Geometry Problems, with emphasis on Brocard

1. Let ABC be an acute angled triangle and $D \in BC, E \in AC, F \in AB$. Prove that circles $\mathcal{C}(AEF) \bigcap \mathcal{C}(BDF) \bigcap \mathcal{C}(CDE) = \{P\}$. In this case P is called the Brocard point for triangle DEF with respect to ABC. Denote this point by P_{DEF}.

2. Prove that $P_{DEF} = P_{D'E'F'} \Longleftrightarrow \Delta DEF \sim \Delta D'E'F'$.

3. Prove that is $DEF, D'E'F'$ are two similar triangles inscribed in triangle ABC then we can think of them in the following way: rotate DEF around P_{DEF} and then use a homothety to make it equal to $D'E'F'$.

4. Let ABC be a triangle and $DEF, D'E'F'$ two inscribed similar triangles. Let $\{M\} = EF \bigcap E'F', \{N\} = FD \bigcap F'D', \{P\} = DE \bigcap D'E'$. Prove that $\mathcal{C}(DD'PN), \mathcal{C}(EE'MP), \mathcal{C}(FF'NM)$ exist and they intersect in a point.

5. Let ABC be a triangle and DEF a triangle inscribed in ABC and similar to it. Prove that $\frac{area(DEF)}{area(ABC)} \geq \frac{1}{4}$. (Hint: take the median triangle. What is its Brocard point?)

6. Let ABC be an acute angled triangle and $DEF, D'E'F'$ two inscribed triangles similar to ABC. Prove that $\frac{DD'}{BC} = \frac{EE'}{AC} = \frac{FF'}{AB}$ if and only if ABC is equilateral. (Hint: same as before)

7. Let $A_1A_2A_3$ be a triangle and $M_{iu} \in (A_jA_k)$ for any permutation $\{i,j,k\} = \{1,2,3\}$ and any $u \in \{1,2,3\}$. Also, for any $u, v \Rightarrow \Delta M_{1u}M_{2u}M_{3u} \sim M_{1v}M_{2v}M_{3v}$.
   a. If $M_{i1}M_{i2} \cdot M_{i2}M_{i3}$ is constant for all i then find the angles of triangles $\Delta M_{1u}M_{2u}M_{3u}$.
   b. If $M_{i1}M_{i2} \cdot M_{i2}M_{i3} \cdot A_jA_k^2$ is constant for all permutations $\{i,j,k\} = \{1,2,3\}$ then find the angles of triangles $\Delta M_{1u}M_{2u}M_{3u}$. (Hint: Consider the podar triangle of P. What is its Brocard point?)

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