Summary of SI2410 Quantum Field Theory

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Sammanfattning

Innehåll

1 Basic Concepts

All About Grassman Variables Grassman variables θ_i are defined such that multiplication of two different variables anticommutes. This means that all functions of a set of Grassman variables can be written as

$$F(\theta) = \sum_{i=0}^{n} \frac{1}{i!} A^{(i),j_1...j_i} \theta_{j_1} \dots \theta_{j_i},$$

with summation over the indices j. The coefficients A are completely antisymmetric.

Functions of Grassman numbers are defined as even $A^k = 0$ for all even k and similarly for odd functions. This implies that $FG = GF$ if either is even and $FG = -GF$ otherwise.

We define derivatives with respect to Grassman variables as

$$\frac{\partial}{\partial \theta_i}(\theta_j) = \delta_{ij}.$$

These derivatives anticommute. They satisfy the product rule

$$\frac{\partial}{\partial \theta_i}(FG) = \frac{\partial^2 F}{\partial \theta_i \partial G} + (-1)^{|F|} F \frac{\partial G}{\partial \theta_i},$$

where $|F|$ is 1 if F is even and 0 if F is odd. Note that this implies the sign convention

$$\frac{\partial}{\partial \theta_i} \theta_i \theta_j = -\frac{\partial}{\partial \theta_i} \theta_j \theta_i.$$

For a composite function

$$f(G(\theta)) = \sum_n f_n G^n(\theta),$$

the chain rule

$$\frac{\partial}{\partial \theta_i} f(G(\theta)) = \frac{\partial G}{\partial \theta_i} \frac{\mathrm{d}f}{\mathrm{d}G}$$

also applies.

Integrals over Grassman variables are defined according to

$$\int \mathrm{d}\theta_i = 0, \quad \int \mathrm{d}\theta_i \, \theta_j = \delta_{ij}.$$

This has the peculiar consequence

$$\int \mathrm{d}\theta_i \, F(\theta) = \frac{\partial F}{\partial \theta_i}.$$

We know that all functions of Grassman variables can be factorized as $F(\theta) = A(\theta) + \theta_i B(\theta)$, where neither A nor B depend on θ_i. This has the consequence that shifts in any one variable does not affect the integral. Scaling of variables does not work that way, however, as we saw that differentiation and integration has the same effect. This produces the general result

$$\theta_j = A_{ij} \phi_j \implies \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n = \frac{1}{\det(A)} \mathrm{d}\phi_1 \dots \mathrm{d}\phi_n \, .$$

The Dirac delta function is defined in this context such that it obeys the relation

$$\int \mathrm{d}\theta_i \, F(\theta) \delta(\theta_i) = F\bigg|_{\theta_i = 0}.$$

From what we have above we see that $\delta(\theta_i) = \theta_i$ satisfies this. This again has consequences of scaling the Dirac delta being different - more specifically

$$\prod_i \delta(A_{ij}\delta_j) = \det(A) \prod_i \delta(\delta_i).$$

Complex Grassman variables can also be defined according to

$$\theta_k = \phi(k) + i\psi(k).$$

Using the relations above we find

$$\mathrm{d}\theta_k \, \mathrm{d}\theta_k^\star = i \, \mathrm{d}\phi_k \, \mathrm{d}\psi_k \, .$$

The Path Integral Approach We will now switch from canonical quantization to path integrals as our basis for quantum field theory. This has a few advantages:

- It switches from the Hamiltonian to the Lagrangian, which can more easily be manifestly Lorentz invariant.
- The switch to the Lagrangian allows for more aesthetic handling of Lagrangians with field derivatives.
- It allows for better handling of non-Abelian gauge theories.

Path Integrals in Quantum Mechanics Consider a system with Lagrangian

$$\mathcal{H} = \frac{p^2}{2m} + V(q).$$

In the Heisenberg pictures, the q operator is given by

$$q(t) = e^{i\mathcal{H}t} q(0) e^{-i\mathcal{H}t},$$

We study states of the form $|q,t\rangle$, which are eigenstates of $q(t)$ with eigenvalue q. We note that

$$q(t) e^{i\mathcal{H}t} |q\rangle = e^{i\mathcal{H}t} q e^{-i\mathcal{H}t} e^{i\mathcal{H}t} |q\rangle = q e^{i\mathcal{H}t} |q\rangle ,$$

hence this is the form of the states we are considering. Eigenstates at reference time t' will be implicitly labelled when convenient.

We might consider the amplitude of the system evolving from q' to q'' over some time. This is given by

$$\left\langle q'', t'' \middle| q', t' \right\rangle = \left\langle q'' \middle| e^{-i\mathcal{H}(t''-t')} \middle| q' \right\rangle ,$$

which coincides with the expression in the Schrödinger picture. To introduce the path integral we will discretize the time interval into N slices of width Δ and write

$$\left\langle q'' \middle| e^{-i\mathcal{H}(t''-t')} \middle| q' \right\rangle = \left\langle q'' \middle| e^{-i\mathcal{H}\Delta} \dots e^{-i\mathcal{H}\Delta} \middle| q' \right\rangle .$$

Between each pair of time evolution operators we now place a completeness relation, reducing the problem to computing

$$\langle q_{i+1} | e^{-i\mathcal{H}\Delta} | q_i \rangle .$$

The final value of the amplitude will be the product of all such factors from 0 to N. We can now split the exponentials according to

$$\langle q_{i+1} | e^{-i\mathcal{H}\Delta} | q_i \rangle = \langle q_{i+1} | e^{-i\Delta \frac{p^2}{2m}} e^{-i\Delta V(q)} e^{-\frac{1}{2}\Delta^2 \left[\frac{p^2}{2m}, V(q)\right]} \dots | q_i \rangle .$$

Due to our discretization we may now ignore the higher-order factors, leaving

$$\langle q_{i+1} | e^{-i\mathcal{H}\Delta} | q_i \rangle = \langle q_{i+1} | e^{-i\Delta \frac{p^2}{2m}} e^{-i\Delta V(q)} | q_i \rangle .$$

The first factor is easily dealt with. To handle the second, we introduce a new completeness relation such that

$$\begin{aligned}
\langle q_{i+1} | e^{-i\mathcal{H}\Delta} | q_i \rangle &= e^{-i\Delta V(q_i)} \int \mathrm{d}p \, \langle q_{i+1} | e^{-i\Delta \frac{p^2}{2m}} | p \rangle \langle p | q_i \rangle \\
&= e^{-i\Delta V(q_i)} \int \mathrm{d}p \, e^{-i\Delta \frac{p^2}{2m}} \langle q_{i+1} | p \rangle \langle p | q_i \rangle \\
&= e^{-i\Delta V(q_i)} \int \mathrm{d}p \, e^{-i\Delta \frac{p^2}{2m} + ip(q_{i+1}-q_i)} \\
&= e^{-i\Delta V(q_i)} \int \mathrm{d}p \, e^{-i\frac{\Delta}{2m}\left(p - \frac{m}{\Delta}(q_{i+1}-q_i)\right)^2} e^{i\frac{m\Delta}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2} \\
&= e^{i\Delta\left(\frac{1}{2}m\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i)\right)} \int \mathrm{d}p \, e^{-i\frac{\Delta}{2m}\left(p - \frac{m}{\Delta}(q_{i+1}-q_i)\right)^2}
\end{aligned}$$

We will leave the integral as is, as it merely provides some normalization factor in the end. Dubbing it A we find

$$\begin{aligned}\left\langle q'', t'' \middle| q', t' \right\rangle &= \int \mathrm{d}q_1 \ldots \mathrm{d}q_N\, A^N e^{i\Delta\left(\frac{1}{2}m\left(\frac{q_1-q'}{\Delta}\right)^2 - V(q')\right)} \ldots e^{i\Delta\left(\frac{1}{2}m\left(\frac{q''-q_N}{\Delta}\right)^2 - V(q_N)\right)} \\ &= \int \mathrm{d}q_1 \ldots \mathrm{d}q_N\, A^N e^{i\sum_{i=0}^{N}\Delta\left(\frac{1}{2}m\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i)\right)}.\end{aligned}$$

Now comes the kicker: We let N go to infinity and Δ to zero such that the sum in the exponent goes to an integral. In this limit we see that what is left is in fact the integral of the Lagrangian, or the action. The action of what, you ask? It's the action corresponding to a particular choice of all the intermediate q.

The integration over an infinite set of such intermediate values corresponds to an integral over function space - over all possible functional forms of $q(t)$. This is what is termed the path integral. It is generally denoted as

$$\int [Dx]\,.$$

Renaming the normalization constant we therefore have

$$\left\langle q'', t'' \middle| q', t' \right\rangle = C \int [Dq]\, e^{iS}.$$

One might ask whether the limit that defines the path integral really exists. Yes, one might very well ask that.

Observables from Path Integrals Let us now consider

$$\left\langle q'', t'' \middle| Tq(t_n) \ldots q(t_1) \middle| q', t' \right\rangle.$$

We may consider the times to be correctly ordered for convenience, allowing us to remove the time ordering. Discretizing time and taking $t_i - t' = k_i\Delta$ we can write

$$q(t_{i+1})q(t_i) = e^{i\mathcal{H}k_{i+1}\Delta} q e^{-i\mathcal{H}k_{i+1}\Delta} e^{i\mathcal{H}k_i\Delta} q e^{-i\mathcal{H}k_i\Delta} = e^{i\mathcal{H}k_{i+1}\Delta} q e^{-i\mathcal{H}(k_{i+1}-k_i)\Delta} q e^{-i\mathcal{H}k_i\Delta}.$$

Using completeness we have

$$e^{-i\mathcal{H}(k_{i+1}-k_i)\Delta} = \int \mathrm{d}q_{k_i+1} \ldots \mathrm{d}q_{k_{i+1}} \left|q_{k_{i+1}}\right\rangle\!\left\langle q_{k_{i+1}}\right| e^{-i\mathcal{H}\Delta} \ldots \left|q_{k_i+1}\right\rangle\!\left\langle q_{k_i+1}\right| e^{-i\mathcal{H}\Delta},$$

hence

$$\begin{aligned}q(t_{i+1})q(t_i) &= \int \mathrm{d}q_{k_i+1} \ldots \mathrm{d}q_{k_{i+1}}\, e^{i\mathcal{H}k_{i+1}\Delta} q \left|q_{k_{i+1}}\right\rangle\!\left\langle q_{k_{i+1}}\right| e^{-i\mathcal{H}\Delta} \ldots \left|q_{k_i+1}\right\rangle\!\left\langle q_{k_i+1}\right| e^{-i\mathcal{H}\Delta} q e^{-i\mathcal{H}k_i\Delta} \\ &= \int \mathrm{d}q_{k_i+1} \ldots \mathrm{d}q_{k_{i+1}}\, q_{k_{i+1}} e^{i\mathcal{H}k_{i+1}\Delta} \left|q_{k_{i+1}}\right\rangle\!\left\langle q_{k_{i+1}}\right| e^{-i\mathcal{H}\Delta} \ldots \left|q_{k_i+1}\right\rangle\!\left\langle q_{k_i+1}\right| e^{-i\mathcal{H}\Delta} q e^{-i\mathcal{H}k_i\Delta}.\end{aligned}$$

On the left end we pair up $e^{-i\mathcal{H}N\Delta}$ with $e^{i\mathcal{H}k_n\Delta}$ and on the right end no modifications are needed, hence

$$\left\langle q'', t'' \middle| Tq(t_n) \ldots q(t_1) \middle| q', t' \right\rangle = \int \mathrm{d}q_1 \ldots \mathrm{d}q_N\, q_{k_1} \ldots q_{k_n} \left\langle q'' \middle| e^{-i\mathcal{H}\Delta} \middle| q_N \right\rangle \ldots \left\langle q_1 \middle| e^{-i\mathcal{H}\Delta} \middle| q' \right\rangle.$$

We can now recognize results from previously, and in the limit of infinitely fine discretization we obtain

$$\left\langle q'', t'' \middle| Tq(t_n) \ldots q(t_1) \middle| q', t' \right\rangle = C \int [Dq]\, q(t_n) \ldots q(t_1) e^{iS}.$$

Note that the ordering in the integration is no longer important.

As a side note, an ad hoc argument for why the path integral converges can be found by going back to the discrete variant, performing a Wick rotation of the time step Δ and thus show that the amplitude of the contributions diverges exponentially as one strays from the path of least action.

Green's Functions from Path Integrals We can use the above to obtain the Green's function from path integrals. We recall that the Green's function is defined as

$$G(t_1, \ldots, t_n) = \frac{\langle \Omega | T q(t_1) \ldots q(t_n) | \Omega \rangle}{\langle \Omega | \Omega \rangle}.$$

Here we will use $t = 0$ as our reference. To get an expression in terms of the ground state we use completion to write

$$\begin{aligned}\left\langle q'', t'' \middle| T q(t_n) \ldots q(t_1) \middle| q', t' \right\rangle &= \sum_{n,m} \left\langle q'' \middle| e^{-i\mathcal{H}t''} \middle| n \right\rangle \langle n | T q(t_n) \ldots q(t_1) | m \rangle \left\langle m \middle| e^{i\mathcal{H}t'} \middle| q' \right\rangle \\ &= \sum_{n,m} e^{i(E_m t' - E_n t'')} \left\langle q'' \middle| n \right\rangle \langle n | T q(t_n) \ldots q(t_1) | m \rangle \left\langle m \middle| q' \right\rangle.\end{aligned}$$

We will now consider the limit of large times (which is what is relevant for QFT contexts anyway). We will also extend time to be complex. In summary we write

$$t' = -\tau(1 - i\varepsilon), \ t'' = \tau(1 - i\varepsilon),$$

where we take τ to be large and let $\varepsilon \to 0$ at the end of our calculations. Thus we have

$$\begin{aligned}\left\langle q'', t'' \middle| T q(t_n) \ldots q(t_1) \middle| q', t' \right\rangle &= \sum_{n,m} e^{-i\tau(1-i\varepsilon)(E_m + E_n)} \left\langle q'' \middle| n \right\rangle \langle n | T q(t_n) \ldots q(t_1) | m \rangle \left\langle m \middle| q' \right\rangle \\ &= \sum_{n,m} e^{-i\tau(E_m + E_n)} e^{-\tau\varepsilon(E_m + E_n)} \left\langle q'' \middle| n \right\rangle \langle n | T q(t_n) \ldots q(t_1) | m \rangle \left\langle m \middle| q' \right\rangle.\end{aligned}$$

In the limit of large times the only contribution thus comes from the ground state, and we have

$$\left\langle q'', t'' \middle| T q(t_n) \ldots q(t_1) \middle| q', t' \right\rangle = \left\langle q'' \middle| \Omega \right\rangle \left\langle \Omega \middle| q' \right\rangle \langle \Omega | T q(t_n) \ldots q(t_1) | \Omega \rangle.$$

The left-hand side is now a path integral and the right-hand side is proportional to the Green's function. We thus have

$$G(t_1, \ldots, t_n) = \frac{\int [Dq] \, q(t_n) \ldots q(t_1) e^{iS}}{\int [Dq] \, e^{iS}}.$$

Path Integrals in Field Theories In moving to field theory there is really nothing new that is introduced. To see this you could simply perform your field theory calculation on a lattice and reuse what we have done above. The eigenstates of q are now replaced by eigenstates of the fields and times are replaced by points in spacetime. We thus arrive at

$$G(x_1, \ldots, x_n) = \frac{\int [D\phi] \, \phi(t_n) \ldots \phi(x_1) e^{iS}}{\int [D\phi] \, e^{iS}}.$$

The path integral is now over all possible forms of the field ϕ throughout spacetime.

The Generating Functional The generating functional is defined as

$$Z = \int [D\phi] \, e^{i\left(S + \int \mathrm{d}^4 x \phi J\right)}.$$

It is a functional of the extra field J. It evidently satisfies

$$Z(0) = \int [D\phi] \, e^{iS}.$$

Furthermore it satisfies

$$-i\,\delta Z = \int [D\phi]\,\phi e^{i\left(S+\int \mathrm{d}^4x\phi J\right)} \cdot \mathrm{d}\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} \int \mathrm{d}^4x\,\phi J$$

$$= \int \mathrm{d}^4x \int [D\phi]\,\phi e^{i\left(S+\int \mathrm{d}^4x\phi J\right)} \cdot \frac{\mathrm{d}}{\mathrm{d}\alpha}\phi\,\delta J\,,$$

hence

$$-i\frac{\delta Z}{\delta J} = \int [D\phi]\,\phi e^{i\left(S+\int \mathrm{d}^4x\phi J\right)} \phi.$$

Similarly we can apply more functional derivatives and consider specific points, setting $J = 0$ at the end, to compute correlation functions.

The Generating Functional for a Free Scalar Field Consider the non-interacting scalar field theory with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2,$$

taken with the field to approach zero at infinity. Before writing the generating functional, we rewrite the action as

$$S = \int \mathrm{d}^4x - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$= \int \mathrm{d}^4x - \frac{1}{2}\left(\partial_\mu\left(\phi\partial^\mu\phi\right) - \phi\partial_\mu\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2$$

$$= \int \mathrm{d}^4x - \frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi^2$$

$$= \int \mathrm{d}^4x\,\frac{1}{2}\phi\left(-\Box - m^2\right)\phi,$$

where we have introduced the d'Alembert operator $-\partial_\mu\partial^\mu$. Next we are going to use a clever trick by introducing the Klein-Gordon propagator $i\tilde{G} = \Delta$, which satisfies

$$\left(-\Box - m^2\right)\tilde{G} = \delta^4(x-y).$$

We then have

$$\int \mathrm{d}^4x - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \phi J = \int \mathrm{d}^4x\,\frac{1}{2}\phi\left(-\Box - m^2\right)\phi + \phi\int \mathrm{d}^4y\,\delta^4(x-y)J(y)$$

$$= \int \mathrm{d}^4x\,\frac{1}{2}\phi\left(-\Box_x - m^2\right)\phi + \phi\int \mathrm{d}^4y\,J(y)\left(-\Box_x - m^2\right)\tilde{G}(x,y).$$

We could do some very neat completing of squares here, but that would require

$$\int \mathrm{d}^4y\,J(y)\tilde{G}(x,y)\left(-\Box_x - m^2\right)\phi = \phi(x)J(x).$$

To prove that we will perform some clever integration by parts. We have

$$
\begin{aligned}
-\int \mathrm{d}^4y\, J(y)\tilde{G}(x,y)\Box_x\phi &= \int \mathrm{d}^4y\, \partial_\mu\left(J(y)\tilde{G}(x,y)\partial^\mu\phi\right) - J(y)\partial_\mu\tilde{G}(x,y)\partial^\mu\phi \\
&= -\int \mathrm{d}^4y\, \partial^\mu(J(y)\partial_\mu\tilde{G}(x,y)\phi) + J(y)\phi\Box_x\tilde{G}(x,y) \\
&= -\int \mathrm{d}^4y\, J(y)\phi\Box_x\tilde{G}(x,y).
\end{aligned}
$$

This completes the proof, and we thus have

$$
\begin{aligned}
\int \mathrm{d}^4x - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \phi J = \int \mathrm{d}^4x\, \frac{1}{2}\phi\left(-\Box_x - m^2\right)\phi \\
+\frac{1}{2}\left(\int \mathrm{d}^4y\, \phi(x)J(y)\left(-\Box_x - m^2\right)\tilde{G}(x,y) + J(y)\tilde{G}(x,y)\left(-\Box_x - m^2\right)\phi(x)\right) \\
=\frac{1}{2}\int \mathrm{d}^4x\left(\phi + \int \mathrm{d}^4y\, J(y)G(x,y)\right)\left(-\Box_x - m^2\right)\left(\phi + \int \mathrm{d}^4z\, J(z)G(x,z)\right) \\
-\int \mathrm{d}^4y \int \mathrm{d}^4z\, J(y)G(x,y)\left(-\Box_x - m^2\right)J(z)G(x,z).
\end{aligned}
$$

The final integral can be simplified as

$$
\begin{aligned}
\int \mathrm{d}^4x \int \mathrm{d}^4y \int \mathrm{d}^4z\, J(y)G(x,y)\left(-\Box_x - m^2\right)J(z)G(x,z) &= \int \mathrm{d}^4x \int \mathrm{d}^4y \int \mathrm{d}^4z\, J(y)G(x,y)J(z)\delta^4(x-z) \\
&= \int \mathrm{d}^4x \int \mathrm{d}^4y\, J(y)G(x,y)J(x).
\end{aligned}
$$

Now define a new field

$$
\chi(x) = \phi(x) + \int \mathrm{d}^4y\, J(y)G(x,y).
$$

This shift has no effect on the integration measure in the path integral, hence we have

$$
\begin{aligned}
Z &= \int [D\chi]\, e^{\frac{i}{2}\int \mathrm{d}^4x\chi(x)\left(-\Box_x - m^2\right)\chi(y) - \int \mathrm{d}^4yJ(y)G(x,y)J(x)} \\
&= Ne^{-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4yJ(y)\Delta(x,y)J(x)}.
\end{aligned}
$$

Let's do a sanity check now, to be sure. We have

$$
\begin{aligned}
G(x_1,x_2) = \frac{1}{Z(0)}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)N_c e^{-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4yJ(y)\Delta(x,y)J(x)} \\
=\frac{\delta}{\delta J(x_1)}\left(\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y\, J(y)\Delta(x,y)\delta^4(x-x_2) + \frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y\, \delta^4(y-x_2)\Delta(x,y)J(x)\right)e^{-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4yJ(y)\Delta(x,y)J(x)} \\
=\frac{\delta}{\delta J(x_1)}\left(\frac{1}{2}\int \mathrm{d}^4y\, J(y)\Delta(x_2,y) + \frac{1}{2}\int \mathrm{d}^4x\, \Delta(x,x_2)J(x)\right)e^{-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4yJ(y)\Delta(x,y)J(x)}.
\end{aligned}
$$

We would in principle need to apply the product rule, but it is now obvious that only the term which applies the functional derivative to the non-exponential part will give a non-zero contribution, hence we find

$$G(x_1, x_2) = \frac{1}{2}(\Delta(x_1, x_2) + \Delta(x_2, x_1)),$$

and using the fact that the propagator is symmetric under switching the two arguments we have

$$G(x_1, x_2) = \Delta(x_1, x_2),$$

as expected.

The Generating Functional for an Interacting Scalar Field Consider the interacting scalar field theory described by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4.$$

The generating functional is given by

$$Z = \int [D\phi]\, e^{i\left(S_{\text{free}} + \int \mathrm{d}^4x\phi J - \frac{1}{4!}\lambda\phi^4\right)}.$$

We will be working in the framework of perturbation theory and will therefore at some point take the parameter λ to be small. We prepare for that by performing a series expansion in λ according to

$$Z = \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m \int [D\phi] \left(\int \mathrm{d}^4x\, \phi^4\right)^m e^{i\left(S_{\text{free}} + \int \mathrm{d}^4x\phi J\right)}.$$

Here comes the kicker: The fields can be brought down from the exponent by functional differentiation. Now, that differentiation has no dependence on ϕ, hence we may extract it from the path integral to find

$$\begin{aligned} Z &= \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m \left(\int \mathrm{d}^4x\, \frac{\delta^m}{\delta J^m}\right)^m \int [D\phi]\, e^{i\left(S_{\text{free}} + \int \mathrm{d}^4x\phi J\right)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m \left(\int \mathrm{d}^4x\, \frac{\delta^m}{\delta J^m}\right)^m Z_{\text{free}}. \end{aligned}$$

This can be related to Feynman diagrams in a very straightforward way.

The Generating Functional in Momentum Space Introducing the Fourier transform

$$\phi(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\phi}(k)e^{ikx},$$

we can write the n-point correlation function as

$$G(x_1, \ldots, x_n) = \frac{1}{Z(0)} \int [D\phi]\, e^{iS} \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4k_n}{(2\pi)^4} e^{i\sum_i k_i x_i} \tilde{\phi}(k_1) \ldots \tilde{\phi}(k_n).$$

Defining

$$G(x_1, \ldots, x_n) = \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4k_n}{(2\pi)^4} e^{i\sum_i k_i x_i} \tilde{G}(k_1, \ldots, k_n)$$

we then see

$$\tilde{G}(k_1, \ldots, k_n) = \frac{1}{Z(0)} \int [D\phi]\, e^{iS} \tilde{\phi}(k_1) \ldots \tilde{\phi}(k_n).$$

To write this even more neatly we note that we could in principle switch the path integral to be over $\tilde{\phi}$. This will at most produce a dimensional factor which is equal in the numerator and denominator, provided that we can appropriately define the generating functional. Let us therefore do this.

Consider the integral

$$\int \mathrm{d}^4 x\, \phi(x) J(x) = \int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} e^{i(k_1 x + k_2 x)} \tilde{\phi}(k_1) \tilde{J}(k_2).$$

The two integrals arise because we are considering two different functions with their own unique Fourier transforms. Performing the x integration we find

$$\begin{aligned}\int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} e^{i(k_1 x + k_2 x)} \tilde{\phi}(k_1) \tilde{J}(k_2) &= \int \frac{\mathrm{d}^4 k_1\, \mathrm{d}^4 k_2}{(2\pi)^4} \delta^4(k_1 + k_2) \tilde{\phi}(k_1) \tilde{J}(k_2) \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\phi}(k) \tilde{J}(-k).\end{aligned}$$

Note the structure of this result, as it will be reused regularly. Pending a rewrite of the action in terms of $\tilde{\phi}$ we have

$$\begin{aligned}Z[J] &= \int [D\phi]\, e^{i\left(S[\phi] + \int \mathrm{d}^4 x \phi J\right)} \\ &= \int [D\phi]\, e^{i\left(S[\tilde{\phi}] + \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\phi}(k) \tilde{J}(-k)\right)}.\end{aligned}$$

The change of integration measure involves a dimensional factor, as previously stated, but because it will always be common for factors in a numerator and denominator, we may define

$$Z[\tilde{J}] = \int \left[D\tilde{\phi}\right] e^{i\left(S[\tilde{\phi}] + \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\phi}(k) \tilde{J}(-k)\right)}.$$

This functional evidently satisfies

$$\tilde{G}(k_1, \ldots, k_n) = \frac{1}{Z[0]} \left(-i(2\pi)^4 \frac{\delta}{\delta \tilde{J}(k_1)}\right) \ldots \left(-i(2\pi)^4 \frac{\delta}{\delta \tilde{J}(k_n)}\right) Z[\tilde{J}]\Big|_{\tilde{J}=0}.$$

Momentum Space Generating Functional for a Free Scalar We once again consider a system with Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2.$$

The action can then be written as

$$\begin{aligned}S &= \int \mathrm{d}^4 x - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \\ &= \int \mathrm{d}^4 x \frac{1}{2} \phi \left(-\Box - m^2\right) \phi \\ &= \frac{1}{2} \int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} e^{ik_1 x} \tilde{\phi}(k_1) \left(-\Box - m^2\right) \left(e^{ik_2 x} \tilde{\phi}(k_2)\right) \\ &= \frac{1}{2} \int \mathrm{d}^4 x \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} e^{ik_1 x} \tilde{\phi}(k_1) \left(-k_2^2 - m^2\right) e^{ik_2 x} \tilde{\phi}(k_2) \\ &= -\frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \tilde{\phi}(-k) \left(k^2 + m^2\right) \tilde{\phi}(k).\end{aligned}$$

The integral in the exponent can once again, this time with a very straight-forward ompleting of the square, be written as

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} - \frac{1}{2}\tilde{\phi}(-k)\left(k^2+m^2\right)\tilde{\phi}(k) + \tilde{\phi}(k)\tilde{J}(-k)$$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4} - \frac{1}{2}\tilde{\phi}(-k)\left(k^2+m^2\right)\tilde{\phi}(k) + \frac{1}{2}(\tilde{\phi}(k)\tilde{J}(-k) + \tilde{\phi}(-k)\tilde{J}(k))$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(\tilde{\phi}(-k) - \frac{1}{k^2+m^2}\tilde{J}(-k)\right)\left(k^2+m^2\right)\left(\tilde{\phi}(k) - \frac{1}{k^2+m^2}\tilde{J}(k)\right) - \frac{1}{k^2+m^2}\tilde{J}(k)\tilde{J}(-k).$$

Defining

$$\tilde{\chi} = \tilde{\phi}(k) - \frac{1}{k^2+m^2}\tilde{J}(k)$$

and switching the integration measure we find

$$Z\left[\tilde{J}\right] = Ne^{\frac{i}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\tilde{J}(k)\tilde{J}(-k)}.$$

Let us once again perform a sanity check. We have

$$\begin{aligned}\tilde{G}(k_1,k_2) &= -(2\pi)^8\frac{\delta}{\delta\tilde{J}(k_1)}\frac{i}{2(2\pi)^4}\left(\frac{1}{k_2^2+m^2}\tilde{J}(-k_2) + \frac{1}{k_2^2+m^2}\tilde{J}(-k_2)\right)e^{\frac{i}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\tilde{J}(k)\tilde{J}(-k)}\Bigg|_{\tilde{J}=0} \\ &= -i(2\pi)^4\frac{\delta}{\delta\tilde{J}(k_1)}\frac{1}{k_2^2+m^2}\tilde{J}(-k_2)e^{\frac{i}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\tilde{J}(k)\tilde{J}(-k)}\Bigg|_{\tilde{J}=0}.\end{aligned}$$

Once again the contribution from the second differentiation of the exponential can be ignored and we find

$$\tilde{G}(k_1,k_2) = -i(2\pi)^4\frac{1}{k_2^2+m^2}\delta^4(k_1+k_2).$$

This is very similar to the momentum-space propagator, except that it only implicitly contains information about G only being a function of the displacement between two points.

Momentum Space Generating Functional for an Interacting Scalar We now consider an interacting scalar field with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4.$$

The interacting part of the action can be rewritten as

$$\begin{aligned}S &= -\frac{1}{4!}\lambda\int \mathrm{d}^4x\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)e^{ix\sum\limits_i k_i} \\ &= -\frac{1}{4!}\lambda\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\delta^4\left(\sum_i k_i\right)\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4).\end{aligned}$$

Similar to the position space case we then have

$$\begin{aligned}&Z[\tilde{J}] \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m\int\left[D\tilde{\phi}\right]\left(\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\delta^4\left(\sum_i k_i\right)\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\right)^m e^{i\left(S_{\text{free}}++\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\phi}(k)\tilde{J}(-k)\right.} \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m\frac{\delta}{\delta\tilde{J}(k_1)}\frac{\delta}{\delta\tilde{J}(k_2)}\frac{\delta}{\delta\tilde{J}(k_3)}\frac{\delta}{\delta\tilde{J}(k_4)}Z_{\text{free}}.\end{aligned}$$

This can also be used to derive Feynman rules.

Multiple Scalar Fields We now consider a system of multiple scalar fields $\phi^{(a)}$. We take them to be real scalar fields, hence

$$\left(\tilde{\phi}^{(a)}\right)^{\star}(k) = \tilde{\phi}^{(a)}(-k).$$

The most general action for such a field theory is

$$S = \frac{1}{2}\int \mathrm{d}^4x\, \phi^{(a)}(x) D_{ab} \phi^{(b)}(x) = \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \tilde{\phi}^{(a)}(-k)\tilde{M}_{ab}(k)\tilde{\phi}^{(b)}(k).$$

D is a matrix of differential operators and $\tilde{M}$ a matrix of functions. The latter can be symmetrized to

$$S = \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \tilde{\phi}^{(a)}(-k) M_{ab}(k)\tilde{\phi}^{(b)}(k),\ \ M_{ab}(k) = \frac{1}{2}(\tilde{M}_{ab}(k) + \tilde{M}_{ba}(-k)).$$

We can now go about finding a generating functional for this kind of system. We have

$$Z = \int [D\phi^a]\, e^{iS + i\int \mathrm{d}^4x \phi^{(a)} J_a}$$

in position space and

$$Z = \int \left[D\tilde{\phi}^a\right] e^{iS + i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\phi}^{(a)}(k)\tilde{J}_a}(-k)$$

in momentum space. The latter exponent can be handled the most easily, so we will do that one. We have

$$\begin{aligned}
S + \int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\phi}^{(a)}(k)\tilde{J}_a(-k) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\tilde{\phi}^{(a)}(-k)M_{ab}(k)\tilde{\phi}^{(b)}(k) + \tilde{\phi}^{(a)}(k)\tilde{J}_a(-k) \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\left(\tilde{\phi}^{(a)}(-k) + M^{-1}_{ac}(-k)\tilde{J}_c(-k)\right) M_{ab}(k)\left(\tilde{\phi}^{(b)}(k) + M^{-1}_{bd}(k)\tilde{J}_d(k)\right) \\
&\quad - \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}M^{-1}_{ac}(-k)\tilde{J}_c(k)M_{ab}(k)M^{-1}_{bd}(k)\tilde{J}_d(k) \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\tilde{\chi}^{(a)}(-k)M_{ab}(k)\tilde{\chi}^{(b)}(k) - \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}M^{-1}_{ac}(-k)\tilde{J}_c(-k)\delta_{ad}\tilde{J}_d(k) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\chi}^{(a)}(-k)M_{ab}(k)\tilde{\chi}^{(b)}(k) - \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{J}_a(-k)M^{-1}_{ac}(-k)\tilde{J}_c(k).
\end{aligned}$$

To do this we have defined the new fields

$$\tilde{\chi}^{(a)}(k) = \tilde{\phi}^{(a)}(k) + M^{-1}_{ac}(-k)\tilde{J}_c(k)$$

and used the fact that

$$M^{-1}(k) = (M^T(k))^{-1} = (M^{-1}(k))^T.$$

This means

$$Z = Ne^{-\frac{i}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{J}_a(-k)M^{-1}_{ac}(-k)\tilde{J}_c(k)}.$$

We now see directly that the two-point correlation function is

$$G_{ab}(k_1, k_2) = \frac{1}{2}(2\pi)^4\left(M^{-1}_{ba}(-k_1) + M^{-1}_{ab}(k_1)\right)\delta(k_1 + k_2).$$

Note that it depends very directly on the existence of M^{-1}, which can turn out to cause issues, as will be seen.

Failing to Quantize the Electromagnetic Field The momentum space Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(k_\mu A_\nu - k_\nu A_\mu\right)\left(k^\mu A^\nu - k^\nu A^\mu\right) \\ &= -\frac{1}{4}\left(A_\nu\left(k^2 A^\nu - k_\mu k^\nu A^\mu\right) - A_\nu\left(k_\mu k^\nu A^\mu - k^2 A^\nu\right)\right) \\ &= \frac{1}{2}A_\nu\left(k^\mu k^\nu A_\mu - k^2 A^\nu\right) \\ &= \frac{1}{2}A_\nu\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right)A_\mu.\end{aligned}$$

This is a bit of a disaster, as M always has at least one eigenvalue of zero. Namely, we have

$$M^{\mu\nu}k_\nu = k^\mu k^\nu k_\nu - k^2 g^{\mu\nu}k_\nu = 0.$$

This means that M^{-1} does not exist for any k. That scenario is a lot worse than for the Klein-Gordon equation, where M^{-1} only didn't exist at the set of k such that $k^2 = -m^2$.

The Free Dirac Field We will quantize Dirac fields by taking

$$\psi(x) = \sum_i \theta_i \phi_i(x),$$

where the ϕ_i are an appropriate basis of functions and the θ_i are complex Grassman variables. We will pair the fields with the Dirac adjoint rather than the normal adjoint, which is allowed as the two are related by a unitary transform. The action is

$$S = \int \mathrm{d}^4x\, \overline{\psi}_\alpha \left(-i\gamma^\mu \partial_\mu - m\right)_{\alpha\beta} \psi_\beta$$

in position space and

$$S = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \tilde{\overline{\psi}}_\alpha(-k) \left(-i\gamma^\mu k_\mu - m\right)_{\alpha\beta} \tilde{\psi}_\beta(k)$$

in momentum space. The indices on the fields are Dirac indices. In the generating functional we pair the fields and their adjoints with fields $\tilde{J}$ and $\tilde{\overline{J}}$. According to our definition of derivatives with respect to Grassman variables we then find

$$\langle\Omega|\tilde{\psi}_{\alpha_1}(k_1)\dots\tilde{\psi}_{\alpha_n}(k_n)\tilde{\overline{\psi}}_{\beta_1}(p_1)\dots\tilde{\overline{\psi}}_{\beta_n}(p_n)|\Omega\rangle = \prod_i \left(-i(2\pi)^4 \frac{\delta}{\delta\tilde{\overline{J}}(k_i)}\right) \prod_i \left(i(2\pi)^4 \frac{\delta}{\delta\tilde{J}(p_i)}\right) Z.$$

We can show that

$$Z = Ne^{-i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{\overline{J}}_\alpha(-k)(\not{k}-m)^{-1}_{\alpha\beta}\tilde{J}_\beta(k)},$$

and from this recover familiar results.

An Interacting Fermion Theory Consider a field theory with a Dirac field, a scalar field and an interaction term

$$\mathcal{L} = \lambda\overline{\psi}\psi\phi.$$

A Go at Quantum Electrodynamics Quantum electrodynamics starts with wanting to extend the global gauge symmetry of the free fermionic field to a local. This is done by replacing the derivative with

$$D_\mu = \partial_\mu - iA_\mu,$$

where A is the so-called gauge field. The full gauge transformation is

$$\Psi \to e^{i\lambda(x)}\Psi,\ A_\mu \to A_\mu + \partial_\mu\lambda.$$

For the field theory to work we now need to add terms describing A to the action, starting with quadratic ones as these are the easiest. The quantity

$$F_{\mu\nu} = [D_\mu, D_\nu]$$

is gauge invariant, so we add

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$$

to the full Lagrangian. This doesn't work, however, as we end up with the momentum space action

$$S = \frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} A_\mu \left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) A_\nu,$$

which was ill-defined. We will have to devise some workaround.

The trick we will use is to for any k construct a basis e for spacetime such that $e^{(1)} = k$ and expand

$$\tilde{A}_\mu = \sum_s \tilde{B}^{(s)}(k) e_\mu^{(s)}(k).$$

We will be interested in the expectation values of some set of observables, which can be written as

$$\begin{aligned} \left\langle \prod_i O_i(k_i) \right\rangle &= \frac{\int \left[D\tilde{A}\right] e^{iS} \prod_i O_i(k_i)}{\int \left[D\tilde{A}\right] e^{iS}} \\ &= \frac{\int \prod_s \left[D\tilde{B}^{(s)}\right] e^{iS} \prod_i O_i(k_i)}{\int \prod_s \left[D\tilde{B}^{(s)}\right] e^{iS}}. \end{aligned}$$

The gauge transform can be written as

$$\delta \tilde{A}_\mu = \sum_s \delta \tilde{B}^{(s)}(k) e_\mu^{(s)}(k) = i k_\mu \tilde{\lambda},$$

implying that the only non-zero variation of B is $\delta \tilde{B}^{(1)}(k) = i\tilde{\lambda}$. This is a very neat trick because it makes the gauge invariance of any observable easy to check - you just need to see if there is dependence on $\tilde{B}^{(1)}$ (something the fields themselves do not satisfy, hence why their expectation values are ill-defined). For any such set of observables $\tilde{B}^{(1)}$ can be integrated out in the numerator and denominator, giving us

$$\left\langle \prod_i O_i(k_i) \right\rangle = \frac{\int \prod_{s>1} \left[D\tilde{B}^{(s)}\right] e^{iS} \prod_i O_i(k_i)}{\int \prod_{s>1} \left[D\tilde{B}^{(s)}\right] e^{iS}}.$$

Note that the action can now be written as

$$S = \frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{s,t>1} \tilde{B}^{(s)}(-k) M^{(s,t)}(k) \tilde{B}^{(t)}(k).$$

Gauge Theories Consider some theory where the action contains derivatives of the field, and suppose that the theory has some global symmetry. From the theory of Lie groups we know that such symmetries can be specified in terms of a set of numbers combined with the generators of the Lie group. Now, what if these numbers were to be replaced by functions, making the symmetry local? This would make the symmetry into a so-called gauge symmetry. Dubbing the fields Ψ, the action of the symmetry produces the outcome

$$\partial_\mu \Psi \to \partial_\mu (U\Psi) = \left(\partial_\mu U + u \partial_\mu\right) \Psi.$$

If we want the local symmetry to be a symmetry of our theory, termed requiring gauge invariance, we are going to have to modify the derivative by introducing a covariant derivative

$$D_\mu = \partial_\mu - iS_\mu.$$

S is a collection of space-dependent matrices, as we are generally working with a field theory with multiple fields. In order for this to work we will have to let these matrices transform along with the fields. The action of a gauge transformation then produces

$$\begin{aligned} D_\mu\Psi \to D'_\mu\Psi' &= (\partial_\mu - iS'_\mu)U\Psi \\ &= (\partial_\mu U + U\partial_\mu - iS'_\mu U)\Psi \\ &= U(U^{-1}\partial_\mu U + \partial_\mu - iU^{-1}S'_\mu U)\Psi. \end{aligned}$$

From this point on we will restrict ourselves to SU(N) transformations, but many of the results are in fact reproducible for a general Lie group. Someone should probably do that. For the local symmetry to be a symmetry of the theory we require $D'_\mu\Psi' = UD_\mu\Psi$, which implies

$$\begin{aligned} U^\dagger\partial_\mu U - iU^\dagger S'_\mu U &= -iS_\mu, \\ S'_\mu &= US_\mu U^\dagger - i(\partial_\mu U)U^\dagger. \end{aligned}$$

Can we impose any restrictions on the matrices S? We can write

$$U = e^{i\theta^a(x)T_a},$$

where T_a are the generators of SU(N) (traceless hermitian matrices). Looking at the action of the covariant derivative close to the identity transform we find

$$D'_\mu(U\Psi) = \left(iT_a\partial_\mu\theta^a + (1 + i\theta^a(x)T_a)\partial_\mu - iS'_\mu(1 + i\theta^a(x)T_a)\right)\Psi.$$

Looking at this we notice that if we were to extract U, all that would be left in the matrix apart from the derivative would be the S matrices and the generators and their products. The generators satisfy

$$[T_a, T_b] = if^c_{ab}T_c,$$

which gives a vague indication that we should choose to expand the S in terms of the generators. This means that the S are traceless and hermitian. We have that

$$(S'_\mu)^\dagger = US^\dagger_\mu U^\dagger + iU(\partial_\mu U^\dagger).$$

Because U is unitary we have

$$U(\partial_\mu U^\dagger) = -U^\dagger(\partial_\mu U),$$

and hermitianity is thus guaranteed by the gauge transformation. Furthermore, as $\mathrm{tr}(\ln(U)) = \ln(\det(U))$ and $\det(U) = 1$ we have

$$\partial_\mu \,\mathrm{tr}(\ln(U)) = \mathrm{tr}\Big(U^\dagger\partial_\mu U\Big) = 0.$$

Because traces are invariant under unitary transformations and cyclic permutations of matrices, tracelessness is also preserved by the gauge transformation. All in all we are left with $N^2 - 1$ degrees of freedom.

Having established this, we can expand the matrices S in terms of the so-called gauge fields B according to

$$S_\mu = B^a_\mu T_a.$$

There are transformation rules associated with these fields as well. Writing

$$UT_bU^\dagger = R^a_b T_a,\ i(\partial_\mu U)U^\dagger = \alpha^a T_a$$

we have

$$(B')^a_\mu = B^b_\mu R^a_b - \alpha^a.$$

Next we introduce the field strength. Noting that

$$\begin{aligned} D_\mu D_\nu &= (\partial_\mu - iS_\mu)(\partial_\nu - iS_\nu) \\ &= \partial_\mu \partial_\nu - i\partial_\mu S_\nu - iS_\mu \partial_\nu - S_\mu S_\nu \\ &= \partial_\mu \partial_\nu - i\left((\partial_\mu S_\nu) + S_\nu \partial_\mu + S_\mu \partial_\nu\right) - S_\mu S_\nu, \end{aligned}$$

meaning

$$\begin{aligned} [D_\mu, D_\nu] &= -i((\partial_\mu S_\nu) - (\partial_\nu S_\mu)) - [S_\mu, S_\nu] \\ &= -i((\partial_\mu S_\nu) - (\partial_\nu S_\mu) + i[S_\mu, S_\nu]). \end{aligned}$$

We define the field strength tensor as

$$G_{\mu\nu} = i[D_\mu, D_\nu].$$

Because the covariant derivatives satisfy $D'_\mu \Psi' = U D_\mu \Psi$ we have

$$G'_{\mu\nu} \Psi' = U G_{\mu\nu} \Psi,$$

hence

$$G'_{\mu\nu} = U G_{\mu\nu} U^\dagger.$$

From this we can construct gauge invariant terms to be added to the action, and the full action is

$$S = -\frac{1}{2g^2} \int \mathrm{d}^4 x \, \mathrm{tr}(G_{\mu\nu} G^{\mu\nu}) + \mathcal{L}(\Psi, \partial \to D).$$