Summary of SI2540 Complex Systems

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Abstract

This is a summary of SI2540 Complex Systems.

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1 Basic Concepts

What is a Complex System? A complex system is a dynamical system characterized by at least one of the following:

- Nonlinearity.
- High sensitivity to initial conditions - the butterfly effect.
- The existence of bifurcations.
- Emergent phenomena - the formation of patterns in the solution.
- Feedback.
- Dissipation.

The Interesting Aspects of Complex Systems The interesting aspects of complex systems are

- long-term behaviour.
- dependence on initial conditions.
- parameter dependence.

Autonomous Systems An autonomous system is described by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x}$ is the state vector of the system.

This is of course not a restriction to first-order systems, as all systems may be written in this form. It is neither a restriction to f being independent of t, as one in this case can simply extend $\mathbf{x}$ to contain t.

Deterministic Systems A deterministic system is a system without random noise. Such systems are entirely specified by $\mathbf{f}$ and an initial condition.

Conservative and Dissipative Systems Conservative systems satisfy $\vec{\boldsymbol{\nabla}} \cdot \mathbf{f} = 0$. Dissipative systems satisfy $\vec{\boldsymbol{\nabla}} \cdot \mathbf{f} < 0$.

Orbits An orbit is a solution to an autonomous system corresponding to some particular initial value. The set of all orbits is the set of flow lines of $\mathbf{f}$. Because the position in phase space fully determines the future solution, flow lines never cross.

Fixed Points A fixed point is a point that satisfies $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Close to such points, non-zero first derivatives produce exponential behaviour locally and first derivatives equal to zero produce evolution slower than exponential.

Bifurcations A bifurcation is a qualitative in the structure of $\mathbf{f}$ as some parameter is varied.

Uniqueness of Solutions The weakest condition for the existence and uniqueness of a solution to

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}),\ \mathbf{x}(t_0) = \mathbf{x}_0$$

in a finite time interval around t_0, which we will assume to hold, is the Lipschitz condition

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq \kappa|\mathbf{x} - \mathbf{y}|$$

for some finite κ. This entails that $\mathbf{f}$ should be continuous and have piecewise continuous derivatives. If this condition holds, the solution is continuous in $\mathbf{x}_0$.

Periodic Motion For a system with a one-dimensional phase space, periodic motion is impossible on the real line or a subset of it. It is possible, however, on spaces with different topologies, such as a circle.

Numerical Integration Numerical integration methods are based around the Taylor expansion

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_t \Delta t + \frac{1}{2!}\left.\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right|_t (\Delta t)^2 + \dots$$

and specific schemes are usually obtained by truncating this expansion.

The Forward Euler Method The forward Euler method is obtained by truncating at the second step. For an autonomous system we have

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \mathbf{f}(t)\Delta t.$$

The error in each step is of ordet $(\Delta t)^2$, and the total error after N steps, which integrate a time τ forward, is of order $N(\Delta t)^2 = \tau\Delta t$. Note that this is equivalent to the approximation

$$\left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_t = \frac{1}{\Delta t}(\mathbf{x}(t+\Delta t) - \mathbf{x}(t)).$$

Runge-Kutta Schemes An improved scheme starts with

$$\frac{1}{\Delta t}(\mathbf{x}(t+\Delta t) - \mathbf{x}(t)) = \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{t+\frac{1}{2}\Delta t} = \mathbf{f}\left(t + \frac{1}{2}\Delta t\right) = \mathbf{f}\left(\mathbf{x}(t) + \frac{1}{2}\Delta t\mathbf{f}\left(\mathbf{x}(t)\right)\right) = \mathbf{f}\left(\mathbf{x}(t)\right) + \frac{1}{2}\Delta t\mathbf{f}'(\mathbf{x}(t))\mathbf{f}\left(\mathbf{x}(t)\right) + \dots$$

From this we devise the second-order Runge-Kutta method

$$\mathbf{k}_1 = \Delta t\mathbf{f}\left(\mathbf{x}(t)\right),\ \mathbf{k}_2 = \Delta t\mathbf{f}\left(\mathbf{x}(t) + \frac{1}{2}\mathbf{k}_1\right),\ \mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \mathbf{k}_2.$$

Similarly there is a fourth-order scheme

$$\mathbf{k}_1 = \Delta t\mathbf{f}\left(\mathbf{x}(t)\right),\ \mathbf{k}_2 = \Delta t\mathbf{f}\left(\mathbf{x}(t) + \frac{1}{2}\mathbf{k}_1\right),\ \mathbf{k}_3 = \Delta t\mathbf{f}\left(\mathbf{x}(t) + \frac{1}{2}\mathbf{k}_2\right),\ \mathbf{k}_4 = \Delta t\mathbf{f}\left(\mathbf{x}(t) + \mathbf{k}_3\right),$$
$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \frac{1}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right),$$

with an accumulated error of order $(\Delta t)^4$.

Symplectic Methods Symplectic methods are numerical integration schemes that respect conservation laws.