Summary of SH2372 General Relativity

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Abstract

This is a summary of SH2372 General Relativity.

The course opens with a discussion of differential geometry. As I have extensive notes on the subject in my summary of SI2360, I only keep the bare minimum in this summary and refer to those notes for details.

Contents

1 Differential Geometry on Manifolds

This is a very brief summary of choice results in differential geometry. For details, please consult my summary of SI2360 Analytical Mechanics and Classical Field Theory.

Manifolds A manifold is a set which is locally isomorphic to $\mathcal{R}^n$. We will take this to mean that we can locally impose coordinates χ^a on the manifold.

More formally, a manifold is described by a number of sets $U_i \subset \mathcal{R}^n$ called charts. To each chart belongs a set of coordinate functions χ_i which map from a subset $M_i \subset M$ to U_i such that χ_i is a smooth bijection. A set of charts such that every point $p \in M$ is found in at least one chart is called an atlas.

Tangent Vectors Tangent vectors describe how scalar fields change with displacement along a curve. In Euclidean space the tangent basis was composed of derivatives with respect to the set of coordinates. In general curved spaces, we define

$$\mathbf{E}_a = \partial_a.$$

Derivatives are linear operators, so at least the set of tangent bases span some vector space and it makes sense to call a derivative a vector. A general tangent vector is now

$$X = X^a \mathbf{E}_a = X^a \partial_a.$$

These live in the tangent space $T_p M$ of the manifold M at the point p.

To get more of a sense of how this can be related to vectors, consider the directional derivative

$$\vec{\boldsymbol{\nabla}}_{\mathbf{n}} = \mathbf{n} \cdot \vec{\boldsymbol{\nabla}} = n^a \partial_a.$$

When applied to Euclidean space, there is a direct correspondence between $\mathbf{n}$ and the directional derivative, as $\vec{\boldsymbol{\nabla}}_{\mathbf{n}} \mathbf{x} = \mathbf{n}$. For more general manifolds, tangent vectors are defined to be directional derivatives. Note that this definition carries with it the same dependence on position as was previously warned about.

Tangent vectors transform according to

$$X^a \partial_a = X^a \partial_a (\chi')^b \partial'_b,$$

implying the transformation rule

$$(X')^a = \partial_b (\chi')^a X^b,$$

which is the same as the transformation rule for contravariant vector components in Euclidean space.

Dual Vectors To define dual vectors, we first introduce the dual space as the set of all linear operations from the tangent space to real numbers. This is also a vector space. The basis for the space is defined such that

$$\mathbf{E}^a (\partial_b) = \delta_a^b.$$

In Euclidean space the dual basis was constructed from the gradient. The only concept here that carries over to manifolds is a definition based on small changes in the coordinates. More specifically, for any smooth scalar field f we define a dual vector field according to

$$df(X) = Xf = X^a \partial_a f$$

and call it the differential. This has a similar structure to an inner product if the dual vector field has components $df_a = \partial_a f$. These components correspond to those of the gradient in Euclidean space. The basis we desire is $\mathbf{E}^a = d\chi^a$. These live in the dual space $T_p^\star M$ of the manifold M at the point p.

The dual basis satisfies satisfies

$$d\chi^a (\partial_b) = \partial_b \chi^a = \delta_a^b,$$

as expected. Using this, we obtain

$$df = (\partial_a f) d\chi^a,$$

which at least looks like the differential of a function.

The components transform according to

$$\partial_a f = \partial'_b f \partial_a (\chi')^b,$$

which is the transformation rule for covariant vector components.

Tensors Having identified a basis for the tangent and dual spaces, we may now construct tensors similarly to what we have previously done. Note now that as the tangent and dual vectors belong to different vector spaces, the notion of type (n, m) tensors is more clear. This also explains why we needed to be careful with indices being up or down when studying Euclidean space, as the difference is huge for manifolds.

Flow of Vector Fields The tangent bundle of a manifold is defined as $TM = \bigcup_p T_pM$. A vector field is a map $X : M \to TM$ such that $X(p) \in T_pM$. Given this, we may define the flow of a vector field as a collection of curves γ_X which given some starting point p satisfy

$$\left.\frac{\mathrm{d}\gamma_X}{\mathrm{d}\tau}\right|_{p,s} = \left.X\right|_{\gamma_X(p,s)}.$$

We may for a fixed s define the function $\gamma_{sX}(p) = \gamma_X(p, s)$, which maps M to itself.

Pushforwards and Pullbacks Consider some function f which maps a manifold M_1 to another manifold M_2, as well as a function $g : M_2 \to \mathcal{R}$. We then define the pullback of g to M_1 by f as $f^\star g = g \circ f$. We also define the pushforward of a vector $V \in T_pM_1$ as $(f_\star V)\phi = V(f^\star\phi)$.

Tangents and the Pushforward f maps the cooordinates χ^a of M_1 to the coordinates η^μ of M_2. By definition we have

$$\begin{aligned}(f_\star V)\phi &= V(f^\star\phi)\\ &= V^a\partial_a(\phi \circ f)\\ &= V^a\partial_\mu\phi\partial_a\eta^\mu,\end{aligned}$$

meaning $f_\star V = V^a\partial_a\eta^\mu\partial_\mu$.

How do we interpret this? Consider some curve γ in M_1 which is mapped to a curve α M_2 by f. If V is the tangent of γ at some particular point, we have

$$\dot{\alpha} = \dot{\eta^\mu}\partial_\mu = \partial_a\eta^\mu\dot{\chi^a}\partial_\mu = \partial_a\eta^\mu V^a\partial_\mu,$$

meaning that the pushforward of a tangent by f is the tangent of the curve produced by f.

The Pullback of Tensors We can now define the pullback of a $(0, m)$ tensor on M_2 according to

$$f^\star\omega(V_1, \ldots, V_m) = \omega(f_\star V_1, \ldots, f_\star V_m).$$

If f is a bijection we may also define the more general pullback of a (n, m) tensor on M_2 as

$$f^\star T(V_1, \ldots, V_m, \omega_1, \ldots, \omega_n) = T(f_\star V_1, \ldots, f_\star V_m, (f^{-1})^\star\omega_1, \ldots, (f^{-1})^\star\omega_n).$$

The Lie Derivative For a tensor field T we define the Lie derivative in the X-direction as

$$\mathcal{L}_X T = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\gamma^\star_{\varepsilon X}T - T\right).$$

An Expression for the Lie Derivative For a general tensor field we have

$$(\mathcal{L}_X T)^{a_1\ldots a_n}_{b_1\ldots b_m} = X^a\partial_a T^{a_1\ldots a_n}_{b_1\ldots b_m} - \sum_{i=1}^{n} T^{a_1\ldots a_{i-1}aa_{i+1}\ldots a_n}_{b_1\ldots b_m}\partial_a X^{a_i} + \sum_{j=1}^{m} T^{a_1\ldots a_n}_{b_1\ldots b_{i-1}ab_{i+1}\ldots b_m}\partial_{b_j}X^a.$$

Connections A connection is an operator on a tensor space that satisfies the following:

- $\vec{\boldsymbol{\nabla}}_X f = Xf = X^a\partial_a f$ for a scalar field f.
- $\vec{\boldsymbol{\nabla}}_{X+Y}T = \vec{\boldsymbol{\nabla}}_X T + \vec{\boldsymbol{\nabla}}_Y T$.
- $\vec{\boldsymbol{\nabla}}_{fX}T = f\vec{\boldsymbol{\nabla}}_X T$.
- $\vec{\boldsymbol{\nabla}}_X(TS) = (\vec{\boldsymbol{\nabla}}_X T)S + T\vec{\boldsymbol{\nabla}}_X S$.

Connections on Manifolds On a manifold, a connection is specified by choosing n independent vectors X_i and defining

$$\vec{\nabla}_{X_i} X_j = \Gamma^k_{ij} X_k,$$

where the expansion coefficients are called connection coefficients or Christoffel symbols. There is no unique way to do this, as the connection then depends on the choice of vectors.

The Connection of a Tensor Field Specify the connection according to $\vec{\nabla}_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c$. We then have

$$(\vec{\nabla}_X T)^{a_1\dots a_n}_{b_1\dots b_m} = X^a \left(\partial_a T^{a_1\dots a_n}_{b_1\dots b_m} + \sum_{i=1}^{n} \Gamma^{a_i}_{ab} T^{a_1\dots a_{i-1} b a_{i+1}\dots a_n}_{b_1\dots b_m} - \sum_{j=1}^{m} \Gamma^b_{ab_j} T^{a_1\dots a_n}_{b_1\dots b_{j-1} b b_{j+1}\dots b_m} \right).$$

The Difference Between Two Connections Given two different connection $\vec{\nabla}$ and $\Box$ with connection coefficients Γ and C we have

$$(\vec{\nabla}_a X - \Box_a X)^b = \partial_a X^b + \Gamma^b_{ac} X^c - (\partial_a X^b + C^b_{ac} X^c) = (\Gamma^b_{ac} - C^b_{ac}) X^c,$$

and by the tensor quotient rule, the difference in the connection coefficients must transform as a tensor.

Parallel Transport A vector X is termed parallel if $\vec{\nabla}_a X = 0$. This defines n^2 equations for the n components of X, meaning that the system is overdetermined, and generally has no solution on a manifold. We may, however, define X to be parallel along a curve γ if

$$\vec{\nabla}_{\dot{\gamma}} X = 0.$$

This allows us to define the parallel transport as the vector field that solves the above equation with the vector X as its initial condition. This defines n equations for the n components, and the system is solvable.

In particular, using the properties of the connection we find

$$\begin{aligned}
\vec{\nabla}_{\dot{\gamma}} X &= \vec{\nabla}_{\dot{\chi}^a \partial_a} X^c \partial_c \\
&= \dot{\chi}^a \vec{\nabla}_a X^c \partial_c \\
&= \dot{\chi}^a ((\vec{\nabla}_a X^c)\partial_c + X^c \vec{\nabla}_a \partial_c) \\
&= \dot{\chi}^a (\partial_a X^b + X^c \Gamma^b_{ac})\partial_b.
\end{aligned}$$

Geodesics and the Geodesic Equation A geodesic is defined as a curve with a tangent vector that is parallel along itself.

By definition a geodesic satisfies

$$\vec{\nabla}_{\dot{\gamma}} \dot{\gamma} = \dot{\chi}^a (\partial_a \dot{\chi}^b + \dot{\chi}^c \Gamma^b_{ac})\partial_b = (\ddot{\chi}^b + \dot{\chi}^a \dot{\chi}^c \Gamma^b_{ac})\partial_b = 0,$$

and thus

$$\ddot{\chi}^b + \dot{\chi}^a \dot{\chi}^c \Gamma^b_{ac} = 0.$$

This is the geodesic equation. Given a starting point and a tangent vector, it is solvable.

Torsion The torsion tensor is a $(1, 2)$ tensor defined as

$$T(X, Y) = \vec{\nabla}_X Y - \vec{\nabla}_Y X - [X, Y].$$

To find its components, we note that

$$\begin{aligned}
T_{ab} = T(\partial_a, \partial_b) &= \vec{\nabla}_a \partial_b - \vec{\nabla}_b \partial_a - [\partial_a, \partial_b] \\
&= \Gamma^c_{ab}\partial_c - \Gamma^c_{ba}\partial_c - (\partial_a\partial_b - \partial_b\partial_a) \\
&= (\Gamma^c_{ab} - \Gamma^c_{ba})\partial_c,
\end{aligned}$$

implying

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}.$$

Curvature Consider some vector Z parallel transported along a small closed loop. The parallel transport is linear, so the result of this process must be connected to some $(1,1)$ tensor. Supposing that the loop is spanned by X and Y, we have

$$Z' - Z = R(X,Y)Z = \vec{\nabla}_X \vec{\nabla}_Y Z - \vec{\nabla}_Y \vec{\nabla}_X Z - \vec{\nabla}_{[X,Y]} Z.$$

We define $R(X,Y)Z$ as the Riemann curvature tensor. It is a $(1,3)$ tensor. Its components are defined by

$$\begin{aligned} R(\partial_a, \partial_b)\partial_c &= R^d{}_{cab}\partial_d \\ &= \vec{\nabla}_a \vec{\nabla}_b \partial_c - \vec{\nabla}_b \vec{\nabla}_a \partial_c \\ &= \vec{\nabla}_a \Gamma^f_{bc} \partial_f - \vec{\nabla}_b \Gamma^f_{ac} \partial_f \\ &= (\vec{\nabla}_a \Gamma^f_{bc})\partial_f + \Gamma^f_{bc} \vec{\nabla}_a \partial_f - (\vec{\nabla}_b \Gamma^f_{ac})\partial_f - \Gamma^f_{ac} \vec{\nabla}_b \partial_f \\ &= (\partial_a \Gamma^d_{bc} + \Gamma^f_{bc}\Gamma^d_{af} - \partial_b \Gamma^d_{ac} - \Gamma^f_{ac}\Gamma^d_{bf})\partial_d, \end{aligned}$$

and thus

$$R^d{}_{cab} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^f_{bc}\Gamma^d_{af} - \Gamma^f_{ac}\Gamma^d_{bf}.$$

Note the placements of the indices.

The Metric Tensor The metric tensor will be taken as the $(0,2)$ tensor that defines inner products on manifolds. The inner product, and therefore also the metric tensor, is a map from $T_pM \times T_pM$ that is symmetric and positive definite. Using this we may extend more of the previously performed work, for instance on curve length.

Metric Compatibility A connection is metric compatible if $\vec{\nabla}_X g = 0$ for all vectors X.

The Levi-Civita Connection For any manifold with some metric there exists a unique connection that is both metric compatible and torsion free. This connection is termed the Levi-Civita connection.

Curves of Minimal Length As the metric defines length, we define the curve length as

$$l_\gamma = \int_\gamma \mathrm{d}s = \int_0^1 \mathrm{d}t\, \sqrt{g_{ab}\dot{\chi}^a \dot{\chi}^b}.$$

Defining $\sqrt{\mathcal{L}} = g_{ab}\dot{\chi}^a\dot{\chi}^b$, the curve that minimizes the distance between the start and end points satisfies

$$\partial_a \sqrt{\mathcal{L}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\sqrt{\mathcal{L}}}{\partial\dot{\chi}^a} = \frac{1}{2\sqrt{\mathcal{L}}}\left(\partial_a \mathcal{L} - \sqrt{\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\mathcal{L}}}\frac{\partial\mathcal{L}}{\partial\dot{\chi}^a}\right)\right) = 0.$$

One can always choose a parametrization such that $\sqrt{\mathcal{L}} = 1$ (the arc length parametrization is one example), yielding

$$\frac{1}{2\sqrt{\mathcal{L}}}\left(\partial_a \mathcal{L} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\chi}^a}\right) = 0,$$

which is equivalent to extremizing the integral of $\mathcal{L}$. In terms of the coordinate functions we thus have

$$\partial_a g_{bc}\dot{\chi}^b\dot{\chi}^c - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\chi}^a} = \partial_a g_{bc}\dot{\chi}^b\dot{\chi}^c - \frac{\mathrm{d}}{\mathrm{d}t}\left(2g_{ab}\dot{\chi}^b\right) = \partial_a g_{bc}\dot{\chi}^b\dot{\chi}^c - 2g_{ab}\ddot{\chi}^b - 2\partial_c g_{ab}\dot{\chi}^b\dot{\chi}^c = 0.$$

Multiplying by $-\frac{1}{2}g^{da}$ we find

$$g^{da}g_{ab}\ddot{\chi}^b - \frac{1}{2}g^{da}\partial_a g_{bc}\dot{\chi}^b\dot{\chi}^c + g^{da}\partial_c g_{ab}\dot{\chi}^b\dot{\chi}^c = \ddot{\chi}^d + \frac{1}{2}g^{da}(2\partial_c g_{ab} - \partial_a g_{bc})\dot{\chi}^b\dot{\chi}^c = 0.$$

Renaming indices for convenience we find

$$\ddot{\chi}^b + \frac{1}{2}g^{bd}(2\partial_c g_{da} - \partial_d g_{ac})\dot{\chi}^a\dot{\chi}^c = 0.$$

Geodesics and Minimum-Length Curves Geodesics and curves of minimal length coincide if

$$\Gamma^b_{ac} = \frac{1}{2} g^{bd} (2\partial_c g_{da} - \partial_d g_{ac}).$$

As the above identification is done based on a quantity that is symmetric in the lower indices, we cannot find any information about the antisymmetric part of the connection coefficients from this.

The Levi-Civita Connection and Geodesics The connection coefficients (or Christoffel symbols) defined by the Levi-Civita connection are symmetric due to the torsion being zero. This implies

$$\Gamma^d_{ab} = \frac{1}{2} g^{dc} (\partial_b g_{ac} + \partial_a g_{cb} - \partial_c g_{ab}).$$

The Induced Metric Given some immersion f of M_1 into M_2 and supposing that the metric g exists on M_2, this induces a metric $\tilde{g} = f^\star g$ on M_1.

Components of the Induced Metric Suppose that there is an immersion $f : \chi^a \to \eta^\mu$ of one manifold into another, and that the metric is g in the outer manifold. By definition the induced metric satisfies $\tilde{g}(U, V) = f^\star g(U, V) = g(f_\star U, f_\star V)$, implying

$$\tilde{g}_{ab} U^a V^b = g_{\mu\nu} U^a \partial_a \eta^\mu V^b \partial_b \eta^\nu,$$

and as this is true for any pair of vectors we recognize

$$\tilde{g}_{ab} = g_{\mu\nu} \partial_a \eta^\mu \partial_b \eta^\nu.$$

Contravariant Curvature Components Using the Levi-Civita connection, we may introduce the contravariant curvature components

$$R_{abcd} = \frac{1}{2} \left(\partial_a \partial_d g_{bc} + \partial_b \partial_c g_{ad} - \partial_a \partial_c g_{bd} - \partial_b \partial_d g_{ac}\right) + g_{fh} (\Gamma^f_{bc} \Gamma^h_{ad} - \Gamma^f_{bd} \Gamma^h_{ac}).$$

As we see, the contravariant components are antisymmetric in the two first and two last indices. Furthermore, a simultaneous switch of the pairs of first and last indices leaves the components invariant, hence $R_{abcd} = R_{cdab} = -R_{abdc} = -R_{bacd}$. In addition we have

$$\begin{aligned} R(X, Y)Z + R(Y, Z)X + R(Z, X)Y =& \vec{\nabla}_X \vec{\nabla}_Y Z - \vec{\nabla}_Y \vec{\nabla}_X Z - \vec{\nabla}_{[X,Y]} Z + \vec{\nabla}_Y \vec{\nabla}_Z X - \vec{\nabla}_Z \vec{\nabla}_Y X - \vec{\nabla}_{[Y,Z]} X \\ &+ \vec{\nabla}_Z \vec{\nabla}_X Y - \vec{\nabla}_X \vec{\nabla}_Z Y - \vec{\nabla}_{[Z,X]} Y \\ =& \cdots = 0. \end{aligned}$$

In component form this becomes $R_{abcd} + R_{acdb} + R_{adbc} = 0$. There also exists a differential Bianchi identity

$$(\vec{\nabla}_Z R)(X, Y) + (\vec{\nabla}_X R)(Y, Z) + (\vec{\nabla}_Y R)(Z, X) = 0,$$

which in coordinate form becoms $\vec{\nabla}_f R_{abcd} + \vec{\nabla}_c R_{abdf} + \vec{\nabla}_d R_{abfc} = 0$, meaning that the total number of independent components is $\frac{1}{12} n^2 (n^2 - 1)$.

Killing Fields K is a Killing field if $\mathcal{L}_K g = 0$.

The Lie Derivative with Killing Fields Let K be a Killing field. We then obtain

$$\mathcal{L}_K g_{ab} = \vec{\nabla}_b K_a + \vec{\nabla}_a K_b = 0,$$

and all Killing fields must satisfy this relation.

The Ricci Tensor The Ricci tensor is defined as $R_{ab} = R^c{}_{acb}$. Its components are

$$R_{ab} = \partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^f_{ba} \Gamma^c_{cf} - \Gamma^f_{ca} \Gamma^c_{bf}.$$

The Ricci Scalar The Ricci scalar is defined as the contraction $\mathcal{R} = g^{ab}R_{ab} = g^{ab}R^{c}{}_{acb}$.

The Einstein Tensor The Einstein tensor is defined as $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}\mathcal{R}$.

The Divergence of the Einstein Tensor With the Levi-Civita connection we have

$$\begin{aligned} g^{ac}g^{bd}(\vec{\boldsymbol{\nabla}}_f R_{abcd} + \vec{\boldsymbol{\nabla}}_c R_{abdf} + \vec{\boldsymbol{\nabla}}_d R_{abfc}) &= \vec{\boldsymbol{\nabla}}_f(g^{ac}g^{bd}R_{abcd}) + \vec{\boldsymbol{\nabla}}_c(g^{ac}g^{bd}R_{abdf}) + \vec{\boldsymbol{\nabla}}_d(g^{ac}g^{bd}R_{abfc}) \\ &= \vec{\boldsymbol{\nabla}}_f(g^{bd}R^{c}{}_{bcd}) - \vec{\boldsymbol{\nabla}}_c(g^{ac}R^{d}{}_{adf}) - \vec{\boldsymbol{\nabla}}_d(g^{bd}R^{c}{}_{bcf}) \\ &= \vec{\boldsymbol{\nabla}}_f\mathcal{R} - \vec{\boldsymbol{\nabla}}_c(g^{ac}R_{af}) - \vec{\boldsymbol{\nabla}}_d(g^{bd}R_{bf}) \\ &= \vec{\boldsymbol{\nabla}}_f\mathcal{R} - 2\vec{\boldsymbol{\nabla}}_c R^{c}{}_{f} = 0. \end{aligned}$$

By comparison we have

$$\vec{\boldsymbol{\nabla}}_a G^{ab} = \vec{\boldsymbol{\nabla}}_a\left(R^{ab} - \frac{1}{2}g^{ab}\mathcal{R}\right),$$

and by lowering the index b we find that this must be zero.

Differential Forms The set of p-forms, or differential forms, is the set of $(0, p)$ tensors that are completely antisymmetric. They are constructed using the wedge product, defined as

$$\bigwedge_{k=1}^{p} d\chi^{a_k} = \sum_{\sigma\in S_p} \operatorname{sgn}(\sigma) \bigotimes_{k=1}^{p} d\chi^{a_{\sigma(k)}}.$$

Here S_p is the set of permutations of p elements. There exists

$$n_p^N = \binom{N}{k}$$

basis elements. We note that the wedge product is antisymmetric under the exchange of two basis elements. Hence, once an ordering of indices has been chosen, any permutation will simply create a linearly dependent map.

Consider now some antisymmetric tensor ω. Introducing the antisymmetrizer

$$\bigotimes_{k=1}^{p} d\chi^{[a_{\sigma(k)}]} = \frac{1}{p!}\sum_{\sigma\in S_p} \operatorname{sgn}(\sigma) \bigotimes_{k=1}^{p} d\chi^{a_{\sigma(k)}},$$

the symmetry yields

$$\omega = \omega_{a_1\dots a_p} \bigotimes_{k=1}^{p} d\chi^{a_{\sigma(k)}} = \omega_{a_1\dots a_p} \bigotimes_{k=1}^{p} d\chi^{[a_{\sigma(k)}]} = \frac{1}{p!}\omega_{a_1\dots a_p} \bigwedge_{k=1}^{p} d\chi^{a_k}.$$

The Exterior Derivative We define the exterior derivative of a differential form according to

$$d\omega = \frac{1}{p!}\partial_{a_1}\omega_{a_2\dots a_{p+1}} \bigwedge_{k=1}^{p+1} d\chi^{a_k},$$

which is a $p+1$-form. This notation makes sense, as at least in the case of a 0-form, we obtain

$$d\omega = \partial_a \omega d\chi^a,$$

which is exactly the form of a dual vector. Somehow this transforms as a tensor.

Integration of Differential Forms Consider a set of p tangent vectors X_i. The corresponding coordinate displacements are $\mathrm{d}\chi_i^a = X_i^a\,\mathrm{d}t_i$, with no sum over i. We would now like to compute the p-dimensional volume defined by the X_i and $\mathrm{d}t_i$. We expect that if any of the X_i are linearly dependent the volume should be zero. We also expect that the volume be linear in the X_i. This implies

$$\mathrm{d}V_p = \omega(X_1, \ldots, X_p)\,\mathrm{d}t_1 \ldots \mathrm{d}t_p$$

for some differential form ω. We now define the integral over the p-volume S over the p-form ω as

$$\int_S \omega = \int \mathrm{d}t_1 \ldots \int \mathrm{d}t_p\, \omega(\dot{\gamma}_1, \ldots, \dot{\gamma}_p).$$

Here the γ_i are the set of curves that span S, the dot symbolizes the derivative with respect to the individual curve parameters and the right-hand integration is performed over the appropriate set of parameter values.

Stokes' Theorem Stokes' theorem relates the integral of a differential form $d\omega$ over some subset V of a manifold to an integral over ∂V of another differential form. It states

$$\int_V d\omega = \oint_{\partial V} \omega.$$

Total Derivatives In the context of a general manifold, a total derivative is of the form $\vec{\boldsymbol{\nabla}}_a T^{aa_1\ldots}_{b_1\ldots}$. Stokes' theorem states

$$\int_V \vec{\boldsymbol{\nabla}}_a T^{aa_1\ldots}_{b_1\ldots} = \oint_{\partial V} T^{aa_1\ldots}_{b_1\ldots}.$$

2 Basic Concepts

A Note on Minkowski Space In special relativity we work with Minkowski space, which is an affine space with a so-called pseudo-metric. This is a metric which is not positive definite, but instead a metric which has only non-zero eigenvalues (and is thus termed non-degenerate). We will work with the signature $(1, 3)$, meaning that there are three eigenvalues of -1 and one eigenvalue 1.

Problems With Gravitation Newton's law of gravitation states that

$$\mathbf{a} = \frac{1}{m_\mathrm{I}}\mathbf{f}_\mathrm{G} = -\frac{m_\mathrm{G}}{m_\mathrm{I}}\vec{\boldsymbol{\nabla}}\Phi,\ \Phi = 4\pi G\rho.$$

There are a few possible problems and peculiarities with this, namely:

- It has no explicit time dependence, and can therefore not hold by itself in special relativity.
- As the mass density does not transform as a Lorentz scalar, we cannot generalize gravitation to special relativity like we did with electromagnetism.
- The ratio between gravitational and inertial mass is the same for all particles. This makes gravitation stand out as a fundamental force.

The Equivalence Principle The equivalence principles states that in a freely-falling, non-rotating- spatially small laboratory, the laws of physics are those of special relativity.

An Extended Description of Spacetime Einstein proposed a solution to the issue of gravitation that also incorporates the equivalence principle.

Einstein's idea was to extend the special theory of relativity to a curved spacetime and propose that this spacetime is bent by matter. This would mean that free particles move along geodesics in this spacetime, explaining the equality of gravitational and inertial mass. For the equivalence principle to hold, i.e. for it to be possible to locally construct a Minkowski spacetime, spacetime must be a pseudo-Riemannian manifold.

This transition is in fact not as big a leap as one might expect. As we have seen in special relativity, simultaneity is no longer guaranteed by the physics, so surfaces of constant time could at least be slanted. As the introduction of spacetime made it possible in principle to introduce curvilinear coordinates on time as well, we have yet to add anything. The only addition in GR is the extension of spacetime from an affine space to a manifold.

In conclusion, we will describe spacetime as a 4-dimensional manifold immersed in 5-dimensional spacetime with a pseudometric of signature $(1, 3)$ and the Levi-Civita connection imposed on it.

Static and Stationary Spacetime If there exists a time-like Killing field of a spacetime, it is stationary. If the spacetime is also orthogonal to a family of 3-surfaces, the spacetime is static. The consequences of the latter is that the metric is block diagonal, with one time block and one space block.

Comoving Observers A comoving observer is one that has fixed spatial coordinates.

Kinematics of Test Particles A test particle is a particle that itself does not affect the spacetime. Such particles can generally move through spacetime, along curves called world lines. With this motion comes the 4-velocity V, defined as the normalized tangent to a world line. In special relativity we could also define a proper acceleration by differentiating with respect to proper time. In general relativity we replace this with the 4-acceleration $A = \vec{\boldsymbol{\nabla}}_V V = \vec{\boldsymbol{\nabla}}_{\dot{\gamma}} \dot{\gamma}$. We may also define the proper acceleration α, which satisfies $\alpha^2 = -A^2 = -g(A, A)$. As $g(V, V) = 1$, we have

$$\vec{\boldsymbol{\nabla}}_V g(V, V) = 2g(V, A) = 0,$$

which implies that A is space-like. Note that the curve parameter we use is τ, which is the proper time and a measure of length in spacetime.

Free Particles A free particle moves along geodesics, meaning $A = 0$. The geodesic equations could in principle be obtained both from a variational principle described by a Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{\chi}^\mu\dot{\chi}^\nu$ or by computing Christoffel symbols, of which both approaches will be used.

4-Momentum and 4-Force We also define the 4-momentum $P = mV$ and the 4-force $F = \vec{\boldsymbol{\nabla}}_V P$.

Simultaneity and Distance Two events are simultaneous if they are on the same hypersurface of constant t. As this depends very much on the choice of coordinates on spacetime, this notion is not at all well-defined.

Similarly, distances are defined along simultaneities and are equally ill-defined. It turns out that the only sense of distance that all observers can agree on is proper time.

The Einstein Field Equations The equations describing the metric are the Einstein field equations. We will derive them from a variational principle, starting with the field equations for vacuum. The action from which we will obtain the field equations is the Einstein-Hilbert action

$$S_{\text{EH}} = -\frac{M_\alpha^2}{2}\int \mathrm{d}^4\chi\, \mathcal{R}\sqrt{-\det(g)}.$$

To derive a set of equations describing the extremum of this action, we will need to differentiate the two factors. We choose to do so with respect to the dual metric components. Considering the metric determinant first, we have

$$\frac{\partial}{\partial g^{\mu\nu}}(g^{\rho\sigma}g_{\rho\sigma}) = \frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}}g_{\rho\sigma} + g^{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial g_{\mu\nu}} = 0.$$

Next, we study the quantity $\operatorname{tr}(\ln(g))$. We have

$$\frac{\partial \operatorname{tr}(\ln(g))}{\partial g^{\mu\nu}} = \operatorname{tr}\left(\frac{\partial \ln(g)}{\partial g^{\mu\nu}}\right) = \operatorname{tr}\left(g^{-1}\frac{\partial g}{\partial g^{\mu\nu}}\right).$$

Using the identity $\mathrm{tr}(\ln(g)) = \ln(\det(g))$, we have

$$\mathrm{tr}\left(g^{-1}\frac{\partial g}{\partial g^{\mu\nu}}\right) = \frac{1}{\det(g)}\frac{\partial \det(g)}{\partial g^{\mu\nu}},$$

hence

$$\frac{\partial \det(g)}{\partial g^{\mu\nu}} = \det(g) g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial g^{\mu\nu}} = -\det(g) g_{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}}$$

and

$$\frac{\partial\sqrt{-\det(g)}}{\partial g^{\mu\nu}} = \frac{1}{2\sqrt{-\det(g)}}\det(g) g^{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-\det(g)}g_{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}}.$$

For the Ricci scalar we have $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$, hence

$$\frac{\partial \mathcal{R}}{\partial g^{\mu\nu}} = R_{\mu\nu} + g^{\rho\sigma}\frac{\partial R_{\rho\sigma}}{\partial g^{\mu\nu}}.$$

To differentiate the Ricci tensor, we first use the fact that differences in connection coefficients, and thus their derivatives, transform as tensors. Thus

$$\vec{\boldsymbol{\nabla}}_\alpha \frac{\partial \Gamma^\beta_{\gamma\delta}}{\partial g^{\mu\nu}} = \partial_\alpha \frac{\partial \Gamma^\beta_{\gamma\delta}}{\partial g^{\mu\nu}} + \Gamma^\beta_{\alpha\rho}\frac{\partial \Gamma^\rho_{\gamma\delta}}{\partial g^{\mu\nu}} - \Gamma^\rho_{\alpha\gamma}\frac{\partial \Gamma^\beta_{\rho\delta}}{\partial g^{\mu\nu}} - \Gamma^\rho_{\alpha\delta}\frac{\partial \Gamma^\beta_{\gamma\rho}}{\partial g^{\mu\nu}}.$$

Now we have

$$\begin{aligned}
\frac{\partial R_{\rho\sigma}}{\partial g^{\mu\nu}} &= \frac{\partial R^\alpha_{\rho\alpha\sigma}}{\partial g^{\mu\nu}} \\
&= \frac{\partial}{\partial g^{\mu\nu}}\left(\partial_\alpha \Gamma^\alpha_{\sigma\rho} - \partial_\sigma \Gamma^\alpha_{\alpha\rho} + \Gamma^\gamma_{\sigma\rho}\Gamma^\alpha_{\alpha\gamma} - \Gamma^\gamma_{\alpha\rho}\Gamma^\alpha_{\sigma\gamma}\right) \\
&= \frac{\partial \partial_\alpha \Gamma^\alpha_{\sigma\rho}}{\partial g^{\mu\nu}} - \frac{\partial \partial_\sigma \Gamma^\alpha_{\alpha\rho}}{\partial g^{\mu\nu}} + \Gamma^\gamma_{\sigma\rho}\frac{\partial \Gamma^\alpha_{\alpha\gamma}}{\partial g^{\mu\nu}} + \Gamma^\alpha_{\alpha\gamma}\frac{\partial \Gamma^\gamma_{\sigma\rho}}{\partial g^{\mu\nu}} - \Gamma^\gamma_{\alpha\rho}\frac{\partial \Gamma^\alpha_{\sigma\gamma}}{\partial g^{\mu\nu}} - \Gamma^\alpha_{\sigma\gamma}\frac{\partial \Gamma^\gamma_{\alpha\rho}}{\partial g^{\mu\nu}} \\
&= \frac{\partial \partial_\alpha \Gamma^\alpha_{\sigma\rho}}{\partial g^{\mu\nu}} + \Gamma^\alpha_{\alpha\gamma}\frac{\partial \Gamma^\gamma_{\sigma\rho}}{\partial g^{\mu\nu}} - \Gamma^\gamma_{\alpha\rho}\frac{\partial \Gamma^\alpha_{\gamma\sigma}}{\partial g^{\mu\nu}} - \Gamma^\gamma_{\alpha\sigma}\frac{\partial \Gamma^\alpha_{\rho\gamma}}{\partial g^{\mu\nu}} - \vec{\boldsymbol{\nabla}}_\sigma \Gamma^\alpha_{\alpha\rho} \\
&= \vec{\boldsymbol{\nabla}}_\alpha \Gamma^\alpha_{\sigma\rho} - \vec{\boldsymbol{\nabla}}_\sigma \Gamma^\alpha_{\alpha\rho}.
\end{aligned}$$

As the Levi-Civita connection is metric compatible, we have

$$\frac{\partial \mathcal{R}}{\partial g^{\mu\nu}} = R_{\mu\nu} + \vec{\boldsymbol{\nabla}}_\alpha (g^{\rho\sigma}\Gamma^\alpha_{\sigma\rho}) - \vec{\boldsymbol{\nabla}}_\sigma (g^{\rho\sigma}\Gamma^\alpha_{\alpha\rho}).$$

The latter terms may be ignored by converting it to a contribution at the boundary.

There are now two ways to proceed. The first is to note that the Lagrangian density has no functional dependence on the derivatives of the metric, implying that its derivatives with respect to the metric are zero, implying

$$\frac{\partial}{\partial g^{\mu\nu}}\left(\mathcal{R}\sqrt{-\det(g)}\right) = R_{\mu\nu}\sqrt{-\det(g)} - \frac{1}{2}\mathcal{R}\sqrt{-\det(g)}g_{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}} = 0,$$

yielding the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = G_{\mu\nu} = 0.$$

The other is to consider variations of the metric and their impact on the action. This will yield

$$\delta S = -\frac{M_\alpha^2}{2}\int \mathrm{d}^4\chi\, \frac{\partial \mathcal{R}\sqrt{-\det(g)}}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu} = -\frac{M_\alpha^2}{2}\int \mathrm{d}^4\chi\,\sqrt{-\det(g)}\left(R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}\right)\delta g^{\mu\nu},$$

and extremization yields the same equations.

The Energy-Momentum Tensor To obtain the field equations for other cases we will add terms to the action. In general, by adding a term S_{matter} to the action with the coefficient $\frac{M_\alpha^2}{2}$ baked into it, we obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu},$$

where

$$T_{\mu\nu} = \frac{2}{\sqrt{-\det(g)}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

is called the energy-momentum tensor.

Alternative Field Equations Starting from the Einstein field equation, we may contract with the metric according to

$$g^{\mu\nu}R_{\nu\sigma} - \frac{1}{2}\delta^\sigma_\mu\mathcal{R} = 8\pi G g^{\mu\nu}T_{\nu\sigma} = 8\pi G T^\mu{}_\sigma,$$

and setting $\mu = \sigma$ and summing yields

$$8\pi G T^\mu{}_\mu = g^{\mu\nu}R_{\nu\mu} - 2\mathcal{R} = -\mathcal{R},$$

which we may insert back into the Einstein field equations to obtain

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\sigma{}_\sigma\right).$$

Example Calculations of the Energy-Momentum Tensor The definition of the energy-momentum tensor is by itself not very elucidating. I therefore show two examples, both to illustrate the results one could obtain and how to do the computation in practice.

The first example is for an electromagnetic field in vacuum, defined by the action and the Lagrangian

$$S = \int \mathrm{d}^4\chi\,\sqrt{-\det(g)}\mathcal{L},\ \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Because electromagnetic theory is a field theory, writing the Lagrangian in a proper way is easy, and as the Lagrangian does not depend on derivatives of the metric we have

$$\begin{aligned}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-\det(g)}g_{\mu\nu}\cdot -\frac{1}{4}g^{\alpha\sigma}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} + \sqrt{-\det(g)}\cdot -\frac{1}{4}F_{\alpha\beta}F_{\rho\sigma}(\delta^\alpha_\mu\delta^\rho_\nu g^{\beta\sigma} + g^{\alpha\rho}\delta^\beta_\mu\delta^\sigma_\nu) \\ &= \frac{1}{4}\sqrt{-\det(g)}\left(\frac{1}{2}g_{\mu\nu}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} - (F_{\mu\beta}F_{\nu\sigma}g^{\beta\sigma} + F_{\alpha\mu}F_{\rho\nu}g^{\alpha\rho})\right).\end{aligned}$$

As the Faraday tensor is antisymmetric, we may rewrite this as

$$\begin{aligned}\frac{\delta S}{\delta g^{\mu\nu}} &= \frac{1}{4}\sqrt{-\det(g)}\left(\frac{1}{2}g_{\mu\nu}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} - (F_{\mu\beta}F_{\nu\sigma}g^{\beta\sigma} + F_{\mu\alpha}F_{\nu\rho}g^{\alpha\rho})\right) \\ &= \frac{1}{4}\sqrt{-\det(g)}\left(\frac{1}{2}g_{\mu\nu}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma} - 2F_{\mu\beta}F_{\nu\sigma}g^{\beta\sigma}\right) \\ &= \frac{1}{4}\sqrt{-\det(g)}\left(\frac{1}{2}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - 2F_{\beta\mu}F^\beta{}_\nu\right),\end{aligned}$$

and arrive at the final expression

$$T_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - F_{\beta\mu}F^\beta{}_\nu.$$

This expression coincides exactly with one derived from other principles in the context of special relativity.

The next example is for a massive free particle, defined by the action and the Lagrangian

$$S = \int \mathrm{d}\tau\,\mathcal{L},\ \mathcal{L} = m\sqrt{g_{\mu\nu}\dot{\chi}^\mu\dot{\chi}^\nu}.$$

An obvious difficulty with this case is that the theory describing the motion of free particles is not a field theory, so we will have to write it as one. We do that by performing the rewrite

$$S = \int \mathrm{d}\tau \int \mathrm{d}^4\eta \, m\delta^4(\eta - \chi(\tau))\sqrt{g_{\mu\nu}\dot{\eta}^\mu\dot{\eta}^\nu},$$

where we have introduced the generalized Dirac delta such that the integral over all spacetime of $\delta^4(\eta)$ is 1. To proceed we will need to integrate out the affine parameter τ. We can perform this elimination by integrating out χ^0 under the assumption that this particular coordinate is also usable as a parameter for the world line. This will yield

$$S = \int \mathrm{d}^4\eta \, m\delta^3(\eta - \chi(\tau))\frac{1}{\dot{\chi}^0}\sqrt{g_{\mu\nu}\dot{\eta}^\mu\dot{\eta}^\nu},$$

where τ may in this expression be taken as the inverse function of χ^0. Having obtained a proper kind of action, we may now compute

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{m\delta^3(\eta - \chi(\tau))\dot{\eta}^\mu\dot{\eta}^\nu}{2\dot{\chi}^0\sqrt{g_{\mu\nu}\dot{\eta}^\mu\dot{\eta}^\nu}},$$

and

$$T^{\mu\nu} = \frac{m\delta^3(\eta - \chi(\tau))\dot{\eta}^\mu\dot{\eta}^\nu}{\dot{\chi}^0\sqrt{-\det(g)g_{\mu\nu}\dot{\eta}^\mu\dot{\eta}^\nu}}.$$

In particular, choosing proper time as the affine parameter we have

$$T^{\mu\nu} = \frac{m\delta^3(\eta - \chi(\tau))\dot{\eta}^\mu\dot{\eta}^\nu}{\dot{\chi}^0\sqrt{-\det(g)}}.$$

To make it even simpler to study, we choose Cartesian coordinates, for which we find

$$T^{00} = m\gamma\delta^3(\eta - \chi(\tau)) = E\delta^3(\eta - \chi(\tau)), \; T^{0i} = m\delta^3(\eta - \chi(\tau))\dot{\eta}^i = p^i\delta^3(\eta - \chi(\tau)),$$

in other words describing a system with the exact energy and momentum of the free particle located at the position of the particle.

Ideal Fluids An ideal fluid is a substance such that its energy-momentum tensor is of the form

$$T^{\mu\nu} = (\rho_0 + p)U^\mu U^\nu - pg^{\mu\nu},$$

where u^μ is the 4-velocity of the rest frame of the fluid and ρ_0 and p are the energy density and pressure of the fluid as measured in the rest frame.

The Weak Field Limit We will now study the Einstein field equations in a limit where the effects of general relativity are weak, in the hopes of finding some limit that reproduces Newtonian gravity. We will do this by linearizing the metric according to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where η is the Minkowski metric and all components of h are small. To raise its components, we use the fact that the metric should produce the Kronecker delta to obtain

$$\delta^\mu_\sigma = g^{\mu\nu}g_{\nu\sigma} = (\eta^{\mu\nu} + f^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) = \eta^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\nu\sigma} + f^{\mu\nu}\eta_{\nu\sigma},$$

where the perturbations of the inverse metric must also necessarily be small, allowing us to neglect higher-order terms. This yields

$$\begin{aligned} \eta^{\mu\nu}h_{\nu\sigma} + f^{\mu\nu}\eta_{\nu\sigma} &= 0, \\ f^{\mu\nu} &= -\eta^{\sigma\nu}\eta^{\mu\rho}h_{\rho\sigma}. \end{aligned}$$

When computing the curvature tensor, we will need the Christoffel symbols, which are given by

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}(\eta^{\lambda\rho} - \eta^{\sigma\rho}\eta^{\lambda\gamma}h_{\gamma\sigma})(\partial_\nu h_{\mu\rho} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}) \\ &= \frac{1}{2}(\eta^{\lambda\rho}\partial_\nu h_{\mu\rho} + \eta^{\lambda\rho}\partial_\mu h_{\rho\nu} - \eta^{\lambda\rho}\partial_\rho h_{\mu\nu}).\end{aligned}$$

We will also need products of the Christoffel symbols. As the symbols themselves are linear in the magnitude of the perturbations, we may however neglect these terms, yielding

$$R^{\mu}{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\lambda\nu}$$

and

$$\begin{aligned}R_{\nu\sigma} &= \partial_\mu \Gamma^{\mu}_{\sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\mu\nu} \\ &= \frac{1}{2}\left(\partial_\mu(\eta^{\mu\rho}\partial_\nu h_{\sigma\rho} + \eta^{\mu\rho}\partial_\sigma h_{\rho\nu} - \eta^{\mu\rho}\partial_\rho h_{\sigma\nu}) - \partial_\sigma(\eta^{\mu\rho}\partial_\nu h_{\mu\rho} + \eta^{\mu\rho}\partial_\mu h_{\rho\nu} - \eta^{\mu\rho}\partial_\rho h_{\mu\nu})\right) \\ &= \frac{1}{2}\left(\partial_\mu(\partial_\nu h_\sigma{}^{\mu} + \partial_\sigma h^{\mu}{}_{\nu} - \eta^{\mu\rho}\partial_\rho h_{\sigma\nu}) - \partial_\sigma(\partial_\nu h^{\mu}{}_{\mu} + \partial_\mu h^{\mu}{}_{\nu} - \eta^{\mu\rho}\partial_\rho h_{\mu\nu})\right) \\ &= \frac{1}{2}\left(\partial_\mu\partial_\nu h_\sigma{}^{\mu} - \partial_\mu \eta^{\mu\rho}\partial_\rho h_{\sigma\nu} - \partial_\sigma\partial_\nu h^{\mu}{}_{\mu} + \partial_\sigma \eta^{\mu\rho}\partial_\rho h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu\partial_\nu h^{\mu}{}_{\sigma} - \partial_\mu\partial^\mu h_{\sigma\nu} - \partial_\sigma\partial_\nu h + \partial_\sigma\partial^\mu h_{\mu\nu}\right) \\ &= \frac{1}{2}(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu}),\end{aligned}$$

where we have introduced the trace of the perturbation $h = h^{\mu}{}_{\mu}$. We can rewrite this further by introducing the d'Alembertian $\Box = \vec{\boldsymbol{\nabla}}^{\mu}\vec{\boldsymbol{\nabla}}_{\mu}$. When applied to the perturbations of the metric, the only terms that are linear in its magnitude are the derivative terms, hence $\Box h_{\mu\nu} = \partial^\sigma\partial_\sigma h_{\mu\nu}$, and

$$R_{\nu\sigma} = \frac{1}{2}(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu}).$$

Next, the Ricci scalar is

$$\begin{aligned}\mathcal{R} &= \frac{1}{2}(\eta^{\nu\sigma} - \eta^{\sigma\nu}\eta^{\mu\rho}h_{\rho\sigma})(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu}) \\ &= \frac{1}{2}\eta^{\nu\sigma}(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu}) \\ &= \frac{1}{2}(\partial^\sigma\partial_\mu h^{\mu}{}_{\sigma} - \Box h - \Box h + \partial^\nu\partial_\mu h^{\mu}{}_{\nu}) \\ &= \partial^\sigma\partial_\mu h^{\mu}{}_{\sigma} - \Box h \\ &= \partial_\mu\partial_\sigma h^{\mu\sigma} - \Box h,\end{aligned}$$

and the Einstein tensor is

$$\begin{aligned}G_{\nu\sigma} &= R_{\nu\sigma} - \frac{1}{2}g_{\nu\sigma}\mathcal{R} \\ &= \frac{1}{2}(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu}) - \frac{1}{2}(\eta_{\nu\sigma} + h_{\nu\sigma})(\partial_\mu\partial_\lambda h^{\mu\lambda} - \Box h) \\ &= \frac{1}{2}(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu} - \eta_{\nu\sigma}(\partial_\mu\partial_\lambda h^{\mu\lambda} - \Box h)) \\ &= \frac{1}{2}(\partial_\nu\partial_\mu h^{\mu}{}_{\sigma} - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^{\mu}{}_{\nu} - \eta_{\nu\sigma}\partial_\mu\partial_\lambda h^{\mu\lambda} + \eta_{\nu\sigma}\Box h).\end{aligned}$$

To proceed, we choose coordinates such that $\Box\chi^\mu = 0$. In these coordinates we have

$$\begin{aligned}\Box\chi^\mu &= \vec{\boldsymbol{\nabla}}^{\nu}\vec{\boldsymbol{\nabla}}_{\nu}\chi^\mu \\ &= \vec{\boldsymbol{\nabla}}^{\nu}(\partial_\nu\chi^\mu) \\ &= \partial^\nu(\partial_\nu\chi^\mu) - g^{\sigma\nu}\Gamma^{\gamma}_{\nu\sigma}\partial_\gamma\chi^\mu \\ &= -g^{\nu\sigma}\Gamma^{\mu}_{\nu\sigma} = 0,\end{aligned}$$

and using the previously obtained Christoffel symbols we find

$$\frac{1}{2}\eta^{\nu\sigma}(\eta^{\mu\rho}\partial_\sigma h_{\nu\rho} + \eta^{\mu\rho}\partial_\nu h_{\rho\sigma} - \eta^{\mu\rho}\partial_\rho h_{\nu\sigma}) = \frac{1}{2}(\partial^\nu h_\nu{}^\mu + \partial_\nu h^{\mu\nu} - \partial^\mu h) = 0,$$

or

$$\partial^\mu h = \partial^\nu h_\nu{}^\mu + \partial_\nu h^{\mu\nu} = 2\partial_\nu h^{\nu\mu}.$$

Lowering the free index yields

$$\partial_\nu h^\nu{}_\mu = \frac{1}{2}\partial_\mu h.$$

The Einstein tensor is now

$$\begin{aligned} G_{\nu\sigma} &= \frac{1}{2}(\partial_\nu\partial_\mu h^\mu{}_\sigma - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \partial_\sigma\partial_\mu h^\mu{}_\nu - \eta_{\nu\sigma}\partial_\mu\partial_\lambda h^{\mu\lambda} + \eta_{\nu\sigma}\Box h) \\ &= \frac{1}{2}\left(\frac{1}{2}\partial_\nu\partial_\sigma h - \Box h_{\nu\sigma} - \partial_\nu\partial_\sigma h + \frac{1}{2}\partial_\sigma\partial_\nu h - \eta_{\nu\sigma}\partial^\lambda\partial_\mu h^\mu{}_\lambda + \eta_{\nu\sigma}\Box h\right) \\ &= \frac{1}{2}\left(-\Box h_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\partial^\lambda\partial_\lambda h + \eta_{\nu\sigma}\Box h\right) \\ &= \frac{1}{2}\left(-\Box h_{\nu\sigma} + \frac{1}{2}\eta_{\nu\sigma}\Box h\right) \\ &= -\frac{1}{2}\left(\Box h_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\Box h\right). \end{aligned}$$

The Einstein field equations are thus

$$G_{\nu\sigma} = -\frac{1}{2}\left(\Box h_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\Box h\right) = 8\pi G T_{\nu\sigma},$$

or alternatively, by defining $\bar{h}_{\nu\sigma} = h_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}h$, the compatibility of the metric implies

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

To proceed we say a little more about the Newtonian limit. It should be found at low speeds, meaning that the energy-momentum tensor will be dominated by T_{00}, which is the energy density, or equivalently the mass density. The corresponding component of $\bar{h}$ will thus also dominate the other side. It should also be obtained when the involved masses are small. If the situation also evolves slowly, we may neglect time derivatives to obtain

$$\nabla^2\bar{h}_{00} = 16\pi G\rho.$$

We may thus identify $\bar{h}_{00} = 4\Phi$, where Φ is the Newtonian gravitational potential. Next, the trace of $\bar{h}$ is

$$\bar{h} = h - \frac{1}{2}\delta^\mu_\mu h = -h,$$

which according to the above should be dominated by the time component. Thus

$$h_{\nu\sigma} = \bar{h}_{\nu\sigma} + \frac{1}{2}\eta_{\nu\sigma}h = \bar{h}_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\bar{h} = 4\Phi\left(\delta^0_\mu\delta^0_\nu - \frac{1}{2}\eta_{\mu\nu}\right)$$

and

$$h_{00} = 2\phi,\ h_{ij} = 2\Phi.$$

The corresponding spacetime length is

$$\mathrm{d}s^2 = (1+2\Phi)\,\mathrm{d}t^2 - (1-2\Phi)\,\mathrm{d}x^i\,\mathrm{d}x^i\,.$$

Next we study geodesics in this spacetime. For example, in a spherically symmetric case we may choose

$$\mathcal{L} = (1+2\Phi)\dot{t}^2 - (1-2\Phi)\left(\dot{r}^2 + r^2(\sin^2(\theta)\dot{\phi}^2 + \dot{\theta}^2)\right).$$

The geodesic equations are

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left((1+2\Phi)\dot{t}\right) = 0,\\
&2\partial_r\Phi\dot{t}^2 + 2\partial_r\Phi\left(\dot{r}^2 + r^2(\sin^2(\theta)\dot{\phi}^2+\dot{\theta}^2)\right) - (1-2\Phi)\left(2r(\sin^2(\theta)\dot{\phi}^2+\dot{\theta}^2)\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}(-2(1-2\Phi)\dot{r}) = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2(\theta)\dot{\phi}\right) = 0,\\
&(1-2\Phi)\left(2r^2\sin(\theta)\cos(\theta)\dot{\phi}^2\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-2(1-2\Phi)r^2\dot{\theta}\right) = 0.
\end{aligned}
$$

We assume to be in a weak-field situation, hence the first equation implies $\dot{t}$ being constant. We may take this constant to be 1, essentially involving a parametrization in terms of t. Next we may by symmetry choose θ to be constant, this constant necessarily being $\frac{\pi}{2}$, to find

$$
\begin{aligned}
&2\partial_r\Phi + 2\partial_r\Phi\left(\dot{r}^2 + \frac{L^2}{r^2}\right) - 2(1-2\Phi)\frac{L^2}{r^3} + \frac{\mathrm{d}}{\mathrm{d}\tau}(2(1-2\Phi)\dot{r}) = 0,\\
&r^2\dot{\phi} = L.
\end{aligned}
$$

Expanding the derivatives yields

$$
\partial_r\Phi + \partial_r\Phi\left(\dot{r}^2+\frac{L^2}{r^2}\right) - (1-2\Phi)\frac{L^2}{r^3} + (1-2\Phi)\ddot{r} - \partial_r\Phi\dot{r}^2 = \partial_r\Phi\left(1+\frac{L^2}{r^2}-\dot{r}^2\right) - (1-2\Phi)\frac{L^2}{r^3} + (1-2\Phi)\ddot{r} = 0.
$$

Assuming slow motion, such that L and $\dot{r}$ are all much smaller than 1, we may simplify the above to

$$
\partial_r\Phi - \frac{L^2}{r^3} + \ddot{r} = \ddot{r} + \partial_r\Phi - r\dot{\phi}^2 = 0,
$$

which is the radial equation of motion for a central potential.

Gravitational Waves We have seen that in the weak field limit the perturbations to the metric in vacuum satisfy the wave equation. The solution is of the form

$$
\bar{h}_{\mu\nu} = A_{\mu\nu}e^{-ik^\sigma x_\sigma}.
$$

This is an eigenfunction of the d'Alembertian with eigenvalue $-k^\mu k_\mu$, hence the wavevector must be light-like and waves in the metric travel at the speed of light.

With the chosen gauge for the coordinates we still have some freedom - namely, the coordinate transform $x^\mu \to x^\mu + B^\mu e^{-ik^\sigma x_\sigma}$ does not change the gauge condition. We may also choose B^μ such that $A^\mu{}_\mu = 0$ and $A_{0\mu} = 0$ in some frame. Furthermore, in this gauge we have

$$
\partial_\mu\bar{h}^{\mu\nu} = \partial_\mu h^{\mu\nu} - \frac{1}{2}\partial_\mu(\eta^{\mu\nu}h) = \partial_\mu h^{\nu\mu} - \frac{1}{2}\partial^\nu h = 0.
$$

This implies $k_\mu A^{\mu\nu} = 0$.

Specializing to a wave propagating in the third Cartesian direction, the only non-zero amplitudes are $A^{11} = -A^{22} = A_+$ and $A^{12} = A^{21} = A_\times$. As gravitational waves propagate between geodesics, we need to consider geodesic deviation. To do this, consider two events close to each other on a simultaneity. Evolution in proper time will then take the events to a new simultaneity. Let X be the vector field that is tangent to the simultaneity and pointing in the direction of the other event for each τ, and U the tangent of one of the geodesics. We then have

$$
\ddot{X} = \vec{\boldsymbol{\nabla}}_U\vec{\boldsymbol{\nabla}}_U X = R(U,X)U,
$$

where we have chosen the torsion such that $[X,U]=0$. In component form:

$$
\ddot{X}^\mu = R^\mu{}_{\lambda\sigma\nu}U^\lambda U^\sigma X^\nu.
$$

For a slowly moving test particle we obtain

$$
\ddot{X}^\mu \approx R^\mu{}_{00\nu}X^\nu.
$$

We have

$$
\begin{aligned}
R_{\mu 0 0 \nu} &= (\eta_{\mu\sigma} + h_{\mu\sigma})(\partial_0 \Gamma^\sigma_{\nu 0} - \partial_\nu \Gamma^\sigma_{00}) \\
&= \frac{1}{2}\eta_{\mu\sigma}\left(\partial_0(\eta^{\sigma\rho}\partial_0 h_{\nu\rho} + \eta^{\sigma\rho}\partial_\nu h_{\rho 0} - \eta^{\sigma\rho}\partial_\rho h_{\nu 0}) - \partial_\nu(\eta^{\sigma\rho}\partial_0 h_{0\rho} + \eta^{\sigma\rho}\partial_0 h_{\rho 0} - \eta^{\sigma\rho}\partial_\rho h_{00})\right) \\
&= \frac{1}{2}\eta_{\mu\sigma}\left(\eta^{\sigma\rho}\partial_0\partial_0 h_{\nu\rho} + \eta^{\sigma\rho}\partial_0\partial_\nu h_{\rho 0} - \eta^{\sigma\rho}\partial_0\partial_\rho h_{\nu 0} - \eta^{\sigma\rho}\partial_\nu\partial_0 h_{0\rho} - \eta^{\sigma\rho}\partial_\nu\partial_0 h_{\rho 0} + \eta^{\sigma\rho}\partial_\nu\partial_\rho h_{00}\right) \\
&= \frac{1}{2}\delta^\rho_\mu\left(\partial_0^2 h_{\nu\rho} - \partial_0\partial_\rho h_{\nu 0} - \partial_\nu\partial_0 h_{\rho 0} + \partial_\nu\partial_\rho h_{00}\right) \\
&= \frac{1}{2}\left(\partial_0^2 h_{\nu\mu} - \partial_0\partial_\mu h_{\nu 0} - \partial_\nu\partial_0 h_{\mu 0} + \partial_\nu\partial_\mu h_{00}\right) \\
&= \frac{1}{2}\partial_0^2 h_{\mu\nu},
\end{aligned}
$$

where the last equality follows from the particular geometry. Writing this out we find

$$
\ddot{X}^\mu = \frac{1}{2}X^\nu \partial_0^2 h^\mu{}_\nu = \frac{1}{2}X^\nu A^\mu{}_\nu \partial_0^2 e^{-ik^\sigma x_\sigma}.
$$

In the case of $A_\times = 0$ we can simplify this to

$$
\ddot{X}^1 = \frac{1}{2}X^\nu A^1{}_\nu \partial_0^2 e^{-ik^\sigma x_\sigma} = -\frac{1}{2}X^1 A_+ \partial_0^2 e^{-ik^\sigma x_\sigma},\ \ddot{X}^2 = \frac{1}{2}X^2 A_+ \partial_0^2 e^{-ik^\sigma x_\sigma}.
$$

To leading order in the amplitude, we thus find

$$
X^1 = X^1(0)\left(1 - \frac{1}{2}A_+ e^{-ik^\sigma x_\sigma}\right),\ X^2 = X^2(0)\left(1 + \frac{1}{2}A_+ e^{-ik^\sigma x_\sigma}\right).
$$

I believe the idea is that because the components of the tangent vector oscillate, so do the distances between the events on the two geodesics. In this case the oscillations are in the two coordinate directions. For the case $A_+ = 0$ we instead have

$$
\ddot{X}^1 = -\frac{1}{2}X^2 A_+ \partial_0^2 e^{-ik^\sigma x_\sigma},\ \ddot{X}^2 = -\frac{1}{2}X^1 A_+ \partial_0^2 e^{-ik^\sigma x_\sigma}.
$$

This produces the same behaviour, but in a coordinate system rotate 45° relative to the original system.

Gravitational Lensing Consider a light source and an observer separated by a distribution of matter producing a gravitational potential Φ. In the weak-field limit the metric is described by

$$
\mathrm{d}s^2 = (1 + 2\Phi)\,\mathrm{d}t^2 - (1 - 2\Phi)\,\mathrm{d}\mathbf{x}^2\,,
$$

and the path of a ray of light travelling between the two is described by

$$
(1 + 2\Phi)\dot{t}^2 - (1 - 2\Phi)\dot{\mathbf{x}}^2 = 0.
$$

In the weak-field limit we expect $\Phi \ll 1$, and thus $\dot{t}^2 \approx \dot{\mathbf{x}}^2$.

In the case where the potential is time-independent, the geodesic equation for the time coordinate yields

$$
(1 + 2\Phi)\dot{t} = c,
$$

and we are free to choose $c = 1$ by rescaling. Next, the equation for the spatial coordinates is

$$
-\ddot{\mathbf{x}} + 2(\dot{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}}\Phi)\dot{\mathbf{x}} = (\dot{t}^2 + \dot{\mathbf{x}}^2)\vec{\boldsymbol{\nabla}}\Phi.
$$

Choosing a coordinate system such that $\dot{\mathbf{x}} = \mathbf{e}_x$ initially, we expect in the weak-field limit

$$
\dot{\mathbf{x}} = (1 + \dots)\mathbf{e}_x + v^2\mathbf{e}_y + v^3\mathbf{e}_z,
$$

where the new velocity components are small and the dots represent perturbations of the path of higher order than the other velocity components. Inserting this into the geodesic equation yields

$$
-\dot{v}^2 = 2(\vec{\boldsymbol{\nabla}}\Phi)^2,
$$

where we refer to vector components on either side. We can solve this differential equation by integrating along the unperturbed path. In particular, by moving both the observer and the source to infinity we find

$$
\dot{v}^2 = -\int\limits_{-\infty}^{\infty} \mathrm{d}t\, 2(\vec{\boldsymbol{\nabla}}\Phi)^2.
$$

The Schwarzschild Solution The Schwarzschild solution is the simplest solution for a spherically symmetric metric.

To derive it, we will use spherical coordinates to describe the space part, implying that we construct spacetime as a combination of spherical shells. The metric will then take the form

$$\mathrm{d}s^2 = f(t,r)\,\mathrm{d}t^2 - g(t,r)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\,,\ \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2$$

in units where $c = 1$. The proceeding work will involve dividing the functions f and g, so we define $f = e^{2\alpha},\ g = e^{2\beta}$. As the metric is diagonal we can write

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2g_{\sigma\sigma}}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}),\ \text{no sum over } \sigma.$$

Again, as the metric is diagonal, at least two of the indices must be equal, immediately identifying any Christoffel symbol with three different indices as zero. The diagonality of the metric and the angle independence of the time and radial components also implies

$$\Gamma^t_{t\theta} = \Gamma^t_{\theta t} = \Gamma^t_{t\phi} = \Gamma^t_{\phi t} = \Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^r_{r\phi} = \Gamma^r_{\phi r} = \Gamma^\theta_{tt} = \Gamma^\theta_{rr} = \Gamma^\phi_{tt} = \Gamma^\phi_{rr} = 0.$$

The time independence of the solid angle also implies $\Gamma^t_{\theta\theta} = \Gamma^t_{\phi\phi} = \Gamma^\theta_{tt} = \Gamma^\phi_{tt} = \Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} = \Gamma^\phi_{\phi t} = \Gamma^\phi_{t\phi} = 0$. Next, as the metric is independent of ϕ, we must also have $\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \Gamma^\phi_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0$. We also find $\Gamma^\theta_{\theta\theta} = 0$ as the θ component is independent of θ. The possibly non-zero ones are

$$\begin{aligned}
&\Gamma^t_{tt} = \frac{1}{2}e^{-2\alpha}\cdot 2\partial_t\alpha e^{2\alpha} = \partial_t\alpha,\ \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}e^{-2\alpha}\cdot 2\partial_r\alpha e^{2\alpha} = \partial_r\alpha,\ \Gamma^t_{rr} = -\frac{1}{2}e^{-2\alpha}\cdot -2\partial_t\beta e^{2\beta} = \partial_t\beta e^{2(\beta-\alpha)},\\
&\Gamma^r_{tt} = -\frac{1}{2}e^{-2\beta}\cdot -2\partial_r\alpha e^{2\alpha} = \partial_r\alpha e^{2(\alpha-\beta)},\ \Gamma^r_{tr} = \Gamma^r_{rt} = -\frac{1}{2}e^{-2\beta}\cdot -2\partial_t\beta e^{2\beta} = \partial_t\beta,\ \Gamma^r_{rr} = -\frac{1}{2}e^{-2\beta}\cdot -2\partial_r\beta e^{2\beta} = \partial_r\beta,\\
&\Gamma^r_{\theta\theta} = -\frac{1}{2}e^{-2\beta}\cdot 2r = -re^{-2\beta},\ \Gamma^r_{\phi\phi} = -\frac{1}{2}e^{-2\beta}\cdot 2r\sin^2(\theta) = -r\sin^2(\theta)e^{-2\beta},\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = -\frac{1}{2r^2}\cdot -2r = \frac{1}{r},\\
&\Gamma^\theta_{\phi\phi} = -\frac{1}{2r^2}\cdot 2r^2\sin(\theta)\cos(\theta) = -\sin(\theta)\cos(\theta),\ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = -\frac{1}{2r^2\sin^2(\theta)}\cdot -2r\sin^2(\theta) = \frac{1}{r},\\
&\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = -\frac{1}{2r^2\sin^2(\phi)}\cdot -2r^2\sin(\theta)\cos(\theta) = \cot(\theta).
\end{aligned}$$

We will now need to solve the Einstein field equations. We will use the formulation purely in terms of the energy-momentum tensor, which yields $R_{\mu\nu} = 0$ in vacuum. We have

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\sigma_{\nu\mu}\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\nu\sigma}.$$

Let us commence by finding restrictions on the functions α and β. Cheating by looking into the future I find

$$\begin{aligned}
R_{tr} =&\partial_\rho\Gamma^\rho_{rt} - \partial_r\Gamma^\rho_{\rho t} + \Gamma^\sigma_{rt}\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\rho t}\Gamma^\rho_{r\sigma}\\
=&\partial_t\partial_r\alpha + \partial_r\partial_t\beta - \partial_r\partial_t\alpha - \partial_r\partial_t\beta + \Gamma^t_{rt}(\Gamma^t_{tt} + \Gamma^r_{rt}) + \Gamma^r_{rt}(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - \Gamma^t_{tt}\Gamma^t_{rt} - \Gamma^t_{rt}\Gamma^r_{rt}\\
&- \Gamma^r_{tt}\Gamma^t_{rr} - \Gamma^r_{rt}\Gamma^r_{rr} - \Gamma^r_{\theta t}\Gamma^\theta_{rr} - \Gamma^r_{\phi t}\Gamma^\phi_{rr} - \Gamma^\theta_{\theta t}\Gamma^\theta_{r\theta} - \Gamma^\phi_{\phi t}\Gamma^\phi_{r\phi}\\
=&\Gamma^r_{rt}(\Gamma^t_{tr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - \Gamma^r_{tt}\Gamma^t_{rr}\\
=&\frac{2}{r}\partial_t\beta = 0,
\end{aligned}$$

hence β is a function of r only. Next:

$$\begin{aligned}
R_{\theta\theta} =&\partial_\rho\Gamma^\rho_{\theta\theta} - \partial_\theta\Gamma^\rho_{\rho\theta} + \Gamma^\sigma_{\theta\theta}\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\rho\theta}\Gamma^\rho_{\theta\sigma}\\
=&\partial_r\Gamma^r_{\theta\theta} - \partial_\theta\Gamma^\phi_{\phi\theta} + \Gamma^r_{\theta\theta}(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - \Gamma^\phi_{\phi\theta}\Gamma^\phi_{\theta\phi} - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} - \Gamma^\theta_{r\theta}\Gamma^r_{\theta\theta}\\
=&-\partial_r(re^{-2\beta}) - \partial_\theta\cot(\theta) - re^{-2\beta}\left(\partial_r\alpha + \partial_r\beta + \frac{2}{r}\right) - \cot^2(\theta) + 2e^{-2\beta}\\
=&-e^{-2\beta} + 2r\partial_r\beta e^{-2\beta} + \csc^2(\theta) - re^{-2\beta}\left(\partial_r\alpha + \partial_r\beta + \frac{2}{r}\right) - \cot^2(\theta) + 2e^{-2\beta}\\
=&1 + e^{-2\beta}\left(-1 + 2r\partial_r\beta - r\partial_r\alpha - r\partial_r\beta - 2 + 2\right)\\
=&1 + e^{-2\beta}\left(r\partial_r\beta - 1 - r\partial_r\alpha\right) = 0.
\end{aligned}$$

On its own it provides little information, but differentiating this with respect to t yields

$$\partial_t \partial_r \alpha = 0,$$

with solution $\alpha = u(t) + v(r)$.

The function u may be eliminated by a change of variables such that $\mathrm{d}t \to e^{-2u(t)}\,\mathrm{d}t$, which yields

$$\mathrm{d}s^2 = e^{2v(r)}\,\mathrm{d}t^2 - 2e^{\beta}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\,.$$

Note that with these simplifications we have $\Gamma^t_{tt} = \Gamma^t_{rr} = \Gamma^r_{rt} = \Gamma^r_{tr} = 0$ and no remaining time-dependent components. Next we compute the components

$$\begin{aligned}
R_{tt} =& \partial_\rho \Gamma^\rho_{tt} - \partial_t \Gamma^\rho_{\rho t} + \Gamma^\sigma_{tt}\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\rho t}\Gamma^\rho_{t\sigma} \\
=& \partial_r \Gamma^r_{tt} + \Gamma^r_{tt}\Gamma^r_{rr} + \Gamma^r_{tt}\Gamma^\theta_{\theta r} + \Gamma^r_{tt}\Gamma^\phi_{\phi r} - \Gamma^\sigma_{tt}\Gamma^t_{t\sigma} - \Gamma^\sigma_{\sigma t}\Gamma^\sigma_{t\sigma} \\
=& \partial_r(\partial_r \alpha e^{2(\alpha-\beta)}) + \partial_r \alpha e^{2(\alpha-\beta)}\partial_r \beta + \partial_r \alpha e^{2(\alpha-\beta)}\frac{1}{r} + \partial_r \alpha e^{2(\alpha-\beta)}\frac{1}{r} - \partial_r \alpha e^{2(\alpha-\beta)}\partial_r \alpha \\
=& e^{2(\alpha-\beta)}\partial_r^2 \alpha + 2\partial_r \alpha(\partial_r \alpha - \partial_r \beta)e^{2(\alpha-\beta)} + e^{2(\alpha-\beta)}\partial_r \alpha \partial_r \beta + e^{2(\alpha-\beta)}\frac{2\partial_r \alpha}{r} - (\partial_r \alpha)^2 e^{2(\alpha-\beta)} \\
=& e^{2(\alpha-\beta)}\left(\partial_r^2 \alpha + \partial_r \alpha\left(\partial_r \alpha - \partial_r \beta + \frac{2}{r}\right)\right) = 0, \\
R_{rr} =& \partial_\rho \Gamma^\rho_{rr} - \partial_r \Gamma^\rho_{\rho r} + \Gamma^\sigma_{rr}\Gamma^\rho_{\rho\sigma} - \Gamma^\sigma_{\rho r}\Gamma^\rho_{r\sigma} \\
=& \partial_r \Gamma^r_{rr} - \partial_r \Gamma^t_{tr} - \partial_r \Gamma^r_{rr} - \partial_r \Gamma^\theta_{\theta r} - \partial_r \Gamma^\phi_{\phi r} + \Gamma^r_{rr}(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - \Gamma^r_{rr}\Gamma^r_{rr} - \Gamma^t_{tr}\Gamma^t_{rt} - \Gamma^r_{rr}\Gamma^r_{rr} \\
&- \Gamma^\theta_{\theta r}\Gamma^\theta_{r\theta} - \Gamma^\phi_{\phi r}\Gamma^\phi_{r\phi} \\
=& -\partial_r^2 \alpha - \partial_r \frac{1}{r} - \partial_r \frac{1}{r} + \partial_r \beta\left(\partial_r \alpha + \partial_r \beta + \frac{2}{r}\right) - (\partial_r \beta)^2 - (\partial_r \alpha)^2 - \frac{2}{r^2} \\
=& -\partial_r^2 \alpha + \partial_r \beta\left(\partial_r \alpha + \frac{2}{r}\right) - (\partial_r \alpha)^2 = 0,
\end{aligned}$$

hence

$$(\partial_r \alpha + \partial_r \beta)\left(\partial_r \alpha + \frac{2}{r}\right) - \partial_r \alpha \partial_r \beta - (\partial_r \alpha)^2 = \frac{2}{r}(\partial_r u + \partial_r \beta) = 0,$$

and $\beta = C - v$. By rescaling $r \to e^{-\frac{1}{2}C} r$ we may set $C = 0$, and find $\beta = -v$. Finally we dig up an old equation and find

$$1 + e^{-2\beta}\left(r\partial_r \beta - 1 - r\partial_r \alpha\right) = 1 + e^{-2\beta}\left(2r\partial_r \beta - 1\right) = 0,$$

which we may solve and write as

$$\partial_r (r e^{-2\beta}) = 1.$$

Integrating this yields

$$r e^{-2\beta} = r - R_{\mathrm{S}},\ e^{-2\beta} = 1 - \frac{R_{\mathrm{S}}}{r}.$$

Finally we may put this together to find

$$\mathrm{d}s^2 = \left(1 - \frac{R_{\mathrm{S}}}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{R_{\mathrm{S}}}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\,,$$

where R_{S} is the Schwarzschild radius. To reobtain Newtonian gravity at large distances we would need $R_{\mathrm{S}} = 2MG$.

This metric has singularities at $r = R_{\mathrm{S}}$ and $r = 0$. If you study the curvature invariant $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, however, you find that it is finite at $r = R_{\mathrm{S}}$ and diverges at $r = 0$. This would imply that there exists a smart choice of coordinates in which the singularity at R_{S} would be eliminated.

This set of coordinates, called Eddington-Finkelstein coordinates, replaces t with a coordinate that has the light-like geodesics as coordinate lines. For a purely radial path the requirement $\mathrm{d}s^2 = 0$ for such a geodesic yields

$$\mathrm{d}t^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right)^{-2} \mathrm{d}r^2\,,$$

with solution

$$t = u - r - R_\mathrm{S} \ln\left(\frac{r}{R_\mathrm{S}} - 1\right),$$

where u is an integration constant labelling the geodesics. This will be the new coordinate. In these coordinates we obtain

$$\mathrm{d}s^2 = \left(1 - \frac{R_\mathrm{S}}{r}\right) \mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r - r^2\,\mathrm{d}\Omega^2\,.$$

Notably, there is now only a singularity at $r = 0$.

For radial light cones in these coordinates, one obtains

$$\left(\left(1 - \frac{R_\mathrm{S}}{r}\right) \mathrm{d}u^2 - 2\,\mathrm{d}r\right) \mathrm{d}u = 0,$$

with solutions $\mathrm{d}u = 0$ and $\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{2}{1 - \frac{R_\mathrm{S}}{r}}$. The first case is as discussed above. The second has $\frac{\mathrm{d}u}{\mathrm{d}r} > 0$ for $r < R_\mathrm{S}$, meaning that world lines moving towards the future are drawn to the singularity at the origin when within the Schwarzschild radius.

To describe space-like world lines, we can use Kruskal-Szekeres coordinates

$$U = \left|\frac{r}{R_\mathrm{S}} - 1\right|^{\frac{1}{2}} e^{\frac{r}{2R_\mathrm{S}}} \sinh\left(\frac{t}{2R_\mathrm{S}}\right),\ V = \left|\frac{r}{R_\mathrm{S}} - 1\right|^{\frac{1}{2}} e^{\frac{r}{2R_\mathrm{S}}} \cosh\left(\frac{t}{2R_\mathrm{S}}\right).$$

One finds that the metric is

$$\mathrm{d}s^2 = \frac{4R_\mathrm{S}^3}{r} e^{-\frac{r}{R_\mathrm{S}}} (\mathrm{d}U^2 - \mathrm{d}V^2) - r^2\,\mathrm{d}\Omega^2\,.$$

In these coordinates geodesics are hyperbolae.

Geometric Symmetries and Conserved Quantities Assume that a spacetime has a Killing field K, and consider a geodesic of the spacetime with tangent U. Along the geodesic we then have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\mu\nu}K^\mu U^\nu) = \vec{\boldsymbol{\nabla}}_U(g_{\mu\nu}K^\mu U^\nu) = (\vec{\boldsymbol{\nabla}}_U g_{\mu\nu})K^\mu U^\nu + g_{\mu\nu}\vec{\boldsymbol{\nabla}}_U K^\mu U^\nu.$$

As the Levi-Civita connection is metric-compatible, the former term vanishes, and we are left with

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\mu\nu}K^\mu U^\nu) = g_{\mu\nu}U^\nu \vec{\boldsymbol{\nabla}}_U K^\mu + g_{\mu\nu}K^\mu \vec{\boldsymbol{\nabla}}_U U^\nu.$$

As the path is a geodesic, the latter term vanishes and all that is left is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\mu\nu}K^\mu U^\nu) = g_{\mu\nu}U^\nu \vec{\boldsymbol{\nabla}}_U K^\mu = g_{\mu\nu}U^\nu U^\sigma \vec{\boldsymbol{\nabla}}_\sigma K^\mu = U^\nu U^\sigma \vec{\boldsymbol{\nabla}}_\sigma K_\nu.$$

The two first factors are symmetric under permutation of indices. The other, on the other hand, is anti-symmetric as K is a Killing field, hence $\frac{\mathrm{d}}{\mathrm{d}\tau}(g_{\mu\nu}K^\mu U^\nu) = 0$. Thus the quantity $g_{\mu\nu}K^\mu U^\nu$ is constant along geodesics.

Symmetries of the Schwarzschild Solution We note that ∂_t and ∂_ϕ are both Killing fields of the Schwarzschild solution. For $r > R_S$ one has

$$g(\partial_t, \partial_t) = 1 - \frac{R_S}{r} > 0,$$

and ∂_t is time-like.

For a geodesic we define

$$\begin{aligned}
\sqrt{2E} &= g(\partial_t, \dot{\gamma}) = \left(1 - \frac{R_S}{r}\right)\dot{t}, \\
L &= -g(\partial_\phi, \dot{\gamma}) = r^2 \sin^2(\theta)\dot{\phi}, \\
\alpha &= g(\dot{\gamma}, \dot{\gamma}),
\end{aligned}$$

which are all constants along the path. The latter is 1 for a time-like geodesic and 0 for a light-like geodesic. By definition we have

$$\alpha = \left(1 - \frac{R_S}{r}\right)\dot{t}^2 - \frac{1}{1 - \frac{R_S}{r}}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2(\theta)\dot{\phi}^2 = \frac{2E}{1 - \frac{R_S}{r}} - \frac{1}{1 - \frac{R_S}{r}}\dot{r}^2 - r^2\dot{\theta}^2 - \frac{L^2}{r^2}.$$

Noting that $\theta = \frac{\pi}{2}$ solves the geodesic equations, we may limit our considerations to such a solution by symmetry. We then have

$$E - \frac{1}{2}\dot{r}^2 = \frac{1}{2}\left(\alpha + \frac{L^2}{r^2}\right)\left(1 - \frac{R_S}{r}\right).$$

This looks like the relation describing a classical particle moving in a potential.

Frequency Shift A wave generally has a phase ϕ which depends on both position and time. For a general world line with tangent V we define

$$\omega = \vec{\nabla}_V \phi = V^\mu \partial_\mu \phi = d\phi(V).$$

By raising the indices of $d\phi$, one obtains the 4-frequency N^μ.

Consider the case of light-like surfaces of constant phase, implying $g^{\mu\nu}(d\phi)_\mu(d\phi)_\nu = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0$. We then find

$$\begin{aligned}
(\vec{\nabla}_N d\phi)_\mu &= N^\nu(\partial_\nu\partial_\mu\phi - \Gamma^\sigma_{\nu\mu}\partial_\sigma\phi) \\
&= g^{\nu\rho}(\partial_\rho\phi\partial_\nu\partial_\mu\phi - \Gamma^\sigma_{\nu\mu}\partial_\rho\phi\partial_\sigma\phi) \\
&= \frac{1}{2}g^{\nu\rho}\left(\partial_\rho\phi\partial_\nu\partial_\mu\phi - \Gamma^\sigma_{\nu\mu}\partial_\rho\phi\partial_\sigma\phi + \partial_\nu\phi\partial_\rho\partial_\mu\phi - \Gamma^\sigma_{\rho\mu}\partial_\nu\phi\partial_\sigma\phi\right) \\
&= \frac{1}{2}g^{\nu\rho}\left(\partial_\mu(\partial_\rho\phi\partial_\nu\phi) - \Gamma^\sigma_{\nu\mu}\partial_\rho\phi\partial_\sigma\phi - \Gamma^\sigma_{\rho\mu}\partial_\nu\phi\partial_\sigma\phi\right) \\
&= \frac{1}{2}g^{\nu\rho}\vec{\nabla}_\mu(\partial_\rho\phi\partial_\nu\phi) \\
&= \frac{1}{2}\vec{\nabla}_\mu(g^{\nu\rho}\partial_\rho\phi\partial_\nu\phi) \\
&= 0.
\end{aligned}$$

In other words, $d\phi$ is parallel along the world line defined by N, implying that these world lines are light-like geodesics.

In general an observer will measure a frequency $f = g(N, V) = d\phi(V)$. The emitted frequency will simply be this inner product, while the frequency observed at a different point will be found by the same process after having parallel transported N. Computing this parallel transport is usually cumbersome, so we will avoid applying it directly in favor of other arguments.

Gravitational Time Dilation and Redshift Consider a static space time with spacetime interval $\mathrm{d}s^2 = \alpha^2\,\mathrm{d}t^2 - h_{ij}\,\mathrm{d}\chi^i\,\mathrm{d}\chi^j$, where the components of the metric are functions of the spatial coordinates only. Now consider two comoving observers in this spacetime observing two events. As ∂_t is a Killing field, the geodesics have time translation symmetry, hence the two observers must observe the same time difference t_0. Each observer observes an elapsed proper time

$$\tau = \int_t^{t+t_0} \mathrm{d}t\,\sqrt{g_{tt}} = \alpha t_0.$$

This means that the elapsed proper time dilates according to

$$\frac{\tau_A}{\tau_B} = \frac{\alpha_A}{\alpha_B}.$$

In particular, if the two events are successive crests of a light pulse, we find the gravitational redshift formula

$$\frac{f_B}{f_A} = \frac{\alpha_A}{\alpha_B}.$$

Experimental Evidence for General Relativity The first example evidence validating general relativity is the perihelion precession of Mercury. Namely, the elliptic axis of the orbit of Mercury precedes with a very long period. Much of this could be attributed to many-body effects, but the remaining contribution matched very closely with the prediction from general relativity.

Next there is strong gravitational lensing, exemplified in an observation of a solar eclipse in 1919. Due to the gravitation from the Sun, a star was visible on Earth despite being behind the Sun.

Then there is so-called Shapiro delay - namely, it takes more time for light signals to travel due to gravitation.

Furthermore there is the Pound-Rebka experiments, which demonstrates gravitational time dilation.

Finally among the major examples discussed is the presence of gravitational effects in binary systems. As they emit gravitational systems they lose energy, causing their orbits to slow. When two stars merge, relatively strong waves are formed. The observation of gravitational waves from such a phenomenon was recently discovered and awarded the Nobel prize.

In addition there are gyroscopic effects and effects due to strong gravitational lensing.

3 Cosmology

The Cosmological Principle The cosmological principle states that the universe is homogenous and isotropic. As a consequence, the spatial part of the universe must be maximally symmetric, meaning that it has 6 Killing fields.

Possible Metrics Given the cosmological principle, there are three possible choices of manifold describing simultaneities. We will describe all of them using polar coordinates, hence they will all have a spatial metric of the form $\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + s^2(r)\,\mathrm{d}\Omega^2$. These are:

- the 3-sphere, with $s(r) = \frac{1}{k}\sin(kr)$.
- flat Euclidean space, with $s(r) = r$.
- a hyperbolic surface, with $s(r) = \frac{1}{k}\sinh(kr)$.

We will treat this options generally and introduce the spacetime interval $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\Sigma^2$. This defines the Robertson-Walker (RW) universe. The factor $a(t)$ is called the scale factor. A similar factor could in principle be found in front of the time interval, but can be eliminated by a change of variables.

The Friedman-Lemaitre-Robertson-Walker Universe The FLRW universe assumes the universe to be filled with an ideal fluid. For the universe to be homogenous and isotropic, this would require ρ and p to be functions of time only. We will also use the general equation of state $p = w\rho$ as an assumption. It can be shown that $w = 0$ for dust and $w = \frac{1}{3}$ for radiation. In order to not break isotropy, the fluid rest frame must also coincide with that of a comoving observer - in other words, we must have $U = \partial_t$. The goal of this section is to find a set of equations describing the evolution of such a universe.

We will need the Christoffel symbols of the RW metric. As the metric is diagonal in the chosen coordinates, the contravariant components of the metric are simply the reciprocals of the covariant ones and we have the familiar

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2g_{\sigma\sigma}}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}), \text{ no sum over } \sigma.$$

Furthermore, we can deduce the following:

- The time component of the metric is constant, hence the $\Gamma^t_{t\mu}$, Γ^μ_{tt} and $\Gamma^t_{\mu t}$ are all zero.
- The radial independence of the radial component implies that Γ^r_{rr} is zero.
- As the metric is diagonal, all symbols with three different indices are zero.
- The angle independence of the time and radial components implies $\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^r_{r\phi} = \Gamma^r_{\phi r} = \Gamma^\theta_{rr} = \Gamma^\phi_{rr} = 0$.
- As the metric is independent of ϕ, we must have $\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \Gamma^\phi_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0$.
- $\Gamma^\theta_{\theta\theta} = 0$ as the θ component is independent of θ.

The possibly non-zero Christoffel symbols are therefore

$$\Gamma^t_{rr} = aa',\ \Gamma^t_{\theta\theta} = \frac{1}{2}s^2aa',\ \Gamma^t_{\phi\phi} = \frac{1}{2}s^2\sin^2(\theta)aa',\ \Gamma^r_{rt} = \Gamma^r_{tr} = -\frac{1}{2a^2}\cdot -2aa' = \frac{a'}{a},\ \Gamma^r_{\theta\theta} = \frac{1}{2a^2}\cdot -2a^2ss' = -ss',$$
$$\Gamma^r_{\phi\phi} = \frac{1}{2a^2}\cdot -2a^2\sin^2(\theta)ss' = -\sin^2(\theta)ss',\ \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{1}{2a^2s^2}s^2aa' = \frac{a'}{a},\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2a^2s^2}a^2ss' = \frac{s'}{s},$$
$$\Gamma^\theta_{\phi\phi} = -\frac{1}{2a^2s^2}\cdot 2a^2s^2\sin(\theta)\cos(\theta) = -\sin(\theta)\cos(\theta),\ \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = \frac{1}{2a^2s^2\sin^2(\theta)}\cdot 2s^2\sin^2(\theta)aa' = \frac{a'}{a},$$
$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{2a^2s^2\sin^2(\theta)}\cdot 2a^2\sin^2(\theta)ss' = \frac{s'}{s},\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2a^2s^2\sin^2(\theta)}\cdot 2a^2s^2\sin(\theta)\cos(\theta) = \cot(\theta).$$

Next we need the components of the Ricci tensor. Due to the diagonality of the metric we only need the diagonal components, and somehow we only need one time component and one spatial component. We thus have

$$\begin{aligned}
R_{tt} &= \partial_\alpha\Gamma^\alpha_{tt} - \partial_t\Gamma^\alpha_{\alpha t} + \Gamma^\beta_{tt}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\alpha t}\Gamma^\alpha_{t\beta} \\
&= -\partial_t\Gamma^\alpha_{\alpha t} - \Gamma^\beta_{\alpha t}\Gamma^\alpha_{\beta t}, \\
R_{rr} &= \partial_\alpha\Gamma^\alpha_{rr} - \partial_r\Gamma^\alpha_{\alpha r} + \Gamma^\beta_{rr}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\alpha r}\Gamma^\alpha_{r\beta} \\
&= \partial_t\Gamma^t_{rr} - \partial_r\Gamma^\alpha_{\alpha r} + \Gamma^t_{rr}\Gamma^\alpha_{\alpha t} - \Gamma^\beta_{\alpha r}\Gamma^\alpha_{\beta r}.
\end{aligned}$$

Summing over the β in the last terms only produces a contribution when $\alpha = \beta$ or β is equal to the third index. Abusing notation we therefore find

$$\begin{aligned}
R_{tt} &= -\partial_t\left(\Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\phi_{\phi t}\right) - \Gamma^\alpha_{\alpha t}\Gamma^\alpha_{\alpha t} - \Gamma^t_{\alpha t}\Gamma^\alpha_{tt} - \Gamma^\alpha_{tt}\Gamma^t_{\alpha t} \\
&= -\partial_t\left(3\frac{a'}{a}\right) - 3\left(\frac{a'}{a}\right)^2 \\
&= -\frac{3}{a^2}(a''a - (a')^2 + (a')^2) \\
&= -3\frac{a''}{a}, \\
R_{rr} &= \partial_t\Gamma^t_{rr} - \partial_r(\Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) + \Gamma^t_{rr}\Gamma^\alpha_{\alpha t} - \Gamma^\alpha_{\alpha r}\Gamma^\alpha_{\alpha r} - \Gamma^r_{\alpha r}\Gamma^\alpha_{rr} - \Gamma^\alpha_{rr}\Gamma^r_{\alpha r} \\
&= \partial_t(aa') - 2\partial_r\frac{s'}{s} + 3aa'\frac{a'}{a} - 2\left(\frac{s'}{s}\right)^2 - 2(a')^2 \\
&= (a')^2 + aa'' - 2\frac{ss'' - (s')^2}{s^2} + 3(a')^2 - 2\left(\frac{s'}{s}\right)^2 - 2(a')^2 \\
&= 2(a')^2 + aa'' - 2\frac{s''}{s}.
\end{aligned}$$

Next we study the energy-momentum tensor. As $\vec{\nabla}_\mu G^{\mu\nu} = 0$, it must hold that $\vec{\nabla}_\mu T^{\mu\nu} = 0$. As the fluid moves along geodesics, implying $\vec{\nabla}_U U = 0$, and the connection is metric compatible we find

$$\begin{aligned}
\vec{\nabla}_\mu T^{\mu\nu} &= \vec{\nabla}_\mu((\rho_0 + p)U^\mu U^\nu - pg^{\mu\nu}) \\
&= U^\mu U^\nu \vec{\nabla}_\mu(\rho_0 + p) + (\rho_0 + p)\vec{\nabla}_\mu(U^\mu U^\nu) - p\vec{\nabla}_\mu g^{\mu\nu} - g^{\mu\nu}\vec{\nabla}_\mu p \\
&= U^\mu U^\nu \partial_\mu(\rho_0 + p) + (\rho_0 + p)(U^\mu \vec{\nabla}_\mu U^\nu + U^\nu \vec{\nabla}_\mu U^\mu) - g^{\mu\nu}\partial_\mu p \\
&= U^\mu U^\nu \partial_\mu(\rho_0 + p) + (\rho_0 + p)(\vec{\nabla}_U U^\nu + U^\nu \vec{\nabla}_\mu U^\mu) - g^{\mu\nu}\partial_\mu p \\
&= U^\mu U^\nu \partial_\mu(\rho_0 + p) + (\rho_0 + p)U^\nu \vec{\nabla}_\mu U^\mu - g^{\mu\nu}\partial_\mu p = 0.
\end{aligned}$$

By assumption the only non-trivial case is $\nu = 0$, for which we find

$$\begin{aligned}
U^\mu U^t \partial_\mu(\rho_0 + p) + (\rho + p)U^t \vec{\nabla}_\mu U^\mu - g^{\mu t}\partial_\mu p &= \rho' + p' + (\rho_0 + p)U^\alpha \Gamma^\mu_{\mu\alpha} - p' \\
&= \rho' + p' + (\rho + p)\Gamma^\mu_{\mu t} - p' \\
&= \rho' + (\rho + p)\cdot 3\frac{a'}{a} \\
&= \rho' + (1 + w)\cdot 3\frac{a'}{a}\rho = 0.
\end{aligned}$$

We can integrate this to find $\rho \propto a^{-3(1+w)}$. This must hold in a universe consistent with the previously developed theory. Next the trace of the energy-momentum tensor is

$$T^\mu{}_\mu = (\rho + p)U^\mu U_\mu - p\delta^\mu_\mu = \rho - 3p.$$

All is ready to write up the Einstein field equations, here in the alternative form. We have

$$\begin{aligned}
R_{tt} &= -3\frac{a''}{a} \\
&= 8\pi G\left(T_{tt} - \frac{1}{2}g_{tt}T^\mu{}_\mu\right) = 8\pi G\left((\rho + p)U_t^2 - pg_{tt} - \frac{1}{2}(\rho - 3p)\right) = 4\pi G\left(2\rho - \rho + 3p\right) = 4\pi G(\rho + 3p), \\
R_{rr} &= 2(a')^2 + aa'' - 2\frac{s''}{s} \\
&= 8\pi G\left(T_{rr} - \frac{1}{2}g_{rr}T^\mu{}_\mu\right) = 8\pi G\left(-pg_{rr} + \frac{1}{2}a^2(\rho - 3p)\right) = 4\pi G\left(2pa^2 + a^2(\rho - 3p)\right) = 4\pi G a^2(\rho - p).
\end{aligned}$$

Defining the Hubble parameter $H = \frac{a'}{a}$ we may rewrite the above as the Friedmann equations

$$\begin{aligned}
\frac{a''}{a} &= -\frac{4\pi}{3}G(\rho + 3p), \\
H^2 &= 2\pi G(\rho - p) - \frac{1}{2}\frac{a''}{a} + \frac{s''}{sa^2} = 2\pi G(\rho - p) + \frac{2\pi}{3}G(\rho + 3p) + \frac{s''}{sa^2} = \frac{8\pi}{3}G\rho + \frac{s''}{sa^2}.
\end{aligned}$$

We may obtain an even nicer form by noting that all the relevant s are eigenfunctions of double differentiation with eigenvalues $\pm k^2$ and 0. We may thus define the curvature parameter $\kappa = -\frac{s''}{s}$, which has the value 0 for flat space, k^2 for a sphere and $-k^2$ for a hyperboloid. We then obtain

$$\begin{aligned}\frac{a''}{a} &= -\frac{4\pi}{3}G(\rho + 3p),\\ H^2 &= \frac{8\pi}{3}G\rho - \frac{\kappa}{a^2}.\end{aligned}$$

A Note on Multi-Component Ideal Fluids In order to study a universe composed of multiple kinds of ideal fluids we will need to use an energy-momentum tensor of the form

$$T = \sum_i T_i,$$

where each term is of the form of an ideal fluid with a particular density and pressure. The statement that the divergence of the energy-momentum tensor is in fact a statement about conservation of energy and momentum, hence it must apply to each of the terms of the sum if the corresponding fluids are non-interacting. This implies that

$$\rho_i \propto a^{-3(1+w_i)}$$

for all the fluids individually. The Friedmann equations may now be written as

$$\begin{aligned}\frac{a''}{a} &= -\frac{4\pi}{3}G\sum_i(\rho_i + 3p_i),\\ H^2 &= -\frac{\kappa}{a^2} + \frac{8\pi}{3}G\sum_i \rho_i.\end{aligned}$$

The Big Bang As we saw above, we have $\frac{\ddot{a}}{a} \propto -(\rho+3p)$. Today we observe that $\dot{a} > 0$. Assuming $\rho+3p > 0$ we have $\ddot{a} < 0$, hence $\dot{a}$ must have been greater in the past. The general shape of a thus implies that there exists a time when a was equal to zero. The time at which this changed, defined to be $t = 0$, is termed the Big Bang.

Cosmological Redshift Consider two comoving observers in a RW universe. We may choose our coordinates such that they are connected by a purely radial path. Supposing that a light signal is sent from one to the other, that light signal satisfies

$$\mathrm{d}t = a(t)\,\mathrm{d}r\,.$$

We may integrate this to find

$$r_2 - r_1 = \int_{t_1}^{t_2} \mathrm{d}t\,\frac{1}{a(t)}.$$

A similar relation will hold for the next pulse of the signal, implying

$$\int_{t_1}^{t_2} \mathrm{d}t\,\frac{1}{a(t)} = \int_{t_1+\delta t_1}^{t_2+\delta t_2} \mathrm{d}t\,\frac{1}{a(t)},$$

which can be rearranged to

$$\int_{t_2}^{t_2+\delta t_2} \mathrm{d}t\,\frac{1}{a(t)} - \int_{t_1}^{t_1+\delta t_1} \mathrm{d}t\,\frac{1}{a(t)} = 0.$$

Assuming the frequency of the light to be high, we may linearize the above. By assumption we have $f_i \propto \frac{1}{\delta t_i}$, and the Doppler shift is thus given by

$$\frac{f_2}{f_1} = \frac{a(t_1)}{a(t_2)}.$$

The Cosmological Redshift We define the cosmological redshift as follows: Choosing comoving coordinates such that $a(t_0) = 1$, we introduce the redshift

$$z = \frac{f_0}{f} - 1.$$

According to the previous example we have

$$z = \frac{1}{a(t)} - 1 \implies a(t) = \frac{1}{1+z}.$$

for some previous time t. The redshift thus acts as a stand-in for cosmological time, larger redshifts corresponding to earlier times.

Evolution of the Universe From the Friedmann equations we see that if the density of the ideal fluid is equal to a critical value $\rho_c = \frac{3H^2}{8\pi G}$, we find $\kappa = 0$, implying that the universe is flat. Introducing the density parameter $\Omega = \frac{\rho}{\rho_c}$ one of the Friedmann equations may be written as

$$\sum_i \Omega_i + \Omega_\kappa = 1,\ \Omega_\kappa = -\frac{\kappa}{H^2 a^2}.$$

In particular, for a flat universe where we define $a(t_0) = 1$ we may reuse one previous solution to write

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G \sum_i \rho_{0,i} a^{-3(w_i+1)},$$

where the densities are all evaluated at t_0.

Accelerating Expansion and Cosmological Constant The universe has been found to have accelerating expansion. Furthermore, there is evidence for the existence of an energy component of the universe corresponding to $w = -1$ and ρ constant. This component is called the cosmological constant and is a kind of dark energy.

Cosmological Inflation The Big Bang model has some problems. One is the existence of a singularity of the metric at the time of the Big Bang. A second is the high degree of flatness of the universe. Finally there is the high degree of homogeneity of the universe.

To elaborate, one of the Friedmann equations may be written as

$$-\kappa = H^2 a^2 - \frac{8\pi G}{3} G a^2 \rho = \frac{8\pi G}{3} a^2 \left(\rho_c - \rho\right) = \frac{8\pi G}{3} a^2 \rho \left(\frac{1}{\Omega} - 1\right)$$

If the universe is dominated by an ideal fluid, we use the fact that $\rho \propto a^{-3(1+w)}$ to write this as

$$1 - \frac{1}{\Omega} \propto a^{1+3w}.$$

This means that if $w > -\frac{1}{3}$ then $1 - \frac{1}{\Omega}$, and thus the curvature of the universe, increases with time. As the universe is currently very flat, $1 - \frac{1}{\Omega}$ must have been close to 1 shortly after the Big Bang, and the universe must have been even flatter.

Furthermore, when studying the cosmic microwave background, regions that are not in causal contact are very close in temperature. This requires explanation.

Cosmological inflation is an attempt at explaining the above. It posits that shortly after the Big Bang there was a rapid expansion dominated by an ideal fluid with $w < -\frac{1}{3}$. It solves the flatness problem by driving $1 - \frac{1}{\Omega}$ towards 0 at these early times. It also solves the problem of homogeneity by allowing the previously described regions to be causally connected at early times, before being separated by expansion. This inflation would of course have to be stopped, a problem that can be solved by dynamic inflation.