Summary of SI2400 Theoretical Particle Physics

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Abstract

This is a summary of SI2400 Theoretical Particle Physics.

Contents

1   Useful Mathematics

The Jacobi Identity for Structure Constants   Applying the Jacobi identity we find

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = i \left(f_{bcd}[X_a, X_d] + f_{cad}[X_b, X_d] + f_{abd}[X_c, X_d] \right)$$
$$= i^2 \left(f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} \right) X_e = 0,$$

which means

$$f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} = 0.$$

Structure Constant Representations   For a group, consider the structure constants

$$[X_a, X_b] = i f_{abc} X_c.$$

Constructing the matrices

$$(T_a)_{bc} = -i f_{abc}$$

we find

$$[T_a, T_b]_{cd} = -(f_{ace} f_{bed} - f_{bce} f_{aed}) = f_{cae} f_{bed} + f_{bce} f_{aed} = -f_{abe} f_{ced} = i f_{abe} (T_e)_{cd},$$

and thus

$$[T_a, T_b] = i f_{abe} T_e,$$

meaning that the structure constants themselves provide a representation of the group. This is a real representation, i.e. is identical to its conjugate representation.

A Closer Look at SU(2)   The Lie algebra of SU(2) is

$$[J_i, J_j] = i \varepsilon_{ijk} J_k.$$

Our goal is to block diagonalize the space on which elements in the group act, starting by looking at the generators. There are three generators, so we can only diagonalize the one. We choose J_3, which will net us the so-called spin representation. Taking the space to be finite dimensional (which in the context of angular momentum is explained by physical arguments), we consider the eigenstate $|j, \alpha\rangle$ with the highest eigenvalue j of J_3. α contains other information, the existence of which is undetermined as of yet. Next we introduce the raising and lowering operators

$$J_\pm = \frac{1}{\sqrt{2}} (J_1 \pm i J_2).$$

We have

$$[J_+, J_-] = \frac{1}{2} i (-[J_1, J_2] + [J_2, J_1]) = J_3,$$
$$[J_3, J_\pm] = \frac{1}{\sqrt{2}} ([J_3, J_1] \pm i [J_3, J_2]) = \frac{1}{\sqrt{2}} (\pm J_1 + i J_2) = \pm J_\pm.$$

For a general eigenstate we find

$$J_3 J_\pm |m, \alpha\rangle = (\pm J_\pm + J_\pm J_3) |m, \alpha\rangle = (m \pm 1) J_\pm |m, \alpha\rangle,$$

and the raising and lowering operators do indeed work as expected. This implies $J_+ |j, \alpha\rangle = 0$. Furthermore, we have

$$\langle j, \beta | J_-^\dagger J_- | j, \alpha \rangle = \langle j, \beta | J_+ J_- | j, \alpha \rangle$$
$$= \langle j, \beta | J_+ J_- - J_- J_+ | j, \alpha \rangle$$
$$= \langle j, \beta | [J_+, J_-] | j, \alpha \rangle$$
$$= \langle j, \beta | J_3 | j, \alpha \rangle$$
$$= j \delta_{\alpha\beta},$$

as we assume the j states to be orthonormal. This implies that the raising and lowering operators preserve orthogonality in terms of the other quantum numbers. Introducing $N_j = \sqrt{j}$ we may thus write

$$J_- \left| j, \alpha \right\rangle = N_j \left| j - 1, \alpha \right\rangle.$$

Similarly we find

$$
\begin{aligned}
J_+ \left| j - 1, \alpha \right\rangle &= \frac{1}{N_j} J_+ J_- \left| j, \alpha \right\rangle \\
&= \frac{1}{N_j} [J_+, J_-] \left| j, \alpha \right\rangle \\
&= N_j \left| j, \alpha \right\rangle.
\end{aligned}
$$

We generalize this to

$$J_- \left| m, \alpha \right\rangle = N_m \left| m - 1, \alpha \right\rangle, \quad J_+ \left| m - 1, \alpha \right\rangle = N_m \left| m, \alpha \right\rangle.$$

These satisfy a recursion relation which we can find by writing

$$
\begin{aligned}
N_m^2 &= \left\langle m, \alpha \right| J_+ J_- \left| m, \alpha \right\rangle \\
&= \left\langle m, \alpha \right| [J_+, J_-] + J_- J_+ \left| m, \alpha \right\rangle \\
&= m + N_{m+1}^2 \left\langle m + 1, \alpha \middle| m + 1, \alpha \right\rangle \\
&= m + N_{m+1}^2.
\end{aligned}
$$

Repeating this we find

$$
\begin{aligned}
N_{j-k}^2 &= \sum_{a=j-k}^{j} a \\
&= \sum_{n=1}^{k+1} j - k + n - 1 \\
&= \frac{k+1}{2} (2(j-k) + k) \\
&= \frac{(2j-k)(k+1)}{2} \\
&= (j - (j-k) + 1)j - \frac{1}{2}(k+1)k = \frac{1}{2}(k+1)(2j-k),
\end{aligned}
$$

or

$$N_m = \frac{1}{\sqrt{2}} \sqrt{(j - m + 1)(j + m)}.$$

From this we can infer two things about the spectrum. First, by the assumption that Hilbert space is finite dimensional, there must exist a maximal number of lowerings l. This case should satisfy

$$N_{j-l} = \frac{1}{\sqrt{2}} \sqrt{(l+1)(2j-l)} = 0.$$

As l is an integer, the possible upper bounds of the spectrum must therefore be half-integer. Next, we can identify the limits of the eigenvalue spectrum: The upper limit corresponds to the fact that $N_{j+1} = 0$. The lower limit comes from $N_{-j} = 0$, meaning the eigenstate at the bottom of the spectrum is $\left| -j, \alpha \right\rangle$. At this point we may ignore the other quantum numbers, as the representation we are working with breaks Hilbert space into invariant subspaces under SU(2). Or, rather, we will replace them simply by the value of j, which together with m specifies the state. We now know that the dimension of each invariant Hilbert space is $2j + 1$.

We can now work out the particulars of this representation by computing the matrix elements

$$
\begin{aligned}
\left\langle j, m \middle| J_3 \middle| j, m' \right\rangle &= m \delta_{m,m'}, \\
\left\langle j, m \middle| J_+ \middle| j, m' \right\rangle &= N_{m'+1} \delta_{m,m'+1} = \frac{1}{\sqrt{2}} \sqrt{(j - m')(j + m' + 1)} \delta_{m,m'+1}, \\
\left\langle j, m \middle| J_- \middle| j, m' \right\rangle &= N_{m'} \delta_{m,m'-1} = \frac{1}{\sqrt{2}} \sqrt{(j - m' + 1)(j + m')} \delta_{m,m'-1},
\end{aligned}
$$

the latter of which can be used to yield

$$\langle j, m | J_1 | j, m' \rangle = \frac{1}{2} \left(\sqrt{(j-m')(j+m'+1)} \delta_{m,m'+1} + \sqrt{(j-m'+1)(j+m')} \delta_{m,m'-1} \right),$$

$$\langle j, m | J_2 | j, m' \rangle = -\frac{i}{2} \left(\sqrt{(j-m')(j+m'+1)} \delta_{m,m'+1} - \sqrt{(j-m'+1)(j+m')} \delta_{m,m'-1} \right).$$

As an example, the representation on the $j = \frac{1}{2}$ subspace is

$$J_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad J_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad J_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This is the simplest representation of SU(2), and is therefore called the fundamental representation. Another example is found on the $j = 1$ subspace, which yields

$$J_1 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Tensor Product of SU(2) Representations   Taking the tensor product of two angular momentum Hilbert spaces corresponds to creating a new space of eigenstates of either angular momentum operator. As we know, however, it is possible to diagonalize this Hilbert space in terms of another operator, namely the total angular momentum. As an example, for two $j = \frac{1}{2}$, the total Hilbert space may be written as the tensor product of two two-dimensional subspaces, or as the direct sum of the eigenspaces of the total angular momentum. The latter has two eigenspaces, one with dimension 3 and one with dimension 1. We often denote this in the ridiculous form

$$2 \otimes 2 = 3 \oplus 1.$$

Young Tableaux   Young tableaux are a useful tool for analyzing the structure of representations. I will present it in the context of SU(3), and therefore need to talk very briefly about that group.

SU(3) is generated by eight matrices $\frac{1}{2}\lambda_a$, which are normalized such that $\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. Two of these commute, hence the group has rank 2.

Corresponding to each representation of SU(3) is a so-called conjugate representation, found by taking the complex conjugate of the first. This representation is denoted by a bar. The generators of this representation are $-\frac{1}{2}\lambda_a^\star$. This representation has not been brought up in the context of SU(2) because it is equivalent to the ones we discussed. This equivalence occurs if there exists a matrix ε which satisfies

$$-\lambda_a^\star = \varepsilon \lambda_a \varepsilon^{-1}.$$

An easy way to treat a conjugate representation with tensor notation is to treat the action of the original representation as acting on contravariant indices and the action of the conjugate representation as acting on covariant indices.

Note that to each antisymmetric rank-$N-1$ tensor in this representation there exists a vector according to

$$t_j = \varepsilon_{i_1 \dots i_{N-1} J} T^{i_1 \dots i_{N-1}},$$

hence we associate such tensors with the conjugate representation.

The next idea would be to take the tensor product of such representations. Young tableaux are useful tools for treating just this. The basic idea is to represent each index with a box, according to $\square$. As the fundamental representation acts on states with single indices, the single box thus represents the fundamental representations. Next, if you have multiple indices, symmetric indices are put in the same row and antisymmetric indices in the same column. The two look like

$$\square\!\square \ , \ \square\square .$$

Higher-rank diagrams are arranged such that the number of boxes in any row is equal to or less than the number of boxes in the rows above.

The power of the Young tableaux is in computing tensor products of representations. This is done with the following steps:

1. Draw tableaux corresponding to each representation.

2. Mark the boxes in the right tableau according to the row it is in.

3. Take one box at a time from the right tableau and attach it to the left one, making sure to respect the rules of the tableaux and not putting two boxes from the same row in the same column.

4. Discard columns of N boxes.

5. For each possible unique combination, compute the direct sum.

How many elements can there be in a column for a general $SU(N)$ tableau? The answer is $N-1$, as there is only a single rank N antisymmetric tensor - the Levi-Civita tensor - which looks the same in all frames and thus transforms analogously to a scalar.

What is the dimensionality of the product representation corresponding to each tableau? To compute this we index each box in the tableau by a row number j and a column number k. Next, for $SU(N)$ we compute the numbers

$$A_{jk} = N + k - j, \ B_{jk} = n_j + m_k + 1 - j - k$$

for every box, where n_j is the number of boxes in the row and m_k is the number of boxes in the column. For a tableau, B_{jk} can also be calculated by drawing an L with the corner in the box in question and the legs extending all the way down and to the right and counting the number of boxes in the L. By combinatorics it can somehow be shown that

$$d = \prod \frac{A_{jk}}{B_{jk}}.$$

Let us now do some examples. The first is a simple one, namely $3 \otimes 3$. Using tableaux we have

$$\square \otimes \square = \begin{matrix}\square\\\square\end{matrix} \oplus \square\square.$$

The first tableau has dimension 3 and the second part has dimension 6. The first one corresponds to the conjugate representation, hence we find $3 \otimes 3 = 6 \oplus 3^\star$. Next we study $3^\star \otimes 3$, which in tableau form is

$$\begin{matrix}\square\\\square\end{matrix} \otimes \square = \begin{matrix}\square\square\\\square\end{matrix} + \bullet,$$

where the bullet signifies an empty tableau - in other words, a scalar. We thus find $3^\star \otimes 3 = 8 \oplus 1$. Finally, let us do the more involved $3 \otimes 3 \otimes 3$. Using our previous work we find

$$\square \otimes \square \otimes \square = \square \otimes \left(\begin{matrix}\square\\\square\end{matrix} \oplus \square\square \right) = \begin{matrix}\square\square\\\square\end{matrix} \oplus \bullet \oplus \square\square\square \oplus \begin{matrix}\square\square\\\square\end{matrix}.$$

Note that the same tableau appears twice due to the direct sum. We thus find $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$.

2   Basic Concepts

The Standard Model   The standard model is a quantum field theory with the gauge group $SU(3) \times SU(2) \times U(1)$. It is comprised of

- quarks and leptons, represented by Dirac matter fields.

- gauge bosons - spin 1 bosons that mediate the fundamental forces.

- the Higgs boson, represented by a scalar field.

Elementary Particles   An elementary particle is a particle without substructure. It is these that comprise the standard model.

Quarks and Leptons   The fermions comprising the standard model come in two kinds: quarks and leptons. Each fermion also has an antiparticle. Quarks and leptons are found in six variations - so-called flavors.

The quarks have colour charge red, green or blue, which is involved in the strong interaction. In nature they are not found as free particles, but are either joined with an antiquark to create a meson, or joined with two other quarks to make a baryon. In general, combinations of quarks are called hadrons. Leptons, by contrast, have no colour charge.

Both quarks and leptons may have electric charge, and they all have spin and interact with the weak interaction. They are also divided into three so-called generations.

The Fundamental Forces   The fundamental forces are the strong and weak nuclear interactions, the electromagnetic interaction and the gravitational interactions. Only the three former have a corresponding quantum field theory, hence this discussion (and in fact the standard model itself at time of writing) will be restricted to those.

An important aspect about the strong force is the fact that its coupling constant is rolling - that is, it varies in space. It is very large at distances comparable to those of nuclei, but very small at the scale of an individual nucleon. This means, for instance, that quarks in a proton are virtually free within the proton, but that the proton itself is stable.

Internal Quantum Numbers of Antiparticles   Below we will discuss a number of internal symmetries of particles. These are all flipped in antiparticles.

Isospin   Early particle physicists, noting the very similar masses of the proton and neutron and studying nuclear energy levels, conjectured the existence of a symmetry of nuclei under swapping of the two. This symmetry would then need to be respected by the strong force. More specifically, they wrote the state of each nucleon as a proton-neutron doublet which transforms under action by SU(2). The generators of this transformation are called components of isospin. We will do a similar thing, but for quarks instead.

The Eightfold Way

Parity   Parity is a discrete symmetry which corresponds to spatial inversion. Strong and electromagnetic interactions respect parity, but weak interactions do not.

Tensor quantities can generally change sign under parity. We call scalars that preserve signs just that - scalars, whereas scalars that change sign are pseudoscalars. Vectors that change sign under parity are indeed vectors, whereas vectors that keep their sign are called axial vectors.

It turns out that the ground states of hadrons are eigenstates of the parity operator. It also turns out that the parity of fermions (taken to be positive) is opposite to that of their antiparticles, whereas they are the sames for bosons. In addition to the intrinsic parity of a particle, their orbital angular momentum also provides parity - more specifically, a factor of $(-1)^l$.

Chirality and Helicity   As has been discussed in other summaries, helicity and chirality depend on the combination of spin and momentum. It turns out that helicity is frame-dependent, but chirality is intrinsic to the particle. Furthermore, some particles are only found with certain chiralities.

Going From Particles to Antiparticles   Consider the particle and antiparticle states

$$e^{-ipx} \begin{bmatrix} 1 \\ 0 \\ \frac{p_x - ip_y}{p_0 + m} \\ -\frac{p_z}{p_0 + m} \end{bmatrix}, \; e^{ipx} \begin{bmatrix} \frac{p_z}{p_0 + m} \\ \frac{p_x + ip_y}{p_0 + m} \\ 1 \\ 0 \end{bmatrix}.$$

We will try to devise a transformation between the two. The exponent is flipped by complex conjugation, so that will have to be included. Next we need a transformation matrix which is block off-diagonal. The lower left block being

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -i\sigma^2$$

will get the job done. The upper right block should similarly be $i\sigma^2$, meaning that the total matrix should be $-i\gamma^2$. The transformation is thus

$$\Psi' = -i\gamma^2\Psi^\star.$$

This also reveals the need for complex conjugation in changing between and antiparticles.

Charge Conjugation Charge conjugation is a discrete symmetry which corresponds to changing all internal quantum numbers of a state, creating a corresponding antiparticle. It is written as

$$C\ket{p} = \ket{\bar{p}}.$$

It is respected by the strong and electromagnetic interactions, but not the weak one.

In particular, for fermions we can use the transformation rule we derived on the Dirac equation with minimal coupling. For the particle we have

$$\gamma^\mu(\partial_\mu - ieA_\mu)\Psi + im\Psi = 0.$$

For the antiparticle we have

$$-i\gamma^2(\gamma^\mu)^\star(\partial_\mu + ieA_\mu)\Psi^\star - i\gamma^2 \cdot -im\Psi^\star = -i\gamma^2(\gamma^\mu)^\star(\partial_\mu + ieA_\mu)\Psi^\star - m\gamma^2\Psi^\star = 0.$$

In the Dirac representation the only matrix that is changed by complex conjugation is γ^2, which changes sign. All the other gamma matrices anticommute, and for γ^2 we can simply reshuffle the complex conjugation to find

$$\gamma^\mu(\partial_\mu + ieA_\mu) \cdot i\gamma^2\Psi^\star + im \cdot i\gamma^2\Psi^\star = 0.$$

This implies

$$C\Psi = i\gamma^2\Psi^\star.$$

Such symmetries can be used to determine possible reactions. A trivial example is $\pi^0 \to \gamma + \gamma$, which balances charge conjugation, whereas $\pi^0 \to \gamma + \gamma + \gamma$ does not, hence only the former is allowed.

Transformation of Antiquark Isospin We know that quark doublets transform under SU(2). An antiquark doublet, however, is related to the complex conjugate of the quark doublet. The complex conjugate itself does not transform the same way as the original doublet - more specifically, we have

$$(\phi')^\star = e^{\frac{i}{2}\theta_i I_i^\star}\phi^\star,$$

which is the wrong transformation rule. We therefore need to write the antiparticle doublet as $\varepsilon\phi^\star$, and choose ε such that

$$\varepsilon e^{\frac{i}{2}\theta_i I_i^\star} = e^{-\frac{i}{2}\theta_i I_i}\varepsilon,$$

which produces the correct transformation rule. Expanding each side around the identity we find

$$\varepsilon I_i^\star = -I_i\varepsilon.$$

We can choose the fundamental representation, which leaves I_1 and I_3 invariant. Choosing $\varepsilon \propto \sigma_2$ the corresponding relations work out. This would also make the final relation work out. One choice is $\varepsilon = i\sigma_2$. In terms of the up and down quark we denote the two states

$$\phi = \begin{bmatrix} u \\ d \end{bmatrix}, \ \hat{\phi} = \begin{bmatrix} \hat{d} \\ -\hat{u} \end{bmatrix}.$$

With this we can construct mesons as eigenstates of total isospin.

CP   *CP* is the combination of charge conjugation and parity. This combination of symmetries allows for the theory to contain more reactions. For instance, consider

$$\pi^+ \to \mu^+ + \nu_\mu.$$

This reaction works provided that the muon and the neutrino have opposite helicities. A parity transformation, however, will reverse all helicities, producing a left-handed muon, which is impossible. Similarly, charge conjugation of this reaction involves a right-handed antineutrino, which is also not allowed. The combined *CP* symmetry, however, produces the reaction

$$\pi^- \to \mu^- + \bar{\nu}_\mu,$$

where the antineutrino is left-handed, which is allowed. The hope of particle physicists was that the weak force respected *CP*.

CP **and Neutral Kaons**   Requiring *CP* invariance of physics had some peculiar consequences for decays of the neutral kaon. Because the kaon is a parity eigenstate with eigenvalue -1 and has an antiparticle, requiring *CP* invariance would entail that the observed states would be

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \overline{K}^0), \ K_2 = \frac{1}{\sqrt{2}}(K^0 + \overline{K}^0),$$

with *CP* eigenvalues of 1 and -1 respectively. The kaon decays into two or three neutral pions, each contributing $C = 1$ and $P = -1$. This means that K_1 decays into two pions and K_2 to three. Because the former releases more energy, it happens faster. Thus, a beam of neutral kaons, which initially consists an even superposition of the *CP* eigenstates, will after some distance consist of only K_2. This was indeed what was found in experiments.

As an aside, this raises a question of what real particles are - after all, kaons are produced as eigenstates of strangeness, K^0 and $\overline{K}^0$, by the strong force, but decay to leave *CP* eigenstates. Which is the real particle? Being familiar with quantum mechanics, the answer is obvious when you think about it - neither! What we call particles are merely basis states in Fock space, and there is no inherent superiority in any choice of basis.

CP **Violation**   The story doesn't quite end there, though - some 2π decays were still found after a pretty long distance. This is an indication of *CP* violation.

Furthermore, the decay of the kaon can produce leptons too - one process producing an electron, and one producing a positron. *CP* violation would entail that the two should be found in equal amounts, but in fact electrons are produced slightly more often. This is fascinating - because physics does not respect *CP*, processes can exist that favor matter over antimatter. This could explain the domination of matter in the universe.

Time Reversal   Time reversal - or rather, motion reversal - is a discrete symmetry corresponding to running a process backwards. It is respected by the strong and electromagnetic interactions. It is hard to test it for the weak interaction.

CPT   *CPT* is a combination of the discrete symmetries we have studied.

We can use it to draw conclusions about time reversal. Because quantum field theory is based on *CPT* invariance and *CP* is violated by the weak interaction, we expect T to be violated as well in order for *CPT* to be respected.

Conservation Laws for Reactions   The following conservation laws apply for reactions in particle physics, in addition to kinematic laws:

- Electric charge. This is a symmetry respected by all fundamental interactions.

- Color charge. Only the strong interaction is at all concerned with it, and the strong interaction has a corresponding symmetry.

- Baryon number. All primitive vertices conserves the number of quarks, hence the number of quarks is conserved. However, quarks are only found as baryons, with quark number 3, and mesons, with quark number 0, hence we might equally well consider the number of baryons.

- Lepton number. Strong interactions do not include leptons, and electromagnetic interactions only couple lepton fields and quark fields. Weak interactions can change the type of lepton, but if a lepton goes in, another always comes out.

- Generational lepton number. In addition to the above, generational lepton numbers are conserved in most processes, as the electromagnetic force only couples leptonic fields of the same generation and the weak force does the same in most cases.

- Flavor. Strong and electromagnetic interactions are not concerned with flavor, but the weak interaction does not respect it.

- Strangeness, which is the number of strange quarks minus the number of strange antiquarks. This is not respected by weak interactions.

How do we tell which fundamental force is at play? A rule of thumb is that if photons are produced, it is the electromagnetic force, and if neutrinos are produced, it is the weak force.

The Quark Model   The quark model is a simplified model which only considers the up, down and strange quarks, combining them into a triplet which transforms under SU(3). Taking the generators to be $\frac{\lambda_a}{2}$, the generators of transformations on the antiparticle states is $-\frac{\lambda_a^\star}{2}$. We denote the particle representation as 3 and the antiparticle representation as $\bar{3}$.

3   Quantum Field Theory

A Comment on Normalization   We will be using the convention where the integral containing the fields has a division by a factor $\sqrt{2\omega_{\mathbf{k}}}$.

Lorentz Transformations and Dirac Fields   We have already established that Dirac fields transform as

$$\psi(x) \to e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x),$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. This means that the adjoint transforms according to

$$\psi^\dagger(x) \to \psi^\dagger(x)e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}^\dagger}.$$

As $\sigma_{\mu\nu}$ are not self-adjoint, $\psi^\dagger\psi$ is not a Lorentz invariant and cannot be included in any field theory. The hope is, however, that there exists some matrix M which we can put in the middle to solve that issue. For this to work we would need

$$e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}^\dagger}M = Me^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}.$$

Expanding around the identity we find

$$\sigma_{\mu\nu}^\dagger M = M\sigma_{\mu\nu}.$$

To proceed we need to know a little about the gamma matrices. It can be shown in the Dirac basis that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Because different bases are related by a unitary similarity transform, this holds in all bases. This means that

$$(\sigma^{\mu\nu})^\dagger = -\frac{i}{2}\gamma^0[\gamma^\nu, \gamma^\mu]\gamma^0 = \gamma^0\sigma^{\mu\nu}\gamma^0,$$

and $\gamma^0 M$ must commute with $\sigma_{\mu\nu}$. Two choices for M are thus γ^5 and γ^0, the latter of which we will use to define the Dirac adjoint $\overline{\psi} = \psi^\dagger\gamma^0$, which transforms the right way under Lorentz transformations.

What about forms of the type $\overline{\psi}F\psi$? We will need a basis for the space of matrices, and can choose the identity, the gamma matrices, their commutators, γ^5 and $\gamma_\mu\gamma^5$. Of these, only the index-free versions will work. To see this, consider

$$\overline{\psi}\gamma^\rho\psi \to \overline{\psi}e^{\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^\rho e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi$$

$$\approx \overline{\psi}\left(1 + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^\rho\left(1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi$$

$$\approx \overline{\psi}\left(\gamma^\rho + \frac{i}{4}\omega_{\mu\nu}[\sigma^{\mu\nu}, \gamma^\rho]\right)\psi.$$

The commutator can be simplified as

$$[\sigma^{\mu\nu}, \gamma^\rho] = \frac{i}{2} \left(\gamma^\mu[\gamma^\nu, \gamma^\rho] + [\gamma^\mu, \gamma^\rho]\gamma^\nu - \gamma^\nu[\gamma^\mu, \gamma^\rho] - [\gamma^\nu, \gamma^\rho]\gamma^\mu \right)$$

$$= \frac{i}{2} \left(\gamma^\mu(\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu) + (\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu)\gamma^\nu - \gamma^\nu(\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu) - (\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu)\gamma^\mu \right)$$

$$= \frac{i}{2} \left(2g^{\nu\rho}\gamma^\mu - 2g^{\rho\mu}\gamma^\nu + 2g^{\nu\rho}\gamma^\mu - 2g^{\rho\mu}\gamma^\nu \right)$$

$$= 2i(g^{\nu\rho}\gamma^\mu - g^{\rho\mu}\gamma^\nu).$$

Thus we have

$$\omega_{\mu\nu}[\sigma^{\mu\nu}, \gamma^\rho] = 2i\omega_{\mu\nu}(g^{\nu\rho}\gamma^\mu - g^{\rho\mu}\gamma^\nu) = -4i\omega^\rho{}_\mu\gamma^\mu$$

and

$$\overline{\psi}\gamma^\rho\psi \approx \overline{\psi}\left(\delta^\rho_\mu + \omega^\rho{}_\mu\right)\gamma^\mu\psi,$$

which is the same transformation rule as for a vector. Similarly objects with more indices transform like tensors, justifying the use of Lorentz indices despite the gamma matrices being the same in all frames.

Energy Change for Klein-Gordon Fields For a Klein-Gordon field the process with the non-relativistic string can be repeated to find

$$\mathcal{H} = \frac{1}{2} \int \mathrm{d}^3\mathbf{p}\, \omega_\mathbf{p}(a^\dagger_\mathbf{p}a_\mathbf{p} + a_\mathbf{p}a^\dagger_\mathbf{p}).$$

Using the commutation relations we find

$$\left[\mathcal{H}, a^\dagger_\mathbf{k}\right] = \frac{1}{2} \int \mathrm{d}^3\mathbf{p}\, \omega_\mathbf{p}(a^\dagger_\mathbf{p}\left[a_\mathbf{p}, a^\dagger_\mathbf{k}\right] + \left[a^\dagger_\mathbf{p}, a^\dagger_\mathbf{k}\right]a_\mathbf{p} + a_\mathbf{p}\left[a^\dagger_\mathbf{p}, a^\dagger_\mathbf{k}\right] + \left[a_\mathbf{p}, a^\dagger_\mathbf{k}\right]a^\dagger_\mathbf{p})$$

$$= \int \mathrm{d}^3\mathbf{p}\, \delta^3(\mathbf{p} - \mathbf{k})\omega_\mathbf{p}a^\dagger_\mathbf{p}$$

$$= \omega_\mathbf{k}a^\dagger_\mathbf{k},$$

and similarly $\left[\mathcal{H}, a^\dagger_\mathbf{k}\right] = -\omega_\mathbf{k}a^\dagger_\mathbf{k}$. Thus these operators raise and lower the energies of eigenstates of the Hamiltonian, further justifying our interpretations.

The Charge of Charged Klein-Gordon Fields The Lagrangian for the charged Klein-Gordon field is

$$\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi.$$

Corresponding to this Lagrangian is a symmetry $\delta\phi = i\alpha\phi$ which leaves the action unchanged. The corresponding Noether current is

$$j^\mu = i(\phi^\star\partial^\mu\phi - \phi\partial^\mu\phi^\star).$$

Using the field expansion

$$\phi = \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3\mathbf{p}\, \frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(a_\mathbf{p}e^{-ipx} + b^\star_\mathbf{p}e^{ipx}\right)$$

with distinct amplitudes a and b we find

$$\rho = -\frac{1}{2(2\pi)^3} \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{k}\, \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}}\left(\left(a^\star_\mathbf{p}e^{ipx} + b_\mathbf{p}e^{-ipx}\right)\left(-a_\mathbf{k}e^{-ikx} + b^\star_\mathbf{k}e^{ikx}\right) - \left(a_\mathbf{p}e^{-ipx} + b^\star_\mathbf{p}e^{ipx}\right)\left(a^\star_\mathbf{k}e^{ikx} - b_\mathbf{k}e^{-ikx}\right)\right)$$

$$= -\frac{1}{2(2\pi)^3} \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{k}\, \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}}\left(-a^\star_\mathbf{p}a_\mathbf{k}e^{i(p-k)x} + b_\mathbf{p}b^\star_\mathbf{k}e^{i(k-p)x} - a_\mathbf{p}a^\star_\mathbf{k}e^{i(k-p)x} + b^\star_\mathbf{p}b_\mathbf{k}e^{i(p-k)x}\right).$$

The quantized version is

$$\rho = -\frac{1}{2(2\pi)^3} \int d^3\mathbf{p} \int d^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}} \left(-a_\mathbf{p}^\dagger a_\mathbf{k} e^{i(p-k)x} + b_\mathbf{p} b_\mathbf{k}^\dagger e^{i(k-p)x} - a_\mathbf{p} a_\mathbf{k}^\dagger e^{i(k-p)x} + b_\mathbf{p}^\dagger b_\mathbf{k} e^{i(p-k)x} \right).$$

Using the commutation relations we find

$$\rho = -\frac{1}{(2\pi)^3} \int d^3\mathbf{p} \int d^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{\omega_\mathbf{p}}} \left(b_\mathbf{p}^\dagger b_\mathbf{k} e^{i(p-k)x} - a_\mathbf{p}^\dagger a_\mathbf{k} e^{i(p-k)x} \right).$$

Computing the total charge amounts to integrating over space, adding a Dirac delta in momentum space and finally yielding

$$Q = \int d^3\mathbf{p} \, a_\mathbf{p}^\dagger a_\mathbf{k} - b_\mathbf{p}^\dagger b_\mathbf{k}.$$

A Trick for Computing Propagators   A neat trick to find propagators, at least for the standard theories we will consider, is to write the Lagrangian as

$$\mathcal{L} = \overline{\phi} A \phi + \dots$$

where A is some operator and the dots are total derivatives. The bar signifies some relevant conjugate field. For real fields you want a prefactor of $\frac{1}{2}$ as well. The propagator is the inverse of A.

Let us show two examples. First, for a charged scalar field we have

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi = \partial_\mu \left(\phi^\dagger \partial^\mu \phi \right) - \phi^\dagger \left(\Box + m^2 \right) \phi,$$

hence $A = -\Box - m^2$. In momentum space we thus find the familiar formula

$$i\Delta_\mathrm{F}(p) = \frac{i}{p^2 - m^2}.$$

Next, for a Dirac field we have

$$\mathcal{L} = i\overline{\psi} \slashed{\partial} \psi - m\overline{\psi}\psi = \overline{\psi} \left(i\slashed{\partial} - m \right).$$

Again, in momentum space we have

$$(\slashed{p} - m) S_\mathrm{F,}(p) = 1.$$

Using gamma gymnastics we know that $(\slashed{p} - m)(\slashed{p} + m) = p^2 - m^2$, hence we find

$$iS_\mathrm{F,}(p) = \frac{i(\slashed{p} + m)}{p^2 - m^2}.$$

Two-Body Scattering Processes   Consider a scattering process involving two incoming particles. We generally have

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \left(\prod_{i>2} \int d^3\mathbf{p}_i \frac{1}{2(2\pi)^3 E_i} \right) (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i>2} p_i \right) |M|^2$$

for a scattering process starting with two particles. In particular, for two outgoing particles we have

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{1}{4(2\pi)^2} \int d^3\mathbf{p}_3 \, d^3\mathbf{p}_4 \frac{1}{E_3 E_4} \delta^4 \left(p_1 + p_2 - \sum_{i>2} p_i \right) |M|^2.$$

Let us first consider the prefactor. We have

$$p_1 \cdot p_2 = \frac{1}{2}((p_1 + p_2)^2 - m_1^2 - m_2^2) = \frac{1}{2}(s - m_1^2 - m_2^2),$$

10

hence

$$(p_1 \cdot p_2)^2 = \frac{1}{4}(s^2 - 2sm_1^2 - 2sm_2^2 + 2m_1^2 m_2^2 + m_1^4 + m_2^4).$$

Specializing to the case of the incoming particles having identical masses m we find

$$(p_1 \cdot p_2)^2 = \frac{1}{4}(s^2 - 2sm^2 - 2sm^2 + 2m^4 + m^4 + m^4) = \frac{1}{4}(s^2 - 4sm^2 + 4m^4),$$

hence the prefactor is

$$\frac{1}{4\sqrt{\frac{1}{4}(s^2 - 4sm^2 + 4m^4) - m^4}} = \frac{1}{2\sqrt{s^2 - 4sm^2}}.$$

Specializing to the center-of-momentum frame we have

$$p_1 = (E, \mathbf{p}), \ p_2 = (E, -\mathbf{p}),$$

and thus $s = 4E^2$

Let us first consider the prefactor. We have

$$p_1 \cdot p_2 = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2,$$

hence

$$(p_1 \cdot p_2)^2 = E_1^2 E_2^2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2.$$

and

$$\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= E_1^2 E_2^2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2 - (E_1^2 - |\mathbf{p}_1|^2)(E_2^2 - |\mathbf{p}_2|^2) \\
&= (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2 + E_1^2 |\mathbf{p}_2|^2 + E_2^2 |\mathbf{p}_1|^2 - |\mathbf{p}_1|^2 |\mathbf{p}_2|^2 \\
&= E_1^2 E_2^2 \left(\left(\frac{\mathbf{p}_1}{E_1} \cdot \frac{\mathbf{p}_2}{E_2} \right)^2 - 2\frac{\mathbf{p}_1}{E_1} \cdot \frac{\mathbf{p}_2}{E_2} + \left| \frac{\mathbf{p}_1}{E_1} \right|^2 + \left| \frac{\mathbf{p}_2}{E_2} \right|^2 - \left| \frac{\mathbf{p}_1}{E_1} \right|^2 \left| \frac{\mathbf{p}_2}{E_2} \right|^2 \right) \\
&= E_1^2 E_2^2 \left(\left(\frac{\mathbf{p}_1}{E_1} \cdot \frac{\mathbf{p}_2}{E_2} \right)^2 + \left| \frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_2}{E_2} \right|^2 - \left| \frac{\mathbf{p}_1}{E_1} \right|^2 \left| \frac{\mathbf{p}_2}{E_2} \right|^2 \right).
\end{aligned}$$

In particular, in the center-of-momentum frame we find

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = E_1^2 E_2^2 \left| \frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_2}{E_2} \right|^2 = E_1^2 E_2^2 v_{\text{rel}}^2.$$

In this frame we also have $s = (E_1 + E_2)^2$. Now, integrating out one 3-momentum we find

$$\sigma = \frac{1}{64\pi^2 E_1 E_2 v_{\text{rel}}} \int \mathrm{d}^3 \mathbf{p}_3 \frac{1}{E_3(\mathbf{p}_3) E_4(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)} \delta\left(E_1 + E_2 - E_3(\mathbf{p}_3) - E_4(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)\right) |M|^2$$

$$= \frac{1}{64\pi^2 E_1 E_2 v_{\text{rel}}} \int \mathrm{d}^3 \mathbf{p}_3 \frac{1}{E_3(\mathbf{p}_3) E_4(-\mathbf{p}_3)} \delta\left(E_1 + E_2 - E_3(\mathbf{p}_3) - E_4(-\mathbf{p}_3)\right) |M|^2,$$

where we have made the dependence on the various energies explicit. The energy is a function of the magnitude of the momentum only, hence we can convert to spherical coordinates to find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 E_1 E_2 v_{\text{rel}}} \int_0^\infty \mathrm{d}p_3 \, p_3^2 \frac{1}{E_3(p_3) E_4(p_3)} \delta\left(\sqrt{s} - E_3(p_3) - E_4(p_3)\right) |M|^2.$$

Take $f(p_3) = \sqrt{s} - E_3(p_3) - E_4(p_3)$. We then find

$$\frac{\mathrm{d}f}{\mathrm{d}p_3} = -\frac{2p_3}{2\sqrt{m_3^2 + p_3^2}} - \frac{2p_3}{2\sqrt{m_4^2 + p_3^2}} = -p_3 \left(\frac{1}{E_3} + \frac{1}{E_4} \right).$$

To complete this calculation we will need the values of the outgoing energies and momenta given the incoming ones. Conservation of energy implies

$$\sqrt{m_3^2 + p_3^2} + \sqrt{m_4^2 + p_3^2} = \sqrt{s},$$

which squares to

$$m_3^2 + p_3^2 = s + m_4^2 + p_3^2 - 2\sqrt{sm_4^2 + sp_3^2}.$$

Rearranging we find

$$E_4 = \sqrt{m_4^2 + p_3^2} = \frac{1}{2\sqrt{s}}(s + m_4^2 - m_3^2).$$

We then have

$$E_3 = \sqrt{s} - E_3 = \frac{1}{2\sqrt{s}}(2s - s - m_4^2 + m_3^2) = \frac{1}{2\sqrt{s}}(2s + m_3^2 - m_4^2).$$

The value of p_3 is given by

$$m_4^2 + p_3^2 = \frac{1}{4s}(s^2 + 2sm_4^2 - 2sm_3^2 - 2m_4^2 m_3^2 + m_4^4 + m_3^4),$$

and thus

$$p_3 = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2sm_4^2 - 2sm_3^2 - 2m_4^2 m_3^2 + m_4^4 + m_3^4}.$$

We then have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_1 E_2 v_{\text{rel}}} \frac{p_3^2}{E_3(p_3) E_4(p_3)} \frac{1}{p_3 \left(\frac{1}{E_3} + \frac{1}{E_4}\right)} |M|^2$$

$$= \frac{1}{64\pi^2 E_1 E_2 \left|\frac{p_1}{E_1} + \frac{p_1}{E_2}\right|} \frac{p_3}{(E_3 + E_4)} |M|^2$$

$$= \frac{1}{64\pi^2 \sqrt{s}(E_1 + E_2)} \frac{p_3}{p_1} |M|^2$$

$$= \frac{1}{64\pi^2 s} \frac{p_3}{p_1} |M|^2.$$

In particular, for elastic scattering processes we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |M|^2.$$

The Electromagnetic Field and its Complications For a general gauge the Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

for some control parameter ξ. The second term, called the gauge fixing term, turns out to be really useful for doing calculations. To show that we expand the Lagrangian as

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu)\right)$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu(g^{\nu\rho}\partial^\mu A_\rho - g^{\mu\rho}\partial^\nu A_\rho) - \frac{1}{\xi}(\partial^\mu A_\mu)(\partial^\nu A_\nu)\right)$$

$$= \frac{1}{2}\left(A_\nu \partial_\mu(g^{\nu\rho}\partial^\mu A_\rho - g^{\mu\rho}\partial^\nu A_\rho) + \frac{1}{\xi}A_\mu \partial^\mu \partial^\nu A_\nu\right) - \frac{1}{2}\partial_\mu\left(A_\nu(g^{\nu\rho}\partial^\mu A_\rho - g^{\mu\rho}\partial^\nu A_\rho)\right) - \frac{1}{2\xi}\partial^\mu(A_\mu \partial^\nu A_\nu).$$

Ignoring the total derivatives we are left with

$$\mathcal{L} = \frac{1}{2} A_\nu \left(g^{\nu\rho} \partial_\mu \partial^\mu A_\rho - g^{\mu\rho} \partial_\mu \partial^\nu A_\rho + \frac{1}{\xi} \partial^\mu \partial^\nu A_\mu \right)$$

$$= \frac{1}{2} A_\nu \left(g^{\nu\mu} \Box A_\mu - \partial^\mu \partial^\nu A_\mu + \frac{1}{\xi} \partial^\mu \partial^\nu A_\mu \right)$$

$$= \frac{1}{2} A_\nu \left(g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right) A_\mu,$$

hence we have found a matrix operator $A^{\mu\nu}$. Its inverse must be the corresponding inverse matrix. Calling the inverse $D_{\nu\rho}$ we require in Fourier space

$$\left(-g^{\mu\nu} p^2 + \left(1 - \frac{1}{\xi} \right) p^\mu p^\nu \right) D_{\nu\rho} = \delta^\mu_\rho.$$

We can expand D in terms of the metric and the momentum to find

$$\left(-g^{\mu\nu} p^2 + \left(1 - \frac{1}{\xi} \right) p^\mu p^\nu \right) (a g_{\nu\rho} + b p_\nu p_\rho) = -a p^2 \delta^\mu_\rho - b p^2 p^\mu p_\rho + \left(1 - \frac{1}{\xi} \right) a p^\mu p_\rho + b \left(1 - \frac{1}{\xi} \right) p^2 p^\mu p_\rho.$$

A first choice is $a = -\frac{1}{p^2}$. Given that, we would be left with

$$-b p^2 + \left(1 - \frac{1}{\xi} \right) \left(-\frac{1}{p^2} + b p^2 \right) = -\frac{1}{\xi} b p^2 - \frac{1}{p^2} \left(1 - \frac{1}{\xi} \right) = 0,$$

with solution

$$b = \frac{1}{(p^2)^2} (1 - \xi).$$

Thus we have

$$i D_{\mu\nu} = -\frac{i}{p^2} \left(g_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right).$$

At this point we may simplify the propagator by simply choosing $\xi = 1$, netting us

$$i D_{\mu\nu} = -\frac{i g_{\mu\nu}}{p^2}.$$

Dirac Fields For Dirac fields we will use the normalization

$$\bar{u} u = 2mc, \ \bar{v} v = -2mc$$

and the completeness relation

$$\sum_s u_s \bar{u}_s = \not{p} + m, \ \sum_s v_s \bar{v}_s = \not{p} - m.$$

Quantum Electrodynamics The Dirac field is invariant under a U(1) symmetry $\psi \to e^{-ieQ\theta} \psi$. The normalization in the exponent contains a parameter e to be discussed, a parameter Q specific to each particle and a parameter θ accounting for the rest. By convention we take $Q = -1$ for the electron. The corresponding Noether current is

$$j^\mu = eQ \bar{\psi} \gamma^\mu \psi.$$

Extending θ to be a function on spacetime breaks this symmetry for the standard Dirac Lagrangian, however. By extending the derivative operator to

$$D_\mu = \partial_\mu + ie A_\mu$$

and requiring $A_\mu \to A_\mu + \partial_\mu \theta$, however, we have a Lagrangian that respects this symmetry. This is called a gauge symmetry, and corresponds to a gauge theory realized by gauge bosons. We identify this field with the

4-potential and the corresponding transformation as a gauge transformation, adding its contribution to the total Lagrangian. To include the gauge field in the Lagrangian, we will need the field strength tensor. We already know how to get it, but I note the result

$$
\begin{aligned}
[D_\mu, D_\nu] &= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu) \\
&= ie(\partial_\mu A_\nu + A_\mu \partial_\nu - \partial_\nu A_\mu - A_\nu \partial_\mu) \\
&= ie(\partial_\mu(A_\nu) + A_\nu \partial_\mu + A_\mu \partial_\nu - \partial_\nu(A_\mu) - A_\mu \partial_\nu - A_\nu \partial_\mu) \\
&= ieF_{\mu\nu},
\end{aligned}
$$

which will be useful later.

When comparing to the non-interacting case, we find an interaction term

$$
\mathcal{L}_{\text{int}} = -eQ\overline{\psi}\gamma^\mu\psi A_\mu.
$$

This is the interaction term defining quantum electrodynamics. It is a so-called gauge theory based on the U(1) gauge group.

Let us now consider the corresponding interaction vertex. Note that a single of these does not represent a physical process. To see this, consider the spontaneous emission of a photon from a particle. Denoting the momenta of the particle before and after the emission as p and k, and the photon momentum as q, we have

$$
q^2 = 2p \cdot q.
$$

Letting the photon have frequency ω and the particle have energy E we find

$$
\omega^2 - \mathbf{q}^2 = 2(E\omega - \mathbf{p} \cdot \mathbf{q}).
$$

By the properties of the scalar product we thus find

$$
E\omega - |\mathbf{p}||\mathbf{q}| \leq \frac{1}{2}(\omega^2 - \mathbf{q}^2) \leq E\omega + |\mathbf{p}||\mathbf{q}|.
$$

Assuming the photon to be physical, the center part must be zero. But if the particle is massive, the left-hand side is positive, creating a contradiction. The only way this can be resolved is for the created photon to not be physical, and merely mediate between two vertices.

In terms of spins, one sums over incoming spins and averages over outgoing spins.

Casimir's Trick   In QED, at least, we will need to compute expressions of the form

$$
\sum_s (\overline{u}_1 F u_2)^\star \overline{u}_1 G u_2
$$

for some spinor u and some matrices F and G. To calculate this we use the fact that

$$
(\overline{u}_1 F u_2)^\star = u_2^\dagger F^\dagger \gamma_0 u_1 = \overline{u}_2 \gamma_0 F^\dagger \gamma_0 u_1.
$$

Introducing $F^\ddagger = \gamma_0 F^\dagger \gamma_0$ we have $(\overline{u}_1 F u_2)^\star = \overline{u}_2 F^\ddagger u_1$. We can now simplify the above as

$$
\begin{aligned}
\sum_s (\overline{u}_1 F u_2)^\star \overline{u}_1 G u_2 &= \sum_s \overline{u}_{2,a} F^\ddagger_{ab} u_{1,b} \overline{u}_{1,c} G_{cd} u_{2,d} \\
&= F^\ddagger_{ab} G_{cd} \sum_{s_1} \sum_{s_2} u_{1,b} \overline{u}_{1,c} u_{2,d} \overline{u}_{2,a} \\
&= F^\ddagger_{ab} G_{cd} \sum_{s_1} (u_1 \overline{u}_1)_{bc} \sum_{s_2} (u_2 \overline{u}_2)_{da}.
\end{aligned}
$$

In the two spin sums we have elements of $u\overline{u}$, which are given by the normalization, hence

$$
\begin{aligned}
\sum_s (\overline{u}_1 F u_2)^\star \overline{u}_1 G u_2 &= F^\ddagger_{ab} G_{cd} (\slashed{p}_1 + \eta_1 m_1)_{bc} (\slashed{p}_2 + \eta_2 m_1)_{da} \\
&= \text{tr}\Big((\slashed{p}_1 + \eta_1 m_1) G (\slashed{p}_2 + \eta_2 m_1) F^\ddagger\Big),
\end{aligned}
$$

where the η are signs which depend on the relevant fermions being particles or antiparticles.

We will also need a mixing term of the form

$$\sum_s (\overline{u}_1 A u_2 \overline{u}_3 B u_4)^{\star} \overline{u}_3 C u_2 \overline{u}_1 D u_4 = \sum_s \overline{u}_{2,a} A^{\ddagger}_{ab} u_{1,b} \overline{u}_{4,c} B^{\ddagger}_{cd} u_{3,d} \overline{u}_{3,e} C_{ef} u_{2,f} \overline{u}_{1,g} D_{gh} u_{4,h}$$

$$= A^{\ddagger}_{ab} B^{\ddagger}_{cd} C_{ef} D_{gh} \sum_{s_1} \sum_{s_2} \sum_{s_3} \sum_{s_4} u_{1,b} \overline{u}_{1,g} u_{2,f} \overline{u}_{2,a} u_{3,d} \overline{u}_{3,e} u_{4,h} \overline{u}_{4,c}$$

$$= A^{\ddagger}_{ab} B^{\ddagger}_{cd} C_{ef} D_{gh} (\slashed{p}_1 + \eta_1 m_1)_{bg} (\slashed{p}_2 + \eta_2 m_2)_{fa} (\slashed{p}_3 + \eta_3 m_3)_{de} (\slashed{p}_4 + \eta_4 m_4)_{hc}$$

$$= \mathrm{tr}\Big(A^{\ddagger}(\slashed{p}_1 + \eta_1 m_1) D (\slashed{p}_4 + \eta_4 m_4) B^{\ddagger}(\slashed{p}_3 + \eta_3 m_3) C (\slashed{p}_2 + \eta_2 m_2) \Big).$$

These are the formulae we need, and we will handle them using trace theorems.

Fundamental QED Processes The fundamental QED processes are the simplest processes we can make with QED vertices. By their nature, these are processes with two incoming and two outgoing particles. These are good for demonstrating Feynman diagrams as there are no undetermined momenta in the diagram, hence we don't have to perform any integrals. The first is electron-electron scattering, represented by
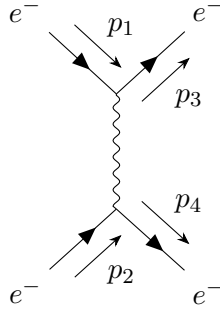

Figure 1: Electron-electron scattering Feynman diagram.

The Feynman rules dictate that

$$iM = \overline{u}(p_3) ie\gamma^{\mu} u(p_1) \frac{-ig_{\mu\nu}}{(p_1 - p_3)^2} \overline{u}(p_4) ie\gamma^{\nu} u(p_2) = ie^2 \frac{\overline{u}(p_3)\gamma^{\mu}u(p_1)\overline{u}(p_4)\gamma_{\mu}u(p_2)}{(p_1 - p_3)^2}.$$

This isn't quite correct, however - as

$$iM = \langle p_3 p_4 | S | p_1 p_2 \rangle$$

and $|p_3 p_4\rangle = -|p_4 p_3\rangle$, M must be antisymmetric under exchange of the outgoing momenta. This amounts to an equivalent diagram adding to M, which means that we must subtract the two. We then find

$$iM = ie^2 \left(\frac{\overline{u}(p_3)\gamma^{\mu}u(p_1)\overline{u}(p_4)\gamma_{\mu}u(p_2)}{(p_1 - p_3)^2} - \frac{\overline{u}(p_4)\gamma^{\mu}u(p_1)\overline{u}(p_3)\gamma_{\mu}u(p_2)}{(p_1 - p_4)^2} \right).$$

Using the Mandelstam variables we write this as

$$iM = ie^2 \left(\frac{T_1}{t} - \frac{T_2}{u} \right).$$

In a final amplitude, how do we handle spin? We typically assume the incoming particles to not be spin-polarized, hence we average over those. For the outgoing spins, we sum over them as we take them to be distinct contributions to the amplitude. As the spin channels can be separated out, this is all done on the level of matrix elements, i.e. in $|M|^2$.

Seeking the spin average of $|M|^2$ we need to calculate spin sums of the products of the Ts, which is handled using Casimir's trick. In the high energy regime, we can neglect the electron mass to find

$$\sum_s T_1^{\star} T_1 = \mathrm{tr}\Big(\slashed{p}_1 \gamma^{\mu} \slashed{p}_3 \gamma^{\nu} \Big) \mathrm{tr}\Big(\slashed{p}_4 \gamma_{\mu} \slashed{p}_2 \gamma_{\nu} \Big)$$

$$= p_{1,\rho} p_{3,\sigma} \mathrm{tr}(\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}) p_2^{\alpha} p_4^{\beta} \mathrm{tr}(\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\gamma_{\nu})$$

$$= 16 p_{1,\rho} p_{3,\sigma} p_2^{\alpha} p_4^{\beta} (g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\sigma\mu})(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\beta\mu})$$

$$= 16 (p_1^{\mu}p_3^{\nu} - p_1^{\sigma}p_{3,\sigma}g^{\mu\nu} + p_1^{\nu}p_3^{\mu})(p_{2,\mu}p_{4,\nu} - p_{2,\beta}p_4^{\beta}g_{\mu\nu} + p_{2,\nu}p_{4,\mu}).$$

The collision is satisfies the laws of relativistic dynamics, hence $p_1 \cdot p_2 = p_3 \cdot p_4$ as well as $p_1 \cdot p_4 = p_3 \cdot p_2$, and

$$\sum_s T_1^\star T_1 = 16((p_1 \cdot p_2)^2 - (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)^2 - (p_1 \cdot p_3)(p_2 \cdot p_4) + 4(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_3)(p_2 \cdot p_4)$$
$$+ (p_1 \cdot p_4)^2 - (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_2)^2)$$
$$= 32((p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2).$$

Similarly, by exchanging p_3 and p_4 we find

$$\sum_s T_2^\star T_2 = 32((p_1 \cdot p_2)^2 + (p_1 \cdot p_3)^2).$$

Finally we have the mixed term, given by

$$\sum_s T_1^\star T_2 = \operatorname{tr}\left(\gamma^\mu \slashed{p}_3 \gamma_\nu \slashed{p}_2 \gamma_\mu \slashed{p}_4 \gamma^\nu \slashed{p}_1\right)$$
$$= \operatorname{tr}\left(\slashed{p}_4 \gamma^\nu \slashed{p}_1 \gamma^\mu \slashed{p}_3 \gamma_\nu \slashed{p}_2 \gamma_\mu\right)$$
$$= -2\operatorname{tr}\left(\slashed{p}_4 \slashed{p}_3 \gamma^\mu \slashed{p}_1 \slashed{p}_2 \gamma_\mu\right)$$
$$= -8\operatorname{tr}\left(\slashed{p}_4 \slashed{p}_3 p_1 \cdot p_2\right)$$
$$= -32(p_1 \cdot p_2)^2.$$

This is real, and therefore equal to its complex conjugate. Finally, neglecting masses we have

$$\overline{M}^2 = 32e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + \frac{2s^2}{tu}\right).$$

Non-Abelian Gauge Theory   The gauge theory of QED is an example of an Abelian gauge theory. For the non-Abelian case, consider a representation R of SU(N) with dimension d_R acting on some multiplet Ψ. To add gauge theory, we suppose that the components of Ψ have some internal structure, either that of bosonic or fermionic fields, all with the same mass. As we saw for the QED case, any constant unitary operator U leaves the laws of physics invariant. Allowing U to have spacetime dependence, however, messes with this, in particular the invariance of derivatives. To remedy this, we will introduce a covariant derivative

$$D_\mu = \partial_\mu + igT_a A_\mu^a = \partial_\mu + ig\mathbf{A}_\mu,$$

which contains some coupling constant g, the generators T_a of SU(N) and the gauge fields A_μ^a. This derivative has a matrix structure, and in the final expression, this structure has been baked into the field. The laws of physics will be invariant if

$$D_\mu \Psi \to U D_\mu' \Psi$$

under a transformation. We have

$$D_\mu'(U\Psi) = (\partial_\mu + igT_a(A')_\mu^a)\left(e^{-igT_a\theta^a}\Psi\right)$$
$$= -igT_a\partial_\mu\theta^a e^{-igT_a\theta^a}\Psi + e^{-igT_a\theta^a}\partial_\mu\Psi + igT_a(A')_\mu^a e^{-igT_a\theta^a}\Psi$$
$$= U\partial_\mu\Psi + \left(igT_a(A')_\mu^a - igT_a\partial_\mu\theta^a\right)U\Psi.$$

For the transformation rule to be fulfilled, we therefore require

$$\left(igT_a(A')_\mu^a - igT_a\partial_\mu\theta^a\right)U = igUT_aA_\mu^a.$$

There are two ways to write the condition. The first is representation dependent, and found by combining the generators and the gauge fields to obtain matrix structure, yielding

$$igA_\mu'U + \partial_\mu U = igUA_\mu,$$

with solution

$$\mathbf{A}_\mu' = U\mathbf{A}_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger.$$

The other is to write the above without the matrix structure in the fields as

$$T_a(A')^a_\mu = UT_aU^\dagger A^a_\mu - \frac{i}{g}(igT_a\partial_\mu\theta^a)UU^\dagger = UT_aU^\dagger A^a_\mu + T_a\partial_\mu\theta^a.$$

We can expand the first term according to

$$UT_aU^\dagger \approx \left(1 - igT_b\theta^b\right)T_a\left(1 + igT_c\theta^c\right)$$
$$\approx T_a + ig\left(T_aT_c\theta^c - T_bT_a\theta^b\right)$$
$$= T_a + ig[T_a, T_b]\theta^b$$
$$= T_a - gf_{abc}T_c\theta^b,$$

hence

$$T_a(A')^a_\mu = T_aA^a_\mu - gf_{abc}T_c\theta^bA^a_\mu + T_a\partial_\mu\theta^a$$
$$= T_a\left(A^a_\mu + gf_{bca}\theta^bA^c_\mu + \partial_\mu\theta^a\right)$$

and

$$(A')^a_\mu = A^a_\mu + gf_{bca}\theta^bA^c_\mu + \partial_\mu\theta^a.$$

We can invoke the adjoint representation, here denoted by elements with small t, to find

$$(A')^a_\mu = A^a_\mu - ig(t_b)_{ac}\theta^bA^c_\mu + \partial_\mu\theta^a.$$

This is interesting, as it means that under global transformations, gauge fields always transform according to the adjoint representation.

The gauge fields should be included in the Lagrangian as well. We choose the same form for the Lagrangian as we did for U(1), but will need an expression for the field strength. We showed a commutation relation previously, and we take that to be defining for the field strength. That is,

$$igT_aF^a_{\mu\nu} = [D_\mu, D_\nu].$$

One of the derivative products is given by

$$(\partial_\mu + igT_aA^a_\mu)(\partial_\nu + igT_bA^b_\nu) = \partial_\mu\partial_\nu + ig(\partial_\mu T_aA^a_\nu + T_bA^b_\mu\partial_\nu) - g^2T_aT_bA^a_\mu A^b_\nu,$$

hence

$$[D_\mu, D_\nu] = igT_a(\partial_\mu A^a_\nu + A^a_\mu\partial_\nu - \partial_\nu A^a_\mu - A^a_\nu\partial_\mu) - g^2A^a_\mu A^b_\nu[T_a, T_b]$$
$$= igT_a(\partial_\mu(A^a_\nu) + A^a_\nu\partial_\mu + A^a_\mu\partial_\nu - \partial_\nu(A^a_\mu) - A^a_\mu\partial_\nu - A^a_\nu\partial_\mu) - ig^2A^a_\mu A^b_\nu f_{abc}T_c$$
$$= igT_a\left(\partial_\mu(A^a_\nu) - \partial_\nu(A^a_\mu) - gf_{bca}A^b_\mu A^c_\nu\right),$$

and

$$F^a_{\mu\nu} = \partial_\mu(A^a_\nu) - \partial_\nu(A^a_\mu) - gf_{bca}A^b_\mu A^c_\nu.$$

Alternatively we may write this as

$$\mathbf{F}_{\mu\nu} = T_aF^a_{\mu\nu} = \partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu) + ig[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

Let us now consider its gauge transformation properties, best demonstrated with the representation-dependent formalism. We have

$$\mathbf{F}'_{\mu\nu} = \partial_\mu\left(U\mathbf{A}_\nu U^\dagger + \frac{i}{g}(\partial_\nu U)U^\dagger\right) - \partial_\nu\left(U\mathbf{A}_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger\right) + ig\left[U\mathbf{A}_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger, U\mathbf{A}_\nu U^\dagger + \frac{i}{g}(\partial_\nu U)U^\dagger\right].$$

Woah, that is a lot of terms. We can eliminate a few of them, however, using the following results:

$$\left[U\mathbf{A}_\mu U^\dagger, U\mathbf{A}_\nu U^\dagger\right] = U[\mathbf{A}_\mu, \mathbf{A}_\nu]U^\dagger,$$
$$\partial_\mu(UU^\dagger) = 0 \implies \partial_\mu(U^\dagger) = -U^\dagger\partial_\mu(U)U^\dagger.$$

Next we have

$$\partial_\mu \left(U\mathbf{A}_\nu U^\dagger + \frac{i}{g}(\partial_\nu U)U^\dagger \right) = \partial_\mu(U)\mathbf{A}_\nu U^\dagger + U\partial_\mu(\mathbf{A}_\nu)U^\dagger - U\mathbf{A}_\nu U^\dagger \partial_\mu(U)U^\dagger + \frac{i}{g}\left((\partial_\mu\partial_\nu U)U^\dagger + (\partial_\nu U)(\partial_\mu U^\dagger) \right)$$

$$= \partial_\mu(U)\mathbf{A}_\nu U^\dagger + U\partial_\mu(\mathbf{A}_\nu)U^\dagger - U\mathbf{A}_\nu U^\dagger \partial_\mu(U)U^\dagger + \frac{i}{g}\left((\partial_\mu\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger \right),$$

$$\left[U\mathbf{A}_\mu U^\dagger, (\partial_\nu U)U^\dagger \right] = U\mathbf{A}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger U\mathbf{A}_\mu U^\dagger$$

$$= U\mathbf{A}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)\mathbf{A}_\mu U^\dagger,$$

$$\left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger \right] = (\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger.$$

We can now look at the terms with and without gauge fields separately. The first add to

$$\partial_\mu(U)\mathbf{A}_\nu U^\dagger + U\partial_\mu(\mathbf{A}_\nu)U^\dagger - U\mathbf{A}_\nu U^\dagger\partial_\mu(U)U^\dagger - \left(\partial_\nu(U)\mathbf{A}_\mu U^\dagger + U\partial_\nu(\mathbf{A}_\mu)U^\dagger - U\mathbf{A}_\mu U^\dagger\partial_\nu(U)U^\dagger \right)$$

$$- \left(U\mathbf{A}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)\mathbf{A}_\mu U^\dagger - \left(U\mathbf{A}_\nu U^\dagger(\partial_\mu U)U^\dagger - (\partial_\mu U)\mathbf{A}_\nu U^\dagger \right) \right)$$

$$= U\left(\partial_\mu(\mathbf{A}_\nu) - \partial_\nu(\mathbf{A}_\mu) \right)U^\dagger.$$

Naturally the commutator also produces a term $U[\mathbf{A}_\mu, \mathbf{A}_\nu]U^\dagger$. The other terms add up to

$$\frac{i}{g}\left((\partial_\mu\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger - (\partial_\nu\partial_\mu U)U^\dagger + (\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger \right) - \frac{i}{g}\left((\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger - (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger \right),$$

which add up to zero. Thus we have

$$\mathbf{F}'_{\mu\nu} = U\mathbf{F}_{\mu\nu}U^\dagger.$$

Constructing $\mathbf{F}^\dagger_{\mu\nu}\mathbf{F}^{\mu\nu}$ guarantees Lorentz invariance, but this object still has matrix structure, and it is not gauge invariant. We can, however, construct $\mathrm{tr}\left(\mathbf{F}^\dagger_{\mu\nu}\mathbf{F}^{\mu\nu} \right)$, with tracing over the multiplet space, which is gauge invariant. We can write this as

$$\mathrm{tr}\left(\mathbf{F}^\dagger_{\mu\nu}\mathbf{F}^{\mu\nu} \right) = (F^a_{\mu\nu})^\star F^{b,\mu\nu}\,\mathrm{tr}(T_a T_b)$$

$$= C(F^a_{\mu\nu})^\star F^{b,\mu\nu}\delta_{ab}$$

$$= C(F^a_{\mu\nu})^\star F^{a,\mu\nu},$$

hence we choose the free-field Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^\star F^{a,\mu\nu}.$$

In general there could be mass terms of the form $A^a_\mu A^{\mu,a}$ in the Lagrangian, but it turns out that these are not gauge invariant. Thus, as long as the gauge symmetry is exact, the gauge bosons must be massless.

The propagator for this kind of gauge field is generally not defined unless you specify a gauge. The Lagrangian has terms of the same form as in QED, but with some self-interaction terms as well. The free part is the same, however, meaning we can use a gauge fixing term

$$\mathcal{L} = -\frac{1}{2\xi}(\partial_\mu A^\mu_a)^\dagger(\partial_\nu A^\nu_a).$$

Now the propagator will be the same as in QED, as will other Feynman rules.

Intrinsic to this kind of field theory is interactions of the gauge field with itself. These may be both cubic and quartic. Introducing $E_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}$ the Feynman rules are

$$gf_{ab\overline{c}}\left(p^\alpha E_{\alpha\mu\nu\lambda} + k^\alpha E_{\alpha\nu\lambda\mu} + q^\alpha E_{\alpha\lambda\mu\nu} \right)$$

for the cubic vertex and

$$-ig^2\left(f_{ab\overline{e}}f_{cde}E_{\mu\nu\lambda\rho} + f_{ac\overline{e}}f_{dbe}E_{\mu\lambda\rho\nu} + f_{ad\overline{e}}f_{bce}E_{\mu\rho\nu\lambda} \right)$$

for the quartic vertex.

Let us now study the interactions of the gauge fields with matter. Taking the fermionic case first, the field multiplet Ψ is described by

$$\mathcal{L} = \sum_{i=1}^{d_R} i\overline{\Psi}^i \slashed{\partial} \Psi^i - m\overline{\Psi}^i \Psi^i.$$

In terms of matrix notation we can write this as

$$\mathcal{L} = i\overline{\Psi} \slashed{\partial} \Psi - m\overline{\Psi}\Psi.$$

The Dirac matrices work on the individual fields in the multiplet, hence in this notation they are diagonal matrices. Replacing the partial derivative with the covariant derivative ensures gauge symmetry. Emerging from this Lagrangian is now an interaction term

$$\mathcal{L}_{\text{int}} = -g\overline{\Psi}\slashed{A}\Psi = -g\overline{\Psi}\gamma^\mu T_a \Psi A_\mu^a.$$

This adds a vertex rule $-ig\gamma^\mu (T_a)_{ij}$, where ij are indices specifying the interacting fields.

Next we consider interactions with scalars. The Lagrangian is

$$
\begin{aligned}
\mathcal{L} &= (D_\mu \Phi)^\dagger D^\mu \Phi - M^2 \Phi^\dagger \Phi \\
&= (\partial_\mu \Phi)^\dagger \partial^\mu \Phi + ig\left((\partial_\mu \Phi)^\dagger T_a A^{a,\mu}\Phi - \Phi^\dagger T_a^\dagger (A_\mu^a)^\dagger \partial^\mu \Phi\right) + g^2 \Phi^\dagger T_a^\dagger (A_\mu^a)^\dagger T_b A^{b,\mu}\Phi - M^2 \Phi^\dagger \Phi \\
&= (\partial_\mu \Phi)^\dagger \partial^\mu \Phi + ig\left((\partial_\mu \Phi)^\dagger T_a A^{a,\mu}\Phi - \Phi^\dagger T_a^\dagger (A_\mu^a)^\dagger \partial^\mu \Phi\right) + g^2 g_{\mu\nu}\Phi^\dagger T_a^\dagger (A_\mu^a)^\dagger T_b A_\nu^b \Phi - M^2 \Phi^\dagger \Phi.
\end{aligned}
$$

We thus see that there are two kinds of vertices, one with two particles and one gauge bosons, and one with two particles and two gauge bosons. The corresponding Feynman rules are $-ig(p + p')_\mu (T_a)_{ij}$ and $ig^2 g_{\mu\nu}(T_a T_b + T_b T_a)_{ij}$. Note that the structure of the generators dictates which particles can interact in both cases.

Rather than the contents of the multiplet, the conserved charges of such theories are in fact closely linked to the generators of the gauge symmetry.

Quantum Chromodynamics Quantum chromodynamics is the theory of the strong nuclear force. It is a quantized SU(3) gauge theory. The number 3 is due to the presence of 3 color charges. We choose SU(3) rather than SU(2), despite the latter having a triplet representation because we need a triplet representation for both the quarks and antiquarks, but SU(2) has only real representations.

The gauge bosons come in eight forms. They are called gluons. Unlike QED, QCD has multiple kinds of vertices as the gluons carry color charge. The one including fermions is exemplified as in figure 2, and symbolizes an incoming up quark with blue colour charge interacting via a gluon and leaving the interaction with red colour charge. The gluon then carries one positive unit of red colour charge and one unit of negative red colour charge.
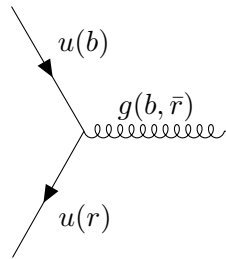

$u(b)$

$g(b, \bar{r})$

$u(r)$

Figure 2: An example of a strong interaction vertex involving a fermion.

In addition there are two pure gluon vertices, shown in figure 3.

The Weak Force The weak force started as an attempt to describe beta decay. The first attempt was to formulate a theory involving an interaction vertex with four fermions, where currents interacted. This turned out not to work, however, as it did not contain parity violation. The fact that neutrinos are only found with certain chiralities made clear the need to introduce this concept as well. Both of these could be remedied by adding $\gamma^\mu (1 - \gamma^5)$ to the Dirac field bilinears. This will bring us closer, but still not leave us with a renormalizable theory, as the coupling constant would have mass dimension -2.
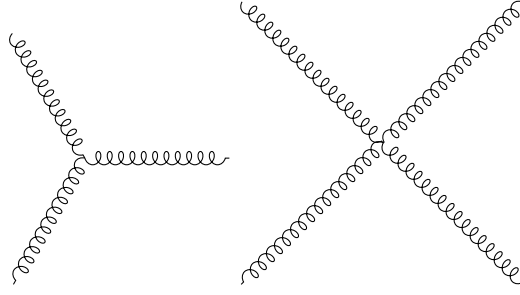
Figure 3: The two strong interaction vertices only involving gluons.

To remedy the above, one could strengthen the analogy to QED by adding a propagator to the interaction. This would remove the four-fermion vertex, replacing it with vertices involving a vector boson. This boson must be heavy, as it has never been observed. The propagator generally takes the form

$$\frac{-ig_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}}{q^2 - M^2 + iM\Gamma},$$

where Γ is the decay width of the boson.

One important question is why the weak force is so weak. One way to understand this is to compare matrix elements with and without the mediating boson. With the vertex factor

$$-\frac{ig_{\mu\nu}}{2\sqrt{2}}\gamma^\mu(1 - \gamma^5),$$

the latter is of the form

$$iM = i\frac{G}{\sqrt{2}}\left(\overline{u}_3\gamma^\mu(1 - \gamma^5)u_1\right)\left(\overline{u}_4\gamma^\mu(1 - \gamma^5)u_2\right),$$

while the former is of the form

$$iM = i\left(\frac{g}{\sqrt{2}}\overline{u}_3\gamma^\mu\frac{1}{2}(1 - \gamma^5)u_1\right)\frac{1}{M^2 - q^2}\left(\frac{g}{\sqrt{2}}\overline{u}_4\gamma^\mu\frac{1}{2}(1 - \gamma^5)u_2\right),$$

and in the low-energy regime

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M^2},$$

hence the heaviness of the gauge boson makes the effective interaction constant so small.

Weak Interaction Vertices   The weak interaction is mediated by a charged W boson and a neutral Z boson. The corresponding vertices are shown in figure 4.
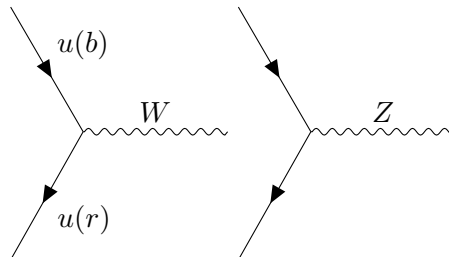
Figure 4: The fundamental vertices of the weak interaction.

Cabibo Mixing   While lepton generations generally do not mix, the same cannot be said for quark families. This is explained through Cabibo mixing. The idea is that while the up, charm and top quarks couple to the gauge bosons, the others do not. Instead primed versions of these quarks, related to the original ones by a unitary transformation, are coupled to the gauge bosons. The matrix is called the mixing matrix, and is parametrized by three angles and one phase. The consequence of this is that generation-mixing processes are suppressed.

The Neutral Gauge Boson The discussion has so far centered on the charged gauge boson, but there is a neutral one as well, with a vertex factor

$$-\frac{ig_{\mu\nu}}{2\sqrt{2}}\gamma^\mu(c_V^f - c_A^f\gamma^5).$$

The factors c^f depend on the involved particles, and the two are generally not equal, meaning that the neutral boson mixes left and right chiral fields. Its propagator is the same as for the charged boson.

A tricky part about this boson being neutral is that it will in certain cases compete with electromagnetic contributions, which are also mediated by a neutral boson. As the latter is much stronger at low energies, there are some difficulties in observing the Z boson.

Spontaneous Symmetry Breaking Consider a theory for a charged scalar field described by

$$\mathcal{L} = (\partial_\mu\phi)^\star\partial^\mu\phi - V(\phi^\star\phi).$$

Evidently the Lagrangian has a $U(1)$ symmetry, but depending on the form of V there might be more to this. For instance, consider

$$V(\phi^\star\phi) = \mu^2\phi^\star\phi + \lambda(\phi^\star\phi)^2.$$

For this Lagrangian to produce reasonable dynamics, λ must be positive. There is no such restriction on μ^2, however, beyond it being real. In fact, if it is negative we find that V has a minimum for a solution $\langle\phi\rangle$ that satisfies

$$|\langle\phi\rangle| = \sqrt{-\frac{\mu^2}{2\lambda}} = \frac{v}{\sqrt{2}}.$$

Let us now consider the dynamics of the field $\phi = \frac{1}{\sqrt{2}}(v+H(x)+i\zeta(x))$, where H and ζ are fields representing small deviations from the minimum. Taking the equalities below to be up to total derivatives we find

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2}\left(\partial_\mu H - i\partial_\mu\zeta\right)\left(\partial^\mu H + i\partial^\mu\zeta\right) - \frac{\mu^2}{2}\left((v+H)^2 + \zeta^2\right) - \frac{\lambda}{4}\left((v+H)^2 + \zeta^2\right)^2 \\
=& \frac{1}{2}\left(\partial_\mu H\partial^\mu H + \partial_\mu\zeta\partial^\mu\zeta\right) - \frac{\mu^2}{2}\left(v^2 + H^2 + 2vH + \zeta^2\right) - \frac{\lambda}{4}\left(v^2 + H^2 + 2vH + \zeta^2\right)^2 \\
=& \frac{1}{2}\left(\partial_\mu H\partial^\mu H + \partial_\mu\zeta\partial^\mu\zeta\right) + \frac{\lambda v^2}{2}\left(H^2 + 2vH + \zeta^2\right) \\
& - \frac{\lambda}{4}\left(6v^2 H^2 + 4v^3 H + 2v^2\zeta^2 + H^4 + 4vH^3 + 2H^2\zeta^2 + 4vH\zeta^2 + \zeta^4\right) \\
=& \frac{1}{2}\left(\partial_\mu H\partial^\mu H + \partial_\mu\zeta\partial^\mu\zeta\right) + \frac{\lambda}{2}\left(v^2 H^2 + 2v^3 H + v^2\zeta^2 - 3v^2 H^2 - 2v^3 H - v^2\zeta^2 - 2vH^3\right. \\
& \left. - H^2\zeta^2 - 2vH\zeta^2 - \frac{1}{2}H^2 - \frac{1}{2}\zeta^4\right) \\
=& \frac{1}{2}\left(\partial_\mu H\partial^\mu H + \partial_\mu\zeta\partial^\mu\zeta\right) - \frac{\lambda}{2}\left(2v^2 H^2 + 2vH^3 + H^2\zeta^2 + 2vH\zeta^2 + \frac{1}{2}H^4 + \frac{1}{2}\zeta^4\right) \\
=& \frac{1}{2}\left(\partial_\mu H\partial^\mu H + \partial_\mu\zeta\partial^\mu\zeta\right) - \lambda\left(v^2 H^2 + vH^3 + \frac{1}{2}H^2\zeta^2 + vH\zeta^2 + \frac{1}{4}H^4 + \frac{1}{4}\zeta^4\right) \\
=& \frac{1}{2}\left(\partial_\mu H\partial^\mu H + \partial_\mu\zeta\partial^\mu\zeta\right) - \lambda\left(v^2 H^2 + vH(H^2 + \zeta^2) + \frac{1}{4}\left(H^2 + \zeta^2\right)^2\right)
\end{aligned}
$$

The interaction term now contains two factors of H that break the symmetry. Note also the appearance of a mass term for H, but not ζ.

What we see here is that while the Lagrangian itself may respect the symmetry, a particular realization of the symmetry may approach a minimum, thereby breaking the symmetry. Because the system finds minima of V spontaneously, this mechanism is called spontaneous symmetry breaking.

Let us study a non-Abelian theory in a similar way. We can construct such a theory by using $\phi^\dagger\phi$ in the previous Lagrangian. Considering fluctuations around a minimum where only ϕ_1 is non-zero we find that the other fields appear in the exact same fashion, meaning that the full $SU(N)$ symmetry has been broken to $SU(N-1)$.

We have thus far considered global symmetries, but what about local ones? Let us consider a system with a local U(1) symmetry. The Lagrangian is

$$\mathcal{L} = (D_\mu \phi)^\star D^\mu \phi - V(\phi^\star \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$

Supposing that the potential has a minimum of the same form as previously seen, the derivative terms will produce one factor

$$\frac{1}{2}(D_\mu v)^\star D^\mu v = \frac{1}{2} v^2 e^2 A^\mu A_\mu.$$

This is a mass term for the gauge fields, produced by the breaking of the local symmetry.

It turns out that the ζ field can be eliminated for the case of local symmetries. To see this, consider the higher-order version

$$\phi = \frac{1}{\sqrt{2}}(v + H)e^{-i\frac{\zeta}{v}}$$

of the previously studied fluctuation. Because we have a local symmetry, the exponential factor can be removed, eliminating ζ from the Lagrangian. The field we removed in fact corresponds to the Goldstone boson. This is one way to view the gauge known as the unitary gauge.

Goldstone's Theorem Goldstone's theorem states that if a system breaks a symmetry with n generators to one with m generators, the theory has $n - m$ massless states. These are called Goldstone bosons.

Electroweak Unification Physicists in the 20th century wanted to unite the electromagnetic and weak interactions into a single theory. The disparity in strength had already been explained by the difference in masses of the mediators, but this still left the question of why the photon was massless but the W and Z bosons were not. Furthermore, the QED vertex has pure vector structure, whereas the weak vertex also has an axial vector part. These were two challenges in uniting the two.

To remedy this, we absorb the VA structure into the spinor, only considering the left-hand projection of the spinor from now on (this is the projection that appears in the interaction vertex). For antiparticles, we similarly consider the right-hand projection (which is still termed left-hand, as its absence from the Dirac sea behaves as being left-handed). Looking at the Dirac structure of a weak interaction vertex we have

$$\begin{aligned} j_\mu^- &= \bar{\nu}\gamma_\mu \frac{1 - \gamma^5}{2} e \\ &= \bar{\nu}\gamma_\mu \left(\frac{1 - \gamma^5}{2}\right)^2 e \\ &= \bar{\nu}\frac{1 + \gamma^5}{2}\gamma_\mu \frac{1 - \gamma^5}{2} e \\ &= \bar{\nu}_\mathrm{L}\gamma_\mu e_\mathrm{L}. \end{aligned}$$

This is a vector coupling, but only between fields of the same handedness. The electromagnetic interaction can similarly be written as

$$j_\mu^{\mathrm{em}} = -\left(\bar{e}_\mathrm{R}\gamma_\mu e_\mathrm{R} + \bar{e}_\mathrm{L}\gamma_\mu e_\mathrm{L}\right),$$

with no cross terms as γ^5 squares to 1. The factor of -1 comes from the electron charge.

We can introduce a more compact notation by introducing the doublet χ_L of left-handed fields, with neutrinos or up-type quarks on the top. Introducing the raising and lowering matrices $\tau^\pm$, obtained from the Pauli matrices, the two charged weak interaction vertices are given by

$$j_\mu^\pm = \bar{\chi}_\mathrm{L}\gamma_\mu \tau^\pm \chi_\mathrm{L}.$$

At this moment it might be tempting to introduce the neutral term

$$j_\mu^3 = \frac{1}{2}\bar{\chi}_\mathrm{L}\gamma_\mu \tau^3 \chi_\mathrm{L},$$

and claim that this corresponds to the neutral current. This does not work, however, because we know that the Z boson mixes chiralities, but this term does not. However, we see that the Pauli matrices generate some symmetry on this doublet. We elevate this to be a gauge symmetry by writing the interaction term as

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}g\overline{\chi}_{\text{L}}\gamma^{\mu}\tau^{a}W_{\mu}^{a}\chi_{\text{L}}.$$

The generators of the symmetry are the Pauli matrices, and we take them to be operators of a so-called weak isospin.

What about right-handed particles and left-handed antiparticles? They are left untouched by this gauge, as they are placed in an isospin singlet. What does this mean? A good question, to be sure.

To unify the theories, we are left with the two neutral gauge bosons, each corresponding to a neutral gauge field. The idea now is to introduce a new U(1) symmetry generated by $\frac{1}{2}Y$, with a corresponding gauge field B_{μ}. The corresponding conserved charge is termed hypercharge. In order for hypercharge to be conserved, particles in the same doublet must have the same hypercharge. The above then implies that

$$Y = \alpha Q + \beta I_{\text{W}}^{3}.$$

As we know there are three gauge fields connected to hypercharge, we surmise that

$$B_{\mu} = A_{\mu}\cos(\theta_{\text{W}}) - Z_{\mu}\sin(\theta_{\text{W}}), \ W_{\mu}^{3} = A_{\mu}\sin(\theta_{\text{W}}) + Z_{\mu}\cos(\theta_{\text{W}}),$$

where A_{μ} and Z_{μ} are the gauge fields corresponding to the photon and the Z boson and we have introduced a weak mixing angle θ_{W}. What we are left with now is a $\text{SU}(2) \times \text{U}(1)$ gauge group, as the two sets of generators commute.

Let us now return to the electron-neutrino pair. Requiring the two to have the same hypercharge, we find

$$-\alpha - \frac{1}{2}\beta = \frac{1}{2}\beta,$$

hence we take $Y = 2(Q - I_{\text{W}}^{3})$ by convention. Certainly, if this is the case, then the corresponding current is

$$\begin{aligned}
j_{\mu}^{Y} &= 2\left(j_{\mu}^{\text{em}} - j_{\mu}^{3}\right) \\
&= -2\left(\overline{e}_{\text{L}}\gamma_{\mu}e_{\text{L}} + \overline{e}_{\text{R}}\gamma_{\mu}e_{\text{R}}\right) - \left(\overline{\nu}_{\text{L}}\gamma_{\mu}\nu_{\text{L}} - \overline{e}_{\text{L}}\gamma_{\mu}e_{\text{L}}\right) \\
&= -2\overline{e}_{\text{R}}\gamma_{\mu}e_{\text{R}} - \overline{\chi}_{\text{L}}\gamma_{\mu}\chi_{\text{L}}.
\end{aligned}$$

We now have what we need to construct the Dirac field part of the Lagrangian without mass fields, which is given by

$$\mathcal{L} = i\left(\overline{\chi}_{\text{L}}\gamma^{\mu}D_{\mu}\chi_{\text{L}} + \overline{\chi}_{\text{R}}\gamma^{\mu}D_{\mu}\chi_{\text{R}}\right).$$

This doesn't quite work, however, as we are still writing the Lagrangian using gauge symmetry. However, for the gauge bosons to be massive, gauge symmetry must be violated. The answer is to have a mechanism of spontaneous symmetry breaking. This is the Higgs mechanism.

The Higgs Mechanism and Electroweak Unification To provide Higgsness to the system we add a scalar doublet to the system, called the Higgs field, which transforms under the SU(2) part of the gauge group. We use the potential

$$V(x) = \mu^{2}x + \lambda x^{2},$$

and introduce $v = \frac{\mu^{2}}{2\lambda}$. This adds coupling to the bosons of the weak force and allows for them to be massive. At the same time the gauge bosons of the U(1) part will be left out to keep the photon massless.

Choosing a particular minimum we are left with four degrees of freedom - three in a phase transformation and one in a shift. A gauge transformation will remove the former, leaving only the latter. This one appears with a mass terms, and turns out to correspond to the actual Higgs boson. This is analogous to the result from the unitary gauge, and is how this gauge manifests in this case.

In addition to the mass terms, we will also pick up self-interaction terms of the gauge bosons.

Tying Electroweak Unification Together   The purpose of this section will be to re-examine the total Lagrangian with the Higgs field, fermion fields and four gauge fields to rediscover all the important bits of the electromagnetic and weak forces. Note that we will only assume the existence of particular kinds of terms and the existence of a SU(2) × U(1) symmetry. The fields we have are the doublet $\psi_{\ell,\mathrm{L}}$, containing the left-handed neutrino and lepton fields, the right-handed lepton fields ℓ_{R}, the field ϕ which has the broken symmetry, and the gauge fields.

First, consider the gauge bosons. Their masses after spontaneous symmetry breaking originate on terms of the form

$$
\begin{aligned}
(D_\mu\phi)^\dagger D^\mu\phi =& \frac{1}{2}\left(\left(\frac{ig}{2}\sigma^a W_\mu^a + ig'Y_\phi B_\mu\right)\begin{bmatrix}0\\v\end{bmatrix}\right)^\dagger\left(\left(\frac{ig}{2}\sigma^a W^{a,\mu} + ig'Y_\phi B^\mu\right)\begin{bmatrix}0\\v\end{bmatrix}\right)\\
=& \frac{1}{2}\left(\frac{ig}{2}\left(W_\mu^1\begin{bmatrix}v\\0\end{bmatrix} + W_\mu^2\begin{bmatrix}-iv\\0\end{bmatrix} + W_\mu^3\begin{bmatrix}0\\-v\end{bmatrix}\right) + ig'Y_\phi B_\mu\begin{bmatrix}0\\v\end{bmatrix}\right)^\dagger\\
&\cdot\left(\frac{ig}{2}\left(W^{1,\mu}\begin{bmatrix}v\\0\end{bmatrix} + W^{2,\mu}\begin{bmatrix}-iv\\0\end{bmatrix} + W^{3,\mu}\begin{bmatrix}0\\-v\end{bmatrix}\right) + ig'Y_\phi B_\mu\begin{bmatrix}0\\v\end{bmatrix}\right)\\
=& \frac{v^2g^2}{8}\left(W_\mu^1 W_{1,\mu} + W_\mu^2 W_{2,\mu}\right) + \frac{v^2}{8}\left(2g'Y_\phi B_\mu - gW_\mu^3\right)\left(2g'Y_\phi B^\mu - gW^{3,\mu}\right)\\
=& \frac{v^2g^2}{8}\left(W_\mu^1 W_{1,\mu} + W_\mu^2 W_{2,\mu}\right) + \frac{v^2}{8}\begin{bmatrix}B_\mu & W_\mu^3\end{bmatrix}\begin{bmatrix}4Y_\phi^2(g')^2 & -2Y_\phi gg'\\-2Y_\phi gg' & g^2\end{bmatrix}\begin{bmatrix}B^\mu\\W^{3,\mu}\end{bmatrix},
\end{aligned}
$$

where we have introduced the hypercharge Y_ϕ of the lower component of the Higgs field. The masses of the charged W bosons are then identified as $\frac{vg}{2}$. Note that the actual bosons are $W^\pm = \frac{1}{\sqrt{2}}(W^1 \pm iW^2)$.

To identify the masses of the neutral bosons, we need to diagonalize the matrix we find in the above. We do that by writing

$$
Z_\mu = \cos(\theta_{\mathrm{W}})W_\mu^3 - \sin(\theta_{\mathrm{W}})B_\mu, \quad A_\mu = \sin(\theta_{\mathrm{W}})W_\mu^3 + \cos(\theta_{\mathrm{W}})B_\mu,
$$

which are the so-called mass eigenstates. We can invert the above to

$$
W_\mu^3 = \cos(\theta_{\mathrm{W}})Z_\mu + \sin(\theta_{\mathrm{W}})A_\mu, \quad B_\mu = -\sin(\theta_{\mathrm{W}})Z_\mu + \cos(\theta_{\mathrm{W}})A_\mu.
$$

The eigenvalues of the mass matrix satisfy

$$
(\lambda - 4Y_{\phi^2}(g')^2)(\lambda - g^2) - 4Y_{\phi^2}g^2(g')^2 = 0,
$$

to which zero is clearly a solution. This means that the system retains a U(1) symmetry after symmetry breaking, meaning that all fields have a U(1) charge Q. The fact that this symmetry remains that Q may be identified with the electric charge. Because the symmetry corresponding to this charge is unbroken by the Higgs field, we must have $Q_\phi = 0$. We use the convention

$$
Q = Y + I_{\mathrm{W}}^3.
$$

For this to be true we must have $Y_\phi = \frac{1}{2}$.

Associating the massless gauge boson with A_μ, the requirement that this be an eigenvector with eigenvalue zero we find

$$
(g')^2\cos(\theta_{\mathrm{W}}) - gg'\sin(\theta_{\mathrm{W}}) = 0,
$$

with solution

$$
\tan(\theta_{\mathrm{W}}) = \frac{g'}{g}.
$$

The other mass eigenvalue is $\frac{v^2(g^2+(g')^2)}{8}$, which is the mass of the Z boson.

Lepton masses originate from terms of the form

$$
y_\ell\overline{\psi}_{\ell,\mathrm{L}}\phi\ell_{\mathrm{R}} + \mathrm{h.\ c.}
$$

ℓ is now the field for a particular fermion. Spontaneous symmetry breaking breaks this to

$$
\frac{y_\ell}{\sqrt{2}}(v\overline{\ell}_{\mathrm{L}}\ell_{\mathrm{R}} + \overline{\ell}_{\mathrm{L}}h\ell_{\mathrm{R}}) + \mathrm{h.c.}
$$

The first, together with its hermitian conjugate, becomes a mass term with mass $\frac{y_\ell v}{\sqrt{2}}$, while the second becomes an interaction term between the lepton and the Higgs boson.

Why do we need fermion mass terms to arise from spontaneous symmetry breaking? The answer is in the nature of the gauge group - because standard Dirac mass terms couple left-to-right, such mass terms will not be uncharged under the SU(2) part of the gauge group. Also, because the gauge group transforms between neutrino and lepton fields, which have very different masses, one could not possibly expect gauge group invariance. The only way to preserve that is with the Higgs mechanism.

Couplings to gauge bosons originate in terms of the form

$$i\overline{\psi}D_\mu\gamma^\mu\psi.$$

We take the fermion field to be an eigenstate of weak isospin and hypercharge. Expanding the part with the neutral gauge field we find

$$\mathcal{L} \supset i\overline{\psi}\left(igT_3 W^3_\mu + ig'YB_\mu\right)\gamma^\mu\psi = -\overline{\psi}\left(gT_3 W^3_\mu + g'YB_\mu\right)\gamma^\mu\psi.$$

Expressed in terms of the physical gauge fields we have

$$\mathcal{L} \supset -\overline{\psi}\left(gT_3(\cos(\theta_W)Z_\mu + \sin(\theta_W)A_\mu) + g'Y(-\sin(\theta_W)Z_\mu + \cos(\theta_W)A_\mu)\right)\gamma^\mu\psi.$$

Collecting the photon coupling terms we find

$$\mathcal{L} \supset -A_\mu\overline{\psi}\left(g\sin(\theta_W)T_3 + g'\cos(\theta_W)Y\right)\gamma^\mu\psi$$
$$= -A_\mu g\sin(\theta_W)\overline{\psi}\left(T_3 + Y\right)\gamma^\mu\psi.$$

This correspond to the electromagnetic interaction vertex provided we make the identification $e = g\sin(\theta_W)$.

Couplings to gauge bosons originate in terms of the form

$$i\overline{\psi}_{\ell,L}D_\mu\gamma^\mu\psi_{\ell,L} + i\overline{\ell}_R D_\mu\gamma^\mu\ell_R.$$

Let us focus on the couplings to the neutral bosons. First we establish that all left-handed fermions have charge -1, and thus hypercharge $Y = -\frac{1}{2}$, whereas neutrons have $Q = 0$ and $Y = -\frac{1}{2}$, and finally right-handed leptons have hypercharge -1. We find

$$\mathcal{L} \supset -\overline{\psi}_{\ell,L}\gamma^\mu\left(\frac{1}{2}g\sigma^3 W^3_\mu + g'YB_\mu\right)\psi_{\ell,L} - \overline{\ell}_R\gamma^\mu\left(g'B_\mu Y\right)\ell_R$$

$$= -\begin{bmatrix}\overline{\nu}_{\ell,L} & \overline{\ell}_L\end{bmatrix}\gamma^\mu\begin{bmatrix}\frac{1}{2}gW^3_\mu + Y_\nu g'B_\mu & 0 \\ 0 & -\frac{1}{2}gW^3_\mu + Y_{\ell,L}g'B_\mu\end{bmatrix}\begin{bmatrix}\nu_{\ell,L} \\ \ell_L\end{bmatrix} - Y_{\ell,R}g'\overline{\ell}_R\gamma^\mu B_\mu\ell_R$$

$$= -\overline{\nu}_{\ell,L}\gamma^\mu\left(\frac{g}{2}W^3_\mu + Y_\nu g'B_\mu\right)\nu_{\ell,L} - \overline{\ell}_L\gamma^\mu\left(-\frac{g}{2}W^3_\mu + Y_{\ell,L}g'B_\mu\right)\ell_L - Y_{\ell,R}g'\overline{\ell}_R\gamma^\mu B_\mu\ell_R$$

$$= -\overline{\nu}_{\ell,L}\gamma^\mu\left(\frac{g}{2}(\cos(\theta_W)Z_\mu + \sin(\theta_W)A_\mu) + Y_\nu g'(-\sin(\theta_W)Z_\mu + \cos(\theta_W)A_\mu)\right)\nu_{\ell,L}$$

$$\quad - \overline{\ell}_L\gamma^\mu\left(-\frac{g}{2}(\cos(\theta_W)Z_\mu + \sin(\theta_W)A_\mu) + Y_{\ell,L}g'(-\sin(\theta_W)Z_\mu + \cos(\theta_W)A_\mu)\right)\ell_L$$

$$\quad - Y_{\ell,R}g'\overline{\ell}_R\gamma^\mu(-\sin(\theta_W)Z_\mu + \cos(\theta_W)A_\mu)\ell_R.$$

Collecting interactions with the photon field first we find

$$\mathcal{L} \supset -\overline{\nu}_{\ell,L}\gamma^\mu A_\mu\left(\frac{g}{2}\sin(\theta_W) + Y_\nu g'\cos(\theta_W)\right)\nu_{\ell,L}$$

$$\quad - \overline{\ell}_L\gamma^\mu A_\mu\left(-\frac{g}{2}\sin(\theta_W) + Y_{\ell,L}g'\cos(\theta_W)\right)\ell_L$$

$$\quad - Y_{\ell,R}g'\overline{\ell}_R A_\mu\gamma^\mu\cos(\theta_W)\ell_R$$

$$= -Q_\nu e\overline{\nu}_{\ell,L}\gamma^\mu A_\mu\nu_{\ell,L} - eQ_\ell\overline{\ell}_L\gamma^\mu A_\mu\ell_L - eQ_\ell\overline{\ell}_R A_\mu\gamma^\mu\ell_R,$$

which are exactly the interaction terms we expect. Next, the interaction terms with the Z boson are given by

$$\mathcal{L} \supset -\overline{\nu}_{\ell,L}Z_\mu\gamma^\mu\left(\frac{g}{2}\cos(\theta_W) - Y_\nu g'\sin(\theta_W)\right)\nu_{\ell,L}$$

$$\quad - \overline{\ell}_L Z_\mu\gamma^\mu\left(-\frac{g}{2}\cos(\theta_W) - Y_{\ell,L}g'\sin(\theta_W)\right)\ell_L$$

$$\quad + Y_{\ell,R}g'\sin(\theta_W)\overline{\ell}_R Z_\mu\gamma^\mu\ell_R.$$

We can rewrite this somewhat by using

$$\frac{1}{\cos(\theta_\mathrm{W})}\sqrt{1+\left(\frac{g'}{g}\right)^2}.$$

Extracting a factor of $\frac{g}{\cos(\theta_\mathrm{W})}$ we find

$$
\begin{aligned}
\mathcal{L} \supset\ & -\frac{g}{\cos(\theta_\mathrm{W})}\bar{\nu}_{\ell,\mathrm{L}}Z_\mu\gamma^\mu\left(\frac{1}{2}\cos^2(\theta_\mathrm{W})-Y_\nu\frac{g'}{g}\sin(\theta_\mathrm{W})\cos(\theta_\mathrm{W})\right)\nu_{\ell,\mathrm{L}} \\
& -\frac{g}{\cos(\theta_\mathrm{W})}\bar{\ell}_\mathrm{L}Z_\mu\gamma^\mu\left(-\frac{1}{2}\cos^2(\theta_\mathrm{W})-Y_{\ell,\mathrm{L}}\frac{g'}{g}\sin(\theta_\mathrm{W})\cos(\theta_\mathrm{W})\right)\ell_\mathrm{L} \\
& +\frac{g}{\cos(\theta_\mathrm{W})}\frac{g'}{g}\sin(\theta_\mathrm{W})\cos(\theta_\mathrm{W})Y_{\ell,\mathrm{R}}\bar{\ell}_\mathrm{R}Z_\mu\gamma^\mu\ell_\mathrm{R} \\
=\ & \sqrt{g^2+(g')^2}Z_\mu\left(-\bar{\nu}_{\ell,\mathrm{L}}\gamma^\mu\left(\frac{1}{2}-\left(\frac{1}{2}+Y_\nu\right)\sin^2(\theta_\mathrm{W})\right)\nu_{\ell,\mathrm{L}}-\bar{\ell}_\mathrm{L}\gamma^\mu\left(-\frac{1}{2}-\left(Y_{\ell,\mathrm{L}}-\frac{1}{2}\right)\sin^2(\theta_\mathrm{W})\right)\ell_\mathrm{L}+\sin^2(\theta_\mathrm{W})Y_{\ell,\mathrm{R}}\bar{\ell}_\mathrm{R} \\
=\ & \frac{\sqrt{g^2+(g')^2}}{2}Z_\mu\left(-\bar{\nu}_\ell\gamma^\mu\left(\frac{1}{2}-\left(\frac{1}{2}+Y_\nu\right)\sin^2(\theta_\mathrm{W})\right)(1-\gamma_5)\nu_\ell-\bar{\ell}\gamma^\mu\left(-\frac{1}{2}-\left(Y_{\ell,\mathrm{L}}-\frac{1}{2}\right)\sin^2(\theta_\mathrm{W})\right)(1-\gamma_5)\ell\right. \\
& +\sin^2(\theta_\mathrm{W})Y_{\ell,\mathrm{R}}\bar{\ell}\gamma^\mu(1+\gamma_5)\ell) \\
=\ & \frac{\sqrt{g^2+(g')^2}}{2}Z_\mu\left(-\bar{\nu}_\ell\gamma^\mu\left(\frac{1}{2}-\left(\frac{1}{2}+Y_\nu\right)\sin^2(\theta_\mathrm{W})\right)(1-\gamma_5)\nu_\ell\right. \\
& \left.-\bar{\ell}\gamma^\mu\left(\left(-\frac{1}{2}-\left(Y_{\ell,\mathrm{L}}-\frac{1}{2}\right)\sin^2(\theta_\mathrm{W})\right)(1-\gamma_5)-\sin^2(\theta_\mathrm{W})Y_{\ell,\mathrm{R}}(1+\gamma_5)\right)\ell\right) \\
=\ & \frac{\sqrt{g^2+(g')^2}}{2}Z_\mu\left(-\bar{\nu}_\ell\gamma^\mu\left(\frac{1}{2}-\left(\frac{1}{2}+Y_\nu\right)\sin^2(\theta_\mathrm{W})\right)(1-\gamma_5)\nu_\ell\right. \\
& \left.-\bar{\ell}\gamma^\mu\left(\left(-\frac{1}{2}-\left(Y_{\ell,\mathrm{L}}-\frac{1}{2}\right)\sin^2(\theta_\mathrm{W})-\sin^2(\theta_\mathrm{W})Y_{\ell,\mathrm{R}}\right)-\left(-\frac{1}{2}-\left(Y_{\ell,\mathrm{L}}-\frac{1}{2}\right)\sin^2(\theta_\mathrm{W})+\sin^2(\theta_\mathrm{W})Y_{\ell,\mathrm{R}}\right)\gamma_5\right)\ell\right).
\end{aligned}
$$

We could now extract the left-handed and right-handed spinor components for leptons, but let us instead investigate more closely why they look like they do. Looking at the electron, where this is more apparent, we see that the $-\frac{1}{2}$ appear as manifestations of weak isospin. In addition the two different factors preceding the sines are all the electric charge. Because the procedure of obtaining couplings will be the same for quarks, we may instantly conclude

$$c_\mathrm{V}=I_\mathrm{W}^3-2Q\sin^2(\theta_\mathrm{W}),\ \ c_\mathrm{A}=I_\mathrm{W}^3.$$