Muirhead's Inequality

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Introduction

Muirhead's inequality requires a little work in order to understand what it actually is, but it is well worth the trouble.

As well as being a useful tool, quoting it is a great way to intimidate people who are not familiar with it. To quote the diary notes (from the UKMT Yearbook) of the UK team leader for the '05 IMO, when a 'jury' decides which questions will be used:

> *'. . . many leaders don't know the technique known as Muirhead's inequality, and some of those who do seem to think that it is an advanced exotic technique . . . '*
>
> *'. . . it turns out that Muirhead's inequality might be better known among the students than parts of the jury . . . '*

A good deal of the material for this article is taken from *Inequalities* by Hardy, Littlewood, Pólya, with examples culled from various sources.

Majorisation

Suppose we have two sequences of reals, $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha_1^{'}, \ldots, \alpha_n^{'})$, which we refer to as (α) and $(\alpha^{'})$.

We say that (α) *majorises* $(\alpha^{'})$, or equivalently that $(\alpha^{'})$ is *majorised by* (α), if the following conditions are met:

- $\alpha_i \geq 0 \ \forall i$, $\alpha_j^{'} \geq 0 \ \forall j$
- $\alpha_1 + \ldots + \alpha_n = \alpha_1^{'} + \ldots + \alpha_n^{'}$
- If the elements of (α) and $(\alpha^{'})$ are arranged in descending order, then

$$\alpha_1 + \ldots + \alpha_i \geq \alpha_1^{'} + \ldots + \alpha_i^{'} \qquad 1 \leq i \leq n$$

Note that there is no requirement that the α_i, $\alpha_i^{'}$ be integers.

If these conditions are satisfied then (α) majorises $(\alpha^{'})$, which we write as

$$(\alpha) \succ (\alpha^{'})$$

For example:

- $(0, 3, 0) \succ (1, 0, 2)$
- $(4, 0, 0, 0) \not\succ (2, 0, 2)$ since the number of elements is different
- $(5, 0, -1) \not\succ (2, 2, 0)$ since terms cannot be negative
- $(2, 1, 1, 1) \not\succ (1, 1, 1, 1)$ since $2 + 1 + 1 + 1 \neq 1 + 1 + 1 + 1$
- $(4, 1, 1, 1) \not\succ (3, 3, 1, 0)$ since $4 + 1 \not\geq 3 + 3$

Symmetric Means

Suppose we have n positive reals, x_i and a sequence (α). Then we can construct a function

$$F(x_1, x_2, \dots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

We can, of course, permute the x_i to give, for example,

$$F(x_2, x_1, \dots, x_n) = x_2^{\alpha_1} x_1^{\alpha_2} \dots x_n^{\alpha_n}$$

We denote the sum over all $n!$ permutations of the x_i by

$$\sum ! F(x_1, x_2, \dots, x_n)$$

Finally, we define the *symmetric mean* as

$$[\alpha] = \frac{1}{n!} \sum ! F(x_1, x_2, \dots, x_n)$$

Obviously, $[\alpha]$ is a function of $x_1, x_2, \dots, x_n$. If we swap any of the x_i, $[\alpha]$ is unchanged. That is why it is called a *symmetric* mean!

For example:

- If $(\alpha) = (0, 3, 1)$ then $F(x_1, x_2, x_3) = x_2^3 x_3$
- If $(\alpha) = (2, 1)$ then $\sum ! F(x, y) = x^2 y + y^2 x$

- If $(\alpha) = (1,3,2)$ then the symmetric mean

$$[\alpha] = \frac{1}{3!}(x_1x_2^3x_3^2 + x_1x_3^3x_2^2 + x_2x_1^3x_3^2 + x_2x_3^3x_1^2 + x_3x_1^3x_2^2 + x_3x_2^3x_1^2)$$

Muirhead's Inequality

Now that we have defined majorisation and symmetric means, the statement of Muirhead's inequality is easy:

Muirhead's Theorem: If $(\alpha) \succ (\alpha^{'})$ then $[\alpha] \geq [\alpha^{'}]$. There is equality if and only if (α) and $(\alpha^{'})$ are identical *or* all the x_i are equal.

For example:

- For positive x, y, prove that $2(x^5 + y^5) \geq (x^2 + y^2)(x^3 + y^3)$.

  Solution: We have:

$$(5,0) \succ (3,2)$$

  therefore by Muirhead's inequality:

$$\frac{1}{2!}(x^5 + y^5) \quad \geq \quad \frac{1}{2!}(x^3y^2 + x^2y^3)$$

  so

$$x^5 + y^5 \quad \geq \quad x^3y^2 + x^2y^3$$

  hence

$$2(x^5 + y^5) \quad \geq \quad x^5 + x^3y^2 + x^2y^3 + y^5 = (x^2 + y^2)(x^3 + y^3)$$

- *BMO1 2002 Round 1 Q3:* For positive real x, y, z such that $x^2 + y^2 + z^2 = 1$ prove that

$$x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}$$

  Solution: We have

$$(4,0,0) \succ (2,1,1)$$

  and

$$(2,2,0) \succ (2,1,1)$$

therefore

$$\frac{2}{3!}(x^4+y^4+z^4) \quad \geq \quad \frac{2}{3!}(x^2yz+xy^2z+xyz^2)$$

and

$$\frac{2}{3!}(x^2y^2+y^2z^2+z^2x^2) \quad \geq \quad \frac{2}{3!}(x^2yz+xy^2z+xyz^2)$$

i.e.

$$x^4+y^4+z^4 \quad \geq \quad x^2yz+xy^2z+xyz^2$$

and

$$x^2y^2+y^2z^2+z^2x^2 \quad \geq \quad x^2yz+xy^2z+xyz^2$$

Adding the first to twice the second gives:

$$(x^4+y^4+z^4)+2(x^2y^2+y^2z^2+z^2x^2) \quad \geq \quad 3(x^2yz+xy^2z+xyz^2)$$

Rearranging:

$$(x^2+y^2+z^2)^2 \quad \geq \quad 3(x^2yz+xy^2z+xyz^2)$$

But $x^2+y^2+z^2=1$, so

$$\begin{aligned} 1 &\geq 3(x^2yz+xy^2z+xyz^2) \\ x^2yz+xy^2z+xyz^2 &\leq \frac{1}{3} \end{aligned}$$

Exercises

- *BMO1 1996 Round 1 Q5:* Let a, b and c be positive real numbers. Prove that

$$4(a^3+b^3) \geq (a+b)^3$$

  and

$$9(a^3+b^3+c^3) \geq (a+b+c)^3$$

- For positive real a, b, c, prove that the RMS Mean is greater than or equal to the Arithmetic Mean, which in turn is greater than or equal to the Geometric Mean, which in turn is greater than or equal to the Harmonic Mean. I.e.:

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3} \geq (abc)^{1/3} \geq \frac{3}{1/a+1/b+1/c}$$