Summary of SH2372 General Relativity

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Abstract

This is a summary of SH2372 General Relativity.

The course opens with a discussion of differential geometry. As I have extensive notes on the subject in my summary of SI2360, I only keep the bare minimum in this summary and refer to those notes for details.

Contents

1 Differential Geometry

For details on much of this, notably the early parts on Euclidean space, please consult my summary of SI2360 Analytical Mechanics and Classical Field Theory.

Euclidean and Affine Spaces A Euclidean space is a set of points such that there to each point can be assigned a position vector. To such spaces we may assign a set of n coordinates χ^a which together uniquely describe each point in the space locally.

Tangent and Dual Bases The tangent and dual bases are defined by

$$\mathbf{E}_a = \partial_{\chi^a}\mathbf{r} = \partial_a\mathbf{r},\ \mathbf{E}^a = \vec{\boldsymbol{\nabla}}\chi^a.$$

Using such bases, we may write

$$\mathbf{v} = v^a\mathbf{E}_a = v_a\mathbf{E}^a.$$

The components of these vectors are called contravariant and covariant components respectively.

Christoffel Symbols When computing the derivative of a vector quantity, one must account both for the change in the quantity itself and the change in the basis vectors. We define the Christoffel symbols according to

$$\partial_b\mathbf{E}_a = \Gamma^c_{ba}\mathbf{E}_c.$$

These can be computed according to

$$\mathbf{E}^c \cdot \partial_b\mathbf{E}_a = \mathbf{E}^c \cdot \Gamma^d_{ba}\mathbf{E}_d = \delta^c_d\Gamma^d_{ba} = \Gamma^c_{ba}.$$

Note that

$$\partial_a\mathbf{E}_b = \partial_a\partial_b\mathbf{r} = \partial_b\partial_a\mathbf{r} = \partial_b\mathbf{E}_a,$$

which implies

$$\Gamma^c_{ba} = \Gamma^c_{ab}.$$

Similarly, we might want to consider $\partial_b\mathbf{E}^a$, which might introduce new symbols. We find, however, that

$$\partial_a\mathbf{E}^b \cdot \mathbf{E}_c = \mathbf{E}^b \cdot \partial_a\mathbf{E}_c + \mathbf{E}_c \cdot \partial_a\mathbf{E}^b = 0,$$

which implies

$$\partial_a\mathbf{E}^b = -\Gamma^b_{ac}\mathbf{E}^c.$$

Covariant Derivatives Covariant derivatives are defined by

$$\vec{\boldsymbol{\nabla}}_a v^b = \partial_a v^b + \Gamma^b_{ac}v^c,$$

and thus satisfy

$$\partial_a\mathbf{v} = \mathbf{E}_b\vec{\boldsymbol{\nabla}}_a v^b.$$

Tensors To define tensors, we first define tensors of the kind $(0, n)$ as maps from n vectors to scalars. Using this, we define tensors of the kind (n, m) as linear maps from $(0, n)$ tensors to $(0, m)$ tensors.

Manifolds Manifolds are sets which are locally isomorphic to an open subset of $\mathbb{R}^n$.

Tangent and Dual Bases The tangent basis for a manifold is $\mathbf{E}_a = \partial_a$. The corresponding dual basis, denoted $d\chi^a$, is defined such that $df(X) = X^a\partial_a f$.

Tensors A general (n, m) tensor is constructed by taking the tensor product of tangent and dual basis vectors.

Pushforwards and Pullbacks

2 Basic Concepts

A Note on Minkowski Space In special relativity we work with Minkowski space, which is an affine space with a so-called pseudo-metric. This is a metric which is not positive definite, but instead a metric which has only non-zero eigenvalues (and is thus termed non-degenerate). We will work with the signature $(1, 3)$, meaning that there are three eigenvalues of -1 and one eigenvalue 1.

The Description of Spacetime In general relativity we will describe spacetime as a 4-dimensional manifold with a pseudometric of signature $(1, 3)$ with a Levi-Civita connection imposed on it.

Kinematics of Test Particles A test particle is a particle that itself does not affect the spacetime. Such particles can generally move through spacetime, along curves called world lines. With this motion comes the 4-velocity V, defined as the normalized tangent to a world line. In special relativity we could also define a proper acceleration by differentiating with respect to proper time. In general relativity we replace this with the 4-acceleration $A = \vec{\boldsymbol{\nabla}}_V V = \vec{\boldsymbol{\nabla}}_{\dot{\gamma}}\dot{\gamma}$. We may also define the proper acceleration α, which satisfies $\alpha^2 = -A^2 = -g(A, A)$, and it can be shown that if V is time-like, then A is space-like. Note that the curve parameter we use is τ, which is the proper time and a measure of length in spacetime.

Free Particles A free particle in special relativity is a particle for which $A = 0$. We take this definition to apply in general relativity as well. This implies that free test particles travel along spacetime geodesics.

4-Momentum and 4-Force We also define the 4-momentum $P = mV$ and the 4-force $F = \vec{\boldsymbol{\nabla}}_V P$.

Frequency Shift A wave generally has a phase which depends on both position and time. We define the frequency of a wave as $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$. For a general world line we define

$$\omega = \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \dot{\chi}^a \partial_a \phi = V\phi = d\phi(V).$$

$d\phi$ is the dual of the 4-frequency N^μ. It can be shown that $\vec{\boldsymbol{\nabla}}_N d\pi = 0$, and thus $\vec{\boldsymbol{\nabla}}_N N = 0$. Rays with tangent N are thus light-like geodesics.

In general an observer will measure a frequency $d\phi(V)$.

Simultaneity Two events are simultaneous if they are on the same hypersurface of constant t. As this depends very much on the choice of coordinates on spacetime, this notion is not at all well-defined.