Summary of SI2390 Relativistic Quantum Physics

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Abstract

This is a summary of SI2390.
We will use units such that $c = \hbar = 1$.

Contents

1 Tasty Bits of Special Relativity

Metric Signature We use the metric signature $(1, -1, -1, -1)$ for the Minkowski metric.

The Levi-Civita Tensor We use the convention $\varepsilon^{0123} = 1$.

The Poincare Group Elements of the Poincare group are specified by a Lorentz transformation Λ and a translation a. Its elements follow the multiplication rule

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2).$$

We may instead construct a representation of the Poincare group with matrices of the form

$$\begin{bmatrix} \Lambda & a \\ 0 & 1 \end{bmatrix},$$

from which the multiplication rule directly follows.

The Lie Algebra of the Lorentz Group The Lorentz group is defined as the set of transformations such that

$$g = \Lambda^T g \Lambda.$$

There are a maximum of 16 generators, meaning we may label them using our index convention. Expanding around the identity we find

$$g = (1 + \omega_{\mu\nu} M^{\mu\nu})^T g (1 + \omega_{\rho\sigma} M^{\rho\sigma}) \approx g + \omega_{\mu\nu} (M^{\mu\nu})^T g + g \omega_{\rho\sigma} M^{\rho\sigma},$$

implying

$$\omega_{\mu\nu}((M^{\mu\nu})^T g + g M^{\mu\nu}) = 0,$$

or

$$M^T g = -gM$$

for all generators. Constructing the generator in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and using the fact that the Minkowski metric is its own universe we find

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} g = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix} = \begin{bmatrix} -A^T & -C^T \\ -B^T & -D^T \end{bmatrix}.$$

The solutions to this have antisymmetric blocks A and D, as well as off-diagonal blocks that are transposes of each other. There are six degrees of freedom for this solution, meaning that the Lorentz group has six degrees of freedom, corresponding to the three rotations and boosts. To preserve the index notation, we may then choose the generators such that $M^{\mu\nu} = -M^{\nu\mu}$. The corresponding choice of parameters must then also be antisymmetric. To get the appropriate amounts of terms we will also divide by 2, as you will see in the following section.

To more explicitly introduce the boosts and rotations, we introduce their generators

$$J^i = -\frac{1}{2}\varepsilon^{ijk} M^{jk},\ K^i = M^{0i},$$

with commutation relations

$$\left[J^i, J^j\right] = i\varepsilon^{ijk} J^k,\ \left[K^i, K^j\right] = -i\varepsilon^{ijk} J^k,\ \left[J^i, K^j\right] = i\varepsilon^{ijk} K^k.$$

We can solve for the original generators as

$$M^{0i} = J^i,\ M^{ij} = \varepsilon^{kij} J^k.$$

Generators of the Poincare Group The generators of the Poincare group are the $M^{\mu\nu}$ of the Lorentz group, as well as the four P^μ that generate translations in spacetime. We will need their Lie algebra, and thus their commutation relations, which are

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho}\right),\ [P^\mu, P^\nu] = 0,\ [M^{\mu\nu}, P^\sigma] = i(g^{\nu\sigma}P^\mu - g^{\mu\sigma}P^\nu).$$

The representations U of the group elements are then

$$U(\Lambda, 0) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}},\ U(1, a) = e^{ia_\mu P^\mu},$$

and to first order

$$U(\Lambda, a) = e^{i\left(a_\mu P^\mu - \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right)}.$$

2 Basic Concepts

Casimir Operators A Casimir operator is an operator that is constructed from the generators of a group and commutes with all generators.

Casimir Operators of the Poincare Group The Casimir operators of the Poincare group are

$$P^2 = P^\mu P_\mu,\ w^2 = w^\mu w_\mu,$$

where we have introduced the Pauli-Lubanski vector

$$w_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma.$$

It can be shown that

$$w_0 = \mathbf{P}\cdot\mathbf{J},\ \mathbf{w} = P_0\mathbf{J} + \mathbf{P}\times\mathbf{K}.$$

The Wigner Classification As we will consider unitary representations of the Poincare group acting on states and the representations can be decomposed into irreps, we will find that we can reduce our considerations to a set of fundamental systems, termed particles. The classification, divided according to the eigenvalues of P^2 and w^2, is according to the Wigner system:

1. $P^2 > 0$, with subclasses:
   - $P^0 < 0$.
   - $P^0 > 0$.
2. $P^2 = 0$, with subclasses:
   - $P^0 < 0$.
   - $P^0 > 0$.
3. $P^2 = 0$ and $P^0 = 0$.
4. $P^2 < 0$, corresponding to tachyons.

Lorentz Covariance and the Schrödinger Equation Using the 4-momentum $P^\mu = (E, \mathbf{p})$ and the correspondence principle $P^\mu = i\partial_\mu$, the quantization of the classical energy $E = \frac{1}{2m}\mathbf{p}^2$ of a free particle is

$$i\partial t\Psi = -\frac{1}{2m}\nabla^2\Psi.$$

This does not in general respect Lorentz transformations, which one might expect given that it is not taken from a Lorentz covariant expression. In other words, the Schrödinger equation is not Lorentz covariant.

The quantization of the relativistic $E^2 = m^2 + \mathbf{p}^2$ is instead

$$-\partial_2 t\phi = m^2\phi - \nabla^2\phi.$$

By introducing the d'Alembertian $\Box = \partial\mu\partial_\mu$ we can write the above as

$$\Box\phi + m^2\phi = 0.$$

This is the Klein-Gordon equation, which is an appropriate quantization of a spinless particle.

A Conserved Current Corresponding to the Klein-Gordon equation there exists a density and a current

$$\rho = \frac{i}{2m}(\phi^* \partial 0 \phi - \phi \partial 0 \phi^*),\ \mathbf{j} = \frac{1}{2im}(\phi^* \vec{\boldsymbol{\nabla}} \phi - \phi \vec{\boldsymbol{\nabla}} \phi^*)$$

such that

$$\partial t \rho + \vec{\boldsymbol{\nabla}} \cdot \mathbf{j} = 0.$$

Alternatively, by combining the two into a 4-current $J^\mu = (\rho, \mathbf{j})$ we find

$$\partial \mu J^\mu = 0.$$

Problems With Stationary States A stationary state is a state such that

$$P^0 \phi = E\phi.$$

For such a state we have

$$J^0 = \frac{E}{m}|\phi|^2.$$

In the classical limit we have $\frac{E}{m} \approx 1$, whereas in the general case we have $E = \pm\sqrt{m^2 + \mathbf{p}^2}$, meaning that J^0 is not positive definite and the conserved Nöether cannot be interpreted as conservation of probability density. This implores us to reinterpret the Klein-Gordon equation as a general field equation.

Plane-Wave Solutions Plane-wave solutions of the Klein-Gordon equation are of the form

$$\phi = N e^{-iP_\mu x^\mu}.$$

In order for these to be solutions, we require

$$P^0 = \pm\sqrt{m^2 + |\mathbf{p}|}.$$

This does not pose a problem in non-interacting cases, as the solutions maintain their signs.

Charged Particles When treating charged particles in external electromagnetic fields, we employ the minimal coupling scheme and perform the replacement $P^\mu \to P^\mu - qA^\mu$. The Klein-Gordon equation then becomes

$$((\partial \mu + iqA_\mu)(\partial_\mu + iqA^\mu) + m^2)\phi = 0.$$

This will cause additional terms

$$J^\mu \to J^\mu - \frac{q}{m}|\phi|^2 A^\mu$$

in the Nöether current, further destroying our hopes of creating a one-particle theory.

The Klein Paradox Consider scattering after normal incidence on a step potential described by $A^\mu = (V\theta(x), \mathbf{0})$. Performing an anzats similar to that in the non-relativistic case, the Klein-Gordon equation predicts the same behaviour as the Schrödinger equation, except for the case where $V > E + m$. In this case the transmitted 4-momentum has a negative space component. Furthermore, the transmission probability becomes negative, but still preserving $T + R = 1$. This peculiar behaviour is known as Klein's paradox.

The Dirac Equation We will now try to develop a theory that remedies the problems with the Klein-Gordon equation. The hope is that this equations has a positive-definite conserved density. An important source of the bad time was the second-order time derivative, so we will try to remedy this with a first-order time derivative. We also make the space derivatives first-order, perhaps because of Lorentz stuff. This leads us to the anzats

$$\partial_t \Psi + (\boldsymbol{\alpha} \cdot \vec{\boldsymbol{\nabla}})\Psi + im\beta\Psi = 0,$$

where β and α^i are matrices and Ψ is a vector. The sizes of these are yet to be determined. The corresponding equation for $\Psi^\dagger$ is

$$\partial_t \Psi^\dagger + (\vec{\boldsymbol{\nabla}} \Psi^\dagger) \cdot \boldsymbol{\alpha}^\dagger - im\Psi^\dagger \beta^\dagger = 0.$$

Considering the quantity $\Psi^\dagger \Psi$ we have

$$\begin{aligned}\partial_t(\Psi^\dagger\Psi) &= (\partial_t\Psi^\dagger)\Psi + \Psi^\dagger \partial_t \Psi \\ &= (im\Psi^\dagger\beta^\dagger - (\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha})\Psi + \Psi^\dagger(-(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi - im\beta\Psi) \\ &= im\Psi^\dagger(\beta^\dagger - \beta) - (\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha}^\dagger\Psi - \Psi^\dagger(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi.\end{aligned}$$

We really want the right-hand side to be the 3-divergence of some 3-current. To do that, we may choose α^i and β to be Hermitian, yielding

$$(\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha}^\dagger\Psi + \Psi^\dagger(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi = \vec{\boldsymbol{\nabla}} \cdot \Psi^\dagger\boldsymbol{\alpha}\Psi.$$

The conserved 4-current is thus $j^\mu = (\Psi^\dagger\Psi, \Psi^\dagger\boldsymbol{\alpha}\Psi)$.

To reobtain something like the 4-vector norm we had when discussing the Klein-Gordon equation, we apply the operator $\partial_t - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}) - im\beta\Psi$ to our anzats to find

$$(\partial_t - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}) - im\beta\Psi)(\partial_t\Psi + (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi + im\beta\Psi) = 0.$$

As the derivatives commute with the matrices, the cross terms vanish, yielding

$$\partial_t^2\Psi - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})^2\Psi - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})im\beta\Psi - im\beta(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi + m^2\beta^2\Psi = 0,$$

or more explicitly

$$\partial_t^2\Psi - (\alpha^i\partial_i)(\alpha^j\partial_j)\Psi - im((\alpha^i\partial_i)\beta + \beta\alpha^i\partial_i)\Psi + m^2\beta^2\Psi = \partial_t^2\Psi - \alpha^i\alpha^j\partial_i\partial_j\Psi - im(\alpha^i\beta + \beta\alpha^i)\partial_i\Psi + m^2\beta^2\Psi = 0.$$

We can symmetrize the second term to find

$$\partial_t^2\Psi - \frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial_i\partial_j\Psi - im(\alpha^i\beta + \beta\alpha^i)\partial_i\Psi + m^2\beta^2\Psi = 0.$$

This produces the same 4-vector norm if

$$\left\{\alpha^i, \alpha^j\right\} = 2\delta^{ij},\ \beta^2 = 1,\ \alpha^i\beta + \beta\alpha^i = 0.$$

Computing the determinant of the last equation, we find

$$\det\left(\alpha^i\beta\right) = (-1)^N \det\left(\beta\alpha^i\right),$$

where N is the length of Ψ. The only way for the above equations to be solvable is then that N be odd. It can also be shown that α^i and β are all traceless. By a series of arguments we find that $N = 4$ is correct.

To complete our discussion, we define $\gamma^0 = \beta,\ \gamma^i = \beta\alpha^i$. With this we multiply our anzats by $-i\beta$ and find

$$(-i\beta\partial_0 - i(\beta\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}))\Psi + m\Psi = 0.$$

Defining the inner product $\gamma^\mu A_\mu = \not{A}$ we arrive at the Dirac equation

$$i \not{\partial}\ \Psi - m\Psi = 0.$$

Properties of the γ^μ The γ^μ satisfy

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Defining the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, we find that it must be Hermitian.

We also have

$$\left\{\gamma^5, \gamma^\mu\right\} = 0.$$

We have

$$\operatorname{tr}\left(\prod_{i=1}^{n}\gamma^{\mu_i}\right) = 0,\ n \text{ odd},$$
$$\operatorname{tr}\left(\gamma^5\right) = 0.$$

The Dirac Algebra The γ^μ are a representation of the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

We may compute explicit representations of this algebra as

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

In this representation we have

$$\gamma^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The Dirac Adjoint The Dirac adjoint is defined as

$$\bar{A} = A^\dagger \gamma^0.$$

Rewriting the 4-Current We may use the properties of the γ^μ to write

$$j^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi = \bar{\Psi} \gamma^\mu \Psi.$$

Plane-Wave Solutions Plane-wave solutions of the Dirac equation are of the form

$$\Psi_P = e^{-iP_\mu x^\mu} u(P^\mu),$$

where $u(P^\mu)$ is a so-called spinor. Inserting it into the Dirac equation we find

$$(-\not{P} + m) u(P^\mu) = 0.$$

Multiplying by $\not{P} + m$ we find

$$(-\not{P}^2 + m^2) u(P^\mu) = (-\gamma^\mu \gamma^\nu P_\mu P_\nu + m^2) u(P^\mu) = 0.$$

We can symmetrize the first term and use the anticommutation relations of the γ^μ to find

$$(-P^2 + m^2) u(P^\mu) = 0.$$

In other words, the solution satisfies the relativistic energy-momentum relation. This also implies that for non-trivial solutions, the 4-momentum is time-like.

In the corresponding rest frame, there are four independent spinor solutions. These are $u_\pm$ and $v_\pm$, and with this representation they are as you would expect.

A Hamiltonian As the plane-wave solutions have time-like 4-momenta, there is a corresponding rest frame. In this rest frame, the Hamiltonian, which is generally

$$\mathcal{H} = \beta m + \boldsymbol{\alpha} \cdot \mathbf{p},$$

reduces to

$$\mathcal{H} = \beta m.$$

Defining

$$\mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix},$$

we have $[\mathcal{H}, \mathbf{\Sigma}] = 0$.