Summary of SI2400 Theoretical Particle Physics

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Abstract

This is a summary of SI2400 Theoretical Particle Physics.

Contents

1 Basic Concepts

The Standard Model The standard model is a quantum field theory with the gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. It is comprised of

- matter fields, which represent fermions.
- gauge bosons - spin 1 bosons that mediate the fundamental forces.
- a scalar field, which represents the Higgs boson.

The Fundamental Forces The fundamental forces are the strong and weak nuclear interactions, the electromagnetic interaction and the gravitational interactions. Only the three former have a corresponding quantum field theory, hence this discussion (and in fact the standard model itself at time of writing) will be restricted to those.

An important aspect about the strong force is the fact that its coupling constant is rolling - that is, it varies in space. It is very large at distances comparable to those of nuclei, but very small at the scale of an individual nucleon. This means, for instance, that quarks in a proton are virtually free within the proton, but that the proton itself is stable.

Elementary Particles An elementary particle is a particle without substructure.

The Building Blocks of the Standard Model The fermions comprising the standard model come in two kinds: quarks and leptons. Each quark also has an antiquark. The quarks have colour charge red, green or blue, which is involved in the strong interaction. In nature they are not found as free particles, but are either joined with an antiquark to create a meson, or joined with two other quarks to make a baryon. In general, combinations of quarks are called hadrons. Leptons, by contrast, have no colour charge. Both quarks and leptons may have electric charge, and they all have spin and interact with the weak interaction. They are also divided into three so-called generations.

Quantum Numbers In addition to charge and momentum, we introduce quantum numbers that must be conserved at vertices in Feynman diagrams. These are:

- lepton number.
- generational lepton number.
- baryon number.
- strangeness (not conserved in weak interactions).
- colour.

These quantum numbers are flipped in the antiparticles.

QCD Vertices QCD, or quantum chromodynamics, is the field theory describing the strong force. Unlike QED, it has different kinds of vertices as the gluons carry colour charge. The one including fermions is exemplified as in figure 1, and symbolizes an incoming up quark with blue colour charge interacting via a gluon and leaving the interaction with red colour charge. The gluon then carries one positive unit of red colour charge and one unit of negative red colour charge.

In addition there are two pure gluon vertices, shown in figure 2.

Weak Interaction Vertices The weak interaction is mediated by a charged W boson and a neutral Z boson. The corresponding vertices are shown in figure 3.

The Jacobi Identity for Structure Constants Applying the Jacobi identity we find

$$\begin{aligned}[X_a,[X_b,X_c]]+[X_b,[X_c,X_a]]+[X_c,[X_a,X_b]] &= i\left(f_{bcd}[X_a,X_d]+f_{cad}[X_b,X_d]+f_{abd}[X_c,X_d]\right)\\ &= i^2\left(f_{bcd}f_{ade}+f_{cad}f_{bde}+f_{abd}f_{cde}\right)X_e=0,\end{aligned}$$

which means

$$f_{bcd}f_{ade}+f_{cad}f_{bde}+f_{abd}f_{cde}=0.$$

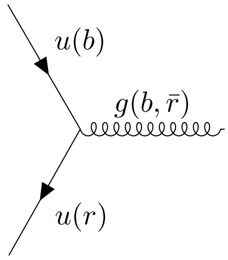


Figure 1: An example of a strong interaction vertex involving a fermion.

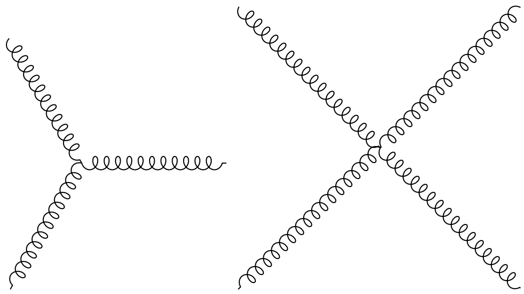


Figure 2: The two strong interaction vertices only involving gluons.

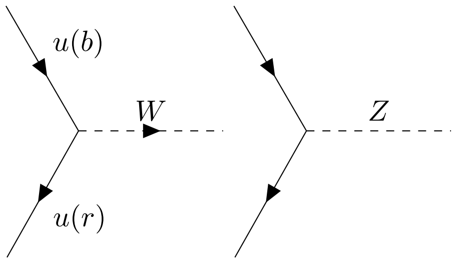


Figure 3: The fundamental vertices of the weak interaction.

Structure Constant Representations For a group, consider the structure constants

$$[X_a, X_b] = if_{abc}X_c.$$

Constructing the matrices

$$(T_a)_{bc} = -if_{abc}$$

we find

$$[T_a, T_b]_{cd} = -(f_{ace}f_{bed} - f_{bce}f_{aed}) = f_{cae}f_{bed} + f_{bce}f_{aed} = -f_{abe}f_{ced} = if_{abe}(T_e)_{cd},$$

and thus

$$[T_a, T_b] = if_{abe}T_e,$$

meaning that the structure constants themselves provide a representation of the group. This is a real representation, i.e. is identical to its conjugate representation.

A Closer Look at SU(2) The Lie algebra of SU(2) is

$$[J_i, J_j] = i\varepsilon_{ijk}J_k.$$

Our goal is to block diagonalize the space on which elements in the group act, starting by looking at the generators. There are three generators, so we can only diagonalize the one. We choose J_3, which will net us the so-called spin representation. Taking the space to be finite dimensional (which in the context of angular momentum is explained by physical arguments), we consider the eigenstate $|j, \alpha\rangle$ with the highest eigenvalue

j of J_3. α contains other information, the existence of which is undetermined as of yet. Next we introduce the raising and lowering operators

$$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2).$$

We have

$$\begin{aligned}
[J_+, J_-] &= \frac{1}{2}i(-[J_1, J_2] + [J_2, J_1]) = J_3, \\
[J_3, J_\pm] &= \frac{1}{\sqrt{2}}([J_3, J_1] \pm i[J_3, J_2]) = \frac{1}{\sqrt{2}}(\pm J_1 + iJ_2) = \pm J_\pm.
\end{aligned}$$

For a general eigenstate we find

$$J_3 J_\pm \left|m, \alpha\right\rangle = (\pm J_\pm + J_\pm J_3) \left|m, \alpha\right\rangle = (m \pm 1) J_\pm \left|m, \alpha\right\rangle,$$

and the raising and lowering operators do indeed work as expected. This implies $J_+ \left|j, \alpha\right\rangle = 0$. Furthermore, we have

$$\begin{aligned}
\langle j, \beta | J_-^\dagger J_- | j, \alpha \rangle &= \langle j, \beta | J_+ J_- | j, \alpha \rangle \\
&= \langle j, \beta | J_+ J_- - J_- J_+ | j, \alpha \rangle \\
&= \langle j, \beta | [J_+, J_-] | j, \alpha \rangle \\
&= \langle j, \beta | J_3 | j, \alpha \rangle \\
&= j \delta_{\alpha\beta},
\end{aligned}$$

as we assume the j states to be orthonormal. This implies that the raising and lowering operators preserve orthogonality in terms of the other quantum numbers. Introducing $N_j = \sqrt{j}$ we may thus write

$$J_- \left|j, \alpha\right\rangle = N_j \left|j - 1, \alpha\right\rangle.$$

Similarly we find

$$\begin{aligned}
J_+ \left|j - 1, \alpha\right\rangle &= \frac{1}{N_j} J_+ J_- \left|j, \alpha\right\rangle \\
&= \frac{1}{N_j} [J_+, J_-] \left|j, \alpha\right\rangle \\
&= N_j \left|j, \alpha\right\rangle.
\end{aligned}$$

We generalize this to

$$J_- \left|m, \alpha\right\rangle = N_m \left|m - 1, \alpha\right\rangle, \; J_+ \left|m - 1, \alpha\right\rangle = N_m \left|m, \alpha\right\rangle.$$

These satisfy a recursion relation which we can find by writing

$$\begin{aligned}
N_m^2 &= \langle m, \alpha | J_+ J_- | m, \alpha \rangle \\
&= \langle m, \alpha | [J_+, J_-] + J_- J_+ | m, \alpha \rangle \\
&= m + N_{m+1}^2 \langle m + 1, \alpha | m + 1, \alpha \rangle \\
&= m + N_{m+1}^2.
\end{aligned}$$

Repeating this we find

$$\begin{aligned}
N_{j-k}^2 &= \sum_{a=j-k}^{j} a \\
&= \sum_{n=1}^{k+1} j - k + n - 1 \\
&= \frac{k+1}{2}(2(j-k) + k) \\
&= \frac{(2j-k)(k+1)}{2} \\
&= (j - (j-k) + 1)j - \frac{1}{2}(k+1)k = \frac{1}{2}(k+1)(2j-k),
\end{aligned}$$

or

$$N_m = \frac{1}{\sqrt{2}}\sqrt{(j-m+1)(j+m)}.$$

From this we can infer two things about the spectrum. First, by the assumption that Hilbert space is finite dimensional, there must exist a maximal number of lowerings l. This case should satisfy

$$N_{j-l} = \frac{1}{\sqrt{2}}\sqrt{(l+1)(2j-l)} = 0.$$

As l is an integer, the possible upper bounds of the spectrum must therefore be half-integer. Next, we can identify the limits of the eigenvalue spectrum: The upper limit corresponds to the fact that $N_{j+1} = 0$. The lower limit comes from $N_{-j} = 0$, meaning the eigenstate at the bottom of the spectrum is $|-j, \alpha\rangle$. At this point we may ignore the other quantum numbers, as the representation we are working with breaks Hilbert space into invariant subspaces under SU(2). Or, rather, we will replace them simply by the value of j, which together with m specifies the state. We now know that the dimension of each invariant Hilbert space is $2j+1$.

We can now work out the particulars of this representation by computing the matrix elements

$$\begin{aligned}
\left\langle j, m \middle| J_3 \middle| j, m' \right\rangle &= m\delta_{m,m'}, \\
\left\langle j, m \middle| J_+ \middle| j, m' \right\rangle &= N_{m'+1}\delta_{m,m'+1} = \frac{1}{\sqrt{2}}\sqrt{(j-m')(j+m'+1)}\delta_{m,m'+1}, \\
\left\langle j, m \middle| J_- \middle| j, m' \right\rangle &= N_{m'}\delta_{m,m'-1} = \frac{1}{\sqrt{2}}\sqrt{(j-m'+1)(j+m')}\delta_{m,m'-1},
\end{aligned}$$

the latter of which can be used to yield

$$\begin{aligned}
\left\langle j, m \middle| J_1 \middle| j, m' \right\rangle &= \frac{1}{2}\left(\sqrt{(j-m')(j+m'+1)}\delta_{m,m'+1} + \sqrt{(j-m'+1)(j+m')}\delta_{m,m'-1}\right), \\
\left\langle j, m \middle| J_2 \middle| j, m' \right\rangle &= -\frac{i}{2}\left(\sqrt{(j-m')(j+m'+1)}\delta_{m,m'+1} - \sqrt{(j-m'+1)(j+m')}\delta_{m,m'-1}\right).
\end{aligned}$$

As an example, the representation on the $j = \frac{1}{2}$ subspace is

$$J_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ J_2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ J_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This is the simplest representation of SU(2), and is therefore called the fundamental representation. Another example is found on the $j = 1$ subspace, which yields

$$J_1 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},\ J_2 = \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{bmatrix},\ J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Tensor Product of SU(2) Representations Taking the tensor product of two angular momentum Hilbert spaces corresponds to creating a new space of eigenstates of either angular momentum operator. As we know, however, it is possible to diagonalize this Hilbert space in terms of another operator, namely the total angular momentum. As an example, for two $j = \frac{1}{2}$, the total Hilbert space may be written as the tensor product of two two-dimensional subspaces, or as the direct sum of the eigenspaces of the total angular momentum. The latter has two eigenspaces, one with dimension 3 and one with dimension 1. We often denote this in the ridiculous form

$$2 \otimes 2 = 3 \oplus 1.$$

Isospin Early particle physicists, noting the very similar masses of the proton and neutron and studying nuclear energy levels, conjectured the existence of a symmetry of nuclei under swapping of the two. This symmetry would then need to be respected by the strong force. More specifically, they wrote the state of each nucleon as a proton-neutron doublet which transforms under action by SU(2). The generators of this transformation are called components of isospin. We will do a similar thing, but for quarks instead.

Parity Parity is a discrete symmetry which corresponds to spatial inversion. Strong and electromagnetic interactions respect parity, but weak interactions do not.

It turns out that the ground states of hadrons are eigenstates of the parity operator. It also turns out that the parity of fermions (taken to be positive) is opposite to that of their antiparticles, whereas they are the sames for bosons. In addition to this there comes parity from their orbital angular momentum.

Chirality and Helicity As has been discussed in other summaries, helicity and chirality depend on the combination of spin and momentum. It turns out that helicity is frame-dependent, but chirality is intrinsic to the particle. Furthermore, some particles are only found with certain chiralities.

Going From Particles to Antiparticles Consider the particle and antiparticle states

$$e^{-ipx}\begin{bmatrix}1\\0\\\frac{p_x-ip_y}{p_0+m}\\-\frac{p_z}{p_0+m}\end{bmatrix},\ e^{ipx}\begin{bmatrix}\frac{p_z}{p_0+m}\\\frac{p_x+ip_y}{p_0+m}\\1\\0\end{bmatrix}.$$

We will try to devise a transformation between the two. The exponent is flipped by complex conjugation, so that will have to be included. Next we need a transformation matrix which is block off-diagonal. The lower left block being

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}=-i\sigma^2$$

will get the job done. The upper right block should similarly be $i\sigma^2$, meaning that the total matrix should be $-i\gamma^2$. The transformation is thus

$$\Psi'=-i\gamma^2\Psi^\star.$$

This also reveals the need for complex conjugation in changing between and antiparticles.

Charge Conjugation Charge conjugation is a discrete symmetry which corresponds to changing all internal quantum numbers of a state, creating a corresponding antiparticle. It is written as

$$C\left|p\right\rangle=\left|\bar{p}\right\rangle.$$

It is respected by the strong and electromagnetic interactions, but not the weak one.

In particular, for fermions we can use the transformation rule we derived on the Dirac equation with minimal coupling. For the particle we have

$$\gamma^\mu(\partial_\mu-ieA_\mu)\Psi+im\Psi=0.$$

For the antiparticle we have

$$-i\gamma^2(\gamma^\mu)^\star(\partial_\mu+ieA_\mu)\Psi^\star-i\gamma^2\cdot-im\Psi^\star=-i\gamma^2(\gamma^\mu)^\star(\partial_\mu+ieA_\mu)\Psi^\star-m\gamma^2\Psi^\star=0.$$

In the Dirac representation the only matrix that is changed by complex conjugation is γ^2, which changes sign. All the other gamma matrices anticommute, and for γ^2 we can simply reshuffle the complex conjugation to find

$$\gamma^\mu(\partial_\mu+ieA_\mu)\cdot i\gamma^2\Psi^\star+im\cdot i\gamma^2\Psi^\star=0.$$

This implies

$$C\Psi=i\gamma^2\Psi^\star.$$

Such symmetries can be used to determine possible reactions. A trivial example is $\pi^0\to\gamma+\gamma$, which balances charge conjugation, whereas $\pi^0\to\gamma+\gamma+\gamma$ does not, hence only the former is allowed.

Transformation of Antiquark Isospin We know that quark doublets transform under SU(2). An antiquark doublet, however, is related to the complex conjugate of the quark doublet. The complex conjugate itself does not transform the same way as the original doublet - more specifically, we have

$$(\phi')^\star = e^{\frac{i}{2}\theta_i I_i^\star}\phi^\star,$$

which is the wrong transformation rule. We therefore need to write the antiparticle doublet as $\varepsilon\phi^\star$, and choose ε such that

$$\varepsilon e^{\frac{i}{2}\theta_i I_i^\star} = e^{-\frac{i}{2}\theta_i I_i}\varepsilon,$$

which produces the correct transformation rule. Expanding each side around the identity we find

$$\varepsilon I_i^\star = -I_i\varepsilon.$$

We can choose the fundamental representation, which leaves I_1 and I_3 invariant. Choosing $\varepsilon \propto \sigma_2$ the corresponding relations work out. This would also make the final relation work out. One choice is $\varepsilon = i\sigma_2$. In terms of the up and down quark we denote the two states

$$\phi = \begin{bmatrix} u \\ d \end{bmatrix},\ \hat{\phi} = \begin{bmatrix} \hat{d} \\ -\hat{u} \end{bmatrix}.$$

With this we can construct mesons as eigenstates of total isospin.

CP CP is the combination of charge conjugation and parity. This combination of symmetries allows for the theory to contain more reactions. For instance, consider

$$\pi^+ \to \mu^+ + \nu_\mu.$$

This reaction works provided that the muon and the neutrino have opposite helicities. A parity transformation, however, will reverse all helicities, producing a left-handed muon, which is impossible. Similarly, charge conjugation of this reaction involves a right-handed antineutrino, which is also not allowed. The combined CP symmetry, however, produces the reaction

$$\pi^- \to \mu^- + \bar{\nu}_\mu,$$

where the antineutrino is left-handed, which is allowed. The hope of particle physicists was that the weak force respected CP.

***CP* Violation** CP violation turns out to be one way to explain the domination of matter in the universe.

Time Reversal Time reversal - or rather, motion reversal - is a discrete symmetry corresponding to running a process backwards. It is respected by the strong and electromagnetic interactions. It is hard to test it for the weak interaction.

CPT CPT is a combination of the discrete symmetries we have studied.

We can use it to draw conclusions about time reversal. Because quantum field theory is based on CPT invariance and CP is violated by the weak interaction, we expect T to be violated as well in order for CPT to be respected.

Conservation Laws for Reactions The following conservation laws apply for reactions in particle physics, in addition to kinematic laws:

- Electric charge. This is a symmetry respected by all fundamental interactions.
- Color charge. Only the strong interaction is at all concerned with it, and the strong interaction has a corresponding symmetry.
- Baryon number. All primitive vertices conserves the number of quarks, hence the number of quarks is conserved. However, quarks are only found as baryons, with quark number 3, and mesons, with quark number 0, hence we might equally well consider the number of baryons.

- Lepton number. Strong interactions do not include leptons, and electromagnetic interactions only couple lepton fields and quark fields. Weak interactions can change the type of lepton, but if a lepton goes in, another always comes out.

- Generational lepton number. In addition to the above, generational lepton numbers are conserved in most processes, as the electromagnetic force only couples leptonic fields of the same generation and the weak force does the same in most cases.

- Flavor. Strong and electromagnetic interactions are not concerned with flavor, but the weak interaction does not respect it.

How do we tell which fundamental force is at play? A rule of thumb is that if photons are produced, it is the electromagnetic force, and if neutrinos are produced, it is the weak force.

Young Tableaux Young tableaux are a useful tool for analyzing the structure of representations. I will present it in the context of SU(3), and therefore need to talk very briefly about that group.

SU(3) is generated by eight matrices $\frac{1}{2}\lambda_a$, which are normalized such that $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. Two of these commute, hence the group has rank 2.

Corresponding to each representation of SU(3) is a so-called conjugate representation, found by taking the complex conjugate of the first. This representation is denoted by a bar. The generators of this representation are $-\frac{1}{2}\lambda_a^\star$. This representation has not been brought up in the context of SU(2) because it is equivalent to the ones we discussed. This equivalence occurs if there exists a matrix ε which satisfies

$$-\lambda_a^\star = \varepsilon\lambda_a\varepsilon^{-1}.$$

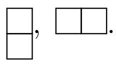
An easy way to treat a conjugate representation with tensor notation is to treat the action of the original representation as acting on contravariant indices and the action of the conjugate representation as acting on covariant indices.

Note that to each antisymmetric rank-$N-1$ tensor in this representation there exists a vector according to

$$t_j = \varepsilon_{i_1 \dots i_{N-1} J} T^{i_1 \dots i_{N-1}},$$

hence we associate such tensors with the conjugate representation.

The next idea would be to take the tensor product of such representations. Young tableaux are useful tools for treating just this. The basic idea is to represent each index with a box, according to $\square$. As the fundamental representation acts on states with single indices, the single box thus represents the fundamental representations. Next, if you have multiple indices, symmetric indices are put in the same row and antisymmetric indices in the same column. The two look like



Higher-rank diagrams are arranged such that the number of boxes in any row is equal to or less than the number of boxes in the rows above.

The power of the Young tableaux is in computing tensor products of representations. This is done with the following steps:

1. Draw tableaux corresponding to each representation.

2. Mark the boxes in the right tableau according to the row it is in.

3. Take one box at a time from the right tableau and attach it to the left one, making sure to respect the rules of the tableaux and not putting two boxes from the same row in the same column.

4. Discard columns of N boxes.

5. For each possible unique combination, compute the direct sum.

How many elements can there be in a column for a general SU(N) tableau? The answer is $N-1$, as there is only a single rank N antisymmetric tensor - the Levi-Civita tensor - which looks the same in all frames and thus transforms analogously to a scalar.

What is the dimensionality of the product representation corresponding to each tableau? To compute this we index each box in the tableau by a row number j and a column number k. Next, for SU(N) we compute the numbers

$$A_{jk} = N + k - j, \; B_{jk} = n_j + m_k + 1 - j - k$$

for every box, where n_j is the number of boxes in the row and m_k is the number of boxes in the column. For a tableau, B_{jk} can also be calculated by drawing an L with the corner in the box in question and the legs extending all the way down and to the right and counting the number of boxes in the L. By combinatorics it can somehow be shown that

$$d = \prod \frac{A_{jk}}{B_{jk}}.$$

Let us now do some examples. The first is a simple one, namely $3 \otimes 3$. Using tableaux we have

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{1,1} \oplus \ydiagram{2}.$$

The first tableau has dimension 3 and the second part has dimension 6. The first one corresponds to the conjugate representation, hence we find $3 \otimes 3 = 6 \oplus 3^\star$. Next we study $3^\star \otimes 3$, which in tableau form is

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} + \bullet,$$

where the bullet signifies an empty tableau - in other words, a scalar. We thus find $3^\star \otimes 3 = 8 \oplus 1$. Finally, let us do the more involved $3 \otimes 3 \otimes 3$. Using our previous work we find

$$\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{1} \otimes \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) = \ydiagram{2,1} \oplus \bullet \oplus \ydiagram{3} \oplus \ydiagram{2,1}.$$

Note that the same tableau appears twice due to the direct sum. We thus find $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$.

The Quark Model The quark model is a simplified model which only considers the up, down and strange quarks, combining them into a triplet which transforms under SU(3). Taking the generators to be $\frac{\lambda_a}{2}$, the generators of transformations on the antiparticle states is $-\frac{\lambda_a^\star}{2}$. We denote the particle representation as 3 and the antiparticle representation as $\bar{3}$.

2 Quantum Field Theory

A Comment on Normalization We will be using the convention where the integral containing the fields has a division by a factor $\sqrt{2\omega_{\mathbf{k}}}$.

Lorentz Transformations and Dirac Fields We have already established that Dirac fields transform as

$$\psi(x) \to e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x),$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. This means that the adjoint transforms according to

$$\psi^\dagger(x) \to \psi^\dagger(x) e^{\frac{i}{4}\omega^{\mu\nu}\sigma^\dagger_{\mu\nu}}.$$

As $\sigma_{\mu\nu}$ are not self-adjoint, $\psi^\dagger\psi$ is not a Lorentz invariant and cannot be included in any field theory. The hope is, however, that there exists some matrix M which we can put in the middle to solve that issue. For this to work we would need

$$e^{\frac{i}{4}\omega^{\mu\nu}\sigma^\dagger_{\mu\nu}} M = M e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}.$$

Expanding around the identity we find

$$\sigma^\dagger_{\mu\nu} M = M\sigma_{\mu\nu}.$$

To proceed we need to know a little about the gamma matrices. It can be shown in the Dirac basis that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Because different bases are related by a unitary similarity transform, this holds in all bases. This means that

$$(\sigma^{\mu\nu})^\dagger = -\frac{i}{2}\gamma^0[\gamma^\nu,\gamma^\mu]\gamma^0 = \gamma^0\sigma^{\mu\nu}\gamma^0,$$

and $\gamma^0 M$ must commute with $\sigma_{\mu\nu}$. Two choices for M are thus γ^5 and γ^0, the latter of which we will use to define the Dirac adjoint $\overline{\psi} = \psi^\dagger\gamma^0$, which transforms the right way under Lorentz transformations.

What about forms of the type $\overline{\psi}F\psi$? We will need a basis for the space of matrices, and can choose the identity, the gamma matrices, their commutators, γ^5 and $\gamma_\mu\gamma^5$. Of these, only the index-free versions will work. To see this, consider

$$\begin{aligned}
\overline{\psi}\gamma^\rho\psi &\to \overline{\psi}e^{\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi \\
&\approx \overline{\psi}\left(1+\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^\rho\left(1-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi \\
&\approx \overline{\psi}\left(\gamma^\rho + \frac{i}{4}\omega_{\mu\nu}[\sigma^{\mu\nu},\gamma^\rho]\right)\psi.
\end{aligned}$$

The commutator can be simplified as

$$\begin{aligned}
[\sigma^{\mu\nu},\gamma^\rho] &= \frac{i}{2}\left(\gamma^\mu[\gamma^\nu,\gamma^\rho] + [\gamma^\mu,\gamma^\rho]\gamma^\nu - \gamma^\nu[\gamma^\mu,\gamma^\rho] - [\gamma^\nu,\gamma^\rho]\gamma^\mu\right) \\
&= \frac{i}{2}\left(\gamma^\mu(\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu) + (\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu)\gamma^\nu - \gamma^\nu(\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu) - (\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu)\gamma^\mu\right) \\
&= \frac{i}{2}\left(2g^{\nu\rho}\gamma^\mu - 2g^{\rho\mu}\gamma^\nu + 2g^{\nu\rho}\gamma^\mu - 2g^{\rho\mu}\gamma^\nu\right) \\
&= 2i(g^{\nu\rho}\gamma^\mu - g^{\rho\mu}\gamma^\nu).
\end{aligned}$$

Thus we have

$$\omega_{\mu\nu}[\sigma^{\mu\nu},\gamma^\rho] = 2i\omega_{\mu\nu}(g^{\nu\rho}\gamma^\mu - g^{\rho\mu}\gamma^\nu) = -4i\omega^\rho{}_\mu\gamma^\mu$$

and

$$\overline{\psi}\gamma^\rho\psi \approx \overline{\psi}\left(\delta^\rho_\mu + \omega^\rho{}_\mu\right)\gamma^\mu\psi,$$

which is the same transformation rule as for a vector. Similarly objects with more indices transform like tensors, justifying the use of Lorentz indices despite the gamma matrices being the same in all frames.

Energy Change for Klein-Gordon Fields For a Klein-Gordon field the process with the non-relativistic string can be repeated to find

$$\mathcal{H} = \frac{1}{2}\int \mathrm{d}^3\mathbf{p}\,\omega_\mathbf{p}(a^\dagger_\mathbf{p}a_\mathbf{p} + a_\mathbf{p}a^\dagger_\mathbf{p}).$$

Using the commutation relations we find

$$\begin{aligned}
\left[\mathcal{H},a^\dagger_\mathbf{k}\right] =& \frac{1}{2}\int \mathrm{d}^3\mathbf{p}\,\omega_\mathbf{p}(a^\dagger_\mathbf{p}\left[a_\mathbf{p},a^\dagger_\mathbf{k}\right] + \left[a^\dagger_\mathbf{p},a^\dagger_\mathbf{k}\right]a_\mathbf{p} + a_\mathbf{p}\left[a^\dagger_\mathbf{p},a^\dagger_\mathbf{k}\right] + \left[a_\mathbf{p},a^\dagger_\mathbf{k}\right]a^\dagger_\mathbf{p}) \\
=& \int \mathrm{d}^3\mathbf{p}\,\delta^3(\mathbf{p}-\mathbf{k})\omega_\mathbf{p}a^\dagger_\mathbf{p} \\
=& \omega_\mathbf{k}a^\dagger_\mathbf{k},
\end{aligned}$$

and similarly $\left[\mathcal{H},a^\dagger_\mathbf{k}\right] = -\omega_\mathbf{k}a^\dagger_\mathbf{k}$. Thus these operators raise and lower the energies of eigenstates of the Hamiltonian, further justifying our interpretations.

The Charge of Charged Klein-Gordon Fields The Lagrangian for the charged Klein-Gordon field is

$$\mathcal{L} = \partial_\mu \phi^\star \partial^\mu \phi - m^2 \phi^\star \phi.$$

Corresponding to this Lagrangian is a symmetry $\delta\phi = i\alpha\phi$ which leaves the action unchanged. The corresponding Noether current is

$$j^\mu = i(\phi^\star \partial^\mu \phi - \phi \partial^\mu \phi^\star).$$

Using the field expansion

$$\phi = \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3\mathbf{p} \, \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(a_\mathbf{p} e^{-ipx} + b^\star_\mathbf{p} e^{ipx}\right)$$

with distinct amplitudes a and b we find

$$\begin{aligned}\rho &= -\frac{1}{2(2\pi)^3} \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}} \left(\left(a^\star_\mathbf{p} e^{ipx} + b_\mathbf{p} e^{-ipx}\right) \left(-a_\mathbf{k} e^{-ikx} + b^\star_\mathbf{k} e^{ikx}\right) - \left(a_\mathbf{p} e^{-ipx} + b^\star_\mathbf{p} e^{ipx}\right) \left(a^\star_\mathbf{k} e^{ikx} - b_\mathbf{k} e^{-ikx}\right) \right) \\ &= -\frac{1}{2(2\pi)^3} \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}} \left(-a^\star_\mathbf{p} a_\mathbf{k} e^{i(p-k)x} + b_\mathbf{p} b^\star_\mathbf{k} e^{i(k-p)x} - a_\mathbf{p} a^\star_\mathbf{k} e^{i(k-p)x} + b^\star_\mathbf{p} b_\mathbf{k} e^{i(p-k)x} \right).\end{aligned}$$

The quantized version is

$$\rho = -\frac{1}{2(2\pi)^3} \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}} \left(-a^\dagger_\mathbf{p} a_\mathbf{k} e^{i(p-k)x} + b_\mathbf{p} b^\dagger_\mathbf{k} e^{i(k-p)x} - a_\mathbf{p} a^\dagger_\mathbf{k} e^{i(k-p)x} + b^\dagger_\mathbf{p} b_\mathbf{k} e^{i(p-k)x} \right).$$

Using the commutation relations we find

$$\rho = -\frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}} \left(b^\dagger_\mathbf{p} b_\mathbf{k} e^{i(p-k)x} - a^\dagger_\mathbf{p} a_\mathbf{k} e^{i(p-k)x} \right).$$

Computing the total charge amounts to integrating over space, adding a Dirac delta in momentum space and finally yielding

$$Q = \int \mathrm{d}^3\mathbf{p} \, a^\dagger_\mathbf{p} a_\mathbf{k} - b^\dagger_\mathbf{p} b_\mathbf{k}.$$

A Trick for Computing Propagators A neat trick to find propagators, at least for the standard theories we will consider, is to write the Lagrangian as

$$\mathcal{L} = \overline{\phi} A \phi + \ldots$$

where A is some operator and the dots are total derivatives. The bar signifies some relevant conjugate field. For real fields you want a prefactor of $\frac{1}{2}$ as well. The propagator is the inverse of A.

Let us show two examples. First, for a charged scalar field we have

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi = \partial_\mu \left(\phi^\dagger \partial^\mu \phi\right) - \phi^\dagger \left(\Box + m^2\right) \phi,$$

hence $A = -\Box - m^2$. In momentum space we thus find the familiar formula

$$i\Delta_\mathrm{F}(p) = \frac{i}{p^2 - m^2}.$$

Next, for a Dirac field we have

$$\mathcal{L} = i\overline{\psi} \partial\!\!\!/ \psi - m\overline{\psi}\psi = \overline{\psi}\left(i\partial\!\!\!/ - m\right).$$

Again, in momentum space we have

$$(p\!\!\!/ - m) S_\mathrm{F},(p) = 1.$$

Using gamma gymnastics we know that $(p\!\!\!/ - m)(p\!\!\!/ + m) = p^2 - m^2$, hence we find

$$iS_\mathrm{F},(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}.$$

The Electromagnetic Field and its Complications For a general gauge the Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

for some control parameter ξ. The second term, called the gauge fixing term, turns out to be really useful for doing calculations. To show that we expand the Lagrangian as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu) \\
&= -\frac{1}{2}\left(\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu)\right) \\
&= -\frac{1}{2}\left(\partial_\mu A_\nu(g^{\nu\rho}\partial^\mu A_\rho - g^{\mu\rho}\partial^\nu A_\rho) - \frac{1}{\xi}(\partial^\mu A_\mu)(\partial^\nu A_\nu)\right) \\
&= \frac{1}{2}\left(A_\nu\partial_\mu(g^{\nu\rho}\partial^\mu A_\rho - g^{\mu\rho}\partial^\nu A_\rho) + \frac{1}{\xi}A_\mu\partial^\mu\partial^\nu A_\nu\right) - \frac{1}{2}\partial_\mu\left(A_\nu(g^{\nu\rho}\partial^\mu A_\rho - g^{\mu\rho}\partial^\nu A_\rho)\right) - \frac{1}{2\xi}\partial^\mu(A_\mu\partial^\nu A_\nu).
\end{aligned}$$

Ignoring the total derivatives we are left with

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}A_\nu\left(g^{\nu\rho}\partial_\mu\partial^\mu A_\rho - g^{\mu\rho}\partial_\mu\partial^\nu A_\rho + \frac{1}{\xi}\partial^\mu\partial^\nu A_\mu\right) \\
&= \frac{1}{2}A_\nu\left(g^{\nu\mu}\Box A_\mu - \partial^\mu\partial^\nu A_\mu + \frac{1}{\xi}\partial^\mu\partial^\nu A_\mu\right) \\
&= \frac{1}{2}A_\nu\left(g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)A_\mu,
\end{aligned}$$

hence we have found a matrix operator $A^{\mu\nu}$. Its inverse must be the corresponding inverse matrix. Calling the inverse $D_{\nu\rho}$ we require in Fourier space

$$\left(-g^{\mu\nu}p^2 + \left(1 - \frac{1}{\xi}\right)p^\mu p^\nu\right)D_{\nu\rho} = \delta^\mu_\rho.$$

We can expand D in terms of the metric and the momentum to find

$$\left(-g^{\mu\nu}p^2 + \left(1 - \frac{1}{\xi}\right)p^\mu p^\nu\right)(ag_{\nu\rho} + bp_\nu p_\rho) = -ap^2\delta^\mu_\rho - bp^2p^\mu p_\rho + \left(1 - \frac{1}{\xi}\right)ap^\mu p_\rho + b\left(1 - \frac{1}{\xi}\right)p^2p^\mu p_\rho.$$

A first choice is $a = -\frac{1}{p^2}$. Given that, we would be left with

$$-bp^2 + \left(1 - \frac{1}{\xi}\right)\left(-\frac{1}{p^2} + bp^2\right) = -\frac{1}{\xi}bp^2 - \frac{1}{p^2}\left(1 - \frac{1}{\xi}\right) = 0,$$

with solution

$$b = \frac{1}{(p^2)^2}(1 - \xi).$$

Thus we have

$$iD_{\mu\nu} = -\frac{i}{p^2}\left(g_{\mu\nu} - (1 - \xi)\frac{p_\mu p_\nu}{p^2}\right).$$

At this point we may simplify the propagator by simply choosing $\xi = 1$, netting us

$$iD_{\mu\nu} = -\frac{ig_{\mu\nu}}{p^2}.$$

Dirac Fields For Dirac fields we will use the normalization

$$\overline{u}u = 2mc,\ \overline{v}v = -2mc$$

and the completeness relation

$$\sum_s u_s\overline{u}_s = \not{p} + m,\ \sum_s v_s\overline{v}_s = \not{p} - m.$$

Quantum Electrodynamics The Dirac field is invariant under a U(1) symmetry $\psi \to e^{-ieQ\theta}\psi$. The normalization in the exponent contains a parameter e to be discussed, a parameter Q specific to each particle and a parameter θ accounting for the rest. By convention we take $Q = -1$ for the electron. The corresponding Noether current is

$$j^\mu = eQ\overline{\psi}\gamma^\mu\psi.$$

Extending θ to be a function on spacetime breaks this symmetry for the standard Dirac Lagrangian, however. By extending the derivative operator to

$$D_\mu = \partial_\mu + ieA_\mu$$

and requiring $A_\mu \to A_\mu + \partial_\mu\theta$, however, we have a Lagrangian that respects this symmetry. This is called a gauge symmetry, and corresponds to a gauge theory realized by gauge bosons. We identify this field with the 4-potential and the corresponding transformation as a gauge transformation, adding its contribution to the total Lagrangian. To include the gauge field in the Lagrangian, we will need the field strength tensor. We already know how to get it, but I note the result

$$\begin{aligned}
[D_\mu, D_\nu] &= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu) \\
&= ie(\partial_\mu A_\nu + A_\mu\partial_\nu - \partial_\nu A_\mu - A_\nu\partial_\mu) \\
&= ie(\partial_\mu(A_\nu) + A_\nu\partial_\mu + A_\mu\partial_\nu - \partial_\nu(A_\mu) - A_\mu\partial_\nu - A_\nu\partial_\mu) \\
&= ieF_{\mu\nu},
\end{aligned}$$

which will be useful later.

When comparing to the non-interacting case, we find an interaction term

$$\mathcal{L}_{\text{int}} = -eQ\overline{\psi}\gamma^\mu\psi A_\mu.$$

This is the interaction term defining quantum electrodynamics. It is a so-called gauge theory based on the U(1) gauge group.

Let us now consider the corresponding interaction vertex. Note that a single of these does not represent a physical process. To see this, consider the spontaneous emission of a photon from a particle. Denoting the momenta of the particle before and after the emission as p and k, and the photon momentum as q, we have

$$q^2 = 2p \cdot q.$$

Letting the photon have frequency ω and the particle have energy E we find

$$\omega^2 - \mathbf{q}^2 = 2(E\omega - \mathbf{p} \cdot \mathbf{q}).$$

By the properties of the scalar product we thus find

$$E\omega - |\mathbf{p}||\mathbf{q}| \leq \frac{1}{2}(\omega^2 - \mathbf{q}^2) \leq E\omega + |\mathbf{p}||\mathbf{q}|.$$

Assuming the photon to be physical, the center part must be zero. But if the particle is massive, the left-hand side is positive, creating a contradiction. The only way this can be resolved is for the created photon to not be physical, and merely mediate between two vertices.

In terms of spins, one sums over incoming spins and averages over outgoing spins.

Casimir's Trick In QED, at least, we will need to compute expressions of the form

$$\sum_s (\overline{u}_1 F u_2)^\star \overline{u}_1 G u_2$$

for some spinor u and some matrices F and G. To calculate this we use the fact that

$$(\overline{u}_1 F u_2)^\star = u_2^\dagger F^\dagger \gamma_0 u_1 = \overline{u}_2 \gamma_0 F^\dagger \gamma_0 u_1.$$

Introducing $F^\ddagger = \gamma_0 F^\dagger \gamma_0$ we have $(\overline{u}_1 F u_2)^\star = \overline{u}_2 F^\ddagger u_1$. We can now simplify the above as

$$\begin{aligned}\sum_s (\overline{u}_1 F u_2)^\star \overline{u}_1 G u_2 &= \sum_s \overline{u}_{2,a} F^\ddagger_{ab} u_{1,b} \overline{u}_{1,c} G_{cd} u_{2,d} \\ &= F^\ddagger_{ab} G_{cd} \sum_{s_1} \sum_{s_2} u_{1,b} \overline{u}_{1,c} u_{2,d} \overline{u}_{2,a} \\ &= F^\ddagger_{ab} G_{cd} \sum_{s_1} (u_1 \overline{u}_1)_{bc} \sum_{s_2} (u_2 \overline{u}_2)_{da}.\end{aligned}$$

In the two spin sums we have elements of $u\overline{u}$, which are given by the normalization, hence

$$\begin{aligned}\sum_s (\overline{u}_1 F u_2)^\star \overline{u}_1 G u_2 &= F^\ddagger_{ab} G_{cd} (\not{p}_1 + \eta_1 m_1)_{bc} (\not{p}_2 + \eta_2 m_1)_{da} \\ &= \operatorname{tr}\Big((\not{p}_1 + \eta_1 m_1) G (\not{p}_2 + \eta_2 m_1) F^\ddagger\Big),\end{aligned}$$

where the η are signs which depend on the relevant fermions being particles or antiparticles.

We will also need a mixing term of the form

$$\begin{aligned}\sum_s (\overline{u}_1 A u_2 \overline{u}_3 B u_4)^\star \overline{u}_3 C u_2 \overline{u}_1 D u_4 =& \sum_s \overline{u}_{2,a} A^\ddagger_{ab} u_{1,b} \overline{u}_{4,c} B^\ddagger_{cd} u_{3,d} \overline{u}_{3,e} C_{ef} u_{2,f} \overline{u}_{1,g} D_{gh} u_{4,h} \\ =& A^\ddagger_{ab} B^\ddagger_{cd} C_{ef} D_{gh} \sum_{s_1} \sum_{s_2} \sum_{s_3} \sum_{s_4} u_{1,b} \overline{u}_{1,g} u_{2,f} \overline{u}_{2,a} u_{3,d} \overline{u}_{3,e} u_{4,h} \overline{u}_{4,c} \\ =& A^\ddagger_{ab} B^\ddagger_{cd} C_{ef} D_{gh} (\not{p}_1 + \eta_1 m_1)_{bg} (\not{p}_2 + \eta_2 m_2)_{fa} (\not{p}_3 + \eta_3 m_3)_{de} (\not{p}_4 + \eta_4 m_4)_{hc} \\ =& \operatorname{tr}\Big(A^\ddagger (\not{p}_1 + \eta_1 m_1) D (\not{p}_4 + \eta_4 m_4) B^\ddagger (\not{p}_3 + \eta_3 m_3) C (\not{p}_2 + \eta_2 m_2)\Big).\end{aligned}$$

These are the formulae we need, and we will handle them using trace theorems.

Fundamental QED Processes The fundamental QED processes are the simplest processes we can make with QED vertices. By their nature, these are processes with two incoming and two outgoing particles. These are good for demonstrating Feynman diagrams as there are no undetermined momenta in the diagram, hence we don't have to perform any integrals. The first is electron-electron scattering, represented by

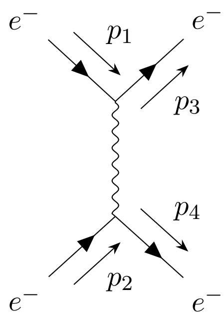


Figure 4: Electron-electron scattering Feynman diagram.

The Feynman rules dictate that

$$iM = \overline{u}(p_3) ie\gamma^\mu u(p_1) \frac{-ig_{\mu\nu}}{(p_1 - p_3)^2} \overline{u}(p_4) ie\gamma^\nu u(p_2) = ie^2 \frac{\overline{u}(p_3)\gamma^\mu u(p_1) \overline{u}(p_4)\gamma_\mu u(p_2)}{(p_1 - p_3)^2}.$$

This isn't quite correct, however - as

$$iM = \langle p_3 p_4 | S | p_1 p_2 \rangle$$

and $|p_3 p_4\rangle = -|p_4 p_3\rangle$, M must be antisymmetric under exchange of the outgoing momenta. This amounts to an equivalent diagram adding to M, which means that we must subtract the two. We then find

$$iM = ie^2 \left(\frac{\overline{u}(p_3)\gamma^\mu u(p_1) \overline{u}(p_4)\gamma_\mu u(p_2)}{(p_1 - p_3)^2} - \frac{\overline{u}(p_4)\gamma^\mu u(p_1) \overline{u}(p_3)\gamma_\mu u(p_2)}{(p_1 - p_4)^2} \right).$$

Using the Mandelstam variables we write this as

$$iM = ie^2 \left(\frac{T_1}{t} - \frac{T_2}{u} \right).$$

Seeking the spin average of $|M|^2$ we need to calculate spin sums of the products of the Ts, which is handled using Casimir's trick. In the high energy regime, we can neglect the electron mass to find

$$\begin{aligned}
\sum_s T_1^\star T_1 &= \mathrm{tr}\Big(\not{p}_1 \gamma^\mu \not{p}_3 \gamma^\nu\Big)\, \mathrm{tr}\Big(\not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu\Big) \\
&= p_{1,\rho} p_{3,\sigma}\, \mathrm{tr}(\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu) p_2^\alpha p_4^\beta\, \mathrm{tr}(\gamma_\alpha \gamma_\mu \gamma_\beta \gamma_\nu) \\
&= 16 p_{1,\rho} p_{3,\sigma} p_2^\alpha p_4^\beta (g^{\rho\mu} g^{\sigma\nu} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\nu} g^{\sigma\mu})(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} + g_{\alpha\nu} g_{\beta\mu}) \\
&= 16 (p_1^\mu p_3^\nu - p_1^\sigma p_{3,\sigma} g^{\mu\nu} + p_1^\nu p_3^\mu)(p_{2,\mu} p_{4,\nu} - p_{2,\beta} p_4^\beta g_{\mu\nu} + p_{2,\nu} p_{4,\mu}).
\end{aligned}$$

The collision is satisfies the laws of relativistic dynamics, hence $p_1 \cdot p_2 = p_3 \cdot p_4$ as well as $p_1 \cdot p_4 = p_3 \cdot p_2$, and

$$\begin{aligned}
\sum_s T_1^\star T_1 =& 16((p_1 \cdot p_2)^2 - (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)^2 - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4) \\
&+ (p_1 \cdot p_4)^2 - (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_2)^2) \\
=& 32((p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2).
\end{aligned}$$

Similarly, by exchanging p_3 and p_4 we find

$$\sum_s T_2^\star T_2 = 32((p_1 \cdot p_2)^2 + (p_1 \cdot p_3)^2).$$

Finally we have the mixed term, given by

$$\begin{aligned}
\sum_s T_1^\star T_2 &= \mathrm{tr}\Big(\gamma^\mu \not{p}_3 \gamma_\nu \not{p}_2 \gamma_\mu \not{p}_4 \gamma^\nu \not{p}_1\Big) \\
&= \mathrm{tr}\Big(\not{p}_4 \gamma^\nu \not{p}_1 \gamma^\mu \not{p}_3 \gamma_\nu \not{p}_2 \gamma_\mu\Big) \\
&= -2\, \mathrm{tr}\Big(\not{p}_4 \not{p}_3 \gamma^\mu \not{p}_1 \not{p}_2 \gamma_\mu\Big) \\
&= -8\, \mathrm{tr}\Big(\not{p}_4 \not{p}_3 p_1 \cdot p_2\Big) \\
&= -32 (p_1 \cdot p_2)^2.
\end{aligned}$$

This is real, and therefore equal to its complex conjugate. Finally, neglecting masses we have

$$\overline{M}^2 = 32 e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + \frac{2s^2}{tu} \right).$$

Non-Abelian Gauge Theory The gauge theory of QED is an example of an Abelian gauge theory. For the non-Abelian case, consider a representation R of $\mathrm{SU}(N)$ with dimension d_R acting on some multiplet Ψ. To add gauge theory, we suppose that the components of Ψ have some internal structure, either that of bosonic or fermionic fields, all with the same mass. As we saw for the QED case, any constant unitary operator U leaves the laws of physics invariant. Allowing U to have spacetime dependence, however, messes with this, in particular the invariance of derivatives. To remedy this, we will introduce a covariant derivative

$$D_\mu = \partial_\mu + igT_a A_\mu^a = \partial_\mu + ig\mathbf{A}_\mu,$$

which contains some coupling constant g, the generators T_a of $\mathrm{SU}(N)$ and the gauge fields A_μ^a. This derivative has a matrix structure, and in the final expression, this structure has been baked into the field. The laws of physics will be invariant if

$$D_\mu \Psi \to U D_\mu' \Psi$$

under a transformation. We have

$$\begin{aligned}
D'_\mu(U\Psi) &= (\partial_\mu + igT_a(A')^a_\mu)\left(e^{-igT_a\theta^a}\Psi\right) \\
&= -igT_a\partial_\mu\theta^a e^{-igT_a\theta^a}\Psi + e^{-igT_a\theta^a}\partial_\mu\Psi + igT_a(A')^a_\mu e^{-igT_a\theta^a}\Psi \\
&= U\partial_\mu\Psi + \left(igT_a(A')^a_\mu - igT_a\partial_\mu\theta^a\right)U\Psi.
\end{aligned}$$

For the transformation rule to be fulfilled, we therefore require

$$\left(igT_a(A')^a_\mu - igT_a\partial_\mu\theta^a\right)U = igUT_aA^a_\mu.$$

There are two ways to write the condition. The first is representation dependent, and found by combining the generators and the gauge fields to obtain matrix structure, yielding

$$igA'_\mu U + \partial_\mu U = igUA_\mu,$$

with solution

$$\mathbf{A}'_\mu = U\mathbf{A}_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger.$$

The other is to write the above without the matrix structure in the fields as

$$T_a(A')^a_\mu = UT_aU^\dagger A^a_\mu - \frac{i}{g}(igT_a\partial_\mu\theta^a)UU^\dagger = UT_aU^\dagger A^a_\mu + T_a\partial_\mu\theta^a.$$

We can expand the first term according to

$$\begin{aligned}
UT_aU^\dagger &\approx \left(1 - igT_b\theta^b\right)T_a\left(1 + igT_c\theta^c\right) \\
&\approx T_a + ig\left(T_aT_c\theta^c - T_bT_a\theta^b\right) \\
&= T_a + ig[T_a, T_b]\theta^b \\
&= T_a - gf_{abc}T_c\theta^b,
\end{aligned}$$

hence

$$\begin{aligned}
T_a(A')^a_\mu &= T_aA^a_\mu - gf_{abc}T_c\theta^bA^a_\mu + T_a\partial_\mu\theta^a \\
&= T_a\left(A^a_\mu + gf_{bca}\theta^bA^c_\mu + \partial_\mu\theta^a\right)
\end{aligned}$$

and

$$(A')^a_\mu = A^a_\mu + gf_{bca}\theta^bA^c_\mu + \partial_\mu\theta^a.$$

We can invoke the adjoint representation, here denoted by elements with small t, to find

$$(A')^a_\mu = A^a_\mu - ig(t_b)_{ac}\theta^bA^c_\mu + \partial_\mu\theta^a.$$

This is interesting, as it means that under global transformations, gauge fields always transform according to the adjoint representation.

The gauge fields should be included in the Lagrangian as well. We choose the same form for the Lagrangian as we did for U(1), but will need an expression for the field strength. We showed a commutation relation previously, and we take that to be defining for the field strength. That is,

$$igT_aF^a_{\mu\nu} = [D_\mu, D_\nu].$$

One of the derivative products is given by

$$(\partial_\mu + igT_aA^a_\mu)(\partial_\nu + igT_bA^b_\nu) = \partial_\mu\partial_\nu + ig(\partial_\mu T_aA^a_\nu + T_bA^b_\mu\partial_\nu) - g^2T_aT_bA^a_\mu A^b_\nu,$$

hence

$$\begin{aligned}
[D_\mu, D_\nu] &= igT_a(\partial_\mu A^a_\nu + A^a_\mu\partial_\nu - \partial_\nu A^a_\mu - A^a_\nu\partial_\mu) - g^2A^a_\mu A^b_\nu[T_a, T_b] \\
&= igT_a(\partial_\mu(A^a_\nu) - \partial_\nu(A^a_\mu)) - ig^2A^a_\mu A^b_\nu f_{abc}T_c \\
&= igT_a\left(\partial_\mu(A^a_\nu) - \partial_\nu(A^a_\mu) - gf_{bca}A^b_\mu A^c_\nu\right),
\end{aligned}$$

hence

$$F^a_{\mu\nu} = \partial_\mu(A^a_\nu) - \partial_\nu(A^a_\mu) - g f_{bca} A^b_\mu A^c_\nu.$$

Alternatively we may write this as

$$\mathbf{F}_{\mu\nu} = T_a F^a_{\mu\nu} = \partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu) + ig[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

Let us now consider its gauge transformation properties, best demonstrated with the representation-dependent formalism. We have

$$\mathbf{F}'_{\mu\nu} = \partial_\mu\left(U\mathbf{A}_\nu U^\dagger + \frac{i}{g}(\partial_\nu U)U^\dagger\right) - \partial_\nu\left(U\mathbf{A}_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger\right) + ig\Big[U\mathbf{A}_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger, U\mathbf{A}_\nu U^\dagger + \frac{i}{g}(\partial_\nu U)U^\dagger\Big].$$

Woah, that is a lot of terms. We can eliminate a few of them, however, using the following results:

$$\begin{aligned}\Big[U\mathbf{A}_\mu U^\dagger, U\mathbf{A}_\nu U^\dagger\Big] &= U[\mathbf{A}_\mu, \mathbf{A}_\nu]U^\dagger,\\ \partial_\mu(UU^\dagger) = 0 \implies \partial_\mu(U^\dagger) &= -U^\dagger\partial_\mu(U)U^\dagger.\end{aligned}$$

Next we have

$$\begin{aligned}\partial_\mu\left(U\mathbf{A}_\nu U^\dagger + \frac{i}{g}(\partial_\nu U)U^\dagger\right) &= \partial_\mu(U)\mathbf{A}_\nu U^\dagger + U\partial_\mu(\mathbf{A}_\nu)U^\dagger - U\mathbf{A}_\nu U^\dagger\partial_\mu(U)U^\dagger + \frac{i}{g}\Big((\partial_\mu\partial_\nu U)U^\dagger + (\partial_\nu U)(\partial_\mu U^\dagger)\Big)\\ &= \partial_\mu(U)\mathbf{A}_\nu U^\dagger + U\partial_\mu(\mathbf{A}_\nu)U^\dagger - U\mathbf{A}_\nu U^\dagger\partial_\mu(U)U^\dagger + \frac{i}{g}\Big((\partial_\mu\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger\Big),\\ \Big[U\mathbf{A}_\mu U^\dagger, (\partial_\nu U)U^\dagger\Big] &= U\mathbf{A}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger U\mathbf{A}_\mu U^\dagger\\ &= U\mathbf{A}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)\mathbf{A}_\mu U^\dagger,\\ \Big[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger\Big] &= (\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger.\end{aligned}$$

We can now look at the terms with and without gauge fields separately. The first add to

$$\begin{aligned}&\partial_\mu(U)\mathbf{A}_\nu U^\dagger + U\partial_\mu(\mathbf{A}_\nu)U^\dagger - U\mathbf{A}_\nu U^\dagger\partial_\mu(U)U^\dagger - \Big(\partial_\nu(U)\mathbf{A}_\mu U^\dagger + U\partial_\nu(\mathbf{A}_\mu)U^\dagger - U\mathbf{A}_\mu U^\dagger\partial_\nu(U)U^\dagger\Big)\\ &-\Big(U\mathbf{A}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)\mathbf{A}_\mu U^\dagger - \Big(U\mathbf{A}_\nu U^\dagger(\partial_\mu U)U^\dagger - (\partial_\mu U)\mathbf{A}_\nu U^\dagger\Big)\Big)\\ &= U\left(\partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu)\right)U^\dagger.\end{aligned}$$

Naturally the commutator also produces a term $U[\mathbf{A}_\mu, \mathbf{A}_\nu]U^\dagger$. The other terms add up to

$$\frac{i}{g}\Big((\partial_\mu\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger - (\partial_\nu\partial_\mu U)U^\dagger + (\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger\Big) - \frac{i}{g}\Big((\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger\Big),$$

which add up to zero. Thus we have

$$\mathbf{F}'_{\mu\nu} = U\mathbf{F}_{\mu\nu}U^\dagger.$$

Constructing $\mathbf{F}^\dagger_{\mu\nu}\mathbf{F}^{\mu\nu}$ guarantees Lorentz invariance, but this object still has matrix structure, and it is not gauge invariant. We can, however, construct $\mathrm{tr}\big(\mathbf{F}^\dagger_{\mu\nu}\mathbf{F}^{\mu\nu}\big)$, with tracing over the multiplet space, which is gauge invariant. We can write this as

$$\begin{aligned}\mathrm{tr}\Big(\mathbf{F}^\dagger_{\mu\nu}\mathbf{F}^{\mu\nu}\Big) &= (F^a_{\mu\nu})^\star F^{b,\mu\nu}\,\mathrm{tr}(T_aT_b)\\ &= C(F^a_{\mu\nu})^\star F^{b,\mu\nu}\delta_{ab}\\ &= C(F^a_{\mu\nu})^\star F^{a,\mu\nu},\end{aligned}$$

hence we choose the free-field Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^\star F^{a,\mu\nu}.$$

In general there could be mass terms of the form $A^a_\mu A^{\mu,a}$ in the Lagrangian, but it turns out that these are not gauge invariant. Thus, as long as the gauge symmetry is exact, the gauge bosons must be massless.

The propagator for this kind of gauge field is generally not defined unless you specify a gauge. The Lagrangian has terms of the same form as in QED, but with some self-interaction terms as well. The free part is the same, however, meaning we can use a gauge fixing term

$$\mathcal{L} = -\frac{1}{2\xi}(\partial_\mu A^\mu_a)^\dagger(\partial_\nu A^\nu_a).$$

Now the propagator will be the same as in QED, as will other Feynman rules.

Intrinsic to this kind of field theory is interactions of the gauge field with itself. These may be both cubic and quartic. Introducing $E_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}$ the Feynman rules are

$$gf_{ab\bar{c}}\left(p^\alpha E_{\alpha\mu\nu\lambda} + k^\alpha E_{\alpha\nu\lambda\mu} + q^\alpha E_{\alpha\lambda\mu\nu}\right)$$

for the cubic vertex and

$$-ig^2\left(f_{ab\bar{e}}f_{cde}E_{\mu\nu\lambda\rho} + f_{ac\bar{e}}f_{dbe}E_{\mu\lambda\rho\nu} + f_{ad\bar{e}}f_{bce}E_{\mu\rho\nu\lambda}\right)$$

for the quartic vertex.

Let us now study the interactions of the gauge fields with matter. Taking the fermionic case first, the field multiplet Ψ is described by

$$\mathcal{L} = \sum_{i=1}^{d_R} i\overline{\Psi}^i \partial\!\!\!/ \Psi^i - m\overline{\Psi}^i\Psi^i.$$

In terms of matrix notation we can write this as

$$\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi.$$

The Dirac matrices work on the individual fields in the multiplet, hence in this notation they are diagonal matrices. Replacing the partial derivative with the covariant derivative ensures gauge symmetry. Emerging from this Lagrangian is now an interaction term

$$\mathcal{L}_{\text{int}} = -g\overline{\Psi}A\!\!\!/\Psi = -g\overline{\Psi}\gamma^\mu T_a\Psi A^a_\mu.$$

This adds a vertex rule $-ig\gamma^\mu(T_a)_{ij}$, where ij are indices specifying the interacting fields.

Next we consider interactions with scalars. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= (D_\mu\Phi)^\dagger D^\mu\Phi - M^2\Phi^\dagger\Phi \\ &= (\partial_\mu\Phi)^\dagger\partial^\mu\Phi + ig\left((\partial_\mu\Phi)^\dagger T_a A^{a,\mu}\Phi - \Phi^\dagger T_a^\dagger (A^a_\mu)^\dagger\partial^\mu\Phi\right) + g^2\Phi^\dagger T_a^\dagger(A^a_\mu)^\dagger T_b A^{b,\mu}\Phi - M^2\Phi^\dagger\Phi \\ &= (\partial_\mu\Phi)^\dagger\partial^\mu\Phi + ig\left((\partial_\mu\Phi)^\dagger T_a A^{a,\mu}\Phi - \Phi^\dagger T_a^\dagger (A^a_\mu)^\dagger\partial^\mu\Phi\right) + g^2 g_{\mu\nu}\Phi^\dagger T_a^\dagger(A^a_\mu)^\dagger T_b A^b_\nu\Phi - M^2\Phi^\dagger\Phi.\end{aligned}$$

We thus see that there are two kinds of vertices, one with two particles and one gauge bosons, and one with two particles and two gauge bosons. The corresponding Feynman rules are $-ig(p+p')_\mu(T_a)_{ij}$ and $ig^2 g_{\mu\nu}(T_aT_b + T_bT_a)_{ij}$. Note that the structure of the generators dictates which particles can interact in both cases.

Quantum Chromodynamics Quantum chromodynamics is the theory of the strong nuclear force. It is a quantized SU(3) gauge theory. The number 3 is due to the presence of 3 color charges. We choose SU(3) rather than SU(2), despite the latter having a triplet representation because we need a triplet representation for both the quarks and antiquarks, but SU(2) has only real representations.

The gauge bosons come in eight forms. They are called gluons.

The Weak Force The weak force started as an attempt to describe beta decay. The first attempt was to formulate a theory involving an interaction vertex with four fermions, where currents interacted. This turned out not to work, however, as it did not contain parity violation. The fact that neutrinos are only found with certain chiralities made clear the need to introduce this concept as well. Both of these could be remedied by adding $\gamma^\mu(1-\gamma^5)$ to the Dirac field bilinears. This will bring us closer, but still not leave us with a renormalizable theory, as the coupling constant would have mass dimension -2.

To remedy the above, one could strengthen the analogy to QED by adding a propagator to the interaction. This would remove the four-fermion vertex, replacing it with vertices involving a vector boson. This boson must be heavy, as it has never been observed. The propagator generally takes the form

$$\frac{-ig_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}}{q^2 - m^2}.$$

One important question is why the weak force is so weak. One way to understand this is to compare matrix elements with and without the mediating boson. The latter is of the form

$$iM = i\frac{G}{\sqrt{2}} \left(\overline{u}_3 \gamma^\mu (1-\gamma^5) u_1\right) \left(\overline{u}_4 \gamma^\mu (1-\gamma^5) u_2\right),$$

while the former is of the form

$$iM = i \left(\frac{g}{\sqrt{2}} \overline{u}_3 \gamma^\mu \frac{1}{2}(1-\gamma^5) u_1\right) \frac{1}{M^2 - q^2} \left(\frac{g}{\sqrt{2}} \overline{u}_4 \gamma^\mu \frac{1}{2}(1-\gamma^5) u_2\right),$$

and in the low-energy regime

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M^2},$$

hence the heaviness of the gauge boson makes the effective interaction constant so small.