Summary of SI2380 Advanced Quantum Mechanics

Yashar Honarmandi
yasharh@kth.se

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Abstract

This is a summary of SI2380 Advanced Quantum Mechanics.

Contents

1 Useful Mathematics

A Useful Commutation Relation You might happen upon commutation relations of the form $[f(A), B]$ show up. We would like to try to simplify it for the particular case where $[A, B] = C$, where C is some operator commuting with A. To do this, we first study $[A^n, B]$ in a general case. We have

$$[A^n, B] = A\big[A^{n-1}, B\big] + [A, B]A^{n-1},$$

prompting us to find this commutator by induction. For $n = 2$ we have

$$\big[A^2, B\big] = A[A, B] + [A, B]A.$$

For $n = 3$ we obtain

$$\big[A^3, B\big] = A\big[A^2, B\big] + [A, B]A^2 = A(A[A, B] + [A, B]A) + [A, B]A^2 = A^2[A, B] + A[A, B]A + [A, B]A^2.$$

A suitable induction hypothesis looks to be

$$[A^n, B] = \sum_{k=1}^{n} A^{n-k}[A, B]A^{k-1}.$$

Assuming it to be true, we have

$$\begin{aligned}
\big[A^{n+1}, B\big] &= A[A^n, B] + [A, B]A^n \\
&= A\sum_{k=1}^{n} A^{n-k}[A, B]A^{k-1} + [A, B]A^n \\
&= \sum_{k=1}^{n} A^{n+1-k}[A, B]A^{k-1} + [A, B]A^n \\
&= \sum_{k=1}^{n+1} A^{n+1-k}[A, B]A^{k-1},
\end{aligned}$$

proving the hypothesis by induction.

Now we write

$$f(A) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n A^n$$

to obtain

$$\begin{aligned}
[f(A), B] &= \left[\sum_{n=0}^{\infty} \frac{1}{n!} f_n A^n, B\right] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f_n [A^n, B] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f_n \sum_{k=1}^{n} A^{n-k}[A, B]A^{k-1}.
\end{aligned}$$

Assuming $[A, B]$ to commute with A, we have

$$\begin{aligned}
[f(A), B] &= \sum_{n=0}^{\infty} \frac{1}{n!} f_n A^{n-1}[A, B] \sum_{k=1}^{n} \\
&= [A, B] \sum_{n=1}^{\infty} \frac{1}{(n-1)!} f_n A^{n-1} \\
&= [A, B] \sum_{m=0}^{\infty} \frac{1}{m!} f_{m+1} A^m \\
&= [A, B] \frac{\mathrm{d}f}{\mathrm{d}A}.
\end{aligned}$$

Symmetries on Hilbert Space A symmetry on Hilbert space is a transformation that leaves all expectation values unaltered.

Wigner's Theorem Wigner's theorem states that any operator that is a symmetry is either unitary or anti-unitary (the latter meaning $\langle\Phi|U^\dagger U|\Psi\rangle = \langle\Psi|\Phi\rangle$).

Transformation of Operators Consider a symmetry operator u. In order for this to be a symmetry, it must also act on all operators according to $A \to uAu^\dagger$.

Time Evolution From Symmetry Consider some system with time translation symmetry - that is, any system for which time translations do not change the theory. Introduce the transformation operator

$$u_\tau |\Psi(t)\rangle = |\Psi(t+\tau)\rangle .$$

This transformation is a smooth map acting on a manifold - namely, Hilbert space. Hence we can use the language of Lie algebra to treat this (if you know nothing about Lie algebra, pretend that I didn't write this and carry on. If you want some reference material, please look at my summary of SI2360). We expand the transformation operator around the identity as

$$u_\tau = 1 - i\frac{\tau}{\hbar}H$$

for some operator H. The requirement that this be unitary yields $H^\dagger - H = 0$, and hence the generator H is self-adjoint. By continuous application of this we obtain

$$u_\tau = e^{-i\frac{\tau}{\hbar}H}.$$

This reproduces the Schrödinger equation, tying it all together neatly. It also demonstrates that the Hamiltonian generates time translation in a mathematical sense.

Tensors A tensor of rank n is a multilinear map from n vectors in some vector space to a scalar.

It is clear that the set of tensors of some rank form a vector space, and so we would like to identify some basis for the space of tensors.

Basis for $n = 1$ We start with rank 1 tensors. The inner product is certainly a rank 1 tensor according to the definition, and so we would like to use that. Now let the set of v_i denote the set of orthonormal basis vectors for the space V. We then choose the basis

$$e_i(v) = \langle v_i|\, v$$

as the basis for the set of rank 1 tensors. This may also be denoted simply as the tensor v_i (the confusion will disappear later).

The Tensor Product To find a basis for tensors of higher order, we first need to introduce the tensor product. We define it for rank 1 tensors as

$$s(v) \otimes t(w) = \langle s|v\rangle \langle t|w\rangle .$$

The tensor product has allowed us to construct a rank 2 tensor from two rank 1 tensors. Repeatedly applying it allows us to construct tensors of any rank. The tensor product is also bilinear, in line with our definition.

The Tensor Product of Operators It follows naturally that

$$(S \otimes T)(v \otimes w) = S(v) \otimes T(w).$$

2 Basic Concepts

Observables An observable is a Hermitian operator whose orthonormal eigenvectors form a basis.

The Postulates of Quantum Mechanics The postulates of quantum mechanics are:

- At any fixed time the state of a physical system is specified by a ket in Hilbert space.
- Every measurable physical quantity corresponds to an operator on Hilbert space. This is a Hermitian observable. The possible outcomes of a measurement are the eigenvalues of A.
- The probability of measuring the value a of operator A in a normalized state $|\Psi\rangle$ is $P(a) = \langle\Psi|P_a|\Psi\rangle$, where P_a is the projector onto the subspace corresponding to the eigenvalue a given by $P_a = |a\rangle\langle a|$.
- If a measurement of an observable A gives an outcome a, the state of the system immediately after the measurement is the projection of the state onto the subspace with eigenvalue a.
- The time evolution of a state is governed by the Schrödinger equation.

Consequences of the Probability Picture The form of writing the projection operator implies $P(a) = |\langle a|\Psi\rangle|^2$, or $P(a)\,\mathrm{d}a = |\langle a|\Psi\rangle|^2\,\mathrm{d}a$ in the continuous case. In order for the probability interpretation to be consistent, i.e. for the sum of all probabilities to amount to 1, it must hold that $\langle\Psi|\Psi\rangle = 1$.

Expectation Values Expectation values are given by

$$\langle A\rangle = \sum aP(a) = \sum a\,\langle\Psi|P_a|\Psi\rangle = \langle\Psi|\sum a\,|a\rangle\,\langle a|\Psi\rangle = \langle\Psi|A|\Psi\rangle\,.$$

Physical States Modifying a state by a phase factor $e^{i\alpha}$ does not change any expectation values.

Pure and Mixed States Pure states are states with a well-defined state vector. Mixed states are states wherein the state vector is not well-defined.

Density Matrix The density matrix is defined as

$$\rho = |\Psi\rangle\langle\Psi|\,.$$

It has some cool properties. For instance:

$$\begin{aligned}
&\mathrm{tr}\{\rho\} = \sum_n \langle n|\rho|n\rangle = \left\langle \psi\middle|\sum_n |n\rangle\langle n|\middle|\psi\right\rangle = \langle\Psi|\Psi\rangle = 1,\\
&\rho^\dagger = \rho,\\
&\langle A\rangle = \sum_{n,m} \langle\Psi|n\rangle\ \langle n|A|m\rangle\,\langle m|\Psi\rangle = \sum_{n,m} \langle m|\Psi\rangle\,\langle\Psi|n\rangle\ \langle n|A|m\rangle = \sum_{n,m} \langle m|\rho|n\rangle\ \langle n|A|m\rangle = \mathrm{tr}(\rho A),\\
&\rho^2 = \rho.
\end{aligned}$$

Note that the latter is only true for pure states. Mixed states have a density matrix of the form

$$\rho = \sum_j P_j\,|\Psi_j\rangle\langle\Psi_j|\,,$$

where the P_j are the probability that the state of the system is $|\Psi_j\rangle$.

The Time Evolution Operator Suppose that there exists an operator $u_{t'}(t)$ which evolves $|\Psi(t')\rangle$ to $|\Psi(t)\rangle$. Such an operator should satisfy

- $u_{t'}(t) = u_{t''}(t)u_{t'}(t'')$ for consistency.
- $u_{t'}(t)$ is unitary to preserve the normalization.
- $u_t(t) = 1$.

Inserting this into the Schrödinger equation yields

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}u_{t'}(t)\left|\Psi(t')\right\rangle = Hu_{t'}(t)\left|\Psi(t')\right\rangle,$$
$$i\hbar\partial_t t' = Hu_{t'}(t).$$

In the case of a time-independent Hamiltonian, the solution must be of the form $u_{t'}(t) = u(t-t')$, and the equation above can be integrated to yield

$$u_{t'}(t) = e^{-i\frac{t-t'}{\hbar}H}.$$

Symmetries in Quantum Mechanics A symmetry in a quantum mechanics context is any transformation that leaves all probabilities invariant. Such symmetries must be symmetries on Hilbert space.

Transformation of Operators Consider a symmetry operator u. In order for this to be a symmetry, it must also act on all operators according to $A \to uAu^\dagger$.

Time Evolution From Symmetry Consider some system with time translation symmetry - that is, any system for which time translations do not change the theory. Introduce the transformation operator

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Space Translation Consider the space operator x^i. A space translation u transforms x^i to $x^i + a^i$, meaning $ux^iu^\dagger = x^i + a^i$. Expanding the translation around the identity yields

$$u = 1 + i\frac{a^i}{\hbar}p_i$$

for some operator p_i. The requirement that u be unitary implies that p is self-adjoint. The transformation rule yields

$$(1 + i\frac{a^i}{\hbar}p_i)x^i(1 - i\frac{a^i}{\hbar}p_i) = x^i + i\frac{a^i}{\hbar}\left\{p_i, x^i\right\}$$

and the requirement

$$\left[p_i, x^i\right] = -i\hbar.$$

Time Evolution of the Density Matrix The time evolution of the density matrix is given by

$$\rho(t) = \sum P_i u_{t_0}(t)\left|\Psi_i\right\rangle\left\langle\Psi_i\right| u_{t_0}(t)^\dagger = u_{t_0}(t)\rho(t_0)u_{t_0}(t)^\dagger.$$

This implies

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho = Hu_{t_0}(t)\rho(t_0)u_{t_0}(t)^\dagger - u_{t_0}(t)\rho(t_0)u_{t_0}(t)^\dagger H = H\rho(t) - \rho(t)H = [H,\rho].$$

The Heisenberg Equation Heisenberg's outlook starts from preserving expectation values under time translations in such a way that all (total) time evolution is contained in the operators, arriving at the transformation rule

$$A_{\mathrm{H}} = u_{t_0}^{\dagger}(t) A_{\mathrm{S}} u_{t_0}(t).$$

A_{H} is the operator according to Heisenberg and A_{S} is the operator according to Schrödinger. We now have

$$\begin{aligned} i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle A_{\mathrm{H}} \rangle &= -u_{t_0}^{\dagger}(t) H A_{\mathrm{S}} u_{t_0}(t) + u_{t_0}^{\dagger}(t)(i\hbar \partial_t A_{\mathrm{S}}) u_{t_0}(t) + u_{t_0}^{\dagger}(t) A_{\mathrm{S}} H u_{t_0}(t) \\ &= -u_{t_0}^{\dagger}(t) H u_{t_0}(t) u_{t_0}^{\dagger}(t) A_{\mathrm{S}} u_{t_0}(t) + u_{t_0}^{\dagger}(t)(i\hbar \partial_t A_{\mathrm{S}}) u_{t_0}(t) + u_{t_0}^{\dagger}(t) A_{\mathrm{S}} u_{t_0}(t) u_{t_0}^{\dagger}(t) H u_{t_0}(t) \\ &= -H_{\mathrm{H}} A_{\mathrm{H}} + u_{t_0}^{\dagger}(t)(i\hbar \partial_t A_{\mathrm{S}}) u_{t_0}(t) + A_{\mathrm{H}} H_{\mathrm{H}} \\ &= -H_{\mathrm{H}} [A_{\mathrm{H}}, +] (i\hbar \partial_t A_{\mathrm{S}})_{\mathrm{H}}. \end{aligned}$$

Propagators The probability amplitude at some point x at time t is given by

$$\Psi(x,t) = \langle x|\Psi(t)\rangle = \langle x|u_0(t)|\Psi(0)\rangle = \int \mathrm{d}x' \, \langle x|u_0(t)|x'\rangle \langle x'|\Psi(0)\rangle .$$

Defining the propagator $G_{x',t'}(x,t) = \langle x|u_{t'}(t)|x'\rangle$, we arrive at

$$\Psi(x,t) = \int \mathrm{d}x' \, G_{x',0}(x,t) \langle x'|\Psi(0)\rangle = \int \mathrm{d}x' \, G_{x',0}(x,t) \Psi(x',0).$$

Hence the propagator acts as a Green's function with respect to time, in some sense.

Arriving at Path Integrals The general propagator of some state is given by

$$G_{x',t'}(x,t) = \sum_{\gamma} G_{\gamma;x',t'}(x,t),$$

where the summation is performed over all possible paths γ between the two points.

Suppose now that the time evolution is divided into steps such that

$$u_{t'}(t) = \prod_{k=1}^{n} u_{t_{k-1}}(t_k), \ t_0 = t', \ t_n = t, \ t_k - t_{k-1} = \delta t.$$

Then

$$G_{x',t'}(x,t) = \left\langle x \left| \prod_{k=1}^{n} u_{t_{k-1}}(t_k) \right| x' \right\rangle .$$

For every k we now introduce an identity according to

$$\begin{aligned} G_{x',t'}(x,t) &= \left\langle x \left| \prod_{k=1}^{n} u_{t_{k-1}}(t_k) \int \mathrm{d}x_k \, |x_{k-1}\rangle\langle x_{k-1}| \right| x' \right\rangle \\ &= \left\langle x \left| \prod_{k=1}^{n} \int \mathrm{d}x_k \, u_{t_{k-1}}(t_k) \, |x_{k-1}\rangle\langle x_{k-1}| \right| x' \right\rangle \\ &= \int \prod_{k=1}^{n} \mathrm{d}x_k \, \langle x_k|u_{t_{k-1}}(t_k)|x_{k-1}\rangle . \end{aligned}$$

The time translation operator has the form $u_{t_{k-1}}(t_k) = e^{-i\frac{\Delta t}{\hbar} H}$. For a Hamiltonian of the form $H = \frac{p^2}{2m} + V(\mathbf{x})$, the terms do not necessarily commute. However, to second order we have

$$\begin{aligned} e^{\alpha A} e^{\alpha B} &= \left(1 + \alpha A + \frac{1}{2}\alpha^2 A^2 + \dots\right)\left(1 + \alpha B + \frac{1}{2}\alpha^2 B^2 + \dots\right), \\ e^{\alpha(A+B)} &= 1 + \alpha A + \alpha B + \frac{1}{2}\alpha^2 (A^2 + B^2 + AB + BA) + \dots, \\ &= e^{A} e^{B} \left(1 - \frac{1}{2}\alpha^2 AB + \frac{1}{2}\alpha^2 BA + \dots\right) \\ &= e^{\alpha A} e^{\alpha B} e^{\frac{1}{2}\alpha^2 [A,B]}. \end{aligned}$$

Ignoring the second-order term yields

$$\begin{aligned}
G_{x',t'}(x,t) &= \int \prod_{k=1}^{n} \mathrm{d}x_k \ \langle x_k | e^{-i\frac{\Delta t}{\hbar}(T+V)} | x_{k-1} \rangle \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} \langle x_k | e^{-i\frac{\Delta t}{\hbar}T} | x_{k-1} \rangle \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} \left\langle x_k \middle| e^{-i\frac{\Delta t}{\hbar}T} \int \mathrm{d}p_k \, |p_k\rangle\langle p_k| \middle| x_{k-1} \right\rangle \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} \left\langle x_k \middle| \int \mathrm{d}p_k \, e^{-i\frac{\Delta t}{\hbar}T} \, |p_k\rangle\langle p_k| \middle| x_{k-1} \right\rangle \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} \int \mathrm{d}p_k \, e^{-i\frac{\Delta t}{2m\hbar}p_k^2} \langle x_k | p_k \rangle \langle p_k | x_{k-1} \rangle \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} \int \mathrm{d}p_k \, e^{-i\frac{\Delta t}{2m\hbar}p_k^2} \frac{1}{2\pi\hbar} e^{i\frac{p_k(x_k - x_{k-1})}{\hbar}} \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} e^{i\frac{m}{2\hbar\Delta t}(x_k - x_{k-1})^2} \frac{1}{2\pi\hbar} \int \mathrm{d}p_k \, e^{-i\frac{\Delta t}{2m\hbar}\left(p_k - \frac{m}{\Delta t}(x_k - x_{k-1})\right)^2} \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} e^{i\frac{m}{2\hbar\Delta t}(x_k - x_{k-1})^2} \sqrt{\frac{m}{2\pi^2\hbar\Delta t i}} \int \mathrm{d}v_k \, e^{-v_k^2} \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \, e^{-i\frac{\Delta t}{\hbar}V(x_{k-1})} e^{i\frac{m}{2\hbar\Delta t}(x_k - x_{k-1})^2} \sqrt{\frac{m}{2\pi\hbar\Delta t i}} \\
&= \int \prod_{k=1}^{n} \mathrm{d}x_k \sqrt{\frac{m}{2\pi\hbar\Delta t i}} e^{i\frac{1}{\hbar}\sum\limits_{k=1}^{n}\left(\frac{1}{2}m\left(\frac{x_k - x_{k-1}}{\Delta t}\right)^2 - V(x_{k-1})\right)\Delta t}.
\end{aligned}$$

In the continuous limit the exponent becomes

$$i\frac{1}{\hbar}\int \mathrm{d}t \, \frac{1}{2}m\dot{x}^2 - V(x) = i\frac{S}{\hbar}$$

where S is the action. The remaining factor, termed the measure, is

$$D(x(t)) = \lim_{\Delta t \to 0} \prod_{k=1}^{n} \mathrm{d}x_k \sqrt{\frac{m}{2\pi\hbar\Delta t i}}.$$

Finally the propagator is given by

$$G_{x',t'}(x,t) = \int D(x(t)) e^{-i\frac{S}{\hbar}}.$$

This is termed the path integral.

As a side note, if the action is large compared to $\hbar$, the action varies strongly, causing destructive interference from all paths except for the one such that

$$\frac{\delta S}{\delta x} = 0.$$

This is Hamilton's principle, the fundamental postulate of classical mechanics.

The Harmonic Oscillator The Hamiltonian of the harmonic oscillator is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2.$$

To diagonalize it we introduce the lowering operator

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \frac{i}{\sqrt{m\omega\hbar}}p\right)$$

and its adjoint, the raising operator. Their commutator is

$$\begin{aligned}\left[a, a^\dagger\right] &= \left[\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \frac{i}{\sqrt{m\omega\hbar}}p\right), \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x - \frac{i}{\sqrt{m\omega\hbar}}p\right)\right] \\ &= \left[\frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar}}x, -\frac{1}{\sqrt{2}}\frac{i}{\sqrt{m\omega\hbar}}p\right] + \left[\frac{1}{\sqrt{2}}\frac{i}{\sqrt{m\omega\hbar}}p, \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar}}x\right] \\ &= \frac{1}{2}\frac{i}{\hbar}\left([x, -p] + [p, x]\right) \\ &= 1.\end{aligned}$$

The definition of the raising and lowering operators may be inverted to obtain

$$x = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}(a^\dagger + a),\ p = \frac{i}{\sqrt{2}}\sqrt{m\omega\hbar}(a^\dagger - a).$$

The Hamiltonian may now be written in terms of these operators as

$$\begin{aligned}H &= \frac{1}{2m}\cdot -\frac{1}{2}m\omega\hbar(a^\dagger - a)^2 + \frac{1}{2}m\omega^2\frac{1}{2}\frac{\hbar}{m\omega}(a^\dagger + a)^2 \\ &= -\frac{1}{4}\hbar\omega(a^\dagger - a)^2 + \frac{1}{4}\hbar\omega(a^\dagger + a)^2 \\ &= \frac{1}{4}\hbar\omega\left((a^\dagger)^2 + a^\dagger a + aa^\dagger + a^2 - \left((a^\dagger)^2 - a^\dagger a - aa^\dagger + a^2\right)\right) \\ &= \frac{1}{2}\hbar\omega\left(a^\dagger a + aa^\dagger\right) \\ &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).\end{aligned}$$

We now define the operator $n = a^\dagger a$. It is Hermitian, meaning that an orthonormal basis of its eigenvectors exists (fortunately, as it constitutes the Hamiltonian). These eigenvectors must be studied next. To do this, we use the commutation relations[1]

$$[n, a] = a^\dagger[a, a] + \left[a^\dagger, a\right]a = -a,\ \left[n, a^\dagger\right] = a^\dagger\left[a, a^\dagger\right] + \left[a^\dagger, a^\dagger\right]a = a^\dagger$$

applied to some eigenvector $|\nu\rangle$ with eigenvalue ν to obtain

$$na\,|\nu\rangle = (an - a)\,|\nu\rangle = (\nu - 1)a\,|\nu\rangle\,.$$

Hence, if some eigenvalue ν exists, we can repeat this argument to show that $\nu-1,\ \nu-2, \ldots$ are also eigenvalues, assuming no value in this sequence is zero. The length of these eigenvectors is given by

$$\langle\nu|a^\dagger a|\nu\rangle = \nu\,\langle\nu|\nu\rangle \geq 0,$$

where the latter is due to the positivity of the inner product. In order for this to work, no negative eigenvalues may exist. This only fits with the previous sequence of eigenvalues if $\nu = 0$ is an eigenvalue.

Having established that, we rename the eigenvalues to n. Next, we have

$$na^\dagger\,|n\rangle = (a^\dagger n + a^\dagger)\,|n\rangle = (n + 1)a^\dagger\,|n\rangle\,.$$

Hence the sequence $n + 1, n + 2, \ldots$ also consists of eigenvalues of n. The length of such vectors is

$$\langle n|aa^\dagger|n\rangle = \langle n|a^\dagger a + 1|n\rangle = (n + 1)\,\langle n|n\rangle > 0.$$

[1] What might inspire this? A suggestion might be the fact that if n and the raising and lowering operators commuted, we would find that they share eigenvectors.

Now the eigenvalues of the Hamiltonian are found to be

$$H_n = \hbar\omega\left(n + \frac{1}{2}\right),\ H\left|n\right\rangle = H_n\left|n\right\rangle.$$

With respect to degeneracy, suppose there is a set of eigenvectors denoted by the index k such that $a\left|0,k\right\rangle = 0$. In the coordinate basis we obtain

$$\left\langle x\middle|\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \frac{i}{\sqrt{m\omega\hbar}}p\right)\middle|0,k\right\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{\mathrm{d}}{\mathrm{d}x}\right)\Psi_{0,k} = 0.$$

The solution to this differential equation is unique, hence the ground state is non-degenerate. The linearity of the raising operator therefore implies that the other eigenvalues are non-degenerate as well.

With respect to normalization, we may require all states to be normalized. Then

$$\begin{aligned} a^\dagger\left|n\right\rangle &= c_{n+1}\left|n+1\right\rangle, \\ \left|c_{n+1}\right|^2 &= \left\langle n|aa^\dagger|n\right\rangle = \left\langle n|n+1|n\right\rangle = n+1, \\ c_n &= \sqrt{n}. \end{aligned}$$

Next we have

$$\begin{aligned} aa^\dagger\left|n-1\right\rangle &= \sqrt{n}a\left|n\right\rangle \\ n\left|n-1\right\rangle &= \sqrt{n}a\left|n\right\rangle, \\ a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle. \end{aligned}$$

Finally, the excited states may be found according to

$$\left|n\right\rangle = \frac{1}{\sqrt{n}}a^\dagger\left|n-1\right\rangle = \cdots = \frac{1}{\sqrt{n!}}(a^\dagger)^n\left|0\right\rangle,$$

which when applied to the ground state will reproduce some special function.

Symmetries and Conserved Quantities Suppose that there exists some unitary transformation $u = e^{-i\frac{\varepsilon}{\hbar}A}$ such that $u^\dagger H u = H$. Expanding the symmetry yields

$$\left(1 + i\frac{\varepsilon}{\hbar}A + \ldots\right)H\left(1 - i\frac{\varepsilon}{\hbar}A + \ldots\right) = H + i\frac{\varepsilon}{\hbar}(-HA + AH) + \cdots = H + i\frac{\varepsilon}{\hbar}[A,H] + \cdots = H,$$

implying that A and H commute. Assuming A to have no explicit time dependence, Heisenberg's equations yield that $\langle A\rangle$ is conserved. These arguments form the basis of some form of Nöether's theorem in quantum mechanics.

Rotations Consider an axis $\mathbf{n}$. The rotation symmetry operator about this axis is termed $u(\theta\mathbf{n})$. There must be three generators of rotations in three dimensions, and if we want them to correspond to the classical notion of rotations, they must satisfy the commutation relation

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k.$$

These generators are the operators for angular momentum, as will be shown.

The total angular momentum is given by $J^2 = J_iJ_i$, and thus commutes with all of its component. Hence we can find a basis for Hilbert space composed of joint eigenvectors of J^2 and any one component of $\mathbf{J}$ (usually J_z).

Properties of Angular Momentum To study the properties of angular momentum, we will use the method of raising and lowering operators. Working in the basis of eigenvectors of J^2 and J_z, the raising and lowering operators are

$$J_+ = J_x + iJ_y,\ J_- = J_x - iJ_y.$$

We have

$$[J_z, J_\pm] = [J_z, J_x \pm iJ_y] = i\hbar(J_y \mp iJ_x) = \hbar(iJ_y \pm J_x) = \pm\hbar(J_x \pm iJ_y) = \pm\hbar J_\pm.$$

Now introduce the eigenstates $|j,m\rangle$ such that $J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$ and $J_z|j,m\rangle = \hbar m|j,m\rangle$. We have

$$J_z J_\pm |j,m\rangle = (J_\pm J_z \pm \hbar J_\pm)|j,m\rangle = (m \pm \hbar)J_\pm |j,m\rangle .$$

Hence the raising and lowering operators do indeed raise and lower the angular momentum. Next we have

$$J^2 J_\pm |j,m\rangle = J_\pm J^2 |j,m\rangle = \hbar^2 j(j+1) J_\pm |j,m\rangle .$$

Hence the raising and lowering operators do not change the value of the total angular momentum. This imposes a constraint on the possible set of angular momenta - namely, all components of the angular momentum are Hermitian, meaning that J_z^2 may not have eigenvalues larger than j^2. Hence states exists such that

$$J_-|j,m\rangle = 0,\ J_+|j,m\rangle = 0.$$

To identify these, consider the operators

$$J_-J_+ = J_x^2 + J_y^2 + i[J_x, J_y] = J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z,$$
$$J_+J_- = J_x^2 + J_y^2 - i[J_x, J_y] = J_x^2 + J_y^2 + \hbar J_z = J^2 - J_z^2 + \hbar J_z$$

Suppose now that we are working on the first eigenstate such that $J_+|j,m\rangle = 0$. Then

$$J_-J_+|j,m\rangle = (J^2 - J_z^2 - \hbar J_z)|j,m\rangle = \hbar^2(j(j+1) - m(m+1))|j,m\rangle = 0.$$

Hence this m satisfies $m = j$. Similarly, for the state such that $J_-|j,m\rangle = 0$, we have

$$J_-J_+|j,m\rangle = (J^2 - J_z^2 + \hbar J_z)|j,m\rangle = \hbar^2(j(j+1) - m^2 + m)|j,m\rangle = \hbar^2(j(j+1) - m(m-1))|j,m\rangle = 0.$$

Hence this m satisfies $m = -j$. Now, as we know that m changes in integer steps between real numbers from $-j$ to $+j$, we must have that $2j$ is an integer.

Next, we choose the basis states to be normalized. Writing $J_+|j,m\rangle = c_{j,m}|j,m+1\rangle$ we have

$$|c_{j,m}|^2 = \langle j,m|J_-J_+|j,m\rangle = \hbar^2(j(j+1) - m(m+1)),\ J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle .$$

Similarly, writing $J_-|j,m\rangle = c_{j,m}|j,m-1\rangle$ we have

$$|c_{j,m}|^2 = \langle j,m|J_+J_-|j,m\rangle = \hbar^2(j(j+1) - m(m-1)),\ J_-|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle .$$

Orbital Angular Momentum Supposing the spatial rotation to be described by some rotation matrix R, we require $u_R|\mathbf{r}\rangle = |R\mathbf{r}\rangle$. This yields

$$\Psi'(\mathbf{r}') = \langle \mathbf{r}|u_R|\Psi\rangle \left\langle R^{-1}\mathbf{r}\middle|\Psi\right\rangle = \Psi(R^{-1}\mathbf{r}).$$

It can be shown that this implies that Ψ was acted on by an operator $e^{-\frac{i}{\hbar}\theta_i L_i}$ where

$$L_i = \varepsilon_{ijk} x_j p_k.$$

This completes the argument that the generators of rotations are indeed angular momenta in the classical sense.

It turns out that the orbital angular momenta may only have integer multiples of $\hbar$ as their eigenvalue, meaning that classical angular momentum does not by itself contain all the properties of angular momenta, as we have seen. It also turns out that the rest comes from spin.

Spin In addition to the transformation of coordinates, the rotation operator could in principle permute the basis. Thus we add an extra factor $D_{\mathbf{n}}(R)$. It commutes with the previously discussed operators as $e^{-i\frac{\theta}{\hbar}L}$ acts equally on all of the basis and $D_{\mathbf{n}}(R)$ is linear in the basis.

Using the machinery of Lie algebra, we may write

$$D_n(R) = e^{-\frac{i}{\hbar}\theta_i S_i}.$$

The total rotation operator is thus

$$e^{-\frac{i}{\hbar}\boldsymbol{\theta}\cdot(\mathbf{L}+\mathbf{S})},$$

prompting us to define the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

$\mathbf{J}$ must generate a rotation, hence $\mathbf{S}$ must satisfy the same commutation relation, implying that $\mathbf{S}$ must also be an angular momentum operator. Furthermore, it does not have the same restrictions as orbital angular momentum, and may therefore correspond to j being a half-integer. This is termed the spin.

Spin-$\frac{1}{2}$ In the particular case of $j = \frac{1}{2}$ we may construct matrices for the spin operator in the eigenbasis of S_z. The resulting matrices are $S_i = \frac{1}{2}\hbar\sigma_i$, where σ_i are the Pauli matrices. The spin factor of the rotation operator may now be written as

$$D_{\mathbf{n}}(R) = e^{-i\frac{\theta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}.$$

To simplify this, we note that

$$\begin{aligned}(\mathbf{n}\cdot\boldsymbol{\sigma})^2 &= n_i n_j \sigma_i \sigma_j \\ &= \sum_{i=j} n_i^2 \sigma_i^2 + \frac{1}{2}\sum_{i\neq j} n_i n_j (\sigma_i\sigma_j + \sigma_j\sigma_i).\end{aligned}$$

The Pauli matrices anticommute and square to identity, hence we have

$$(\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1.$$

This yields

$$\begin{aligned}D_{\mathbf{n}}(R) &= \sum_{m=0} \frac{1}{(2m)!}\left(-i\frac{\theta}{2}\right)^{2m}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2m} + \frac{1}{(2m+1)!}\left(-i\frac{\theta}{2}\right)^{2m+1}(\mathbf{n}\cdot\boldsymbol{\sigma})^{2m+1} \\ &= \sum_{m=0} \frac{1}{(2m)!}(-1)^m\left(\frac{\theta}{2}\right)^{2m} - \frac{i}{(2m+1)!}(-1)^m\left(\frac{\theta}{2}\right)^{2m+1}\mathbf{n}\cdot\boldsymbol{\sigma} \\ &= \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\mathbf{n}\cdot\boldsymbol{\sigma}.\end{aligned}$$

Addition of Angular Momenta We would like to identify eigenstates of the total angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ in terms of the eigenstates of its components. The basis we are starting with is the product basis

$$|l_1, l_2; m_1, m_2\rangle = |l_1, m_1\rangle \otimes |l_2, m_2\rangle .$$

The components of the angular momentum are given by

$$J_i = (L_1)_i \otimes 1 + 1 \otimes (L_2)_i,$$

and we therefore have

$$J_z |l_1, l_2; m_1, m_2\rangle = \hbar(m_1 + m_2) |l_1, l_2; m_1, m_2\rangle .$$

As $\mathbf{J}$ is indeed an angular momentum, we may now use what we now to introduce the eigenbasis of $\mathbf{J}$ as $|l_1, l_2; j, m\rangle$. It may be expressed in terms of the previous basis as

$$|l_1, l_2; j, m\rangle = \sum_{m_1,m_2} \langle l_1, l_2; m_1, m_2 | l_1, l_2; j, m\rangle |l_1, l_2; m_1, m_2\rangle .$$

The $\langle l_1, l_2; m_1, m_2 | l_1, l_2; j, m\rangle$ are called Clebsch-Gordan coefficients, and may be found in tables.

Using what we found in a previous step, we must have $m \leq l_1 + l_2$ and $j = l_1 + l_2$. This also allows us to identify one eigenstate

$$|l_1, l_2; l_1 + l_2, l_1 + l_2\rangle = |l_1, l_2; l_1, l_2\rangle .$$

To identify other states, one simply applies the lowering operator $J_\pm = J_x \pm iJ_y = (L_1)_- + (L_2)_-$. This will produce $2(l_1 + l_2) + 1$ new states. Next, we identify

Quantum Hall Effect The Hamiltonian of a charged particle in a magnetic field is

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2.$$

We recall that gauge transformations of the electromagnetic field are defined according to $A^\mu \to A^\mu - \partial^\mu \chi$ for some function χ. While Maxwell's equations are gauge invariant, Schrödinger's equation is not. We try to remedy this by combining the gauge transformation with a transformation

$$|\Psi\rangle \to e^{i\frac{q}{\hbar}\chi}|\Psi\rangle .$$

The inverse transformation yields

$$(\mathbf{p} - q\mathbf{A})|\Psi\rangle = (\mathbf{p} - q(\mathbf{A} - \vec{\boldsymbol{\nabla}}\chi))e^{-i\frac{q}{\hbar}\chi}|\Psi\rangle ,$$

which is given by

$$(\mathbf{p} - q(\mathbf{A} - \vec{\boldsymbol{\nabla}}\chi))e^{-i\frac{q}{\hbar}\chi}|\Psi\rangle = e^{i\frac{q}{\hbar}\chi}(\mathbf{p} - q\vec{\boldsymbol{\nabla}}\chi - q(\mathbf{A} - \vec{\boldsymbol{\nabla}}\chi))|\Psi\rangle = e^{-i\frac{q}{\hbar}\chi}(\mathbf{p} - q\mathbf{A})|\Psi\rangle ,$$

implying that the transformation does not change the Schrödinger equation.

We will start by studying two-dimensional motion in a rectangular domain with a constant magnetic field in the z-direction. In the Landau gauge we choose the vector potential $\mathbf{A} = Bx\mathbf{e}_x$. The Hamiltonian is thus

$$H = \frac{1}{2m}\left(p_x^2 + (p_y - qBx)^2\right).$$

We see that the Hamiltonian commutes with p_y, but not with p_x, implying that the solution is of the form

$$\Psi(\mathbf{x}) = \frac{1}{\sqrt{2\pi\hbar}}e^{-ik_y y}f(x),$$

where $k_y = \frac{1}{\hbar}p_y$ may be taken to have a definite value. We are thus left with

$$H = \frac{1}{2m}\left(p_x^2 + \hbar^2\left(k_y - \frac{1}{l_B^2}x\right)^2\right),\ l_B^2 = \frac{\hbar}{qB}.$$

We solve this by introducing raising and lowering operators

$$a_{k_y} = \frac{1}{\sqrt{2}}\left(\frac{x - k_y l_B^2}{l_B} + i\frac{l_B}{\hbar}p_x\right)$$

such that

$$H = \hbar\omega(a_{k_y}^\dagger a_{k_y} + \frac{1}{2}),$$

yielding a harmonic oscillator with the classical cyclotron frequency

$$\omega = \frac{\hbar}{ml_B^2} = \frac{eB}{m}.$$

The energy levels of this harmonic oscillator are called Landau levels.

To study the degeneracy, we impose periodic boundary conditions in the y-direction, implying

$$k_y = \frac{2\pi}{L_y}m.$$

If you have a large but finite sample of length L_x in the x-direction, the fact that the state is localized around $k_y l_B^2$ implies that the maximum value of m is

$$N = \frac{L_x}{l_B^2\frac{2\pi}{L_y}} = \frac{qBA}{h}.$$

In particular, for $q = e$ we have

$$H = \frac{\Phi}{\Phi_0},$$

where $\Phi_0 = \frac{h}{e}$ is the flux quantum.

To study samples at the edge of the sample, add a potential to represent the edge and assume that it varies slowly when compared to the length scale l_B. In this case the eigenvalues are modified to

$$E_n = \hbar\omega(n + \frac{1}{2}) + V(x == -k_y l_B^2)$$

The edge states (perhaps) carry the current, meaning that such systems that we have studied will display steps in their Hall coefficient. This is the quantum Hall effect.

Aharanov-Bohm Effect To demonstrate the principle, consider a metal ring connected to two terminals with magnetic flux through the middle and suppose that current flows from one terminal to the other. The vector potential is non-zero in the ring, and may in fact be written as $\mathbf{A} = \vec{\boldsymbol{\nabla}} f$ here. Performing a gauge transformation preserves the Schrödinger equation, as before. Now, as we may write

$$f = \int_{\gamma} \mathrm{d}\mathbf{x} \cdot \mathbf{A},$$

this implies that the phase of the state is determined by the path taken. Combining this with our knowledge of path integrals yields the transmittion probability

$$T = |t_{\text{upper}}|^2 + |t_{\text{lower}}|^2 + 2\,\mathrm{Re}\left(t_{\text{upper}} t^{\star}_{\text{upper}} e^{iq\left(\int_{\gamma_{\text{lower}}} \mathrm{d}\mathbf{x}\cdot\mathbf{A} - \int_{\gamma_{\text{upper}}} \mathrm{d}\mathbf{x}\cdot\mathbf{A} \right)} \right) = |t_{\text{upper}}|^2 + |t_{\text{lower}}|^2 + 2\,\mathrm{Re}\left(t_{\text{upper}} t^{\star}_{\text{upper}} e^{-iq\Phi} \right).$$

This oscillating transmission probability is the Aharanov-Bohm effect.