\delta A_\mu} = \frac{1}{8\pi}\left(\frac{M}{|M|} - \frac{\Lambda}{|\Lambda|}\right)\left(\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho - \partial_\nu\left(-\varepsilon^{\mu\nu\rho}A_\rho\right)\right) = \frac{1}{4\pi}\left(\frac{M}{|M|} - \frac{\Lambda}{|\Lambda|}\right)\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho,$$

and in particular, in the limit of $\Lambda \to -\infty$,

$$J^i = \frac{1}{2\pi}\varepsilon^{ij}\left(-\partial_0 A_j + \partial_j A_0\right) = -\frac{1}{2\pi}\varepsilon^{ij}E_j,$$

which also reproduces the Hall effect.

Theories With Topological Response We will now consider some field theories with auxillary fields. The effective actions of these theories contain topological terms, which are metric-independent. The significance of this metric-independence is that correlation functions, and therefore the theory, is stable under deformations of spacetime or generalized coordinate transformations. It also implies that all correlations lengths are zero.

The scheme for constructing such theories was laid out by Abanov and Wiegmann. The idea is to introduce a tuple of slowly varying fields V. We also introduce the matrices $\Gamma_i^{(2k+1)}$, which are Hermitian Dirac matrices representing the Clifford algebra with $2k+1$ generators. From these we construct the operators

$$m^{(l)} = \begin{cases} \sum\limits_{i=1}^{l} m^i \Gamma_i^{(l)}, & l = 2n+1, \\ m_l + i\gamma_5 \sum\limits_{i=1}^{l-1} m^i \Gamma_i^{(l-1)}, & l = 2n. \end{cases}$$

γ_5 is $i^{\frac{d-1}{2}}$ times the product of all Dirac matrices working on Dirac structure, distinguished from the Γ_i, which work in flavor space. Models with mass terms on S^d in $d+1$ dimensions are now given by

$$\mathcal{L} = -\overline{\psi}(\partial\!\!\!/ + Mm^{(d+1)})\psi,$$

and models with mass terms on S^{d+1} are given by

$$\mathcal{L} = -\overline{\psi}(\partial\!\!\!/ + Mm^{(d+3)})\psi.$$

For this latter case we set $m_l = \cos(\nu)$ for some constant ν. Abanov and Wiegmann confine the m to some unit sphere, which we will refrain from doing, and so we can simply set $m_l = 1$. These two classes will be termed A and B. We have also introduced an overall mass scale M. Note that the confinement to a manifold is performed by the restriction of m. The models we will study are particular examples of this construction, as well as some modified versions.

For even d we make the factorization

$$\mathcal{L} = -\overline{\psi}\Gamma_{d+3}^{(d+3)}(\partial\!\!\!/\Gamma_{d+3}^{(d+3)} + \Gamma_{d+3}^{(d+3)}Mm^{(d+1)})\psi,$$

and redefine the conjugate field as $\overline{\psi} \to \overline{\psi}\Gamma_{d+3}^{(d+3)}$, such that

$$-\overline{\psi}\Gamma_{d+3}^{(d+3)}(\partial\!\!\!/\Gamma_{d+3}^{(d+3)} + \Gamma_{d+3}^{(d+3)}Mm^{(d+1)})\psi.$$

The point of this redefinition will be made apparent.

The Dirac equation for these cases is

$$(\partial\!\!\!/ + Mm^{(l)})\psi = 0.$$

The corresponding Hamiltonian is

$$\mathcal{H} = i\alpha^i\partial_i + i\gamma^0 Mm^{(l)} = \alpha^i p_i - \beta Mm^{(l)},$$

where the α^i are hermitian and satisfy

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \ \{\alpha^i, \beta\} = 0.$$

This implies that this Hamiltonian belongs to class AIII.

Let us consider the other metric convention too, with

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - Mm^{(l)})\psi.$$

The Dirac equation is

$$(i\partial\!\!\!/ - Mm^{(l)})\psi = 0.$$

The corresponding Hamiltonian is given by

$$\mathcal{H} = i\alpha^i\partial_i + \beta Mm^{(l)} = \beta(\alpha^i p_i + Mm^{(l)}).$$

Some Examples The first is the $1+1$-dimensional theory

$$\mathcal{L} = -i\overline{\psi}\left(\partial\!\!\!/ + M_1 + iM_2\gamma_5\right)\psi.$$

This is the class-A model in $d = 1$. Noting that

$$(\gamma_5)^2 = 1 \implies e^{i\phi\gamma_5} = \cos(\phi) + i\sin(\phi)\gamma_5,$$

we can write

$$M_1 + iM_2\gamma_5 = M\left(\cos(\alpha) + i\sin(\alpha)\gamma_5\right) = Me^{i\alpha\gamma_5},$$

with M and α being the magnitude and argument of the complex number $M_1 + iM_2$ (note that hermiticity implies that both parameters be real). Adding the minor modification of coupling the fermion field to a gauge field, the full Lagrangian for this theory is

$$\mathcal{L} = -i\overline{\psi}\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

To compute the effective action, we integrate out the fermion and perform a perturbation expansion treating $D\!\!\!\!/ + Me^{i\alpha\gamma_5}$ as a perturbed version of $\partial\!\!\!/ + M$. The effective action for the gauge field is

$$\begin{aligned}\Gamma =& -i\ln\big(\det\big(-i\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\big)\big) + \int \mathrm{d}^2x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ =& -i\,\mathrm{tr}\big(\ln\big(-i\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\big)\big) + \int \mathrm{d}^2x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}$$

The trace can be computed by summing over some basis with respect to both the field and Dirac structures. In particular, the new term is

$$\begin{aligned}-i\,\mathrm{tr}\big(\ln\big(-i\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\big)\big) \approx& -i\,\mathrm{tr}\left(\ln\left(-i\left(\partial\!\!\!/ + M\right)\left(1 + \frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\right)\right)\right) \\ =& C_0 - i\,\mathrm{tr}\left(\ln\left(1 + \frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\right)\right) \\ \approx& C_0 - i\,\mathrm{tr}\left(\frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M} - \frac{1}{2}\frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\right).\end{aligned}$$

Let us first compute the inverse of the denominator. We have

$$(\partial\!\!\!/ + M)(\partial\!\!\!/ - M) = \partial\!\!\!/^2 - M^2 = \partial^2 - M^2 \implies \frac{1}{\partial\!\!\!/ + M} = \frac{\partial\!\!\!/ - M}{\partial^2 - M^2}.$$

Computing the trace in momentum space, applying the correspondence principle $p = -i\partial$ and using the systematics of Feynman diagrams we find

$$
\begin{aligned}
&- i\,\mathrm{tr}\left(-ie\frac{\not{A}+iM\alpha\gamma_5}{\not{\partial}+M} - \frac{1}{2}\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\right)\\
={}&- i\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\left(\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\right) + \frac{i}{2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_3}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\,\mathrm{tr}\left(\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\right)\\
={}&- i\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\left(\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\right)(2\pi)^2\delta_{p_1+p_2}\\
&+\frac{i}{2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_3}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\,\mathrm{tr}\left(\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\frac{i\not{p}_3-M}{-p_3^2-M^2}(-ie\not{A}(p_4)+iM\alpha(p_4)\gamma_5)\right)\\
&\cdot(2\pi)^4\delta_{p_1+p_2-p_3}\delta_{-p_1+p_3+p_4}\\
={}&- i\int\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\left(\frac{-i\not{p}_2-M}{-p_2^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\right)\\
&+\frac{i(2\pi)^2}{2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\\
&\cdot\,\mathrm{tr}\left(\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\frac{i(\not{p}_1+\not{p}_2)-M}{-(p_1+p_2)^2-M^2}(-ie\not{A}(p_4)+iM\alpha(p_4)\gamma_5)\right)\delta_{p_2+p_4}\\
={}&- i\int\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\left(\frac{-i\not{p}_2-M}{-p_2^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\right)\\
&+\frac{i}{2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\cdot\mathrm{tr}\left((-ie\not{A}(p_4)+iM\alpha(p_4)\gamma_5)\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(-p_4)+iM\alpha(-p_4)\gamma_5)\frac{i(\not{p}_1-\not{p}_4)-M}{-(p_1-p_4)^2-M^2}\right).
\end{aligned}
$$

The corresponding Feynman diagram is shown in figure 2.

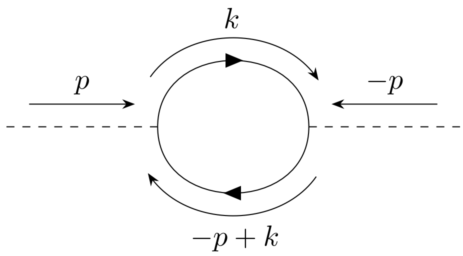

Figure 2: Feynman diagram for the second-order term in the effective action.

The second line produces the lowest-order topological terms. There we need only consider the case where the number of Dirac matrices is even, as the odd-numbered cases vanish when tracing the matrices. Absorbing the coupling constant into the gauge field, these terms are

$$
\begin{aligned}
&\frac{i}{2}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\operatorname{tr}\biggl((-i\not{A}(p)+iM\alpha(k)\gamma_5)\frac{i\not{k}-M}{-k^2-M^2}(-i\not{A}(-p)+iM\alpha(-p)\gamma_5)\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\biggr)\\
\supset&\frac{iM}{2}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\operatorname{tr}\biggl(\not{A}(p)\frac{i\not{k}-M}{-k^2-M^2}\alpha(-p)\gamma_5\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\biggr)\\
&+\frac{iM}{2}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\operatorname{tr}\biggl(\alpha(p)\gamma_5\frac{i\not{k}-M}{-k^2-M^2}\not{A}(-p)\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\biggr)\\
=&\frac{iM}{2}\Biggl(\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\alpha(-p)A_\mu(p)\operatorname{tr}\biggl(\frac{i\not{k}-M}{-k^2-M^2}\gamma^\mu\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\gamma_5\biggr)\\
&+\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\alpha(p)A_\mu(-p)\operatorname{tr}\biggl(\frac{i\not{k}-M}{-k^2-M^2}\gamma_5\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\gamma^\mu\biggr)\Biggr)\\
\supset&\frac{M^2}{2}\Biggl(\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)A_\mu(p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}\bigl(\bigl(\not{k}\gamma^\mu+\gamma^\mu(-\not{p}+\not{k})\bigr)\gamma_5\bigr)\\
&+\frac{\alpha(p)A_\mu(-p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}\bigl(\bigl(\not{p}\gamma_5+\gamma_5(-\not{p}+\not{k})\bigr)\gamma^\mu\bigr)\Biggr)\\
=&\frac{M^2}{2}\Biggl(\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)A_\mu(p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}(k_\nu\gamma^\nu\gamma^\mu\gamma_5+(-p+k)_\nu\gamma^\mu\gamma^\nu\gamma_5)\\
&+\frac{\alpha(p)A_\mu(-p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}(k_\nu\gamma^\nu\gamma_5\gamma^\mu+(-p+k)_\nu\gamma_5\gamma^\nu\gamma^\mu)\Biggr).
\end{aligned}
$$

Because we take the mass fields to be slowly varying, we can remove all contributions over order 1 in $\frac{p}{k}$. Next, because we are integrating over k explicitly and due to the Levi-Civita symbol, we may remove contributions proportional to k_μ to any order. Inverting the p-integral in the second term leaves us with

$$
\begin{aligned}
&\frac{M^2}{2}\Biggl(\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)A_\mu(p)}{(-k^2-M^2)^2}\operatorname{tr}(k_\nu\gamma^\nu\gamma^\mu\gamma_5+(-p+k)_\nu\gamma^\mu\gamma^\nu\gamma_5+k_\nu\gamma^\nu\gamma_5\gamma^\mu+(p+k)_\nu\gamma_5\gamma^\nu\gamma^\mu)\Biggr)\\
=&-M^2\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)p_\nu A_\mu(p)}{(-p^2-M^2)^2}\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma_5)\\
=&-2M^2\int \frac{\mathrm{d}^2p}{(2\pi)^2}\varepsilon^{\mu\nu}\alpha(-p)p_\nu A_\mu(p)\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(k^2+M^2)^2}.
\end{aligned}
$$

Let us consider the innermost integral. Performing a Wick rotation and a substitution we have

$$
\begin{aligned}
\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(k^2+M^2)^2}=&\frac{1}{M^2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{(-q_0^2+q_1^2+1)^2}\\
=&\frac{i}{M^2}\int\frac{\mathrm{d}^2\ell}{(2\pi)^2}\frac{1}{(\ell_0^2+\ell_1^2+1)^2}\\
=&\frac{i}{2\pi M^2}\int_0^\infty \mathrm{d}r\,\frac{r}{(r^2+1)^2}\\
=&-\frac{i}{2\pi M^2}\,\frac{1}{2(r^2+1)}\bigg|_0^\infty\\
=&\frac{i}{4\pi M^2}.
\end{aligned}
$$

The final expression for the momentum space effective action is then

$$\Gamma = -\frac{i}{2\pi}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\varepsilon^{\mu\nu}\alpha(-p)p_\nu A_\mu(p),$$

and switching to real space we have

$$\Gamma = \frac{1}{2\pi}\int \mathrm{d}^2 x\, \varepsilon^{\mu\nu}\alpha\partial_\mu A_\nu = \frac{1}{2\pi}\int \alpha \wedge F.$$

The α appearing here is of course only the prefactor for γ_5, and so we can infer the true structure of this term to be

$$\Gamma = \frac{1}{2\pi}\int \sin(\alpha)\wedge F.$$

At this point we can also note the existence of terms involving $1-\cos(\alpha)$, as the full effective action is

$$\Gamma = -\,i\,\mathrm{tr}\left(\ln\left(1+\frac{1-\cos(\alpha)-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\right)\right).$$

As such, the above procedure can be extended to higher order. The result will be terms with arbitrarily high powers of $1-\cos(\alpha)$, as well as new numerical constants.

Another model to study in one dimension is

$$\mathcal{L} = -i\overline{\Psi}\left(\not{\partial}+M\left(1+\sum_{a=2,3,5} i m_a(x)\gamma^a\right)\right)\Psi.$$

We once again employ the perturbation approach to write the effective action as

$$\begin{aligned}\Gamma =& -\,i\,\mathrm{tr}\left(\ln\left(-i(\not{\partial}+M)\left(1+\frac{iM\sum_{a=2,3,5}{}_{,a}\gamma^a}{\not{\partial}+M}\right)\right)\right)\\ =&C_0 - i\,\mathrm{tr}\left(\ln\left(1+\frac{i\sum_{a=2,3,5} m_a\gamma^a}{\not{\partial}+M}\right)\right).\end{aligned}$$

Because we are working with all Dirac matrices, we note that the only shot at obtaining a topological term is to consider a term of (at least) order three in the expansion. The Feynman diagram is shown in figure 3.

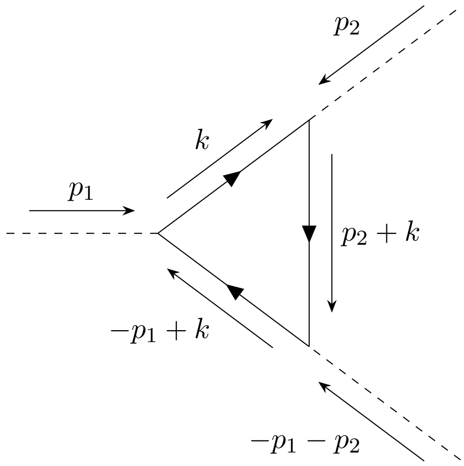

Figure 3: Feynman diagram for the third-order term in the effective action.

This term is given by

$$
\begin{aligned}
\Gamma =& -\frac{iM^3}{3}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{tr}\left(\frac{i\gamma m^a(p_1)\Gamma^{(3)}_a}{\partial\!\!\!/ + M}\frac{i\gamma m^b(p_2)\Gamma^{(3)}_b}{\partial\!\!\!/ + M}\frac{i\gamma m^c(-p_1-p_2)\Gamma^{(3)}_c}{\partial\!\!\!/ + M}\right)\\
=& -\frac{M^3}{3}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{m_a(p_1)m_b(p_2)m_c(-p_1-p_2)}{(-k^2-M^2)(-(p_2+k)^2-M^2)(-(-p_1+k)^2-M^2)}\\
&\cdot\mathrm{tr}\Big(\gamma\Gamma^{(3)}_a(ik\!\!\!/ - M)\gamma\Gamma^{(3)}_b(i(p\!\!\!/_2 + k\!\!\!/) - M)\gamma\Gamma^{(3)}_c(i(-p\!\!\!/_1 + k\!\!\!/) - M)\Big),
\end{aligned}
$$

where a summation convention has now been adopted for the greek indices.

The above calculation can be systematized by adopting aliases for the momenta. Denoting the external momenta from the background as p_i and the fermion momentum from the line starting at p_i as k_i, these vectors satisfy.

$$
\sum_{i=1}^{3} p_i = 0,\ \ k_i = k_1 + \sum_{j>1}^{i} p_j
$$

by construction. The integrand can thus be written as

$$
\begin{aligned}
&\frac{m_a(p_1)m_b(p_2)m_c(-p_1-p_2)}{(-k^2-M^2)(-(p_2+k)^2-M^2)(-(-p_1+k)^2-M^2)}\\
&\cdot\mathrm{tr}\Big(\gamma\Gamma^{(3)}_a(ik\!\!\!/ - M)\gamma\Gamma^{(3)}_b(i(p\!\!\!/_2 + k\!\!\!/) - M)\gamma\Gamma^{(3)}_c(i(-p\!\!\!/_1 + k\!\!\!/) - M)\Big)\\
=&\frac{m_a(p_1)m_b(p_2)m_c(p_3)}{\prod_{i=1}^{3}(-k_i^2-M^2)}\,\mathrm{tr}\Big(\gamma\Gamma^{(3)}_a(ik\!\!\!/_1 - M)\gamma\Gamma^{(3)}_b(ik\!\!\!/_2 - M)\gamma\Gamma^{(3)}_c(ik\!\!\!/_3 - M)\Big).
\end{aligned}
$$

The topological term corresponds to exactly one of the factors from the fermion loop contributing a factor $-M$. Because k_1 is explicitly integrated over and the rest of the fermion loop will contribute a factor symmetric in k_1, the only contribution comes from when the first bracket contributes this factor. Furthermore, anticipating the Levi-Civita symbol in the topological term, any term with higher-order products of components of k_1 - or indeed, products of components of any momentum vector - can be ignored. These facts combined allow for the replacement

$$
k_{i,\mu}k_{j,\nu} \to \sum_{\alpha>1}^{i}\ \sum_{\beta>1,\beta\neq\alpha}^{j} p_{\alpha,\mu}p_{\beta,\nu}.
$$

A similar relation can be derived for higher-order product. As the isospace and Dirac structure traces may be computed separately, the integrand simplifies to

$$
\begin{aligned}
&Mk_{2,\mu}k_{3,\nu}\frac{m_a(p_1)m_b(p_2)m_c(p_3)}{\prod_{i=1}^{3}(-k_i^2-M^2)}\,\mathrm{tr}\Big(\gamma\Gamma^{(3)}_a\gamma\Gamma^{(3)}_b\gamma^\mu\gamma\Gamma^{(3)}_c\gamma^\nu\Big)\\
=& -M\frac{m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)}{\prod_{i=1}^{3}(-k_i^2-M^2)}\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma)\,\mathrm{tr}\Big(\Gamma^{(3)}_a\Gamma^{(3)}_b\Gamma^{(3)}_c\Big)\\
=&4iMp_{2,\mu}p_{3,\nu}\varepsilon^{\mu\nu}\varepsilon^{abc}\frac{m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)}{(-k_1^2-M^2)(-k_2^2-M^2)(-k_3^2-M^2)}.
\end{aligned}
$$

The effective action in the slowly-varying limit, for which $k_i \approx k$ for all i, is thus

$$
\Gamma = -\frac{4M^4}{3}\frac{1}{8\pi M^4}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\varepsilon^{\mu\nu}\varepsilon^{abc}m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3),
$$

or in real space,

$$
\Gamma = \frac{1}{6\pi}\sum_{a,b,c=2,3,5}\int \mathrm{d}^2x\,\varepsilon^{\mu\nu}\varepsilon^{abc}m_a\partial_\mu m_b\partial_\nu m_c.
$$

To realize that this is indeed a Weiss-Zumino-Witten term, note that the mass fields comprise a map from spacetime to the target space. Introducing the 2-form $\omega = \frac{1}{6\pi}\varepsilon_{abc}m^a \wedge dm^b \wedge dm^c$, the effective action can be written as

$$\Gamma = \int \mathrm{d}^2x\, \varepsilon^{\mu\nu}\omega_{ab}\partial_\mu m^a \partial_\nu m^b = \int m^\star \omega.$$

At this point we can make some notes about the topological terms for all models in class B. In any topological term, all mass fields appear exactly once, as do momenta with each index. The Feynman diagrams also have identical shape, as we will see later. This will produce the pullback of a $d+1$-form, the components of which are linear in the fields, in the effective action.

Let us reproduce this topological term using the other Lagrangian. We have

$$\begin{aligned}
\Gamma =& -\frac{iM^3}{3}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{tr}\left(\frac{i\gamma_5 \sum\limits_{a=2,3,5} m^a\Gamma^{(3)}_a}{\not{\partial}+M}\frac{i\gamma_5 \sum\limits_{b=2,3,5} m^b\Gamma^{(3)}_b}{\not{\partial}+M}\frac{i\gamma_5 \sum\limits_{c=2,3,5} m^c\Gamma^{(3)}_c}{\not{\partial}+M}\right)\\
=& -\frac{M^3}{3}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\\
&\cdot \sum_{a,b,c=2,3,5} m_a(p_1)m_b(p_2)m_c(p_3)\,\mathrm{tr}\left(\gamma_5\frac{i\not{k}_1 - M}{-k_1^2-M^2}\gamma_5\frac{i\not{k}_2 - M}{-k_2^2-M^2}\gamma_5\frac{i\not{k}_3 - M}{-k_3^2-M^2}\right)\mathrm{tr}\left(\Gamma^{(3)}_a\Gamma^{(3)}_b\Gamma^{(3)}_c\right)\\
=& \frac{M^3}{3}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\sum_{a,b,c=2,3,5}\frac{m_a(p_1)m_b(p_2)m_c(p_3)}{(k_1^2+M^2)(k_2^2+M^2)(k_3^2+M^2)}\\
&\cdot \mathrm{tr}(\gamma_5(i\not{k}_1 - M)\gamma_5(i\not{k}_2 - M)\gamma_5(i\not{k}_3 - M))\cdot -2i\varepsilon^{abc}\\
\approx& -\frac{2iM^4}{3}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\sum_{a,b,c=2,3,5}\frac{\varepsilon^{abc}m_a(p_1)m_b(p_2)m_c(p_3)}{(k_1^2+M^2)^3}\,\mathrm{tr}(\not{k}_2\gamma_5\not{k}_3)\\
=& -\frac{2M^4}{3}\frac{\Gamma(2)}{4\pi\Gamma(3)M^4}\sum_{a,b,c=2,3,5}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\varepsilon^{abc}m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma_5)\\
=& -\frac{1}{12\pi}\sum_{a,b,c=2,3,5}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\varepsilon^{abc}m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)\cdot 2\varepsilon^{\mu\nu}\\
=& -\frac{1}{6\pi}\sum_{a,b,c=2,3,5}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\varepsilon^{\mu\nu}\varepsilon^{abc}m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3),
\end{aligned}$$

which in real space is equal to

$$\Gamma = \frac{1}{6\pi}\sum_{a,b,c=2,3,5}\int \mathrm{d}^2x\,\varepsilon^{\mu\nu}\varepsilon^{abc}m_a\partial_\mu m_b\partial_\nu m_c.$$

Let us demonstrate the appearance of all mass fields in one dimension. To that end we consider the theory

$$\mathcal{L} = -i\overline{\psi}\left(\not{\partial} + M_0 + i\gamma_5\sum_{a=1}^{3} M^a\Gamma^{(3)}_a\right)\psi.$$

The Feynman diagram dictating the term we are looking for is shown in figure 4, and the effective action is

$$\begin{aligned}
\Gamma = -\frac{i}{5}\int \frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_3}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{tr}\Bigg(&\frac{M_0(p_2) - M + i\gamma_5\sum\limits_{a=1}^{3} M^a(p_2)\Gamma^{(3)}_a}{i\not{k}+M}\frac{M_0(p_3) - M + i\gamma_5\sum\limits_{b=1}^{3} M^b(p_3)\Gamma^{(3)}_b}{i(\not{p}_2+\not{k})+M}\\
&\cdot\frac{M_0(p_4) - M + i\gamma_5\sum\limits_{c=1}^{3} M^c(p_4)\Gamma^{(3)}_c}{i(\not{p}_2+\not{p}_3+\not{k})+M}\frac{M_0(p_1) - M + i\gamma_5\sum\limits_{d=1}^{3} M^d(p_1)\Gamma^{(3)}_d}{i(-\not{p}_1+\not{k})+M}\Bigg).
\end{aligned}$$

The trace part, taking account what we know of these theories, is

$$M \operatorname{tr}\left(\left(M_0(p_2) - M + i\gamma_5 \sum_{a=1}^{3} M^a(p_2)\Gamma_a^{(3)}\right)(i\not{k}_2 + M)\left(M_0(p_3) - M + i\gamma_5 \sum_{b=1}^{3} M^b(p_3)\Gamma_b^{(3)}\right)\right.$$
$$\left.\cdot\,(i\not{k}_3 + M)\left(M_0(p_4) - M + i\gamma_5 \sum_{c=1}^{3} M^c(p_4)\Gamma_c^{(3)}\right)(i\not{k}_4 + M)\left(M_0(p_1) - M + i\gamma_5 \sum_{d=1}^{3} M^d(p_1)\Gamma_d^{(3)}\right)\right).$$

Depending on the placement of the usual mass field, we get four terms, each of which in turn produce three more. The first is

$$\begin{aligned}
&iM^2(M_0(p_2) - M)\sum_{b,c,d=1}^{3} M^b(p_3)M^c(p_4)M^d(p_1)\operatorname{tr}\left(\Gamma_b^{(3)}\Gamma_c^{(3)}\Gamma_d^{(3)}\right)(\operatorname{tr}\left(\not{k}_2\gamma_5\not{k}_3\gamma_5^2\right) + \operatorname{tr}\left(\not{k}_2\gamma_5^2\not{k}_4\gamma_5\right) + \operatorname{tr}(\gamma_5\not{k}_3\gamma_5\not{k}_4\gamma_5))\\
=&4M^2(M_0(p_2) - M)\sum_{b,c,d=1}^{3} \varepsilon_{bcd}M^b(p_3)M^c(p_4)M^d(p_1)\\
&\cdot\varepsilon^{\mu\nu}(-p_{2,\mu}p_{3,\nu} + p_{2,\mu}(p_{3,\nu} + p_{4,\nu}) - p_{2,\mu}(p_{3,\nu} + p_{4,\nu}) - p_{3,\mu}(p_{2,\nu} + p_{4,\nu}))\\
=&-4M^2(M_0(p_2) - M)\sum_{b,c,d=1}^{3} \varepsilon_{bcd}M^b(p_3)M^c(p_4)M^d(p_1)\varepsilon^{\mu\nu}p_{3,\mu}p_{4,\nu}.
\end{aligned}$$

The second is

$$\begin{aligned}
&iM^2(M_0(p_3) - M)\sum_{a,c,d=1}^{3} M^a(p_2)M^c(p_4)M^d(p_1)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_c^{(3)}\Gamma_d^{(3)}\right)(\operatorname{tr}\left(\gamma_5\not{k}_2\not{k}_3\gamma_5^2\right) + \operatorname{tr}(\gamma_5\not{k}_2\gamma_5\not{k}_4\gamma_5) + \operatorname{tr}(\gamma_5\not{k}_3\gamma_5\not{k}_4\gamma_5))\\
=&4M^2(M_0(p_3) - M)\sum_{a,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_4)M^d(p_1)\\
&\cdot\varepsilon^{\mu\nu}(p_{2,\mu}p_{3,\nu} - p_{2,\mu}(p_{3,\nu} + p_{4,\nu}) - p_{2,\mu}(p_{3,\nu} + p_{4,\nu}) - p_{3,\mu}(p_{2,\nu} + p_{4,\nu}))\\
=&-4M^2(M_0(p_3) - M)\sum_{a,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_4)M^d(p_1)\varepsilon^{\mu\nu}(2p_{2,\mu}p_{4,\nu} + p_{3,\mu}p_{4,\nu})\\
=&-4M^2(M_0(p_3) - M)\sum_{a,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_4)M^d(p_1)\varepsilon^{\mu\nu}(2p_{2,\mu}p_{4,\nu} - (p_{1,\mu} + p_{2,\mu})p_{4,\nu})\\
=&-4M^2(M_0(p_3) - M)\sum_{a,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_4)M^d(p_1)\varepsilon^{\mu\nu}(p_{2,\mu}p_{4,\nu} - p_{1,\mu}p_{4,\nu}).
\end{aligned}$$

Note the rewrite in terms of the momenta appearing within the sum. The third is

$$\begin{aligned}
&iM^2(M_0(p_4) - M)\sum_{a,b,d=1}^{3} M^a(p_2)M^b(p_3)M^d(p_1)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_d^{(3)}\right)(\operatorname{tr}(\gamma_5\not{k}_2\gamma_5\not{k}_3\gamma_5) + \operatorname{tr}(\gamma_5\not{k}_2\gamma_5\not{k}_4\gamma_5) + \operatorname{tr}\left(\gamma_5^2\not{k}_3\not{k}_4\gamma_5\right))\\
=&4M^2(M_0(p_4) - M)\sum_{b,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_3)M^d(p_1)\\
&\cdot\varepsilon^{\mu\nu}(-p_{2,\mu}p_{3,\nu} - p_{2,\mu}(p_{3,\nu} + p_{4,\nu}) + p_{2,\mu}(p_{3,\nu} + p_{4,\nu}) + p_{3,\mu}(p_{2,\nu} + p_{4,\nu}))\\
=&4M^2(M_0(p_4) - M)\sum_{b,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_3)M^d(p_1)\varepsilon^{\mu\nu}(-2p_{2,\mu}p_{3,\nu} + p_{3,\mu}p_{4,\nu})\\
=&-4M^2(M_0(p_4) - M)\sum_{b,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_3)M^d(p_1)\varepsilon^{\mu\nu}(2p_{2,\mu}p_{3,\nu} + p_{3,\mu}(p_{1,\nu} + p_{2,\nu}))\\
=&-4M^2(M_0(p_4) - M)\sum_{b,c,d=1}^{3} \varepsilon_{bcd}M^b(p_2)M^c(p_3)M^d(p_1)\varepsilon^{\mu\nu}(p_{2,\mu}p_{3,\nu} + p_{3,\mu}p_{1,\nu}).
\end{aligned}$$

The final term is

$$
\begin{aligned}
& iM^2(M_0(p_1)-M)\sum_{a,b,c=1}^{3} M^a(p_2)M^b(p_3)M^c(p_4)\,\mathrm{tr}\Big(\Gamma^{(3)}_a\Gamma^{(3)}_b\Gamma^{(3)}_c\Big)(\mathrm{tr}(\gamma_5 \not{k}_2\gamma_5\not{k}_3\gamma_5)+\mathrm{tr}\big(\gamma_5\not{k}_2\gamma_5^2\not{k}_4\big)+\mathrm{tr}\big(\gamma_5^2\not{k}_3\gamma_5\not{k}_4\big))\\
=&4M^2(M_0(p_1)-M)\sum_{a,b,c=1}^{3}\varepsilon_{abc}M^a(p_2)M^b(p_3)M^c(p_4)\\
&\cdot\varepsilon^{\mu\nu}(-p_{2,\mu}p_{3,\nu}+p_{2,\mu}(p_{3,\nu}+p_{4,\nu})-p_{2,\mu}(p_{3,\nu}+p_{4,\nu})-p_{3,\mu}(p_{2,\nu}+p_{4,\nu}))\\
=&-4M^2(M_0(p_1)-M)\sum_{b,c,d=1}^{3}\varepsilon_{bcd}M^b(p_2)M^c(p_3)M^d(p_4)\varepsilon^{\mu\nu}p_{3,\mu}p_{4,\nu}.
\end{aligned}
$$

Transforming back to real space, these add up to

$$
\begin{aligned}
&-4M^2(M_0-M)\sum_{b,c,d=1}^{3}\varepsilon_{bcd}\varepsilon^{\mu\nu}\Big(-\partial_\mu M^b\partial_\nu M^c M^d-\partial_\mu M^b\partial_\nu M^c M^d+M^b\partial_\nu M^c\partial_\mu M^d\\
&\quad-\partial_\mu M^b\partial_\nu M^c M^d-M^b\partial_\mu M^c\partial_\nu M^d-M^b\partial_\mu M^c\partial_\nu M^d\Big)\\
=&24M^2(M_0-M)\sum_{b,c,d=1}^{3}\varepsilon_{bcd}\varepsilon^{\mu\nu}M^b\partial_\mu M^c\partial_\nu M^d.
\end{aligned}
$$

This can of course only be done in the long-wavelength limit, for which the effective action is

$$
\begin{aligned}
\Gamma=&-\frac{i}{5}\sum_{b,c,d=1}^{3}\int \mathrm{d}^2x\,24M^2(M_0-M)\varepsilon_{bcd}\varepsilon^{\mu\nu}M^b\partial_\mu M^c\partial_\nu M^d\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(-k^2-M^2)^4}\\
=&\frac{24M^2}{5}\frac{\Gamma\left(3\right)\Gamma\left(1\right)}{4\pi\Gamma\left(4\right)\Gamma\left(1\right)}M^{2(1-4)}\sum_{b,c,d=1}^{3}\int\mathrm{d}^2x\,(M_0-M)\varepsilon_{bcd}\varepsilon^{\mu\nu}M^b\partial_\mu M^c\partial_\nu M^d\\
=&\frac{2}{5\pi M^4}\sum_{b,c,d=1}^{3}\int\mathrm{d}^2x\,(M_0-M)\varepsilon_{bcd}\varepsilon^{\mu\nu}M^b\partial_\mu M^c\partial_\nu M^d.
\end{aligned}
$$

This integral is in turn given by the pullback of the 2-form

$$
\omega=\frac{2}{5\pi}(M_0-M)\varepsilon_{bcd}M^b dM^c dM^d.
$$

Let us look more closely at ω by introducing an explicit parametrization. Namely, we use

$$
m^2=\cos(\phi)\sin(\theta)\sin(\alpha),\ m^3=\sin(\phi)\sin(\theta)\sin(\alpha),\ m^5=\cos(\theta)\sin(\alpha),\ m^0=\cos(\alpha).
$$

The full form of ω is

$$
\omega=\frac{4}{5\pi}(m^5dm^2dm^3+m^3dm^5dm^2+m^2dm^3dm^5).
$$

We have

$$
\begin{aligned}
&dm^2=c_\phi s_\theta c_\alpha d\alpha+c_\phi c_\theta s_\alpha d\theta-s_\phi s_\theta s_\alpha d\phi,\ dm^3=s_\phi s_\theta c_\alpha d\alpha+s_\phi c_\theta s_\alpha d\theta+c_\phi s_\theta s_\alpha d\phi,\\
&dm^5=c_\theta c_\alpha d\alpha-s_\theta s_\alpha d\theta,\ dm^0=-s_\alpha d\alpha,
\end{aligned}
$$

adopting an abbreviated trigonometric notation, and thus

$$
\begin{aligned}
\omega =&\frac{4}{5\pi}\left(c_\theta s_\alpha\left(c_\phi s_\theta c_\alpha d\alpha + c_\phi c_\theta s_\alpha d\theta - s_\phi s_\theta s_\alpha d\phi\right)\left(s_\phi s_\theta c_\alpha d\alpha + s_\phi c_\theta s_\alpha d\theta + c_\phi s_\theta s_\alpha d\phi\right)\right.\\
&+ s_\phi s_\theta s_\alpha\left(c_\theta c_\alpha d\alpha - s_\theta s_\alpha d\theta\right)\left(c_\phi s_\theta c_\alpha d\alpha + c_\phi c_\theta s_\alpha d\theta - s_\phi s_\theta s_\alpha d\phi\right)\\
&\left.+ c_\phi s_\theta s_\alpha\left(s_\phi s_\theta c_\alpha d\alpha + s_\phi c_\theta s_\alpha d\theta + c_\phi s_\theta s_\alpha d\phi\right)\left(c_\theta c_\alpha d\alpha - s_\theta s_\alpha d\theta\right)\right)\\
=&\frac{4}{5\pi}\left(c_\theta s_\alpha\left(s_\theta^2 c_\alpha s_\alpha d\alpha d\phi + c_\theta s_\theta s_\alpha^2 d\theta d\phi\right)\right.\\
&+ s_\phi s_\theta s_\alpha\left(c_\phi c_\alpha s_\alpha d\alpha d\theta - s_\phi s_\theta c_\theta c_\alpha s_\alpha d\alpha d\phi + s_\phi s_\theta^2 s_\alpha^2 d\theta d\phi\right)\\
&\left.+ c_\phi s_\theta s_\alpha\left(-s_\phi c_\alpha s_\alpha d\alpha d\theta - c_\phi s_\theta c_\theta s_\alpha c_\alpha d\alpha d\phi + c_\phi s_\theta^2 s_\alpha^2 d\theta d\phi\right)\right)\\
=&\frac{4}{5\pi}\left(c_\theta s_\theta^2 c_\alpha s_\alpha^2 d\alpha d\phi + c_\theta^2 s_\theta s_\alpha^3 d\theta d\phi\right.\\
&+ s_\phi c_\phi s_\theta c_\alpha s_\alpha^2 d\alpha d\theta - s_\phi^2 s_\theta^2 c_\theta c_\alpha s_\alpha^2 d\alpha d\phi + s_\phi^2 s_\theta^3 s_\alpha^3 d\theta d\phi\\
&\left.- c_\phi s_\phi s_\theta c_\alpha s_\alpha^2 d\alpha d\theta - c_\phi^2 s_\theta^2 c_\theta s_\alpha^2 c_\alpha d\alpha d\phi + c_\phi^2 s_\theta^3 s_\alpha^3 d\theta d\phi\right)\\
=&2\sin(\theta)\sin^3(\alpha)d\theta d\phi.
\end{aligned}
$$

Adding the other mass field we instead end up with

$$\omega = 2(\cos(\alpha) - 1)\sin(\theta)\sin^3(\alpha)d\theta d\phi.$$

What if Hsin et al are correct in that $d\omega \propto \eta$? The volume form on S^3 is given by

$$\eta = \sin^2(\alpha)\sin(\theta)d\alpha d\theta d\phi.$$

A naive guess is then

$$\omega = \sin^2(\alpha)\cos(\theta)d\alpha d\phi,$$

but this is singular at $\theta = 0$ and $\theta = \pi$. The trick is to remove singularities at either pole - namely, we split S^3 in two patches on which we take

$$\omega_1 = \sin^2(\alpha)(\cos(\theta) - 1)d\alpha d\phi,\ \omega_2 = \sin^2(\alpha)(\cos(\theta) + 1)d\alpha d\phi.$$

As we have shifted ω on either patch by an exact form, ω still satisfies the same property. We then have

$$\Gamma = \omega_1 - \omega_2 = -2\sin^2(\alpha)d\alpha d\phi,$$

and

$$
\begin{aligned}
\int_\gamma \Gamma =& -2\int_0^{2\pi}\int_0^{\pi} \mathrm{d}\alpha\,\mathrm{d}\phi\sin^2(\alpha)\\
=& -2\pi,
\end{aligned}
$$

with γ being some cycle at fixed θ.

What if we were to add coupling to a gauge field too? The effective action would be

$$\Gamma = -i\,\mathrm{tr}\left(\ln\left(1 + \frac{-i\not{A} + i\sum\limits_{a=2,3,5} M_a\gamma^a}{\not{\partial} + M}\right)\right).$$

To order 3, the structure of the Feynman diagram is identical, hence the trace part of the effective action is

$$-i\,\mathrm{tr}\big((i\not{k}_1 - M)(e\not{A} + M_a\gamma^a)(i(-\not{k}_1 - \not{p}) - M)(e\not{A} + M_b\gamma^a)(i(\not{k}_2 - \not{k}_1) - M)(e\not{A} + M_c\gamma^c)\big).$$

Because all of the latin-indiced Dirac matrices are needed, however, the gauge field does not appear in any topological terms. To get a topological term with a gauge field, we can instead consider a fourth-order term, represented by the Feynman diagram in figure 4.

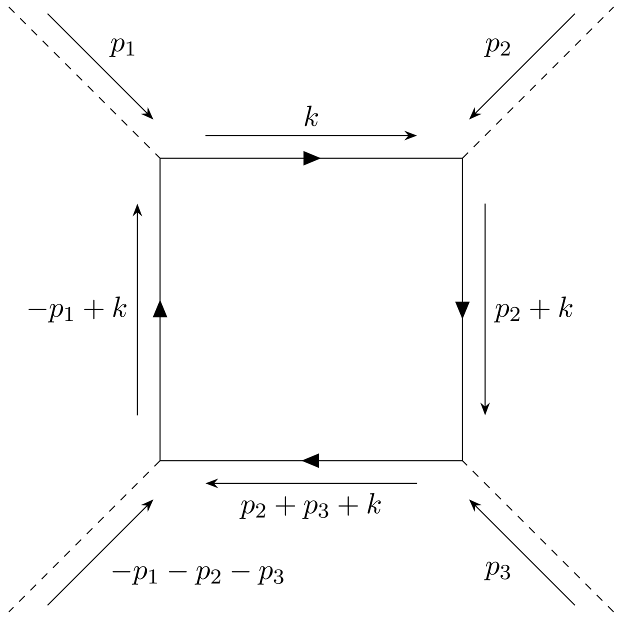

Figure 4: Feynman diagram for the fourth-order term in the effective action.

At this point we can deduce what will happen. The replacement of a momentum with a gauge field in the trace (which is needed to produce a topological term) while leaving the same number of mass fields causes the integrand to be symmetric in the mass indices and makes the topological term vanish.

This argument also implies the necessary conditions for the gauge fields to appear in response terms. We expect all the fields corresponding to anticommuting mass terms to appear exactly once, as well as all the derivatives. Removing a derivative from a mass field nets you the possibility of creating a symmetric expression in mass indices. In order for the gauge fields to appear in topological terms, the spacetime dimension must therefore be equal to the number of mass fields. If it is greater, the spacetime Levi-Civita will cancel the derivative product. If it is lesser, the mass index Levi-Civita will cancel the now symmetric product of mass fields.

A counterpoint to the above is that this model does not explicitly apply transformations in isospace, forcing out the Levi-Civita symbol, but that this could be avoided by sticking more rigorously to Abanov and Wiegmann's method. The fact that the Dirac matrices square to the identity then implies that there would be some topological terms that are pullbacks of symmetric tensors on the mass manifold. Of course, these all vanish because the topological terms necessarily contain a Levi-Civita symbol on spacetime, and as such only fully antisymmetric tensors on the mass manifold produce topological terms. This legitimizes the above approach.

Let us also consider two models in two dimensions. The first is

$$\mathcal{L} = -i\overline{\psi}\left(\not{D} + \sum_{a=1}^{3} M^a \Gamma_a^{(3)}\right)\psi.$$

The effective action is

$$\Gamma = -i\,\mathrm{tr}\left(\ln\left(1 + \frac{-M - i\not{A} + \sum_{a=1}^{3} M^a \Gamma_a^{(3)}}{\not{\partial} + M}\right)\right).$$

The topological term, which contains the gauge field, corresponds to figure 4 (to lowest order). We may drop the term proportional to the identity, and the effective action is given by

$$
\begin{aligned}
\Gamma =&\frac{i}{4}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{tr}\left(\frac{-i\not{A}+\sum_{a=1}^3 M^a\Gamma_a^{(3)}}{\not{\partial}+M}\frac{-i\not{A}+\sum_{b=1}^3 M^b\Gamma_b^{(3)}}{\not{\partial}+M}\frac{-i\not{A}+\sum_{c=1}^3 M^c\Gamma_c^{(3)}}{\not{\partial}+M}\frac{-i\not{A}+\sum_{d=1}^3 M^d\Gamma_d^{(3)}}{\not{\partial}+M}\right)\\
=&\frac{i}{4}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{tr}\left(\left(-i\not{A}(p_1)+\sum_{a=1}^3 M^a(p_1)\Gamma_a^{(3)}\right)\frac{i\not{k}-M}{-k^2-M^2}\right.\\
&\cdot\left(-i\not{A}(p_2)+\sum_{b=1}^3 M^b(p_2)\Gamma_b^{(3)}\right)\frac{i(\not{p}_2+\not{k})-M}{-(p_2+k)^2-M^2}\\
&\cdot\left(-i\not{A}(p_3)+\sum_{b=1}^3 M^b(p_3)\Gamma_b^{(3)}\right)\frac{i(\not{p}_2+\not{p}_3+\not{k})-M}{-(p_2+p_3+k)^2-M^2}\\
&\left.\cdot\left(-i\not{A}(-p_1-p_2-p_3)+\sum_{c=1}^3 M^d(-p_1-p_2-p_3)\Gamma_d^{(3)}\right)\frac{i(-\not{p}_1+\not{k})-M}{-(-p_1+k)^2-M^2}\right).
\end{aligned}
$$

Let us consider the trace part first. The relevant terms have exactly one gauge field appearing and two momenta. Adopting a shorthand, the first term of the trace can be written as

$$
\begin{aligned}
&-i\sum_{b,c,d=1}^3 M^b(p_2)M^c(p_3)M^d(p_4)\,\mathrm{tr}\Big(\not{A}(p_1)(i\not{k}_1-M)\Gamma_b^{(3)}(i\not{k}_2-M)\Gamma_c^{(3)}(i\not{k}_3-M)\Gamma_d^{(3)}(i\not{k}_4-M)\Big)\\
=&-i\sum_{b,c,d=1}^3 M^b(p_2)M^c(p_3)M^d(p_4)\,\mathrm{tr}(A_\mu(p_1)\gamma^\mu(i\not{k}_1-M)(i\not{k}_2-M)(i\not{k}_3-M)(i\not{k}_4-M))\,\mathrm{tr}\Big(\Gamma_b^{(3)}\Gamma_c^{(3)}\Gamma_d^{(3)}\Big)\\
=&-2M\sum_{a,b,c=1}^3 \varepsilon_{abc}M^a(p_2)M^b(p_3)M^c(p_4)\,\mathrm{tr}(A_\mu(p_1)\gamma^\mu(i\not{k}_2-M)(i\not{k}_3-M)(i\not{k}_4-M)).
\end{aligned}
$$

This is the point at which we can still generalize. At a glance it might seem like we could do it all in one go by fixing the position of the gauge field and cyclically permuting the momenta, but this is not the case due to the loop having been assigned a set of momenta. Using the previously established properties of the shorthand, we write the trace as

$$
\begin{aligned}
\mathrm{tr}(\gamma^\mu(i\not{k}_2-M)(i\not{k}_3-M)(i\not{k}_4-M)) =&M\,\mathrm{tr}(\gamma^\mu\not{k}_2\not{k}_3+\gamma^\mu\not{k}_2\not{k}_4+\gamma^\mu\not{k}_3\not{k}_4)\\
=&2M\varepsilon^{\mu\nu\rho}\left(k_{2,\nu}k_{3,\rho}+k_{2,\nu}k_{4,\rho}+k_{3,\nu}k_{4,\rho}\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(p_{2,\nu}p_{3,\rho}+p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}(p_{2,\rho}+p_{4,\rho})\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(2p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}p_{4,\rho}\right).
\end{aligned}
$$

The next is identical to the previous due to the k_1 not contributing. Third is

$$
\begin{aligned}
\mathrm{tr}((i\not{k}_2-M)\gamma^\mu(i\not{k}_3-M)(i\not{k}_4-M)) =&M\,\mathrm{tr}(\not{k}_2\gamma^\mu\not{k}_3+\gamma^\mu\not{k}_3\not{k}_4+\not{k}_2\gamma^\mu\not{k}_4)\\
=&2M\varepsilon^{\mu\nu\rho}(-k_{2,\nu}k_{3,\rho}+k_{3,\nu}k_{4,\rho}-k_{2,\nu}k_{4,\rho})\\
=&2M\varepsilon^{\mu\nu\rho}(-p_{2,\nu}p_{3,\rho}+p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}(p_{2,\rho}+p_{4,\rho})-p_{2,\nu}(p_{3,\rho}+p_{4,\rho}))\\
=&2M\varepsilon^{\mu\nu\rho}(p_{3,\nu}p_{4,\rho}-2p_{2,\nu}p_{3,\rho}).
\end{aligned}
$$

Finally there is

$$
\begin{aligned}
\mathrm{tr}((i\not{k}_2-M)(i\not{k}_3-M)\gamma^\mu(i\not{k}_4-M)) =&M\,\mathrm{tr}(\not{k}_2\not{k}_3\gamma^\mu+\not{k}_2\gamma^\mu\not{k}_4+\not{k}_3\gamma^\mu\not{k}_4)\\
=&2M\varepsilon^{\mu\nu\rho}\left(k_{2,\nu}k_{3,\rho}-k_{2,\nu}k_{4,\rho}-k_{3,\nu}k_{4,\rho}\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(p_{2,\nu}p_{3,\rho}-p_{2,\nu}(p_{3,\rho}+p_{4,\rho})-p_{2,\nu}(p_{3,\rho}+p_{4,\rho})-p_{3,\nu}(p_{2,\rho}+p_{4,\rho})\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(-2p_{2,\nu}p_{4,\rho}-p_{3,\nu}p_{2,\rho}-p_{3,\nu}p_{4,\rho}\right).
\end{aligned}
$$

At this point we may integrate k out, and the rest of the effective action is

$$
\frac{i}{4}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(-k^2-M^2)^4}=-\frac{\Gamma(4-2)\,\Gamma(2)}{64\pi^2\Gamma(4)\,\Gamma(2)}M^{2(2-4)}=-\frac{1}{384\pi^2M^4}.
$$

All that remains is sorting out the terms from the trace. Each one is preceded by a factor $-4M^2\varepsilon^{\mu\nu\rho}\varepsilon_{abc}$, and the correspondence principle translates the remainder to

$$
\begin{aligned}
&A_\mu(p_1)M^a(p_2)M^b(p_3)M^c(p_4)\left(2p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}p_{4,\rho}\right)\\
\to &-2A_\mu\partial_\nu M^a(M^b\partial_\rho M^c+\partial_\rho M^b M^c)-A_\mu M^a\partial_\nu M^b\partial_\rho M^c,\\
&M^a(p_1)A_\mu(p_2)M^b(p_3)M^c(p_4)\left(2p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}p_{4,\rho}\right)\\
\to &-2M^a\partial_\nu A_\mu(\partial_\rho M^b M^c+M^b\partial_\rho M^c)-M^aA_\mu\partial_\nu M^b\partial_\rho M^c,\\
&M^a(p_1)M^b(p_2)A_\mu(p_3)M^c(p_4)(p_{3,\nu}p_{4,\rho}-2p_{2,\nu}p_{3,\rho})\\
\to &-M^aM^b\partial_\nu A_\mu\partial_\rho M^c+2M^a\partial_\nu M^b\partial_\rho A_\mu M^c,\\
&M^a(p_1)M^b(p_2)M^c(p_3)A_\mu(p_4)\left(-2p_{2,\nu}p_{4,\rho}-p_{3,\nu}p_{2,\rho}-p_{3,\nu}p_{4,\rho}\right)\\
\to &2M^a\partial_\nu M^bM^c\partial_\rho A_\mu+M^a\partial_\rho M^b\partial_\nu M^cA_\mu+M^aM^b\partial_\nu M^c\partial_\rho A_\mu.
\end{aligned}
$$

The only terms surviving contraction with the mass field Levi-Civita are those with two derivatives on the mass fields. The final effective action is then

$$
\begin{aligned}
\Gamma=&-\frac{1}{96\pi^2}\varepsilon^{\mu\nu\rho}\varepsilon_{abc}\int \mathrm{d}^3x\,A_\mu\left(m^a\partial_\rho m^b\partial_\nu m^c-m^a\partial_\nu m^b\partial_\rho m^c-2\partial_\nu m^a(m^b\partial_\rho m^c+\partial_\rho m^bm^c)-m^a\partial_\nu m^b\partial_\rho m^c\right)\\
=&\frac{1}{32\pi}\varepsilon^{\mu\nu\rho}\varepsilon_{abc}\int \mathrm{d}^3x\,A_\mu m^a\partial_\rho m^b\partial_\nu m^c\\
=&\frac{1}{32\pi}\int \mathrm{d}^3x\,A\wedge\omega,
\end{aligned}
$$

with $\omega=\varepsilon_{abc}m^adm^bdm^c$.

The case for $d=2$ is given by

$$
\mathcal{L}_{2+1}=-\overline{\Psi}\left(\partial\!\!\!/\Gamma_a^{(5)}+M\left(1+\sum_{a=1}^{4}m^a\Gamma_5^{(5)}\Gamma_a^{(5)}\right)\right)\Psi.
$$

The masses are now confined to S^3.

The Feynman diagram corresponding to the topological term is given in figure 5.

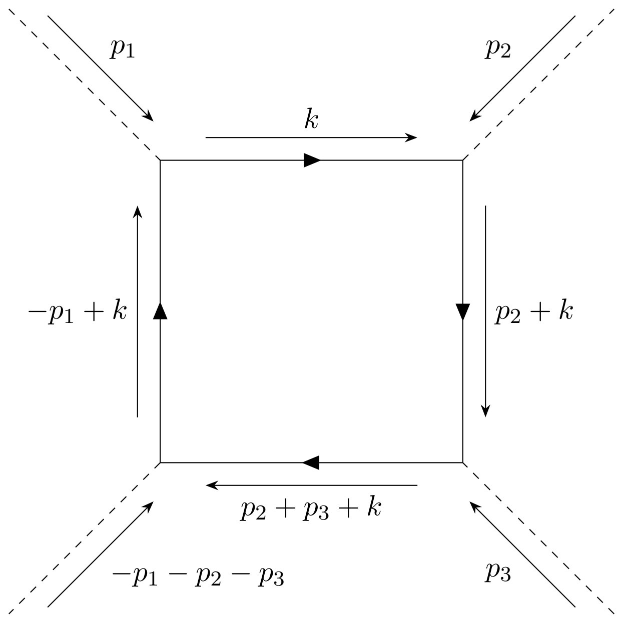

Figure 5: Feynman diagram for the fourth-order term in the effective action.

This term is given by

$$\begin{aligned}
\Gamma \supset& \frac{iM^4}{4}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{tr}\left(\frac{m^a\Gamma^{(5)}_5\Gamma^{(5)}_a}{\partial\!\!\!/ + M}\frac{m^b\Gamma^{(5)}_5\Gamma^{(5)}_b}{\partial\!\!\!/ + M}\frac{m^c\Gamma^{(5)}_5\Gamma^{(5)}_c}{\partial\!\!\!/ + M}\frac{m^d\Gamma^{(5)}_5\Gamma^{(5)}_d}{\partial\!\!\!/ + M}\right)\\
=& \frac{iM^4}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{m^a(p_1)m^b(p_2)m^c(p_3)m^d(p_4)}{\prod\limits_{i=1}^{4}(-k_i^2 - M^2)}\,\mathrm{tr}\left(\Gamma^{(5)}_5\Gamma^{(5)}_a(i\gamma^\mu\Gamma^{(5)}_5 k_{1,\mu} - M)\Gamma^{(5)}_5\Gamma^{(5)}_b\right.\\
&\left.\cdot\,(i\gamma^\nu\Gamma^{(5)}_5 k_{2,\nu} - M)\Gamma^{(5)}_5\Gamma^{(5)}_c(i\gamma^\rho\Gamma^{(5)}_5 k_{3,\rho} - M)\Gamma^{(5)}_5\Gamma^{(5)}_d(i\gamma^\sigma\Gamma^{(5)}_5 k_{4,\sigma} - M)\right),
\end{aligned}$$

where the previously introduced aliases have been adopted. Using similar arguments as for $d = 1$, this can be simplified to

$$\begin{aligned}
\Gamma \supset& -\frac{M^5}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}k_{2,\mu}k_{3,\nu}k_{4,\rho}\frac{m^a(p_1)m^b(p_2)m^c(p_3)m^d(p_4)}{\prod\limits_{i=1}^{4}(-k_i^2 - M^2)}\\
&\cdot\,\mathrm{tr}\left(\Gamma^{(5)}_5\Gamma^{(5)}_a\Gamma^{(5)}_5\Gamma^{(5)}_b\gamma^\mu\Gamma^{(5)}_5\Gamma^{(5)}_5\Gamma^{(5)}_c\gamma^\nu\Gamma^{(5)}_5\Gamma^{(5)}_5\Gamma^{(5)}_d\gamma^\rho\Gamma^{(5)}_5\right)\\
=& -\frac{M^5}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}k_{2,\mu}k_{3,\nu}k_{4,\rho}\frac{m^a(p_1)m^b(p_2)m^c(p_3)m^d(p_4)}{\prod\limits_{i=1}^{4}(k_i^2 + M^2)}\\
&\cdot\,\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho)\,\mathrm{tr}\left(\Gamma^{(5)}_5\Gamma^{(5)}_a\Gamma^{(5)}_5\Gamma^{(5)}_b\Gamma^{(5)}_c\Gamma^{(5)}_d\Gamma^{(5)}_5\right)\\
=& -\frac{i}{32\pi}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\varepsilon^{\mu\nu\rho}\varepsilon^{abcd}m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)p_{4,\rho}m_d(p_4)\\
=& \frac{1}{32\pi}\int \mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}\varepsilon^{abcd}m_a\partial_\mu m_b\partial_\nu m_c\partial_\rho m_d.
\end{aligned}$$

This is also a Weiss-Zumino-Witten term corresponding to a 3-form

$$\omega = \frac{1}{32\pi}\varepsilon_{abcd}m^a \wedge dm^b \wedge dm^c \wedge dm^d.$$

A New Model The models studied above have connected d-dimensional systems to response terms described by integrals of up to $d+1$-forms. To relate the study of these theories to the higher Berry curvature, we will instead need to find a response term described by the integral of a $d+2$-form. Let us consider spatial dimension 1 and introduce mass terms according to

$$M = M_0 + i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l.$$

The matrices Γ represent a Clifford algebra and act in flavor space, and we take the mass fields to lie on S^5. The Lagrangian of the model is

$$\mathcal{L} = -i\overline{\Psi}\left(\partial\!\!\!/ + M\right)\Psi = -i\overline{\Psi}\left(\partial\!\!\!/ + M_0 + i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l\right)\Psi.$$

The idea underlying this model follows Abanov and Wiegmann. They construct models with mass fields confined to S^d or S^{d+1} and show that the topological response terms are related to (the pullbacks of) d-forms and $d+1$-forms respectively. A first obvious attempt in one dimension is therefore to use mass fields on S^3 (and the simple way to do this just so happens to be Abanov and Wiegmann's A-series model in $d = 3$), but we saw that this only produced a topological term given by a 2-form. This model attempts to fix this by extending the mass fields in a way such that, had you done it in $d = 3$, it would take the response term from being given by a 3-form to a 4-form. The hope is that it will achieve a similar result.

At this point it is pertinent to ask whether this attempt really stood any chance. The answer is no, and for a very simple reason. Looking at the above, the appearance of the pullback was no coincidence; it arrived precisely because of the form of the effective action. By its definition the pullback does not affect the rank of any tensor. As the effective action is given by an integral over spacetime, it follows that any form appearing in it must exist on spacetime, and the highest form in $d+1$-dimensional spacetime is a $d+1$-form. Note that this argument has no reliance on the structure of the mass fields. Nevertheless, we show the attempt below.

The effective action is

$$\begin{aligned}
\Gamma =& -i\ln\left(\det\left(-i\left(\not{\partial}+M_0+i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l\right)\right)\right)\\
=& -i\,\mathrm{tr}\left(\ln\left(-i\left(\not{\partial}+M_0+i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l\right)\right)\right)\\
=& C_0 - i\,\mathrm{tr}\left(\ln\left(1+\frac{m+i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l}{\not{\partial}+M}\right)\right)\\
=& C_0 - i\,\mathrm{tr}\left(\ln\left(1+\frac{(\not{\partial}-M)\left(m+i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l\right)}{\partial^2-M^2}\right)\right),
\end{aligned}$$

where we have introduced the mass perturbation $m = M_0 - M$, with M being a fixed mass scale parameter. The topological term comes from the fifth-order expansion, with the Feynman diagram shown in figure 6.

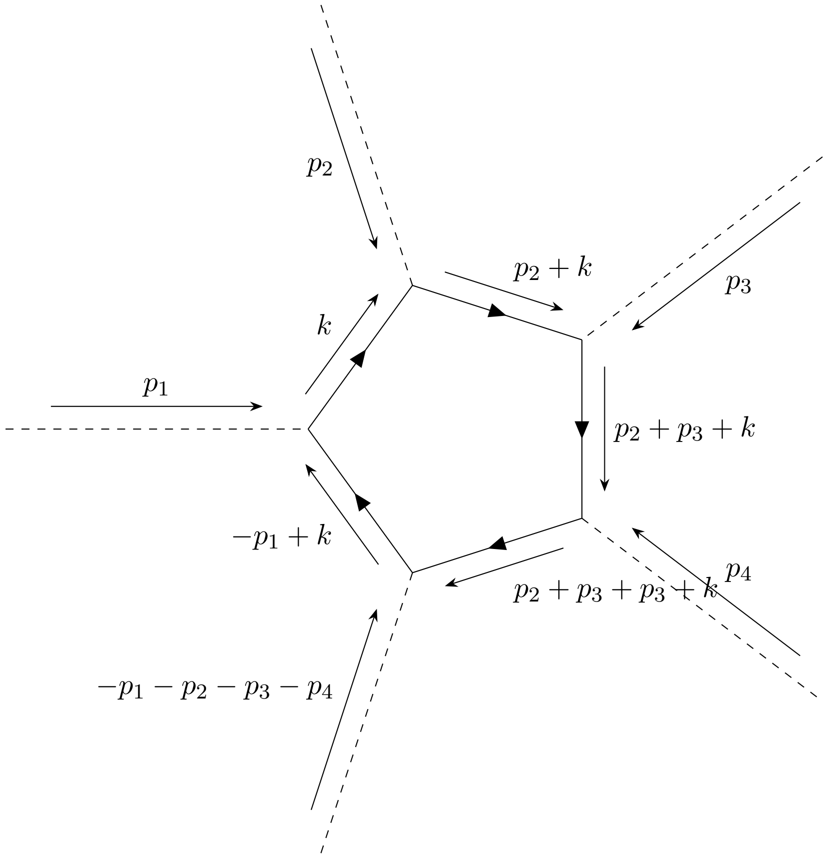

Figure 6: Feynman diagram for the third-order term in the effective action.

This translates to

$$\Gamma = -\frac{i}{5}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\left(\left(\frac{(\partial\!\!\!/ - M)\left(m + i\gamma^{01}\sum_{l=1}^{5} M_l\Gamma^l\right)}{\partial^2 - M^2}\right)^5\right)$$

$$\supset \frac{1}{5}\sum_{l_i=1}^{5}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{tr}\left(\frac{(ik\!\!\!/ - M)\gamma^{01}M_{l_1}\Gamma^{l_1}}{-k^2 - M^2}\frac{(i(p\!\!\!/_2 + k\!\!\!/) - M)\gamma^{01}M_{l_2}\Gamma^{l_2}}{-(p_2+k)^2 - M^2}\frac{(i(p\!\!\!/_2 + p\!\!\!/_3 + k\!\!\!/) - M)\gamma^{01}M_{l_3}\Gamma^{l_3}}{-(p_2+p_3+k)^2 - M^2}\right.$$
$$\left.\cdot\frac{(i(p\!\!\!/_2 + p\!\!\!/_3 + p\!\!\!/_4 + k\!\!\!/) - M)\gamma^{01}M_{l_4}\Gamma^{l_4}}{-(p_2+p_3+p_4+k)^2 - M^2}\frac{(i(-p\!\!\!/_1 + k\!\!\!/) - M)\gamma^{01}M_{l_5}\Gamma^{l_5}}{(-p_1+k)^2 - M^2}\right).$$

Let us now consider the contents of the trace. We are looking for topological terms, which appear in the presence of all Γ and all γ appearing exactly once. The matrices will produce a Levi-Civita tensor, meaning any contributions of orders 1 or 2 in k will vanish. As such the topological term is given by

$$\Gamma \supset -\frac{M}{5}\sum_{l_i=1}^{5}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{tr}\left(\frac{M_{l_1}\Gamma^{l_1}}{-k^2 - M^2}\frac{(i(p\!\!\!/_2 + k\!\!\!/) - M)\gamma^{01}M_{l_2}\Gamma^{l_2}}{-(p_2+k)^2 - M^2}\frac{(i(p\!\!\!/_2 + p\!\!\!/_3 + k\!\!\!/) - M)\gamma^{01}M_{l_3}\Gamma^{l_3}}{-(p_2+p_3+k)^2 - M^2}\right.$$
$$\left.\cdot\frac{(i(p\!\!\!/_2 + p\!\!\!/_3 + p\!\!\!/_4 + k\!\!\!/) - M)\gamma^{01}M_{l_4}\Gamma^{l_4}}{-(p_2+p_3+p_4+k)^2 - M^2}\frac{(i(-p\!\!\!/_1 + k\!\!\!/) - M)M_{l_5}\Gamma^{l_5}}{(-p_1+k)^2 - M^2}\right).$$

The contents of the trace are

$$\mathrm{tr}\Big(\Gamma^{l_1}(i(p\!\!\!/_2 + k\!\!\!/) - M)\gamma^{01}\Gamma^{l_2}(i(p\!\!\!/_2 + p\!\!\!/_3 + k\!\!\!/) - M)\gamma^{01}\Gamma^{l_3}(i(p\!\!\!/_2 + p\!\!\!/_3 + p\!\!\!/_4 + k\!\!\!/) - M)\gamma^{01}\Gamma^{l_4}(i(-p\!\!\!/_1 + k\!\!\!/) - M)\Gamma^{l_5}\Big)$$
$$= \mathrm{tr}\Big(\Gamma^{l_1}\Gamma^{l_2}\Gamma^{l_3}\Gamma^{l_4}\Gamma^{l_5}\Big)\,\mathrm{tr}\Big((i(p\!\!\!/_2 + k\!\!\!/) - M)\gamma^{01}(i(p\!\!\!/_2 + p\!\!\!/_3 + k\!\!\!/) - M)\gamma^{01}(i(p\!\!\!/_2 + p\!\!\!/_3 + p\!\!\!/_4 + k\!\!\!/) - M)\gamma^{01}(i(-p\!\!\!/_1 + k\!\!\!/) - M)\Big),$$

exploiting the product structure of the operators. The case where all Dirac matrices appear exactly once correspond to exactly two momenta appearing, meaning this topological term too will have a pullback of a 2-form onto spacetime.

Extending to Synthetic Dimensions Thus far we have seen that the effective actions in class B involve the pullback of a $d+1$-form. To relate this to the higher Berry curvature, we can imagine the following: If we can write spacetime as the boundary of some other manifold Y and extend the mass fields to Y, then Stokes' theorem allows us to write

$$\Gamma = \int m^\star\omega = \int_Y d(m^\star\omega) = \int_Y m^\star(d\omega).$$

The form $d\omega$ is then a $d+2$-form which plays the role of the higher Berry curvature.

A first question is whether $d\omega$ exists. Certainly the containment of mass fields in class B to S^{d+1} implies that the answer is no. There are, however, two possible ways to solve that. The first, as proposed by Abanov and Wiegmann, is to simply not confine the mass fields at all. The alternative, which is what is done by Hsin et al, is to extend the usual mass term to a field. The consequence of this choice is that the response terms we have considered are lowest-order terms in the mass perturbation $M_0(x) - M$.

Extension Schemes We will consider some slightly different theories next. The construction is as follows: Suppose spacetime is some manifold X. We can then construct a higher manifold by assuming X to be the boundaries of two different manifold $Y_\pm$. Extending spacetime with a synthetic time-like dimension to produce the manifold $Y_L = X \times I$, the full manifold on which the theory lives is

$$Y = \overline{Y}_- \cup Y_L \cup Y_+.$$

The bar indicates orientation reversal, and the I dimension goes from y_- to y_+ on the journey between the two borders. We assume that close to the intersections this separability applies too, and so the union of Y_L with the regions close to the intersections forms what is called the collar neighborhood of X in Y. Specializing to even d, the Lagrangian on Y is

$$\mathcal{L}_Y = \overline{\Psi}(\not{D} + M_1 + i\Gamma_5 M_2)\Psi.$$

The corresponding Dirac equation is

$$(\not{D} + M_1 + i\Gamma_5 M_2)\Psi = 0.$$

Compare this to the "true" Dirac equation

$$(\not{D} + M)\Psi = 0,$$

with M being some hermitian matrix. M_1 will vary smoothly between constant values $\pm m_0$ in $Y_\pm$ as a function of y only, and M_2 depends only on x in the collar neighborhood and can be extended to $Y_\pm$ in some fashion. On Y_L we have $M_2 = M$. For even d, M_2 may be any hermitian matrix, whereas for odd d we have $M_2 = Q + i\gamma_5 P$ for two hermitian matrices Q and P. We also impose the restriction that $A = A_i(x)dx^i$.

We now introduce the two Dirac operators

$$D_d = \gamma^i D_i + M_2,\ \ D_{d+1} = \Gamma^\mu D_\mu + M_1 + i\Gamma.$$

The latter can be written as

$$D_{d+1} = -iK + M_1,$$

where

$$K = i(\Gamma^\mu D_\mu + i\Gamma_5 M_2) = i\Gamma^y(\partial_y + D_X)$$

is a hermitian operator.

The expression we would like to show is

$$\frac{Z_d(X)}{Z_d^{(0)}(X)} \cdot \frac{1}{\left|\frac{Z_d(X)}{Z_d^{(0)}(X)}\right|} = \exp(2\pi i(\eta_K(Y_+, \mathrm{APS}) - \eta_{K^{(0)}}(Y_+, \mathrm{APS}))).$$

On the right-hand side we have the η-invariant of K given the boundary condition, which is the number of positive eigenvalues minus the number of negative eigenvalues divided by two. On the left-hand side are the partition functions for the full case, as well as the version with a superscript zero, indicating the case where M is just a constant. This can be interpreted as a Berry phase of the system, so let us therefore review the important aspects of the construction that ensure this.

The properties of K arise when computing certain state overlaps. A boundary condition is needed to guarantee hermicity. The condition used in the original work is implemented by considering the eigenstates of D_X, which come in pairs as $\{\Gamma^y, D_X\} = 0$, and is given by

$$P_{<0}^{D_X}\psi = 0$$

on ∂Y_+, where $P_{<0}^{D_X}$ is the projection operator onto the negative eigenstates of D_X. The hermiticity of K then allows for expansion of states in terms of the eigenstates of K. These facts combine to imply the above formula.

In the case of odd d things are slightly different. On X the mass matrix we want to realize is of the form $M_1 + i\gamma_5 M_2$, where M_1 and M_2 are both Hermitian. On Y we extend the mass matrix, producing the Lagrangian

$$\mathcal{L} = \overline{\Psi}\left(\not{D} + M_k \otimes \tau^k\right)\Psi,$$

with the τ^k being Pauli matrices. The added mass matrix M_3 is a function of y only, going from $-m_0$ to m_0. The above can be factorized as

$$\mathcal{L} = \Psi^\dagger \beta \otimes \tau^3 \left(\not{D} \otimes \tau^3 + M_3 + i(M_1 \otimes \tau^2 - M_2 \otimes \tau^1)\right)\Psi,$$

and by choosing $\Psi^\dagger\beta\otimes\tau^3$ as the adjoint field, the above can be written as

$$\mathcal{L}=\overline{\Psi}\left(\not{D}\otimes\tau^3+M_3+i(M_1\otimes\tau^2-M_2\otimes\tau^1)\right)\Psi.$$

The corresponding Dirac operators are

$$D_d=\gamma^i D_i+M_1+i\gamma_5 M_2,\;\; D_{d+1}=\Gamma^\mu D_\mu+M_1\tau^1+M_2\tau^2+M_3\tau^3,$$

and the relation $D_{d+1}=\tau^3(-iK+M_3)$ implies

$$K=i\left(\tau^3\Gamma^\mu D_\mu+iM_1\otimes\tau^2-iM_2\otimes\tau^1\right).$$

Along the collar neighborhood we have

$$K=i\tau^3\Gamma^y\left(\partial_y+D_X\right),\;\; D_X=\Gamma^y\left(\Gamma^i D_i+M_1\otimes\tau^1+M_2\otimes\tau^2\right).$$

By a somewhat more involved argument involving the same requirements on K, the formula for the Berry phase applies equally well to this case.

Let us next consider some examples. First is the Lagrangian

$$\mathcal{L}_{2+1}=-\overline{\Psi}\left(\not{\partial}+M\left(1+\sum_{a=3}^{6}m^a\Gamma_a^{(5)}\right)\right)\Psi.$$

More explicitly, this is a model with isospin and Dirac structure, and the above matrices are

$$\Gamma^\mu=\gamma^\mu\otimes 1_{\text{isospin}},\;\; \Gamma^5=\gamma^\mu\otimes 1_{\text{isospin}},\;\; \Gamma_a^{(5)}=1_{\text{Dirac}}\otimes\gamma_a.$$

The extended Lagrangian is

$$\mathcal{L}_{3+1}=-\overline{\Psi}\left(\not{\partial}+M_1+iM\Gamma_5\left(1+\sum_{a=3}^{6}m^a\Gamma_a^{(5)}\right)\right)\Psi.$$

The topological term in the effective action arises from

$$\begin{aligned}
\Gamma\supset&-\frac{i}{5}\int \mathrm{d}^4p\,\mathrm{d}^4k\,\mathrm{tr}\left(\left(\frac{M_1-M+iM\Gamma_5+iM\Gamma_5\sum\limits_{a=3}^{7}m^a\Gamma_a}{\not{\partial}+M}\right)^5\right)\\
\supset&-\frac{i}{5}\int \mathrm{d}^4p\,\mathrm{d}^4k\,\frac{1}{\prod\limits_{i=1}^{5}(-k_i^2-M^2)}\,\mathrm{tr}\Big((i\not{k}_1-M)iM\Gamma_5m^a(p_1)\Gamma_a^{(5)}(i\not{k}_2-M)iM\Gamma_5m^b(p_2)\Gamma_b^{(5)}\\
&\cdot(i\not{k}_3-M)iM\Gamma_5m^c(p_3)\Gamma_c^{(5)}(i\not{k}_4-M)iM\Gamma_5m^d(p_4)\Gamma_d^{(5)}(i\not{k}_5-M)iM\Gamma_5m^e(p_5)\Gamma_e^{(5)}\Big)\\
\supset&-\frac{M^6}{5}\int \mathrm{d}^4p\,\mathrm{d}^4k\,\frac{m^a(p_1)m^b(p_2)m^c(p_3)m^d(p_4)m^e(p_5)}{\prod\limits_{i=1}^{5}(-k_i^2-M^2)}\,\mathrm{tr}\left(\Gamma_5\Gamma_a^{(5)}\not{k}_2\Gamma_5\Gamma_b^{(5)}\not{k}_3\Gamma_5\Gamma_c^{(5)}\not{k}_4\Gamma_5\Gamma_d^{(5)}\not{k}_5\Gamma_5\Gamma_e^{(5)}\right)\\
\approx&\frac{M^6}{5}\int \mathrm{d}^4p\,\mathrm{d}^4k\,\frac{m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)p_{4,\rho}m_d(p_4)p_{5,\sigma}m_e(p_5)}{(k^2+M^2)^5}\,\mathrm{tr}(\Gamma^\mu\Gamma^\nu\Gamma^\rho\Gamma^\sigma\Gamma_5)\,\mathrm{tr}\left(\Gamma_a^{(5)}\Gamma_b^{(5)}\Gamma_c^{(5)}\Gamma_d^{(5)}\Gamma_e^{(5)}\right)\\
=&\frac{M^6}{5}\cdot i\frac{\Gamma(3)}{(4\pi)^2\Gamma(5)\,M^6}\int \mathrm{d}^4p\,\mathrm{d}^4k\,m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)p_{4,\rho}m_d(p_4)p_{5,\sigma}m_e(p_5)\cdot-4i\varepsilon^{\mu\nu\rho\sigma}\cdot-4\varepsilon_{abcde}\\
=&-\frac{1}{60\pi^2}\int \mathrm{d}^4p\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{abcde}m_a(p_1)p_{2,\mu}m_b(p_2)p_{3,\nu}m_c(p_3)p_{4,\rho}m_d(p_4)p_{5,\sigma}m_e(p_5)\\
=&-\frac{1}{60\pi^2}\int \mathrm{d}^3x\,\mathrm{d}y\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{abcde}m_a\partial_\mu m_b\partial_\nu m_c\partial_\rho m_d\partial_\sigma m_e.
\end{aligned}$$

This is a familiar Weiss-Zumino-Witten term with

$$\omega_{abcd} = -\frac{1}{60\pi^2}\varepsilon_{eabcd}m^e.$$

The first example with $d = 1$ is given by

$$\mathcal{L}_{1+1} = -\overline{\Psi}\left(\partial\!\!\!/ + M\left(m_4 + i\gamma\sum_{a=1}^{3} m^a\Gamma_a^{(3)}\right)\right)\Psi,$$

with m_{d+3} now being allowed to vary such that the target space of this model is S^3. The extended Lagrangian is

$$\mathcal{L}_{1+2} = -\overline{\Psi}\left(\partial\!\!\!/\tau^3 + M + iM\left(m_4\tau^2 - \sum_{a=1}^{3} m^a\Gamma_a^{(3)}\tau^1\right)\right)\Psi.$$

The number of anticommuting matrices in this theory is seven, and so the previously performed computation for $d = 2$ can be repeated. The topological term in the effective action is given by

$$\begin{aligned}
\Gamma \supset& \frac{iM^4}{4}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{tr}\left(\frac{m_4\tau^2 - \sum_{a=1}^{3} m^a\Gamma_a^{(3)}\tau^1}{\partial\!\!\!/\tau^3 + M}\frac{m_4\tau^2 - \sum_{b=1}^{3} m^b\Gamma_b^{(3)}\tau^1}{\partial\!\!\!/\tau^3 + M}\right.\\
&\left.\cdot\frac{m_4\tau^2 - \sum_{c=1}^{3} m^c\Gamma_c^{(3)}\tau^1}{\partial\!\!\!/\tau^3 + M}\frac{m_4\tau^2 - \sum_{d=1}^{3} m^d\Gamma_d^{(3)}\tau^1}{\partial\!\!\!/\tau^3 + M}\right)\\
=&\frac{iM^4}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\prod_{i=1}^{4}(-k_i^2 - M^2)}\,\mathrm{tr}\left(\left(m_4\tau^2 - \sum_{a=1}^{3} m^a\Gamma_a^{(3)}\tau^1\right)(i\gamma^\mu\tau^3k_{1,\mu} - M)\right.\\
&\cdot\left(m_4\tau^2 - \sum_{a=1}^{3} m^b\Gamma_b^{(3)}\tau^1\right)(i\gamma^\nu\tau^3k_{2,\nu} - M)\left(m_4\tau^2 - \sum_{a=1}^{3} m^c\Gamma_c^{(3)}\tau^1\right)(i\gamma^\rho\tau^3k_{3,\rho} - M)\\
&\left.\cdot\left(m_4\tau^2 - \sum_{d=1}^{3} m^d\Gamma_d^{(3)}\tau^1\right)(i\gamma^\sigma\tau^3k_{4,\sigma} - M)\right)\\
=&-\frac{M^5}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_{2,\mu}k_{3,\nu}k_{4,\rho}}{\prod_{i=1}^{4}(-k_i^2 - M^2)}\,\mathrm{tr}\left(\left(m_4\tau^2 - \sum_{a=1}^{3} m^a\Gamma_a^{(3)}\tau^1\right)\right.\\
&\cdot\left(m_4\tau^2 - \sum_{a=1}^{3} m^b\Gamma_b^{(3)}\tau^1\right)\gamma^\mu\tau^3\left(m_4\tau^2 - \sum_{a=1}^{3} m^c\Gamma_c^{(3)}\tau^1\right)\gamma^\nu\tau^3\\
&\left.\cdot\left(m_4\tau^2 - \sum_{d=1}^{3} m^d\Gamma_d^{(3)}\tau^1\right)\gamma^\rho\tau^3\right)\\
=&-\frac{M^5}{4}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_{2,\mu}k_{3,\nu}k_{4,\rho}}{\prod_{i=1}^{4}(k_i^2 + M^2)}\,\mathrm{tr}\left(\left(m_4\tau^2 - \sum_{a=1}^{3} m^a\Gamma_a^{(3)}\tau^1\right)\right.\\
&\left.\cdot\left(m_4\tau^2 - \sum_{a=1}^{3} m^b\Gamma_b^{(3)}\tau^1\right)\tau^3\left(m_4\tau^2 - \sum_{a=1}^{3} m^c\Gamma_c^{(3)}\tau^1\right)\tau^3\left(m_4\tau^2 - \sum_{d=1}^{3} m^d\Gamma_d^{(3)}\tau^1\right)\tau^3\right)\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho).
\end{aligned}$$

The trace has four terms depending on the placement of m_4. The first is

$$\begin{aligned}
&-\sum_{b,c,d=1}^{3}\operatorname{tr}\left(m_4\tau^2 m^b\Gamma_b^{(3)}\tau^1\tau^3 m^c\Gamma_c^{(3)}\tau^1\tau^3 m^d\Gamma_d^{(3)}\tau^1\tau^3\right)\\
=&-\sum_{a,b,c=1}^{3} m^4(p_1)m^a(p_2)m^b(p_3)m^c(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^2\tau^1\tau^3\tau^1\tau^3\tau^1\tau^3\right)\\
=&-\sum_{a,b,c=1}^{3} m^4(p_1)m^a(p_2)m^b(p_3)m^c(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^1\tau^2\tau^3\right).
\end{aligned}$$

The second is

$$\begin{aligned}
&-\sum_{a,c,d=1}^{3}\operatorname{tr}\left(m^a\Gamma_a^{(3)}\tau^1 m^4\tau^2\tau^3 m^c\Gamma_c^{(3)}\tau^1\tau^3 m^d\Gamma_d^{(3)}\tau^1\tau^3\right)\\
=&-\sum_{a,b,c=1}^{3} m^a(p_1)m^4(p_2)m^b(p_3)m^c(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^1\tau^2\tau^3\tau^1\tau^3\tau^1\tau^3\right)\\
=&\sum_{a,b,c=1}^{3} m^a(p_1)m^4(p_2)m^b(p_3)m^c(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^1\tau^2\tau^3\right).
\end{aligned}$$

The third is

$$\begin{aligned}
&-\sum_{a,b,d=1}^{3}\operatorname{tr}\left(m^a\Gamma_a^{(3)}\tau^1 m^b\Gamma_b^{(3)}\tau^1\tau^3 m^4\tau^2\tau^3 m^d\Gamma_d^{(3)}\tau^1\tau^3\right)\\
=&-\sum_{a,b,c=1}^{3} m^a(p_1)m^b(p_2)m^4(p_3)m^c(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^3\tau^2\tau^3\tau^1\tau^3\right)\\
=&-\sum_{a,b,c=1}^{3} m^a(p_1)m^b(p_2)m^4(p_3)m^c(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^1\tau^2\tau^3\right).
\end{aligned}$$

The fourth is

$$\begin{aligned}
&-\sum_{a,b,c=1}^{3}\operatorname{tr}\left(m^a\Gamma_a^{(3)}\tau^1 m^b\Gamma_b^{(3)}\tau^1\tau^3 m^c\Gamma_c^{(3)}\tau^1\tau^3 m^4\tau^2\tau^3\right)\\
=&-\sum_{a,b,c=1}^{3} m^a(p_1)m^b(p_2)m^c(p_3)m^4(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^3\tau^1\tau^3\tau^2\tau^3\right)\\
=&\sum_{a,b,c=1}^{3} m^a(p_1)m^b(p_2)m^c(p_3)m^4(p_4)\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right)\operatorname{tr}\!\left(\tau^1\tau^2\tau^3\right).
\end{aligned}$$

The traces evaluate to

$$\operatorname{tr}\left(\Gamma_a^{(3)}\Gamma_b^{(3)}\Gamma_c^{(3)}\right) = -2i\varepsilon_{abc},\ \operatorname{tr}\!\left(\tau^1\tau^2\tau^3\right) = 2i,$$

and in combination with the four terms we have

$$\begin{aligned}
\Gamma =& -\frac{M^5}{4}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}k_{2,\mu}k_{3,\nu}k_{4,\rho}\frac{m^a(p_1)m^b(p_2)m^c(p_3)m^d(p_4)}{\prod_{i=1}^{4}(k_i^2+M^2)}\cdot -4\varepsilon_{abcd}\cdot 2\varepsilon^{\mu\nu\rho}\\
=&\frac{i}{32\pi}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\varepsilon^{\mu\nu\rho}\varepsilon_{abcd}m^a(p_1)p_{2,\mu}m^b(p_2)p_{3,\nu}m^c(p_3)p_{4,\rho}m^d(p_4)\\
=&-\frac{1}{32\pi}\int\mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}\varepsilon_{abcd}m^a\partial_\mu m^b\partial_\nu m^c\partial_\rho m^d.
\end{aligned}$$