Notes for the Master Thesis

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Sammanfattning

This is a collection of notes pertaining to concepts I needed to learn for my master's thesis.

Innehåll

1 Topology

Topological Spaces Let X be a set and $T = \{U_i | i \in I\}$ be a collection of subsets of X (I is some set of indices). The pair (X, T) (sometimes we only explicitly write X) is defined as a topological space if

- $\emptyset,\ X \in T$.
- If J is any subcollection of I, the family $\{U_j | j \in J\}$ satisfies

$$\bigcup_{j \in J} U_j \in T.$$

- If J is any finite subcollection of I, the family $\{U_j | j \in J\}$ satisfies

$$\bigcap_{j \in J} U_j \in T.$$

If the two satisfy the definition, we say that T gives a topology to X. The U_i are called its open sets.

Two cases of little interest are $T = \{\emptyset,\ X\}$ and T being the collection of all subsets of X. The two are called the trivial and discrete topologies respectively.

Metrics A metric is a map $d : X \times X \to \mathbb{R}$ that satisfies

- $d(x, y) = d(y, x)$.
- $d(x, x) \geq 0$, with equality applying if and only if $x = y$.
- $d(x, y) + d(y, z) \geq d(x,)z$.

Metric Space Suppose X is endowed with a metric. The collection of open disks

$$U_\varepsilon = \{x \in X | d(x, x_0) < \varepsilon\}$$

then gives a topology to X called the metric topology. The pair forms a metric space.

Continuous Maps A map between two topological spaces X and Y is continuous if its inverse maps an open set in Y to an open set in X.

Neighborhoods N is a neighborhood of x if it is a subset of X and x belongs to at least one open set contained within N.

Hausdorff Spaces A topological space is a Hausdorff space if, for any two points x, y, there exists neighborhoods U_x, U_y of the two points that do not intersect. This is an important type of topological space, as examples in physics are practically always within this category.

Homeomorphisms A homeomorphism (not to be confused with a homomorphism, however hard it may be) is a continuous map between two topological spaces with a continuous inverse. Two topological spaces are homeomorphic if there exists a homeomorphism between them.

Homotopy Types Two topological spaces belong to the same homotopy type if there exists a continuous map from one to the other. This is a more relaxed version of homeomorphicity, as we no longer require the map to be invertible.

Donuts and Coffee Mugs Homeomorphicity defines an equivalence relation between topological spaces. This means that we can define topological spaces into categories based on homeomorphicity.

We are now in a position to introduce the poor man's notion of topology, which considers two bodies as equivalent if one can be deformed into the other without touching two parts of the surface or tearing a part of the body. These continuous deformations correspond to homeomorphisms, but we will try to keep the discussions more to the abstract.

Topological Invariants An important question pertaining to this division of topological spaces is what separates the different categories. One possible answer is so-called topological invariants, quantities which are invariant under homeomorphism. The issue with this answer is that the full set of topological invariants has not been identified, hence they can only be used to verify that two topological spaces belong to different categories.

2 Tensor Gauge Theories

Important Aspects

Noether's Theorem For Higher-Order Problems Noether's theorem as proved in previous summaries is restricted to field theories up to first order in the derivatives. To study these theories we will need a version that works on higher-order theories too.

We may follow a previous derivation up to a certain point. Assuming that

$$\int\limits_{\Omega'} \mathrm{d}(x')^\mu\, \mathcal{L}' - \int\limits_{\Omega} \mathrm{d}x^\mu\, \mathcal{L} = \int\limits_{\Omega} \mathrm{d}x^\mu\, \partial_\mu V^\mu$$

for some particular variational transform, we have

$$\int\limits_{\Omega} \mathrm{d}x^\mu\, \mathcal{L}((\phi')^a, x^\mu) - \mathcal{L}(\phi^a, x^\mu) + \frac{\partial}{\partial x^\nu}(\mathcal{L}(\phi^a, x^\mu)\, \delta x^\nu) = \int\limits_{\Omega} \mathrm{d}x^\mu\, \partial_\nu V^\nu.$$

The two first terms can now be expanded according to

$$\mathcal{L}((\phi')^a, x^\mu) - \mathcal{L}(\phi^a, x^\mu) = \frac{\partial \mathcal{L}}{\partial \phi^a}\bar{\delta}\phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\bar{\delta}\partial_\mu \phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\bar{\delta}\partial_\mu \partial_\nu \phi^a.$$

Using the equations of motion and the fact that barred variations commute with derivatives, we have

$$\begin{aligned}
\mathcal{L}((\phi')^a, x^\mu) - \mathcal{L}(\phi^a, x^\mu) = &\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\right)\bar{\delta}\phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\partial_\mu \bar{\delta}\phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\mu \partial_\nu \bar{\delta}\phi^a \\
= &\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\bar{\delta}\phi^a\right) - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\bar{\delta}\phi^a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu \bar{\delta}\phi^a\right) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu \bar{\delta}\phi^a \\
= &\partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu\right)\bar{\delta}\phi^a\right) - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\bar{\delta}\phi^a \\
&- \partial_\nu \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\bar{\delta}\phi^a\right) + \partial_\nu \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\bar{\delta}\phi^a \\
= &\partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\right)\bar{\delta}\phi^a\right).
\end{aligned}$$

This nets us

$$\int\limits_{\Omega} \mathrm{d}x\, \partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\right)\bar{\delta}\phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu \bar{\delta}\phi^a + \mathcal{L}\,\delta x^\mu - V^\mu\right) = 0.$$

Using the expansion of the full variation we can write this as

$$\begin{aligned}
&\int\limits_{\Omega} \mathrm{d}x\, \partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\right)(\delta\phi^a - \partial_\rho \phi^a\, \delta x^\rho) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu\left(\delta\phi^a - \partial_\rho \phi^a\, \delta x^\rho\right) + \mathcal{L}\,\delta x^\mu - V^\mu\right) \\
= &\int\limits_{\Omega} \mathrm{d}x\, \partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu\right)\delta\phi^a - \partial_\rho \phi^a \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\right)\delta x^\rho\right. \\
&\left. - \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\left(\partial_\nu \partial_\rho \phi^a + \partial_\rho \phi^a \partial_\nu\right)\delta x^\rho + \mathcal{L}\,\delta x^\mu - V^\mu\right) \\
= &\int\limits_{\Omega} \mathrm{d}x\, \partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\partial_\nu\right)\delta\phi^a\right. \\
&\left. + \left(\mathcal{L}\delta^\mu_\rho - \partial_\rho \phi^a \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\right) - \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a}\left(\partial_\nu \partial_\rho \phi^a + \partial_\rho \phi^a \partial_\nu\right)\right)\delta x^\rho - V^\mu\right) = 0.
\end{aligned}$$

The full conserved current is then

$$
\begin{aligned}
j^\mu =& \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} \partial_\nu \right) \delta \phi^a \\
&+ \left(\mathcal{L} \delta^\mu_\rho - \partial_\rho \phi^a \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} \right) - \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} \left(\partial_\nu \partial_\rho \phi^a + \partial_\rho \phi^a \partial_\nu \right) \right) \delta x^\rho - V^\mu .
\end{aligned}
$$

A First Theory in 2D The first theory we will study is defined by the Lagrangian

$$
\mathcal{L} = \frac{1}{2} \mu_0 (\partial_0 \phi)^2 - \frac{1}{2\mu} (\partial_x \partial_y \phi)^2 ,
$$

which we can symmetrize to

$$
\mathcal{L} = \frac{1}{2} \mu_0 (\partial_0 \phi)^2 - \frac{1}{8\mu} (\partial_x \partial_y \phi + \partial_y \partial_x \phi)^2 .
$$

The equation of motion is

$$
\mu_0 \partial_0^2 \phi + \frac{1}{4\mu} \left(\partial_x \partial_y (\partial_x \partial_y \phi + \partial_y \partial_x \phi) + \partial_y \partial_x (\partial_x \partial_y \phi + \partial_y \partial_x \phi) \right) = \mu_0 \partial_0^2 \phi + \frac{1}{\mu} \partial_x^2 \partial_y^2 \phi = 0.
$$

Note the minus sign appearing in the term originating from derivatives of order 2.

Assuming μ and μ_0 to be constant parameters of the theory, there exists a symmetry $\delta\phi = f(x) + g(y)$. The full Noether current is

$$
j^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} \partial_\nu \right) \delta \phi^a .
$$

Its Noether charge is

$$
j^0 = \mu_0 (f+g) \partial^0 \phi ,
$$

and its current is

$$
\begin{aligned}
j^x =& - (f+g) \partial_y \left(-\frac{1}{4\mu} (\partial_x \partial_y \phi + \partial_y \partial_x \phi) \right) - \frac{1}{4\mu} (\partial_x \partial_y \phi + \partial_y \partial_x \phi) g' = \frac{1}{2\mu} \left((f+g) \partial_y \partial^y \partial^x \phi - g' \partial^x \partial^y \phi \right) , \\
j^y =& \frac{1}{2\mu} \left((f+g) \partial_x \partial^x \partial^y \phi - f' \partial^x \partial^y \phi \right) .
\end{aligned}
$$

The continuity equation is thus

$$
\partial_0 j^0 + \partial_x j^x + \partial_y j^y = 0.
$$

Note that the non-trivial contributions to the space components arise from the derivatives of the first terms acting on the field part. There is a factor $f+g$ which arises in both the charge and these terms in the current, hence we conclude that

$$
\partial_0 J^0 + \partial_x J^x + \partial_y J^y = 0
$$

for

$$
J^0 = \mu_0 \partial^0 \phi , \ J^x = \frac{1}{2\mu} \partial_y \partial^y \partial^x \phi , \ J^y = \frac{1}{2\mu} \partial_x \partial^x \partial^y \phi .
$$

It will later be argued that the first derivatives of ϕ are not well-defined, meaning that the current will not be well-defined (although the continuity equation will still be). Therefore it will be desirable to write the current in terms of higher-order derivatives of the field. To that end, we note that the symmetry that produces the above current has one term with a derivative guaranteed to be factored out. More specifically, the family of indices that appear to a second order has a term proportional to

$$
J^\mu = - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} = - \partial_\nu J^{\mu\nu} .
$$

The symmetrized version of the theory has

$$J^{xy} = J^{yx} = -\frac{1}{2\mu}\partial^x \partial^y \phi.$$

We thus have

$$\partial_0 J^0 = \partial_x \partial_y (J^{xy} + J^y).$$

We are definitely arriving at something interesting here, so let us try to generalize the above.

Consider a system with a Lagrangian such that some set of indices appear only to first order and the other only to second order. The second-order indices are assumed to appear symmetrically. Suppose next that there exists a family of quasisymmetries satisfying

- the fields are varied in a way that does not depend on themselves.
- the variation depends only on the second-order coordinates.
- the coordinates themselves are not varied.

The corresponding Noether current is

$$j^\mu = \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi^a} \partial_\nu \right) \delta\phi^a - V^\mu.$$

For the family of first order indices only the first and last terms are non-trivial, and we write them as

$$j^\mu = J^\mu \, \delta\phi^a - V^\mu.$$

For the family of second-order indices we have two distinct contributions

$$j^\mu = J^{\mu\nu} \partial_\nu \, \delta\phi^a - \delta\phi^a \, \partial_\nu J^{\mu\nu}.$$

Introducing implicit summation over the two disjoint categories of indices, the continuity equation now reads

$$\partial_\mu j^\mu + \partial_\mu \left(J^{\mu\nu} \partial_\nu \, \delta\phi^a - \delta\phi^a \, \partial_\nu J^{\mu\nu} \right) = 0.$$

Expanding the bracket we find

$$\partial_\mu (J^\mu \, \delta\phi^a) + \partial_\mu J^{\mu\nu} \partial_\nu \, \delta\phi^a + J^{\mu\nu} \partial_\mu \partial_\nu \, \delta\phi^a - \partial_\mu \, \delta\phi^a \, \partial_\nu J^{\mu\nu} - \delta\phi^a \, \partial_\mu \partial_\nu J^{\mu\nu} = 0,$$

which can be simplified to

$$\delta\phi^a \, \partial_\mu J^\mu - \partial_\mu V^\mu + J^{\mu\nu} \partial_\mu \partial_\nu \, \delta\phi^a - \delta\phi^a \, \partial_\mu \partial_\nu J^{\mu\nu} = 0.$$