Summary of SH2372 General Relativity

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Abstract

This is a summary of SH2372 General Relativity.

The course opens with a discussion of differential geometry. As I have extensive notes on the subject in my summary of SI2360, I only keep the bare minimum in this summary and refer to those notes for details.

Contents

1 Differential Geometry

For details on much of this, notably the early parts on Euclidean space, please consult my summary of SI2360 Analytical Mechanics and Classical Field Theory.

Euclidean and Affine Spaces A Euclidean space is a set of points such that there to each point can be assigned a position vector. To such spaces we may assign a set of n coordinates χ^a which together uniquely describe each point in the space locally.

Tangent and Dual Bases The tangent and dual bases are defined by

$$\mathbf{E}_a = \partial_{\chi^a}\mathbf{r} = \partial_a\mathbf{r},\ \mathbf{E}^a = \vec{\boldsymbol{\nabla}}\chi^a.$$

Using such bases, we may write

$$\mathbf{v} = v^a\mathbf{E}_a = v_a\mathbf{E}^a.$$

The components of these vectors are called contravariant and covariant components respectively.

Christoffel Symbols When computing the derivative of a vector quantity, one must account both for the change in the quantity itself and the change in the basis vectors. We define the Christoffel symbols according to

$$\partial_b\mathbf{E}_a = \Gamma^c_{ba}\mathbf{E}_c.$$

These can be computed according to

$$\mathbf{E}^c \cdot \partial_b\mathbf{E}_a = \mathbf{E}^c \cdot \Gamma^d_{ba}\mathbf{E}_d = \delta^c_d\Gamma^d_{ba} = \Gamma^c_{ba}.$$

Note that

$$\partial_a\mathbf{E}_b = \partial_a\partial_b\mathbf{r} = \partial_b\partial_a\mathbf{r} = \partial_b\mathbf{E}_a,$$

which implies

$$\Gamma^c_{ba} = \Gamma^c_{ab}.$$

Similarly, we might want to consider $\partial_b\mathbf{E}^a$, which might introduce new symbols. We find, however, that

$$\partial_a\mathbf{E}^b \cdot \mathbf{E}_c = \mathbf{E}^b \cdot \partial_a\mathbf{E}_c + \mathbf{E}_c \cdot \partial_a\mathbf{E}^b = 0,$$

which implies

$$\partial_a\mathbf{E}^b = -\Gamma^b_{ac}\mathbf{E}^c.$$

Covariant Derivatives Covariant derivatives are defined by

$$\vec{\boldsymbol{\nabla}}_a v^b = \partial_a v^b + \Gamma^b_{ac}v^c,$$

and thus satisfy

$$\partial_a\mathbf{v} = \mathbf{E}_b\vec{\boldsymbol{\nabla}}_a v^b.$$

Tensors To define tensors, we first define tensors of the kind $(0, n)$ as maps from n vectors to scalars. Using this, we define tensors of the kind (n, m) as linear maps from $(0, n)$ tensors to $(0, m)$ tensors.

Manifolds Manifolds are sets which are locally isomorphic to an open subset of $\mathcal{R}^n$.

Tangent and Dual Bases The tangent basis for a manifold is $\mathbf{E}_a = \partial_a$. The corresponding dual basis, denoted $d\chi^a$, is defined such that $df(X) = X^a\partial_a f$.

Tensors A general (n, m) tensor is constructed by taking the tensor product of tangent and dual basis vectors.

Pushforwards and Pullbacks

2 Basic Concepts

A Note on Minkowski Space In special relativity we work with Minkowski space, which is an affine space with a so-called pseudo-metric. This is a metric which is not positive definite, but instead a metric which has only non-zero eigenvalues (and is thus termed non-degenerate). We will work with the signature $(1, 3)$, meaning that there are three eigenvalues of -1 and one eigenvalue 1.

The Description of Spacetime In general relativity we will describe spacetime as a 4-dimensional manifold with a pseudometric of signature $(1, 3)$ with a Levi-Civita connection imposed on it.

Comiving Observers A comoving observer is one that has fixed spatial coordinates.

Kinematics of Test Particles A test particle is a particle that itself does not affect the spacetime. Such particles can generally move through spacetime, along curves called world lines. With this motion comes the 4-velocity V, defined as the normalized tangent to a world line. In special relativity we could also define a proper acceleration by differentiating with respect to proper time. In general relativity we replace this with the 4-acceleration $A = \vec{\boldsymbol{\nabla}}_V V = \vec{\boldsymbol{\nabla}}_{\dot{\gamma}}\dot{\gamma}$. We may also define the proper acceleration α, which satisfies $\alpha^2 = -A^2 = -g(A, A)$, and it can be shown that if V is time-like, then A is space-like. Note that the curve parameter we use is τ, which is the proper time and a measure of length in spacetime.

Free Particles A free particle in special relativity is a particle for which $A = 0$. We take this definition to apply in general relativity as well. This implies that free test particles travel along spacetime geodesics.

4-Momentum and 4-Force We also define the 4-momentum $P = mV$ and the 4-force $F = \vec{\boldsymbol{\nabla}}_V P$.

Simultaneity Two events are simultaneous if they are on the same hypersurface of constant t. As this depends very much on the choice of coordinates on spacetime, this notion is not at all well-defined.

Static and Stationary Spacetime If there exists a stime-like Killing field of a spacetime, it is stationary. If the spacetime is also orthogonal to a family of 3-surfaces, the spacetime is static. The consequences of the latter is that the metric has no off-diagonal components.

The Einstein Field Equations The equations describing the metric are the Einstein field equations. We will derive them from a variational principle, starting with the field equations for vacuum. The action from which we will obtain the field equations is the Einstein-Hilbert action

$$S_{\text{EH}} = -\frac{M_\alpha^2}{2}\int \mathrm{d}^4\chi\, \mathcal{R}\sqrt{-\det(g)}.$$

The final equations are

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R} = 0.$$

To obtain the field equations for other cases we will add terms to the action. In general, by adding a term S_{matter} to the action, we obtain

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R} = 8\pi G T_{\mu\nu},$$

where

$$T_{\mu\nu} = \frac{2}{\sqrt{-\det(g)}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

is called the energy-momentum tensor.

Frequency Shift A wave generally has a phase which depends on both position and time. We define the frequency of a wave as $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$. For a general world line we define

$$\omega = \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \dot{\chi}^a \partial_a \phi = V\phi = d\phi(V).$$

$d\phi$ is the dual of the 4-frequency N^μ. It can be shown that $\vec{\nabla}_N d\pi = 0$, and thus $\vec{\nabla}_N N = 0$. Rays with tangent N are thus light-like geodesics.

In general an observer will measure a frequency $d\phi(V)$.

The Schwarzschild Solution The Schwarzschild solution is the simplest solution for a spherically symmetric metric. It is of the form

$$\mathrm{d}s^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\,,$$

where R_S is the Schwarzschild radius and we work in units where $c = 1$. To reobtain Newtonian gravity at large distances we would need $R_\mathrm{S} = 2MG$.

This metric has singularities at $r = R_\mathrm{S}$ and $r = 0$. If you study the curvature invariant $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, however, you find that it is finite at $r = R_\mathrm{S}$ and diverges at $r = 0$. This would imply that there exists a smart choice of coordinates in which the singularity at R_S would be eliminated.

This set of coordinates, called Eddington-Finkelstein coordinates, replaces t with a coordinate that has the light-like geodesics as coordinate lines. For a purely radial path the requirement $\mathrm{d}s^2 = 0$ for such a geodesic yields

$$\mathrm{d}t^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-2}\mathrm{d}r^2\,,$$

with solution

$$t = u - r - R_\mathrm{S}\ln\left(\frac{r}{R_\mathrm{S}} - 1\right),$$

where u is an integration constant labelling the geodesics. This will be the new coordinate. In these coordinates we obtain

$$\mathrm{d}s^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right)\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r - r^2\,\mathrm{d}\Omega^2\,.$$

Notably, there is now only a singularity at $r = 0$.

For radial light cones in these coordinates, one obtains

$$\left(\left(1 - \frac{R_\mathrm{S}}{r}\right)\mathrm{d}u^2 - 2\,\mathrm{d}r\right)\mathrm{d}u = 0,$$

with solutions $\mathrm{d}u = 0$ and $\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{2}{1-\frac{R_\mathrm{S}}{r}}$. The first case is as discussed above. The second has $\frac{\mathrm{d}u}{\mathrm{d}r} > 0$ for $r < R_\mathrm{S}$, meaning that world lines moving towards the future are drawn to the singularity at the origin when within the Schwarzschild radius.

To describe space-like world lines, we can use Kruskal-Szekeres coordinates

$$U = \left|\frac{r}{R_\mathrm{S}} - 1\right|^{\frac{1}{2}} e^{\frac{r}{2R_\mathrm{S}}}\sinh\left(\frac{t}{2R_\mathrm{S}}\right),\quad V = \left|\frac{r}{R_\mathrm{S}} - 1\right|^{\frac{1}{2}} e^{\frac{r}{2R_\mathrm{S}}}\cosh\left(\frac{t}{2R_\mathrm{S}}\right).$$

One finds that the metric is

$$\mathrm{d}s^2 = \frac{4R_\mathrm{S}^3}{r}e^{-\frac{r}{R_\mathrm{S}}}(\mathrm{d}U^2 - \mathrm{d}V^2) - r^2\,\mathrm{d}\Omega^2\,.$$

In these coordinates geodesics are hyperbolae.

Symmetries and Conserved Quantities Assume that a spacetime has a Killing field K, and consider a geodesic of the spacetime with tangent U. Along the geodesic we then have

$$\mathrm{d}\tau\, g_{\mu\nu}K^\mu U^\nu = \vec{\nabla}_U g_{\mu\nu}K^\mu U^\nu = (\vec{\nabla}_U g_{\mu\nu})K^\mu U^\nu + g_{\mu\nu}\vec{\nabla}_U K^\mu U^\nu.$$

As the Levi-Civita connection is metric-compatible, the former term vanishes, and we are left with

$$\mathrm{d}\tau\, g_{\mu\nu}K^\mu U^\nu = g_{\mu\nu}U^\nu\vec{\nabla}_U K^\mu + g_{\mu\nu}K^\mu\vec{\nabla}_U U^\nu.$$

As the path is a geodesic, the latter term vanishes and all that is left is

$$\mathrm{d}\tau\, g_{\mu\nu}K^\mu U^\nu = g_{\mu\nu}U^\nu\vec{\nabla}_U K^\mu = g_{\mu\nu}U^\nu U^\sigma\vec{\nabla}_\sigma K^\mu = U^\nu U^\sigma\vec{\nabla}_\sigma K_\nu.$$

The two first factors are symmetric under permutation of indices, whereas the latter is not, hence this is just 0. Thus the quantity $g_{\mu\nu}K^\mu U^\nu$ is a constant of motion along the path.

Symmetries of the Schwarzschild Solution We note that ∂_t and ∂_ϕ are both Killing fields of the Schwarzschild solution. For $r > R_\mathrm{S}$ one find that ∂_t is time-like.

For a general path we define

$$\begin{aligned}\sqrt{2E} &= g(\partial_t, \dot{\gamma}) = \left(1 - \frac{R_\mathrm{S}}{r}\right)\dot{t},\\ L &= g(\partial_\phi, \dot{\gamma}) = r^s\sin^2(\theta)\dot{\phi},\\ \alpha &= g(\dot{\gamma}, \dot{\gamma}).\end{aligned}$$

The latter is 1 for a time-like path and 0 for a light-like path. By definition we have

$$\alpha = \left(1 - \frac{R_\mathrm{S}}{r}\right)\dot{t}^2 - \frac{1}{1 - \frac{R_\mathrm{S}}{r}}\dot{r}^2 - r^2\sin^2(\theta)\dot{\phi}^2 = \frac{2E}{1 - \frac{R_\mathrm{S}}{r}} - \frac{1}{1 - \frac{R_\mathrm{S}}{r}}\dot{r}^2 - \frac{L^2}{r^2},$$

which may be written as

$$E - \frac{1}{2}\dot{r}^2 = \frac{1}{2}\left(\alpha + \frac{L^2}{r^2}\right)\left(1 - \frac{R_\mathrm{S}}{r}\right).$$

This looks like the relation describing the potential energy of a particle in classical mechanics.

Gravitational Time Dilation and Redshift Consider a static space time. This means that there exists coordinates χ^a such that $\mathrm{d}s^2 = \alpha^2\,\mathrm{d}t^2 - h_{ij}\,\mathrm{d}\chi^i\,\mathrm{d}\chi^j$, where the components of the metric are functions of the spatial coordinates only. Now consider two static observers in this spacetime observing two events. As the geodesics have time translation symmetry due to the metric being static, the two observers must observe the same time difference t_0. Each observer observes an elapsed proper time

$$\tau = \int_t^{t+t_0}\mathrm{d}t\,\sqrt{g_{tt}} = \alpha t_0.$$

This means that

$$\frac{\tau_A}{\tau_B} = \frac{\alpha_A}{\alpha_B}.$$

From this the gravitational redshift follows. If the two events are successive crests of a light pulse, we find

$$\frac{f_B}{f_A} = \frac{\alpha_A}{\alpha_B}.$$

Ideal Fluids An ideal fluid is a substance such that its energy-momentum tensor is of the form

$$T^{\mu\nu} = (\rho_0 + p)u^\mu u^\nu - pg^{\mu\nu},$$

where u^μ is the 4-velocity of the rest frame of the fluid and ρ_0 and p are the energy density and pressure of the fluid as measured in the rest frame.

The Weak Field Limit We will now study the Einstein field equations in a limit where the effects of general relativity are weak, in the hopes of finding some limit that reproduces Newtonian gravity. We will do this by linearizing the metric according to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where η is the Minkowski metric and all components of h are small. To raise its components, we use the fact that the metric should produce the Kronecker delta to obtain

$$\delta^\sigma_\mu = g^{\mu\nu} g_{\nu\sigma} = (\eta^{\mu\nu} + f^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) = \eta^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu} h_{\nu\sigma} + f^{\mu\nu}\eta_{\nu\sigma},$$

where the perturbations of the inverse metric must also necessarily be small. This yields

$$f^{\mu\nu} = -\eta^{\sigma\nu}\eta^{\rho\mu} h_{\rho\sigma}.$$

When computing the curvature tensor, we will need squares of the Christoffel symbols. These are linear in the perturbations and may therefore be neglected, yielding

$$R^\mu_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\lambda\nu}$$

and

$$R_{\nu\sigma} = \frac{1}{2}(\partial_\nu \partial_\mu h^\mu{}_\sigma - \Box h_{\nu\sigma} - \partial_\nu \partial_\sigma h + \partial_\sigma \partial_\mu h^\mu{}_\nu),$$

where we have introduced the d'Alembertian $\eta^{\mu\nu}\partial_\mu\partial_\nu$ and $h = h^\mu{}_\mu$. The Einstein tensor is

$$G_{\nu\sigma} = \frac{1}{2}(\partial_\nu \partial_\mu h^\mu{}_\sigma - \Box h_{\nu\sigma} - \partial_\nu \partial_\sigma h + \partial_\sigma \partial_\mu h^\mu{}_\nu + \eta_{\nu\sigma}\Box h).$$

To proceed, we choose coordinates such that $\Box \chi^\mu = 0$. It may be shown that this leads to

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0 \implies \partial_\mu h^\mu{}_\lambda = \frac{1}{2}\partial_\lambda h.$$

This will yield the Einstein field equations

$$G_{\nu\sigma} = -\frac{1}{2}(\Box h_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\Box h) = 8\pi G T_{\nu\sigma},$$

or alternatively, by defining $\bar{h}_{\nu\sigma} = h_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma} h$:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

To proceed we say a little more about the Newtonian limit. It should be found at low speeds, meaning that the energy-momentum tensor will be dominated by T_{00}, which is the energy density, or equivalently the mass density. The corresponding component of $\bar{h}$ will thus also dominate the other side. It should also be obtained when the involved masses are small. If the situation also evolves slowly, we may neglect time derivatives to obtain

$$\nabla^2 \bar{h}_{00} = 16\pi G\rho.$$

We may thus identify $\bar{h}_{00} = 4\Phi$, where Φ is the Newtonian gravitational potential. The corresponding spacetime length is

$$\mathrm{d}s^2 = (1 + 2\Phi)\,\mathrm{d}t^2 - (1 - 2\Phi)\,\mathrm{d}x^i\,\mathrm{d}x^i\,.$$

It can also be shown that the geodesics of this metric corresponds to Newton's laws for such a case.

Gravitational Waves We have seen that in the weak field limit the perturbations to the metric in vacuum satisfy the wave equation. The solution is of the form

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{-ik^\sigma x_\sigma}.$$

This is an eigenfunction of the d'Alembertian with eigenvalue $k^\mu k_\mu$, hence the wavevector must be light-like and waves in the metric travel at the speed of light.

With the chosen gauge for the coordinates we still have some freedom - namely, the coordinate transform $x^\mu \to x^\mu + B^\mu e^{-k^\sigma x_\sigma}$ gives the same gauge. We may also choose B^μ such that $A^\mu{}_\mu = 0$ and $A_{0\mu} = 0$ in some frame. Furthermore, in this gauge we have $\partial_\mu \bar{h}^{\mu\nu} = k_\mu A^{\mu\nu} = 0$. For a wave propagating in the third Cartesian direction the only non-zero amplitudes are $A^{11} = -A^{22} = A_+$ and $A^{12} = A^{21} = A_\times$.

Gravitational waves propagate between geodesics, so we need to consider geodesic deviation. To do this, consider two events close to each other on a simultaneity. Evolution in proper time will then take the events to a new simultaneity. Let X be the vector field that is tangent to the simultaneity and pointing in the direction of the other event for each τ, and U the tangent of the geodesic. We then have:

$$\ddot{X} = \vec{\boldsymbol{\nabla}}_U \vec{\boldsymbol{\nabla}}_U X = R(U, X)U,$$

where we have chosen the torsion such that $[X, U] = 0$. In component form:

$$\ddot{X}^\mu = R^\mu_{\lambda\sigma\nu} U^\lambda U^\sigma X^\nu.$$

For a slowly moving test particle we obtain

$$\ddot{X}^\mu = R^\mu_{00\nu} X^\nu.$$

As we have $R_{\mu 00\nu} = \frac{1}{2}\partial_0^2 h_{\mu\nu}$, the solution is

$$X^1 = X^1(0)\left(1 + \frac{1}{2}A_+ e^{-ik^\sigma x_\sigma}\right), \; X^2 = X^2(0)\left(1 - \frac{1}{2}A_+ e^{-ik^\sigma x_\sigma}\right).$$

Hence spatial distances oscillate due to the oscillating metric. I think.

Gravitational Lensing Consider an object and an observer separated by a distribution of matter producing a gravitational potential Φ. In the weak-field limit the metric is described by

$$\mathrm{d}s^2 = (1 + 2\Phi)\,\mathrm{d}t^2 - (1 - 2\Phi)\,\mathrm{d}\mathbf{x}^2\,,$$

and the path of a ray of light travelling between the two is described by

$$(1 + 2\Phi)\dot{t}^2 - (1 - 2\Phi)\dot{\mathbf{x}}^2 = 0.$$

In the weak-field limit we expect $\Phi \ll 1$, and thus $\dot{t}^2 \approx \dot{\mathbf{x}}^2$.

In the case where the potential is time-independent, the geodesic equation for the time coordinate yields

$$(1 + 2\Phi)\dot{t} = c,$$

and we are free to choose $c = 1$ by rescaling the parameter. Next, the equation for the spatial coordinates is

$$-\ddot{\mathbf{x}} + 2(\dot{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}\Phi)\dot{\mathbf{x}} = (\dot{t}^2 + \dot{\mathbf{x}}^2)\vec{\boldsymbol{\nabla}}\Phi.$$

Choosing a coordinate system such that $\dot{\mathbf{x}} = \mathbf{e}_x$ initially. In the weak-field we expect

$$\dot{\mathbf{x}} = (1 + \dots)\mathbf{e}_x + v^2\mathbf{e}_y + v^3\mathbf{e}_z,$$

where the new velocity components are small and the dots represent perturbations of the path of higher order than the other velocity components. Inserting this into the geodesic equation yields

$$-\dot{v}^2 = 2(\vec{\boldsymbol{\nabla}}\Phi)^2,$$

where we refer to vector components on either side. We can solve this differential equation by integrating along the unperturbed path. In particular, by moving both the observer and the object to infinity we find

$$\dot{v}^2 = -\int_{-\infty}^{\infty} \mathrm{d}t\, 2(\vec{\boldsymbol{\nabla}}\Phi)^2.$$

3 Cosmology

The Cosmological Principle The cosmological principle states that the universe is homogenous and isotropic. As a consequence, the spatial part of the universe must be maximally symmetric, meaning that it has 6 Killing fields.

Possible Metrics Given the cosmological principle, there are three possible choices of manifold describing simultaneities. We will describe all of them using polar coordinates, hence they will all have a spatial metric of the form $\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + s^2(r)\,\mathrm{d}\Omega^2$. These are:

- the 3-sphere, with $s(r) = \frac{1}{k}\sin(kr)$.
- flat Euclidean space, with $s(r) = r$.
- a hyperbolic surface, with $s(r) = \frac{1}{k}\sinh(kr)$.

We will treat this options generally and introduce the spacetime interval $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\Sigma^2$. This defines the Robertson-Walker (RW) universe. The factor $a(t)$ is called the scale factor. A similar factor could in principle be found in front of the time interval, but can be eliminated by a change of variables.

The Friedman-Lemaitre-Robertson-Walker Universe The FLRW universe assumes the universe to be filled with an ideal fluid. For the universe to be homogenous and isotropic, this would require ρ and p to be functions of time only. We will also use the general equation of state $p = w\rho$ as an assumption. It can be shown that $w = 0$ for dust and $w = \frac{1}{3}$ for radiation. In order to not break isotropy, the fluid rest frame must also coincide with that of a comoving observer.

To proceed, we use the fact that $\vec{\nabla}_\mu G^{\mu\nu} = 0$, implying $\vec{\nabla}_\mu T^{\mu\nu} = 0$. As the fluid moves along geodesics, implying $\vec{\nabla}_U U = 0$, we find

$$U^\mu U^\nu \partial_\mu(p+\rho) + (p+\rho)U^\nu \vec{\nabla}_\mu U^\mu - g^{\mu\nu}\partial_\mu p = 0.$$

For $\nu = 0$ we find

$$\dot{p} + \dot{\rho} + (p+\rho)\Gamma^\mu_{\mu 0} - \dot{p} = \dot{\rho} + 3(1+w)\frac{\dot{a}}{a}\rho = 0.$$

We can integrate this to find $\rho \propto a^{-3(1+w)}$. Next, the Einstein field equations imply the so-called Friedmann equations

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T^\sigma{}_\sigma\right).$$

One equation of note is

$$R_{00} = 4\pi G(\rho + 3p) = -3\frac{\ddot{a}}{a}.$$

The Big Bang As we saw above, we have $\frac{\ddot{a}}{a} \propto -(\rho + 3p)$. Now, we have observed that $\dot{a} > 0$, and as $\ddot{a} < 0$ we must have that $\dot{a}$ was greater before. The general shape of the corresponding solution implies that there exists a time when a was equal to zero. The time at which this changed is termed the Big Bang.