Summary of SI2390 Relativistic Quantum Physics

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Abstract

This is a summary of SI2390.
We will use units such that $c = \hbar = 1$.

Contents

1 Tasty Bits of Special Relativity

Metric Signature We use the metric signature $(1, -1, -1, -1)$ for the Minkowski metric.

The Levi-Civita Tensor We use the convention $\varepsilon^{0123} = 1$.

The Poincare Group Elements of the Poincare group are specified by a Lorentz transformation Λ and a translation a. Its elements follow the multiplication rule

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2).$$

We may instead construct a representation of the Poincare group with matrices of the form

$$\begin{bmatrix} \Lambda & a \\ 0 & 1 \end{bmatrix},$$

from which the multiplication rule directly follows.

The Lie Algebra of the Lorentz Group The Lorentz group is defined as the set of transformations such that

$$g = \Lambda^T g \Lambda.$$

There are a maximum of 16 generators, meaning we may label them using our index convention. Expanding around the identity we find

$$g = (1 + \omega_{\mu\nu} M^{\mu\nu})^T g (1 + \omega_{\rho\sigma} M^{\rho\sigma}) \approx g + \omega_{\mu\nu} (M^{\mu\nu})^T g + g\omega_{\rho\sigma} M^{\rho\sigma},$$

implying

$$\omega_{\mu\nu}((M^{\mu\nu})^T g + g M^{\mu\nu}) = 0,$$

or

$$M^T g = -gM$$

for all generators. Constructing the generator in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and using the fact that the Minkowski metric is its own universe we find

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} g = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix} = \begin{bmatrix} -A^T & -C^T \\ -B^T & -D^T \end{bmatrix}.$$

The solutions to this have antisymmetric blocks A and D, as well as off-diagonal blocks that are transposes of each other. There are six degrees of freedom for this solution, meaning that the Lorentz group has six degrees of freedom, corresponding to the three rotations and boosts. To preserve the index notation, we may then choose the generators such that $M^{\mu\nu} = -M^{\nu\mu}$. The corresponding choice of parameters must then also be antisymmetric. To get the appropriate amounts of terms we will also divide by 2, as you will see in the following section.

To more explicitly introduce the boosts and rotations, we introduce their generators

$$J^i = -\frac{1}{2}\varepsilon^{ijk} M^{jk},\ K^i = M^{0i},$$

with commutation relations

$$\left[J^i, J^j\right] = i\varepsilon^{ijk} J^k,\ \left[K^i, K^j\right] = -i\varepsilon^{ijk} J^k,\ \left[J^i, K^j\right] = i\varepsilon^{ijk} K^k.$$

We can solve for the original generators as

$$M^{0i} = J^i,\ M^{ij} = \varepsilon^{kij} J^k.$$

Generators of the Poincare Group The generators of the Poincare group are the $M^{\mu\nu}$ of the Lorentz group, as well as the four P^μ that generate translations in spacetime. We will need their Lie algebra, and thus their commutation relations, which are

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho}\right), \; [P^\mu, P^\nu] = 0, \; [M^{\mu\nu}, P^\sigma] = i(g^{\nu\sigma}P^\mu - g^{\mu\sigma}P^\nu).$$

The representations U of the group elements are then

$$U(\Lambda, 0) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}, \; U(1, a) = e^{ia_\mu P^\mu},$$

and to first order

$$U(\Lambda, a) = e^{i\left(a_\mu P^\mu - \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}\right)}.$$

2 Basic Concepts

Casimir Operators A Casimir operator is an operator that is constructed from the generators of a group and commutes with all generators.

Casimir Operators of the Poincare Group The Casimir operators of the Poincare group are

$$P^2 = P^\mu P_\mu, \; w^2 = w^\mu w_\mu,$$

where we have introduced the Pauli-Lubanski vector

$$w_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma.$$

It can be shown that

$$w_0 = \mathbf{P}\cdot\mathbf{J}, \; \mathbf{w} = P_0\mathbf{J} + \mathbf{P}\times\mathbf{K}.$$

The Wigner Classification As we will consider unitary representations of the Poincare group acting on states and the representations can be decomposed into irreps, we will find that we can reduce our considerations to a set of fundamental systems, termed particles. The classification, divided according to the eigenvalues of P^2 and w^2, is according to the Wigner system:

1. $P^2 > 0$, with subclasses:
   - $P^0 < 0$.
   - $P^0 > 0$.
2. $P^2 = 0$, with subclasses:
   - $P^0 < 0$.
   - $P^0 > 0$.
3. $P^2 = 0$ and $P^0 = 0$.
4. $P^2 < 0$, corresponding to tachyons.

Lorentz Covariance and the Schrödinger Equation Using the 4-momentum $P^\mu = (E, \mathbf{p})$ and the correspondence principle $P^\mu = i\partial^\mu$, the quantization of the classical energy $E = \frac{1}{2m}\mathbf{p}^2$ of a free particle is

$$i\partial_t\Psi = -\frac{1}{2m}\nabla^2\Psi.$$

This does not in general respect Lorentz transformations, which one might expect given that it is not taken from a Lorentz covariant expression. In other words, the Schrödinger equation is not Lorentz covariant.

The quantization of the relativistic $E^2 = m^2 + \mathbf{p}^2$ is instead

$$-\partial_t^2\phi = m^2\phi - \nabla^2\phi.$$

By introducing the d'Alembertian $\Box = \partial_\mu\partial^\mu$ we can write the above as

$$\Box\phi + m^2\phi = 0.$$

This is the Klein-Gordon equation, which is an appropriate quantization of a spinless particle.

A Conserved Current Corresponding to the Klein-Gordon equation there exists a density and a current

$$\rho = \frac{i}{2m}({}^\star\phi\partial_0\phi - \phi\partial_0{}^\star\phi),\ \mathbf{j} = \frac{1}{2im}({}^\star\phi\vec{\boldsymbol{\nabla}}\phi - \phi\vec{\boldsymbol{\nabla}}^\star\phi)$$

such that

$$\partial_t\rho + \vec{\boldsymbol{\nabla}} \cdot \mathbf{j} = 0.$$

Alternatively, by combining the two into a 4-current $J^\mu = (\rho, \mathbf{j})$ we find

$$\partial_\mu J^\mu = 0.$$

Problems With Stationary States A stationary state is a state such that

$$P^0\phi = E\phi.$$

For such a state we have

$$J^0 = \frac{E}{m}|\phi|^2.$$

In the classical limit we have $\frac{E}{m} \approx 1$, whereas in the general case we have $E = \pm\sqrt{m^2 + \mathbf{p}^2}$, meaning that J^0 is not positive definite and the conserved Nöether cannot be interpreted as conservation of probability density. This implores us to reinterpret the Klein-Gordon equation as a general field equation.

Plane-Wave Solutions Plane-wave solutions of the Klein-Gordon equation are of the form

$$\phi = Ne^{-iP_\mu x^\mu}.$$

In order for these to be solutions, we require

$$P^0 = \pm\sqrt{m^2 + |\mathbf{p}|}.$$

This does not pose a problem in non-interacting cases, as the solutions maintain their signs.

Charged Particles When treating charged particles in external electromagnetic fields, we employ the minimal coupling scheme and perform the replacement $P^\mu \to P^\mu - qA^\mu$. The Klein-Gordon equation then becomes

$$((\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2)\phi = 0.$$

This will cause additional terms

$$J^\mu \to J^\mu - \frac{q}{m}|\phi|^2 A^\mu$$

in the Nöether current, further destroying our hopes of creating a one-particle theory.

The Klein Paradox Consider scattering after normal incidence on a step potential described by $A^\mu = (V\theta(x), \mathbf{0})$. Performing an anzats similar to that in the non-relativistic case, the Klein-Gordon equation predicts the same behaviour as the Schrödinger equation, except for the case where $V > E + m$. In this case the transmitted 4-momentum has a negative space component. Furthermore, the transmission probability becomes negative, but still preserving $T + R = 1$. This peculiar behaviour is known as Klein's paradox.

The Dirac Equation We will now try to develop a theory that remedies the problems with the Klein-Gordon equation. The hope is that this equations has a positive-definite conserved density. An important source of the bad time was the second-order time derivative, so we will try to remedy this with a first-order time derivative. We also make the space derivatives first-order, perhaps because of Lorentz stuff. This leads us to the anzats

$$\partial^t_\Psi + (\boldsymbol{\alpha} \cdot \vec{\boldsymbol{\nabla}})\Psi + im\beta\Psi = 0,$$

where β and α^i are matrices and Ψ is a vector. The sizes of these are yet to be determined. The corresponding equation for $\Psi^\dagger$ is

$$\partial^t_{\Psi^\dagger} + (\vec{\boldsymbol{\nabla}}\Psi^\dagger) \cdot \boldsymbol{\alpha}^\dagger - im\Psi^\dagger\beta^\dagger = 0.$$

Considering the quantity $\Psi^\dagger\Psi$ we have

$$\begin{aligned}\partial^t_{(\Psi^\dagger\Psi)} &= (\partial^t_{\Psi^\dagger})\Psi + \Psi^\dagger\partial^t_\Psi \\ &= (im\Psi^\dagger\beta^\dagger - (\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha})\Psi + \Psi^\dagger(-(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi - im\beta\Psi) \\ &= im\Psi^\dagger(\beta^\dagger - \beta) - (\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha}^\dagger\Psi - \Psi^\dagger(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi.\end{aligned}$$

We really want the right-hand side to be the 3-divergence of some 3-current. To do that, we may choose α^i and β to be Hermitian, yielding

$$(\vec{\boldsymbol{\nabla}}\Psi^\dagger)\cdot\boldsymbol{\alpha}^\dagger\Psi + \Psi^\dagger(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi = \vec{\boldsymbol{\nabla}}\cdot\Psi^\dagger\boldsymbol{\alpha}\Psi.$$

The conserved 4-current is thus $j^\mu = (\Psi^\dagger\Psi, \Psi^\dagger\boldsymbol{\alpha}\Psi)$.

To reobtain something like the 4-vector norm we had when discussing the Klein-Gordon equation, we apply the operator $\partial^t - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}) - im\beta\Psi$ to our anzats to find

$$(\partial^t - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}) - im\beta\Psi)(\partial^t_\Psi + (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi + im\beta\Psi) = 0.$$

As the derivatives commute with the matrices, the cross terms vanish, yielding

$$\partial^[_2]t\Psi - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})^2\Psi - (\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})im\beta\Psi - im\beta(\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}})\Psi + m^2\beta^2\Psi = 0,$$

or more explicitly

$$\partial_t^2\Psi - (\alpha^i\partial_i)(\alpha^j\partial_j)\Psi - im((\alpha^i\partial_i)\beta + \beta\alpha^i\partial_i)\Psi + m^2\beta^2\Psi = \partial_t^2\Psi - \alpha^i\alpha^j\partial_i\partial_j\Psi - im(\alpha^i\beta + \beta\alpha^i)\partial_i\Psi + m^2\beta^2\Psi = 0.$$

We can symmetrize the second term to find

$$\partial_t^2\Psi - \frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial_i\partial_j\Psi - im(\alpha^i\beta + \beta\alpha^i)\partial_i\Psi + m^2\beta^2\Psi = 0.$$

This produces the same 4-vector norm if

$$\left\{\alpha^i, \alpha^j\right\} = 2\delta^{ij},\ \beta^2 = 1,\ \alpha^i\beta + \beta\alpha^i = 0.$$

Computing the determinant of the last equation, we find

$$\det\left(\alpha^i\beta\right) = (-1)^N\det\left(\beta\alpha^i\right),$$

where N is the length of Ψ. The only way for the above equations to be solvable is then that N be odd. It can also be shown that α^i and β are all traceless. By a series of arguments we find that $N = 4$ is correct.

To complete our discussion, we define $\gamma^0 = \beta,\ \gamma^i = \beta\alpha^i$. With this we multiply our anzats by $-i\beta$ and find

$$(-i\beta\partial^0 - i(\beta\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}}))\Psi + m\Psi = 0.$$

Defining the inner product $\gamma^\mu A_\mu = \not{A}$ we arrive at the Dirac equation

$$i\not{\partial}\Psi - m\Psi = 0.$$

Properties of the γ^μ The γ^μ satisfy

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

Defining the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, we find that it must be Hermitian.

We also have

$$\left\{\gamma^5, \gamma^\mu\right\} = 0.$$

We have

$$\operatorname{tr}\left(\prod_{i=1}^n \gamma^{\mu_i}\right) = 0,\ n \text{ odd},$$
$$\operatorname{tr}\left(\gamma^5\right) = 0.$$

The Dirac Algebra The γ^μ are a representation of the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

We may compute explicit representations of this algebra as

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

In this representation we have

$$\gamma^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The Dirac Adjoint The Dirac adjoint is defined as

$$\bar{A} = A^\dagger \gamma^0.$$

Rewriting the 4-Current We may use the properties of the γ^μ to write

$$j^\mu = \Psi^\dagger \gamma^0 \gamma^\mu \Psi = \bar{\Psi} \gamma^\mu \Psi.$$

A Free Dirac Particle For a free we multiply the operator $\not{p} - m$ by its conjugate to find

$$(\not{p} + m)(\not{p} - m) = \not{p}^2 - m^2.$$

We have

$$\not{p}^2 = \gamma^\mu \gamma^\nu p_\mu p_\nu = \gamma^\mu \gamma^\nu p_\nu p_\mu = \gamma^\nu \gamma^\mu p_\mu p_\nu,$$

hence

$$\not{p}^2 - m^2 = \frac{1}{2}\{\gamma^\mu, \gamma_\nu\} p_\mu p_\nu - m^2 = g^{\mu\nu} p_\mu p_\nu - m^2 = p^2 - m^2.$$

The Dirac particle has now been diagonalized, and it is in fact separable in terms of its spacetime dependence. The solution is thus a plane wave times a spinor - more specifically,

$$\Psi = e^{-ipx} u,$$

where u is some spinor. The 4-momentum satisfies the previously derived relation. It can also be shown for such a particle that $\mathcal{H} = i\partial_0$, and thus has eigenvalue $p_0 = E$. The sign of this eigenvalue is yet to be determined, and will in fact produce solutions with positive and negative energy.

Given the energy eigenvalues, the form of the spinors is determined by the Dirac equation itself. As we have

$$\gamma^\mu p_\mu = E \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} - \mathbf{p} \cdot \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} = \begin{bmatrix} E & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & -E \end{bmatrix},$$

hence we may write the spinor in block form to find the equation

$$\begin{bmatrix} E - m & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & -E - m \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} (E - m)u_A - \mathbf{p} \cdot \boldsymbol{\sigma} u_B \\ \mathbf{p} \cdot \boldsymbol{\sigma} u_A - (E + m)u_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Inserting the two equations into the other, we find

$$u_A = \frac{1}{E^2 - m^2} (\mathbf{p} \cdot \boldsymbol{\sigma})^2 u_A.$$

The matrix quantity on the right is given by

$$(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = \sigma^i \sigma^j p_i p_j = (2\delta^{ij} - \sigma^j \sigma^i) p_i p_j = 2\mathbf{p}^2 - \sigma^j \sigma^i p_j p_i,$$

implying

$$u_A = \frac{\mathbf{p}^2}{E^2 - m^2} u_A,$$

which in combination with the previously derived energy-momentum relation implies that we may freely choose a basis for the top half. As the same holds for the lower parts, we will for each p find a set of solutions by choosing one half of the spinor and computing the other from that. We have

$$\mathbf{p} \cdot \sigma = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix},$$

and the four solutions are thus

$$\begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{bmatrix}, \begin{bmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{p_x-ip_y}{E-m} \\ -\frac{p_z}{E-m} \\ 0 \\ 1 \end{bmatrix}.$$

At this point we must return to the sign of E. The two former solutions blow up if E is chosen to be negative, whereas the latter ones blow up if E is positive. Thus we have achieved the assignment of energy signs to the different spinors.

What is the interpretation of this? Dirac's first interpretation involved the so-called Dirac sea, but the current understanding is that the negative energy solutions represent antiparticle states with positive energy.

Plane-Wave Solutions Plane-wave solutions of the Dirac equation are of the form

$$\Psi_P = e^{-iP_\mu x^\mu} u(P^\mu),$$

where $u(P^\mu)$ is a so-called spinor. Inserting it into the Dirac equation we find

$$(-\not{P} + m)u(P^\mu) = 0.$$

Multiplying by $\not{P} + m$ we find

$$(-\not{P}^2 + m^2)u(P^\mu) = (-\gamma^\mu\gamma^\nu P_\mu P_\nu + m^2)u(P^\mu) = 0.$$

We can symmetrize the first term and use the anticommutation relations of the γ^μ to find

$$(-P^2 + m^2)u(P^\mu) = 0.$$

In other words, the solution satisfies the relativistic energy-momentum relation. This also implies that for non-trivial solutions, the 4-momentum is time-like.

In the corresponding rest frame, there are four independent spinor solutions. These are $u_\pm$ and $v_\pm$, and with this representation they are as you would expect.

A Hamiltonian As the plane-wave solutions have time-like 4-momenta, there is a corresponding rest frame. In this rest frame, the Hamiltonian, which is generally

$$\mathcal{H} = \beta m + \boldsymbol{\alpha} \cdot \mathbf{p},$$

reduces to

$$\mathcal{H} = \beta m.$$

Defining

$$\mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix},$$

we have $[\mathcal{H}, \mathbf{\Sigma}] = 0$.

Lorentz Transformation of Spinors Suppose that Ψ is a solution to the Dirac equation. Under a Lorentz transform it should transform according to

$$\Psi'(x') = S(\Lambda)\Psi(x).$$

We would like to identify the transformation matrix S.

Because the Dirac equation is Lorentz covariant, it looks the same in all frames. We can use the chain rule to find

$$\partial^\mu = \Lambda^{\mu'}{}_\mu \partial^{\mu'}$$

and thus

$$(-i\gamma^\mu\partial^\mu + m)\Psi = (-i\gamma^\mu\Lambda^{\mu'}{}_\mu\partial^{\mu'} + m)S^{-1}\Psi' = 0.$$

Note that as m is a Lorentz scalar, it is also the same in all frames. Multiplication by S yields

$$(-i\Lambda^{\mu'}{}_\mu S\gamma^\mu S^{-1}\partial^{\mu'} + m)\Psi' = 0.$$

Comparing this with the Dirac equation in the primed frame, we must therefore have

$$\Lambda^{\mu'}{}_\mu S\gamma^\mu S^{-1} = \gamma^{\mu'} \iff \Lambda^{\mu'}{}_\mu\gamma^\mu = S^{-1}\gamma^{\mu'}S.$$

To proceed, we apply the expansion

$$\Lambda^{\mu'}{}_\mu = \delta^{\mu'}_\mu + \varepsilon\omega^{\mu'}{}_\mu,\ \omega^{\mu\nu} = -\omega^{\nu\mu},$$

and expand S in terms of ε as

$$S = 1 - \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu},\ \sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

for some set of matrices $\sigma_{\mu\nu}$. We see that its inverse to first order must be

$$S^{-1} = 1 + \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu}.$$

We now have

$$\begin{aligned}(\delta^{\mu'}_\mu + \varepsilon\omega^{\mu'}{}_\mu)\gamma^\mu &= \left(1 + \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu}\right)\gamma^{\mu'}\left(1 - \frac{i}{4}\varepsilon\omega^{\mu\nu}\sigma_{\mu\nu}\right) \\ &= \gamma^{\mu'} + \frac{i}{4}\varepsilon\omega^{\mu\nu}(\sigma_{\mu\nu}\gamma^{\mu'} - \gamma^{\mu'}\sigma_{\mu\nu}),\end{aligned}$$

or

$$\frac{i}{4}\omega^{\mu\nu}\left[\sigma_{\mu\nu}, \gamma^{\mu'}\right] = \omega^{\mu'}{}_\mu\gamma^\mu = \omega^{\mu'\mu}\gamma_\mu.$$

To remove the $\omega^{\mu\nu}$ we will use the fact that

$$\omega^{\mu'}{}_\nu\gamma^\nu = g^{\mu'\mu}\omega_{\mu\nu}\gamma^\nu = -g^{\mu\mu'}\omega_{\nu\mu}\gamma^\nu = -g^{\nu\mu'}\omega_{\mu\nu}\gamma^\mu = -g^{\mu'\nu}\omega_{\mu\nu}\gamma^\mu,$$

hence

$$\frac{i}{4}\omega^{\mu\nu}\left[\sigma_{\mu\nu}, \gamma^{\mu'}\right] = \frac{1}{2}\omega_{\mu\nu}(g^{\mu'\mu}\gamma^\nu - g^{\mu'\nu}\gamma^\mu) = \frac{1}{2}\omega^{\mu\nu}(g^{\mu'}{}_\mu\gamma_\nu - g^{\mu'}{}_\nu\gamma_\mu),$$

which must finally imply

$$\left[\gamma^{\mu'}, \sigma_{\mu\nu}\right] = 2i(g^{\mu'}{}_\mu\gamma_\nu - g^{\mu'}{}_\nu\gamma_\mu).$$

The solution to this is

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu].$$

Charged Particles When introducing charge, we employ the minimal coupling scheme. The Dirac equation then becomes

$$(-\gamma^\mu(i\partial_\mu - qA_\mu) + m)\Psi = (\not{p} - q\not{A} - m)\Psi = 0.$$

To proceed we write the solution as

$$\Psi = (\not{p} - q\not{A} + m)\chi,$$

yielding

$$(\not{p} - q\not{A} - m)(\not{p} - q\not{A} + m)\chi = ((\not{p} - q\not{A})^2 - m^2)\chi = 0.$$

We investigate the operator appearing in this modified solution as

$$\begin{aligned}
(\not{p} - q\not{A})^2 &= \gamma^\mu\gamma^\nu(p_\mu - qA_\mu)(p_\nu - qA_\nu) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu(p_\mu - qA_\mu)(p_\nu - qA_\nu) + \gamma^\nu\gamma^\mu(p_\nu - qA_\nu)(p_\mu - qA_\mu)\right) \\
&= \frac{1}{2}\left(\gamma^\mu\gamma^\nu(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) + \gamma^\nu\gamma^\mu(p_\nu p_\mu - q(p_\nu A_\mu + A_\nu p_\mu) + q^2 A_\nu A_\mu)\right) \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\}(p_\mu p_\nu + q^2 A_\mu A_\nu) - q(\gamma^\mu\gamma^\nu(p_\mu A_\nu + A_\mu p_\nu) + \gamma^\nu\gamma^\mu(p_\nu A_\mu + A_\nu p_\mu))\right) \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\}(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) - q\gamma^\nu\gamma^\mu(p_\nu A_\mu + A_\nu p_\mu - p_\mu A_\nu - A_\mu p_\nu)\right) \\
&= \frac{1}{2}\left(2g^{\mu\nu}(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) - q\gamma^\nu\gamma^\mu([p_\nu, A_\mu] - [p_\mu, A_\nu])\right).
\end{aligned}$$

We have

$$[p_\mu, A_\nu] = i((\partial_\mu A_\nu) + A_\nu\partial_\mu - A_\nu\partial_\mu) = i(\partial_\mu A_\nu),$$

and thus

$$\begin{aligned}
(\not{p} - q\not{A})^2 &= \frac{1}{2}\left(2g^{\mu\nu}(p_\mu p_\nu - q(p_\mu A_\nu + A_\mu p_\nu) + q^2 A_\mu A_\nu) - iq\gamma^\nu\gamma^\mu(\partial_\nu A_\mu - \partial_\mu A_\nu)\right) \\
&= (p - qA)^2 + \frac{i}{2}q\gamma^\nu\gamma^\mu F_{\mu\nu}.
\end{aligned}$$

As

$$\gamma^\nu\gamma^\mu = \frac{1}{2}([\gamma^\nu, \gamma^\mu] + \{\gamma^\nu, \gamma^\mu\}),$$

where the first term is symmetric and the second is antisymmetric, we have

$$(\not{p} - q\not{A})^2 = (p - qA)^2 + \frac{1}{2}q\sigma^{\nu\mu}F_{\mu\nu} = (p - qA)^2 - \frac{1}{2}q\sigma^{\mu\nu}F_{\mu\nu}.$$

It can be shown that

$$\frac{1}{2}q\sigma^{\mu\nu}F_{\mu\nu} = -q(\mathbf{\Sigma}\cdot\mathbf{B} - i\boldsymbol{\alpha}\cdot\mathbf{E}),$$

where the first term is interpreted as a magnetic dipole contribution and the second as an electric monopole contribution.

The Hydrogenic Atom A system with a Coulomb potential $V(r) = -\frac{Ze^2}{4\pi r}$ is an example of an exactly solvable model with the Dirac formalism. The corresponding Hamiltonian is

$$H = \beta m + \boldsymbol{\alpha}\cdot\mathbf{p} + V(r),$$

which commutes with $\mathbf{J}$ and the parity operator P. We thus seek simultaneous eigenstates of $H, \mathbf{J}^2, J^3$ and P. The corresponding eigenvalues are $E, j(j+1), m$ and $(-1)^{j+\frac{\varpi}{2}}$, where

$$\varpi = \begin{cases} 1, & P = (-1)^{j+\frac{1}{2}}, \\ -1, & P = (-1)^{j-\frac{1}{2}}. \end{cases}$$

As the problem is divided into blocks, we write the desired states as

$$\Psi = \begin{bmatrix} \phi \\ \chi \end{bmatrix}.$$

The angular part may be separated out, and corresponding to them are two sets of solutions $\mathcal{Y}_{lj}^m$, where l is the eigenvalue of L and takes on values $j \pm \frac{1}{2}\bar{\omega}$ for the two sets of solutions. Explicitly the solutions are

$$\mathcal{Y}_{lj}^m = \begin{bmatrix} \sqrt{\frac{j+m}{2j}} Y_{j-\frac{1}{2}, m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}} Y_{j-\frac{1}{2}, m+\frac{1}{2}} \end{bmatrix}, \ j = l + \frac{1}{2}$$

and

$$\mathcal{Y}_{lj}^m = \begin{bmatrix} \sqrt{\frac{j-m+1}{2(j+1)}} Y_{j+\frac{1}{2}, m-\frac{1}{2}} \\ -\sqrt{\frac{j+m+1}{2(j+1)}} Y_{j+\frac{1}{2}, m+\frac{1}{2}} \end{bmatrix}, \ j = l - \frac{1}{2}, \ l > 0.$$

To proceed, we make the anzats

$$\phi = \frac{1}{r} F(r) \mathcal{Y}_{lj}^m, \ \chi = \frac{1}{r} G(r) \mathcal{Y}_{lj}^m.$$

We also introduce

$$\mathbf{r} \cdot \mathbf{p} = -ir\partial_r, \ p_r = -\frac{i}{r}\partial_r = \frac{1}{r}(\mathbf{r} \cdot \mathbf{p} - i),$$

as well as

$$\alpha_r = \frac{1}{r} \boldsymbol{\alpha} \cdot \mathbf{r},$$

from which one can show

$$\boldsymbol{\alpha} \cdot \mathbf{p} = \alpha_r (p_r + \frac{i}{r} \beta K),$$

where $K = \beta(\boldsymbol{\Sigma} \cdot \mathbf{L} + 1)$, with eigenvalues $-\bar{\omega}\left(j + \frac{1}{2}\right)$. The Hamiltonian can now be written as

$$H = \alpha_r (p_r - \frac{i\bar{\omega}\left(j + \frac{1}{2}\right)}{r} \beta) + \beta m + V(r).$$

The eigenvalue equation then becomes

$$\left(\begin{bmatrix} 0 & -\partial_r \\ \partial_r & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\bar{\omega}(j+\frac{1}{2})}{r} \\ \frac{i\bar{\omega}(j+\frac{1}{2})}{r} & 0 \end{bmatrix} \right) \begin{bmatrix} F \\ G \end{bmatrix} = (E - m - V(r)) \begin{bmatrix} F \\ G \end{bmatrix}.$$

In the particular case of the hydrogenic atom, we introduce the following notation:

$$\kappa = \sqrt{m^2 - E^2}, \ \rho = \kappa r, \ \tau = \bar{\omega}\left(j + \frac{1}{2}\right), \nu = \sqrt{\frac{m - E}{m + E}}, \ Z' = \frac{Ze^2}{4\pi}$$

to find

$$\left(\begin{bmatrix} 0 & -\partial_\rho \\ \partial_\rho & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\tau}{\rho} \\ \frac{\tau}{\rho} & 0 \end{bmatrix} \right) \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} -\nu + \frac{Z'}{\rho} & 0 \\ 0 & \frac{1}{\nu} + \frac{Z'}{\rho} \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}.$$

We can show that close to the origin the functions behave like power laws, and at infinity they decay exponentially. We thus make the anzats

$$F = e^{-\rho} f, \ G = e^{-\rho} g$$

and introduce

$$w = \begin{bmatrix} f \\ g \end{bmatrix}, \ A = \begin{bmatrix} -\tau & Z' \\ Z' & \tau \end{bmatrix}, \ B = \begin{bmatrix} 1 & \frac{1}{\nu} \\ \nu & 1 \end{bmatrix},$$

from which we obtain

$$\rho\partial_\rho w = (A + \rho B)w.$$

We solve this problem with a Frobenius anzats

$$w = \rho^\mu \sum_{s=0}^{N} w_s \rho^s,$$

where N is some number to be determined. The recursion relation one obtains is

$$\mu w_0 = A w_0,\ (s + \lambda - A)w_s = B w_{s-1},$$

where $\lambda = \sqrt{\tau^2 - (Z')^2}$ is the magnitude of the eigenvalues of A. It can be shown that w converges and that N is finite. More specifically it holds that

$$N + \lambda - \frac{1}{2}Z'\left(\frac{1}{\nu} - \nu\right) = 0,$$

implying

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \frac{Z^2\alpha^2}{N + \sqrt{\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2}}}},\ N + j + \frac{1}{2} = 1, \dots, n,$$

where n is termed the little quantum number and α is the fine structure constant.

Expanding this we find

$$\frac{E}{m} \approx 1 - \frac{1}{2}\frac{Z^2\alpha^2}{\left(j+\frac{1}{2}\right)^2} + \dots = 1 - \frac{1}{2}\frac{Z^2\alpha^2}{n^2}\left(1 + \frac{Z^2\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right),$$

which is similar to the result found with perturbation theory of the non-relativistic problem, except for the angular momentum-dependence of the perturbation.

Chirality For a massless particle we have

$$i\partial\!\!\!/\Psi = 0.$$

Choosing the so-called chiral representation

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

we find the Dirac equation to be block off-diagonal, meaning that the upper and lower halves of the spinor are decoupled. The Dirac equation for the two halves becomes

$$i\partial_0\Psi_2 - i\boldsymbol{\sigma}\cdot\vec{\boldsymbol{\nabla}}\Psi_2 = 0,\ i\partial_0\Psi_1 + i\boldsymbol{\sigma}\cdot\vec{\boldsymbol{\nabla}}\Psi_1 = 0,$$

or alternatively, using the correspondence principle and introducing $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\tilde{\sigma}^\mu = (1, -\boldsymbol{\sigma})$,

$$\sigma^\mu p_\mu \Psi_2 = 0,\ \tilde{\sigma}^\mu p_\mu \Psi_1 = 0.$$

The spinors constructed with only one of these are eigenstates of γ^5, Ψ_1 having eigenvalue -1 and Ψ_2 having eigenvalue 1. These eigenvalues are called the chirality of the state. The chosen representation is called the chiral representation because in this representation the Dirac equation decouples with respect to chiral eigenstates. Introducing the total state

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$

and the chiral states

$$\Psi_+ = \begin{bmatrix} 0 \\ \Psi_2 \end{bmatrix}, \ \Psi_- = \begin{bmatrix} \Psi_1 \\ 0 \end{bmatrix}$$

we find

$$\Psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\Psi,$$

which defines the chiral projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5).$$

Inserting a plane-wave solution $u_p e^{-ipx}$ we find for the negative-chirality part that

$$p^0 u_p = -\boldsymbol{\sigma} \cdot \mathbf{p} u_p,$$

which squares to imply

$$(p^0)^2 - \mathbf{p}^2 = 0,$$

with solutions $p^0 = \pm|\mathbf{p}|$. Similarly we find

$$p^0 u_p = \boldsymbol{\sigma} \cdot \mathbf{p} u_p$$

for the positive-chirality state, which has the same solution. The case $p^0 = |\mathbf{p}|$ is called a Weyl state, whereas the other case is called an anti-Weyl state.

Helicity (and Chirality) Helicity is defined as the scalar product between the 3-momentum and the spin. This definition holds for both massive and massless particles. States with parallel spin and 3-momentum are said to have right helicity, and the opposite is termed left helicity. Repeating the derivation we did for massless particles, we find

$$\sigma^\mu p_\mu \Psi_\mathrm{R} = m\Psi_\mathrm{L}, \ \tilde{\sigma}^\mu p_\mu \Psi_1 = m\Psi_\mathrm{R},$$

where we now denote the lower and upper halves of the spinor with what will manifest as the helicity.

For massless particles we now reconsider chirality in terms of helicity. For such particles, helicity manifests as left-chiral and right-chiral objects. Massless particles distinguish themselves in that the Dirac equation completely decouples the left- and right chiral parts, hence helicity is preserved by Lorentz transformations. Massive particles, by contrast, oscillate between the two, as indicated by the form of the Dirac equation.

We now consider massless particle solutions, which satisfy $p^0 = |\mathbf{p}|$. We find

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|}\Psi_\mathrm{R} = \Psi_\mathrm{R}, \ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|}\Psi_\mathrm{L} = -\Psi_\mathrm{L}.$$

These states are thus eigenstates of the helicity operator

$$h = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|}.$$

For antiparticles we instead have $p^0 = -|\mathbf{p}|$, hence right-handed antiparticles have negative helicity and vice versa.

What does helicity mean? It can be shown that the helicity indicates whether the spin and momentum are parallel or anti-parallel.

3 Introductory Quantum Field Theory

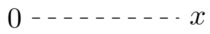
Quantum Field Theory The basic idea of quantum field theory is that operators are generated by operator-valued fields, meaning that operators are localized in this formalism. The interpretation of these fields is that they are creation and annihilation operators at different points. The plan is to obtain a Lagrangian density for a particular classical theory, find the equations of motions and solve the inhomogenous versions to find the propagators (to be discussed).

Propagators The propagator is the amplitude of probability from travelling between two points in spacetime. They act as Green's functions for the equations of motion to which they correspond.

One propagator is the Klein-Gordon propagator, defined as

$$i\Delta_{\mathrm{F}}\left(x\right) = \left\langle 0|T\phi(x)\phi(0)|0\right\rangle .$$

It can be represented as a Feynman diagram according to figure **??**m and is interpreted as a particle being



created at 0, propagated to x and annihilated there. Equivalently, it may be interpreted as an antiparticle doing the same in the opposite order.

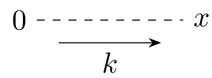
In Fourier space we may also introduce a propagator according to

$$i\Delta_{\mathrm{F}}\left(x\right) = \int \mathrm{d}^4k\,\frac{1}{(2\pi)^4}e^{-ik^\mu x_\mu}\frac{i}{k^2-m^2-i0},$$

and thus define

$$i\Delta_{\mathrm{F}}\left(k\right) = \frac{i}{k^2-m^2-i0}.$$

Note the constant $i0 = \lim_{\varepsilon=0}\varepsilon i$. In a Feynman diagram it looks like this:



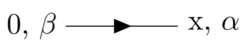
Another is the Dirac propagator, defined as

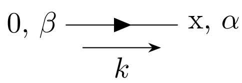
$$iS_{\mathrm{F},\alpha\beta}\left(x\right) = \left\langle 0|T\psi_\alpha(x)\bar{\Psi}_\beta(0)|0\right\rangle$$

in real space and

$$iS_{\mathrm{F},\alpha\beta}\left(k\right) = \frac{i}{k^2-m^2+i0}(\not{k}+m)_{\alpha\beta}$$

which is illustrated as in real space and in momentum space.





Finally there is the photon propagator

$$iD_{\mathrm{F},\mu\nu}\left(x\right) = \left\langle 0|TA_\mu(x)A_\nu(0)|0\right\rangle ,$$

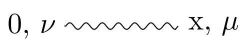
in real space and

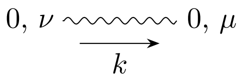
$$iD_{\mathrm{F},\mu\nu}\left(x\right) == -\frac{i}{k^2+i0}g_{\mu\nu} \tag{1}$$

in momentum space. Note that in other gauges one will have to perform the replacement

$$g_{\mu\nu} \to g_{\mu\nu} + k_\mu F_\nu(x),\ k_\nu F_\mu(x),$$

where the new terms depend on the gauge. These are represented by in real space and in momentum space.





Gauge Interactions There are three fundamental gauge interactions we will consider: electromagnetic, weak and strong interactions. These will all be mediated by spin-1 particles. We will use a minimal coupling scheme with a general coupling parameter g. We will generally find that this adds an interaction term to the Lagrangian density. As an example, for the Dirac Lagrangian density we will find

$$\mathcal{L} \to \mathcal{L} - g\bar{\Psi}\gamma_\mu \Psi A^\mu.$$

The Normal-Ordered Product The normal-ordered product of a set of fields is the product rearranged such that for each term, creation operators are found to the right. This implies that the normal-ordered product of fermionic fields carry sign information, whereas bosonic fields do not.

The Time-Ordered Product The time-ordered product of a set of fields is the product rearranged such that they appear in order of decreasing time from the left. This implies that the time-ordered product of fermionic fields carry sign information, whereas bosonic fields do not.

The Non-Relativistic String Consider a non-relativistic string with a coordinate x along its equilibrium position. To study it, we impose periodic boundary conditions and discretize the string, dividing it into N points separated by a distance a. Denoting the displacements at each point as ϕ_i and assigning each point a mass m, the total kinetic energy of the string is

$$T = \frac{1}{2}m \sum_{i=0}^{N-1} \left(\frac{\mathrm{d}\phi_i}{\mathrm{d}t}\right)^2.$$

Implementing a simple Hook-like tension in the string we also have a total potential energy

$$V = \frac{1}{2}k \sum_{i=0}^{N-1} (\phi_{i+1} - \phi_i)^2.$$

We will now consider a continuum limit, where N goes to infinity and a to zero such that $l = Na$ is constant. Introducing

$$\mu = \frac{m}{a}, \ \tau = ka$$

and relabelling the displacements by noting that

$$\phi_i(t) = \phi(t, x_i) = \phi(t, ia),$$

allowing us to use the label $\phi(t, x)$ in the continuum limit, we find that the kinetic energy is

$$T = \frac{1}{2}\mu \sum_{i=0}^{N-1} a \left(\frac{\mathrm{d}\phi_i}{\mathrm{d}t}\right)^2 \to \frac{1}{2}\mu \int_0^l \mathrm{d}z \, (\partial_t \phi)^2$$

and the potential energy is

$$V = \frac{1}{2}\frac{\tau}{a} \sum_{i=0}^{N-1} a^2 \left(\frac{\phi_{i+1} - \phi_i}{a}\right)^2 \to \frac{1}{2}\tau \int_0^l \mathrm{d}x \, (\partial_x \phi)^2.$$

The system is then described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mu \, (\partial_t \phi)^2 - \frac{1}{2}\tau (\partial_x \phi)^2.$$

Rescaling the field by a factor $\frac{1}{\sqrt{\tau}}$ and introducing $v^2 = \frac{\tau}{\mu}$ we have

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{v^2}(\partial_t\phi)^2 - (\partial_x\phi)^2\right).$$

The equations of motion is the discrete case are

$$-k(-\phi_{i+1} + 2\phi_i - \phi_{i-1}) - m\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} = 0,$$

which can be written as

$$\frac{m}{k}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} - (\phi_{i+1} - 2\phi_i + \phi_{i-1}) = \frac{m}{k}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} - a^2\frac{1}{a}\left(\frac{\phi_{i+1}-\phi_i}{a} - \frac{\phi_i - \phi_{i-1}}{a}\right) = 0.$$

We note that

$$\frac{m}{k} = \frac{a^2\mu}{\tau} = \frac{a^2}{v^2},$$

hence

$$\frac{1}{v^2}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2} - \frac{1}{a}\left(\frac{\phi_{i+1}-\phi_i}{a} - \frac{\phi_i - \phi_{i-1}}{a}\right) = 0.$$

The equations for the continuum limit are

$$\frac{1}{v^2}\partial_t^2\phi - \partial_x^2\phi = 0,$$

which are indeed the continuum limits of the discretized solutions.

The solutions of the equations of motion are so-called normal modes. They are denoted as having positive or negative frequency (or equivalently, energy). The solutions are

$$\phi_n = \frac{1}{\sqrt{l}}e^{i(k_n x - \omega_n t)},\ \phi_n^\star = \frac{1}{\sqrt{l}}e^{-i(k_n x - \omega_n t)},\ k_n = \frac{2\pi n}{l},\ \omega_n^2 = v^2 k_n^2.$$

Their normalization is

$$\int_0^l \mathrm{d}x\, \phi_n^\star \phi_m = \delta_{nm}.$$

A general solution can then be written as

$$\phi = \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{l}}\left(a_n(t)e^{ik_n x} + a_n^\star(t)e^{-ik_n x}\right),\ a_n(t) = a_n(0)e^{-i\omega_n t}.$$

The two terms correspond to positive and negative frequencies. The expansion coefficients thus satisfy

$$\frac{\mathrm{d}^2 a_n}{\mathrm{d}t^2} + \omega_n^2 a_n = 0,$$

and we treat them as simple harmonic oscillators.

This treatment is a bit unclear, so let us one-up it by computing the Lagrangian using the above expansion.

We have

$$L = \int_0^l \mathrm{d}x \, \frac{1}{2} \left(\frac{1}{v^2} \left(\sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{l}} \left(\dot{a}_n(t) e^{ik_n x} + \dot{a}_n^\star(t) e^{-ik_n x} \right) \right)^2 - \left(\sum_{n=-\infty}^{\infty} \frac{k_n c_n}{\sqrt{l}} \left(a_n(t) e^{ik_n x} - a_n^\star(t) e^{-ik_n x} \right) \right)^2 \right)$$

$$= \frac{1}{2l} \int_0^l \mathrm{d}x \, \frac{1}{v^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \left(\dot{a}_n(t) e^{ik_n x} + \dot{a}_n^\star(t) e^{-ik_n x} \right) \left(\dot{a}_m(t) e^{ik_m x} + \dot{a}_m^\star(t) e^{-ik_, x} \right)$$

$$- \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} k_n k_m c_n c_m \left(a_n(t) e^{ik_n x} - a_n^\star(t) e^{-ik_n x} \right) \left(a_m(t) e^{ik_m x} - a_m^\star(t) e^{-ik_m x} \right)$$

$$= \frac{1}{2l} \int_0^l \mathrm{d}x \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{c_n c_m}{v^2} \left(\dot{a}_n \dot{a}_m e^{i(k_n + k_m)x} + \dot{a}_n \dot{a}_m^\star e^{i(k_n - k_m)x} + \dot{a}_n^\star \dot{a}_m e^{-i(k_m - k_n)x} + \dot{a}_n^\star \dot{a}_m^\star e^{-i(k_m + k_n)x} \right)$$

$$- k_n k_m c_n c_m \left(a_n a_m e^{i(k_n + k_m)x} - a_n a_m^\star e^{i(k_n - k_m)x} - a_m a_n^\star e^{i(k_m - k_n)x} + a_n^\star a_m^\star e^{-i(k_n + k_m)x} \right)$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{c_n c_m}{v^2} \left(\dot{a}_n \dot{a}_m \delta_{n,-m} + \dot{a}_n \dot{a}_m^\star \delta_{n,m} + \dot{a}_n^\star \dot{a}_m \delta_{n,m} + \dot{a}_n^\star \dot{a}_m^\star \delta_{n,-m} \right)$$

$$- k_n k_m c_n c_m \left(a_n a_m \delta_{n,-m} - a_n a_m^\star \delta_{n,m} - a_m a_n^\star \delta_{n,m} + a_n^\star a_m^\star \delta_{n,-m} \right)$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{v^2} \left(c_n c_{-n} \dot{a}_n \dot{a}_{-n} + 2c_n^2 \dot{a}_n \dot{a}_n^\star + c_n c_{-n} \dot{a}_n^\star \dot{a}_{-n}^\star \right) - \left(k_n k_{-n} c_n c_{-n} a_n a_{-n} - 2k_n^2 c_n^2 a_n a_n^\star + k_n k_{-n} c_n c_{-n} a_n^\star a_{-n}^\star \right).$$

Restricting ourselves to real solutions we find

$$a_{-n} = a_n^\star, \; c_{-n} = c_n,$$

and the Lagrangian may be written as

$$L = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{4c_n^2}{v^2} \dot{a}_n \dot{a}_{-n} - 4k_n^2 c_n^2 a_n a_{-n} = 4 \sum_{n=0}^{\infty} \frac{c_n^2}{v^2} \dot{a}_n \dot{a}_{-n} - k_n^2 c_n^2 a_n a_{-n},$$

which is a Lagrangian in a set of discrete degrees of freedom. The corresponding canonical momenta are

$$p_{\pm n} = \frac{4c_n^2}{v^2} \dot{a}_{\mp n},$$

and the Hamiltonian is

$$H = \sum_{n=0}^{\infty} \frac{4c_n^2}{v^2} \dot{a}_{\mp n} - 4 \sum_{n=0}^{\infty} \frac{c_n^2}{v^2} \dot{a}_n \dot{a}_{-n} - k_n^2 c_n^2 a_n a_{-n} = 4 \sum_{n=0}^{\infty} k_n^2 c_n^2 a_n a_{-n} = 4 \sum_{n=0}^{\infty} k_n^2 c_n^2 a_n a_n^\star,$$

where we switched to the notation of complex conjugates to emphasize the distinctness of the degrees of freedom. We move the dimensionanlity away from the degrees of freedom by choosing

$$c_n^2 = \frac{v}{2k_n},$$

yielding

$$H = \sum_{n=-\infty}^{\infty} v k_n a_n^\star a_n = \sum_{n=-\infty}^{\infty} \omega_n a_n^\star a_n.$$

Introducing new canonical coordinates and momenta

$$q_n = \frac{1}{\sqrt{2\omega_n}} (a_n + a_n^\star), \; p_n = i \sqrt{\frac{\omega_n}{2}} (a_n - a_n^\star),$$

which invert to

$$a_n = \frac{1}{\sqrt{2\omega_n}} (\omega_n q_n + i p_n), \; a_n^\star = \frac{1}{\sqrt{2\omega_n}} (\omega_n q_n - i p_n),$$

we find

$$
\begin{aligned}
H &= \sum_{n=0}^{\infty} \frac{1}{2}(\omega_n q_n + i p_n)(\omega_n q_n - i p_n) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} p_n^2 + \omega_n^2 q_n^2,
\end{aligned}
$$

which is that of a system of harmonic oscillators. The coordinates and momenta satisfy canonical commutation relations, which means

$$
\begin{aligned}
\{a_n, a_m\} &= \frac{1}{2\sqrt{\omega_n\omega_m}} \left(\omega_n\omega_m\{q_n, q_m\} + i(\omega_n\{q_n, p_m\} + \omega_m\{p_n, q_m\}) - \{p_n, p_m\}\right) \\
&= \frac{i}{2\sqrt{\omega_n\omega_m}}(\omega_n\delta_{nm} - \omega_m\delta_{nm}) = 0, \\
\{a_n, a_m^\star\} &= \frac{1}{2\sqrt{\omega_n\omega_m}} \left(\omega_n\omega_m\{q_n, q_m\} + i(-\omega_n\{q_n, p_m\} + \omega_m\{p_n, q_m\}) + \{p_n, p_m\}\right) \\
&= \frac{i}{2\sqrt{\omega_n\omega_m}} \left(-\omega_n\delta_{nm} - \omega_m\delta_{nm}\right) = -i\delta_{nm}, \\
\{a_n^\star, a_m^\star\} &= \frac{1}{2\sqrt{\omega_n\omega_m}} \left(\omega_n\omega_m\{q_n, q_m\} - i(\omega_n\{q_n, p_m\} + \omega_m\{p_n, q_m\}) - \{p_n, p_m\}\right) \\
&= -\frac{i}{2\sqrt{\omega_n\omega_m}} \left(\omega_n\delta_{nm} - \omega_m\delta_{nm}\right) = 0,
\end{aligned}
$$

We also make a brief note of the Hamiltonian. As we know for the discrete system we have $\mathcal{H} = T + V$, so we would find the Hamiltonian density to be

$$
\mathcal{H} = \frac{1}{2}\mu\left(\partial_t\phi\right)^2 + \frac{1}{2}\tau(\partial_x\phi)^2.
$$

This doesn't really work, however. To do this properly, we will need the canonical momentum density

$$
\pi = \frac{\partial\mathcal{L}}{\partial\partial_t\phi} = \mu\partial_t\phi.
$$

The Hamiltonian density is thus

$$
\mathcal{H} = \frac{1}{\mu}\pi^2 - \frac{1}{2\mu}\pi^2 + \frac{1}{2}\tau(\partial_x\phi)^2 = \frac{1}{2\mu}\pi^2 + \frac{1}{2}\tau(\partial_x\phi)^2.
$$

Note that these also have canonical Poisson brackets

$$
\{\phi(t,x), \phi(t,y)\} = \{\pi(t,x), \pi(t,y)\} = 0,\ \{\pi(t,x), \pi(t,y)\} = \delta(x-y).
$$

Quantizing the Non-Relativistic String To quantize the relativistic string, we replace the expansion coefficients with operators and impose the canonical commutation relations - that is, the p and q commute and $[q_m, p_m] = i\delta_{nm}$. This leads to the a and $a^\dagger$ commuting, and $\left[a_n, a_m^\dagger\right] = \delta_{nm}$. The so-called quantum field is then

$$
\phi = v\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\omega_n l}} \left(a_n e^{i(k_n x - \omega_n t)} + a_n^\dagger e^{-i(k_n x - \omega_n t)}\right) = \phi^{(+)} + \phi^{(-)},
$$

where the two terms contain only creation and annihilation operators respectively. The Hamiltonian is now somehow

$$
H = \sum_{n=-\infty}^{\infty} \frac{1}{2}\omega_n(a_n^\dagger a_n + a_n a_n^\dagger) = \sum_{n=-\infty}^{\infty} \omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right).
$$

We may subtract the latter term, which is a vacuum energy contribution. Of course, this term does not come when directly quantizing the Hamiltonian we derived above. We may also introduce the number operators $\mathcal{N}_n = a_n^\dagger a_n$.

Quantizing the Klein-Gordon Field We start with the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi).$$

Expanding into eigenfunctions of the space parts, a general solution may be written as

$$\phi = \sum_n q_n(t) u_n(\mathbf{x}).$$

As we saw in the case of the non-relativistic string, the expansion coefficients evolve according to some Lagrangian. We may thus impose the canonical commutation relations on these by also introducing

$$p_m = \int \mathrm{d}^3\mathbf{x}\, u_m \pi,$$

where π is the momentum density. From these we introduce creation and annihilation operators according to

$$a_n = \frac{1}{\sqrt{2}}(q_n + i p_m),$$

which satisfy the correct commutation relations.

To proceed we start with the expansion

$$\phi = \int \mathrm{d}^4k\, a(k) e^{-ikx} \delta(k^2 - m^2).$$

We have introduced a Dirac delta to guarantee that only solutions of the Klein-Gordon equation are included. We can use the properties of the Dirac delta to obtain

$$\phi = \int \mathrm{d}^4k\, a(k) e^{-ikx} \frac{1}{|2k^0|} \left(\delta(k^0 - \omega_{\mathbf{k}}) + \delta(k^0 + \omega_{\mathbf{k}}) \right),$$

at least for the purposes of integration, where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Integrating we find

$$\phi = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \frac{1}{2\omega_{\mathbf{k}}} \left(a_k(t) e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^\star(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right),$$

where we have introduced a particular choice of normalization and done some renaming and relabelling.

We can obtain a nicer expression by introducing coefficients in terms of 3-momentum. We will take $a_k = c_k a_{\mathbf{k}}$. The canonical Poisson bracket is

$$\left\{ a_{\mathbf{k}}, a_{\mathbf{q}}^\star \right\} = -i\delta^3(\mathbf{k} - \mathbf{q}).$$

Taking this to be true, we compute the momentum density

$$\pi = \frac{i}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}\, \frac{1}{2} \left(-a_k(t) e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^\star(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right),$$

and find

$$\begin{aligned}
\{\phi(t, \mathbf{x}), \pi(t, \mathbf{y})\} &= \frac{i}{(2\pi)^6} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{q}\, \frac{1}{4\omega_{\mathbf{k}}} \left\{ a_k(t) e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^\star(t) e^{-i\mathbf{k}\cdot\mathbf{x}}, -a_q(t) e^{i\mathbf{q}\cdot\mathbf{y}} + a_q^\star(t) e^{-i\mathbf{q}\cdot\mathbf{y}} \right\} \\
&= \frac{i}{(2\pi)^6} \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}^3\mathbf{q}\, \frac{1}{2\omega_{\mathbf{k}}} e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} \left\{ a_k(t), a_q^\star(t) \right\} \\
&= \frac{1}{(2\pi)^6} \int \mathrm{d}^3\mathbf{k}\, \frac{|c_k|^2}{2\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{x} - \mathbf{y})},
\end{aligned}$$

and the choice

$$c_k = (2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\mathbf{k}}}$$

yields the correct Poisson bracket. We then rewrite the field as

$$\phi = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathbf{k}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + a^\star_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right).$$

The above expressions are ripe for quantization. To separate the time dependence from the creation and annihilation operators, we note that the Hamiltonian does not act on functions. As the creation and annihilation operators create eigenstates of the Hamiltonian, we have

$$\begin{aligned}
a_{\mathbf{k}}(t) &= e^{iHt}a_{\mathbf{k}}(0)e^{-iHt}\left|n_{\mathbf{k}},\ \text{rest}\right\rangle \\
&= e^{iHt}a_{\mathbf{k}}(0)e^{-it(n_{\mathbf{k}}\omega_{\mathbf{k}}+\text{rest})}\left|n_{\mathbf{k}},\ \text{rest}\right\rangle \\
&= \sqrt{n_{\mathbf{q}}}e^{it((n_{\mathbf{k}}-1)\omega_{\mathbf{k}}+\text{rest})}e^{-it(n_{\mathbf{k}}\omega_{\mathbf{k}}+\text{rest})}\left|n_{\mathbf{k}}-1,\ \text{rest}\right\rangle \\
&= e^{-it\omega_{\mathbf{k}}}\left|n_{\mathbf{k}},\ \text{rest}\right\rangle.
\end{aligned}$$

A similar argument can be performed for the annihilation operators, meaning that the creation and annihilation terms of the field are

$$\phi_+ = \int \mathrm{d}^3\mathbf{k}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}a_{\mathbf{k}}e^{-ikx},\ \ \phi_- = \int \mathrm{d}^3\mathbf{k}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}a^\dagger_{\mathbf{k}}e^{ikx}.$$

It is now clear that the two terms commute, even when evaluated at different points. We also have

$$\begin{aligned}
[\phi_+(x),\phi_-(y)] &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{q}\,\frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{q}}}}\left[a(\mathbf{k}),a^\dagger(\mathbf{q})\right]e^{-i(kx-qy)} \\
&= \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{q}\,\frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{q}}}}\delta(\mathbf{k}-\mathbf{q})e^{-i(kx-qy)} \\
&= \frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\,\frac{1}{\omega_{\mathbf{k}}}e^{-ik(x-y)},
\end{aligned}$$

which we define as $\Delta_+(x-y)$. Similarly we define

$$[\phi_-(x),\phi_+(y)] = \Delta_{\text{-}}(x-y) = -\frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\,\frac{1}{\omega_{\mathbf{k}}}e^{ik(x-y)}.$$

This implies

$$[\phi(x),\phi(y)] = -\frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\,\frac{1}{k^0}\left(e^{-ik(x-y)} - e^{ik(x-y)}\right),$$

defined to be $i\Delta(x-y)$. It can also be shown that

$$i\Delta(x) = -\frac{1}{2\pi}\mathrm{sgn}(x^0)(\delta(x^\mu x\mu) - \frac{1}{2}m^2\theta(x^\mu x\mu)).$$

The propagator solves the inhomogenous Klein-Gordon equation, with inhomogeneity $-\delta(x-y)$ for convention. We define its Fourier transform as

$$G(x-y) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,e^{-ik(x-y)}\tilde{G}(k).$$

The Klein-Gordon equation implies

$$\tilde{G} = \frac{1}{k^2-m^2}.$$

This function has a pole at $k^0 = \pm\omega$, where $\omega^2 = \mathbf{k}^2 + m^2$. To remedy this, we add a term $-i\varepsilon$ to the denominator, shifting the poles from the real axis. Writing

$$G(x-y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int \mathrm{d}k^0 \, \frac{1}{(k^0)^2 - \omega^2 - i\varepsilon} e^{-ik^0(x^0-y^0)}$$

for $x^0 > y^0$ and switching the sign of the last integral if the opposite is true, we find that the last integral is

$$-2\pi i \frac{e^{\pm ik^0(x^0-y^0)}}{2k^0} \tag{2}$$

for the two cases. The two corresponding functions are dubbed $\pm i\Delta_\pm(x-y)$, and we finally have

$$i\Delta_{\mathrm{F}}\,(x-y) = i(\theta(x^0-y^0)\Delta_+(x-y) - \theta(y^0-x^0)\Delta_+(x-y)).$$

Corresponding to the Klein-Gordon field is an energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\pi\partial^\nu\pi + \frac{1}{2}g^{\mu\nu}(m^2\phi^2 - \Box\phi),$$

which is symmetric and divergence-free. We define the 4-momentum

$$P^\mu = \int \mathrm{d}^3\mathbf{x}\, T^{0\mu}.$$

It can somehow be shown that

$$P^\mu = \int \mathrm{d}^3 q\, \frac{1}{q^0} q^\mu a^\dagger(q) a(q),$$

where $\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = q^0\delta(\mathbf{p}-\mathbf{q})$. It can be shown that $a^\dagger(p)\left|0\right\rangle$ is an eigenstate of this operator.

Quantizing the Dirac Field We start with the Lagrangian of a free Dirac field

$$\mathcal{L} = -\frac{1}{2}\bar{\Psi}(-i\partial\!\!\!/ + m)\Psi - \frac{1}{2}(i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi})\Psi.$$

Another Lagrangian is

$$\mathcal{L}_{\mathrm{D}} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi.$$

The corresponding momentum densities are

$$\pi = \frac{i}{2}\Psi^\dagger, \ \bar{\pi} = -\frac{i}{2}\gamma^0\Psi$$

for the first choice and

$$\pi = i\Psi^\dagger, \ \bar{\pi} = 0$$

for the second. The corresponding energy-momentum tensor is

$$T^{\mu\nu} = \frac{i}{2}(\bar{\Psi}\gamma^\mu\gamma^\nu\Psi - (\partial^\nu\Psi\gamma^\mu\Psi)),$$

with corresponding momentum densities

$$P^\mu = i\int \mathrm{d}^3\mathbf{x}\, \Psi^\dagger\gamma^0\gamma^\mu\Psi.$$

To quantize the field, we expand it as

$$\Psi = \sum_s \int \mathrm{d}^3\mathbf{p}\, \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E(\mathbf{p})}} (b(\mathbf{p}, s)u(\mathbf{p}, s)e^{-ipx} + d^\dagger(\mathbf{p}, s)v(\mathbf{p}, s)e^{ipx}),$$

$$\Psi^\dagger = \sum_s \int \mathrm{d}^3\mathbf{p}\, \frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{E(\mathbf{p})}} (b^\dagger(\mathbf{p}, s)u^\dagger(\mathbf{p}, s)e^{ipx} + d(\mathbf{p}, s)v^\dagger(\mathbf{p}, s)e^{-ipx}),$$

where we have introduced the particle and antiparticle spinors and creation and annihilation operators b and d for particles and antiparticles. This leads to the Hamiltonian becoming

$$\mathcal{H} = \int \mathrm{d}^3\mathbf{p}\, E(\mathbf{p}) \sum_s (b^\dagger b(\mathbf{p}, s) - dd^\dagger(\mathbf{p}, s)).$$

In order to produce positive values, we must therefore choose anticommutation relations

$$\left\{d(\mathbf{p}, s), d^\dagger(\mathbf{p}', s')\right\} = \delta(\mathbf{p} - \mathbf{p}')\delta_{ss'}.$$

Inspired by previous work, we employ the anzats $S_{\mathrm{F},}\,(x - y) = (i\not{\partial}_x + m)F(x - y)$, implying

$$(\Box + m^2)F(x - y) = -\delta(x - y),$$

meaning F is one of the propagators of the Klein-Gordon equation.

To introduce coupling to an electromagnetic field, the Dirac equation becomes

$$(i\not{\partial} - m)\Psi = q\not{A}\Psi$$

in the minimal-coupling scheme. This somehow produces the solution

$$\Psi = \Psi_{\text{in}} + q \int \mathrm{d}^4y\, S_{\mathrm{R}}(x - y)\not{A}(y)\Psi(y),\ \Psi = \Psi_{\text{out}} + q \int \mathrm{d}^4y\, S_{\mathrm{A}}(x - y)\not{A}(y)\Psi(y).$$

To simplify this we will employ a perturbation scheme

$$\Psi = \Psi^{(0)} + q\Psi^{(1)} + q^2\Psi^{(2)} + \ldots,$$

which yields

$$\Psi^{(0)} = \Psi_{\text{in}},\ \Psi^{(1)} = \int \mathrm{d}^4y\, S_{\mathrm{R}}(x - y)\not{A}(y)\Psi_{\text{in}}(y),\ \ldots$$

We also introduce a unitary operator S such that

$$\Psi_{\text{out}} = S^\dagger\Psi_{\text{in}}S,$$

and expand according to

$$S = 1 + qS^{(1)} + q^2S^{(2)} + \ldots$$

The unitarity implies that $S^{(1)}$ is anti-Hermitian. Introducing the function $K = S_{\mathrm{R}} - S_{\mathrm{A}}$ we find

$$\Psi_{\text{out}} = \Psi_{\text{in}} + q \int \mathrm{d}^4y\, K(x - y)\not{A}\Psi.$$

A first-order expansion of the left-hand side yields

$$\Psi_{\text{in}} + q((S^{(1)})^\dagger\Psi_{\text{in}} - \Psi_{\text{in}}(S^{(1)})^\dagger) = \Psi_{\text{in}} + q\left[(S^{(1)})^\dagger, \Psi_{\text{in}}\right],$$

implying

$$\left[(S^{(1)})^\dagger, \Psi_{\text{in}}\right] = \int \mathrm{d}^4y\, K(x - y)\not{A}\Psi,$$

with solution

$$(S^{(1)})^\dagger = i \int \mathrm{d}^4z : \bar{\Psi}_{\text{in}} : \not{A}\Psi_{\text{in}}.$$

Quantizing the Electromagnetic Field The quantization of the electromagnetic field carries a few more difficulties with it. First, the photons, which build up the quantum theory, are massless. Second, photons have a vector character to them.

While the electric and magnetic fields are measurable, and should at first glance be the quantities we replace by operators, we instead do this to the 4-potential. The question of gauge then arises. Starting with Maxwell's equations, written in the two forms

$$\partial_\mu F^{\mu\nu} = J^\nu, \; \Box A^\nu - \partial^\nu \partial_\mu A^\mu = J^\nu.$$

One choice of gauge is $\partial_\mu A^\mu = 0$, called the Lorenz gauge. Another is the temporal gauge $A^0 = 0$. Third there is the axial gauge $A^3 = 0$. Finally there is the Coulomb gauge $\partial_i A^i = 0$. For a free electromagnetic field the temporal and Coulomb gauges can be satisfied simultaneously. The combination of these is known as the radiation gauge. In this gauge we have for a free field that

$$\Box A^\mu = 0.$$

The Coulomb gauge requirement in momentum space is $\mathbf{k} \cdot \mathbf{A} = 0$. This means that this gauge has no longitudinal photons.

We will quantize the electromagnetic field in four attempts. The first is to employ the familiar non-relativistic scheme

$$\mathbf{A} = \frac{1}{\sqrt{2(2\pi)^3}} \int \mathrm{d}^3\mathbf{k} \, \frac{1}{\sqrt{\omega}} \sum_{\eta=\pm 1} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \boldsymbol{\varepsilon}(\mathbf{k},\eta) a_{\mathrm{nr}}(\mathbf{k},\eta) + e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \boldsymbol{\varepsilon}^\star(\mathbf{k},\eta) a_{\mathrm{nr}}^\dagger(\mathbf{k},\eta),$$

which guarantees that $\mathbf{A}$ is Hermitian. The classical interpretation of this is that the field is composed of harmonic oscillators, with commutation relations

$$\left[a_{\mathrm{nr}}(\mathbf{k},\eta), a_{\mathrm{nr}}^\dagger(\mathbf{k}',\eta')\right] = \delta_{\eta\eta'} \delta(\mathbf{k}-\mathbf{k}').$$

In our next attempt we introduce $a = \sqrt{k^0} a_{\mathrm{nr}}$. This yields

$$\mathbf{A} = \frac{1}{\sqrt{2(2\pi)^3}} \int \mathrm{d}^3\mathbf{k} \, \frac{1}{k_0} \sum_{\eta=\pm 1} e^{-ikx} \boldsymbol{\varepsilon}(\mathbf{k},\eta) a(\mathbf{k},\eta) + e^{ikx} \boldsymbol{\varepsilon}^\star(\mathbf{k},\eta) a^\dagger(\mathbf{k},\eta),$$

as well as the commutation relation

$$\left[a(\mathbf{k},\eta), a^\dagger(\mathbf{k}',\eta')\right] = k^0 \delta_{\eta\eta'} \delta(\mathbf{k}-\mathbf{k}').$$

These formulations have not been covariant. Strochi discovered that one cannot have a covariant quantization and a positive definite metric on Hilbert space at the same time. Our third attempt will be

$$A^\mu = \frac{1}{\sqrt{2(2\pi)^3}} \int \mathrm{d}^3\mathbf{k} \, \frac{1}{k_0} \sum_{\eta=1}^{2} \left(e^{-ikx} a(\mathbf{k},\eta) + e^{ikx} a^\dagger(\mathbf{k},\eta)\right) e^\mu(\mathbf{k},\eta).$$

Finally we try to extend this to something that has an inner product, with the result

$$A^\mu = \frac{1}{\sqrt{2(2\pi)^3}} \int \mathrm{d}^3\mathbf{k} \, \frac{1}{k_0} \sum_{\lambda=0}^{3} -g_{\lambda\lambda} \left(e^{-ikx} a(\mathbf{k},\lambda) + e^{ikx} a^\dagger(\mathbf{k},\lambda)\right) e^\mu(\mathbf{k},\lambda).$$

However, this contains creation and annihilation operators for longitudinal and scalar photons, which have not been found. We need to get rid of them. We also have the commutation relations

$$\left[a^\mu(\mathbf{k}), (a^\nu)^\dagger(\mathbf{k}')\right] = -g^{\mu\nu} k^0 \delta(\mathbf{k}-\mathbf{k}')$$

for the operators

$$a^\mu(\mathbf{k}) = -\sum_{\lambda=0}^{3} g_{\lambda\lambda} e^\mu(\mathbf{k},\lambda) a(\mathbf{k},\lambda).$$

This implies

$$\left|(a^0)^\dagger |0\rangle\right| = -g^{00} k^0 \delta(\mathbf{0}),$$

which is negative. Fermi's solution was to switch the interpretations of $a(\mathbf{k},0)$ and $a^\dagger(\mathbf{k},0)$. Another idea is to just keep the indefinite metric.

More on the Electromagnetic Field The classical Lagrangian for an electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu.$$

It turns out that if j is a conserved current, this Lagrangian is invariant under gauge transformations $A^\mu \to A^\mu + \partial^\mu \chi$.

By adding a mass term to the Lagrangian, making it of the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu - j_\mu A^\mu,$$

we find the equations of motion to be

$$\partial_\mu F^{\mu\nu} + M^2 A^\nu = j^\nu.$$

This implies that the current is only conserved in the Lorenz gauge. Proceeding with this gauge, the equation of motion becomes

$$(\Box + M^2)A^\mu = j^\mu.$$

Another possible extension is with a term

$$\mathcal{L} = G_{\mu\nu}G^{\mu\nu}, G_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu,$$

which also has gauge issues.

When performing a gauge transformation we assume the field and potentials to transform according to

$$A_\mu \to A_\mu + \partial_\mu \chi, \ \Psi \to e^{i\alpha(x)}\Psi,$$

where $\alpha = -e\chi$. Introducing the so-called covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu.$$

By applying the gauge transformation we find

$$D_\mu \Psi \to e^{i\alpha} D_\mu \Psi,$$

or $D_\mu \to e^{i\alpha} D_\mu$, which the ordinary derivative does not satisfy. We can also show that

$$\left[D_\mu, D_\nu\right] = e^{i\alpha}\left[D_\mu, D_\nu\right]\Psi,$$

although this commutator is not a covariant derivative. We also find

$$\left[D_\mu, D_\nu\right] = ieF_{\mu\nu}.$$

Classical Yang-Mills Theory The idea behind Yang-Mills theory is to construct a doublet of Dirac fields, from now termed Ψ, which transforms according to

$$\Psi \to e^{\frac{i}{2}\sigma^j \alpha^j(x)}\Psi$$

under gauge transformations. We introduce a covariant derivative

$$D_\mu = \partial_\mu - \frac{i}{2}gA_\mu^j \sigma^j,$$

which satisfies

$$\left[D_\mu, D_\nu\right] = -\frac{i}{2}gF_{\mu\nu}^j \sigma^j,$$

where

$$\frac{i}{2}F_{\mu\nu}^j \sigma^j = \frac{1}{2}\left(\partial_\mu A_\nu^j - \partial_\nu A_\mu^j\right)\sigma^j - ig\left[\frac{1}{2}A_\mu^j \sigma^j, \frac{1}{2}A_\nu^k \sigma^k\right].$$

This can be solved to yield

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon^{ijk} A^j_\mu A^k_\nu,$$

a so-called non-abelian field strength. It transforms according to

$$\frac{i}{2} F^j_{\mu\nu}\sigma^j \to \frac{i}{2} F^j_{\mu\nu}\sigma^j + \left[\frac{i}{2}\alpha^j\sigma^j, \frac{1}{2}F^k_{\mu\nu}\sigma^k\right].$$

The Lagrangian corresponding to this theory is

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}\left(\left(\frac{1}{2}F^j_{\mu\nu}\sigma^j\right)^2\right) = -\frac{1}{4}F^j_{\mu\nu}F^{\mu\nu,j},$$

which is gauge invariant.

4 Perturbation Theory

The S-Matrix Consider a set of particles moving from infinity to some point, where they all interact before being scattered and moving back to infinity. Denoting incoming states at infinity with in and outgoing states with out, we are interested in probability amplitudes of the form

$$A = \langle \mathbf{q}_1, \dots, \mathbf{q}_m, \text{out} | \mathbf{p}_1, \dots, \mathbf{p}_n, \text{in} \rangle .$$

These are states of the real-world Hamiltonian, but the states at infinity should be similar to those of an interaction-free theory. Denoting states of the corresponding interaction-free theory without the text, we define the operator S, or the S-matrix, as

$$\langle \mathbf{q}_1, \dots, \mathbf{q}_m, \text{out} | \mathbf{p}_1, \dots, \mathbf{p}_n, \text{in} \rangle = \langle \mathbf{q}_1, \dots, \mathbf{q}_m | S | \mathbf{p}_1, \dots, \mathbf{p}_n \rangle .$$

Asymptotic Fields Consider some theory described by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + \mathcal{L}_{\text{int}},$$

where the interaction term contains no derivatives and we have used the so-called bare mass m_0. The equation of motion is

$$(\Box + m_0^2)\phi = j(x),$$

where $j(x)$ arises from the interaction term. We want to find a so-called asymptotic field ϕ_{in}, which is a free Klein-Gordon field with mass m, that transforms the same as ϕ under the Poincare group. This field is then to be associated with the vacuum $|0\rangle$ of the free theory, which is distinct from the physical vacuum $|\Omega\rangle$ of the interacting theory.

The two fields can be related by adding a term $(m^2 - m_0^2)\phi$ to the equation of motion, which modifies the right-hand side to $\tilde{j}(x)$. We then find

$$\sqrt{Z}\phi_{\text{in}} = \phi + \int \mathrm{d}^4 y\, \Delta_{\text{R}}\left(x - y, m\right)\tilde{j}(y),$$

where Z is a normalization constant. We want $\tilde{j}$ to satisfy the strong asymptotic condition

$$\lim_{x^0 \to -\inf} \phi(x) = \sqrt{Z}\phi_{\text{in}}(x).$$

This would imply that when particles are at large separations, they do not interact. However, this assumption neglects self-interactions as well, and can therefore never truly hold. In fact, if it were to hold by assumption, the S matrix would be trivial. The claim of Lehmann, Symanzik and Zimmermann is that we can consider two normalizable states $|\alpha\rangle$ and $|\beta\rangle$ and introduce the functional

$$\phi(t, f) = i\int \mathrm{d}^3\mathbf{x}\, f^\star(t, \mathbf{x})\overleftrightarrow{\partial_0}\phi(t, \mathbf{x}),\ a\overleftrightarrow{\partial}b = a\partial b - (\partial a)b,$$

where f is a normalizable Klein-Gordon field (a function, not an operator) with mass m, and thus acts as a test function in the Schwarz class. We then have the weak asymptotic condition

$$\lim_{x^0\to-\infty} \langle\alpha|\phi(t,f)|\beta\rangle = \sqrt{Z}\,\langle\alpha|\phi_{\text{in}}(t,f)|\beta\rangle\,,$$

which is more well-defined.

At this point we introduce another asymptotic free field ϕ_{out}, which should satisfy the same conditions as ϕ_{in}. We can expand ϕ_{in} in normal modes as

$$\phi_{\text{in}} = \int \mathrm{d}^3\mathbf{p}\,\frac{1}{p_0}(f_p(x)a_{\text{in}}(\mathbf{p}) + f_p^\star(x)a_{\text{in}}^\dagger(\mathbf{p})),\;\; f_p(x) = \frac{1}{\sqrt{2(2\pi)^3}}e^{-ipx},$$

with a similar expansion for ϕ_{out}. This can somehow be inverted to

$$a_{\text{in}}(\mathbf{p}) = i\int \mathrm{d}^3\mathbf{x}\,f_p^\star(x)\overleftrightarrow{\partial_0}\phi_{\text{in}}(x).$$

The LSZ Formalism In the LSZ formalism we write the elements of the S-matrix as vacuum expectation values of products of the field operators.

The LSZ Formula for Klein-Gordon Fields To demonstrate it, consider a real scalar Klein-Gordon field ϕ and states $|\alpha,\mathbf{p}_n,\text{in}\rangle$ and $|\beta,\text{out}\rangle$, where α is a shorthand for a set of momenta $\mathbf{p}$ and β similarly a shorthand for momenta $\mathbf{q}$. We then write

$$\begin{aligned}\langle\beta,\text{out}|\alpha,\mathbf{p}_n,\text{in}\rangle &= \langle\beta,\text{out}|a_{\text{in}}^\dagger(\mathbf{p_n})|\alpha,\text{in}\rangle\\ &= \langle\beta,\text{out}|a_{\text{out}}^\dagger(\mathbf{p_n})|\alpha,\text{in}\rangle + \langle\beta,\text{out}|a_{\text{in}}^\dagger(\mathbf{p_n})|\alpha,\text{in}\rangle - \langle\beta,\text{out}|a_{\text{out}}^\dagger(\mathbf{p_n})|\alpha,\text{in}\rangle\\ &= \langle\beta,\hat{\mathbf{p}}_n,\text{out}|\alpha,\text{in}\rangle - i\,\langle\beta,\text{out}|\int\mathrm{d}^3\mathbf{x}\,f_{p_n}\overleftrightarrow{\partial_0}(\phi_{\text{in}}^\dagger-\phi_{\text{out}}^\dagger)|\alpha,\text{in}\rangle\\ &= \langle\beta,\hat{\mathbf{p}}_n,\text{out}|\alpha,\text{in}\rangle - i\,\langle\beta,\text{out}|\int\mathrm{d}^3\mathbf{x}\,f_{p_n}\overleftrightarrow{\partial_0}(\phi_{\text{in}}-\phi_{\text{out}})|\alpha,\text{in}\rangle\,,\end{aligned}$$

assuming the asymptotic fields to be self-adjoint. The hat signifies that $\mathbf{p}_n$ has been removed, which might mean annihilation depending on the structure of β. Using the asymptotic limit we have

$$\begin{aligned}\langle\beta,\text{out}|\alpha,\mathbf{p}_n,\text{in}\rangle &= \langle\beta,\hat{\mathbf{p}}_n,\text{out}|\alpha,\text{in}\rangle + \frac{i}{\sqrt{Z}}\left(\lim_{x^0\to\infty}-\lim_{x^0\to-\infty}\right)\langle\beta,\text{out}|\int\mathrm{d}^3\mathbf{x}\,f_{p_n}\overleftrightarrow{\partial_0}\phi|\alpha,\text{in}\rangle\\ &= \langle\beta,\hat{\mathbf{p}}_n,\text{out}|\alpha,\text{in}\rangle + \frac{i}{\sqrt{Z}}\langle\beta,\text{out}|\int\mathrm{d}^4x\,\partial_0\left(f_{p_n}\overleftrightarrow{\partial_0}\phi\right)|\alpha,\text{in}\rangle\\ &= \langle\beta,\hat{\mathbf{p}}_n,\text{out}|\alpha,\text{in}\rangle + \frac{i}{\sqrt{Z}}\int\mathrm{d}^4x\,\partial_0\left(f_{p_n}\overleftrightarrow{\partial_0}\langle\beta,\text{out}|\phi|\alpha,\text{in}\rangle\right)\\ &= \langle\beta,\hat{\mathbf{p}}_n,\text{out}|\alpha,\text{in}\rangle + \frac{i}{\sqrt{Z}}\int\mathrm{d}^4x\,f_{p_n}\partial_0^2\langle\beta,\text{out}|\phi|\alpha,\text{in}\rangle - (\partial_0^2 f_{p_n})\langle\beta,\text{out}|\phi|\alpha,\text{in}\rangle\,,\end{aligned}$$

where we have used the fact that

$$\partial(a\overleftrightarrow{\partial}b) = \partial(a\partial b-(\partial a)b) = (\partial a)\partial b + a\partial^2 b - (\partial^2 a)b - (\partial a)\partial b = a\partial^2 b - (\partial^2 a)b.$$

The first term is the so-called elastic term, and is zero if $\mathbf{p}_n$ is not in β, i.e. if there is no forward scattering. In this case we can use the fact that the normal modes solve the Klein-Gordon equation to find

$$\langle\beta,\text{out}|\alpha,\mathbf{p}_n,\text{in}\rangle = \frac{i}{\sqrt{Z}}\int\mathrm{d}^4x\,f_{p_n}(\partial_0^2+m^2)\langle\beta,\text{out}|\phi|\alpha,\text{in}\rangle - \langle\beta,\text{out}|\phi|\alpha,\text{in}\rangle\,\nabla^2 f_{p_n}.$$

Studying the last term we find

$$\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle\, \nabla^2 f_{p_n} &= \int \mathrm{d}\mathbf{S}\cdot\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle\, \vec{\boldsymbol{\nabla}} f_{p_n} - \int \mathrm{d}^3\mathbf{x}\, \vec{\boldsymbol{\nabla}}\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle \cdot \vec{\boldsymbol{\nabla}} f_{p_n} \\
&= -\int \mathrm{d}\mathbf{S}\cdot f_{p_n} \vec{\boldsymbol{\nabla}}\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle + \int \mathrm{d}^3\mathbf{x}\, f_{p_n} \nabla^2\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle \\
&= \int \mathrm{d}^3\mathbf{x}\, f_{p_n} \nabla^2\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle\,,
\end{aligned}$$

assuming the field to vanish at infinity. Hence

$$\langle\beta,\mathrm{out}|\alpha,\mathbf{p}_n,\mathrm{in}\rangle = \frac{i}{\sqrt{Z}} \int \mathrm{d}^4x\, f_{p_n} (\Box + m^2)\, \langle\beta,\mathrm{out}|\phi|\alpha,\mathrm{in}\rangle\,.$$

We continue with the matrix element in the integrand. Rewriting it slightly we find

$$\begin{aligned}
\langle\gamma,\mathbf{q}_m,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle &= \langle\gamma,\mathrm{out}|a_{\mathrm{out}}(\mathbf{q}_m)\phi(x)|\alpha,\mathrm{in}\rangle - \langle\gamma,\mathrm{out}|\phi(x)a_{\mathrm{in}}(\mathbf{q}_m)|\alpha,\mathrm{in}\rangle + \langle\gamma,\mathrm{out}|\phi(x)a_{\mathrm{in}}(\mathbf{q}_m)|\alpha,\mathrm{in}\rangle \\
&= \langle\gamma,\mathrm{out}|\phi(x)|\hat{\mathbf{q}}_m,\alpha,\mathrm{in}\rangle \\
&\quad + \langle\gamma,\mathrm{out}|i\int \mathrm{d}^3\mathbf{y}\, f^\star_{q_m}(y)\overleftrightarrow{\partial}_{y,0}\phi_{\mathrm{out}}(y)\phi(x) - i\phi(x)\int \mathrm{d}^3\mathbf{y}\, f^\star_{q_m}(y)\overleftrightarrow{\partial}_{y,0}\phi_{\mathrm{in}}(y)|\alpha,\mathrm{in}\rangle\,.
\end{aligned}$$

Because the normal modes commute with the fields, we find

$$\begin{aligned}
f^\star_{q_m}(y)\overleftrightarrow{\partial}_{y,0}\phi_{\mathrm{out}}(y) &= f^\star_{q_m}(y)\partial_{y,0}\phi_{\mathrm{out}}(y) - (\partial_{y,0}f^\star_{q_m}(y))\phi_{\mathrm{out}}(y) \\
&= \phi_{\mathrm{out}}(y)\partial_{y,0}f^\star_{q_m}(y) + (\partial_{y,0}\phi_{\mathrm{out}}(y))f^\star_{q_m}(y) \\
&= -\phi_{\mathrm{out}}(y)\overleftrightarrow{\partial}_{y,0}f^\star_{q_m}(y),
\end{aligned}$$

and thus

$$\begin{aligned}
\langle\gamma,\mathbf{q}_m,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle &= \langle\gamma,\mathrm{out}|\phi(x)|\hat{\mathbf{q}}_m,\alpha,\mathrm{in}\rangle \\
&\quad - i\,\langle\gamma,\mathrm{out}|\int \mathrm{d}^3\mathbf{y}\, \phi_{\mathrm{out}}(y)\phi(x)\overleftrightarrow{\partial}_{y,0}f^\star_{q_m}(y) - \phi(x)\phi_{\mathrm{in}}(y)\overleftrightarrow{\partial}_{y,0}f^\star_{q_m}(y)|\alpha,\mathrm{in}\rangle \\
&= \langle\gamma,\mathrm{out}|\phi(x)|\hat{\mathbf{q}}_m,\alpha,\mathrm{in}\rangle - i\int \mathrm{d}^3\mathbf{y}\, \langle\gamma,\mathrm{out}|\phi_{\mathrm{out}}(y)\phi(x) - \phi(x)\phi_{\mathrm{in}}(y)|\alpha,\mathrm{in}\rangle\, \overleftrightarrow{\partial}_{y,0}f^\star_{q_m}(y).
\end{aligned}$$

Assuming no forward scattering, we use the limit procedure to find

$$\langle\gamma,\mathbf{q}_m,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle = -\frac{i}{\sqrt{Z}} \int \mathrm{d}^4y\, \partial_0 \left(\langle\gamma,\mathrm{out}|T\phi(y)\phi(x)|\alpha,\mathrm{in}\rangle\, \overleftrightarrow{\partial}_{y,0}f^\star_{q_m}(y) \right),$$

where we have introduced the time-ordered product to account for the switching of the order when evaluated in the two limits. Repeating the final steps of the previous derivation we find

$$\langle\gamma,\mathbf{q}_m,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle = \frac{i}{\sqrt{Z}} \int \mathrm{d}^4y\, (\Box_y + m^2)\, \langle\gamma,\mathrm{out}|T\phi(x)\phi(y)|\alpha,\mathrm{in}\rangle\, f^\star_{q_n}(y).$$

What we have seen now is how removing single momenta from each side of the matrix element. Generalizing this argument to repetitions we find

$$\begin{aligned}
\langle\mathbf{q}_1,\ldots,\mathbf{q}_m,\mathrm{out}|\mathbf{p}_1,\ldots,\mathbf{p}_n,\mathrm{in}\rangle = \left(\frac{i}{\sqrt{Z}}\right)^{n+m} \prod_{i=1}^{n}\prod_{j=1}^{m} \int \mathrm{d}^4x_i\, \mathrm{d}^4y_j\, f_{p_i}(x_i) f^\star_{q_j}(y_j)(\Box_{x_i}+m^2)(\Box_{y_j}+m^2) \\
\times\, \langle\Omega|T\phi(y_1)\ldots\phi(y_m)\phi(x_1)\ldots\phi(x_n)|\Omega\rangle\,.
\end{aligned}$$

The Interaction Picture The most basic pictures of quantum mechanics are the Schrödinger picture, where states evolve by multiplying by the time evolution operator e^{-iHt} and operators are time-independent, and the Heisenberg picture, where operators evolve according to $A(t) = e^{iHt}A(0)e^{-iHt}$. A third picture, which will be used in the proceeding discussion of perturbation theory, is the interaction picture. In this picture, both the states and the operators will evolve in time.

Consider a Hamiltonian of the form

$$H = H_{\text{in}} + H_{\text{I}}.$$

The two terms are the free part and the interaction part. The operators evolve according to

$$O(t) = e^{iH_{\text{in}}t}O(0)e^{-iH_{\text{in}}t},$$

implying an equation of motion

$$i\frac{\mathrm{d}O}{\mathrm{d}t} = [O, H_{\text{in}}].$$

To preserve expectation values when compared to the Schrödinger picture, we require

$$|\Psi_{\text{I}}(t)\rangle = e^{iH_{\text{in}}t}|\Psi(t)\rangle.$$

Differentiating with respect to time

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_{\text{I}}(t)\rangle = iH_{\text{in}}e^{iH_{\text{in}}t}|\Psi(t)\rangle + e^{iH_{\text{in}}t}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = e^{iH_{\text{in}}t}(iH_{\text{in}} - iH)|\Psi(t)\rangle.$$

We can write this as

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_{\text{I}}(t)\rangle = e^{iH_{\text{in}}t}H_{\text{I}}e^{-iH_{\text{in}}t}|\Psi_{\text{I}}(t)\rangle,$$

or alternatively, by defining $H' = e^{iH_{\text{in}}t}H_{\text{I}}e^{-iH_{\text{in}}t}$, we have

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_{\text{I}}(t)\rangle = H'|\Psi_{\text{I}}(t)\rangle.$$

A Field Transformation Operator ϕ and ϕ_{in}, together with their conjugate momenta, are complete fields that satisfy the same commutation relations. Treating the interacting fields in the Heisenberg picture and the asymptotic fields in the interaction picture, there must be a unitary operator $U(t)$ such that

$$\phi = U^{-1}\phi_{\text{in}}U, \;\; \pi = U^{-1}\pi_{\text{in}}U.$$

Our task is to find this operator.

We note that the total Hamiltonian is a sum of the free-field Hamiltonian H_{in} and the interaction Hamiltonian H_{I}. Differentiating the above relation we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\text{in}} = \dot{U}\phi U^{-1} + U\partial_t\phi U^{-1} + U\phi\dot{U}^{-1}.$$

The unitarity of U yields

$$\dot{U}U^{-1} + U\dot{U}^{-1} = 0 \iff \dot{U}^{-1} = -U^{-1}\dot{U}U^{-1},$$

hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\text{in}} &= \dot{U}\phi U^{-1} + U\partial_t\phi U^{-1} - U\phi U^{-1}\dot{U}U^{-1}\\ &= \dot{U}U^{-1}\phi_{\text{in}} + iU[H(\phi_{\text{in}}, \pi_{\text{in}}), \phi]U^{-1} - \phi_{\text{in}}\dot{U}U^{-1}\\ &= \left[\dot{U}U^{-1}, \phi_{\text{in}}\right] + i[H(\phi_{\text{in}}, \pi_{\text{in}}), \phi_{\text{in}}],\end{aligned}$$

assuming the Hamiltonian to commute with U. Now, in the interaction picture the time evolution of the asymptotic field is dictated only by the free-field Hamiltonian, hence

$$\begin{aligned}&\left[\dot{U}U^{-1}, \phi_{\text{in}}\right] + i[H(\phi_{\text{in}}, \pi_{\text{in}}), \phi_{\text{in}}] = i[H_{\text{in}}(\phi_{\text{in}}, \pi_{\text{in}}), \phi_{\text{in}}],\\ &\left[\dot{U}U^{-1}, \phi_{\text{in}}\right] + i[H_{\text{I}}(\phi_{\text{in}}, \pi_{\text{in}}), \phi_{\text{in}}] = 0.\end{aligned}$$

Because this commutator is zero for any free field, we must have

$$\dot{U}U^{-1} + iH_{\mathrm{I}} = -iE_0,$$

yielding the differential equation

$$i\dot{U} = (H_{\mathrm{I}} + E_0)U = H'_{\mathrm{I}}U,$$

where we introduced the shifted interaction Hamiltionian $H'_{\mathrm{I}} = H_{\mathrm{I}} + E_0$.

We note that the above equation is very similar to that describing the time evolution operator in non-relativistic quantum mechanics. We therefore introduce the operator

$$U(t,t') = U(t)U^{-1}(t'),$$

which satisfies $U(t,t') = U(t,\tau)U(\tau,t')$ and

$$i\partial_t U(t,t') = H'_{\mathrm{I}}U(t,t').$$

We must have $U(t',t') = 1$, and we thus obtain the integral equation

$$U(t,t') = 1 - i\int_{t'}^{t} \mathrm{d}t_1\, H'_{\mathrm{I}}(t_1)U(t_1,t').$$

This can be solved using an iteration scheme

$$\begin{aligned} U(t,t') =& 1 - i\int_{t'}^{t} \mathrm{d}t_1\, H'_{\mathrm{I}}(t_1) + (-i)^2\int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t_1} \mathrm{d}t_2\, H'_{\mathrm{I}}(t_1)H'_{\mathrm{I}}(t_2) \\ &+ (-i)^n \int_{t'}^{t} \mathrm{d}t_1 \dots \int_{t'}^{t_{n-1}} \mathrm{d}t_n \prod_{i=1}^{n} H'_{\mathrm{I}}(t_i) + \dots . \end{aligned}$$

Now we note that this setup is such that the different times appear in order, meaning we can use the time-ordered product. A sneaky trick we can use is that operators commute within time-ordered products, as the time ordering fixes the order. We thus have

$$\int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t_1} \mathrm{d}t_2\, TH'_{\mathrm{I}}(t_1)H'_{\mathrm{I}}(t_2) = \int_{t'}^{t} \mathrm{d}t_2 \int_{t'}^{t_2} \mathrm{d}t_1\, TH'_{\mathrm{I}}(t_2)H'_{\mathrm{I}}(t_1) = \int_{t'}^{t} \mathrm{d}t_2 \int_{t'}^{t_2} \mathrm{d}t_1\, TH'_{\mathrm{I}}(t_1)H'_{\mathrm{I}}(t_2),$$

and we can symmetrize according to

$$\int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t_1} \mathrm{d}t_2\, TH'_{\mathrm{I}}(t_1)H'_{\mathrm{I}}(t_2) = \frac{1}{2}\int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t} \mathrm{d}t_2\, TH'_{\mathrm{I}}(t_1)H'_{\mathrm{I}}(t_2).$$

We thus end up with

$$U(t,t') = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} \mathrm{d}t_1 \dots \int_{t'}^{t} \mathrm{d}t_n\, T\prod_{i=1}^{n} H'_{\mathrm{I}}(t_i) = T\exp\left(-i\int_{t'}^{t} \mathrm{d}\tau\, H'_{\mathrm{I}}(\tau)\right) = T\exp\left(-i\int \mathrm{d}^4x\, \mathcal{H}'_{\mathrm{I}}(\phi_{\mathrm{in}},\pi_{\mathrm{in}})\right),$$

where we have introduced the time-ordered exponential and the Hamiltonian density corresponding to the interaction. This expansion, in particular, the second-to-last expression, is called the Dyson expansion.

Back To VEVs We now return to the issue of computing vacuum expectation values. Consider the expectation value

$$\tau(x_1,\dots,x_n) = \langle\Omega|T\prod_{i=1}^{n}\phi(x_i)|\Omega\rangle\,.$$

We have

$$\begin{aligned}\tau(x_1,\dots,x_n) &= \langle\Omega|T\prod_{i=1}^{n}U^{-1}(t_i)\phi_{\text{in}}(x_i)U(t_i)|\Omega\rangle\\ &= \langle\Omega|TU^{-1}(t_1)\phi_{\text{in}}(x_1)U(t_1,t_2)\phi_{\text{in}}(x_2)\dots U(t_{n-1},t_n)\phi_{\text{in}}(x_n)U(t_n)|\Omega\rangle\,.\end{aligned}$$

Introducing a t that is greater than all involved times (implying that $-t$ is smaller than all times) we have

$$\tau(x_1,\dots,x_n) = \langle\Omega|TU^{-1}(t)U(t,t_1)\phi_{\text{in}}(x_1)U(t_1,t_2)\phi_{\text{in}}(x_2)\dots U(t_{n-1},t_n)\phi_{\text{in}}(x_n)U(t_n,-t)U(-t)|\Omega\rangle\,.$$

In the limit of $t\to\infty$ we can somehow extract the U and rearrange the operators to find

$$\begin{aligned}\tau(x_1,\dots,x_n) &= \langle\Omega|U^{-1}(t)T(\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2)\dots\phi_{\text{in}}(x_n)U(t,t_1)U(t_1,t_2)\dots U(t_{n-1},t_n)U(t_n,-t))U(-t)|\Omega\rangle\\ &= \langle\Omega|U^{-1}(t)T\left(\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)U(t,-t)\right)U(-t)|\Omega\rangle\\ &= \langle\Omega|U^{-1}(t)T\left(\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)e^{-i\int_{-t}^{t}\mathrm{d}\tau H_{\mathrm{I}'(\tau)}}\right)U(-t)|\Omega\rangle\end{aligned}$$

Next we want to show that $|\Omega\rangle$ is an eigenstate of $U(-t)$, as that will allow us to remove the Us. We therefore look at the state $\langle\alpha,\mathbf{p},\text{in}|U(-t)|\Omega\rangle$. It can be shown that as t goes to infinity, this matrix element approaches zero, meaning

$$\lim_{t\to\infty}U(\pm t)\,|\Omega\rangle = \lambda_\pm\,|\Omega\rangle\,.$$

This means that for the physical vacuum, adding the completeness relation amounts to just multiplying by $|\Omega\rangle\langle\Omega|$. In the limit we thus have

$$\begin{aligned}\tau(x_1,\dots,x_n) &= \langle\Omega|U^{-1}(t)T\left(\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)e^{-i\int_{-t}^{t}\mathrm{d}\tau H_{\mathrm{I}'(\tau)}}\right)U(-t)|\Omega\rangle\\ &= \langle\Omega|U^{-1}(t)|\Omega\rangle\;\langle\Omega|T\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)e^{-i\int_{-t}^{t}\mathrm{d}\tau H_{\mathrm{I}'(\tau)}}|\Omega\rangle\;\langle\Omega|U(-t)|\Omega\rangle\\ &= \lambda_+^\star\lambda_-\,\langle\Omega|T\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)e^{-i\int_{-t}^{t}\mathrm{d}\tau H_{\mathrm{I}'(\tau)}}|\Omega\rangle\,.\end{aligned}$$

To prepare for the final step, we note that

$$\lambda_+^\star\lambda_- = \langle\Omega|U(-t,t)|\Omega\rangle$$

according to the definition, hence

$$\tau(x_1,\dots,x_n) = \frac{\langle\Omega|T\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)e^{-i\int_{-t}^{t}\mathrm{d}\tau H_{\mathrm{I}'(\tau)}}|\Omega\rangle}{\langle\Omega|Te^{-i\int_{-t}^{t}\mathrm{d}\tau H_{\mathrm{I}'(\tau)}}|\Omega\rangle}.$$

Switching Vacuua The physical and free vacuua are not orthogonal in the case of small perturbations. Expanding in eigenstates of the total Hamiltonian we have

$$e^{-iHt}\,|0\rangle = e^{-iE_0t}\,|\Omega\rangle\,\langle\Omega|0\rangle + \sum_{n>0}e^{-iE_nt}\,|n\rangle\,\langle n|0\rangle\,.$$

Because the energies are ordered, we can approach infinite time as $t \to (1+i\varepsilon)\infty$ for some small ε to find that all the exponentials corresponding to excited states approach zero faster than the others, meaning

$$|\Omega\rangle = \lim_{t\to\infty} \frac{1}{\langle\Omega|0\rangle} e^{iE_0 t} e^{-iHt} |0\rangle .$$

In this limit we may add a small time shift without changing the result. Defining energy such that the ground state of the free Hamiltonian has zero energy, we may write

$$\begin{aligned}
|\Omega\rangle &= \lim_{t\to\infty} \frac{1}{\langle\Omega|0\rangle} e^{iE_0(t+t_0)} e^{-iH(t+t_0)} |0\rangle \\
&= \lim_{t\to\infty} \frac{1}{\langle\Omega|0\rangle} e^{iE_0(t_0-(-t))} e^{-iH(t_0-(-t))} e^{iH_{\text{in}}(t_0-(-t))} |0\rangle \\
&= \lim_{t\to\infty} \frac{1}{\langle\Omega|0\rangle} e^{iE_0(t_0-(-t))} e^{-i(H-H_{\text{in}})(t_0-(-t))} |0\rangle \\
&= \lim_{t\to\infty} \frac{1}{\langle\Omega|0\rangle} e^{iE_0(t-(-t_0))} U(t_0,-t) |0\rangle .
\end{aligned}$$

By similar arguments it can be shown that

$$\langle\Omega| = \lim_{t\to\infty} \frac{1}{\langle 0|\Omega\rangle} e^{iE_0(t-t_0)} \langle 0| U(t,t_0).$$

Returning to the issue of the time-ordered product we have

$$\tau(x_1,\dots,x_n) = \lim_{t\to\infty} \frac{1}{|\langle\Omega|0\rangle|^2} e^{2iE_0 t} \langle 0|U(t,t_0)U^{-1}(t)T(\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n))U(t,-t)U(-t)U(t_0,-t)|0\rangle .$$

Choosing $t_0 = 0$ we have

$$\begin{aligned}
\tau(x_1,\dots,x_n) &= \lim_{t\to\infty} \frac{1}{|\langle\Omega|0\rangle|^2} e^{2iE_0 t} \langle 0|U(t)U^{-1}(t)T(\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n))U(t,-t)U(-t)U(-t)^{-1}|0\rangle \\
&= \lim_{t\to\infty} \frac{1}{|\langle\Omega|0\rangle|^2} e^{2iE_0 t} \langle 0|T\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)U(t,-t)|0\rangle .
\end{aligned}$$

Requiring the physical vacuum to be normalized we find

$$\lim_{t\to\infty} \frac{1}{|\langle\Omega|0\rangle|^2} e^{2iE_0 t} \langle 0|U(t,-t)|0\rangle = 1,$$

and thus

$$\tau(x_1,\dots,x_n) = \lim_{t\to\infty} \frac{\langle 0|T\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)|0\rangle}{\langle 0|U(t,-t)|0\rangle}.$$

We have thus reduced the problem of computing expectation values in the interacting theory to that of computing them in the free theory.

Relating Time and Normal Ordering To compute expectation values of the form $\langle 0|T\phi_{\text{in}}(x_1)\dots\phi_{\text{in}}(x_n)|0\rangle$, we divide the fields into two terms $\phi_{\text{in}}(x) = \phi_{\text{in},+}(x) + \phi_{\text{in},}(x_1)$ where the two terms contain all positive and negative frequencies respectively, i.e. all annihilation and creation operators. The purpose is to express the time-ordered product as something containing a normal ordered product, which has zero vacuum expectation value. Assuming $x_1^0 > x_2^0$ we have

$$T\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2) = \phi_{\text{in},+}(x_1)\phi_{\text{in},+}(x_2) + \phi_{\text{in},+}(x_1)\phi_{\text{in},-}(x_2) + \phi_{\text{in},-}(x_1)\phi_{\text{in},+}(x_2) + \phi_{\text{in},-}(x_1)\phi_{\text{in},-}(x_2).$$

We reorder the terms by writing

$$\begin{aligned}
T\phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2) = &\phi_{\text{in},+}(x_1)\phi_{\text{in},+}(x_2) + [\phi_{\text{in},+}(x_1), \phi_{\text{in},-}(x_2)] + \phi_{\text{in},-}(x_2)\phi_{\text{in},+}(x_1) \\
&+ \phi_{\text{in},-}(x_1)\phi_{\text{in},+}(x_2) + \phi_{\text{in},-}(x_1)\phi_{\text{in},-}(x_2).
\end{aligned}$$

We now define the Wick contraction of two fields

$$\overbracket{\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)} = \phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2) - :\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2):,$$

and from the above we see that

$$\overbracket{\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)} = \begin{cases} [\phi_{\mathrm{in},+}(x_1), \phi_{\mathrm{in},-}(x_2)], & x_1^0 > x_2^0 \\ [\phi_{\mathrm{in},+}(x_2), \phi_{\mathrm{in},-}(x_1)], & x_2^0 > x_1^0, \end{cases}$$

This yields

$$T\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2) =: \phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2) + \overbracket{\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)} : .$$

What is this contraction? Using the expression for the two kinds of modes we have

$$\begin{aligned}
[\phi_{\mathrm{in},+}(x_1), \phi_{\mathrm{in},-}(x_2)] &= \frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{q}\,\frac{1}{k^0 q^0} e^{i\mathbf{q}\cdot\mathbf{x}_2 - i\mathbf{k}\cdot\mathbf{x}_1}\left[a(\mathbf{k}), a^\dagger(\mathbf{q})\right] \\
&= \frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\int \mathrm{d}^3\mathbf{q}\,\frac{1}{k^0} e^{i\mathbf{q}\cdot\mathbf{x}_2 - i\mathbf{k}\cdot\mathbf{x}_1}\delta(\mathbf{k}-\mathbf{q}) \\
&= \frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\,\frac{1}{k^0} e^{-i\mathbf{k}\cdot(\mathbf{x}_1 - \cdot\mathbf{x}_2)} \\
&= i\Delta_+\left(x_1 - x_2\right), \\
[\phi_{\mathrm{in},+}(x_2), \phi_{\mathrm{in},-}(x_1)] &= \frac{1}{2(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\,\frac{1}{k^0} e^{-i\mathbf{k}\cdot(\mathbf{x}_2 - \cdot\mathbf{x}_1)} = -i\Delta_{\text{-}}\left(x - y\right),
\end{aligned}$$

hence we conclude

$$\overbracket{\phi_{\mathrm{in}}(x_1)\phi_{\mathrm{in}}(x_2)} = i\Delta_{\mathrm{F}}\left(x - y\right).$$

Other contractions may also be computed. We have

$$\overbracket{\psi_{\mathrm{in}}(x_1)\overline{\psi}_{\mathrm{in}}(x_2)} = -\overbracket{\overline{\psi}_{\mathrm{in}}(x_2)\psi_{\mathrm{in}}(x_1)} = iS_{\mathrm{F},F}\left(x_1 - x_2\right),\ \overbracket{A_\mu(x_1)A_\nu(x_2)} = iD_{\mathrm{F},\mu\nu}\left(x_1 - x_2\right).$$

Wick's Theorem Wick's theorem states that

$$T\phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_n) =: \phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_n) + \text{ all possible contractions }:$$

for bosonic fields. For fermionic fields, the connection terms are multiplied by a factor of -1 to the power of the number of times the fields must be reordered to appear in the order in which they are contracted.

We will prove this theorem for the case of a bosonic field. We have already shown it to be true for the product of two fields, setting us up for a proof by induction. Assume Wick's theorem to hold for n fields, and add an extra field. Because operators commute within time ordering, we may relabel the coordinates such that they are in order, implying

$$\begin{aligned}
T\phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_{n+1}) &= \phi_{\mathrm{in}}(x_1)T\phi_{\mathrm{in}}(x_2)\ldots\phi_{\mathrm{in}}(x_{n+1}) \\
&= \phi_{\mathrm{in}}(x_1)\left(: \phi_{\mathrm{in}}(x_2)\ldots\phi_{\mathrm{in}}(x_{n+1}) + \sum \text{contractions} :\right).
\end{aligned}$$

To prove the theorem, we must bring this expression to normal ordered form. It might be useful to strategize first, however, so let us do that. Consider the terms containing exclusively contractions of the last $n-k$ fields as an example. In the n case, these contain the normal ordered product of the first k fields. If Wick's theorem is to hold for the $n+1$ case, we expect that the corresponding terms should contain a normal ordered product of the first $k+1$ fields, as well as terms with a single contractions of the new field with all the old ones. The truth of this would imply that all contractions are both preserved and augmented by an extra contraction, proving the induction statement. We would therefore like to prove that

$$\phi_{\mathrm{in}}(x_1) : \phi_{\mathrm{in}}(x_2)\ldots\phi_{\mathrm{in}}(x_{k+1}) :=: \phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_{k+1}) + \sum_{m=2}^{k+1} \overbracket{\phi_{\mathrm{in}}(x_1)\ldots\phi_{\mathrm{in}}(x_m)}\cdots : .$$

Note that this guarantees the truth of the theorem because this is done for all k and by swapping the first $k+1$ fields with other fields that are contracted in this case.

On the left-hand side we may split $\phi_{\text{in}}(x_1)$ into its creation and annihilation parts. By introducing the commutator, we find

$$\begin{aligned}\phi_{\text{in}}(x_1) : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :=& \phi_{\text{in},-}(x_1) : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) : + [\phi_{\text{in},+}(x_1), : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :] \\ &+ : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) : \phi_{\text{in},+}(x_1) \\ =& : \phi_{\text{in},-}(x_1)\phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) : + : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1})\phi_{\text{in},+}(x_1) : \\ &+ [\phi_{\text{in},+}(x_1), : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :] \\ =& : \phi_{\text{in}}(x_1)\phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) : + [\phi_{\text{in},+}(x_1), : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :].\end{aligned}$$

The hope is then that the commutator term contains all the connections.

To whittle it down, we note that

$$\begin{aligned}[A_1, A_2 \ldots A_{k+1}] &= [A_1, A_2 \ldots A_k]A_{k+1} + A_2 \ldots A_k[A_1, A_{k+1}] \\ &= \left([A_1, A_2 \ldots A_{k-1}]A_k + A_2 \ldots A_{k-1}[A_1, A_k]\right) A_{k+1} + A_2 \ldots A_k[A_1, A_{k+1}] \qquad = \ldots \\ &= \sum_{i=2}^{k+1} A_2 \ldots A_{i-1}[A_1, A_i]A_{i+1} \ldots A_{k+1}.\end{aligned}$$

Writing the normal ordered product as

$$\begin{aligned}: \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :=& \sum_{P(2,\ldots,k+1)} \sum_{m=2}^{k} \phi_{\text{in},-}(x_{P(2)}) \ldots \phi_{\text{in},-}(x_{P(m)})\phi_{\text{in},+}(x_{P(m+1)}) \ldots \phi_{\text{in},+}(x_{P(k+1)}) \\ &+ \phi_{\text{in},+}(x_2) \ldots \phi_{\text{in},+}(x_{k+1}),\end{aligned}$$

we have

$$\begin{aligned}&[\phi_{\text{in},+}(x_1), : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :] \\ =& \sum_{P(2,\ldots,k+1)} \sum_{m=2}^{k} \left[\phi_{\text{in},+}(x_1), \phi_{\text{in},-}(x_{P(2)}) \ldots \phi_{\text{in},-}(x_{P(m)})\phi_{\text{in},+}(x_{P(m+1)}) \ldots \phi_{\text{in},+}(x_{P(k+1)})\right] \\ =& \sum_{P(2,\ldots,k+1)} \sum_{m=2}^{k} \sum_{i=2}^{m} \phi_{\text{in},-}(x_{P(2)}) \ldots \phi_{\text{in},-}(x_{P(i)}) \left[\phi_{\text{in},+}(x_1), \phi_{\text{in},-}(x_{P(i)})\right] \phi_{\text{in},-}(x_{P(i+1)}) \ldots \phi_{\text{in},-}(x_{P(m)})\phi_{\text{in},+}(x_{P(m+1)}) \ldots\end{aligned}$$

We can extend the summation over i to stop at $k+1$, yielding

$$[\phi_{\text{in},+}(x_1), : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{k+1}) :] = \sum_{i=2}^{k+1} : \phi_{\text{in}}(x_2) \ldots \phi_{\text{in}}(x_{i-1})[\phi_{\text{in},+}(x_1), \phi_{\text{in}}(x_i)]\phi_{\text{in}}(x_{i+1}) \ldots \phi_{\text{in}}(x_{k+1}) : .$$

We recognize the commutator in the sum as the desired contraction, and the proof is thus complete.

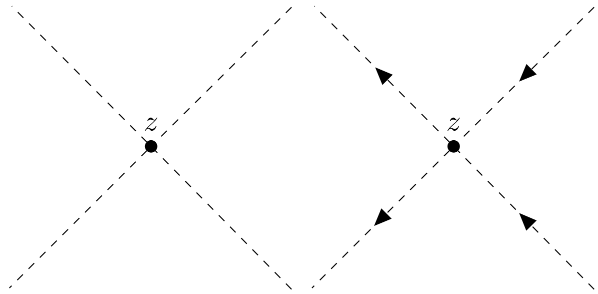
More on Feynman Diagrams We will now illustrate some more uses of Feynman diagrams in obtaining mathematical expressions that can be computed. First, we consider a Klein-Gordon theory with perturbation

$$H_{\text{I}}' = \frac{\lambda}{4!}\phi^4.$$

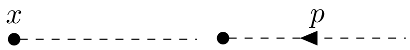
We will write the expectation value

$$\langle 0|T\phi_{\text{in}}(x)\phi_{\text{in}}(y)e^{-i\int\limits_{-\infty}^{\infty} \mathrm{d}\tau H_{\text{I}}'(\tau)}|0\rangle$$

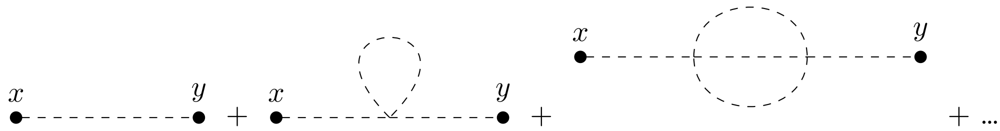
as a sum of Feynman diagrams with two external points. We have already seen the propagator in both position and momentum space, corresponding to multiplication by a propagator. In addition we introduce the vertex which corresponds to adding $-i\lambda \int \mathrm{d}^4 z$ and $-i\lambda$. There are also external points which correspond to

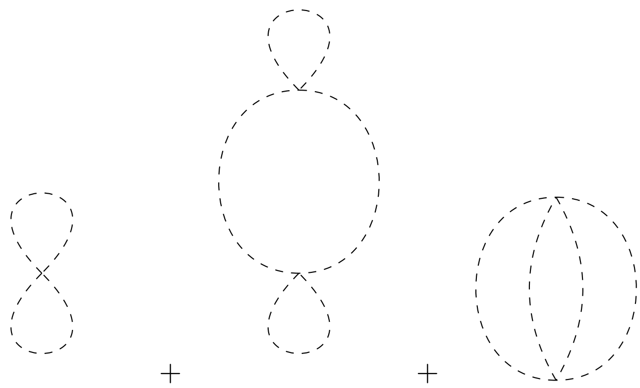


(a) Position space. (b) Momentum space.



(a) Position space. (b) Momentum space.





multiplication by 1 and e^{-ipx} respectively. One also has to divide by symmetry factors. In addition, p must be conserved at all vertices and undetermined momenta add $\frac{1}{(2\pi)^4}\int \mathrm{d}^4p$ in momentum space.

Returning to the expectation value in question, we find to different orders in the perturbation that it is equal to multiplied by the exponential of the sum of the diagrams which is equal to

$$\langle 0|Te^{-i\int_{-\infty}^{\infty}\mathrm{d}\tau H_{\mathrm{I}}'(\tau)}|0\rangle\,.$$

This implies

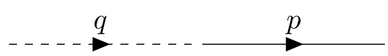
$$\langle T\phi_{\mathrm{in}}(x)\phi_{\mathrm{in}}(y)|\Omega|=\rangle\ \langle 0|T\phi_{\mathrm{in}}(x)\phi_{\mathrm{in}}(y)e^{-i\int_{-\infty}^{\infty}\mathrm{d}\tau H_{\mathrm{I}}'(\tau)}|0\rangle_{\mathrm{c}}\,,$$

where the subscript denotes a division by $\langle 0|Te^{-i\int_{-\infty}^{\infty}\mathrm{d}\tau H_{\mathrm{I}}'(\tau)}|0\rangle$.

Next we consider Yukawa theory, which is a simplified version of QED consisting of a Dirac field and a Klein-Gordon field interacting with

$$H_{\mathrm{I}}' = g\bar{\Psi}\phi\Psi.$$

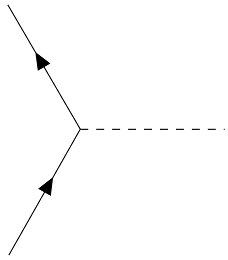
In momentum space we represent $\overline{\phi(x)\phi(y)}$ and $\Psi\bar{\Psi}$ by the Feynman diagrams which correspond to multipli-



cation by

$$\frac{1}{q^2 - m_\phi^2 + i\varepsilon}, \quad i\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

respectively, where m_ϕ is the bare mass of the Klein-Gordon field. We also have the vertex which corresponds

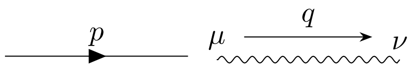


to a multiplication by $-ig$.

Finally there is quantum electrodynamics, which contains of a Dirac field interacting with the electromagnetic field. The interaction Hamiltonian is

$$H_{\mathrm{I}}' = \int \mathrm{d}^3\mathbf{x}\, e\bar{\Psi}\gamma^\mu \Psi A_\mu.$$

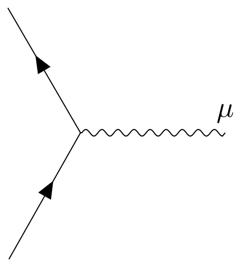
The two relevant propagators are represented as where the momentum arrow is supposed to point left. This



corresponds to multiplication by

$$i\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}, \quad -i\frac{g^{\mu\nu}}{q^2 + i\varepsilon}$$

respectively. There is also the vertex which corresponds to multiplication by $-ie\gamma^\mu$.



Obtaining Feynman Rules: An Example We will here demonstrate how to obtain Feynman rules using the results we have derived, using ϕ^4 theory as an example.

We start with the interaction Hamiltonian

$$H_{\mathrm{I}} = \frac{\lambda}{4!}\phi^4.$$

Suppose now that we want to study incoming and outgoing states with defined momenta. We are looking for

$$\langle q|S|p\rangle = (2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}\,\langle 0|a_{\mathbf{q}}Sa_{\mathbf{p}}^\dagger|0\rangle\,.$$

Next we use the Dyson expansion

$$S = T\left(e^{-i\int \mathrm{d}^4x H_{\mathrm{I}}}\right) = T\left(1 - i\int \mathrm{d}^4x\, H_{\mathrm{I}}(x) - \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\, H_{\mathrm{I}}(x)H_{\mathrm{I}}(y) + \dots\right)$$

to write the desired matrix element as

$$\begin{aligned}\langle q|S|p\rangle &= (2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}\,\langle 0|a_{\mathbf{q}}T\left(1 - i\int \mathrm{d}^4x\, H_{\mathrm{I}}(x) - \frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y\, H_{\mathrm{I}}(x)H_{\mathrm{I}}(y) + \dots\right)a^\dagger_{\mathbf{p}}|0\rangle \\ &= (2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}\,\langle 0|a_{\mathbf{q}}T\left(1 - \frac{i\lambda}{4!}\int \mathrm{d}^4x\, \phi^4(x) - \frac{1}{2}\left(\frac{i\lambda}{4!}\right)^4\int \mathrm{d}^4x\int \mathrm{d}^4y\, \phi^4(x)\phi^4(y) + \dots\right)a^\dagger_{\mathbf{p}}|0\rangle\,.\end{aligned}$$

This is a perturbative expansion of the matrix element. The next step is to use Wick's theorem to simplify this. Writing the different orders as $S^{(n)}_{qp}$, we first study the first-order correction

$$S^{(1)}_{qp} = -\frac{i\lambda}{4!}(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}\,\langle 0|a_{\mathbf{q}}T\left(\int \mathrm{d}^4x\,\phi^4(x)\right)a^\dagger_{\mathbf{p}}|0\rangle = -\frac{i\lambda}{4!}(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}\,\langle 0|T\int \mathrm{d}^4x\, a_{\mathbf{q}}\phi^4(x)a^\dagger_{\mathbf{p}}|0\rangle\,.$$

Looking at this, Wick's theorem produces a term which separates the single operators from the fields. There are three such terms. The other contains contractions of the fields with the single operators. We somehow find

$$\contraction{}{\phi}{(x)}{a}\phi(x)a^\dagger(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega(\mathbf{p})}}e^{-ipx},\quad \contraction{}{a}{(\mathbf{p})}{\phi}a(\mathbf{p})\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega(\mathbf{p})}}e^{ipx},\quad \contraction{}{a}{(\mathbf{p})}{a}a(\mathbf{p})a^\dagger(\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q}).$$

There are twelve terms of this kind, and it is given by

$$12(2\pi)^3\sqrt{4\omega(\mathbf{p})\omega(\mathbf{q})}\frac{-i\lambda}{4!}\int \mathrm{d}^4z\,\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega(\mathbf{p})}}e^{-ipz}\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega(\mathbf{p})}}e^{iqz}\int \mathrm{d}^4k\,\frac{1}{(2\pi)^4}\frac{e^{-ik(z-z)}}{k^2-m^2+i\varepsilon}$$

$$=\frac{-i\lambda}{2}\int \mathrm{d}^4z\, e^{-i(p-q)z}\int \mathrm{d}^4k\,\frac{1}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon}$$

Finally we try to make sense of the individual terms in Wick's theorem by drawing a Feynman diagram according to the following rules:

- Draw a number of vertices equal to the number of interactions you are considering.
- Contractions between the incoming state and a field produce a leg with an external point and incoming direction. Similarly, contractions with the outgoing state create outgoing legs.
- Contractions between fields create loops in the diagram.

Scattering Cross-Sections Consider two volumes of particles with lengths l_A and l_B and particle densities ρ_A and ρ_B normal to the length approaching one another. If Φ is the cross-sectional area common to the volumes we define the scattering cross-section as

$$\sigma = \frac{\text{number of scattering events}}{l_A\rho_A l_B\rho_B\Phi}.$$

The number of scattering events is then

$$\sigma l_A l_B\int \mathrm{d}^2\mathbf{x}\,\rho_A\rho_B.$$

To study this, we will expand the S-matrix as $S = 1 + iT$. In particular, for the case of two particles, there exists an invariant element M such that

$$\langle \mathbf{p}_a, p_b|iT|p_A, p_B\rangle = (2\pi)^4\delta\left(p_A + p_B - \sum_f p_f\right)iM(p_A + p_B \to p_f).$$

We can on this basis show that

$$\mathrm{d}\sigma = \frac{1}{4E_AE_B|v_A - v_B|}\prod_f \mathrm{d}^3\mathbf{p}_f\,\frac{1}{(2\pi)^3}\frac{1}{2E_f}|M|^2\cdot(2\pi)^4\delta^4\left(p_A + p_B - \sum_f p_f\right).$$

Decay Rates Consider a reaction $a \to b + c + \ldots$. It can be shown that decay rates can be computed according to

$$\mathrm{d}\Gamma = \frac{1}{2m_a}(2\pi)^4\delta^4\left(p_a - \sum_{\text{products}} p_f\right)|M(a \to b + c + \ldots)|^2\left(\prod_{\text{products}} \mathrm{d}^3\mathbf{p}_f\,\frac{1}{(2\pi)^3}\right).$$

From this, lifetimes are given by $\tau = \frac{1}{\Gamma}$.