Notes for the Master Thesis

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Abstract

This is a collection of notes pertaining to concepts I needed to learn for my master's thesis.

Contents

1 Mathematical Prior

The Residue Theorem The residue theorem states that for a function $f(z)$ with a pole of order n at z_0, the integral of f about a positively oriented contour around z_0 satisfies

$$\oint \frac{\mathrm{d}z}{2\pi i} f(z) = \mathrm{Res}(f, z_0),$$

with

$$\mathrm{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left((z - z_0)^n f(z) \right).$$

Pushforwards and Pullbacks Consider some function f which maps a manifold M_1 to another manifold M_2, as well as a function $g : M_2 \to \mathbb{R}$. We then define the pullback of g to M_1 by f as $f^\star g = g \circ f$. We also define the pushforward of a vector $V \in T_pM_1$ by f as the map $T_pM_1 \to T_pM_2$ such that $(f_\star V)\phi = V(f^\star \phi)$.

We can also define the pullback of a $(0, m)$ tensor on M_2 by f according to

$$f^\star \omega(V_1, \ldots, V_m) = \omega(f_\star V_1, \ldots, f_\star V_m).$$

Its components are

$$(f^\star \omega)_{a_1 \ldots a_m} = \omega_{\mu_1 \ldots \mu_m} \prod_{i=1}^{m} \partial_{a_i} f^{\mu_i}.$$

Finally, if f is a bijection we may also define the more general pullback of a (n, m) tensor on M_2 as

$$f^\star T(V_1, \ldots, V_m, \omega_1, \ldots, \omega_n) = T(f_\star V_1, \ldots, f_\star V_m, (f^{-1})^\star \omega_1, \ldots, (f^{-1})^\star \omega_n).$$

Its components are

$$(f^\star T)^{b_1 \ldots b_n}_{a_1 \ldots a_m} = T^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_m} \prod_{i=1}^{m} \partial_{a_i} f^{\nu_i} \prod_{j=1}^{n} \partial_{\mu_j} (f^{-1})^{b_j}.$$

Properties of the Pullback One important property of the pullback is its symmetry preservation. To study it, it will be sufficient to study the pullback of a $(0, 2)$ tensor, as the proof generalizes to an arbitrary number of indices.

We have

$$\begin{aligned}(f^\star T)_{ab} \pm (f^\star T)_{ba} =& T_{\mu\nu} \partial_a f^\mu \partial_b f^\nu \pm T_{\mu\nu} \partial_b f^\mu \partial_a f^\nu \\ =& (T_{\mu\nu} \pm T_{\nu\mu}) \partial_b f^\mu \partial_a f^\nu,\end{aligned}$$

which implies that (anti)symmetry is preserved by the pullback. The important property is that the pullback is linear in the tensor that is pulled back, and as such the argument generalizes to (anti)symmetry with respect to an arbitrary number of indices.

Differential Forms The set of p-forms, or differential forms, is the set of $(0, p)$ tensors that are completely antisymmetric. They are constructed using the wedge product, defined as

$$\bigwedge_{k=1}^{p} d\chi^{\mu_k} = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) \bigotimes_{k=1}^{p} d\chi^{\mu_{\sigma(k)}}.$$

Here S_p is the set of permutations of p elements. There exists

$$n_p^N = \binom{N}{k}$$

basis elements. We note that the wedge product is antisymmetric under the exchange of two basis elements. Hence, once an ordering of indices has been chosen, any permutation will simply create a linearly dependent map.

Consider now some antisymmetric tensor ω. Introducing the antisymmetrizer

$$\bigotimes_{k=1}^{p} d\chi^{[\mu_{\sigma(k)}]} = \frac{1}{p!}\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)\bigotimes_{k=1}^{p} d\chi^{\mu_{\sigma(k)}},$$

the symmetry yields

$$\omega = \omega_{\mu_1\ldots\mu_p}\bigotimes_{k=1}^{p} d\chi^{\mu_{\sigma(k)}} = \omega_{\mu_1\ldots\mu_p}\bigotimes_{k=1}^{p} d\chi^{[\mu_{\sigma(k)}]} = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\bigwedge_{k=1}^{p} d\chi^{\mu_k}.$$

In other words, we can antisymmetrize the components of ω to write it as a differential form.

Two important classes of differential forms are exact and closed forms. An exact form is the exterior derivative of another, while a closed form has zero external derivative.

The Exterior Derivative We define the exterior derivative of a differential form according to

$$d\omega = \frac{1}{p!}\partial_{\mu_1}\omega_{\mu_2\ldots\mu_{p+1}}\bigwedge_{k=1}^{p+1} d\chi^{\mu_k},$$

which is a $p+1$-form.

Differential Forms and Pullbacks We will now briefly show some properties of differential forms related to the pullback.

The first is, for two maps $f : M_1 \to M_2$ and $g : M_2 \to M_3$, that

$$((g\circ f)^\star\omega)(X_1,\ldots,X_n) = \omega((g\circ f)_\star X_1,\ldots,(g\circ f)_\star X_n),$$

with ω being an n-form on M_3 and X_i vectors on M_1. For a vector we have

$$((g\circ f)_\star X)(\phi) = X((g\circ f)^\star\phi) = X(\phi\circ(g\circ f)).$$

As function compositions are associative, this implies

$$(g\circ f)_\star X = g_\star f_\star X, \tag{1}$$

and thus

$$\begin{aligned}((g\circ f)^\star\omega)(X_1,\ldots,X_n) =& \omega(g_\star f_\star X_1,\ldots,g_\star f_\star X_n)\\ =& g^\star\omega(f_\star X_1,\ldots,f_\star X_n)\\ =& f^\star g^\star\omega(X_1,\ldots,X_n),\end{aligned}$$

and thus $(g\circ f)^\star = f^\star\circ g^\star$.

Next,

$$\begin{aligned}f^\star(\omega\wedge\eta)(X_1,\ldots,X_n,X_{n+1},\ldots,X_{n+m}) =& \omega(f_\star X_1,\ldots,f_\star X_n)\wedge\eta(f_\star X_{n+1},\ldots,f_\star X_{n+m})\\ =& f^\star\omega(X_1,\ldots,X_n)\wedge f^\star\eta(X_{n+1},\ldots,X_{n+m}),\end{aligned}$$

and thus $f^\star(\omega\wedge\eta) = f^\star\omega\wedge f^\star\eta$.

This can be used to study

$$f^\star\left(\phi d\psi_1\wedge\cdots\wedge d\psi_n\right) = f^\star\phi\cdot f^\star(d\psi_1\wedge\cdots\wedge d\psi_n).$$

We have

$$f^\star d\psi(X) = d\psi(f_\star X) = (f_\star X)(\psi) = X(f_\star\psi) = X(\psi\circ f) = d(\psi\circ f)(X),$$

and thus $f^\star d\psi = d(\psi\circ f)$, implying

$$f^\star\left(\phi d\psi_1\wedge\cdots\wedge d\psi_n\right) = (\phi\circ f)d(\psi_1\circ f)\wedge\cdots\wedge d(\psi_n\circ f).$$

A final important property comes from studying

$$\begin{aligned}f^\star(d\omega) =& f^\star\left(\frac{1}{n!}\partial_{\mu_1}\omega_{\mu_2\ldots\mu_{n+1}}d\chi^{\mu_2}\wedge\cdots\wedge d\chi^{\mu_{n+1}}\right)\\ =& d\left(\frac{1}{n!}\omega_{\mu_2\ldots\mu_{n+1}}\circ f\right)d(\chi^{\mu_2}\circ f)\wedge\cdots\wedge d(\chi^{\mu_{n+1}}\circ f).\end{aligned}$$

The pullback of the 1-forms nets you basis vectors in the manifold to which you pull back, and thus you have

$$f^\star(d\omega) = d(f^\star\omega).$$

Interior Multiplication The interior multiplication of a differential form with a vector is defined as

$$i_X\omega(\dots) = \omega(X, \dots).$$

Its action on a 0-form is defined to yield zero. In index notation it is the contration of the first index of ω by X. It satisfies

$$i_X(\eta \wedge \omega) = i_X(\eta) \wedge \omega + (-1)^p \eta \wedge (i_X\omega),$$

where p is the degree of η.

Matrix-Valued Differential Forms A matrix-valued differential form is a differential form whose components are matrices. For these we need to define a slightly different wedge product according to

$$(A \wedge B)^a{}_b = A^a{}_c \wedge B^c{}_b.$$

In words, its components are found by computing the matrix product of the corresponding components of A and B, but using the wedge product instead of the normal multiplication. The output of this is then a new matrix-valued differential form. Their exterior derivatives are defined as for normal differential forms. We will use greek indices for the differential form structure and latin indices for the matrix structure.

Non-Abelian Gauge Theory - an Example We define the field strength 2-form

$$F = dA + A^2,$$

where we now suppress wedge products, as tensor products will not appear. By definition we have

$$A^2 = (A_\mu d\chi^\mu) \wedge (A_\nu d\chi^\nu) = \frac{1}{2}(A_\mu A_\nu - A_\nu A_\mu) d\chi^\mu d\chi^\nu = \frac{1}{2}[A_\mu, A_\nu] d\chi^\mu d\chi^\nu.$$

Now, F is a differential form, meaning we can write $F = \frac{1}{2} F_{\mu\nu} d\chi^\mu d\chi^\nu$. As for dA we have

$$dA = \partial_\mu A_\nu d\chi^\mu d\chi^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) d\chi^\mu d\chi^\nu,$$

where we in the last step explicitly antisymmetrized the result. Thus we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Next, defining the gauge covariant derivative

$$\vec{\nabla}_\mu = \partial_\mu + A_\mu.$$

We then have

$$\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right] = [\partial_\mu, \partial_\nu] + [A_\mu, A_\nu] + [\partial_\mu, A_\nu] - [\partial_\nu, A_\mu].$$

For the first commutator, all components have the same matrix structure, so they commute. For the last two terms we will need to use the product rule to find

$$[\partial_\mu, A_\nu] = \partial_\mu A_\nu - A_\nu \partial_\mu = (\partial_\mu A_\nu) + A_\nu \partial_\mu - A_\nu \partial_\mu = (\partial_\mu A_\nu),$$

with the brackets highlighting the terms that are self-contained and do not act as operators. Thus we have

$$\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right] = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + [A_\mu, A_\nu] = F_{\mu\nu}.$$

Next, for two vector fields we have

$$F(X, Y) = \frac{1}{2}\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right] d\chi^\mu(X) d\chi^\nu(Y) = \frac{1}{2}\left[\vec{\nabla}_\mu, \vec{\nabla}_\nu\right](X^\mu Y^\nu - X^\nu Y^\mu).$$

We have

$$\vec{\nabla}_\rho X^\mu Y^\nu = \partial_\rho X^\mu Y^\nu + A_\rho (X^\mu Y^\nu) = Y^\nu \partial_\rho (X^\mu) + X^\mu \partial_\rho (Y^\nu) + X^\mu Y^\nu \partial_\rho + A_\rho X^\mu Y^\nu,$$

and in particular

$$\begin{aligned}\vec{\nabla}_\nu X^\mu Y^\nu =& Y^\nu\partial_\nu(X^\mu) + X^\mu\partial_\nu(Y^\nu) + X^\mu Y^\nu\partial_\nu + A_\nu X^\mu Y^\nu = Y^\nu\partial_\nu(X^\mu) + X^\mu\partial_\nu(Y^\nu) + X^\mu\vec{\nabla}_Y,\\ \vec{\nabla}_\mu X^\mu Y^\nu =& Y^\nu\partial_\mu(X^\mu) + X^\mu\partial_\mu(Y^\nu) + Y^\nu\vec{\nabla}_X.\end{aligned}$$

Thus we have

$$\begin{aligned}\vec{\nabla}_\mu\vec{\nabla}_\nu X^\mu Y^\nu =& \partial_\mu Y^\nu\partial_\nu(X^\mu) + \partial_\mu X^\mu\partial_\nu(Y^\nu) + \partial_\mu X^\mu\vec{\nabla}_Y + A_\mu X^\mu\vec{\nabla}_Y\\ =& \partial_\mu Y^\nu\partial_\nu(X^\mu) + \partial_\mu X^\mu\partial_\nu(Y^\nu) + (\partial_\mu X^\mu)\vec{\nabla}_Y + \vec{\nabla}_X\vec{\nabla}_Y,\\ \vec{\nabla}_\nu\vec{\nabla}_\mu X^\mu Y^\nu =& \partial_\mu Y^\nu\partial_\nu(X^\mu) + \partial_\nu X^\mu\partial_\mu(Y^\nu) + (\partial_\nu Y^\nu)\vec{\nabla}_X + \vec{\nabla}_Y\vec{\nabla}_X\end{aligned}$$

The final result is found by first computing the difference of the above. One then notes down the result of swapping X and Y in that difference and subtracting that from what you have. First, the two connections net their commutator. The lone connections are found twice after the subtaction. Next, for the other terms the derivative can act on either factor or move to the right. The two former have terms with opposite sign cancelling them, and

$$\begin{aligned}\left[\vec{\nabla}_\mu,\vec{\nabla}_\nu\right](X^\mu Y^\nu - X^\nu Y^\mu) =& 2\left[\vec{\nabla}_X,\vec{\nabla}_Y\right] + 2\left((\partial_\mu X^\mu)\vec{\nabla}_Y - (\partial_\nu Y^\nu)\vec{\nabla}_X\right)\\ &+ 2(Y^\nu(\partial_\nu X^\mu) + X^\mu(\partial_\nu Y^\nu))\partial_\mu - 2(Y^\nu(\partial_\mu X^\mu) + X^\mu(\partial_\mu Y^\nu))\partial_\nu\\ =& 2\left[\vec{\nabla}_X,\vec{\nabla}_Y\right] + 2\left((\partial_\mu X^\mu)\vec{\nabla}_Y - (\partial_\nu Y^\nu)\vec{\nabla}_X\right)\\ &+ 2[Y,X]^\mu\partial_\mu + 2(X^\mu(\partial_\nu Y^\nu)\partial_\mu - Y^\nu(\partial_\mu X^\mu)\partial_\nu)\\ =& 2\left[\vec{\nabla}_X,\vec{\nabla}_Y\right] + 2\left((\partial_\mu X^\mu)Y^\nu A_\nu - (\partial_\nu Y^\nu)X^\mu A_\mu\right) + 2[Y,X]^\mu\partial_\mu\\ =& 2\left(\left[\vec{\nabla}_X,\vec{\nabla}_Y\right] + [Y,X]^\mu A_\mu + [Y,X]^\mu\partial_\mu\right),\end{aligned}$$

and thus

$$F(X,Y) = \left[\vec{\nabla}_X,\vec{\nabla}_Y\right] - \vec{\nabla}_{[X,Y]}.$$

As for the exterior derivative of the field strength, we have

$$\begin{aligned}dF =& \frac{1}{2}\partial_\mu F_{\nu\rho}d\chi^\mu d\chi^\nu d\chi^\rho\\ =& \frac{1}{2}\left(\partial_\mu(A_\nu A_\rho) - \partial_\mu(A_\rho A_\nu)\right)d\chi^\mu d\chi^\nu d\chi^\rho\\ =& \frac{1}{2}\left(\partial_\mu(A_\nu)A_\rho + A_\nu\partial_\mu(A_\rho) - \partial_\mu(A_\rho)A_\nu - A_\rho\partial_\mu(A_\nu)\right)d\chi^\mu d\chi^\nu d\chi^\rho,\end{aligned}$$

as we have for any differential form F that $ddF = \partial_\mu\partial_\nu F_I d\chi^\mu d\chi^\nu d\chi^I = 0$, using the notation I to refer to some set of indices. Now, by comparison we have

$$FA - AF = \frac{1}{2}[F_{\mu\nu}, A_\rho]d\chi^\mu d\chi^\nu d\chi^\rho = \frac{1}{2}\left(F_{\mu\nu}A_\rho - A_\mu F_{\nu\rho}\right)d\chi^\mu d\chi^\nu d\chi^\rho,$$

using the properties of the indices under cyclic permutation. We find

$$\begin{aligned}F_{\mu\nu}A_\rho - A_\mu F_{\nu\rho} =& \left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)A_\rho - A_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu + [A_\nu, A_\rho]\right)\\ =& \partial_\mu(A_\nu)A_\rho - \partial_\nu(A_\mu)A_\rho - A_\mu\partial_\nu A_\rho + A_\mu\partial_\rho A_\nu,\end{aligned}$$

and permuting indices cyclically we find

$$dF + AF - FA = 0,$$

which is a so-called Bianchi identity.

Next we study the form F^2. We have

$$dF^2 = d(F)F + FdF = (FA - AF)F + F(FA - AF).$$

Computing the trace of this involves computing the trace of objects of the form

$$F_{\mu_1\mu_2}A_{\mu_3}F_{\mu_4\mu_5}\bigwedge_{i=1}^{5}d\chi^{\mu_i}.$$

We note that if the first factor of F can be moved to the right in the trace, we will have shown that it is zero. Traces are invariant under cyclic permutation of matrices, so we can certainly move the components themselves. As for the differential forms, moving the first and second ones to the right requires either passing through 3 others, providing no net sign and yielding

$$\operatorname{tr}\big(dF^2\big) = d\operatorname{tr}\big(F^2\big) = 0.$$

Similarly we may consider the form F^n. Its trace is

$$d\operatorname{tr}(F^n) = n\operatorname{tr}\big(F^{n-1}dF\big) = n\operatorname{tr}\big(F^{n-1}(FA - AF)\big).$$

By the exact same argument we must have

$$d\operatorname{tr}(F^n) = 0.$$

Alternatively, let us prove that

$$dF^n = F^nA - AF^n.$$

Evidently it holds for $n = 1$. Next, assuming it to hold for some particular n, it follows that

$$\begin{aligned} dF^{n+1} =& d(F^n)F + F^n dF \\ =& (F^nA - AF^n)F + F^n(FA - AF) \\ =& F^{n+1}A - AF^{n+1}, \end{aligned}$$

completing the proof. Using the properties of the trace and the cyclic permutivity of the indices, we reobtain the same result.

The form F^2 is of some more interest. We have

$$F^2 = (dA)^2 + A^2dA + d(A)A^2 + A^4.$$

Let us compute its trace. The last term is the easiest to handle as it is a contraction of a symmetric matrix product with an antisymmetric differential form, and is thus zero. Antisymmetrizing the components does not help due to the trace, which allows you to cyclically permute the matrices without changing the sign. Antisymmetrizing they start with the opposite sign of another term related to the first one by cyclic permutation, and it must be zero. As for the others we have

$$\begin{aligned} \operatorname{tr}\big((dA)^2 + A^2dA + d(A)A^2\big) =& \operatorname{tr}\left(d(AdA) + \frac{2}{3}(A^2dA + d(A)A^2 + Ad(A)A)\right) \\ =& \operatorname{tr}\left(d(AdA) + \frac{2}{3}dA^3\right) \\ =& d\operatorname{tr}\left((AdA) + \frac{2}{3}A^3\right). \end{aligned}$$

Let us finally investigate the topological invariance of integrals of F^2. Under a small deformation of A, we have

$$\begin{aligned} \delta\operatorname{tr}(F^n) =& n\operatorname{tr}\big(F^{n-1}\,\delta F\big) \\ =& n\left(\operatorname{tr}\big(F^{n-1}d\,\delta A\big) + \operatorname{tr}\big(F^{n-1}\,\delta A\,A\big) + \operatorname{tr}\big(F^{n-1}A\,\delta A\big)\right) \\ =& n\left(\operatorname{tr}\big(F^{n-1}d\,\delta A\big) - \operatorname{tr}\big(AF^{n-1}\,\delta A\big) + \operatorname{tr}\big(F^{n-1}A\,\delta A\big)\right) \\ =& n\operatorname{tr}\big(dF^{n-1}\,\delta A\big). \end{aligned}$$

Integration of Differential Forms Consider a set of p tangent vectors X_i. The corresponding coordinate displacements are $\mathrm{d}\chi_i^a = X_i^a \, \mathrm{d}t_i$, with no sum over i. We would now like to compute the p-dimensional volume defined by the X_i and $\mathrm{d}t_i$. We expect that if any of the X_i are linearly dependent the volume should be zero. We also expect that the volume be linear in the X_i. This implies

$$\mathrm{d}V_p = \omega(X_1, \dots, X_p) \, \mathrm{d}t_1 \dots \mathrm{d}t_p$$

for some differential form ω. We now define the integral over the p-volume S over the p-form ω as

$$\int\limits_S \omega = \int \mathrm{d}t_1 \dots \int \mathrm{d}t_p \, \omega(\dot{\gamma}_1, \dots, \dot{\gamma}_p).$$

Here the γ_i are the set of curves that span S, the dot symbolizes the derivative with respect to the individual curve parameters and the right-hand integration is performed over the appropriate set of parameter values.

Stokes' Theorem Stokes' theorem relates the integral of a differential form $d\omega$ over some subset V of a manifold to an integral over ∂V of another differential form. It states that

$$\int\limits_V d\omega = \oint\limits_{\partial V} \omega.$$

2 Topology

Topological Spaces Let X be a set and $T = \{U_i | i \in I\}$ be a collection of subsets of X (I is some set of indices). The pair (X, T) (sometimes we only explicitly write X) is defined as a topological space if

- $\emptyset, \ X \in T$.
- If J is any subcollection of I, the family $\{U_j | j \in J\}$ satisfies

$$\bigcup_{j \in J} U_j \in T.$$

- If J is any finite subcollection of I, the family $\{U_j | j \in J\}$ satisfies

$$\bigcap_{j \in J} U_j \in T.$$

If the two satisfy the definition, we say that T gives a topology to X. The U_i are called its open sets.

Two cases of little interest are $T = \{\emptyset, \ X\}$ and T being the collection of all subsets of X. The two are called the trivial and discrete topologies respectively.

Metrics A metric is a map $d : X \times X \to \mathbb{R}$ that satisfies

- $d(x, y) = d(y, x)$.
- $d(x, x) \geq 0$, with equality applying if and only if $x = y$.
- $d(x, y) + d(y, z) \geq d(x,)z$.

Metric Spaces Suppose X is endowed with a metric. The collection of open disks

$$U_\varepsilon = \{x \in X | d(x, x_0) < \varepsilon\}$$

then gives a topology to X called the metric topology. The pair forms a metric space.

Continuous Maps A map between two topological spaces X and Y is continuous if its inverse maps an open set in Y to an open set in X.

Neighborhoods N is a neighborhood of x if it is a subset of X and x belongs to at least one open set contained within N.

Hausdorff Spaces A topological space is a Hausdorff space if, for any two points x, y, there exists neighborhoods U_x, U_y of the two points that do not intersect. This is an important type of topological space, as examples in physics are practically always within this category.

Homeomorphisms A homeomorphism (not to be confused with a homomorphism, however hard it may be) is a continuous map between two topological spaces with a continuous inverse. Two topological spaces are homeomorphic if there exists a homeomorphism between them.

A specific case if homeomorphicity is diffeomorphicity, which applies between two manifolds between which there is a smooth map.

Homotopy Types Two topological spaces belong to the same homotopy type if there exists a continuous map from one to the other. This is a more relaxed version of homeomorphicity, as we no longer require the map to be invertible.

Retractability A domain Ω is retractable to O if there exists a smooth map ϕ_t on Ω parametrized by t such that $\phi_1(x) = x$ and $\phi_0(x) = O$.

Inverting the Exterior Derivative It holds that $d^2 = 0$, and thus one might believe that for every form ω such that $d\omega = 0$ there exists a form χ such that $\omega = d\chi$. This will, however, turn out to depend on the topological properties of the underlying space.

Assuming the underlying space to be retractable to O, we have $\phi_1^\star\omega = \omega$ and $\phi_0^\star\omega = 0$. Define η to be the vector tangential to the coordinate flow as t is varied. The fact that $d\omega = 0$ implies that $d\phi_t^\star\omega = 0$, and one can show that $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^\star\omega = \mathcal{L}_\eta(\phi_t^\star\omega)$. We then have

$$\mathcal{L}_\eta(\phi_t^\star\omega) = (i_\eta d + d i_\eta)(\phi_t^\star\omega) = d(i_\eta(\phi_t^\star\omega)).$$

Integrating with respect to t we finally find

$$\omega = d\int\limits_0^1 \mathrm{d}t\, i_\eta(\phi_t^\star\omega).$$

Donuts and Coffee Mugs Homeomorphicity defines an equivalence relation between topological spaces. This means that we can define topological spaces into categories based on homeomorphicity.

We are now in a position to introduce the poor man's notion of topology, which considers two bodies as equivalent if one can be deformed into the other without touching two parts of the surface or tearing a part of the body. These continuous deformations correspond to homeomorphisms, but we will try to keep the discussions more to the abstract.

Topological Invariants An important question pertaining to this division of topological spaces is what separates the different categories. One possible answer is so-called topological invariants, quantities which are invariant under homeomorphism. The issue with this answer is that the full set of topological invariants has not been identified, hence they can only be used to verify that two topological spaces belong to different categories.

Subgroups and Equivalence Let H be a subgroup of an (Abelian) group G. We say that x and y are equivalent if

$$x - y \in H.$$

The addition operation is the group operation, a smart choice of notation as G is Abelian. We denote the corresponding equivalence class as $[x]$ and the set of all such classes is the quotient space $\frac{G}{H}$. From this we have a natural operation

$$[x] + [y] = [x + y]$$

on the quotient space.

Homotopy and Homotopy Classes Consider maps between two manifolds. Maps that can be continuously deformed into each other are said to be homotopic. This concept is related to that of homeomorphicity, and as such there exists topological invariants for such maps too. These divide maps into homotopy classes. A particular case is $\pi_n(M)$, which is the set of homotopy classes of maps from S^n to M.

Homology The homology of a topological space is a set of topological invariants of the space, represented by its homology groups $H_0(M), H_1(M), \ldots$

Cohomology To define (de Rham) cohomology, we first introduce the set of closed r-forms $Z^r(M)$ and the set of exact r-forms $B^r(M)$. The cohomology group is then

$$H^r(M) = \frac{Z^r(M)}{B^r(M)}.$$

This divides up the forms in $Z^r(M)$ into classes which differ by an exact form.

3 Quantum Pumps

Quantum Phases Quantum phases are low-temperature phases of matter. For parametrized system, the existence of quantum fluctuations entails that these systems exhibit phase transitions, realized by varying parameters of the Hamiltonian. The phase transition describes an abrupt change in the ground state.

Berry Phase, Connection and Curvature Consider a system with a Hamiltonian and eigenstates parametrized by some set of parameters χ - that is, we have for each value of χ a set of eigenstates

$$\mathcal{H}(\chi) |n(\chi)\rangle = E_n(\chi) |n(\chi)\rangle .$$

The adiabatic theorem tells us that if R is varied such that the Hamiltonian changes sufficiently slowly, a state which is initialized to an eigenstate at $t = 0$ will evolve to a corresponding eigenstate at a later time. In the general case we have

$$|\psi_n(t)\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t \mathrm{d}\tau E_n(\tau)} |n(\chi(t))\rangle .$$

The former factor is the complex exponential of the so-called Berry phase. Inserting this into the Schrödinger equation we find

$$\gamma_n = i \int_0^t \mathrm{d}\tau \, \langle n(\chi(\tau))| \frac{\partial}{\partial \tau} |n(\chi(\tau))\rangle .$$

Noting that

$$\frac{\partial}{\partial \tau} |n(\chi(\tau))\rangle = \frac{\mathrm{d}R}{\mathrm{d}\tau} \cdot \vec{\boldsymbol{\nabla}}_R |\chi(R)\rangle ,$$

we can define the Berry connection

$$A_n = i \langle n(\chi)| \vec{\boldsymbol{\nabla}}_\chi |n(\chi)\rangle$$

and find

$$\gamma_n = i \int_C \mathrm{d}\chi \cdot A_n.$$

C is the orbit in parameter space traversed during the time evolution.

In a slightly more sophisticated manner, the Berry connection may be taken to be a 1-form

$$A_n = i \langle n(R)| \partial_\mu |n(R)\rangle d\chi^\mu.$$

Due to Stokes' theorem, the line integral of the Berry connection about some closed path is related to the surface integral of its exterior derivative, termed the Berry curvature. Its components are

$$\Omega^{(2)}_{n,\mu\nu} = \partial_\mu A_{n,\nu} - \partial_\nu A_{n,\mu},$$

and we have

$$\int_{\partial S} A_n = \frac{1}{2}\int_S \mathrm{d}\chi^\mu \wedge \mathrm{d}\chi^\nu \, \Omega^{(2)}_{n,\mu\nu}.$$

Applications of the Above What is the use of the Berry connection and curvature? From the point of view of computing responses in quantum systems, the answer might seem to be not much. More broadly, however, it turns out that the Berry curvature in particular is useful in characterizing systems. The idea is as follows: An important assumption of the above arguments is the existence of a non-degenerate and gapped ground state for all parameter values. If the integral $\Omega^{(2)}$ over some closed surface is zero, it follows by Stokes' theorem that there are no degeneracy points enclosed by the surface, as $\Omega^{(2)}$ is closed everywhere. As such, the Berry curvature can be used to detect degeneracy points. It also turns out that the degeneracy points are stable under deformations of the Hamiltonian, as integrals of the Berry curvature are quantized.

A More Sophisticated Definition From this point on we switch to the more compact notation

$$\partial_\mu \left|n\right\rangle = \left|\partial_\mu n\right\rangle$$

and suppress the parameter dependence. The Berry curvature is given by

$$\Omega^{(2)} = dA_n = \frac{1}{2}(\partial_\mu A_{n,\nu} - \partial_\nu A_{n,\mu}) d\chi^\mu \wedge d\chi^\nu,$$

and we find

$$\Omega^{(2)}_{\mu\nu} = i\left(\left\langle \partial_\mu n \middle| \partial_\nu n\right\rangle + \left\langle n \middle| \partial_\mu \partial_\nu n\right\rangle - \left\langle \partial_\nu n \middle| \partial_\mu n\right\rangle - \left\langle n \middle| \partial_\nu \partial_\mu n\right\rangle\right) = i\left(\left\langle \partial_\mu n \middle| \partial_\nu n\right\rangle - \left\langle \partial_\nu n \middle| \partial_\mu n\right\rangle\right).$$

This can be expressed without derivatives of the state. To do that we differentiate the eigenvalue expression to yield

$$\partial_\mu \mathcal{H} \left|n\right\rangle + \mathcal{H}\left|\partial_\mu n\right\rangle = \partial_\mu E_n \left|n\right\rangle + E_n \left|\partial_\mu n\right\rangle.$$

Using the orthogonality of the eigenstates, we have for some $n \neq m$ that

$$\left\langle m \middle| \partial_\mu \mathcal{H} \middle| n\right\rangle = (E_n - E_m)\left\langle m \middle| \partial_\mu n\right\rangle.$$

We can now solve for the inner product on the left-hand side and its complex conjugate, as well as sum over m, to find

$$\Omega^{(2)}_{\mu\nu} = i\sum_{m\neq n} \frac{\left\langle n \middle| \partial_\mu \mathcal{H} \middle| m\right\rangle \left\langle m \middle| \partial_\nu \mathcal{H} \middle| n\right\rangle - \text{c.c}}{(E_n - E_m)^2}.$$

Finally we introduce a third definition

$$\Omega^{(2)} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(G d\mathcal{H} G^2 d\mathcal{H}\big).$$

G is given by $(z - \mathcal{H})^{-1}$ and the integral is a counter-clockwise contour integral around the energy of the state in consideration. There is also the appearance of the exterior derivative of the Hamiltonian.

Does this correspond to our previous notion of the Berry curvature? To investigate, let us rewrite the above operators as

$$\mathcal{H} = \sum_n E_n \left|n\middle\rangle\middle\langle n\right|, \ G = \sum_n \frac{1}{z - E_n}\left|n\middle\rangle\middle\langle n\right|.$$

Next we note that

$$GdG^{-1} = -dGG^{-1} = -Gd\mathcal{H},$$

hence

$$GdG^{-1}GdG^{-1}G = Gd\mathcal{H}Gd\mathcal{H}G,$$

and by cyclic permutation we have

$$\begin{aligned}\mathrm{tr}\big(GdG^{-1}GdG^{-1}G\big) &= \mathrm{tr}(Gd\mathcal{H}Gd\mathcal{H}G)\\ &= \mathrm{tr}\big(G\partial_\mu\mathcal{H}G\partial_\nu\mathcal{H}G\big)d\chi^\mu d\chi^\nu\\ &= \mathrm{tr}\big(G\partial_\nu\mathcal{H}G^2\partial_\mu\mathcal{H}\big)d\chi^\mu d\chi^\nu\\ &= -\,\mathrm{tr}\big(Gd\mathcal{H}G^2d\mathcal{H}\big).\end{aligned}$$

As a warmup to the final computation, consider a case where the spectrum is parameter-independent. In the eigenbasis of the Hamiltonian we generally have

$$\begin{aligned}dG^{-1} &= \sum_n \left(-\partial_\mu E_n\,|n\rangle\langle n| + (z-E_n)\left(|\partial_\mu n\rangle\langle n| + |n\rangle\langle\partial_\mu n|\right)\right)d\chi^\mu,\\ dG &= \sum_n \left(\frac{\partial_\mu E_n}{(z-E_n)^2}\,|n\rangle\langle n| + \frac{1}{z-E_n}\left(|\partial_\mu n\rangle\langle n| + |n\rangle\langle\partial_\mu n|\right)\right)d\chi^\mu,\end{aligned}$$

and thus in this case

$$GdG^{-1}GdG^{-1}G = \sum \frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\,|1\rangle\langle 1|\left(|\partial_\mu 2\rangle\langle 2| + |2\rangle\langle\partial_\mu 2|\right)|3\rangle\langle 3|\left(|\partial_\nu 4\rangle\langle 4| + |4\rangle\langle\partial_\nu 4|\right)|5\rangle\langle 5|\,e^{\mu\nu},$$

where the natural numbers are summed over and we abbreviate the differential form basis vector. Multiplying this out we have

$$\begin{aligned}GdG^{-1}GdG^{-1}G &= \sum\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\,|1\rangle\left(\langle 1|\partial_\mu 2\rangle\,\delta_{23} + \delta_{12}\,\langle\partial_\mu 2|3\rangle\right)\left(\langle 3|\partial_\nu 4\rangle\,\delta_{45} + \delta_{34}\,\langle\partial_\nu 4|5\rangle\right)\langle 5|\,e^{\mu\nu}\\ &= \sum|1\rangle\Bigg(\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\,\langle 1|\partial_\mu 2\rangle\,\delta_{23}\,\langle 3|\partial_\nu 4\rangle\,\delta_{45}\\ &\quad+\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\,\langle 1|\partial_\mu 2\rangle\,\delta_{23}\delta_{34}\,\langle\partial_\nu 4|5\rangle\\ &\quad+\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\delta_{12}\,\langle\partial_\mu 2|3\rangle\,\langle 3|\partial_\nu 4\rangle\,\delta_{45}\\ &\quad+\frac{(z-E_2)(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\delta_{12}\,\langle\partial_\mu 2|3\rangle\,\delta_{34}\,\langle\partial_\nu 4|5\rangle\Bigg)\langle 5|\,e^{\mu\nu}\\ &= \sum|1\rangle\Bigg(\frac{1}{z-E_1}\,\langle 1|\partial_\mu 2\rangle\,\langle 2|\partial_\nu 4\rangle\,\langle 4| + \frac{z-E_2}{(z-E_1)(z-E_5)}\,\langle 1|\partial_\mu 2\rangle\,\langle\partial_\nu 2|5\rangle\,\langle 5|\\ &\quad+\frac{1}{z-E_3}\,\langle\partial_\mu 1|3\rangle\,\langle 3|\partial_\nu 4\rangle\,\langle 4| + \frac{1}{z-E_5}\,\langle\partial_\mu 1|3\rangle\,\langle\partial_\nu 3|5\rangle\,\langle 5|\Bigg)\,e^{\mu\nu},\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\big(GdG^{-1}GdG^{-1}G\big) &= \sum\Bigg(\frac{1}{z-E_1}\,\langle 1|\partial_\mu 2\rangle\,\langle 2|\partial_\nu 4\rangle\,\langle 4|1\rangle + \frac{z-E_2}{(z-E_1)(z-E_5)}\,\langle 1|\partial_\mu 2\rangle\,\langle\partial_\nu 2|5\rangle\,\langle 5|1\rangle\\ &\quad+\frac{1}{z-E_3}\,\langle\partial_\mu 1|3\rangle\,\langle 3|\partial_\nu 4\rangle\,\langle 4|1\rangle + \frac{1}{z-E_5}\,\langle\partial_\mu 1|3\rangle\,\langle\partial_\nu 3|5\rangle\,\langle 5|1\rangle\Bigg)\,e^{\mu\nu}\\ 1 &= \sum\Bigg(\frac{1}{z-E_1}\,\langle 1|\partial_\mu 2\rangle\,\langle 2|\partial_\nu 4\rangle\,\delta_{41} + \frac{z-E_2}{(z-E_1)(z-E_5)}\,\langle 1|\partial_\mu 2\rangle\,\langle\partial_\nu 2|5\rangle\,\delta_{51}\\ &\quad+\frac{1}{z-E_3}\,\langle\partial_\mu 1|3\rangle\,\langle 3|\partial_\nu 4\rangle\,\delta_{41} + \frac{1}{z-E_5}\,\langle\partial_\mu 1|3\rangle\,\langle\partial_\nu 3|5\rangle\,\delta_{51}\Bigg)\,e^{\mu\nu}\\ &= \sum\Bigg(\frac{1}{z-E_1}\,\langle 1|\partial_\mu 2\rangle\,\langle 2|\partial_\nu 1\rangle + \frac{z-E_2}{(z-E_1)^2}\,\langle 1|\partial_\mu 2\rangle\,\langle\partial_\nu 2|1\rangle\\ &\quad+\frac{1}{z-E_3}\,\langle\partial_\mu 1|3\rangle\,\langle 3|\partial_\nu 1\rangle + \frac{1}{z-E_1}\,\langle\partial_\mu 1|3\rangle\,\langle\partial_\nu 3|1\rangle\Bigg)\,e^{\mu\nu}.\end{aligned}$$

Let us now perform the contour integral about a particular energy E_n. All of them are equal to 1 if and only if n is equal to the index that appears in the denominator, hence

$$\begin{aligned}\Omega^{(2)} &= -\frac{i}{2}\sum\left(\langle n|\partial_\mu 1\rangle\langle 1|\partial_\nu n\rangle + \langle n|\partial_\mu 1\rangle\langle\partial_\nu 1|n\rangle + \langle\partial_\mu 1|n\rangle\langle n|\partial_\nu 1\rangle + \langle\partial_\mu n|1\rangle\langle\partial_\nu 1|n\rangle\right)e^{\mu\nu}\\ &= -\frac{i}{2}\sum\left(-\langle\partial_\mu n|1\rangle\langle 1|\partial_\nu n\rangle + \langle\partial_\mu n|1\rangle\langle 1|\partial_\nu n\rangle + \langle 1|\partial_\mu n\rangle\langle\partial_\nu n|1\rangle - \langle\partial_\mu n|1\rangle\langle 1|\partial_\nu n\rangle\right)e^{\mu\nu}\\ &= \frac{i}{2}\left(\langle\partial_\mu n|\partial_\nu n\rangle - \langle\partial_\nu n|\partial_\mu n\rangle\right)e^{\mu\nu},\end{aligned}$$

and thus

$$\Omega^{(2)}_{\mu\nu} = i\left(\langle\partial_\mu n|\partial_\nu n\rangle - \langle\partial_\nu n|\partial_\mu n\rangle\right).$$

Let us now go to the general case. It will contain an operator product

$$\begin{aligned}&|1\rangle\langle 1|\left(-\partial_\mu E_2\,|2\rangle\langle 2| + (z-E_2)\left(|\partial_\mu 2\rangle\langle 2| + |2\rangle\langle\partial_\mu 2|\right)\right)|3\rangle\langle 3|\left(-\partial_\nu E_4\,|4\rangle\langle 4| + (z-E_4)\left(|\partial_\nu 4\rangle\langle 4| + |4\rangle\langle\partial_\nu 4|\right)\right)|5\rangle\langle 5|\\ &= |1\rangle\left(-\partial_\mu E_2\delta_{12}\delta_{23} + (z-E_2)\left(\langle 1|\partial_\mu 2\rangle\delta_{23} + \delta_{12}\langle\partial_\mu 2|3\rangle\right)\right)\left(-\partial_\nu E_4\delta_{34}\delta_{45} + (z-E_4)\left(\langle 3|\partial_\nu 4\rangle\delta_{45} + \delta_{34}\langle\partial_\nu 4|5\rangle\right)\right)\langle 5|,\end{aligned}$$

and the trace will turn this to

$$\left(-\partial_\mu E_2\delta_{12}\delta_{23} + (z-E_2)\left(\langle 1|\partial_\mu 2\rangle\delta_{23} + \delta_{12}\langle\partial_\mu 2|3\rangle\right)\right)\left(-\partial_\nu E_4\delta_{34}\delta_{45} + (z-E_4)\left(\langle 3|\partial_\nu 4\rangle\delta_{45} + \delta_{34}\langle\partial_\nu 4|5\rangle\right)\right)\delta_{15}.$$

Each bracket has three terms, so let us denote their products (after adding the extra factors) as a_{ij}, with i and j denoting which terms from each of the brackets are multiplied. We know that when tracing $a_{22}+a_{23}+a_{32}+a_{33}$, we get the result. We will thus have completed the proof if we can show that the others yield no net contribution. First we have

$$\begin{aligned}\sum a_{11} &= \sum\frac{1}{(z-E_1)(z-E_3)(z-E_5)}\left(-\partial_\mu E_2\delta_{12}\delta_{23}\right)\left(-\partial_\nu E_4\delta_{34}\delta_{45}\right)\delta_{15}e^{\mu\nu}\\ &= \sum\frac{\partial_\mu(E_2)\partial_\nu(E_4)}{(z-E_1)(z-E_3)(z-E_5)}\delta_{12}\delta_{23}\delta_{34}\delta_{45}\delta_{15}e^{\mu\nu}\\ &= \sum\frac{\partial_\mu(E_1)\partial_\nu(E_1)}{(z-E_1)^3}e^{\mu\nu}.\end{aligned}$$

This is identically zero as it contains a contraction of symmetric components with the antisymmetric differential form basis. As for the others we have

$$\begin{aligned}\sum a_{12} &= -\sum\frac{\partial_\mu E_2(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\delta_{12}\delta_{23}\langle 3|\partial_\nu 4\rangle\delta_{45}\delta_{15}e^{\mu\nu} = -\sum\frac{\partial_\mu E_1}{(z-E_1)^2}\langle 1|\partial_\nu 1\rangle e^{\mu\nu},\\ \sum a_{13} &= -\sum\frac{\partial_\mu E_2(z-E_4)}{(z-E_1)(z-E_3)(z-E_5)}\delta_{12}\delta_{23}\delta_{34}\langle\partial_\nu 4|5\rangle\delta_{15}e^{\mu\nu} = -\sum\frac{\partial_\mu E_1}{(z-E_1)^2}\langle\partial_\nu 1|1\rangle e^{\mu\nu},\\ \sum a_{21} &= -\sum\frac{\partial_\nu E_4(z-E_2)}{(z-E_1)(z-E_3)(z-E_5)}\langle 1|\partial_\mu 2\rangle\delta_{23}\delta_{34}\delta_{45}\delta_{15}e^{\mu\nu} = -\sum\frac{\partial_\nu E_1}{(z-E_1)^2}\langle 1|\partial_\mu 1\rangle e^{\mu\nu},\\ \sum a_{31} &= -\sum\frac{\partial_\nu E_4(z-E_2)}{(z-E_1)(z-E_3)(z-E_5)}\delta_{12}\langle\partial_\mu 2|3\rangle\delta_{34}\delta_{45}\delta_{15}e^{\mu\nu} = -\sum\frac{\partial_\nu E_1}{(z-E_1)^2}\langle\partial_\mu 1|1\rangle e^{\mu\nu},\end{aligned}$$

and these all cancel each other exactly, completing the proof.

Properties of Parametrized States We will now derive some useful properties of derivatives of states of parametrized systems. Because orthogonality is preserved we have

$$\partial_\mu\langle m|n\rangle = \langle\partial_\mu m|n\rangle + \langle m|\partial_\mu n\rangle = 0.$$

Because the identity is also preserved we have

$$\sum|\partial_\mu n\rangle\langle n| + |n\rangle\langle\partial_\mu n| = 0.$$

The Single Spin Consider a single spin-$\frac{1}{2}$ in an external field of length 1. The Hamiltonian is

$$\mathcal{H} = h_x\sigma_x + h_y\sigma_y + h_z\sigma_z,$$

with the external field being restricted in length. With respect to the σ_z eigenstates at $\theta = \phi = 0$, which are of course angle-independent, we have

$$|{\downarrow}\rangle_{\theta,\phi} = \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}, \ |{\uparrow}\rangle_{\theta,\phi} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \tag{2}$$

and thus

$$A_{-,\theta} = 0, \ A_{-,\phi} = \sin^2\left(\frac{\theta}{2}\right), \ A_{+,\theta} = 0, \ A_{+,\phi} = \cos^2\left(\frac{\theta}{2}\right).$$

The Berry curvature is then

$$\Omega^{(2)}_{\pm,\theta\phi} = \mp\frac{1}{2}\sin(\theta).$$

This implies that the Berry phase induced after an adiabatic cycle is equal to half the subtended solid angle.

Higher Berry Curvature and the KS Invariant For an infinite $1d$ system, $\Omega^{(2)}$ might diverge. A convergent quantity might instead be found by splitting the Hamiltonian into a sum of local terms working at a finite range, i.e.

$$\mathcal{H} = \sum_{p\in\mathbb{Z}} \mathcal{H}_p.$$

The quantity

$$F^{(2)}_{pq} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H}_pG^2d\mathcal{H}_q\big)$$

then decays exponentially with respect to $|p-q|$ if the Hamiltonian is gapped, and is thus well-defined. Next we can construct the two-form

$$F^{(2)}_q = \sum_{p\in\mathbb{Z}} F^{(2)}_{pq} = \frac{i}{2}\oint \frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(Gd\mathcal{H}G^2d\mathcal{H}_q\big).$$

Its exterior derivative is given by

$$dF^{(2)}_q = \sum_{p\in\mathbb{Z}} F^{(3)}_{pq}.$$

We have

$$\begin{aligned}
&\partial_\rho(G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q)e^{\rho\mu\nu} \\
&= \big(\partial_\rho G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\rho\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p\partial_\rho G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_pG^2\partial_\rho\partial_\nu\mathcal{H}_q\big)\,e^{\rho\mu\nu} \\
&= \big(\partial_\rho G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p(\partial_\rho GG + G\partial_\rho G)\partial_\nu\mathcal{H}_q\big)\,e^{\rho\mu\nu} \\
&= \big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p(G\partial_\rho\mathcal{H}G^2 + G^2\partial_\rho\mathcal{H}G)\partial_\nu\mathcal{H}_q\big)\,e^{\rho\mu\nu} \\
&= \big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_p(G\partial_\rho\mathcal{H}G^2 + G^2\partial_\rho\mathcal{H}G)\partial_\nu\mathcal{H}_q\big)\,e^{\rho\mu\nu} \\
&= \big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_pG\partial_\rho\mathcal{H}G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_pG^2\partial_\rho\mathcal{H}G\partial_\nu\mathcal{H}_q\big)\,e^{\rho\mu\nu},
\end{aligned}$$

and thus

$$\begin{aligned}
&\mathrm{tr}\big(\partial_\rho(G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q)\big)e^{\rho\mu\nu} \\
&= \mathrm{tr}\big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_pG\partial_\rho\mathcal{H}G^2\partial_\nu\mathcal{H}_q + G\partial_\mu\mathcal{H}_pG^2\partial_\rho\mathcal{H}G\partial_\nu\mathcal{H}_q\big)e^{\rho\mu\nu} \\
&= \mathrm{tr}\big(\partial_\rho\mathcal{H}G\partial_\mu\mathcal{H}_pG^2\partial_\nu\mathcal{H}_q - G\partial_\rho\mathcal{H}G^2\partial_\mu\mathcal{H}_qG\partial_\nu\mathcal{H}_p + G\partial_\mu\mathcal{H}_pG^2\partial_\rho\mathcal{H}G\partial_\nu\mathcal{H}_q\big)e^{\rho\mu\nu}.
\end{aligned}$$

Somehow we are to find

$$F_{pq}^{(3)} = \frac{i}{6}\oint\frac{\mathrm{d}z}{2\pi i}\,\mathrm{tr}\big(G^2 d\mathcal{H} G d\mathcal{H}_p G d\mathcal{H}_q - G d\mathcal{H} G^2 d\mathcal{H}_p G d\mathcal{H}_q\big) - (p\leftrightarrow q).$$

To compute this we expand in eigenstates of the Hamiltonian according to

$$\begin{aligned} G^2 d\mathcal{H} G d\mathcal{H}_p G d\mathcal{H}_q &= \sum \frac{|1\rangle\langle 1|\,|2\rangle\langle 2|\,d\mathcal{H}\,|3\rangle\langle 3|\,d\mathcal{H}_p\,|4\rangle\langle 4|\,d\mathcal{H}_q}{(z-E_1)(z-E_2)(z-E_3)(z-E_4)} \\ &= \sum \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(z-E_1)^2(z-E_2)(z-E_3)}. \end{aligned}$$

Let us now compute the contour integral around the ground state. The contributions from where only one number is zero is

$$\begin{aligned} &\sum -|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q\left(\frac{1}{(E_0-E_2)^2(E_0-E_3)}+\frac{1}{(E_0-E_2)(E_0-E_3)^2}\right) \\ &\quad + \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_1)^2(E_0-E_3)} + \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)^2(E_0-E_2)}. \end{aligned}$$

Introducing

$$G_0 = \sum_{n\neq 0}\frac{1}{E_0-E_n}\,|n\rangle\langle n|\,,$$

this can be written as

$$-|0\rangle\langle 0|\left(d\mathcal{H} G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q + d\mathcal{H} G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_q\right) + G_0^2 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0 d\mathcal{H}_q + G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q.$$

Similarly, when two of the numbers are zero we get the contribution

$$\begin{aligned} &\sum\frac{1}{2}\left(2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_3)^3} + 2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_2)^3}\right) - 2\frac{|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)^3} \\ &= |0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0^3 d\mathcal{H}_q + |0\rangle\langle 0|\,d\mathcal{H} G_0^3 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q - 2G_0^3 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q. \end{aligned}$$

Finally, if none or all of them are zero there is no contribution. Next, we have

$$G d\mathcal{H} G^2 d\mathcal{H}_p G d\mathcal{H}_q = \sum\frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(z-E_1)(z-E_2)^2(z-E_3)}.$$

The contributions after computing the contour integral are

$$\begin{aligned} &\sum -|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q\left(\frac{1}{(E_0-E_1)^2(E_0-E_3)}+\frac{1}{(E_0-E_1)(E_0-E_3)^2}\right) \\ &\quad + \frac{|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_2)^2(E_0-E_3)} + \frac{|1\rangle\langle 1|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)(E_0-E_2)^2} \\ &= -G_0^2 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0 d\mathcal{H}_q - G_0 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0^2 d\mathcal{H}_q + |0\rangle\langle 0|\,d\mathcal{H} G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q + G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q \end{aligned}$$

when one number is zero and

$$\begin{aligned} &\sum\frac{1}{2}\left(2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|3\rangle\langle 3|\,d\mathcal{H}_q}{(E_0-E_3)^3} + 2\frac{|1\rangle\langle 1|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_1)^3}\right) - 2\frac{|0\rangle\langle 0|\,d\mathcal{H}\,|2\rangle\langle 2|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q}{(E_0-E_2)^2} \\ &= |0\rangle\langle 0|\,d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p G_0^3 d\mathcal{H}_q + G_0^3 d\mathcal{H}\,|0\rangle\langle 0|\,d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q - 2\,|0\rangle\langle 0|\,d\mathcal{H} G_0^3 d\mathcal{H}_p\,|0\rangle\langle 0|\,d\mathcal{H}_q \end{aligned}$$

when two are. The final result is thus

$$\begin{aligned} F_{pq}^{(3)} &= \frac{i}{6}\Big(-\left\langle d\mathcal{H} G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q\right\rangle - \left\langle d\mathcal{H} G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_q\right\rangle + \left\langle d\mathcal{H}_p G_0 d\mathcal{H}_q G_0^2 d\mathcal{H}\right\rangle \\ &\quad + \left\langle d\mathcal{H}_q G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\right\rangle + \langle d\mathcal{H}\rangle\left\langle d\mathcal{H}_p G_0^3 d\mathcal{H}_q\right\rangle + \left\langle d\mathcal{H} G_0^3 d\mathcal{H}_p\right\rangle\langle d\mathcal{H}_q\rangle \\ &\quad - 2\left\langle d\mathcal{H}_q G_0^3 d\mathcal{H}\right\rangle\langle d\mathcal{H}_p\rangle + \left\langle d\mathcal{H}_p G_0 d\mathcal{H}_q G_0^2 d\mathcal{H}\right\rangle + \left\langle d\mathcal{H}_p G_0^2 d\mathcal{H}_q G_0 d\mathcal{H}\right\rangle \\ &\quad - \left\langle d\mathcal{H} G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q\right\rangle - \left\langle d\mathcal{H}_q G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\right\rangle - \langle d\mathcal{H}\rangle\left\langle d\mathcal{H}_p G_0^3 d\mathcal{H}_q\right\rangle \\ &\quad - \langle d\mathcal{H}_p\rangle\left\langle d\mathcal{H}_q G_0^3 d\mathcal{H}\right\rangle + 2\left\langle d\mathcal{H} G_0^3 d\mathcal{H}_p\right\rangle\langle d\mathcal{H}_q\rangle\Big) - (p\leftrightarrow q) \\ &= \frac{i}{6}\Big(-2\left\langle d\mathcal{H} G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_q\right\rangle - \left\langle d\mathcal{H} G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_q\right\rangle + 2\left\langle d\mathcal{H}_p G_0 d\mathcal{H}_q G_0^2 d\mathcal{H}\right\rangle \\ &\quad + \left\langle d\mathcal{H}_p G_0^2 d\mathcal{H}_q G_0 d\mathcal{H}\right\rangle + \left\langle d\mathcal{H}_q G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\right\rangle - \left\langle d\mathcal{H}_q G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\right\rangle \\ &\quad + 3\left\langle d\mathcal{H} G_0^3 d\mathcal{H}_p\right\rangle\langle d\mathcal{H}_q\rangle - 3\left\langle d\mathcal{H}_q G_0^3 d\mathcal{H}\right\rangle\langle d\mathcal{H}_p\rangle\Big) - (p\leftrightarrow q), \end{aligned}$$

where all the expectation values are computed in the ground state.

This quantity is somewhat difficult to manage, but one can reduce it somewhat. First, states excited outside of the support of $\mathcal{H}_p$ and $\mathcal{H}_q$ do not contribute, as they are orthogonal to the ground state and can pass through $\mathcal{H}_p$ and $\mathcal{H}_q$, as well as their exterior derivatives. By a similar token, $F^{(3)}_{pq}$ is non-zero only if $\mathcal{H}_p$ and $\mathcal{H}_q$ have overlapping support. This also implies that the only terms in the Hamiltonian that contribute are the ones with support overlapping with both $\mathcal{H}_p$ and $\mathcal{H}_q$.

Using these quantities we can construct a 3-form Berry curvature

$$\Omega^{(3)}(f) = \frac{1}{2}\sum_{p,q\in\mathbb{Z}} F^{(3)}_{pq}(f(q) - f(p)).$$

f is some sigmoid function, its particular shape turning out to be unimportant. A simple choice is $f(p) = \Theta(p-a)$ for some $a \in \mathbb{Z} + \frac{1}{2}$. For this particular choice we have

$$\begin{aligned}
\Omega^{(3)}(f) =& \frac{1}{2}\sum_{p,q\in\mathbb{Z}} F^{(3)}_{pq}(\Theta(q-a) - \Theta(p-a)) \\
=& \frac{1}{2}\sum_{p\in\mathbb{Z},\ q>a} F^{(3)}_{pq}(1-\Theta(p-a)) - \frac{1}{2}\sum_{p\in\mathbb{Z},\ q<a} F^{(3)}_{pq}\Theta(p-a) \\
=& \frac{1}{2}\sum_{p<a,\ q>a} F^{(3)}_{pq} - \frac{1}{2}\sum_{p>a,\ q<a} F^{(3)}_{pq} \\
=& \sum_{p<a,\ q>a} F^{(3)}_{pq},
\end{aligned}$$

using the antisymmetry of $F^{(3)}_{pq}$.

Finally we can define the KS invariant

$$Q_{\mathrm{KS}} = \int \Omega^{(3)}(f),$$

which is performed over the full parameter space of the Hamiltonian. This is a topological invariant.

The Dimerized Spin Chain Consider an infinite spin chain with Hamiltonian

$$\mathcal{H}_{1d} = \sum_{p\in\mathbb{Z}} \mathcal{H}^1_p(w) + \sum_{p\in 2\mathbb{Z}+1} \mathcal{H}^{2,+}_{p,p+1}(w) + \sum_{p\in 2\mathbb{Z}} \mathcal{H}^{2,-}_{p,p+1}(w).$$

The parameter takes values on S^3. There are three kinds of terms here. The first is

$$\mathcal{H}^1_p(w) = (-1)^p(w_1\sigma^1_p + w_2\sigma^2_p + w_3\sigma^3_p),$$

which is some fluctuating on-site term. The two others are

$$\mathcal{H}^{2,\pm}_{p,p+1}(w) = g^{\pm}(w)\sum_{\mu=1,2,3} \sigma^\mu_p\sigma^\mu_{p+1},$$

with two functions

$$g^+(w) = \begin{cases} w_4, & 0 \le w_4 \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad g^-(w) = \begin{cases} -w_4, & -1 \le w_4 \le 0, \\ 0, & \text{otherwise}. \end{cases}$$

This type of interaction defines five distinct regimes:

- $w_4 = 1$, where there is only odd-even bonding.
- $0 < w_4 < 1$, where there is odd-even bonding and on-site interactions.
- $w_4 = 0$, where there is only on-site interaction.
- $-1 < w_4 < 0$, where there is even-odd bonding and on-site interactions.

- $w_4 = 1$, where there is only even-odd bonding.

To compute the 3-form Berry curvature and KS invariant, we rewrite the Hamiltonian as a sum of local terms. These are

$$\mathcal{H}_p(w) = \mathcal{H}^1_p(w) + x\mathcal{H}^{2,\pm}_{p,p+1}(w) + (1-x)\mathcal{H}^{2,\mp}_{p-1,p}(w).$$

The top sign is for odd p. The new parameter x is an extra control parameter, taken to be fixed. Its introduction is an explicit representation of the ambiguity of the choice of local terms.

For the sigmoid function f we choose a Heaviside function, this time leaving us with two variants - f with $a \in 2\mathbb{Z} - \frac{1}{2}$ and f' with $a \in 2\mathbb{Z} + \frac{1}{2}$. To see how they differ, consider the regime $w_4 > 0$. In this case f splits the dimer in two and f' switches on between two dimers.

Because the local terms in the Hamiltonian only interact at range 1 in either direction, the eigenstates of the system for any parameter choice are product states over each dimer. This means

$$\Omega^{(3)}(f) = \Omega^{(3)}(f') = F^{(3)}_{a-\frac{1}{2},a+\frac{1}{2}},$$

with the particular choice of a distinguishing the two cases. $\Omega^{(3)}(f)$ is only non-trivial if the sites $a \pm \frac{1}{2}$ belong to the same dimer, hence $\Omega^{(3)}(f) = 0$ unless $w_4 > 0$ and $\Omega^{(3)}(f') = 0$ unless $w_4 < 0$.

We will need to diagonalize the dimer, so we first transform the basis from an angle-independent one into one parallel with the Zeeman field using a unitary operator U. This transforms states according to $|\psi\rangle \to |\psi\rangle_{\theta,\phi} = U|\psi\rangle$ and any operator according to $A \to a = UAU^\dagger$, the explicit angle dependence having been removed from the left-hand side of both equalities. This angle dependence is instead baked into the basis. The small-letter notation will be useful for clarification when a matrix representation is invoked. Having applied this transformation we choose simultaneous eigenstates of $S'_{z,p} + S'_{z,p+1}$ and $(S'_p + S'_{p+1})^2$, which are also eigenstates of $(S'_p)^2$ and $(S'_{p+1})^2$. The vector appearing in the Zeeman term has length $\sqrt{1-w_4^2}$, meaning

$$h_p = -2\sqrt{1-w_4^2}S'_{z,p} + 4xw_4 S'_p \cdot S'_{p+1}, \; h_{p+1} = 2\sqrt{1-w_4^2}S_{z,p+1} + 4(1-x)w_4 S'_p \cdot S'_{p+1}$$

for $p = a - \frac{1}{2}$. Furthermore, as

$$S'_p \cdot S'_{p+1} = \frac{1}{2}((S'_p + S'_{p+1})^2 - (S'_p)^2 - (S'_{p+1})^2),$$

we have

$$h_p = -\sqrt{1-w_4^2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + 2xw_4 \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix},$$

$$h_{p+1} = \sqrt{1-w_4^2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + (1-x)w_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

in the eigenbasis of total spin, and the total dimer Hamiltonian is

$$h = w_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} - 2\sqrt{1-w_4^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenstates $|1,1\rangle$ and $|1,-1\rangle$ are still eigenstates of the total Hamiltonian, with energy w_4. In addition there are two eigenstates found by diagonalizing

$$\begin{bmatrix} w_4 & -2\sqrt{1-w_4^2} \\ -2\sqrt{1-w_4^2} & -3w_4 \end{bmatrix}.$$

The energies are $\pm 2 - w_4$, with eigenstates

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -\sqrt{1+w_4} & \sqrt{1-w_4} \\ \sqrt{1-w_4} & \sqrt{1+w_4} \end{bmatrix}.$$

We proceed by introducing hyperspherical coordinates

$$w_1 = \sin(\alpha)\cos(\theta),\ w_2 = \sin(\alpha)\sin(\theta)\cos(\phi),\ w_3 = \sin(\alpha)\sin(\theta)\sin(\phi),\ w_4 = \cos(\alpha),$$

for which we have

$$h_p = -\sin(\alpha)\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&-1&0\\0&1&0&0\end{bmatrix} + x\cos(\alpha)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{bmatrix},$$

$$h_{p+1} = \sin(\alpha)\begin{bmatrix}1&0&0&0\\0&0&0&-1\\0&0&-1&0\\0&-1&0&0\end{bmatrix} + (1-x)\cos(\alpha)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{bmatrix},$$

$$h_p + h_{p+1} = h = \cos(\alpha)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{bmatrix} - 2\sin(\alpha)\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{bmatrix}.$$

The eigenstates of individual spin are given in equation 2, and we then have

$$|1,1\rangle_{\theta,\phi} = \begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right)e^{-2i\phi}\\ \frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi}\\ \sin^2\left(\frac{\theta}{2}\right)\\ 0\end{bmatrix},\ |1,0\rangle_{\theta,\phi} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\sin(\theta)e^{-2i\phi}\\ \cos(\theta)e^{-i\phi}\\ \frac{1}{\sqrt{2}}\sin(\theta)\\ 0\end{bmatrix},\ |1,-1\rangle_{\theta,\phi} = \begin{bmatrix}\sin^2\left(\frac{\theta}{2}\right)e^{-2i\phi}\\ -\frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi}\\ \cos^2\left(\frac{\theta}{2}\right)\\ 0\end{bmatrix},\ |0,0\rangle_{\theta,\phi} = \begin{bmatrix}0\\0\\0\\e^{-i\phi}\end{bmatrix}$$

with respect to the total spin basis for $\theta = \phi = 0$. We can then explicitly write

$$U = \begin{bmatrix}\cos^2\left(\frac{\theta}{2}\right)e^{-2i\phi} & -\frac{1}{\sqrt{2}}\sin(\theta)e^{-2i\phi} & \sin^2\left(\frac{\theta}{2}\right)e^{-2i\phi} & 0\\ \frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi} & \cos(\theta)e^{-i\phi} & -\frac{1}{\sqrt{2}}\sin(\theta)e^{-i\phi} & 0\\ \sin^2\left(\frac{\theta}{2}\right) & \frac{1}{\sqrt{2}}\sin(\theta) & \cos^2\left(\frac{\theta}{2}\right) & 0\\ 0&0&0&e^{-i\phi}\end{bmatrix}.$$

Let us also derive an expression for g_0. The eigenstates of the Hamiltonian in the angle-dependent basis are

$$v_{-2-\cos(\alpha)} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ \sqrt{1-\cos(\alpha)}\\ 0\\ \sqrt{1+\cos(\alpha)}\end{bmatrix},\ v_{\cos(\alpha),1} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ v_{\cos(\alpha),2} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ v_{2-\cos(\alpha)} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ -\sqrt{1+\cos(\alpha)}\\ 0\\ \sqrt{1-\cos(\alpha)}\end{bmatrix}.$$

Forming these into a matrix V and computing VDV^{-1} for

$$D = \begin{bmatrix}-\frac{1}{2(1+\cos(\alpha))} & 0 & 0 & 0\\ 0&0&0&0\\ 0&0&-\frac{1}{2(1+\cos(\alpha))}&0\\ 0&0&0&-\frac{1}{4}\end{bmatrix}$$

nets us

$$g_0 = \begin{bmatrix}-\frac{1}{2(1+\cos(\alpha))} & 0 & 0 & 0\\ 0 & -\frac{1}{8}(1+\cos(\alpha)) & 0 & \frac{1}{8}\sin(\alpha)\\ 0&0&-\frac{1}{2(1+\cos(\alpha))}&0\\ 0&\frac{1}{8}\sin(\alpha)&0&-\frac{1}{8}(1-\cos(\alpha))\end{bmatrix}.$$

The three angles are now neatly separated, as ϕ and θ only enter in U and α only enters in the combination of eigenstates after U has been applied. Using the explicit formula we then have

$$\begin{aligned}F^{(3)}_{p,p+1} =& \frac{i}{6}\Big(-2\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1}\right\rangle - \left\langle d\mathcal{H}G_0 d\mathcal{H}_p G_0^2 d\mathcal{H}_{p+1}\right\rangle + 2\left\langle d\mathcal{H}_p G_0 d\mathcal{H}_{p+1} G_0^2 d\mathcal{H}\right\rangle\\ &+ \left\langle d\mathcal{H}_p G_0^2 d\mathcal{H}_{p+1} G_0 d\mathcal{H}\right\rangle + \left\langle d\mathcal{H}_{p+1} G_0^2 d\mathcal{H} G_0 d\mathcal{H}_p\right\rangle - \left\langle d\mathcal{H}_{p+1} G_0 d\mathcal{H} G_0^2 d\mathcal{H}_p\right\rangle\\ &+ 3\left\langle d\mathcal{H}G_0^3 d\mathcal{H}_p\right\rangle\left\langle d\mathcal{H}_{p+1}\right\rangle - 3\left\langle d\mathcal{H}_{p+1}G_0^3 d\mathcal{H}\right\rangle\left\langle d\mathcal{H}_p\right\rangle\Big) - (p \leftrightarrow p+1).\end{aligned}$$

In order to get non-trivial results, we must compute these expectation values in an angle-independent basis. To that end, we note that all the operators involved only depend on α in the angle-dependent basis. We then write $A = U^\dagger a U$ and consider its expectation value in some angle-dependent state $|\psi\rangle = U^\dagger |\psi\rangle_{\theta,\phi}$. We then have

$$\langle\psi|A|\psi\rangle = \langle\psi|UU^\dagger aUU^\dagger|\psi\rangle_{\theta,\phi} = \langle\psi|a|\psi\rangle_{\theta,\phi}\,.$$

This means, for instance, that

$$\begin{aligned}\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1}\right\rangle &= \left\langle U(d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1})_{\theta,\phi}U^\dagger\right\rangle_{\theta,\phi} \\ &= \left\langle Ud(U^\dagger hU)U^\dagger g_0^2 Ud(U^\dagger h_p U)U^\dagger g_0 Ud(U^\dagger h_{p+1}U)U^\dagger\right\rangle_{\theta,\phi}.\end{aligned}$$

This should generally be a polynomial of order up to 2 in x. To simplify the calculation we could start by eliminating terms. For example, the order-2 terms come from the contributions where h_p and h_{p+1} look identical. By antisymmetry with respect to the lattice sites these terms vanish, and the remaining contributions are of order 1 at most.

By antisymmetry we may compute any component of this form, so we choose $F^{(3)}_{p,p+1,\alpha\theta\phi}$, for which the two latter terms vanish.

This is going to take some time, so let's start. We have

$$\begin{aligned}&\left\langle d\mathcal{H}G_0^2 d\mathcal{H}_p G_0 d\mathcal{H}_{p+1}\right\rangle \\ =&-\frac{i}{64}\cos(\theta)\left(\cos(\alpha)-1\right)\left(2\sin(\alpha)\cos^2(\alpha)-2\cos^3(\alpha)-5\sin(\alpha)\cos(\alpha)+3\cos(\alpha)^2+\sin(\alpha)+4\cos(\alpha)+3\right)\end{aligned}$$

Using the explicit formula above we somehow find

$$\Omega^{(3)}_{\alpha\theta\phi} = \frac{1}{2}(2+\cos(\alpha))\tan^2\left(\frac{\alpha}{2}\right)\sin(\theta)$$

for $0 < \alpha < \frac{\pi}{2}$. We then have

$$\begin{aligned}Q_{\mathrm{KS}} =& \int_{S^3}\Omega^{(3)}_{\alpha\theta\phi} = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{2\pi}\mathrm{d}\alpha\,\mathrm{d}\theta\,\mathrm{d}\phi\,(2+\cos(\alpha))\tan^2\left(\frac{\alpha}{2}\right)\sin(\theta) \\ =&2\pi\int_0^{\frac{\pi}{2}}\mathrm{d}\alpha\,(2+\cos(\alpha))\tan^2\left(\frac{\alpha}{2}\right) = 2\pi.\end{aligned}$$

What if we were to use a different sigmoid? Swapping to f', which is zero for $w_4 > 0$. Elsewhere we find

$$\Omega^{(3)}_{\alpha\theta\phi} = \frac{1}{2}(2-\cos(\alpha))\cot^2\left(\frac{\alpha}{2}\right)\sin(\theta),$$

and thus

$$\begin{aligned}Q_{\mathrm{KS}} =&2\pi\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}\alpha\,(2-\cos(\alpha))\cot^2\left(\frac{\alpha}{2}\right) \\ =&-2\pi\int_{\frac{\pi}{2}}^{0}\mathrm{d}\beta\,(2-\cos(\pi-\beta))\cot^2\left(\frac{\pi-\beta}{2}\right) \\ =&2\pi\int_0^{\frac{\pi}{2}}\mathrm{d}\beta\,(2+\cos(\beta))\tan^2\left(\frac{\beta}{2}\right) \\ =&2\pi,\end{aligned}$$

and indeed the choice of sigmoid was irrelevant.

Dimensional Reduction in Topological Insulators Topological insulators are generally described by some fermion model. Two classes of such models are A and AIII. The two are defined by not respecting time reversal and charge conjugation as symmetries. In addition, AIII has some unitary operator Γ that anticommutes with the Hamiltonian, also known as having a chiral symmetry. The existence of the chiral symmetry implies symmetry of the energy spectrum about the Fermi level.

It turns out that every model of class A in even spatial dimension can be related to a model of class AIII in one lower dimension. The process of relating the two is termed dimensional reduction. The scheme is as follows: Start with a Dirac model in $d = 2n + 2$ dimensions given by

$$\mathcal{H} = m\Gamma^{2n+3}_{(2n+3)} + \sum_{a=1}^{2n+2} k_a \Gamma^a_{(2n+3)},$$

with $\Gamma^a_{(2n+3)}$ being generators of the $2n + 3$-dimensional Clifford algebra. This is the class A model. The dimensional reduction is performed by setting $k_{2n+2} = 0$, yielding a model in $d = 2n + 2$ in class AIII, as the Hamiltonian now anticommutes with $\Gamma^{2n+2}_{(2n+3)}$.

Generally for these models we introduce Bloch states $|u^{\pm}_a(k)\rangle$, with k being confined to the first Brillouin zone, a being a band index and the sign dictating whether the state is occupied. From this we can define a non-Abelian Berry curvature for the occupied state according to

$$A_{ab,\mu} = \left\langle u^-_a(k) \middle| \partial_\mu \middle| u^-_b(k) \right\rangle .$$

The Berry curvature is then $F = dA + A^2$. Models in class A in $d = 2n+2$ can be characterized by the $n+1$th Chern character

$$\mathrm{ch}_{n+1}(F) = \frac{1}{(n+1)!}\,\mathrm{tr}\left(\left(\frac{iF}{2\pi}\right)^{n+1}\right),$$

as well as its integral, called the Chern number. The Chern character is the exterior derivative of the $2(n + 1) - 1$th Chern-Simons form, defined as

$$Q_{2n+1} = \frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1} \int_0^1 \mathrm{d}t\,\mathrm{tr}(AF_t^n),\;\; F_t = tdA + t^2A^2 = tF + (t^2 - t)A^2.$$

Note that the Chern-Simons form is not gauge invariant. Instead, for a gauge transformation

$$A' = g^{-1}Ag + g^{-1}dg,\;\; F' = g^{-1}Fg,$$

we have

$$\begin{aligned}
Q'_{2n+1} =& \frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1} \int_0^1 \mathrm{d}t\,\mathrm{tr}\big((g^{-1}Ag + g^{-1}dg)\,\big(tg^{-1}Fg + (t^2 - t)(g^{-1}Ag + g^{-1}dg)^2\big)^n\big)\\
=& \frac{1}{n!}\left(\frac{i}{2\pi}\right)^{n+1} \int_0^1 \mathrm{d}t\,\mathrm{tr}\big((g^{-1}Ag + g^{-1}dg)\,\big(tg^{-1}Fg + (t^2 - t)(g^{-1}A^2g + g^{-1}Adg + +g^{-1}dgg^{-1}Ag + g^{-1}dgg^{-1}dg)\big)
\end{aligned}$$

One term here is of course Q_{2n+1}. The other is the result of setting $A = g^{-1}dg$ and $F = 0$ in the Chern-Simons form. The remaining terms are the exterior derivative of some $2n$-form, somehow.

To introduce a similar concept for class AIII, we first introduce the projection matrix $P(k)$ onto the occupied bands and $Q = 1 - 2P$. The chiral symmetry somehow implies that the Q matrix can be written as

$$Q = \begin{bmatrix} 0 & q \\ q^\dagger & 0 \end{bmatrix},$$

with q being a unitary matrix, in the basis where Γ is diagonal. We can now introduce the winding number for a model in AIII and $d = 2n + 1$ as

$$\nu_{2n+1} = \int \omega_{2n+1}(q),\;\; \omega_{2n+1}(q) = \frac{(-1)^n n!}{(2n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \mathrm{tr}\big((q^{-1}dq)^{2n+1}\big).$$

The Chern-Simons forms define a characteristic class on odd-dimensional manifolds, meaning we can use them to characterize systems. We therefore introduce the Chern-Simons invariant

$$\mathrm{CS}_{2n+1} = \int Q_{2n+1},$$

with the integral being over the first Brillouin zone. Returning to the issue of gauge invariance, evaluating the Chern-Simons form at $g^{-1}dg$ gives the same result as the winding number evaluated at g. The winding number is an integer which in this case measures the winding g about the Brillouin zone. Note that $\pi_{\mathrm{BZ}}(U(N)) = \mathbb{Z}$. It therefore follows that the Wilson loop

$$W_{2n+1} = e^{2\pi i \mathrm{CS}_{2n+1}}$$

is gauge invariant.

Let us move on to a more specific example. The simplest would be starting in $d = 2$ with the Hamiltonian

$$\mathcal{H} = k_x\sigma_x + k_y\sigma_y + m\sigma_z.$$

Introducing $\lambda = \sqrt{k^2 + m^2}$, the eigenstates are given by

$$\left|u^-(k)\right\rangle = \frac{1}{\sqrt{2\lambda(\lambda+m)}}\begin{bmatrix} -k_x + ik_y \\ \lambda + m \end{bmatrix}, \quad \left|u^+(k)\right\rangle = \frac{1}{\sqrt{2\lambda(\lambda-m)}}\begin{bmatrix} k_x - ik_y \\ \lambda - m \end{bmatrix}.$$

Writing

$$\partial_\mu f(\sqrt{k^2+m^2}) = f' \frac{k_\mu}{\sqrt{k^2+m^2}}$$

we find

$$\begin{aligned}
A_x =& \frac{1}{\sqrt{2\lambda(\lambda+m)}}\begin{bmatrix} -k_x - ik_y & \lambda + m \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2\lambda(\lambda+m)}} - \frac{(-k_x+ik_y)(4\lambda+2m)}{2(2\lambda(\lambda+m))^{\frac{3}{2}}}\frac{k_x}{\lambda} \\ -\frac{\frac{m}{2\lambda^2}}{2\sqrt{\frac{1}{2}+\frac{m}{2\lambda}}}\frac{k_x}{\lambda} \end{bmatrix} \\
=& \frac{1}{2\lambda(\lambda+m)}\begin{bmatrix} -k_x - ik_y & \lambda + m \end{bmatrix} \begin{bmatrix} -1 - \frac{(-k_x+ik_y)(2\lambda+m)}{2\lambda(\lambda+m)}\frac{k_x}{\lambda} \\ -\frac{mk_x}{2\lambda^2} \end{bmatrix} \\
=& \frac{1}{2\lambda(\lambda+m)}\left(k_x + ik_y - \frac{k^2(2\lambda+m)k_x}{2\lambda^2(\lambda+m)} - \frac{m(\lambda+m)k_x}{2\lambda^2}\right) \\
=& \frac{ik_y}{2\lambda(\lambda+m)} + \frac{k_x}{4\lambda^3(\lambda+m)^2}\left(2\lambda^2(\lambda+m) - k^2(2\lambda+m) - m(\lambda+m)^2\right) \\
=& \frac{ik_y}{2\lambda(\lambda+m)} + \frac{k_x}{4\lambda^3(\lambda+m)^2}\left(2\lambda^3 + 2\lambda^2 m - 2k^2\lambda - k^2 m - m(\lambda^2 + m^2 + 2\lambda m)\right) \\
=& \frac{ik_y}{2\lambda(\lambda+m)}.
\end{aligned}$$

By symmetry we have

$$\begin{aligned}
A_y =& \frac{1}{2\lambda(\lambda+m)}\begin{bmatrix} -k_x - ik_y & \lambda + m \end{bmatrix} \begin{bmatrix} i - \frac{(-k_x+ik_y)(2\lambda+m)}{2\lambda(\lambda+m)}\frac{k_y}{\lambda} \\ -\frac{mk_y}{2\lambda^2} \end{bmatrix} \\
=& -\frac{i}{2\lambda(\lambda+m)}\begin{bmatrix} -k_x - ik_y & \lambda + m \end{bmatrix} \begin{bmatrix} -1 - \frac{(-k_x+ik_y)(2\lambda+m)}{2\lambda(\lambda+m)}\frac{ik_y}{\lambda} \\ -\frac{mik_y}{2\lambda^2} \end{bmatrix} \\
=& -\frac{i}{2\lambda(\lambda+m)}\left(k_x + ik_y - \frac{k^2(2\lambda+m)ik_y}{2\lambda^2(\lambda+m)} - \frac{m(\lambda+m)ik_y}{2\lambda^2}\right) \\
=& -\frac{ik_x}{2\lambda(\lambda+m)}.
\end{aligned}$$

The Berry curvature is then

$$
\begin{aligned}
F_{xy} &= -\frac{i}{\lambda(\lambda+m)} + \frac{i(2\lambda+m)}{2\lambda^3(\lambda+m)^2}(k_x^2+k_y^2) \\
&= -\frac{i}{\lambda(\lambda+m)}\left(1-\frac{(2\lambda+m)}{2\lambda^2(\lambda+m)}(\lambda^2-m^2)\right) \\
&= -\frac{i}{2\lambda^3(\lambda+m)}\left(2\lambda^2-(\lambda-m)(2\lambda+m)\right) \\
&= -\frac{im}{2\lambda^3}.
\end{aligned}
$$

Approximating the integral over the first Brillouin zone to an integral over all momenta we have

$$
\begin{aligned}
\mathrm{Ch}_1 =& \frac{i}{2\pi}\int F = \frac{m}{4\pi}\int \mathrm{d}^2k\,\frac{1}{(k^2+m^2)^{\frac{3}{2}}} \\
=& \frac{m}{4\pi|m|}\frac{4\pi^2\Gamma\left(\frac{1}{2}\right)\Gamma(1)}{4\pi\Gamma\left(\frac{3}{2}\right)\Gamma(1)} \\
=& \frac{m}{2|m|}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \\
=& \frac{m}{2|m|}.
\end{aligned}
$$

Setting $k_y = 0$ instead nets us a model in class AIII. To define a topological invariant for this model we reparametrize it and switch form such that the Hamiltonian is

$$
\mathcal{H} = k_x\sigma_x - m\sigma_y.
$$

The eigenstates are

$$
\left|u^-(k)\right\rangle = \frac{1}{\sqrt{2}\lambda}\begin{bmatrix}-k_x-im\\ \lambda\end{bmatrix},\ \left|u^+(k)\right\rangle = \frac{1}{\sqrt{2}\lambda}\begin{bmatrix}k_x+im\\ \lambda\end{bmatrix}..
$$

The projection operator onto occupied states is therefore

$$
P = \frac{1}{2}\begin{bmatrix}1 & \frac{-k_x-im}{\lambda}\\ \frac{-k_x+im}{\lambda} & 1\end{bmatrix},
$$

from which we find

$$
Q = \frac{1}{\lambda}\begin{bmatrix}0 & k_x+im\\ k_x-im & 0\end{bmatrix}.
$$

The Hamiltonian commutes with σ_z, which is diagonal in the chosen basis, and thus we find

$$
q = \frac{k_x+im}{\lambda}.
$$

The winding number is given by the form

$$
\begin{aligned}
\omega_1 =& \frac{i}{2\pi}\,\mathrm{tr}\left(q^{-1}dq\right) \\
=& \frac{i}{2\pi}\frac{\lambda}{k_x+im}\left(\frac{1}{\lambda}-\frac{(k_x+im)k_x}{\lambda^3}\right)dk_x \\
=& \frac{i}{2\pi}\frac{k_x-im}{\lambda}\frac{\lambda^2-(k_x+im)k_x}{\lambda^3}dk_x \\
=& \frac{im}{2\pi}\frac{k_x-im}{\lambda}\frac{m-ik_x}{\lambda^3}dk_x \\
=& \frac{m}{2\pi}\frac{(k_x-im)(k_x+im)}{\lambda^4}dk_x \\
=& \frac{m}{2\pi\lambda^2}dk_x,
\end{aligned}
$$

and we have

$$\begin{aligned}
\nu_1 =& \int \omega_1 \\
=& \int_{-\infty}^{\infty} \mathrm{d}k_x \, \frac{m}{2\pi\lambda^2} \\
=& m \frac{\Gamma\left(\frac{1}{2}\right)^2}{2\sqrt{\pi}\Gamma\left(1\right)\Gamma\left(\frac{1}{2}\right)} (m^2)^{-\frac{1}{2}} \\
=& \frac{m}{2|m|}.
\end{aligned}$$

To compute the Chern-Simons invariant, we need the Berry connection for the occupied states. Reusing a previous calculation we find

$$A = -\frac{im}{2\lambda^2} dk_x.$$

The Chern-Simons form is then given by

$$Q_1 = \frac{i}{2\pi} \int_0^1 \mathrm{d}t \, \mathrm{tr}(A) = \frac{i}{2\pi} \, \mathrm{tr}(A),$$

and the Chern-Simons invariant is

$$\begin{aligned}
\mathrm{CS}_1 =& \int Q_1 \\
=& \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_x \, \frac{m}{2\lambda^2} \\
=& \frac{m}{4|m|}.
\end{aligned}$$

This is half the winding number, and this is in fact not a coincidence. The Wilson loop is then given by

$$W = e^{\pm i \frac{\pi}{2}}.$$

Summary of Articles The first article is about the Berry phase in gapped systems, studied through effective field theory. Promoting parameters of the theory to background fields nets new terms to the effective action, so-called Weiss-Zumino-Witten terms. Considering such terms yields constraints on the phase diagram of the theory, and in particular implies the existence of gapless points which are stable under (some kinds of) deformations. Such points are called diabolical points. For theories with finite degrees of freedom, the existence of such points is identified using the Berry curvature. For field theories one generalization of the Berry phase is the WZW terms appearing in an effective action.

The WZW terms can be related to the ordinary Berry phase by considering interfaces in the model and performing dimensional reduction along these. The result is that the integrals of the Berry curvature are the same for the full model and the effective model along the interface. By that argument these terms also distinguish themselves from the higher Berry curvature, which is mathematically distinct from the usual Berry curvature.

The second article is about dimensional reduction of topological insulators. It demonstrates how Hamiltonians that differ with respect to dimensionality and symmetry properties can be related via dimensional reduction.

4 Quantum Field Theory

Effective Actions To define the effective action, we first introduce

$$E[J] = -i \ln(Z),$$

where Z is the generating functional of some quantum filed theory. E is essentially a measure of the vacuum energy as a function of the source J. There is also a strong analogy to statistical mechanics at play, with Z playing the role of the partition function and E the role of the Helmholtz free energy. Its functional derivatives are given by

$$\frac{\delta E}{\delta J(x)} = -\frac{i}{Z}\frac{\delta Z}{\delta J(x)} = \frac{\int [\mathrm{D}\phi]\, \phi(x) e^{i\left(S+\int \mathrm{d}^d y J(y)\phi(y)\right)}}{\int [\mathrm{D}\phi]\, e^{i\left(S+\int \mathrm{d}^d y J(y)\phi(y)\right)}}.$$

In analogy with statistical mechanics, this can be considered a classical vacuum expectation value in the presence of a source, hence we term it $\phi_J(x)$. Its evaluation at $J = 0$ nets us the familiar correlation function. In analogy with statistical mechanics we can now perform a Legendre transform according to

$$\Gamma[\phi] = E[J_\phi] - \int \mathrm{d}^d x\, \phi(x) J_\phi(x).$$

As we are used to, the above defines J_ϕ as a functional satisfying

$$\frac{\delta\Gamma}{\delta\phi(x)} = -J_\phi(x).$$

The quantity Γ is the effective action. Note that E and Γ coincide for $J = 0$.

Let us first study properties of E. We have

$$\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)} E = -\, i \left(i^2 \langle \phi(x)\phi(y)\rangle - i^2 \langle \phi(x)\rangle \langle \phi(y)\rangle\right).$$

This demonstrates the explicit removal of disconnected Feynman diagrams in E. The general result is

$$\left(\prod_{i=1}^{n} \frac{\delta}{\delta J(x_i)}\right) E = -i^{n+1} \left\langle \prod_{i=1}^{n} \phi(x_i) \right\rangle_{\mathrm{conn}},$$

which will be useful for computing terms in the effective action.

One can also introduce a partial effective action, where only some fields are integrated out. This is the case that is important to us. It also turns out that the systematics of computing traces in this case are equivalent to connected Feynman diagrams. We will see more of this when studying concrete examples.

An ad hoc argument for why one can use Feynman diagrams as we do is the following: The effective action will typically involve some series in operators. Introducing a position space basis implies that one starts and ends at the same place. From there one can start to split up the operators with more and more identities added. Fourier transforming nets you diagrams with loops.

Theories With Topological Response We will now consider some field theories with background fields. The effective actions of these theories contain topological terms, which are metric-independent. The significance of this metric-independence is that correlation functions, and therefore the theory, is stable under deformations of spacetime or generalized coordinate transformations. It also implies that all correlations lengths are zero.

The scheme for constructing such theories was laid out by Abanov and Wiegmann. The idea is to introduce a tuple of slowly varying fields V. We also introduce the matrices $\Gamma_i^{(2k+1)}$, which are Hermitian Dirac matrices representing the Clifford algebra with $2k + 1$ generators. From these we construct the operators

$$V^{(l)} = \begin{cases} \sum\limits_{i=1}^{l} V_i \Gamma_i^{(l)}, & l = 2n+1, \\ V_l + i\gamma_5 \sum\limits_{i=1}^{l-1} V_i \Gamma_i^{(l-1)}, & l = 2n, \end{cases}$$

with the parametrization $V_l = \cos(\nu)$ for some constant ν. Abanov and Wiegmann confine the V to some unit sphere, which we will refrain from doing, and so we can simply set $V_l = 1$. γ_5 is $i^{\frac{d-1}{2}}$ times the product of all

Dirac matrices working on Dirac structure, distinguished from the Γ_i, which work in flavor space. Models with mass terms on S^d in $d+1$ dimensions are now given by

$$\mathcal{L} = -i\overline{\psi}(\partial\!\!\!/ + MV^{(d+1)})\psi,$$

and models with mass terms on S^{d+1} are given by

$$\mathcal{L} = -i\overline{\psi}(\partial\!\!\!/ + MV^{(d+3)})\psi.$$

These two classes will be termed A and B. We have also introduced an overall mass scale M. Note that the confinement to a manifold is performed by the restriction of V. The models we will study are particular examples of this construction, as well as some modified versions. The essential property of the Dirac matrices is

$$\mathrm{tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{d+1}}\gamma_5) = -i^{\frac{d-1}{2}}\,(-2)^{\frac{d+1}{2}}\,\varepsilon^{\mu_1\dots\mu_{d+1}}$$

in odd-dimensional space and

$$\mathrm{tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{d+1}}) = -i^{\frac{d}{2}-1}\,(-2)^{\frac{d}{2}}\,\varepsilon^{\mu_1\dots\mu_{d+1}}$$

in even-dimensional space.

Some Examples The first is the $1+1$-dimensional theory

$$\mathcal{L} = -i\overline{\psi}\left(\partial\!\!\!/ + M_1 + iM_2\gamma_5\right)\psi.$$

This is the class-A model in $d=1$. Noting that

$$(\gamma_5)^2 = 1 \implies e^{i\phi\gamma_5} = \cos(\phi) + i\sin(\phi)\gamma_5,$$

we can write

$$M_1 + iM_2\gamma_5 = M\left(\cos(\alpha) + i\sin(\alpha)\gamma_5\right) = Me^{i\alpha\gamma_5},$$

with M and α being the magnitude and argument of the complex number $M_1 + iM_2$ (note that hermiticity implies that both parameters be real). Adding the minor modification of coupling the fermion field to a gauge field, the full Lagrangian for this theory is

$$\mathcal{L} = -i\overline{\psi}\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

To compute the effective action, we integrate out the fermion and perform a perturbation expansion treating $D\!\!\!\!/ + Me^{i\alpha\gamma_5}$ as a perturbed version of $\partial\!\!\!/ + M$. The effective action for the gauge field is

$$\begin{aligned}\Gamma =& -i\ln\!\left(\det\!\left(-i\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\right)\right) + \int \mathrm{d}^2x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ =& -i\,\mathrm{tr}\!\left(\ln\!\left(-i\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\right)\right) + \int \mathrm{d}^2x - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}$$

The trace can be computed by summing over some basis with respect to both the field and Dirac structures. In particular, the new term is

$$\begin{aligned}-i\,\mathrm{tr}\!\left(\ln\!\left(-i\left(D\!\!\!\!/ + Me^{i\alpha\gamma_5}\right)\right)\right) \approx& -i\,\mathrm{tr}\!\left(\ln\!\left(-i\left(\partial\!\!\!/ + M\right)\left(1 + \frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\right)\right)\right)\\ =& C_0 - i\,\mathrm{tr}\!\left(\ln\!\left(1 + \frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\right)\right)\\ \approx& C_0 - i\,\mathrm{tr}\!\left(\frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M} - \frac{1}{2}\frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\frac{-ieA\!\!\!/ + iM\alpha\gamma_5}{\partial\!\!\!/ + M}\right).\end{aligned}$$

Let us first compute the inverse of the denominator. We have

$$(\not{\partial}+M)(\not{\partial}-M)=\not{\partial}^2-M^2=\partial^2-M^2 \implies \frac{1}{\not{\partial}+M}=\frac{\not{\partial}-M}{\partial^2-M^2}.$$

Computing the trace in momentum space, applying the correspondence principle $p=-i\partial$ and using the systematics of Feynman diagrams we find

$$\begin{aligned}
&-i\,\mathrm{tr}\bigg(-ie\frac{\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}-\frac{1}{2}\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\bigg)\\
=&-i\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\bigg(\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\bigg)+\frac{i}{2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_3}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\,\mathrm{tr}\bigg(\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\frac{-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\bigg)\\
=&-i\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\bigg(\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\bigg)\delta_{p_1+p_2}\\
&+\frac{i}{2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_3}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\,\mathrm{tr}\bigg(\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\frac{i\not{p}_3-M}{-p_3^2-M^2}(-ie\not{A}(p_4)+iM\alpha(p_4)\gamma_5)\bigg)\\
&\cdot\delta_{p_1+p_2-p_3}\delta_{-p_1+p_3+p_4}\\
=&-\frac{i}{(2\pi)^2}\int\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\bigg(\frac{-i\not{p}_2-M}{-p_2^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\bigg)\\
&+\frac{i}{2(2\pi)^2}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\\
&\cdot\mathrm{tr}\bigg(\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\frac{i(\not{p}_1+\not{p}_2)-M}{-(p_1+p_2)^2-M^2}(-ie\not{A}(p_4)+iM\alpha(p_4)\gamma_5)\bigg)\delta_{p_2+p_4}\\
=&-\frac{i}{(2\pi)^2}\int\frac{\mathrm{d}^2p_2}{(2\pi)^2}\,\mathrm{tr}\bigg(\frac{-i\not{p}_2-M}{-p_2^2-M^2}(-ie\not{A}(p_2)+iM\alpha(p_2)\gamma_5)\bigg)\\
&+\frac{i}{2(2\pi)^4}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_4}{(2\pi)^2}\cdot\mathrm{tr}\bigg((-ie\not{A}(p_4)+iM\alpha(p_4)\gamma_5)\frac{i\not{p}_1-M}{-p_1^2-M^2}(-ie\not{A}(-p_4)+iM\alpha(-p_4)\gamma_5)\frac{i(\not{p}_1-\not{p}_4)-M}{-(p_1-p_4)^2-M^2}\bigg).
\end{aligned}$$

The corresponding Feynman diagram is shown in figure 1.

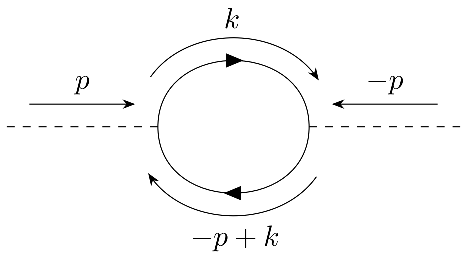

Figure 1: Feynman diagram for the second-order term in the effective action.

The second line produces the lowest-order topological terms. There we need only consider the case where the number of Dirac matrices is even, as the odd-numbered cases vanish when tracing the matrices. Removing some normalization factors and absorbing the coupling constant into the gauge field, these terms are

$$
\begin{aligned}
&\frac{i}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\operatorname{tr}\left((-i\not{A}(p)+iM\alpha(k)\gamma_5)\frac{i\not{k}-M}{-k^2-M^2}(-i\not{A}(-p)+iM\alpha(-p)\gamma_5)\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\right)\\
\supset&\frac{iM}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\operatorname{tr}\left(\not{A}(p)\frac{i\not{k}-M}{-k^2-M^2}\alpha(-p)\gamma_5\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\right)\\
&+\frac{iM}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\operatorname{tr}\left(\alpha(p)\gamma_5\frac{i\not{k}-M}{-k^2-M^2}\not{A}(-p)\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\right)\\
=&\frac{iM}{2}\left(\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\alpha(-p)A_\mu(p)\operatorname{tr}\left(\frac{i\not{k}-M}{-k^2-M^2}\gamma^\mu\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\gamma_5\right)\right.\\
&\left.+\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\alpha(p)A_\mu(-p)\operatorname{tr}\left(\frac{i\not{k}-M}{-k^2-M^2}\gamma_5\frac{i(-\not{p}+\not{k})-M}{-(-p+k)^2-M^2}\gamma^\mu\right)\right)\\
\supset&\frac{M^2}{2}\left(\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)A_\mu(p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}\left(\left(\not{k}\gamma^\mu+\gamma^\mu(-\not{p}+\not{k})\right)\gamma_5\right)\right.\\
&\left.+\frac{\alpha(p)A_\mu(-p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}\left(\left(\not{p}\gamma_5+\gamma_5(-\not{p}+\not{k})\right)\gamma^\mu\right)\right)\\
=&\frac{M^2}{2}\left(\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)A_\mu(p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}(k_\nu\gamma^\nu\gamma^\mu\gamma_5+(-p+k)_\nu\gamma^\mu\gamma^\nu\gamma_5)\right.\\
&\left.+\frac{\alpha(p)A_\mu(-p)}{(-k^2-M^2)(-(-p+k)^2-M^2)}\operatorname{tr}(k_\nu\gamma^\nu\gamma_5\gamma^\mu+(-p+k)_\nu\gamma_5\gamma^\nu\gamma^\mu)\right).
\end{aligned}
$$

Because we take the mass fields to be slowly varying, we can remove all contributions over order 1 in $\frac{p}{k}$. Next, because we are integrating over k explicitly and due to the Levi-Civita symbol, we may remove contributions proportional to k_μ to any order. Inverting the p-integral in the second term leaves us with

$$
\begin{aligned}
&\frac{M^2}{2}\left(\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)A_\mu(p)}{(-k^2-M^2)^2}\operatorname{tr}(k_\nu\gamma^\nu\gamma^\mu\gamma_5+(-p+k)_\nu\gamma^\mu\gamma^\nu\gamma_5+k_\nu\gamma^\nu\gamma_5\gamma^\mu+(p+k)_\nu\gamma_5\gamma^\nu\gamma^\mu)\right)\\
=&-M^2\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\alpha(-p)p_\nu A_\mu(p)}{(-p^2-M^2)^2}\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma_5)\\
=&-2M^2\int\frac{\mathrm{d}^2p}{(2\pi)^2}\varepsilon^{\mu\nu}\alpha(-p)p_\nu A_\mu(p)\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(k^2+M^2)^2}.
\end{aligned}
$$

Let us consider the innermost integral. Performing a Wick rotation and a substitution we have

$$
\begin{aligned}
\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{(k^2+M^2)^2}=&\frac{1}{M^2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{(-q_0^2+q_1^2+1)^2}\\
=&\frac{i}{M^2}\int\frac{\mathrm{d}^2\ell}{(2\pi)^2}\frac{1}{(\ell_0^2+\ell_1^2+1)^2}\\
=&\frac{i}{2\pi M^2}\int_0^\infty\mathrm{d}r\,\frac{r}{(r^2+1)^2}\\
=&-\frac{i}{2\pi M^2}\left.\frac{1}{2(r^2+1)}\right|_0^\infty\\
=&\frac{i}{4\pi M^2}.
\end{aligned}
$$

Let us also note the general result

$$\int \frac{\mathrm{d}^{d+1}k}{(2\pi)^d}\frac{(k^2)^a}{(k^2+\Delta)^b} = i\frac{\Gamma\left(b-a-\frac{d+1}{2}\right)\Gamma\left(a+\frac{d+1}{2}\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(b\right)\Gamma\left(\frac{d+1}{2}\right)}\Delta^{a+\frac{d+1}{2}-b}.$$

The final expression for the momentum space effective action is then

$$\Gamma = -\frac{i}{2\pi}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\varepsilon^{\mu\nu}\alpha(-p)p_\nu A_\mu(p),$$

and switching to real space we have

$$\Gamma = \frac{1}{2\pi}\int \mathrm{d}^2x\,\varepsilon^{\mu\nu}\alpha\partial_\mu A_\nu = \frac{1}{2\pi}\int \alpha\wedge F.$$

The α appearing here is of course only the prefactor for γ_5, and so we can infer the true structure of this term to be

$$\Gamma = \frac{1}{2\pi}\int \sin(\alpha)F.$$

At this point we can also note the existence of terms involving $1-\cos(\alpha)$, as the full effective action is

$$\Gamma = -\,i\,\mathrm{tr}\left(\ln\left(1+\frac{1-\cos(\alpha)-ie\not{A}+iM\alpha\gamma_5}{\not{\partial}+M}\right)\right).$$

As such, the above procedure can be extended to higher order. The result will be terms with arbitrarily high powers of $1-\cos(\alpha)$, as well as new numerical constants.

Another model to study in one dimension is

$$\mathcal{L} = -i\overline{\Psi}\left(\not{\partial}+M+\sum_{a=2,3,5}iM_a(x)\gamma^a\right)\Psi.$$

This is an example of a model from class B. We once again employ the perturbation approach to write the effective action as

$$\begin{aligned}\Gamma =& -\,i\,\mathrm{tr}\left(\ln\left(-i(\not{\partial}+M)\left(1+\frac{i\sum\limits_{a=2,3,5}M_a\gamma^a}{\not{\partial}+M}\right)\right)\right)\\ =& C_0-i\,\mathrm{tr}\left(\ln\left(1+\frac{i\sum\limits_{a=2,3,5}M_a\gamma^a}{\not{\partial}+M}\right)\right).\end{aligned}$$

Because we are working with all Dirac matrices, we note that the only shot at obtaining a topological term is to consider a term of (at least) order three in the expansion. The Feynman diagram is shown in figure 2.

This term is given by

$$\begin{aligned}\Gamma =& -\frac{i}{3}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{tr}\left(\frac{i\sum\limits_{a=2,3,5}M_a\gamma^a}{\not{\partial}+M}\frac{i\sum\limits_{b=2,3,5}M_b\gamma^b}{\not{\partial}+M}\frac{i\sum\limits_{c=2,3,5}M_c\gamma^c}{\not{\partial}+M}\right)\\ =& -\frac{1}{3}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\\ &\cdot\sum_{a,b,c=2,3,5}M_a(p_1)M_b(p_2)M_c(-p_1-p_2)\,\mathrm{tr}\left(\gamma^a\frac{i\not{k}-M}{-k^2-M^2}\gamma^b\frac{i(\not{p}_2+\not{k})-M}{-(p_2+k)^2-M^2}\gamma^c\frac{i(-\not{p}_1+\not{k})-M}{-(-p_1+k)^2-M^2}\right)\\ =& -\frac{1}{3}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\sum_{a,b,c=2,3,5}\frac{M_a(p_1)M_b(p_2)M_c(-p_1-p_2)}{(-k^2-M^2)(-(p_2+k)^2-M^2)(-(-p_1+k)^2-M^2)}\\ &\cdot\,\mathrm{tr}\left(\gamma^a(i\not{k}-M)\gamma^b(i(\not{p}_2+\not{k})-M)\gamma^c(i(-\not{p}_1+\not{k})-M)\right).\end{aligned}$$

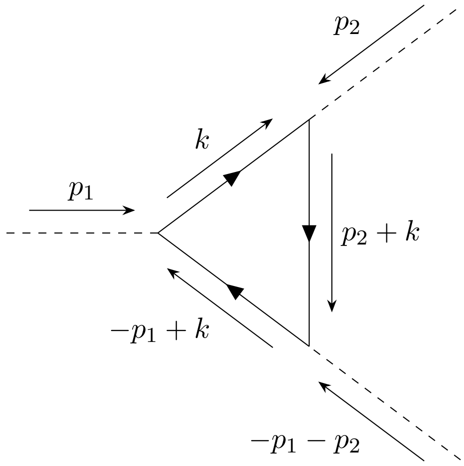

Figure 2: Feynman diagram for the third-order term in the effective action.

The topological term corresponds to exactly one of a, b, c being 5 and one of the numerators in the fraction being a mass. We immediately note that because of the antisymmetry inherent in the topological term and the fact that k is explicitly integrated over, this must be the first factor. We can also ignore terms of any order in k_μ. Setting $a = 5$ as an example nets you

$$\begin{aligned} M \operatorname{tr}\Big(\gamma^5\gamma^b(\not{p}_2 + \not{k})\gamma^c(-\not{p}_1 + \not{k})\Big) =& - M \operatorname{tr}\Big((\not{p}_2 + \not{k})(-\not{p}_1 + \not{k})\gamma^b\gamma^c\gamma^5\Big) \\ =& - 4iM(p_2 + k)_\mu(-p_1 + k)_\nu\varepsilon^{\mu\nu bc} \\ =& 4iMp_{2,\mu}p_{1,\nu}\varepsilon^{\mu\nu bc}. \end{aligned}$$

In the long-wavelength limit the effective action is

$$\begin{aligned} \Gamma \supset& -\frac{4iM}{3}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\varepsilon^{\mu\nu bc}\sum_{b,c=2,3}\frac{p_{2,\mu}p_{1,\nu}M_5(p_1)M_b(p_2)M_c(-p_1-p_2)}{(-k^2-M^2)^3} \\ =& \frac{4M}{3}\frac{\Gamma(2)\Gamma(1)}{4\pi\Gamma(3)\Gamma(1)}M^{-4}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\varepsilon^{\mu\nu bc}\sum_{b,c=2,3}p_{2,\mu}p_{1,\nu}M_5(p_1)M_b(p_2)M_c(-p_1-p_2) \\ =& \frac{1}{6\pi M^3}\int\frac{\mathrm{d}^2p_1}{(2\pi)^2}\frac{\mathrm{d}^2p_2}{(2\pi)^2}\varepsilon^{\mu\nu bc}\sum_{b,c=2,3}p_{2,\mu}p_{1,\nu}M_5(p_1)M_b(p_2)M_c(-p_1-p_2). \end{aligned}$$

Using the facts that the values of b, c are disjoint from those of μ, ν, we have that the real space term is

$$\Gamma \supset -\frac{1}{6\pi M^3}\sum_{b,c=2,3}\varepsilon^{\mu\nu}\varepsilon^{bc}\int \mathrm{d}^2x\, M_c\partial_\mu M_b\partial_\nu M_5.$$

To understand the systematics of the above calculations, let us adopt aliases for the momenta and write the integrand as

$$\begin{aligned} &\sum_{a,b,c=2,3,5}\frac{M_a(p_1)M_b(p_2)M_c(-p_1-p_2)}{(-k^2-M^2)(-(p_2+k)^2-M^2)(-(-p_1+k)^2-M^2)} \\ &\cdot \operatorname{tr}\Big(\gamma^a(i\not{k} - M)\gamma^b(i(\not{p}_2 + \not{k}) - M)\gamma^c(i(-\not{p}_1 + \not{k}) - M)\Big) \\ =& \sum_{a,b,c=2,3,5}\frac{M_a(p_1)M_b(p_2)M_c(p_3)}{(-k_1^2-M^2)(-k_2^2-M^2)(-k_3^2-M^2)}\operatorname{tr}\Big(\gamma^a(i\not{k}_1 - M)\gamma^b(i\not{k}_2 - M)\gamma^c(i\not{k}_3 - M)\Big). \end{aligned}$$

These vectors satisfy

$$\sum_{i=1}^{3}p_i = 0,\ k_i = k_1 + \sum_{j>1}^{i}p_j.$$

Because k_1 is explicitly integrated over and we anticipate the appearance of the Levi-Civita symbol, we can make the identification

$$k_{i,\mu}k_{j,\nu} \to \sum_{\alpha>1}^{i} \sum_{\beta>1,\beta\neq\alpha}^{j} p_{\alpha,\mu}p_{\beta,\nu}.$$

This implies that including the momentum from the first factor adds a zero, hence in addition to fixing exactly one latin index to 5 we may also take the first factor to be the one contributing the factor M. Thus the above amounts to

$$\begin{aligned}
&Mk_{2,\mu}k_{3,\nu} \sum_{a,b,c=2,3,5} \frac{M_a(p_1)M_b(p_2)M_c(p_3)}{(-k_1^2 - M^2)(-k_2^2 - M^2)(-k_3^2 - M^2)} \operatorname{tr}\Big(\gamma^a\gamma^b\gamma^\mu\gamma^c\gamma^\nu\Big) \\
&= -Mp_{2,\mu}p_{3,\nu} \sum_{a,b,c=2,3,5} \frac{M_a(p_1)M_b(p_2)M_c(p_3)}{(-k_1^2 - M^2)(-k_2^2 - M^2)(-k_3^2 - M^2)} \operatorname{tr}\Big(\gamma^\mu\gamma^\nu\gamma^a\gamma^b\gamma^c\Big) \\
&= -Mp_{2,\mu}p_{3,\nu} \sum_{a,b=2,3} \varepsilon^{\mu\nu ab} \frac{M_5(p_1)M_a(p_2)M_b(p_3) - M_a(p_1)M_5(p_2)M_b(p_3) + M_a(p_1)M_b(p_2)M_5(p_3)}{(-k_1^2 - M^2)(-k_2^2 - M^2)(-k_3^2 - M^2)} \\
&= -Mp_{2,\mu}p_{3,\nu}\varepsilon^{\mu\nu} \sum_{a,b=2,3,5} \varepsilon^{abc} \frac{M_a(p_1)M_b(p_2)M_c(p_3)}{(-k_1^2 - M^2)(-k_2^2 - M^2)(-k_3^2 - M^2)}.
\end{aligned}$$

The full topological term is then

$$\Gamma = -\frac{1}{6\pi M^3} \sum_{a,b,c=2,3,5} \int \mathrm{d}^2x\, \varepsilon^{\mu\nu}\varepsilon^{abc} M_a \partial_\mu M_b \partial_\nu M_c.$$

We can take this one step further by taking the mass fields to be a map from spacetime to some manifold without the mass scale and introducing the 2-form ω with components $\omega_{ab} = \varepsilon_{cab}m^a$ to write the above as

$$\Gamma = -\frac{1}{6\pi} \int m^\star \omega.$$

At this point we can make some notes about the topological terms for all models in class B. In any topological term, all mass fields appear exactly once, as do momenta with each index. The Feynman diagrams also have identical shape, as we will see later. This will produce the pullback of a $d+1$-form, the components of which are linear in the fields, in the effective action.

What if we were to add coupling to a gauge field too? The effective action would be

$$\Gamma = -i\operatorname{tr}\left(\ln\left(1 + \frac{-i\not{A} + i\sum\limits_{a=2,3,5} M_a\gamma^a}{\not{\partial} + M}\right)\right).$$

To order 3, the structure of the Feynman diagram is identical, hence the trace part of the effective action is

$$-i\operatorname{tr}\big((i\not{k}_1 - M)(e\not{A} + M_a\gamma^a)(i(-\not{k}_1 - \not{p}) - M)(e\not{A} + M_b\gamma^a)(i(\not{k}_2 - \not{k}_1) - M)(e\not{A} + M_c\gamma^c)\big).$$

Because all of the latin-indiced Dirac matrices are needed, however, the gauge field does not appear in any topological terms. To get a topological term with a gauge field, we can instead consider a fourth-order term, represented by the Feynman diagram in figure 3.

At this point we can deduce what will happen. The replacement of a momentum with a gauge field in the trace (which is needed to produce a topological term) while leaving the same number of mass fields causes the integrand to be symmetric in the mass indices and makes the topological term vanish.

This argument also implies the necessary conditions for the gauge fields to appear in response terms. We expect all the fields corresponding to anticommuting mass terms to appear exactly once, as well as all the derivatives. Removing a derivative from a mass field nets you the possibility of creating a symmetric expression in mass indices. In order for the gauge fields to appear in topological terms, the spacetime dimension must therefore be equal to the number of mass fields. If it is greater, the spacetime Levi-Civita will cancel the derivative product. If it is lesser, the mass index Levi-Civita will cancel the now symmetric product of mass fields.

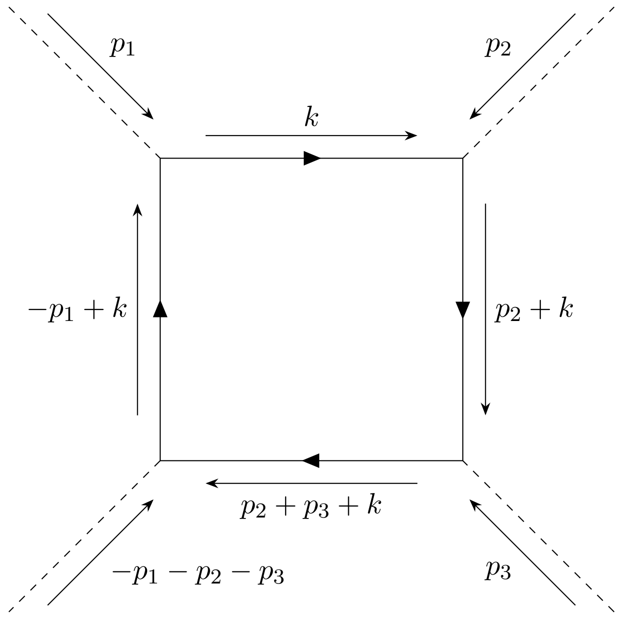

Figure 3: Feynman diagram for the fourth-order term in the effective action.

A counterpoint to the above is that this model does not explicitly apply transformations in isospace, forcing out the Levi-Civita symbol, but that this could be avoided by sticking more rigorously to Abanov and Wiegmann's method. The fact that the Dirac matrices square to the identity then implies that there would be some topological terms that are pullbacks of symmetric tensors on the mass manifold. Of course, these all vanish because the topological terms necessarily contain a Levi-Civita symbol on spacetime, and as such only fully antisymmetric tensors on the mass manifold produce topological terms. This legitimizes the above approach.

Let us also consider two models in two dimensions. The first is

$$\mathcal{L} = -i\overline{\psi}\left(\not{D} + M + \sum_{a=1}^{3} M^a \Gamma_a^{(3)}\right)\psi.$$

The effective action is

$$\Gamma = -i\,\mathrm{tr}\left(\ln\left(1 + \frac{-i\not{A} + \sum_{a=1}^{3} iM^a\Gamma_a^{(3)}}{\not{\partial} + M}\right)\right).$$

There are two topological terms here, corresponding to figures 2 and 3. Concentrating on the latter, which contains the gauge field, it is given by

$$\begin{aligned}
\Gamma =& \frac{i}{4}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^3}\frac{\mathrm{d}^2 p_2}{(2\pi)^3}\frac{\mathrm{d}^2 p_3}{(2\pi)^3}\frac{\mathrm{d}^2 k}{(2\pi)^3}\,\mathrm{tr}\left(\frac{-\not{A} + \sum_{a=1}^{3} M^a\Gamma_a^{(3)}}{\not{\partial} + M}\frac{-\not{A} + \sum_{b=1}^{3} M^b\Gamma_b^{(3)}}{\not{\partial} + M}\frac{-\not{A} + \sum_{c=1}^{3} M^c\Gamma_c^{(3)}}{\not{\partial} + M}\frac{-\not{A} + \sum_{d=1}^{3} M^d\Gamma_d^{(3)}}{\not{\partial} + M}\right)\\
=& \frac{i}{4}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^3}\frac{\mathrm{d}^2 p_2}{(2\pi)^3}\frac{\mathrm{d}^2 p_3}{(2\pi)^3}\frac{\mathrm{d}^2 k}{(2\pi)^3}\,\mathrm{tr}\left(\left(-\not{A}(p_1) + \sum_{a=1}^{3} M^a(p_1)\Gamma_a^{(3)}\right)\frac{i\not{k} - M}{-k^2 - M^2}\right.\\
&\cdot\left(-\not{A}(p_2) + \sum_{b=1}^{3} M^b(p_2)\Gamma_b^{(3)}\right)\frac{i(\not{p}_2 + \not{k}) - M}{-(p_2 + k)^2 - M^2}\\
&\cdot\left(-\not{A}(p_3) + \sum_{b=1}^{3} M^b(p_3)\Gamma_b^{(3)}\right)\frac{i(\not{p}_2 + \not{p}_3 + \not{k}) - M}{-(p_2 + p_3 + k)^2 - M^2}\\
&\cdot\left.\left(-\not{A}(-p_1 - p_2 - p_3) + \sum_{c=1}^{3} M^d(-p_1 - p_2 - p_3)\Gamma_d^{(3)}\right)\frac{i(-\not{p}_1 + \not{k}) - M}{-(-p_1 + k)^2 - M^2}\right).
\end{aligned}$$

Let us consider the trace part first. The relevant terms have exactly one gauge field appearing and two momenta. Adopting a shorthand, the first term of the trace can be written as

$$
\begin{aligned}
&-\sum_{b,c,d=1}^{3} M^b(p_2)M^c(p_3)M^d(p_4)\,\mathrm{tr}\Big(\not{A}(p_1)(i\not{k}_1-M)\Gamma_b^{(3)}(i\not{k}_2-M)\Gamma_c^{(3)}(i\not{k}_3-M)\Gamma_d^{(3)}(i\not{k}_4-M)\Big)\\
&=-\sum_{b,c,d=1}^{3} M^b(p_2)M^c(p_3)M^d(p_4)\,\mathrm{tr}(A_\mu(p_1)\gamma^\mu(i\not{k}_1-M)(i\not{k}_2-M)(i\not{k}_3-M)(i\not{k}_4-M))\,\mathrm{tr}\Big(\Gamma_b^{(3)}\Gamma_c^{(3)}\Gamma_d^{(3)}\Big)\\
&=-M\sum_{a,b,c=1}^{3}\varepsilon_{abc}M^a(p_2)M^b(p_3)M^c(p_4)\,\mathrm{tr}(A_\mu(p_1)\gamma^\mu(i\not{k}_2-M)(i\not{k}_3-M)(i\not{k}_4-M)).
\end{aligned}
$$

This is the point at which we can still generalize. At a glance it might seem like we could do it all in one go by fixing the position of the gauge field and cyclically permuting the momenta, but this is not the case due to the loop having been assigned a set of momenta. Using the previously established properties of the shorthand, we write the trace as

$$
\begin{aligned}
\mathrm{tr}(\gamma^\mu(i\not{k}_2-M)(i\not{k}_3-M)(i\not{k}_4-M)) =&M\,\mathrm{tr}(\gamma^\mu\not{k}_2\not{k}_3+\gamma^\mu\not{k}_2\not{k}_4+\gamma^\mu\not{k}_3\not{k}_4)\\
=&2M\varepsilon^{\mu\nu\rho}\left(k_{2,\nu}k_{3,\rho}+k_{2,\nu}k_{4,\rho}+k_{3,\nu}k_{4,\rho}\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(p_{2,\nu}p_{3,\rho}+p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}(p_{2,\rho}+p_{4,\rho})\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(2p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}p_{4,\rho}\right).
\end{aligned}
$$

The next is identical to the previous due to the k_1 not contributing. Third is

$$
\begin{aligned}
\mathrm{tr}((i\not{k}_2-M)\gamma^\mu(i\not{k}_3-M)(i\not{k}_4-M)) =&M\,\mathrm{tr}(\not{k}_2\gamma^\mu\not{k}_3+\gamma^\mu\not{k}_3\not{k}_4+\not{k}_2\gamma^\mu\not{k}_4)\\
=&2M\varepsilon^{\mu\nu\rho}\,\mathrm{tr}(-k_{2,\nu}k_{3,\rho}+k_{3,\nu}k_{4,\rho}-k_{2,\nu}k_{4,\rho})\\
=&2M\varepsilon^{\mu\nu\rho}\,\mathrm{tr}(-p_{2,\nu}p_{3,\rho}+p_{2,\nu}(p_{3,\rho}+p_{4,\rho})+p_{3,\nu}(p_{2,\rho}+p_{4,\rho})-p_{2,\nu}(p_{3,\rho}+p_{4,\rho}))\\
=&2M\varepsilon^{\mu\nu\rho}\,\mathrm{tr}(p_{3,\nu}p_{4,\rho}-2p_{2,\nu}p_{3,\rho}).
\end{aligned}
$$

Finally there is

$$
\begin{aligned}
\mathrm{tr}((i\not{k}_2-M)(i\not{k}_3-M)\gamma^\mu(i\not{k}_4-M)) =&M\,\mathrm{tr}(\not{k}_2\not{k}_3\gamma^\mu+\not{k}_2\gamma^\mu\not{k}_4+\not{k}_3\gamma^\mu\not{k}_4)\\
=&2M\varepsilon^{\mu\nu\rho}\left(k_{2,\nu}k_{3,\rho}-k_{2,\nu}k_{4,\rho}-k_{3,\nu}k_{4,\rho}\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(p_{2,\nu}p_{3,\rho}-p_{2,\nu}(p_{3,\rho}+p_{4,\rho})-p_{2,\nu}(p_{3,\rho}+p_{4,\rho})-p_{3,\nu}(p_{2,\rho}+p_{4,\rho})\right)\\
=&2M\varepsilon^{\mu\nu\rho}\left(-2p_{2,\nu}p_{4,\rho}-p_{3,\nu}p_{2,\rho}-p_{3,\nu}p_{4,\rho}\right).
\end{aligned}
$$

Adding them all together we find

A New Model The models studied above have connected d-dimensional systems to response terms described by integrals of up to $d+1$-forms. To relate the study of these theories to the higher Berry curvature, we will instead need to find a response term described by the integral of a $d+2$-form. Let us consider spatial dimension 1 and introduce mass terms according to

$$M=M_0+i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l.$$

The matrices Γ represent a Clifford algebra and act in flavor space, and we take the mass fields to lie on S^5. The Lagrangian of the model is

$$\mathcal{L}=-i\overline{\Psi}\left(\not{\partial}+M\right)\Psi=-i\overline{\Psi}\left(\not{\partial}+M_0+i\gamma^{01}\sum_{l=1}^{5}M_l\Gamma^l\right)\Psi.$$

The idea underlying this model follows Abanov and Wiegmann. They construct models with mass fields confined to S^d or S^{d+1} and show that the topological response terms are related to (the pullbacks of) d-forms and $d+1$-forms respectively. A first obvious attempt in one dimension is therefore to use mass fields on S^3 (and the simple way to do this just so happens to be Abanov and Wiegmann's A-series model in $d=3$), but we saw that this only produced a topological term given by a 2-form. This model attempts to fix this by extending the mass fields in a way such that, had you done it in $d = 3$, it would take the response term from being given by a 3-form to a 4-form. The hope is that it will achieve a similar result.

At this point it is pertinent to ask whether this attempt really stood any chance. The answer is no, and for a very simple reason. Looking at the above, the appearance of the pullback was no coincidence; it arrived precisely because of the form of the effective action. By its definition the pullback does not affect the rank of any tensor. As the effective action is given by an integral over spacetime, it follows that any form appearing in it must exist on spacetime, and the highest form in $d+1$-dimensional spacetime is a $d+1$-form. Note that this argument has no reliance on the structure of the mass fields. Nevertheless, we show the attempt below.

The effective action is

$$\begin{aligned}\Gamma =& -i\ln\left(\det\left(-i\left(\partial\!\!\!/ + M_0 + i\gamma^{01}\sum_{l=1}^{5} M_l\Gamma^l\right)\right)\right)\\ =& -i\,\mathrm{tr}\left(\ln\left(-i\left(\partial\!\!\!/ + M_0 + i\gamma^{01}\sum_{l=1}^{5} M_l\Gamma^l\right)\right)\right)\\ =& C_0 - i\,\mathrm{tr}\left(\ln\left(1 + \frac{m + i\gamma^{01}\sum_{l=1}^{5} M_l\Gamma^l}{\partial\!\!\!/ + M}\right)\right)\\ =& C_0 - i\,\mathrm{tr}\left(\ln\left(1 + \frac{(\partial\!\!\!/ - M)\left(m + i\gamma^{01}\sum_{l=1}^{5} M_l\Gamma^l\right)}{\partial^2 - M^2}\right)\right),\end{aligned}$$

where we have introduced the mass perturbation $m = M_0 - M$, with M being a fixed mass scale parameter. The topological term comes from the fifth-order expansion, with the Feynman diagram shown in figure 4.

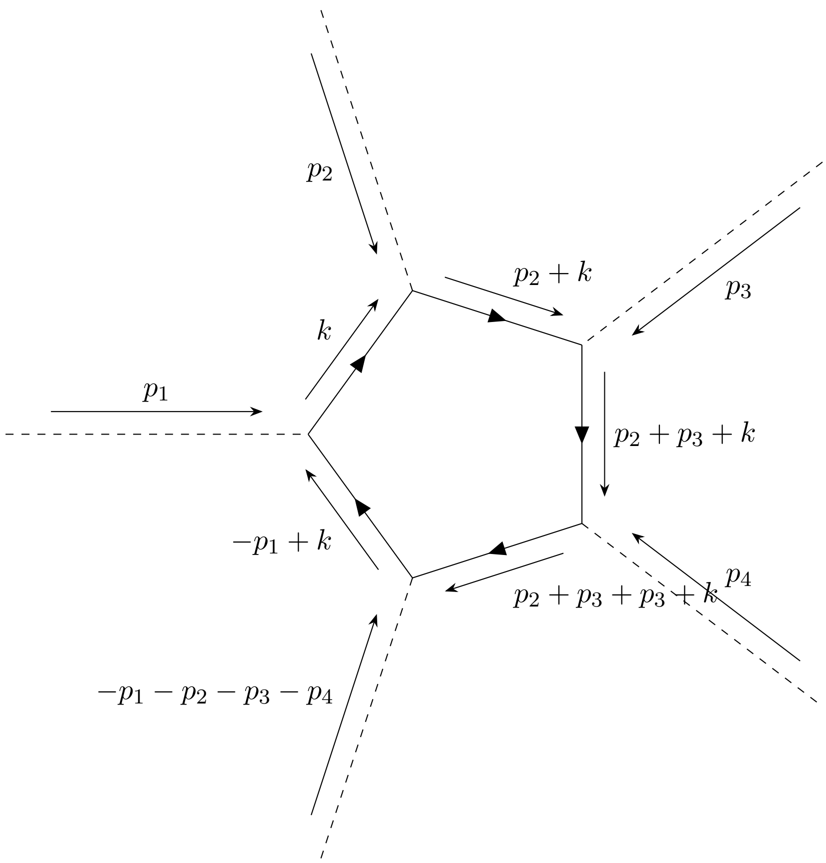

Figure 4: Feynman diagram for the third-order term in the effective action.

This translates to

$$\Gamma = -\frac{i}{5}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2 k}{(2\pi)^2}\,\mathrm{tr}\left(\left(\frac{(\not{\partial} - M)\left(m + i\gamma^{01}\sum_{l=1}^{5} M_l\Gamma^l\right)}{\partial^2 - M^2}\right)^5\right)$$

$$\supset \frac{1}{5}\sum_{l_i=1}^{5}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{tr}\left(\frac{(i\not{k} - M)\gamma^{01}M_{l_1}\Gamma^{l_1}}{-k^2 - M^2}\frac{(i(\not{p}_2 + \not{k}) - M)\gamma^{01}M_{l_2}\Gamma^{l_2}}{-(p_2+k)^2 - M^2}\frac{(i(\not{p}_2 + \not{p}_3 + \not{k}) - M)\gamma^{01}M_{l_3}\Gamma^{l_3}}{-(p_2+p_3+k)^2 - M^2}\right.$$
$$\left.\cdot\frac{(i(\not{p}_2 + \not{p}_3 + \not{p}_4 + \not{k}) - M)\gamma^{01}M_{l_4}\Gamma^{l_4}}{-(p_2+p_3+p_4+k)^2 - M^2}\frac{(i(-\not{p}_1 + \not{k}) - M)\gamma^{01}M_{l_5}\Gamma^{l_5}}{(-p_1+k)^2 - M^2}\right).$$

Let us now consider the contents of the trace. We are looking for topological terms, which appear in the presence of all Γ and all γ appearing exactly once. The matrices will produce a Levi-Civita tensor, meaning any contributions of orders 1 or 2 in k will vanish. As such the topological term is given by

$$\Gamma \supset -\frac{M}{5}\sum_{l_i=1}^{5}\int \frac{\mathrm{d}^2 p_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2 k}{(2\pi)^2}\mathrm{tr}\left(\frac{M_{l_1}\Gamma^{l_1}}{-k^2 - M^2}\frac{(i(\not{p}_2 + \not{k}) - M)\gamma^{01}M_{l_2}\Gamma^{l_2}}{-(p_2+k)^2 - M^2}\frac{(i(\not{p}_2 + \not{p}_3 + \not{k}) - M)\gamma^{01}M_{l_3}\Gamma^{l_3}}{-(p_2+p_3+k)^2 - M^2}\right.$$
$$\left.\cdot\frac{(i(\not{p}_2 + \not{p}_3 + \not{p}_4 + \not{k}) - M)\gamma^{01}M_{l_4}\Gamma^{l_4}}{-(p_2+p_3+p_4+k)^2 - M^2}\frac{(i(-\not{p}_1 + \not{k}) - M)M_{l_5}\Gamma^{l_5}}{(-p_1+k)^2 - M^2}\right).$$

The contents of the trace are

$$\mathrm{tr}\Big(\Gamma^{l_1}(i(\not{p}_2 + \not{k}) - M)\gamma^{01}\Gamma^{l_2}(i(\not{p}_2 + \not{p}_3 + \not{k}) - M)\gamma^{01}\Gamma^{l_3}(i(\not{p}_2 + \not{p}_3 + \not{p}_4 + \not{k}) - M)\gamma^{01}\Gamma^{l_4}(i(-\not{p}_1 + \not{k}) - M)\Gamma^{l_5}\Big)$$
$$= \mathrm{tr}\Big(\Gamma^{l_1}\Gamma^{l_2}\Gamma^{l_3}\Gamma^{l_4}\Gamma^{l_5}\Big)\,\mathrm{tr}\Big((i(\not{p}_2 + \not{k}) - M)\gamma^{01}(i(\not{p}_2 + \not{p}_3 + \not{k}) - M)\gamma^{01}(i(\not{p}_2 + \not{p}_3 + \not{p}_4 + \not{k}) - M)\gamma^{01}(i(-\not{p}_1 + \not{k}) - M)\Big),$$

exploiting the product structure of the operators. The case where all Dirac matrices appear exactly once correspond to exactly two momenta appearing, meaning this topological term too will have a pullback of a 2-form onto spacetime.

Extending to Synthetic Dimensions Thus far we have seen that the effective actions in class B involve the pullback of a $d+1$-form. To relate this to the higher Berry curvature, we can imagine the following: If we can write spacetime as the boundary of some other manifold Y and extend the mass fields to Y, then Stokes' theorem allows us to write

$$\Gamma = \int m^\star\omega = \int_Y d(m^\star\omega) = \int_Y m^\star(d\omega).$$

The form $d\omega$ is then a $d+2$-form which plays the role of the higher Berry curvature.

A first question is whether $d\omega$ exists. Certainly the containment of mass fields in class B to S^{d+1} implies that the answer is no. There are, however, two possible ways to solve that. The first, as proposed by Abanov and Wiegmann, is to simply not confine the mass fields at all. The alternative, which is what is done by Hsin et al, is to extend the usual mass term to a field. The consequence of this choice is that the response terms we have considered are lowest-order terms in the mass perturbation $M_0(x) - M$.