Summary of SI2390 Relativistic Quantum Physics

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Abstract

This is a summary of SI2390.
We will use units such that $c = \hbar = 1$.

Contents

1 Tasty Bits of Special Relativity

Metric Signature We use the metric signature $(1, -1, -1, -1)$ for the Minkowski metric.

The Levi-Civita Tensor We use the convention $\varepsilon^{0123} = 1$.

The Poincare Group Elements of the Poincare group are specified by a Lorentz transformation Λ and a translation a. Its elements follow the multiplication rule

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2).$$

We may instead construct a representation of the Poincare group with matrices of the form

$$\begin{bmatrix} \Lambda & a \\ 0 & 1 \end{bmatrix},$$

from which the multiplication rule directly follows.

The Lie Algebra of the Lorentz Group The Lorentz group is defined as the set of transformations such that

$$g = \Lambda^T g \Lambda.$$

There are a maximum of 16 generators, meaning we may label them using our index convention. Expanding around the identity we find

$$g = (1 + \omega_{\mu\nu} M^{\mu\nu})^T g (1 + \omega_{\rho\sigma} M^{\rho\sigma}) \approx g + \omega_{\mu\nu} (M^{\mu\nu})^T g + g \omega_{\rho\sigma} M^{\rho\sigma},$$

implying

$$\omega_{\mu\nu}((M^{\mu\nu})^T g + g M^{\mu\nu}) = 0,$$

or

$$M^T g = -gM$$

for all generators. Constructing the generator in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and using the fact that the Minkowski metric is its own universe we find

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} g = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix} = \begin{bmatrix} -A^T & -C^T \\ -B^T & -D^T \end{bmatrix}.$$

The solutions to this have antisymmetric blocks A and D, as well as off-diagonal blocks that are transposes of each other. There are six degrees of freedom for this solution, meaning that the Lorentz group has six degrees of freedom, corresponding to the three rotations and boosts. To preserve the index notation, we may then choose the generators such that $M^{\mu\nu} = -M^{\nu\mu}$. The corresponding choice of parameters must then also be antisymmetric. To get the appropriate amounts of terms we will also divide by 2, as you will see in the following section.

To more explicitly introduce the boosts and rotations, we introduce their generators

$$J^i = -\frac{1}{2}\varepsilon^{ijk} M^{jk},\ K^i = M^{0i},$$

with commutation relations

$$\left[J^i, J^j\right] = i\varepsilon^{ijk} J^k,\ \left[K^i, K^j\right] = -i\varepsilon^{ijk} J^k,\ \left[J^i, K^j\right] = i\varepsilon^{ijk} K^k.$$

We can solve for the original generators as

$$M^{0i} = J^i,\ M^{ij} = \varepsilon^{kij} J^k.$$

Generators of the Poincare Group The generators of the Poincare group are the $M^{\mu\nu}$ of the Lorentz group, as well as the four P^μ that generate translations in spacetime. We will need their Lie algebra, and thus their commutation relations, which are

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho}\right),\ [P^\mu, P^\nu] = 0,\ [M^{\mu\nu}, P^\sigma] = i(g^{\nu\sigma}P^\mu - g^{\mu\sigma}P^\nu).$$

The representations U of the group elements are then

$$U(\Lambda, 0) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}},\ U(1, a) = e^{ia_\mu P^\mu},$$

and to first order

$$U(\Lambda, a) = e^{i\left(a_\mu P^\mu - \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right)}.$$

2 Basic Concepts

Casimir Operators A Casimir operator is an operator that is constructed from the generators of a group and commutes with all generators.

Casimir Operators of the Poincare Group The Casimir operators of the Poincare group are

$$P^2 = P^\mu P_\mu,\ w^2 = w^\mu w_\mu,$$

where we have introduced the Pauli-Lubanski vector

$$w_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma.$$

It can be shown that

$$w_0 = \mathbf{P}\cdot\mathbf{J},\ \mathbf{w} = P_0\mathbf{J} + \mathbf{P}\times\mathbf{K}.$$

The Wigner Classification As we will consider unitary representations of the Poincare group acting on states and the representations can be decomposed into irreps, we will find that we can reduce our considerations to a set of fundamental systems, termed particles. The classification, divided according to the eigenvalues of P^2 and w^2, is according to the Wigner system:

1. $P^2 > 0$, with subclasses:
   - $P^0 < 0$.
   - $P^0 > 0$.
2. $P^2 = 0$, with subclasses:
   - $P^0 < 0$.
   - $P^0 > 0$.
3. $P^2 = 0$ and $P^0 = 0$.
4. $P^2 < 0$, corresponding to tachyons.

Lorentz Covariance and the Schrödinger Equation Using the 4-momentum $P^\mu = (E, \mathbf{p})$ and the correspondence principle $P^\mu = i\partial^\mu$, the quantization of the classical energy $E = \frac{1}{2m}\mathbf{p}^2$ of a free particle is

$$i\partial_t\Psi = -\frac{1}{2m}\nabla^2\Psi.$$

This does not in general respect Lorentz transformations, which one might expect given that it is not taken from a Lorentz covariant expression. In other words, the Schrödinger equation is not Lorentz covariant.

The quantization of the relativistic $E^2 = m^2 + \mathbf{p}^2$ is instead

$$-\partial_t^2\phi = m^2\phi - \nabla^2\phi.$$

By introducing the d'Alembertian $\Box = \partial_\mu\partial^\mu$ we can write the above as

$$\Box\phi + m^2\phi = 0.$$

This is the Klein-Gordon equation, which is an appropriate quantization of a spinless particle.

A Conserved Current Corresponding to the Klein-Gordon equation there exists a density and a current

$$\rho = \frac{i}{2m}(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*),\ \mathbf{j} = \frac{1}{2im}(\phi^* \vec{\boldsymbol{\nabla}} \phi - \phi \vec{\boldsymbol{\nabla}} \phi^*)$$

such that

$$\partial_t \rho + \vec{\boldsymbol{\nabla}} \cdot \mathbf{j} = 0.$$

Alternatively, by combining the two into a 4-current $J^\mu = (\rho, \mathbf{j})$ we find

$$\partial_\mu J^\mu = 0.$$

Problems With Stationary States A stationary state is a state such that

$$P^0 \phi = E\phi.$$

For such a state we have

$$J^0 = \frac{E}{m}|\phi|^2.$$

In the classical limit we have $\frac{E}{m} \approx 1$, whereas in the general case we have $E = \pm\sqrt{m^2 + \mathbf{p}^2}$, meaning that J^0 is not positive definite and the conserved Nöether cannot be interpreted as conservation of probability density. This implores us to reinterpret the Klein-Gordon equation as a general field equation.

Plane-Wave Solutions Plane-wave solutions of the Klein-Gordon equation are of the form

$$\phi = Ne^{-iP_\mu x^\mu}.$$

In order for these to be solutions, we require

$$P^0 = \pm\sqrt{m^2 + |\mathbf{p}|}.$$

This does not pose a problem in non-interacting cases, as the solutions maintain their signs.

Charged Particles When treating charged particles in external electromagnetic fields, we employ the minimal coupling scheme and perform the replacement $P^\mu \to P^\mu - qA^\mu$. The Klein-Gordon equation then becomes

$$((\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2)\phi = 0.$$

This will cause additional terms

$$J^\mu \to J^\mu - \frac{q}{m}|\phi|^2 A^\mu$$

in the Nöether current, further destroying our hopes of creating a one-particle theory.

The Klein Paradox Consider scattering after normal incidence on a step potential described by $A^\mu = (V\theta(x), \mathbf{0})$. Performing an anzats similar to that in the non-relativistic case, the Klein-Gordon equation predicts the same behaviour as the Schrödinger equation, except for the case where $V > E + m$. In this case the transmitted 4-momentum has a negative space component. Furthermore, the transmission probability becomes negative, but still preserving $T + R = 1$. This peculiar behaviour is known as Klein's paradox.

The Dirac Equation We will now try to develop a theory that remedies the problems with the Klein-Gordon equation. The hope is that this equations has a positive-definite conserved density. An important source of the bad time was the second-order time derivative, so we will try to remedy this with a first-order time derivative. We also make the space derivatives first-order, perhaps because of Lorentz stuff. This leads us to the anzats

$$\partial_\Psi^t + (\boldsymbol{\alpha} \cdot \vec{\boldsymbol{\nabla}})\Psi + im\beta\Psi = 0,$$

where β and α^i are matrices and Ψ is a vector. The sizes of these are yet to be determined. The corresponding equation for $\Psi^\dagger$ is

$$\partial^t_{\Psi^\dagger} + (\vec{\boldsymbol{\nabla}}\Psi^\dagger) \cdot \boldsymbol{\alpha}^\dagger - im\Psi^\dagger\beta^\dagger = 0.$$

Considering the quantity $\Psi^\dagger\Psi$ we have

$$\begin{aligned}
\partial^t_{(\Psi^\dagger\Psi)} &= (\partial^t_{\Psi^\dagger})\Psi + \Psi^\dagger\partial^t_\Psi \\
&= (im\Psi^\dagger\beta^\dagger - (\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha})\Psi + \Psi^\dagger(-(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi - im\beta\Psi) \\
&= im\Psi^\dagger(\beta^\dagger - \beta) - (\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha}^\dagger\Psi - \Psi^\dagger(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi.
\end{aligned}$$

We really want the right-hand side to be the 3-divergence of some 3-current. To do that, we may choose α^i and β to be Hermitian, yielding

$$(\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha}^\dagger\Psi + \Psi^\dagger(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi = \vec{\boldsymbol{\nabla}}\cdot\Psi^\dagger\boldsymbol{\alpha}\Psi.$$

The conserved 4-current is thus $j^\mu = (\Psi^\dagger\Psi, \Psi^\dagger\boldsymbol{\alpha}\Psi)$.

To reobtain something like the 4-vector norm we had when discussing the Klein-Gordon equation, we apply the operator $\partial^t - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}) - im\beta\Psi$ to our anzats to find

$$(\partial^t - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}) - im\beta\Psi)(\partial^t_\Psi + (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi + im\beta\Psi) = 0.$$

As the derivatives commute with the matrices, the cross terms vanish, yielding

$$\partial^{[}_{2]}t\Psi - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})^2\Psi - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})im\beta\Psi - im\beta(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi + m^2\beta^2\Psi = 0,$$

or more explicitly

$$\partial_t^2\Psi - (\alpha^i\partial_i)(\alpha^j\partial_j)\Psi - im((\alpha^i\partial_i)\beta + \beta\alpha^i\partial_i)\Psi + m^2\beta^2\Psi = \partial_t^2\Psi - \alpha^i\alpha^j\partial_i\partial_j\Psi - im(\alpha^i\beta + \beta\alpha^i)\partial_i\Psi + m^2\beta^2\Psi = 0.$$

We can symmetrize the second term to find

$$\partial_t^2\Psi - \frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial_i\partial_j\Psi - im(\alpha^i\beta + \beta\alpha^i)\partial_i\Psi + m^2\beta^2\Psi = 0.$$

This produces the same 4-vector norm if

$$\left\{\alpha^i, \alpha^j\right\} = 2\delta^{ij},\ \beta^2 = 1,\ \alpha^i\beta + \beta\alpha^i = 0.$$

Computing the determinant of the last equation, we find

$$\det\left(\alpha^i\beta\right) = (-1)^N\det\left(\beta\alpha^i\right),$$

where N is the length of Ψ. The only way for the above equations to be solvable is then that N be odd. It can also be shown that α^i and β are all traceless. By a series of arguments we find that $N = 4$ is correct.

To complete our discussion, we define $\gamma^0 = \beta,\ \gamma^i = \beta\alpha^i$. With this we multiply our anzats by $-i\beta$ and find

$$(-i\beta\partial^0 - i(\beta\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}))\Psi + m\Psi = 0.$$

Defining the inner product $\gamma^\mu A_\mu = \not{A}$ we arrive at the Dirac equation

$$i\not{\partial}\Psi - m\Psi = 0.$$

Properties of the γ^μ The γ^μ satisfy

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Defining the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, we find that it must be Hermitian.
We also have

$$\left\{\gamma^5, \gamma^\mu\right\} = 0.$$

We have

$$\operatorname{tr}\left(\prod_{i=1}^n \gamma^{\mu_i}\right) = 0,\ n \text{ odd},$$
$$\operatorname{tr}\left(\gamma^5\right) = 0.$$

The Dirac Algebra The γ^μ are a representation of the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

We may compute explicit representations of this algebra as

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

In this representation we have

$$\gamma^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The Dirac Adjoint The Dirac adjoint is defined as

$$\bar{A} = A^\dagger \gamma^0.$$

Rewriting the 4-Current We may use the properties of the γ^μ to write

$$j^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi = \bar{\Psi} \gamma^\mu \Psi.$$

A Free Dirac Particle For a free we multiply the operator $\not{p} - m$ by its conjugate to find

$$(\not{p} + m)(\not{p} - m) = \not{p}^2 - m^2.$$

We have

$$\not{p}^2 = \gamma^\mu \gamma^\nu p_\mu p_\nu = \gamma^\mu \gamma^\nu p_\nu p_\mu = \gamma^\nu \gamma^\mu p_\mu p_\nu,$$

hence

$$\not{p}^2 - m^2 = \frac{1}{2}\{\gamma^\mu, \gamma_\nu\} p_\mu p_\nu - m^2 = g^{\mu\nu} p_\mu p_\nu - m^2 = p^2 - m^2.$$

The Dirac particle has now been diagonalized, and it is in fact separable in terms of its spacetime dependence. The solution is thus a plane wave times a spinor - more specifically,

$$\Psi = e^{-ipx} u,$$

where u is some spinor. The 4-momentum satisfies the previously derived relation. It can also be shown for such a particle that $\mathcal{H} = i\partial_0$, and thus has eigenvalue $p_0 = E$. The sign of this eigenvalue is yet to be determined, and will in fact produce solutions with positive and negative energy.

Given the energy eigenvalues, the form of the spinors is determined by the Dirac equation itself. As we have

$$\gamma^\mu p_\mu = E \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \mathbf{p} \cdot \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} = \begin{bmatrix} E & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & -E \end{bmatrix},$$

hence we may write the spinor in block form to find the equation

$$\begin{bmatrix} E - m & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & -E - m \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} (E - m)u_A - \mathbf{p} \cdot \boldsymbol{\sigma} u_B \\ \mathbf{p} \cdot \boldsymbol{\sigma} u_A - (E + m) u_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Inserting the two equations into the other, we find

$$u_A = \frac{1}{E^2 - m^2} (\mathbf{p} \cdot \boldsymbol{\sigma})^2 u_A.$$

The matrix quantity on the right is given by

$$(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = \sigma^i \sigma^j p_i p_j = (2\delta^{ij} - \sigma^j \sigma^i) p_i p_j = 2\mathbf{p}^2 - \sigma^j \sigma^i p_j p_i,$$

implying

$$u_A = \frac{\mathbf{p}^1}{E^2 - m^2} u_A,$$

which in combination with the previously derived energy-momentum relation implies that we may freely choose a basis for the top half. As the same holds for the lower parts, we will for each p find a set of solutions by choosing one half of the spinor and computing the other from that. We have

$$\mathbf{p} \cdot \sigma = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix},$$

and the four solutions are thus

$$\begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{bmatrix}, \begin{bmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{p_x-ip_y}{E-m} \\ -\frac{p_z}{E-m} \\ 0 \\ 1 \end{bmatrix}.$$

At this point we must return to the sign of E. The two former solutions blow up if E is chosen to be negative, whereas the latter ones blow up if E is positive. Thus we have achieved the assignment of energy signs to the different spinors.

What is the interpretation of this? Dirac's first interpretation involved the so-called Dirac sea, but the current understanding is that the negative energy solutions represent antiparticle states with positive energy.

Plane-Wave Solutions Plane-wave solutions of the Dirac equation are of the form

$$\Psi_P = e^{-iP_\mu x^\mu} u(P^\mu),$$

where $u(P^\mu)$ is a so-called spinor. Inserting it into the Dirac equation we find

$$(-\not{P} + m)u(P^\mu) = 0.$$

Multiplying by $\not{P} + m$ we find

$$(-\not{P}^2 + m^2)u(P^\mu) = (-\gamma^\mu\gamma^\nu P_\mu P_\nu + m^2)u(P^\mu) = 0.$$

We can symmetrize the first term and use the anticommutation relations of the γ^μ to find

$$(-P^2 + m^2)u(P^\mu) = 0.$$

In other words, the solution satisfies the relativistic energy-momentum relation. This also implies that for non-trivial solutions, the 4-momentum is time-like.

In the corresponding rest frame, there are four independent spinor solutions. These are $u_\pm$ and $v_\pm$, and with this representation they are as you would expect.

A Hamiltonian As the plane-wave solutions have time-like 4-momenta, there is a corresponding rest frame. In this rest frame, the Hamiltonian, which is generally

$$\mathcal{H} = \beta m + \boldsymbol{\alpha} \cdot \mathbf{p},$$

reduces to

$$\mathcal{H} = \beta m.$$

Defining

$$\mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix},$$

we have $[\mathcal{H}, \mathbf{\Sigma}] = 0$.

Lorentz Transformation of Spinors Suppose that Ψ is a solution to the Dirac equation. Under a Lorentz transform it should transform according to

$$\Psi'(x') = S(\Lambda)\Psi(x).$$

We would like to identify the transformation matrix S.

Because the Dirac equation is Lorentz covariant, it looks the same in all frames. We can use the chain rule to find

$$\partial^\mu = \Lambda^{\mu'}{}_\mu \partial^{\mu'}$$

and thus

$$(-i\gamma^\mu\partial^\mu + m)\Psi = (-i\gamma^\mu\Lambda^{\mu'}{}_\mu\partial^{\mu'} + m)S^{-1}\Psi' = 0.$$

Note that as m is a Lorentz scalar, it is also the same in all frames. Multiplication by S yields

$$(-i\Lambda^{\mu'}{}_\mu S\gamma^\mu S^{-1}\partial^{\mu'} + m)\Psi' = 0.$$

Comparing this with the Dirac equation in the primed frame, we must therefore have

$$\Lambda^{\mu'}{}_\mu S\gamma^\mu S^{-1} = \gamma^{\mu'} \iff \Lambda^{\mu'}{}_\mu\gamma^\mu = S^{-1}\gamma^{\mu'}S.$$

To proceed, we apply the expansion

$$\Lambda^{\mu'}{}_\mu = \delta^{\mu'}_\mu + \varepsilon\omega^{\mu'}{}_\mu,\ \omega^{\mu\nu} = -\omega^{\nu\mu},$$

and expand S in terms of ε as

$$S = 1 - \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu},\ \sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

for some set of matrices $\sigma_{\mu\nu}$. We see that its inverse to first order must be

$$S^{-1} = 1 + \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu}.$$

We now have

$$\begin{aligned}(\delta^{\mu'}_\mu + \varepsilon\omega^{\mu'}{}_\mu)\gamma^\mu &= \left(1 + \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu}\right)\gamma^{\mu'}\left(1 - \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu}\right)\\ &= \gamma^{\mu'} + \frac{i}{4}\varepsilon\omega^{\mu\nu}(\sigma_{\mu\nu}\gamma^{\mu'} - \gamma^{\mu'}\sigma_{\mu\nu}),\end{aligned}$$

or

$$\frac{i}{4}\omega^{\mu\nu}\left[\sigma_{\mu\nu}, \gamma^{\mu'}\right] = \omega^{\mu'}{}_\mu\gamma^\mu = \omega^{\mu'\mu}\gamma_\mu.$$

To remove the $\omega^{\mu\nu}$ we will use the fact that

$$\omega^{\mu'}{}_\nu\gamma^\nu = g^{\mu'\mu}\omega_{\mu\nu}\gamma^\nu = -g^{\mu\mu'}\omega_{\nu\mu}\gamma^\nu = -g^{\nu\mu'}\omega_{\mu\nu}\gamma^\mu = -g^{\mu'\nu}\omega_{\mu\nu}\gamma^\mu,$$

hence

$$\frac{i}{4}\omega^{\mu\nu}\left[\sigma_{\mu\nu}, \gamma^{\mu'}\right] = \frac{1}{2}\omega_{\mu\nu}(g^{\mu'\mu}\gamma^\nu - g^{\mu'\nu}\gamma^\mu) = \frac{1}{2}\omega^{\mu\nu}(g^{\mu'}{}_\mu\gamma_\nu - g^{\mu'}{}_\nu\gamma_\mu),$$

which must finally imply

$$\left[\gamma^{\mu'}, \sigma_{\mu\nu}\right] = 2i(g^{\mu'}{}_\mu\gamma_\nu - g^{\mu'}{}_\nu\gamma_\mu).$$

The solution to this is

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu].$$

Charged Particles When introducing charge, we employ the minimal coupling scheme. The Dirac equation then becomes

$$(-\gamma^\mu(i\partial_\mu - qA_\mu) + m)\Psi = (\not{p} - q\not{A} - m)\Psi = 0.$$

To proceed we write the solution as

$$\Psi = (\not{p} - q\not{A} + m)\chi,$$

yielding

$$(\not{p} - q\not{A} - m)(\not{p} - q\not{A} + m)\chi = ((\not{p} - q\not{A})^2 - m^2)\chi = 0.$$

We investigate the operator appearing in this modified solution as

$$\begin{aligned}
(\not{p} - q\not{A})^2 &= \gamma^\mu\gamma^\nu(p_\mu - qA_\mu)(p_\nu - qA_\nu) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu(p_\mu - qA_\mu)(p_\nu - qA_\nu) + \gamma^\nu\gamma^\mu(p_\nu - qA_\nu)(p_\mu - qA_\mu)\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) + \gamma^\nu\gamma^\mu(p_\nu p_\mu - q(p_\nu A_\mu + A_\nu p_\mu) + q^2 A_\nu A_\mu)\right) \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\}(p_\mu p_\nu + q^2 A_\mu A_\nu) - q(\gamma^\mu\gamma^\nu(p_\mu A_\nu + A_\mu p_\nu) + \gamma^\nu\gamma^\mu(p_\nu A_\mu + A_\nu p_\mu))\right) \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\}(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) - q\gamma^\nu\gamma^\mu(p_\nu A_\mu + A_\nu p_\mu - p_\mu A_\nu - A_\mu p_\nu)\right) \\
&= \frac{1}{2}\left(2g^{\mu\nu}(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) - q\gamma^\nu\gamma^\mu([p_\nu, A_\mu] - [p_\mu, A_\nu]\right).
\end{aligned}$$

We have

$$[p_\mu, A_\nu] = i((\partial_\mu A_\nu) + A_\nu\partial_\mu - A_\nu\partial_\mu) = i(\partial_\mu A_\nu),$$

and thus

$$\begin{aligned}
(\not{p} - q\not{A})^2 &= \frac{1}{2}\left(2g^{\mu\nu}(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) - iq\gamma^\nu\gamma^\mu(\partial_\nu A_\mu - \partial_\mu A_\nu)\right) \\
&= (p - qA)^2 + \frac{i}{2}q\gamma^\nu\gamma^\mu F_{\mu\nu}.
\end{aligned}$$

As

$$\gamma^\nu\gamma^\mu = \frac{1}{2}([\gamma^\nu, \gamma^\mu] + \{\gamma^\nu, \gamma^\mu\}),$$

where the first term is symmetric and the second is antisymmetric, we have

$$(\not{p} - q\not{A})^2 = (p - qA)^2 + \frac{1}{2}q\sigma^{\nu\mu}F_{\mu\nu} = (p - qA)^2 - \frac{1}{2}q\sigma^{\mu\nu}F_{\mu\nu}.$$

It can be shown that

$$\frac{1}{2}q\sigma^{\mu\nu}F_{\mu\nu} = -q(\mathbf{\Sigma}\cdot\mathbf{B} - i\boldsymbol{\alpha}\cdot\mathbf{E}),$$

where the first term is interpreted as a magnetic dipole contribution and the second as an electric monopole contribution.

The Hydrogenic Atom A system with a Coulomb potential $V(r) = -\frac{Ze^2}{4\pi r}$ is an example of an exactly solvable model with the Dirac formalism. The corresponding Hamiltonian is

$$H = \beta m + \boldsymbol{\alpha}\cdot\mathbf{p} + V(r),$$

which commutes with $\mathbf{J}$ and the parity operator P. We thus seek simultaneous eigenstates of $H, \mathbf{J}^2, J^3$ and P. The corresponding eigenvalues are $E, j(j+1), m$ and $(-1)^{j+\frac{\varpi}{2}}$, where

$$\varpi = \begin{cases} 1, & P = (-1)^{j+\frac{1}{2}}, \\ -1, & P = (-1)^{j-\frac{1}{2}}. \end{cases}$$

As the problem is divided into blocks, we write the desired states as

$$\Psi = \begin{bmatrix} \phi \\ \chi \end{bmatrix}.$$

The angular part may be separated out, and corresponding to them are two sets of solutions $\mathcal{Y}_{lj}^m$, where l is the eigenvalue of L and takes on values $j \pm \frac{1}{2}\bar{\omega}$ for the two sets of solutions. Explicitly the solutions are

$$\mathcal{Y}_{lj}^m = \begin{bmatrix} \sqrt{\frac{j+m}{2j}} Y_{j-\frac{1}{2}, m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}} Y_{j-\frac{1}{2}, m+\frac{1}{2}} \end{bmatrix}, \; j = l + \frac{1}{2}$$

and

$$\mathcal{Y}_{lj}^m = \begin{bmatrix} \sqrt{\frac{j-m+1}{2(j+1)}} Y_{j+\frac{1}{2}, m-\frac{1}{2}} \\ -\sqrt{\frac{j+m+1}{2(j+1)}} Y_{j+\frac{1}{2}, m+\frac{1}{2}} \end{bmatrix}, \; j = l - \frac{1}{2}, \; l > 0.$$

To proceed, we make the anzats

$$\phi = \frac{1}{r} F(r) \mathcal{Y}_{lj}^m, \; \chi = \frac{1}{r} G(r) \mathcal{Y}_{lj}^m.$$

We also introduce

$$\mathbf{r} \cdot \mathbf{p} = -ir\partial_r, \; p_r = -\frac{i}{r}\partial_r = \frac{1}{r}(\mathbf{r} \cdot \mathbf{p} - i),$$

as well as

$$\alpha_r = \frac{1}{r} \boldsymbol{\alpha} \cdot \mathbf{r},$$

from which one can show

$$\boldsymbol{\alpha} \cdot \mathbf{p} = \alpha_r (p_r + \frac{i}{r} \beta K),$$

where $K = \beta(\boldsymbol{\Sigma} \cdot \mathbf{L} + 1)$, with eigenvalues $-\bar{\omega}\left(j + \frac{1}{2}\right)$. The Hamiltonian can now be written as

$$H = \alpha_r (p_r - \frac{i\bar{\omega}\left(j + \frac{1}{2}\right)}{r} \beta) + \beta m + V(r).$$

The eigenvalue equation then becomes

$$\left(\begin{bmatrix} 0 & -\partial_r \\ \partial_r & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\bar{\omega}(j+\frac{1}{2})}{r} \\ \frac{i\bar{\omega}(j+\frac{1}{2})}{r} & 0 \end{bmatrix} \right) \begin{bmatrix} F \\ G \end{bmatrix} = (E - m - V(r)) \begin{bmatrix} F \\ G \end{bmatrix}.$$

In the particular case of the hydrogenic atom, we introduce the following notation:

$$\kappa = \sqrt{m^2 - E^2}, \; \rho = \kappa r, \; \tau = \bar{\omega}\left(j + \frac{1}{2}\right), \nu = \sqrt{\frac{m-E}{m+E}}, \; Z' = \frac{Ze^2}{4\pi}$$

to find

$$\left(\begin{bmatrix} 0 & -\partial_\rho \\ \partial_\rho & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\tau}{\rho} \\ \frac{\tau}{\rho} & 0 \end{bmatrix} \right) \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} -\nu + \frac{Z'}{\rho} & 0 \\ 0 & \frac{1}{\nu} + \frac{Z'}{\rho} \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}.$$

We can show that close to the origin the functions behave like power laws, and at infinity they decay exponentially. We thus make the anzats

$$F = e^{-\rho} f, \; G = e^{-\rho} g$$

and introduce

$$w = \begin{bmatrix} f \\ g \end{bmatrix}, \; A = \begin{bmatrix} -\tau & Z' \\ Z' & \tau \end{bmatrix}, \; B = \begin{bmatrix} 1 & \frac{1}{\nu} \\ \nu & 1 \end{bmatrix},$$

from which we obtain

$$\rho \partial_\rho w = (A + \rho B) w.$$

We solve this problem with a Frobenius anzats

$$w = \rho^\mu \sum_{s=0}^{N} w_s \rho^s,$$

where N is some number to be determined. The recursion relation one obtains is

$$\mu w_0 = A w_0,\ (s + \lambda - A) w_s = B w_{s-1},$$

where $\lambda = \sqrt{\tau^2 - (Z')^2}$ is the magnitude of the eigenvalues of A. It can be shown that w converges and that N is finite. More specifically it holds that

$$N + \lambda - \frac{1}{2} Z' \left(\frac{1}{\nu} - \nu \right) = 0,$$

implying

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \frac{Z^2 \alpha^2}{N + \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2}}}},\ N + j + \frac{1}{2} = 1, \dots, n,$$

where n is termed the little quantum number and α is the fine structure constant.

Expanding this we find

$$\frac{E}{m} \approx 1 - \frac{1}{2} \frac{Z^2 \alpha^2}{\left(j + \frac{1}{2}\right)^2} + \dots = 1 - \frac{1}{2} \frac{Z^2 \alpha^2}{n^2} \left(1 + \frac{Z^2 \alpha^2}{n^2} \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \right),$$

which is similar to the result found with perturbation theory of the non-relativistic problem, except for the angular momentum-dependence of the perturbation.

3 Introductory Quantum Field Theory

The Non-Relativistic String Consider a non-relativistic string with a coordinate x along its equilibrium position. To study it, we impose periodic boundary conditions and discretize the string, dividing it into N points separated by a distance a. Denoting the displacements at each point as ϕ_i and assigning each point a mass m, the total kinetic energy of the string is

$$T = \frac{1}{2} m \sum_{i=0}^{N-1} \left(\frac{\mathrm{d}\phi_i}{\mathrm{d}t} \right)^2.$$

Implementing a simple Hook-like tension in the string we also have a total potential energy

$$V = \frac{1}{2} k \sum_{i=0}^{N-1} \left(\phi_{i+1} - \phi_i \right)^2.$$

We will now consider a continuum limit, where N goes to infinity and a to zero such that $l = Na$ is constant. Introducing

$$\mu = \frac{m}{a},\ \tau = ka$$

and relabelling the displacements by noting that

$$\phi_i(t) = \phi(t, x_i) = \phi(t, ia),$$

allowing us to use the label $\phi(t,x)$ in the continuum limit, we find that the kinetic energy is

$$T = \frac{1}{2}\mu \sum_{i=0}^{N-1} a \left(\frac{\mathrm{d}\phi_i}{\mathrm{d}t}\right)^2 \to \frac{1}{2}\mu \int_0^l \mathrm{d}z \, (\partial_t \phi)^2$$

and the potential energy is

$$V = \frac{1}{2}\frac{\tau}{a} \sum_{i=0}^{N-1} a^2 \left(\frac{\phi_{i+1}-\phi_i}{a}\right)^2 \to \frac{1}{2}\tau \int_0^l \mathrm{d}x \, (\partial_x \phi)^2.$$

The system is then described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mu \, (\partial_t \phi)^2 - \frac{1}{2}\tau (\partial_x \phi)^2.$$

Rescaling the field by a factor $\frac{1}{\sqrt{\tau}}$ and introducing $v^2 = \frac{\tau}{\mu}$ we have

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{v^2}(\partial_t \phi)^2 - (\partial_x \phi)^2\right).$$

The equations of motion is the discrete case are

$$-k(-\phi_{i+1} + 2\phi_i - \phi_{i-1}) - m\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} = 0,$$

which can be written as

$$\frac{m}{k}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} - (\phi_{i+1} - 2\phi_i + \phi_{i-1}) = \frac{m}{k}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} - a^2\frac{1}{a}\left(\frac{\phi_{i+1}-\phi_i}{a} - \frac{\phi_i - \phi_{i-1}}{a}\right) = 0.$$

We note that

$$\frac{m}{k} = \frac{a^2\mu}{\tau} = \frac{a^2}{v^2},$$

hence

$$\frac{1}{v^2}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} - \frac{1}{a}\left(\frac{\phi_{i+1}-\phi_i}{a} - \frac{\phi_i - \phi_{i-1}}{a}\right) = 0.$$

The equations for the continuum limit are

$$\frac{1}{v^2}\partial_t^2\phi - \partial_x^2\phi = 0,$$

which are indeed the continuum limits of the discretized solutions.

The solutions of the equations of motion are so-called normal modes. They are denoted as having positive or negative frequency (or equivalently, energy). The solutions are

$$\phi_n = \frac{1}{\sqrt{l}}e^{i(k_n x - \omega_n t)}, \ \phi_n^* = \frac{1}{\sqrt{l}}e^{-i(k_n x - \omega_n t)}, \ k_n = \frac{2\pi n}{l}, \ \omega_n^2 = v^2 k_n^2.$$

Their normalization is

$$\int_0^l \mathrm{d}x \, \phi_n^* \phi_m = \delta_{nm}.$$

A general solution can then be written as

$$\phi = \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{l}}\left(a_n(t)e^{ik_n x} + a_n^*(t)e^{-ik_n x}\right), \ a_n(t) = a_n(0)e^{-i\omega_n t}.$$

The expansion coefficients thus satisfy

$$\frac{\mathrm{d}^2 a_n}{\mathrm{d}t^2} + \omega_n^2 a_n = 0,$$

and we treat them as simple harmonic oscillators.

We also make a brief note of the Hamiltonian. As we know for the discrete system we have $\mathcal{H} = T + V$, so we would find the Hamiltonian density to be

$$\mathcal{H} = \frac{1}{2}\mu\left(\partial_t \phi\right)^2 + \frac{1}{2}\tau(\partial_x \phi)^2.$$

This doesn't really work, however. To do this properly, we will need the canonical momentum density

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_t \phi} = \mu \partial_t \phi.$$

The Hamiltonian density is thus

$$\mathcal{H} = \frac{1}{\mu}\pi^2 - \frac{1}{2\mu}\pi^2 + \frac{1}{2}\tau(\partial_x \phi)^2 = \frac{1}{2\mu}\pi^2 + \frac{1}{2}\tau(\partial_x \phi)^2.$$

When quantizing the non-relativistic strings, we will only quantize the simple harmonic oscillators. It can be shown that

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} c_n^2 \frac{2\omega_n^2}{v^2} a_n^* a_n,$$

which with an appropriate choice of c_n becomes

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} \omega_n a_n^* a_n.$$

Alternatively, by introducing generalized coordinates and momenta

$$q_m = \frac{1}{\sqrt{2\omega_n}}(a_n + a_n^*),\ p_m = -i\sqrt{\frac{\omega_n}{2}}(a_n - a_n^*)$$

the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}\sum_{n=-\infty}^{\infty}(p_n^2 + \omega_n^2 q_n^2).$$

Quantizing the Non-Relativistic String To quantize the relativistic string, we replace the expansion coefficients with operators and impose the canonical commutation relations - that is, the p and q commute and $[q_m, p_m] = i\delta_{nm}$. This leads to the a and $a^\dagger$ commuting, and $\left[a_n, a_m^\dagger\right] = \delta_{nm}$. The so-called quantum field is then

$$\phi = v\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\omega_n l}}\left(a_n e^{i(k_n x - \omega_n t)} + a_n^\dagger e^{-i(k_n x - \omega_n t)}\right) = \phi^{(+)} + \phi^{(-)},$$

where the two terms contain only creation and annihilation operators respectively. The Hamiltonian is now somehow

$$H = \sum_{n=-\infty}^{\infty} \frac{1}{2}\omega_n (a_n^\dagger a_n + a_n a_n^\dagger) = \sum_{n=-\infty}^{\infty} \omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right).$$

We may subtract the latter term, which is a vacuum energy contribution. We may also introduce the number operators $\mathcal{N}_n = a_n^\dagger a_n$.

With the introduction of the canonical momentum density comes a canonical commutation relation

$$\left[\phi(x,t), \pi(x',t)\right] = i\delta(x - x').$$

The Normal-Ordered Product The normal-ordered product is defined as

$$N\phi^1\phi^2 =: \phi^1\phi^2 := \phi^1\phi^2 - \langle 0|\phi^1\phi^2|0\rangle \,.$$

One usually splits up the two fields in their creation and annihilation terms. In this division one makes sure to move annihilation operators to the left.

Quantum Field Theory The basic idea of quantum field theory is that operators are generated by operator-valued fields, meaning that operators are localized in this formalism. The interpretation of these fields is that they are creation and annihilation operators at different points.

The Time-Ordered Product The time-ordered product is defined as

$$T\phi_1\phi_2 = \theta(t_1 - t_2)\phi_1\phi_2 + \theta(t_2 - t_1)\phi_2\phi_1 \tag{1}$$

for bosonic fields and

$$T\psi_1\psi_2 = \theta(t_1 - t_2)\psi_1\psi_2 - \theta(t_2 - t_1)\psi_2\psi_1$$

for fermionic fields. The time subscripts denote the time coordinates at which the two fields are evaluated.

Propagators The propagator is the amplitude of probability from travelling between two points in spacetime. They act as Green's functions for the equations of motion to which they correspond.

One propagator is the Klein-Gordon propagator, defined as

$$i\Delta_{\mathrm{F}}\,(x) = \langle 0|T\phi(x)\phi(0)|0\rangle \,.$$

It can be represented as a Feynman diagram according to and is interpreted as a particle being created at 0,

$$0 \;\text{- - - - - - - - - -}\; x$$

propagated to x and annihilated there. Equivalently, it may be interpreted as an antiparticle doing the same in the opposite order.

In Fourier space we may also introduce a propagator according to

$$i\Delta_{\mathrm{F}}\,(x) = \int \mathrm{d}^4k\,\frac{1}{(2\pi)^4}e^{-ik^\mu x_\mu}\frac{i}{k^2 - m^2 - i0},$$

and thus define

$$i\Delta_{\mathrm{F}}\,(k) = \frac{i}{k^2 - m^2 - i0}.$$

Note the constant $i0 = \lim\limits_{\varepsilon=0}\varepsilon i$. In a Feynman diagram it looks like this:

$$\begin{array}{c} 0 \;\text{- - - - - - - - - -}\; x \\ \underset{k}{\longrightarrow} \end{array}$$

Another is the Dirac propagator, defined as

$$iS_{\mathrm{F},\alpha\beta}\,(x) = \langle 0|T\psi_\alpha(x)\bar{\Psi}_\beta(0)|0\rangle$$

in real space and

$$iS_{\mathrm{F},\alpha\beta}\,(k) = \frac{i}{k^2 - m^2 + i0}(\not{k} + m)_{\alpha\beta}$$

which is illustrated as in real space and in momentum space.

Finally there is the photon propagator

$$iD_{\mathrm{F},\mu\nu}\,(x) = \langle 0|TA_\mu(x)A_\nu(0)|0\rangle \,,$$

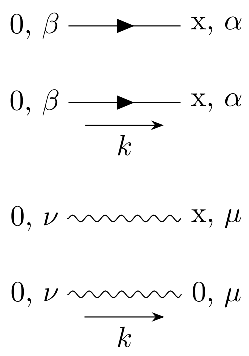
in real space and

$$iD_{\mathrm{F},\mu\nu}\,(x) == -\frac{i}{k^2 + i0}g_{\mu\nu} \tag{2}$$

in momentum space. Note that in other gauges one will have to perform the replacement

$$g_{\mu\nu} \to g_{\mu\nu} + k_\mu F_\nu(x),\; k_\nu F_\mu(x),$$

where the new terms depend on the gauge. These are represented by in real space and in momentum space.



The Plan The plan is to now obtain a Lagrangian density for our theory, find the equations of motions and solve the inhomogenous versions to find the propagators.

Gauge Interactions There are three fundamental gauge interactions we will consider: electromagnetic, weak and strong interactions. These will all be mediated by spin-1 particles. We will use a minimal coupling scheme with a general coupling parameter g. We will generally find that this adds an interaction term to the Lagrangian density. As an example, for the Dirac Lagrangian density we will find

$$\mathcal{L} \to \mathcal{L} - g\bar{\Psi}\gamma_\mu \Psi A^\mu.$$

Quantizing the Klein-Gordon Field We start with the Klein-Gordon equation

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi).$$

Expanding into eigenfunctions of the parts not involving time a general solution may be written as

$$\phi = \sum_n q_n(t)u_n(\mathbf{x}).$$

The Klein-Gordon equation implies that the expansion coefficients satisfy Euler-Lagrange equations for some Lagrangian. We may thus impose the canonical commutation relations on these by also introducing

$$p_m = \int \mathrm{d}^3\mathbf{x}\, u_m \pi,$$

where π is the momentum density.

Corresponding to the Klein-Gordon field is an energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu \pi \partial^\nu \pi + \frac{1}{2}g^{\mu\nu}(m^2\phi^2 - \Box\phi),$$

which is symmetric and divergence-free. We define the 4-momentum

$$P^\mu = \int \mathrm{d}^3\mathbf{x}\, T^{0\mu}.$$

It can somehow be shown that

$$P^\mu = \int \mathrm{d}^3 q\, \frac{1}{q^0} q^\mu a^\dagger(q)a(q),$$

where $\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = q^0\delta(\mathbf{p}-\mathbf{q})$. It can be shown that $a^\dagger(p)\left|0\right\rangle$ is an eigenstate of this operator.

Let us now consider the commutator between the field at different points. We first introduce the creation and annihilation terms

$$\phi_+ = \frac{1}{\sqrt{2(2\pi)^3}} \int \mathrm{d}^3\mathbf{k}\, \frac{1}{k^0} a(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}},\ \phi_- = \frac{1}{\sqrt{2(2\pi)^3}} \int \mathrm{d}^3\mathbf{k}\, \frac{1}{k^0} a^\dagger(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

From this one can show that

$$[\phi(x), \phi(y)] = -\frac{1}{2(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \frac{1}{k^0} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right),$$

defined to be $i\Delta(x-y)$. It can also be shown that

$$i\Delta(x) = -\frac{1}{2\pi} \operatorname{sgn}(x^0)(\delta(x^\mu x\mu) - \frac{1}{2} m^2 \theta(x^\mu x\mu)).$$

The propagator solves the inhomogenous Klein-Gordon equation, with inhomogeneity $-\delta(x-y)$ for convention. We define its Fourier transform as

$$G(x-y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k\, e^{-ik(x-y)} \tilde{G}(k).$$

The Klein-Gordon equation implies

$$\tilde{G} = \frac{1}{k^2 - m^2}.$$

This function has a pole at $k^0 = \pm\omega$, where $\omega^2 = \mathbf{k}^2 + m^2$. To remedy this, we add a term $-i\varepsilon$ to the denominator, shifting the poles from the real axis. Writing

$$G(x-y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int \mathrm{d}k^0\, \frac{1}{(k^0)^2 - \omega^2 - i\varepsilon} e^{-ik^0(x^0-y^0)}$$

for $x^0 > y^0$ and switching the sign of the last integral if the opposite is true, we find that the last integral is

$$-2\pi i \frac{e^{\pm ik^0(x^0-y^0)}}{2k^0} \tag{3}$$

for the two cases. The two corresponding functions are dubbed $\pm i\Delta_\pm(x-y)$, and we finally have

$$i\Delta_{\mathrm{F}}(x-y) = i(\theta(x^0-y^0)\Delta_+(x-y) - \theta(y^0-x^0)\Delta_+(x-y)).$$

Quantizing the Dirac Field We start with the Lagrangian of a free Dirac field

$$\mathcal{L} = -\frac{1}{2}\bar{\Psi}(-i\partial\!\!\!/ + m)\Psi - \frac{1}{2}(i\partial_\mu \bar{\Psi}\gamma^\mu + m\bar{\Psi})\Psi.$$

Another Lagrangian is

$$\mathcal{L}_{\mathrm{D}} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi.$$

The corresponding momentum densities are

$$\pi = \frac{i}{2}\Psi^\dagger,\ \bar{\pi} = -\frac{i}{2}\gamma^0\Psi$$

for the first choice and

$$\pi = i\Psi^\dagger,\ \bar{\pi} = 0$$

for the second. The corresponding energy-momentum tensor is

$$T^{\mu\nu} = \frac{i}{2}(\bar{\Psi}\gamma^\mu\gamma^\nu\Psi - (\partial^\nu\Psi\gamma^\mu\Psi)),$$

with corresponding momentum densities

$$P^\mu = i\int \mathrm{d}^3\mathbf{x}\, \Psi^\dagger\gamma^0\gamma^\mu\Psi.$$

To quantize the field, we expand it as

$$\Psi = \sum_s \int \mathrm{d}^3\mathbf{p}\, \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E(\mathbf{p})}} (b(\mathbf{p},s)u(\mathbf{p},s)e^{-ipx} + d^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ipx}),$$

$$\Psi^\dagger = \sum_s \int \mathrm{d}^3\mathbf{p}\, \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E(\mathbf{p})}} (b^\dagger(\mathbf{p},s)u^\dagger(\mathbf{p},s)e^{ipx} + d(\mathbf{p},s)v^\dagger(\mathbf{p},s)e^{-ipx}),$$

where we have introduced the particle and antiparticle spinors and creation and annihilation operators b and d for particles and antiparticles. This leads to the Hamiltonian becoming

$$\mathcal{H} = \int \mathrm{d}^3\mathbf{p}\, E(\mathbf{p}) \sum_s (b^\dagger b(\mathbf{p},s) - dd^\dagger(\mathbf{p},s)).$$

In order to produce positive values, we must therefore choose anticommutation relations

$$\left\{d(\mathbf{p},s), d^\dagger(\mathbf{p}',s')\right\} = \delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}.$$

Inspired by previous work, we employ the anzats $S_{\mathrm{F},}(x-y) = (i\not{\partial}_x + m)F(x-y)$, implying

$$(\Box + m^2)F(x-y) = -\delta(x-y),$$

meaning F is one of the propagators of the Klein-Gordon equation.

To introduce coupling to an electromagnetic field, the Dirac equation becomes

$$(i\not{\partial} - m)\Psi = q\not{A}\Psi$$

in the minimal-coupling scheme. This somehow produces the solution

$$\Psi = \Psi_{\mathrm{in}} + q\int \mathrm{d}^4y\, S_{\mathrm{R}}(x-y)\not{A}(y)\Psi(y),\ \Psi = \Psi_{\mathrm{out}} + q\int \mathrm{d}^4y\, S_{\mathrm{A}}(x-y)\not{A}(y)\Psi(y).$$

To simplify this we will employ a perturbation scheme

$$\Psi = \Psi^{(0)} + q\Psi^{(1)} + q^2\Psi^{(2)} + \ldots,$$

which yields

$$\Psi^{(0)} = \Psi_{\mathrm{in}},\ \Psi^{(1)} = \int \mathrm{d}^4y\, S_{\mathrm{R}}(x-y)\not{A}(y)\Psi_{\mathrm{in}}(y),\ \ldots$$

We also introduce a unitary operator S such that

$$\Psi_{\mathrm{out}} = S^\dagger\Psi_{\mathrm{in}}S,$$

and expand according to

$$S = 1 + qS^{(1)} + q^2S^{(2)} + \ldots$$

The unitarity implies that $S^{(1)}$ is anti-Hermitian. Introducing the function $K = S_{\mathrm{R}} - S_{\mathrm{A}}$ we find

$$\Psi_{\mathrm{out}} = \Psi_{\mathrm{in}} + q\int \mathrm{d}^4y\, K(x-y)\not{A}\Psi.$$

A first-order expansion of the left-hand side yields

$$\Psi_{\mathrm{in}} + q((S^{(1)})^\dagger\Psi_{\mathrm{in}} - \Psi_{\mathrm{in}}(S^{(1)})^\dagger) = \Psi_{\mathrm{in}} + q\Big[(S^{(1)})^\dagger, \Psi_{\mathrm{in}}\Big],$$

implying

$$\Big[(S^{(1)})^\dagger, \Psi_{\mathrm{in}}\Big] = \int \mathrm{d}^4y\, K(x-y)\not{A}\Psi,$$

with solution

$$(S^{(1)})^\dagger = i\int \mathrm{d}^4z : \bar{\Psi}_{\mathrm{in}}\not{A}\Psi_{\mathrm{in}} : .$$