

Summary of SH2372 General Relativity

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Abstract

This is a summary of SH2372 General Relativity.

The course opens with a discussion of differential geometry. As I have extensive notes on the subject in my summary of SI2360, I only keep the bare minimum in this summary and refer to those notes for details.

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1 Differential Geometry

For details on much of this, notably the early parts on Euclidean space, please consult my summary of SI2360 Analytical Mechanics and Classical Field Theory.

Euclidean and Affine Spaces A Euclidean space is a set of points such that there to each point can be assigned a position vector. To such spaces we may assign a set of n coordinates χ^a which together uniquely describe each point in the space locally.

Tangent and Dual Bases The tangent and dual bases are defined by

$$\mathbf{E}_a = \partial_{\chi^a} \mathbf{r} = \partial_a \mathbf{r}, \quad \mathbf{E}^a = \vec{\nabla} \chi^a.$$

Using such bases, we may write

$$\mathbf{v} = v^a \mathbf{E}_a = v_a \mathbf{E}^a.$$

The components of these vectors are called contravariant and covariant components respectively.

Christoffel Symbols When computing the derivative of a vector quantity, one must account both for the change in the quantity itself and the change in the basis vectors. We define the Christoffel symbols according to

$$\partial_b \mathbf{E}_a = \Gamma_{ba}^c \mathbf{E}_c.$$

These can be computed according to

$$\mathbf{E}^c \cdot \partial_b \mathbf{E}_a = \mathbf{E}^c \cdot \Gamma_{ba}^d \mathbf{E}_d = \delta_d^c \Gamma_{ba}^d = \Gamma_{ba}^c.$$

Note that

$$\partial_a \mathbf{E}_b = \partial_a \partial_b \mathbf{r} = \partial_b \partial_a \mathbf{r} = \partial_b \mathbf{E}_a,$$

which implies

$$\Gamma_{ba}^c = \Gamma_{ab}^c.$$

Similarly, we might want to consider $\partial_b \mathbf{E}^a$, which might introduce new symbols. We find, however, that

$$\partial_a \mathbf{E}^b \cdot \mathbf{E}_c = \mathbf{E}^b \cdot \partial_a \mathbf{E}_c + \mathbf{E}_c \cdot \partial_a \mathbf{E}^b = 0,$$

which implies

$$\partial_a \mathbf{E}^b = -\Gamma_{ac}^b \mathbf{E}^c.$$

Covariant Derivatives Covariant derivatives are defined by

$$\vec{\nabla}_a v^b = \partial_a v^b + \Gamma_{ac}^b v^c,$$

and thus satisfy

$$\partial_a \mathbf{v} = \mathbf{E}_b \vec{\nabla}_a v^b.$$

Tensors To define tensors, we first define tensors of the kind $(0, n)$ as maps from n vectors to scalars. Using this, we define tensors of the kind (n, m) as linear maps from $(0, n)$ tensors to $(0, m)$ tensors.

Manifolds Manifolds are sets which are locally isomorphic to an open subset of \mathbb{R}^n .

Tangent and Dual Bases The tangent basis for a manifold is $\mathbf{E}_a = \partial_a$. The corresponding dual basis, denoted $d\chi^a$, is defined such that $df(X) = X^a \partial_a f$.

Tensors A general (n, m) tensor is constructed by taking the tensor product of tangent and dual basis vectors.