

Summary of SI2510 Statistical Mechanics

Yashar Honarmandi
yasharh@kth.se

September 29, 2020

Abstract

This is a summary of SI2510 Statistical Mechanics.

Contents

1	Basic Concepts	1
2	Mean-Field Theory	5
3	Landau Theory	9
4	Renormalization	14
5	Percolation	18

1 Basic Concepts

Phase Transitions Landau introduced the concept that phase transitions are defined by spontaneous symmetry breaking.

Order Parameters An order parameter is a quantity that describes spontaneous symmetry breaking. It is zero in one phase and non-zero in another. An order parameter need not have a physical interpretation.

First- and Second-Order Phase Transitions Second-order phase transitions have continuous free energy and order parameter, whereas first-order phase transitions have discontinuous order parameter.

Critical Exponents Many phenomena exhibit a behaviour of the form $|T - T_c|^{-c}$ close to phase transitions. The exponent c is termed the critical exponent.

Some specific critical exponents that may appear are γ for χ , α for C , $-\beta$ for the order parameter, $-\delta\beta$ for the conjugate field h , γ for the corresponding susceptibility χ and ν for the correlation length.

Density Matrices The probability distribution is of the form

$$p_n = \frac{1}{\sum_m P_m} P_n = \frac{1}{Z} P_n,$$

where the summation is performed over some set of states. We now introduce the density matrix

$$\rho = \frac{1}{Z} \sum_n P_n |n\rangle\langle n|,$$

yielding

$$\langle O \rangle = \sum_n p_n O_{nn} = \sum_n \frac{1}{Z} P_n \langle n|O|n \rangle = \frac{1}{Z} \sum_n \sum_m P_n \langle n|m \rangle \langle m|O|n \rangle = \frac{1}{Z} \sum_n \sum_m \langle n|\rho|m \rangle \langle m|O|n \rangle = \text{tr}(O\rho).$$

As a side note, ρ takes the form

$$\rho = \frac{1}{Z} e^{-\beta K},$$

where K is the Hamiltonian in the canonical ensemble and $H - \mu N$ in the grand canonical ensemble. In these cases, the partition function Z may be computed according to

$$Z = \sum_m P_m = \sum_m e^{-\beta K_m} = \sum_m \langle m|e^{-\beta K}|m \rangle = \text{tr}(e^{-\beta K}).$$

The Ising Model The Ising model is a simple model of magnets. In this model, a magnet is a collection of interacting spins on a lattice under the influence of an external field. Its generalized coordinates are σ_i , which may take the values ± 1 , signifying a particular spin pointing up or down. The Hamiltonian is

$$\mathcal{H} = -J \sum_i \sum_{j=\text{nn}(i)} \sigma_i \sigma_j - h \sum_i \sigma_i.$$

The inner summation is carried out over the nearest neighbours of site i in the Ising model, but is generally a sum over the whole lattice. The order parameter defining its phase transition for the case $J > 0$ is $m = \langle \sigma_i \rangle$.

This model will be used to demonstrate core concepts in the course.

Scaling Laws While we will use the Ising model as a reference in this summary, it has been found that very different systems display similar critical exponents. Such exponents are also found to obey so-called scaling laws. As an example, it may be shown that

$$\chi(C_H - C_M) = T \left(\left(\frac{\partial M}{\partial T} \right)^2 \right)_H.$$

This can only be satisfied if

$$\chi C_H > T \left(\left(\frac{\partial M}{\partial T} \right)^2 \right)_H.$$

Introducing the critical exponent of the involved quantities, we must therefore have

$$-\gamma - \alpha \leq 2(\beta - 1), \quad \gamma + \alpha + 2\beta \geq 2.$$

The intriguing part is that each side is in fact almost equal, meaning that we may treat them as such. Three other scaling laws are:

$$\beta(\delta - 1) = \gamma, \quad \nu d = 2 - \alpha, \quad \gamma = \nu(2 - \eta).$$

We have also included the exponent $\Gamma(r, T_c) \propto r^{-(d-2+\eta)}$.

Exact Solution in One Dimension To solve the Ising model in one dimension we will impose periodic boundary conditions $\sigma_N = \sigma_0$. The Hamiltonian may be written as

$$\mathcal{H} = -J \sum_{i=0}^{N-1} \sigma_i \sigma_{i+1} - \frac{1}{2} h \sum_{i=0}^{N-1} \sigma_i + \sigma_{i+1}.$$

The partition function is thus

$$Z = \sum_{\sigma_0=\pm 1} \cdots \sum_{\sigma_{N-1}=\pm 1} e^{\beta \left(J \sum_{i=0}^{N-1} \sigma_i \sigma_{i+1} + \frac{1}{2} h \sum_{i=0}^{N-1} \sigma_i + \sigma_{i+1} \right)}.$$

Introducing the transfer matrix with elements $t_{\sigma\sigma'} = e^{\beta J \sigma \sigma' + \frac{1}{2} h (\sigma + \sigma')}$ we have

$$Z = \sum_{\sigma_0=\pm 1} \cdots \sum_{\sigma_{N-1}=\pm 1} \prod_{i=0}^{N-1} t_{\sigma_i \sigma_{i+1}}.$$

Now consider some particular spin and perform the summation over this one first. We obtain

$$\sum_{\sigma_j=\pm 1} t_{\sigma_{j-1} \sigma_j} t_{\sigma_j \sigma_{j+1}} = t_{\sigma_{j-1} \sigma_{j+1}}^2.$$

This process is repeated until you obtain

$$Z = \text{tr}(t^N).$$

The matrix representation of the transfer matrix is

$$t = \begin{bmatrix} e^{\beta(J+h)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-h)} \end{bmatrix}.$$

Its eigenvalues are the solutions to

$$\left(\lambda - e^{\beta(J+h)} \right) \left(\lambda - e^{\beta(J-h)} \right) - e^{-2\beta J} = 0,$$

and are given by

$$\begin{aligned} \lambda^2 - 2e^{\beta J} \cosh(\beta h) \lambda + 2 \sinh(2\beta J) &= 0, \\ \lambda_{\pm} &= e^{\beta J} \cosh(\beta h) \pm \sqrt{e^{2\beta J} \cosh^2(\beta h) - 2 \sinh(2\beta J)} \\ &= e^{\beta J} \cosh(\beta h) \pm \sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}}. \end{aligned}$$

Now that we have the eigenvalues, we identify the partition function as

$$Z = \lambda_+^N + \lambda_-^N.$$

This can be further simplified to

$$Z = \lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right) \approx \lambda_+^N.$$

Next, the free energy is given by

$$G = -k_B T \left(N \ln(\lambda_+) + \ln \left(1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right) \right).$$

The magnetization is given by

$$\begin{aligned} m &= -\frac{1}{N} \left(\frac{\partial G}{\partial \beta h} \right)_T \\ &\approx \frac{e^{\beta J} \sinh(\beta h) + \frac{e^{2\beta J} \sinh(\beta h) \cosh(\beta h)}{\sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}}}}{e^{\beta J} \cosh(\beta h) + \sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}}} \\ &= \sinh(\beta h) \frac{1 + \frac{\cosh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}}{\cosh(\beta h) + \sqrt{\sinh^2(\beta h) + e^{-4\beta J}}} \\ &= \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta J}}}. \end{aligned}$$

If $h = 0$ there is no spontaneous magnetization. However, at low temperatures a very small field will produce saturation magnetization. This corresponds to a phase transition at $T = 0$.

Next consider the pair distribution function

$$g(j) = \langle \sigma_0 \sigma_j \rangle.$$

The error introduced by assuming uncorrelated spins, as will be done later, is

$$\Gamma(j) = \langle \sigma_i \sigma_{i+j} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+j} \rangle.$$

In a general case with different couplings between spins and without an external field we have

$$\begin{aligned} \langle \sigma_i \sigma_{i+j} \rangle &= \frac{1}{Z} \sum \sigma_i \sigma_{i+j} e^{\beta \sum_{i=0}^{N-1} J_i \sigma_i \sigma_{i+1}} \\ &= \frac{1}{Z} \sum \sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_{i+2} \dots \sigma_{i+j-1} \sigma_{i+j} e^{\beta \sum_{i=0}^{N-1} J_i \sigma_i \sigma_{i+1}} \\ &= \frac{1}{Z} \frac{\partial}{\partial \beta J_i} \dots \frac{\partial}{\partial \beta J_{i+j}} Z. \end{aligned}$$

Using the fact that

$$Z = 2 \prod_{i=1}^N 2 \cosh(\beta J_i)$$

we obtain

$$\langle \sigma_i \sigma_{i+j} \rangle = \prod_{k=i}^{i+j} \tanh(\beta J_k).$$

In one dimension there is no magnetization. In the case where all couplings are the same we obtain

$$\Gamma(j) = \tanh^j(\beta J) = e^{-\frac{j}{\xi}},$$

where we have introduced the correlation length

$$\xi = -\frac{1}{\ln(\tanh(\beta J))}.$$

Kadanoff Block Spins Consider the Ising model on a d -dimensional hypercubic lattice with lattice constant a . Divide the lattice into blocks with l spins in each direction, meaning that each block contains l^d lattice sites. These blocks then also form a lattice with lattice constant la . If we assume that $la \ll \xi$, most of the spins in a block are correlated, allowing us to introduce a scaling hypothesis. Denoting the total spin of some block as $S_{I,\text{tot}}$ we introduce the new variables

$$S_I = l^{-x_S} S_{I,\text{tot}}.$$

Assume now that the Hamiltonian looks the same when expressed in terms of the block spins. The free energy should then be unchanged by the choice of new variables. Introducing the variable

$$t = \frac{T - T_c}{T_c},$$

we expect both this parameter and h to scale with the transformation of variables according to

$$t \rightarrow tl^{x_t}, \quad h \rightarrow hl^{x_h}.$$

We now expect the free energy to scale according to

$$G = Ng(t, h) = N_l g(t_l, h_l) = \frac{N}{l^d} g(t_l, h_l).$$

The correlation length is also expected to scale as

$$\xi \rightarrow \frac{\xi}{l}.$$

This implies that in our new variables, the reduced temperature is increased. Similarly the field is given by

$$h = h \sum S_i = h \sum S_{I,\text{tot}} = hl^{x_S} \sum S_I,$$

and has thus increased. The scaling hypothesis is now

$$g(t, h) = l^{-d} g(tl^{x_t}, hl^{x_h}).$$

In other words, the free energy per particle is a homogenous function of t and h .

Near $t = 0$ the correlation length is the only characteristic length scale. As it diverges in this limit, the system is invariant under scale transformations. This implies that all thermodynamic functions are homogenous, somehow.

Let us now derive some critical exponents in this way. For instance, we compute the order parameter as

$$m(t, h) = \frac{\partial}{\partial h} \left(l^{-d} g(tl^{x_t}, hl^{x_h}) \right) = l^{x_h-d} m(tl^{x_t}, hl^{x_h}).$$

This is true for any scaling factor according to the hypothesis, meaning that we may choose $l = |t|^{-\frac{1}{x_t}}$ (and $h = 0$, I believe), which one can squint to see as

$$\beta = -\frac{y_h - d}{y_t}.$$

To determine δ , we choose $t = 0$, $l = |h|^{-\frac{1}{y_h}}$ to obtain

$$\delta = -\frac{y_h}{y_h - d}.$$

To determine γ we instead use

$$\chi(t, h) = \frac{\partial}{\partial h} \left(l^{x_h-d} m(tl^{x_t}, hl^{x_h}) \right) = l^{2x_h-d} \chi(tl^{x_t}, hl^{x_h}).$$

Choosing the scale factor $l = |t|$ nets

$$\gamma = \frac{2y_h - d}{y_t}.$$

To determine α , we use

$$C(t, h) = -t \frac{\partial^2 g}{\partial t^2} = l^{2y_h - d} C(tl^{x_t}, hl^{x_h}).$$

This yields

$$\alpha = \frac{2y_t - d}{y_t}.$$

The scaling assumptions must be verified experimentally. For one method, consider the order parameter. Choosing $l = |t|^{-\frac{1}{y_t}}$ we obtain

$$|t|^{-\beta} m(t, h) = m(\pm 1, h|t|^{-\Delta}),$$

where Δ is the gap exponent, given by $\frac{y_h}{y_t}$. In other words, plotting $|t|^{-\beta} m(t, h)$ against $h|t|^{-\Delta}$, one should see different curves depending on the sign of t .

We now turn to the correlation length. We expect

$$\xi(t, h) = l \xi(tl^{x_t}, hl^{x_h}).$$

Setting $l = |t|^{-\frac{1}{x_t}}$ we find

$$\nu = \frac{1}{x_t}.$$

This yields

$$\nu d = 2 - \alpha.$$

Next, the correlation function. We have

$$\begin{aligned} \Gamma(r_l, t_l) &= \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle \\ &= l^{-2x_h} \sum_{i \in I, j \in J} \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \\ &= l^{d-2x_h} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle) \\ &= l^{d-2x_h} \Gamma(r, t), \end{aligned}$$

implying

$$\Gamma(r, t) = l^{2x_h - d} \Gamma\left(\frac{r}{l}, l^{x_t}\right).$$

Setting $l = r$ and $t = 0$ we obtain

$$\Gamma(r, 0) = r^{2x_h - d} \Gamma(1, 0).$$

By the order-of-magnitude estimate $\Gamma \propto r^{d-2+\nu}$ for the correlation function we thus have

$$2 - \eta = 2x_h - d = \frac{\gamma}{\nu}$$

and finally

$$\gamma = \nu(2 - \eta).$$

2 Mean-Field Theory

The Idea The idea between mean-field theory is to separate the Hamiltonian by for any component of the system replace the ones with which it is interacting by an expectation value. Alternatively one may also replace the interactions themselves by mean values. This way a partition function may be computed more easily.

One usually requires self-consistency, meaning that expectation values should be computable from this partition function. This implies that the partition function implicitly gives the expectation values which were used to separate the Hamiltonian.

Mean-Field Theory of the Ising Model To construct a mean-field theory for the Ising model, we write the Hamiltonian as

$$\mathcal{H} = - \sum_i \sigma_i \left(J \sum_{j=\text{nn}(i)} \sigma_j + h \right).$$

We note that the replacement $\sigma_j \rightarrow m$ will make the Hamiltonian separable, hence we propose the mean-field Hamiltonian

$$\mathcal{H} = - \sum_i \sigma_i \left(J \sum_{j=\text{nn}(i)} m + h \right) = - \sum_i \sigma_i (qJm + h) = -h' \sum_i \sigma_i$$

where we have introduced the number q of nearest neighbours to any one site and the effective field

$$h' = h + qJm.$$

Using this Hamiltonian, the partition function is now given by

$$Z = \text{tr} \left(e^{-\beta \mathcal{H}} \right) = \sum_{\{\sigma\}} e^{\beta h' \sum_i \sigma_i} = \left(\sum_{\sigma=\pm 1} e^{\beta h' \sigma} \right)^N = 2^N \cosh^N(\beta h'),$$

$$m = \frac{1}{N} \left\langle \sum_i \sigma_i \right\rangle = \frac{1}{NZ} \sum_i \sum_{\sigma_i} \sigma_i e^{\beta h' \sum_j \sigma_j} = \frac{1}{N} \frac{d}{d\beta h'} \ln(2^N \cosh^N(\beta h')) = \tanh(\beta h').$$

Note that this implies that all spins are expected to point in the same direction. Baked into the process there is a specific idea of the structure of the solution, and it is therefore important to make a good guess about this. We proceed with the ferromagnetic case, where the guess is good, and obtain

$$m = \tanh(\beta (h + qJm)).$$

This equation can be solved graphically to yield the magnetization, but we will discuss it qualitatively here. The expression above is anti-symmetric in m , meaning we may consider $m \geq 0$. Depending on the parameters, the number of solutions is between one and three. In the case where $h = 0$, one solution is $m = 0$, and two other solutions may be found at $m = \pm m_0$. At low temperatures the right-hand side approaches ± 1 , yielding $m_0 = 1$. For $h = 0$ solutions are only found where the right-hand side grows faster than the left-hand side at zero. This is satisfied if

$$\beta qJ > 1.$$

This defines the critical temperature

$$T_c = \frac{qJ}{k_B},$$

above which no non-zero solutions are found.

As the temperature approaches the critical temperature, above which no spontaneous magnetism is found, m_0 is small (but non-zero) and we obtain

$$m_0 \approx \beta qJm_0 - \frac{1}{3} (\beta qJm_0)^3,$$

$$\left(\frac{T_c}{T} \right)^3 m_0^2 = 3 \left(\frac{T_c}{T} - 1 \right),$$

$$m_0 = \left(\frac{T}{T_c} \right)^{\frac{3}{2}} \sqrt{3 \left(\frac{T_c}{T} - 1 \right)}$$

$$= \sqrt{\frac{3}{T_c} \left(\frac{T}{T_c} \right)^2 (T_c - T)}.$$

This identifies the critical exponent $\beta = 1$. One of the uses of mean-field theory is determining asymptotic behaviour close to phase transitions.

One more thing should be mentioned, namely the assertion that there actually is spontaneous magnetization. After all, if three solutions are possible, who is to say that one of the non-zero ones are found? To do this, we consider the entropy of an ideal paramagnet, which can be shown to be

$$S = -Nk_B \left(\frac{1-m}{2} \ln \left(\frac{1-m}{2} \right) + \frac{1+m}{2} \ln \left(\frac{1+m}{2} \right) \right).$$

We see that the non-zero solutions maximize entropy - a relief.

Critical Behaviour Using the mean-field result, we may now study other quantities close to the phase transition. The susceptibility is given by

$$\chi = \left(\frac{\partial m}{\partial h} \right)_T.$$

The implicit equation for the magnetization yields

$$\begin{aligned} \chi &= \frac{\beta}{\cosh^2(\beta(qJm + h))} (qJ\chi + 1), \\ \chi &= \frac{\beta}{\cosh^2(\beta(qJm + h)) - \beta qJ} = \frac{1}{k_B \left(T \cosh^2(\beta(qJm + h)) - \frac{qJ}{k_B} \right)}. \end{aligned}$$

Introducing the critical temperature, we write this as

$$\chi = \frac{1}{k_B (T \cosh^2(\beta(qJm + h)) - T_c)}.$$

In particular, for $h = 0$ and temperatures above T_c , where there is no magnetization, we obtain

$$\chi = \frac{1}{k_B (T - T_c)}.$$

When approaching the phase transition from below for $h = 0$, we use the asymptotic expression for the magnetization to obtain

$$\begin{aligned} \chi &= \frac{1}{k_B \left(T \cosh^2 \left(\beta qJ \sqrt{\frac{3}{T_c}} \left(\frac{T}{T_c} \right)^2 (T_c - T) \right) - T_c \right)} \\ &= \frac{1}{k_B \left(T \cosh^2 \left(\sqrt{3} \left(1 - \frac{T}{T_c} \right) \right) - T_c \right)} \\ &= \frac{1}{k_B T_c \left(\frac{T}{T_c} \left(1 + 3 \left(1 - \frac{T}{T_c} \right) \right) - 1 \right)} \\ &= \frac{1}{k_B T_c \left(4 \frac{T}{T_c} - 3 \left(\frac{T}{T_c} \right)^2 - 1 \right)} \\ &= \frac{1}{k_B T_c \left(3 \frac{T}{T_c} \left(1 - \frac{T}{T_c} \right) + \frac{T}{T_c} - 1 \right)} \\ &\approx \frac{1}{k_B T_c \left(3 \left(1 - \frac{T}{T_c} \right) + \frac{T}{T_c} - 1 \right)} \\ &= \frac{1}{2k_B (T_c - T)}. \end{aligned}$$

The Bragg-Williams Approximation The Bragg-Williams approximation to mean-field theory starts with constructing the availability in terms of the order parameter. In the case of the Ising model, we introduce the

numbers N_{\pm} of spins with values ± 1 . Furthermore, we introduce the numbers $N_{\pm\pm}$ of spin pairs of any kind. The Hamiltonian is thus

$$\mathcal{H} = -J(N_{++} + N_{--} - N_{+-}) - h(N_+ - N_-).$$

Treating the spins as independent allows us to write

$$S = -k_B (N_+ \ln(N_+) + N_- \ln(N_-)).$$

The number of pairs is given by

$$N_{\pm\pm} = \frac{qN_{\pm}^2}{2N}, \quad N_{+-} = \frac{qN_+N_-}{N}.$$

To proceed, we re-express the spin numbers in terms of the order parameter by using $N = N_+ + N_-$ and $\sigma = N_+ - N_-$ to obtain

$$N_+ = \frac{1}{2}N(1+m), \quad N_- = \frac{1}{2}N(1-m).$$

The Hamiltonian is now given by

$$\begin{aligned} \mathcal{H} &= -\frac{qJ}{2N}(N_+^2 + N_-^2 - 2N_+N_-) - Nhm \\ &= -\frac{qJN}{8}((1+m)^2 + (1-m)^2 - 2(1+m)(1-m)) - Nhm \\ &= -\frac{qJN}{2}m^2 - Nhm, \end{aligned}$$

and the free energy is somehow

$$\begin{aligned} G(h, T) &= \mathcal{H} - TS \\ &= -\frac{qJN}{2}m^2 - Nhm + \frac{1}{2}Nk_B T \left((1+m) \ln\left(\frac{1}{2}N(1+m)\right) + (1-m) \ln\left(\frac{1}{2}N(1-m)\right) \right). \end{aligned}$$

Minimizing it with respect to the order parameter yields

$$\begin{aligned} -qJNm - Nh + \frac{1}{2}Nk_B T \left(\ln\left(\frac{1}{2}N(1+m)\right) + 1 - \ln\left(\frac{1}{2}N(1-m)\right) - 1 \right) &= 0, \\ -qJm - h + \frac{1}{2}k_B T \ln\left(\frac{1+m}{1-m}\right) &= 0. \end{aligned}$$

Its solution is

$$\begin{aligned} \frac{1+m}{1-m} &= e^{2\beta(qJm+h)}, \\ 1+m &= (1-m)e^{2\beta(qJm+h)}, \\ m(1+e^{2\beta(qJm+h)}) &= e^{2\beta(qJm+h)} - 1, \\ m &= \tanh(\beta(qJm+h)), \end{aligned}$$

as expected.

Inaccuracies of Mean-Field Theories The mean-field arguments predict the existence of a phase transition, but this cannot be the case in one dimension. To see this, consider a chain in its ground state and the set of excitations that flips all spins to the right of some spin k . The change in energy is $2J$, and the number of possible states corresponding to this energy is $N-1$, hence the free energy changes by $2J - k_B \ln(N-1)$. For large N such states are thus always preferable. Their removal of translation invariance implies that there is no magnetization, in contradiction of the mean-field results.

A slightly better result is obtained for a $N \times N$ lattice in two dimensions. The set of excitations now consists of excitations that split the system in two distinct magnetic domains. Any particular excitation is described by a chain running through the bonds. Each segment crosses one bond, and the energy change due to the excitation is $2LJ$, where L is the number of segments. The typical length is $2N$. The next segment may

always be placed in at least two sites, neglecting the boundaries. Including the N possible starting points, the multiplicity of the chain is $N2^L$, and the free energy change of the excitation is

$$\Delta G = 4NJ - k_B T \ln(2^{2N} N).$$

The phase transition occurs when this energy change is negative, i.e. when

$$k_B T \ln(2^{2N} N) > 4NJ, \quad k_B T (2N \ln(2) + \ln(N)) > 4NJ, \quad T < T_c \approx \frac{2J}{k_B \ln(2)},$$

which is decently close to the analytically obtained results.

Antiferromagnetism The case of $J < 0$ is another interesting case, and gives rise to some interesting phases. One phenomenon which may occur is frustration, where the lattice structure is such that for any group of spins, the state of lowest energy is degenerate and not such that all interactions are beneficial. Assuming this not to be the case, we find that the ground state corresponds to an ordered phase such that two lattice translations combined leave the system invariant. This symmetry is broken by the phase transition, hence we introduce the order parameter

$$m = \frac{1}{N} \sum (-1)^j \sigma_j,$$

which may be written as

$$m = \frac{1}{2}(m_A - m_B),$$

where the lattice has been divided into two sublattices.

The mean-field Hamiltonian is

$$\mathcal{H} = \mathcal{H}_A + \mathcal{H}_B = -Jq \left(m_B \sum_A \sigma_i + m_A \sum_B \sigma_i \right),$$

and by the same methods as previously, the mean-field partition function is

$$Z = 2^N \cosh^{\frac{1}{2}N}(\beta J q m_B) \cosh^{\frac{1}{2}N}(\beta J q m_A).$$

From this we obtain

$$m_A = \tanh(\beta J q m_B).$$

By construction we have $m_A = -m_B$, which yields

$$m_A = -\tanh(\beta J q m_A).$$

This is the same as we obtained for the ferromagnetic case, and we may immediately identify the critical temperature

$$T_c = -\frac{qJ}{k_B}.$$

3 Landau Theory

The Idea Landau's theory is a general theory of phase transitions. The core idea is to series expand the free energy in terms of the order parameter close to the phase transition. This isn't really valid, but this method works nevertheless.

The general form of the series expansion is

$$G(m, T) = a_0(T) + \sum_i \frac{1}{i} a_i(T) m^i,$$

where certain terms may be zero depending on the symmetry of the system. We assume the order parameter to be finite at equilibrium, meaning that the free energy must be bounded from below. One generally truncates this sum to make it possible to handle, and the free energy being bounded is guaranteed by the highest-order term to be even in m and positive.

Landau Theory of the Ising Model For the Ising model we expect the system to be invariant with respect to flipping all spins, hence we have the series expansion

$$G(m, T) = a_0(T) + \sum_i \frac{1}{2i} a_{2i}(T) m^{2i}.$$

At $T = 0$ you will have $|\mathbf{m}| = 1$.

Suppose now that as the temperature is lowered, a_2 is the first coefficient to change sign (at least one coefficient must do this in order for a phase transition to exist). Close to the temperature T_c at which it changes sign, it may be linearized as

$$a_2(T) = a_{2,0}(T - T_c).$$

At equilibrium we have

$$\frac{\partial G}{\partial m} = \sum_i a_{2i}(T) m^{2i-1} = 0.$$

Assuming that other coefficients are approximately constant close to the transition temperature and approaching T_c from below, where m is small but non-zero, we have

$$a_{2,0}(T - T_c) + \sum_{i>2} a_{2i}(T_c) m^{2i-2} = 0.$$

We truncate this sum at order 2 in m to obtain

$$m = \sqrt{\frac{a_{2,0}}{a_4(T_c)}(T_c - T)},$$

which reproduces the correct critical exponent.

We see that the series expansion and our assumption about which coefficients change sign produce a theory with a second-order phase transition. This is profound, and both the nature of the phase transition and the critical exponent are general. This is part of the power of Landau theory. This is also expected as the truncated free energy series is expected to transition between having a single minimum and two minima.

The heat capacity is given by

$$C = T \left(\frac{\partial S}{\partial T} \right)_h.$$

In the case of a second-order phase transition we have

$$S = -\frac{\partial G}{\partial T} = -\frac{da_0}{dT} - \sum_i \frac{1}{2i} \left(\frac{da_{2i}}{dT} m^{2i} + a_{2i} \frac{dm^{2i}}{dT} \right).$$

Close to and below the critical temperature we have

$$\begin{aligned} C &= T \left(-\frac{d^2 a_0}{dT^2} - \sum_i \frac{1}{2i} \left(\frac{d^2 a_{2i}}{dT^2} m^{2i} + \frac{da_{2i}}{dT} \frac{dm^{2i}}{dT} + \frac{da_{2i}}{dT} \frac{dm^{2i}}{dT} + a_{2i} \frac{d^2 m^{2i}}{dT^2} \right) \right) \\ &= T \left(-\frac{d^2 a_0}{dT^2} - \sum_i \frac{1}{2i} \left(\frac{d^2 a_{2i}}{dT^2} m^{2i} + 2 \frac{da_{2i}}{dT} \frac{dm^{2i}}{dT} + a_{2i} \frac{d^2 m^{2i}}{dT^2} \right) \right). \end{aligned}$$

We have

$$\begin{aligned} \frac{dm^{2i}}{dT} &= i m^{2(i-1)} \frac{dm^2}{dT} = -i m^{2(i-1)} \frac{a_{2,0}}{a_4(T_c)}, \\ \frac{d^2 m^{2i}}{dT^2} &= i(i-1) m^{2(i-2)} \frac{a_{2,0}^2}{a_4^2(T_c)}, \end{aligned}$$

and thus

$$\begin{aligned}
C &= T \left(-\frac{d^2 a_0}{dT^2} - \sum_{i=1}^{\infty} \frac{1}{2i} \left(\frac{d^2 a_{2i}}{dT^2} m^{2i} - 2i \frac{da_{2i}}{dT} \frac{a_{2,0}}{a_4(T_c)} m^{2(i-1)} + a_{2i} i(i-1) m^{2(i-2)} \frac{a_{2,0}^2}{a_4^2(T_c)} \right) \right) \\
&= -T \frac{d^2 a_0}{dT^2} + T \frac{a_{2,0}^2}{a_4(T_c)} - T \sum_{i=2}^{\infty} \frac{1}{2i} \left(\frac{d^2 a_{2i}}{dT^2} m^{2i} - 2i \frac{da_{2i}}{dT} \frac{a_{2,0}}{a_4(T_c)} m^{2(i-1)} + a_{2i} i(i-1) m^{2(i-2)} \frac{a_{2,0}^2}{a_4^2(T_c)} \right) \\
&= -T \frac{d^2 a_0}{dT^2} + T \frac{a_{2,0}^2}{a_4(T_c)} - \frac{1}{4} T \left(\frac{d^2 a_4}{dT^2} m^4 - 4 \frac{da_4}{dT} \frac{a_{2,0}}{a_4(T_c)} m^2 + 2a_4 \frac{a_{2,0}^2}{a_4^2(T_c)} \right) \\
&\quad - T \sum_{i=3}^{\infty} \frac{1}{2i} \left(\frac{d^2 a_{2i}}{dT^2} m^{2i} - 2i \frac{da_{2i}}{dT} \frac{a_{2,0}}{a_4(T_c)} m^{2(i-1)} + a_{2i} i(i-1) m^{2(i-2)} \frac{a_{2,0}^2}{a_4^2(T_c)} \right) \\
&\approx -T \frac{d^2 a_0}{dT^2} + T \frac{a_{2,0}^2}{2a_4(T_c)},
\end{aligned}$$

where we have ignored terms containing the magnetization. Above the critical temperature the magnetization is instead identically zero, netting

$$C = -T \frac{d^2 a_0}{dT^2}.$$

Hence there is a step in the phase transition of

$$-\frac{a_{2,0}^2 T_c}{2a_4(T_c)}$$

when approaching the phase transition from below.

Suppose instead that a_4 is the first coefficient to change sign. In this case the free energy might have several local minima, meaning that a discontinuous step in the order parameter might occur. To show that such a step exists, we need to show that $a_2(T_c) > 0$. We investigate this by comparing $G(m_0, T_c)$ to $G(0, T_c)$, where m_0 is the magnetization at the minimum. The phase transition occurs when the two are equal, i.e. when

$$G(m_0, T_c) - G(0, T_c) = \sum_i \frac{1}{2i} a_{2i}(T_c) m_0^{2i} = 0.$$

The magnetization corresponds to a minimum of G , and thus satisfies

$$\sum_i a_{2i}(T_c) m_0^{2i-1} = 0.$$

Truncating at order 6 we have

$$a_2(T_c) m_0 + a_4(T_c) m_0^3 + a_6(T_c) m_0^5 = 0, \quad \frac{1}{2} a_2(T_c) m_0^2 + \frac{1}{4} a_4(T_c) m_0^4 + \frac{1}{6} a_6(T_c) m_0^6 = 0.$$

The non-trivial value of the order parameter satisfies

$$a_2(T_c) + a_4(T_c) m_0^2 + a_6(T_c) m_0^4 = 0, \quad \frac{1}{2} a_2(T_c) + \frac{1}{4} a_4(T_c) m_0^2 + \frac{1}{6} a_6(T_c) m_0^4 = 0.$$

Combining the equation nets

$$\begin{aligned}
\frac{1}{2} a_4(T_c) m_0^2 + \frac{2}{3} a_6(T_c) m_0^4 &= 0, \\
\frac{1}{2} a_4(T_c) + \frac{2}{3} a_6(T_c) m_0^2 &= 0, \\
m_0^2 &= -\frac{3a_4(T_c)}{4a_6(T_c)}.
\end{aligned}$$

For this to work, we must have $a_6(T_c) > 0$ to keep the global minimum at finite magnetization and $a_4(T_c) < 0$ per our assumption of the existence of a local minimum, making the magnetization real. Inserting this into a previous expression yields

$$a_2(T_c) - a_4(T_c) \frac{3a_4(T_c)}{4a_6(T_c)} + a_6(T_c) \frac{9a_4^2(T_c)}{16a_6^2(T_c)} = a_2(T_c) - \frac{3}{4} \frac{a_4^2(T_c)}{a_6(T_c)} + \frac{9}{16} \frac{a_4^2(T_c)}{a_6(T_c)} = a_2(T_c) - \frac{3}{16} \frac{a_4^2(T_c)}{a_6(T_c)} = 0,$$

and thus

$$a_2(T_c) = \frac{3}{16} \frac{a_4^2(T_c)}{a_6(T_c)} > 0,$$

as we wanted to show.

Non-Symmetric Cases Suppose we have some case where this symmetry does not hold. Then we would instead use the series expansion

$$G(m, T) = a_0(T) + \sum_i \frac{1}{i} a_i(T) m^i.$$

It might be of interest to remove linear terms. This can be done by introducing a new order parameter $\tilde{m} = m + \Delta$ (the tilde will be omitted from now) where Δ is chosen appropriately so that

$$G(m, T) = a_0(T) + \sum_{i=2} \frac{1}{i} a_i(T) m^i.$$

The coefficients have implicitly been modified as well. Truncating the sum at a_4 , the equilibrium magnetization satisfies

$$a_2(T_c) m_0 + a_3(T_c) m_0^2 + a_4(T_c) m_0^3 = 0.$$

In addition, at the transition point the free energy is equal at the post-transition equilibrium and zero, yielding

$$\frac{1}{2} a_2(T_c) m_0^2 + \frac{1}{3} a_3(T_c) m_0^3 + \frac{1}{4} a_4(T_c) m_0^4 = 0.$$

The non-zero solution satisfies

$$\frac{1}{3} a_3(T_c) + \frac{1}{2} a_4(T_c) m_0 = 0, \quad m_0 = -\frac{2}{3} \frac{a_3(T_c)}{a_4(T_c)}$$

and

$$a_2(T_c) - \frac{2}{3} \frac{a_3^2(T_c)}{a_4(T_c)} + \frac{4}{9} \frac{a_3^2(T_c)}{a_4(T_c)} = 0, \quad a_2(T_c) = \frac{2}{9} \frac{a_3^2(T_c)}{a_4(T_c)}$$

Ginzburg-Landau Theory Landau theory characterizes a system in terms of a single order parameter. Ginzburg-Landau theory instead characterizes the system in terms of a field $m(\mathbf{r})$. This field could be thought of as at any particular point describing the order parameter when calculated based only on the vicinity of that point. In other words, it is a more resolved version of Landau theory.

The order parameter extremizes the free energy, which in this theory is given by

$$F = \int d^d \mathbf{x} a_0(T) + \sum_i \frac{1}{2i} a_{2i}(T) m^{2i} + \frac{1}{2} f (\vec{\nabla} m)^2.$$

The series expansion generalize Landau theory, whereas the last term is a simple extra term that gives non-trivial behaviour of m . We assume $f > 0$ as fluctuations should add to the free energy.

The corresponding extensive variable (the external field in the Ising model) is in this theory given by

$$h = \frac{\delta F}{\delta m}.$$

We have

$$\delta F = \int d^d \mathbf{x} \delta m \sum_{2i} a_{2i}(T) m^{2i-1} + f \vec{\nabla}(\delta m) \cdot \vec{\nabla} m.$$

Fixing boundary conditions and integrating by parts yields

$$\delta F = \int d^d \mathbf{x} \delta m \left(\sum_i a_{2i}(T) m^{2i-1} - f \nabla^2 m \right),$$

and finally

$$h = \sum_i a_{2i}(T)m^{2i-1} - f\nabla^2 m.$$

Close to a second-order phase transition we may re-obtain the result from Landau theory, namely

$$m_0^2 = -\frac{a_2(T)}{a_4(T)},$$

by setting $h = 0$.

Ginzburg-Landau theory allow us to study fluctuations, which is what we will do next. Suppose we add some small perturbation $h_0\delta(\mathbf{x})$ from $h = 0$, which changes the field to $m_0 + \phi$, where m_0 depends on temperature and ϕ on position. Truncating the sum at $i = 2$ we obtain

$$h_0\delta(\mathbf{x}) = a_2(T)(m_0 + \phi) + a_4(T)(m_0 + \phi)^3 - f\nabla^2 m_0 - f\nabla^2 \phi.$$

Neglecting higher-order terms in ϕ we obtain

$$\nabla^2 \phi - \frac{a_2(T)}{f}\phi - \frac{a_2(T)}{f}m_0 - \frac{3a_4(T)m_0^2}{f}\phi - \frac{a_4(T)}{f}m_0^3 = -\frac{h_0}{f}\delta(\mathbf{x}).$$

This simplifies to

$$\nabla^2 \phi + \frac{2a_2(T)}{f}\phi = -\frac{h_0}{f}\delta(\mathbf{x})$$

below the critical temperature and

$$\nabla^2 \phi - \frac{a_2(T)}{f}\phi = -\frac{h_0}{f}\delta(\mathbf{x})$$

above the critical temperature. The solution to these equations is

$$\phi = \frac{h_0}{4\pi f} \frac{e^{-\frac{r}{\xi}}}{r},$$

where

$$\xi = \begin{cases} \sqrt{\frac{f}{a_2(T)}}, & T > T_c, \\ \sqrt{-\frac{f}{2a_2(T)}}, & T < T_c. \end{cases}$$

ξ is the correlation length, and according to the linearization $a_2 = a_{2,0}(T - T_c)$ it diverges when approaching the phase transition.

The perturbation ϕ may be related to a correlation function. To see this, we add a term

$$- \int d^3\mathbf{x} m h$$

to the Hamiltonian, yielding

$$\langle m \rangle = \frac{\text{tr} \left(m e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right)}{\text{tr} \left(e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right)}.$$

This implies

$$\begin{aligned} \frac{\delta \langle m \rangle}{\delta h_0} &= \frac{\beta \text{tr} \left(m(\mathbf{0}) m e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right) \text{tr} \left(e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right) - \beta \text{tr} \left(m e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right) \text{tr} \left(m(\mathbf{0}) e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right)}{\left(\text{tr} \left(e^{-\beta \left(\mathcal{H} - \int d^3\mathbf{x} m h \right)} \right) \right)^2} \\ &= \beta (\langle m(\mathbf{0}) m \rangle - \langle m(\mathbf{0}) \rangle \langle m \rangle) \\ &= \beta \Gamma(\mathbf{r}). \end{aligned}$$

As ϕ contains the entire effect of the perturbation, this is equal to $\frac{\phi}{h_0}$, hence ϕ is an order parameter correlation function. The susceptibility is given by

$$\chi = \int d^3\mathbf{x} \beta \Gamma(\mathbf{r}),$$

and this implies that the mean-field result $\chi \propto |T_c - T|^{-1}$ is obtained.

The Ginzburg Criterion The Ginzburg criterion is a self-consistency criterion for mean-field or Landau theories.

To obtain it, we generalize to d dimensions. In such cases ϕ will generally have a different form, but we may still use the order-of-magnitude approximation

$$\phi = \frac{e^{-\frac{r}{\xi}}}{r^{d-2}}.$$

We would like to crudely approximate the correlation function at large distances. This is expected to be valid if the correlation function is small compared to the overall order parameter, which is satisfied if

$$\frac{\int_{\Omega(\xi)} d^d\mathbf{x} \langle m(\mathbf{0})m \rangle - \langle m(\mathbf{0}) \rangle \langle m \rangle}{\int_{\Omega(\xi)} d^d\mathbf{x} m_0^2} \ll 1,$$

where $\Omega(\xi)$ is the d -dimensional hypersphere of radius ξ . I believe this is the Ginzburg criterion.

We will now use the Ginzburg criterion to try to estimate the dimensionality for which Landau theory correctly predicts the critical behaviour. Using our estimate of the correlation function and introducing the critical exponent for the order parameter, we have

$$\frac{\int_{\Omega(\xi)} d^d\mathbf{x} \frac{e^{-\frac{r}{\xi}}}{r^{d-2}}}{\int_{\Omega(\xi)} d^d\mathbf{x} |T_c - T|^{2\beta}} \ll 1$$

when close to criticality. Computing this in spherical coordinates yields

$$\frac{d \int_0^\xi dr r e^{-\frac{r}{\xi}}}{\xi^d |T_c - T|^{2\beta}} = \frac{d \xi^2 \int_0^1 du u e^{-u}}{\xi^d |T_c - T|^{2\beta}} = \xi^{2-d} |T_c - T|^{-2\beta} d \int_0^1 du u e^{-u}.$$

Introducing the critical exponent ν for the correlation length, the left-hand side is proportional to

$$|T_c - T|^{2\beta + (d-2)\nu}.$$

The inequality is thus satisfied if and only if

$$d > 2 + \frac{2\beta}{\nu}.$$

4 Renormalization

Decimation of the Ising Chain Before proceeding, we define the dimensionless operator

$$H = -\beta\mathcal{H} = K \sum_i \sum_{j=\text{nn}(i)} \sigma_i \sigma_j + \frac{1}{2}h \sum_i \sigma_i + \sigma_{i+1}.$$

When computing the partition function, we may do this by first summing over spins with odd indices. For any one of these, we obtain the (partial) sum

$$2 \cosh(K(\sigma_{i-1} + \sigma_{i+1}) + h).$$

Next we may write

$$2e^{\frac{1}{2}h(\sigma_{i-1}+\sigma_{i+1})} \cosh(K(\sigma_{i-1} + \sigma_{i+1}) + h) = e^{2g+K'\sigma_{i-1}\sigma_{i+1}+\frac{1}{2}h'(\sigma_{i-1}+\sigma_{i+1})}$$

for quantities

$$\begin{aligned} K' &= \frac{1}{4} \ln \left(\frac{\cosh(2K+h) \cosh(2K-h)}{\cosh^2(h)} \right), \\ h' &= h + \frac{1}{2} \ln \left(\frac{\cosh(2K+h)}{\cosh(2K-h)} \right), \\ g &= \frac{1}{8} \ln(16 \cosh(2K+h) \cosh(2K-h) \cosh^2(h)). \end{aligned}$$

Thus the trace over odd-numbered spins is

$$e^{Ng+K' \sum_i \sigma_{2i} \sigma_{2i+2} + h' \sum_i \sigma_{2i}}.$$

When computing the trace over the even-numbered spins, we notice that it has the same structure as for the total chain, apart from the modified constants. We thus obtain

$$Z(N, K, h) = e^{Ng(K,h)} Z\left(\frac{1}{2}N, K', h'\right).$$

We may repeat this procedure indefinitely. Hence we obtain

$$-\frac{\beta G}{N} = \sum_{j=0}^{\infty} \frac{1}{2^j} g(K_j, h_j).$$

A More General Discussion The more general approach is to consider a model on a d -dimensional lattice with a set of degrees of freedom σ_i and coupling constants K_α , and suppose that a transform of the kind we have discussed, now termed a renormalization transform, preserves the form of the Hamiltonian. We describe the system with the dimensionless quantity

$$H = -\beta \mathcal{H} = \sum_{\alpha=1}^n K_\alpha \psi_\alpha(\sigma)$$

where the functions $\psi_\alpha(\sigma)$ describe one particular kind of interaction. As an example, the Ising model with coupling all over the lattice may have one function ψ_1 which contains all nearest-neighbour interactions, one function ψ_2 which describes next-to-nearest neighbours and so on. The renormalization transforms the Hamiltonian to

$$H' = Ng(K_\alpha) + \sum_{\alpha=1}^n K'_\alpha \psi_\alpha(\sigma'),$$

where σ' is the set of reduced degrees of freedom, which must have the same algebraic property as the non-reduced ones, and we have new coupling coefficients which are functions of the old. Supposing that the number of degrees of freedom is reduced by a factor b^d we obtain

$$-\frac{\beta}{N} G(N, K_\alpha) = f(K_\alpha) = g(K_\alpha) + b^{-d} f(K'_\alpha).$$

We will now study the critical points of such model. To do this, consider a model described by two coupling constants. The set of critical points for this model will form a line in coupling constant space. Under a renormalization transform, points in this space are made to flow. This flow cannot approach the critical line, as the correlation length there is infinite and generally decreased by a renormalization transform. The long-range order may not be changed by such transforms either, hence the flow cannot make points cross the critical line. Certain points on the critical line may be fixed points, but generally all of them are not. To proceed, consider some particular fixed point K_1^*, K_2^* and a flow close to this point. We introduce displacements ΔK from the critical point, and obtain to first order

$$K'_1 = R_1(K_1^* + \Delta K_1, K_2^* + \Delta K_2) = K_1^* + \Delta K_1 \frac{\partial R_1}{\partial K_1} + \Delta K_2 \frac{\partial R_1}{\partial K_2},$$

with a similar expression for K_2' . Introducing the matrix

$$M_{ij} = \frac{\partial R_i}{\partial K_j}$$

and the displacements

$$\Delta K_1' = K_1' - K_1^*$$

we have

$$\Delta K_1' = M_{11}\Delta K_1 + M_{12}\Delta K_2, \quad \Delta K_2' = M_{21}\Delta K_1 + M_{22}\Delta K_2.$$

To continue, we change variables as determined by the eigenvectors of M . Naming these ϕ_1 and ϕ_2 , the new variables, called scaling fields, are

$$U_i = \phi_{1,i}\Delta K_1 + \phi_{2,i}\Delta K_2.$$

One somehow obtains

$$U_i' = b^{y_i}U_i.$$

As we have assumed that points on the critical line flow towards the fixed point, one of the y_i must be negative while the other is positive. This also means that one of the eigenvectors must be tangential to the critical line at the fixed point. The possibility $y_i = 0$ corresponds to a line of fixed points.

Returning to the energy function, we expect g to be analytic, which somehow implies that f satisfies the recursion relation

$$f(K_\alpha) = b^{-d}f(K_\alpha').$$

Close to the critical point, this may be written as

$$f(U_1', U_2') = b^{-d}f(b^{y_1}U_1', b^{y_2}U_2')$$

Suppose now that $U_1 \propto t$ while U_2 is approximately unchanged by the temperature. Then

$$f(t, U_2') = b^{-d}f(b^{y_1}t, b^{y_2}U_2').$$

Hence $f \propto |t|^{\frac{d}{y_1}}$.

Two things may be concluded from this. First, the critical exponents are determined by the character of the fixed point. Second, it turns out that the flow is independent of U_2 . Hence, all systems whose Hamiltonians flow to the same critical fixed points have the same critical exponents, which is reminiscent of the previously studied universality arguments.

The previous analysis generalizes to higher-dimensional coupling constant spaces.

The Cumulant Method For more examples, we study the Ising model on a two-dimensional triangular lattice. The dimensionless Hamiltonian is

$$H = \frac{1}{2} \sum_{i,j} K_{ij} \sigma_i \sigma_j + h \sum_i \sigma_i,$$

where we now allow interactions throughout the lattice.

The triangular lattice may be divided into triangular blocks of three spins. The lattice of such blocks is a triangular lattice with a lattice constant greater by a factor $\sqrt{3}$. We will perform a renormalization procedure on these blocks defined by the projection operator $P(\mu_I, \{\sigma_I\})$. μ_I are the new variables and $\{\sigma_I\}$ are the spin configurations in the blocks. We somehow have

$$e^{Ng(K,h)+H'(\{\mu\},K',h')} = \text{tr} \left(\left(\prod_I P(\mu_I, \{\sigma_I\}) \right) e^{H(\{\sigma\},K,h)} \right).$$

If we require the projection operators to have trace 1, this somehow implies

$$\text{tr}_{\{\mu\}}(e^{Ng+H'}) = \text{tr}_{\{\sigma\}}(e^H),$$

and the free energy is preserved.

We wish to treat this system approximately, and do this by dividing the Hamiltonian into two terms

$$H(\{\sigma\}, K, h) = H_0(\{\sigma\}, K, h) + V(\{\sigma\}, K, h).$$

H_0 contains all terms that operate within block and V contains all interactions between blocks. We introduce K_n as the coupling constant n -th nearest neighbours and thus have

$$H_0 = \sum_I K_1(\sigma_{I,1}\sigma_{I,2} + \sigma_{I,1}\sigma_{I,3} + \sigma_{I,2}\sigma_{I,3}) + h(\sigma_{I,1} + \sigma_{I,2} + \sigma_{I,3}), \quad V = \sum_{I,J,n} K_n \sum_{\alpha,\beta} \sigma_{I,\alpha} \sigma_{J,\beta}.$$

This yields

$$\text{tr} \left(\left(\prod_I P(\mu_I, \{\sigma_I\}) \right) e^{H(\{\sigma\}, K, h)} \right) = Z_0 \langle e^V \rangle$$

where we have defined expectation values in terms of considering only interactions within the block. This is not an approximation.

Let us now approximate by using a so-called truncated cumulant expansion

$$\langle e^V \rangle = e^{\langle V \rangle + \frac{1}{2!} \langle (V - \langle V \rangle)^2 \rangle + \frac{1}{3!} \langle (V - \langle V \rangle)^3 \rangle + \dots}.$$

We name the terms in the exponent C_i .

Let us now include only nearest-neighbour interactions. We thus write the partition function as

$$Z_0 = \text{tr} \left(\left(\prod_I P(\mu_I, \{\sigma_I\}) \right) e^{H_0} \right) = \text{tr} \left(\left(\prod_I P(\mu_I, \{\sigma_I\}) \right) e^{\sum_I K_1(\sigma_{I,1}\sigma_{I,2} + \sigma_{I,1}\sigma_{I,3} + \sigma_{I,2}\sigma_{I,3}) + h(\sigma_{I,1} + \sigma_{I,2} + \sigma_{I,3})} \right).$$

We may somehow write this as

$$Z_0 = \text{tr} \left(\left(\prod_I P(\mu_I, \{\sigma_I\}) \right) e^{A+B\mu_i} \right).$$

Asssuming the block spins to be designated according to a majority rule, this somehow implies

$$e^{A+B} = e^{3K+3h} + 3e^{-K+h}, \quad e^{A-B} = e^{3K-3h} + 3e^{-K-h}.$$

Using block independence, the expectation value of $\langle V \rangle$ is easy to compute. We may write

$$\langle \sigma_{I,\alpha} \rangle = C + D\mu_I.$$

This is somehow combined to

$$Ng(K, h) + H'(\{\mu\}, K', h') = \frac{1}{3}NA(K, h) + B \sum_I \mu_I + 2K \sum_{\text{nn}(I,J)} (C + D\mu_I)(C + D\mu_J),$$

yielding the recursion relations

$$K' = 2KD^2(K, h), \quad h' = B(K, h) + 12KC(K, h)D(K, h), \quad g(K, h) = \frac{1}{3}NA(K, h) + 2KC^2(K, h).$$

For $h = 0$, this reduces to

$$K' = 2K \left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}} \right)^2.$$

It turns out that this has a fixed point corresponding to $K = 0.3356$. One may also obtain $y_t = 0.882$, $y_h = 2.034$. These results do not agree particularly well with the exact results, but nevertheless demonstrate the power of the framework.

Next we consider the second-order approximation.

5 Percolation

The Idea In many contexts the existence of certain behaviour depends on the structure of microscopic clusters. The study of such clusters is called percolation theory.

Percolation models come in two forms: site percolation, in which lattice sites are occupied with some probability p , and bond percolation, in which nearest-neighbour bonds are formed with some probability p .

Definition of Quantities We now define the following quantities for such models:

- the occupation probability p .
- the site percolation probability p_c , which is the lowest probability such that infinite clusters form in the thermodynamic limit.
- $n_s(p)$, which is the number of clusters of size s per lattice site given the occupation probability.
- $P(p)$, which is the fraction of occupied sites belonging to a spanning cluster (a cluster which goes from one end of the lattice to the other).
- $S(p)$, which is the mean size of finite clusters.
- the pair connectedness $C(p, r)$, which is the probability that two occupied sites separated by a distance r belong to the same cluster.

Given these definitions, we may infer the following: First, the probability that a particular site is occupied and belongs to a cluster of size s is equal to $sn_s(p)$. Next, we must have $P(1) = 1$ and $P(p) = 0$ for $p < p_c$ in the thermodynamic limit. In addition, all sites either belong to a spanning cluster or a cluster of finite size, implying

$$pP(p) + \sum_s sn_s(p) = p,$$

where the summation is over clusters of finite size. Finally, we have

$$S(p) = \frac{\sum_s s^2 n_s(p)}{\sum_s sn_s(p)}.$$

Scaling Theory For doing scaling theory, we postulate that the quantity

$$G(p) = \sum_s n_s(p)$$

is analogous to the free energy per site in such models. According to this analogy, $P(p)$ plays the role of the order parameter, $S(p)$ the role of the susceptibility and $C(p, r)$ that of the correlation function. Based on this, we define critical exponents as previously. The interesting part is that the critical exponents will not depend on details of our work.

To this we add an extra piece of conjecture, namely that there exists a typical cluster size s_ξ close to $p = p_c$ which dominates the critical behaviour of G, P and S . We define its critical exponent according to

$$s_\xi \propto |p - p_c|^{-\frac{1}{\sigma}}.$$

We also assume that

$$n_s(p) = n_s(p_c) f\left(\frac{s}{s_\xi}\right).$$

f must, by construction, approach zero as its argument increases to infinity and approach 1 as its argument approaches zero. For large s , it has been observed that n_s behaves as a power law in s with power $-\tau$, which is dependent on dimensionality. Thus we have

$$n_s(p) = s^{-\tau} \phi\left(s|p - p_c|^{\frac{1}{\sigma}}\right).$$

We may now write G as

$$G = \sum_s s^{-\tau} \phi\left(s|p - p_c|^{\frac{1}{\sigma}}\right) \approx |p - p_c|^{\frac{\tau-1}{\sigma}} \int_0^\infty dx x \phi(x).$$

Assuming the integral to converge, we have

$$\alpha = 2 - \frac{\tau - 1}{\sigma}.$$

In similar ways one obtains

$$\gamma = \frac{3 - \tau}{\sigma}, \quad \beta = \frac{2 - \tau}{\sigma},$$

and thus the scaling relationship $\alpha + 2\beta + \gamma = 2$ is satisfied.