Sammanfattning av SI2360 Analytisk mekanik och klassisk fältteori

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Sammanfattning

Innehåll

1 Coordinates

Coordinates A general set of coordinates on $\mathbb{R}^n$ is n numbers $x^a, a = 1, \dots, n$ that uniquely define a point in the space.

Example: Cartesian coordinates In cartesian coordinates we introduce an orthonormal basis $\mathbf{e}_i$. Vi kan då skriva $\mathbf{x} = x^i \mathbf{e}_i$.

Basis vectors There are two different choices of coordinate bases.

The first is the tangent basis of vectors

$$\mathbf{E}_a = \partial_{x_a} \mathbf{x} = \partial_a \mathbf{x}.$$

The second is the dual basis

$$\mathbf{E}^a = \vec{\boldsymbol{\nabla}} x^a.$$

Vector coordinates Any vector can now be written as

$$\mathbf{v} = v^a \mathbf{E}_a = v_a \mathbf{E}^a.$$

The v^a are called contravariant components and the v_a are called covariant components.

We can now compute the scalar product

$$\mathbf{E}_a \cdot \mathbf{E}^b = \partial_a \mathbf{x} \cdot \vec{\boldsymbol{\nabla}} x^a = \delta_a^b.$$

Coordinate transformations Suppose that a vector can be written as

$$\mathbf{v} = v^a \mathbf{E}_a = v^{a'} \mathbf{E}'_{a'}.$$

How do we transform between these? A single component is given by

$$v^{a'} = \mathbf{E}'_{a'} \cdot v^a \mathbf{E}_a = v^a (\vec{\boldsymbol{\nabla}} x'^{a'} \cdot \partial_a \mathbf{x}) = v^a \partial_a x'^{a'}.$$

Tangents to curves The tangent to a curve is given by

$$\dot{\gamma} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \partial_a \mathbf{x} \frac{\mathrm{d}x^a}{\mathrm{d}t} = \dot{x}^a \mathbf{E}_a.$$

Gradients The gradient of a curve is given by

$$\vec{\boldsymbol{\nabla}} f = \partial_a f \vec{\boldsymbol{\nabla}} x^a = \mathbf{E}^a \partial_a f.$$

Rates of change along a curve The rate of change of a quantity along a path is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \partial_a f \frac{\mathrm{d}x^a}{\mathrm{d}t} = \vec{\boldsymbol{\nabla}} f \cdot \dot{\gamma}.$$

2 Tensors

Definition A tensor of rank N is a multilinear map from N vectors to a scalar.

Components of a tensor The components of a tensor are defined by

$$T(\mathbf{E}^{a_1}, \dots, \mathbf{E}^{a_N}) = T^{a_1, \dots, a_N}.$$

These are called the contravariant components of the tensor, and the covariant components are defined similarly. Mixed components can also be defined.

Rules for tensors Tensors obey the following rules:

$$(T_1+T_2)(\mathbf{w}_1,\dots,\mathbf{w}_{n_2}) = T_1(\mathbf{w}_1,\dots,\mathbf{w}_{n_2}) + T_2(\mathbf{w}_1,\dots,\mathbf{w}_{n_2}),$$
$$(kT)(\mathbf{w}_1,\dots,\mathbf{w}_{n_2}) = kT(\mathbf{w}_1,\dots,\mathbf{w}_{n_2}).$$

In component form:

$$(T_1+T_2)^{a_1\dots a_n} = T_1^{a_1\dots a_n} + T_2^{a_1\dots a_n},$$
$$(kT)^{a_1\dots a_n} = kT^{a_1\dots a_n}.$$

The metric tensor The metric tensor g is a rank 2 tensor defined by $g(\mathbf{v},\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$. Its components satisfy

$$v_a = \mathbf{E}_a \cdot v^b \mathbf{E}_b = g(\mathbf{E}_a, \mathbf{E}_b) v_b = g_{ab} v^b,$$

and likewise

$$v^a = g^{ab} v_b.$$

Tensor product Given two tensors T_1 and T_2 of ranks n_1 and n_2, we kan define the rank n_1+n_2 tensor $T_1 \otimes T_2$ as

$$(T_1 \otimes T_2)(\mathbf{v}_1,\dots,\mathbf{v}_{n_1},\mathbf{w}_1,\dots,\mathbf{w}_{n_2}) = T_1(\mathbf{v}_1,\dots,\mathbf{v}_{n_1}) T_2(\mathbf{w}_1,\dots,\mathbf{w}_{n_2}).$$

In component form:

$$(T_1 \otimes T_2)^{a_1\dots a_{n_1+n_2}} {=} T_1^{a_1\dots a_{n_1}} T_2^{a_{n_1+1}\dots a_{n_1+n_2}}.$$

Tensors as linear combinations Using the tensor product, all tensors can be written as linear combinations of certain basis elements due to their bilinearity. Define

$$e_{a_1\dots a_n} = \mathbf{E}_{a_1} \otimes \dots \otimes \mathbf{E}_{a_n}$$

to be the tensor that satisfies

$$e_{a_1\dots a_n}(\mathbf{E}^{b_1},\dots,\mathbf{E}^{b_n}) = (\mathbf{E}_{a_1}\cdot\mathbf{E}^{b_1})\dots(\mathbf{E}_{a_n}\cdot\mathbf{E}^{b_n}) = \delta_{a_1}^{b_1}\dots\delta_{a_n}^{b_n}.$$

Then any tensor can be written as

$$T = T^{a_1\dots a_n} e_{a_1\dots a_n}$$

where the $T^{a_1\dots a_n}$ are exactly the contravariant components of T.

Tensors as linear transforms on tensors A rank n tensor can also be viewed as a linear map from rank m tensors to rank $n-m$ tensors. To do this, we first define, given T, the rank $n-m$ tensor $\tilde{T}(\mathbf{w}_1 \otimes \dots \otimes \mathbf{w}_m)$ such that

$$(\tilde{T}(\mathbf{w}_1 \otimes \dots \otimes \mathbf{w}_m))(\mathbf{v}_1,\dots,\mathbf{v}_{n-m}) = T(\mathbf{w}_1,\dots,\mathbf{w}_m,\mathbf{v}_1,\dots,\mathbf{v}_{n-m}).$$

This map is also linear in all the $\mathbf{w}_i$. Next, given a rank $n-m$ tensor $\tilde{T}$, one can define the rank $n-m$ tensor $T(\mathbf{w}_1,\dots,\mathbf{w}_m)$ such that

$$T(\mathbf{w}_1,\dots,\mathbf{w}_m,\mathbf{v}_1,\dots,\mathbf{v}_{n-m}) = (\tilde{T}(\mathbf{w}_1 \otimes \dots \otimes \mathbf{w}_m))(\mathbf{v}_1,\dots,\mathbf{v}_{n-m}).$$

This is a linear rank n tensor.

Tensor contraction Given a complete set of vectors $\mathbf{v}_i$ and their dual $\mathbf{v}^i$ such that $\mathbf{v}_i \cdot \mathbf{v}^i = \delta_i^j$, the contraction $e_{12}T$ of two arguments of a rank n tensor is the tensor of rank $n-2$ satisfying

$$(e_{12}T)(\mathbf{w}_1,\dots,\mathbf{w}_{n-2}) = T(\mathbf{v}_i,\mathbf{v}^i,\mathbf{w}_1,\dots,\mathbf{w}_{n-2}).$$

In component form:

$$(e_{12}T)^{a_1\dots a_{n-2}} = T_c^{\ c\, a_1\dots a_{n-2}}.$$

The definition is similar (I assume) for the contraction of other arguments.

3 Geometry

Covariant derivatives and Christoffel symbols When computing a derivative, one must account both for the change in the quantity itself and the change of basis. We have

$$\partial_b \mathbf{E}_a = \Gamma^c{}_{ba} \mathbf{E}_c$$

where the $\Gamma^c{}_{ba}$ are called Christoffel symbols. These satisfy

$$\mathbf{E}^c \cdot \partial_b \mathbf{E}_a = \mathbf{E}^c \cdot \Gamma^d{}_{ba} \mathbf{E}_d = \delta^c_d \Gamma^d{}_{ba} = \Gamma^c{}_{ba}.$$

Note that

$$\partial_a \mathbf{E}_b = \partial_a \partial_b \mathbf{x} = \partial_b \partial_a \mathbf{x} = \partial_b \mathbf{E}_a,$$

which implies

$$\Gamma^c{}_{ba} = \Gamma^c{}_{ab}.$$

Using this, we can compute the partial derivate of $\mathbf{v} = v^a \mathbf{E}_a$ with respect to χ^a as

$$\partial_a \mathbf{v} = \mathbf{E}_b \partial_a v^b + v^b \partial_a \mathbf{E}_b = \mathbf{E}_b \partial_a v^b + v^b \Gamma^c{}_{ab} \mathbf{E}_c.$$

Renaming the summation indices yields

$$\partial_a \mathbf{v} = \mathbf{E}_b (\partial_a v^b + v^c \Gamma^b{}_{ac}),$$

which contains one term from the change in the coordinates and one term from the change in basis. We now define the covariant derivative of the contravariant components of $\mathbf{v}$ as

$$\vec{\boldsymbol{\nabla}}_a v^b = \partial_a v^b + v^c \Gamma^b{}_{ac}.$$

We would also like to define the covariant derivative of the covariant components of a vector field. To do this, we use the fact that

$$\partial_a \mathbf{E}_b \cdot \mathbf{E}^c = \partial_a \delta^c_b = 0.$$

The product rule yields

$$\mathbf{E}_b \cdot \partial_a \mathbf{E}^c + \mathbf{E}^c \cdot \partial_a \mathbf{E}_b = \mathbf{E}_b \cdot \partial_a \mathbf{E}^c + \mathbf{E}^c \cdot \Gamma^d{}_{ab} \mathbf{E}_d = \mathbf{E}_b \cdot \partial_a \mathbf{E}^c + \delta^c_d \cdot \Gamma^d{}_{ab} = \mathbf{E}_b \cdot \partial_a \mathbf{E}^c + \Gamma^c{}_{ab},$$

which implies

$$\partial_a \mathbf{E}^c = -\Gamma^c{}_{ab} \mathbf{E}^b.$$

Repeating the steps above now yields

$$\vec{\boldsymbol{\nabla}}_a v_b = \partial_a v_b - \Gamma^c{}_{ab} v_c.$$

Curve length Consider some curve parametrized by t, and let $\dot{\gamma}$ denote its tangent. The curve length is given by

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = g(\dot{\gamma}, \dot{\gamma})\, \mathrm{d}t^2 = g_{ab} \dot{\chi^a} \dot{\chi^b}\, \mathrm{d}t^2\,.$$

The curve length is now given by

$$L = \int \mathrm{d}t\, \sqrt{g_{ab} \dot{\chi^a} \dot{\chi^b}}.$$

Geodesics A geodesic is a curve that extremises the curve length between two points. From variational calculus, it is known that such curves satisfy the Euler-Lagrange equations, and we would like a differential equation that describes such a curve. By defining $\mathcal{L} = \sqrt{g_{ab}\dot{\chi}^a\dot{\chi}^b}$, the Euler-Lagrange equations for the curve length becomes

$$\partial_{\chi^a}\mathcal{L} - \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{\chi}^a}\mathcal{L} = 0.$$

Computing the derivatives yields

$$\frac{1}{2\mathcal{L}}\left(\dot{\chi}^a\dot{\chi}^b\partial_c g_{ab} - \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{\chi}^a g_{ac})\right) = 0.$$

We note that the expression in the paranthesis is the Euler-Lagrange equation for the integral of $\mathcal{L}^2$. Expanding the derivative yields

$$\frac{1}{\mathcal{L}}\left(\frac{1}{2}\dot{\chi}^a\dot{\chi}^b\partial_c g_{ab} - g_{ac}\ddot{\chi}^a - \dot{\chi}^a\dot{\chi}^b\partial_b g_{ac}\right) = 0.$$

Multiplying this by $-g^{cd}\mathcal{L}$ yields

$$g_{ac}g^{cd}\ddot{\chi}^a - \frac{1}{2}\dot{\chi}^a\dot{\chi}^b g^{cd}(2\partial_b g_{ac} - \partial_c g_{ab}) = g_{ac}g^{cd}\ddot{\chi}^a - \frac{1}{2}\dot{\chi}^a\dot{\chi}^b g^{cd}(\partial_b g_{ac} + \partial_b g_{ac} - \partial_c g_{ab}) = 0.$$

The a and b indices are summed over, and may thus be swapped. Combined with the symmetry of the metric tensor, this yields

$$g_{ac}g^{cd}\ddot{\chi}^a - \frac{1}{2}\dot{\chi}^a\dot{\chi}^b g^{cd}(\partial_b g_{ac} + \partial_a g_{cb} - \partial_c g_{ab}) = 0.$$

For some very magical reason, this can be simplified to yield the geodesic equation

$$\ddot{\chi}^a - \frac{1}{2}\dot{\chi}^a\dot{\chi}^b g^{cd}(\partial_b g_{ac} + \partial_a g_{cb} - \partial_c g_{ab}) = 0.$$

Christoffel symbols and the geodesic equation Consider a straight line with a tangent vector of constant magnitude. In euclidean space, this is a geodesic. This curve satisfies

$$\frac{\mathrm{d}\dot{\gamma}}{\mathrm{d}t} = (\dot{\gamma}\cdot\vec{\boldsymbol{\nabla}})\dot{\gamma} = \dot{\chi}^a\partial_a\dot{\gamma} = \dot{\chi}^a(\vec{\boldsymbol{\nabla}}_a\dot{\chi}^d)\mathbf{E}_d = (\dot{\chi}^a\partial_a\dot{\chi}^d + \dot{\chi}^a\dot{\chi}^c\Gamma^d{}_{ac})\mathbf{E}_d.$$

Comparison to the chain rule yields

$$\frac{\mathrm{d}\dot{\gamma}}{\mathrm{d}t} = (\ddot{\chi}^a + \dot{\chi}^a\dot{\chi}^c\Gamma^d{}_{ac})\mathbf{E}_d.$$

Comparing this to the geodesic equation yields

$$\Gamma^d{}_{ab} = \frac{1}{2}g^{dc}(\partial_b g_{ac} + \partial_a g_{cb} - \partial_c g_{ab}).$$

A better approach would have been to go through the derivation of the geodesic equation again, identifying the Christoffel symbols as you go, but I have no idea how to do that.

The geometry of curved space In curved space, we face the restriction that there is no position vector. All vectors in curved space are instead restricted to the tangent space. It turns out that tangent vectors at a point have coordinates $\dot{\chi}^a$ and that the tangent vectors consist of the tangent vectors to the coordinate lines.

We can also impose a metric tensor such that $\mathbf{v}\cdot\mathbf{w} = g_{ab}v^a w^b$, where the metric tensor is symmetric and positive definite.

Dual vectors can be defined as linear maps from tangent vectors to scalars. In particular, the dual vector $\mathrm{d}f$ can be defined as

$$\mathrm{d}f\,(\mathbf{v}) = v^a\partial_a f = \frac{\mathrm{d}f}{\mathrm{d}t}$$

along a curve with $\mathbf{v}$ as a tangent. A basis for the space of dual vectors is $e^a = \mathrm{d}\chi^a$.

Curve lengths are defined and computed as before. By defining geodesics as curves that extremize path length, this gives a set of Christoffel symbols and therefore a covariant derivative and a sense of what it means for a vector to change along a curve.

4 Classical mechanics

In classical mechanics, configuration space is the space of all possible configurations of a system. We can impose coordinates χ^a on this space in order to use what we know.

Kinetic energy Kinetic energy is defined by a rank 2 tensor as

$$E_\mathrm{k} = \frac{1}{2} T_{ab} \dot{\chi}^a \dot{\chi}^b,$$

where the dot now really represents the time derivative.

Hamilton's principle We define the Lagrangian of a system as $\mathcal{L} = E_\mathrm{k} - V$, where V is the potential energy and taken to be a function on coordinate space. The action of a system over time is defined as

$$S = \int \mathrm{d}t \, \mathcal{L}.$$

Hamilton's principle states that for the motion of the system in configuration space, $\delta S = 0$. This can be expressed as

$$\delta S = \int \mathrm{d}t \, \delta \mathcal{L} = \int \mathrm{d}t \left(\partial_{\chi^a} \mathcal{L} - \frac{\mathrm{d}}{\mathrm{d}t} \partial_{\dot{\chi}^a} \mathcal{L} \right) \delta \chi^a = 0.$$

The kinetic metric Consider a system with no potential energy. The Lagrangian simply becomes $\mathcal{L} = \frac{1}{2} T_{ab} \dot{\chi}^a \dot{\chi}^b$. This is very similar to the integral of curve length (or, rather its square, the extremum of which was noted to be the same), except g_{ab} has been replaced by T_{ab}. This inspires us to define T_{ab} as the kinetic metric, with corresponding Christoffel symbols.

Motion of a classical system By defining $a^b = \dot{\chi}^a \vec{\boldsymbol{\nabla}}_a \dot{\chi}^b$, the previous work leads us to a system with no potential satisfying $a^b = \ddot{\chi}^b + \Gamma^b{}_{ac} \dot{\chi}^a \dot{\chi}^c = 0$. In other words, a system with no potential moves along the geodesics of the kinetic metric.

For a system with a potential, only the $\partial_{\chi^a} \mathcal{L}$ term is affected, and

$$a^b = -T^{ba} \partial_a V = T^{ba} F,$$

which is a generalization of Newton's second law.