A Point with many Properties

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The following problem is taken from the IMO 1985:

Problem 1 (IMO 1985)**.** *A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.*

Moreover, the following problem was given recently at the APMO 2008:

Problem 2 (APMO 2008)**.** *Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.*

When solving these two problems, one will notice that the point in question in Problem 2 coincides with the point M in Problem 1. In this note, we will show some more properties of this particular point.

Proposition 1. *Let ABC be a non-isosceles triangle and let k be a circle with center O which passes through B and C (k being different from the circumcircle of $\triangle ABC$). Let k meet AB in D (with $D \neq B$) and AC in E ($E \neq C$). The lines CD and BE meet the circumcircle of ABC in G and H respectively ($G \neq C$ and $H \neq B$). Let the line DE meet the line BC at K and the tangent lines of B and C to the circumcircle of the triangle ABC at M and L respectively. BE and CD intersect at S. Then*

(a) the circumcircle of $\triangle ABC$,

(b) the circumcircle of $\triangle ADE$,

(c) the line AK,

(d) the perpendicular line to AK through the point O,

(e) the line OS,

(f) the line MG and

(g) the line LH

all concur.

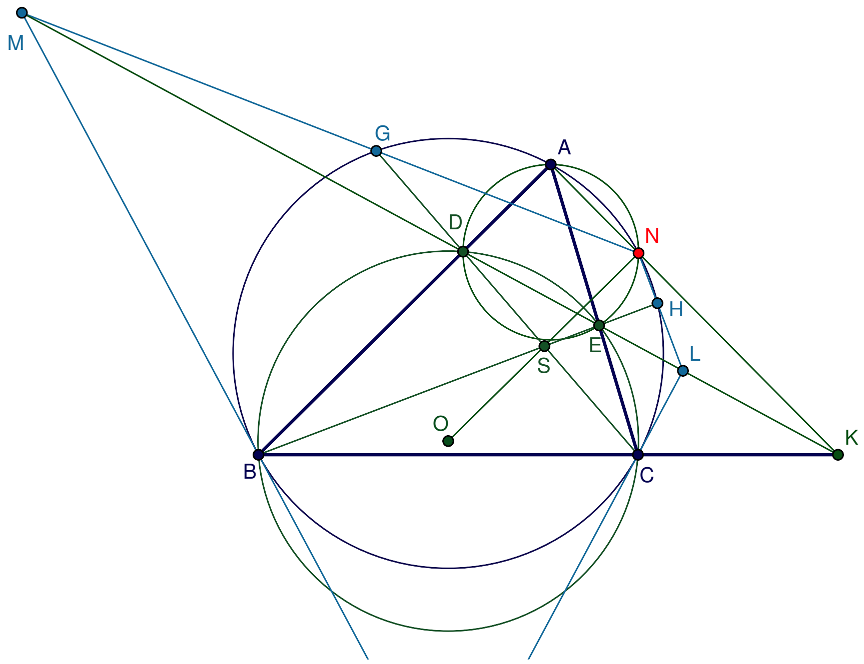

Figure 1: Figure to Proposition 1

We start off with the rather easy proof of the properties (a), (b) and (c).

Lemma 1. *In the configuration of Proposition 1, the circumcircle of $\triangle ABC$, the circumcircle of $\triangle ADE$ and AK concur.*

Proof. Let N be the point where AK meets the circumcircle of $\triangle ABC$ again (apart from A). Then, using the power of K with respect to the circumcircle of $\triangle ABC$ and the cyclic quadrilateral $BCED$, we have

$$KD \cdot KE = KB \cdot KC = KA \cdot KN,$$

so N lies on the circumcircle of $\triangle ADE$. $\square$

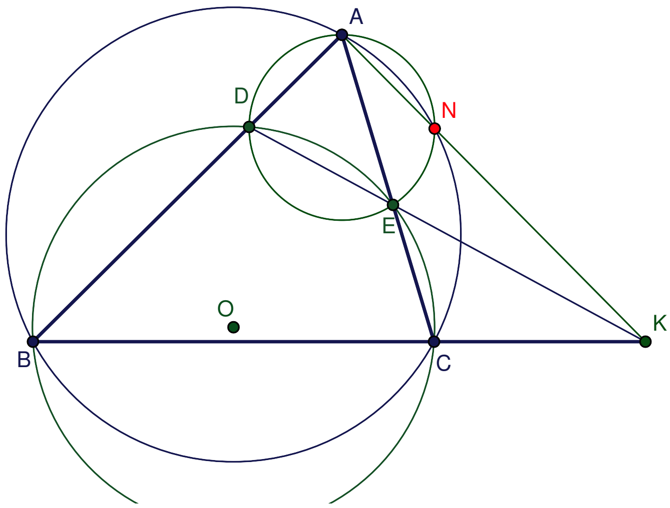

Figure 2: Figure to Lemma 1

Next, we are going to handle the property (d) of Proposition 1 (which is also the problem statement of Problem 1).

Lemma 2. *In the configuration of Proposition 1, the line ON is perpendicular to AK, where N is the second point of intersection of the circumcircles of $\triangle ABC$ and $\triangle ADE$.*

We already know from Lemma 1 that AN, DE and BC concur at K. I will give two proofs of Lemma 2.

First Proof of Lemma 2. To avoid different cases for different arrangements of points, I will make use of directed angles modulo 180°.

Since $BCED$ and $ADEN$ are cyclic we have

$$\measuredangle KCE = \measuredangle BCE = \measuredangle BDE = \measuredangle ADE = \measuredangle ANE = \measuredangle KNE,$$

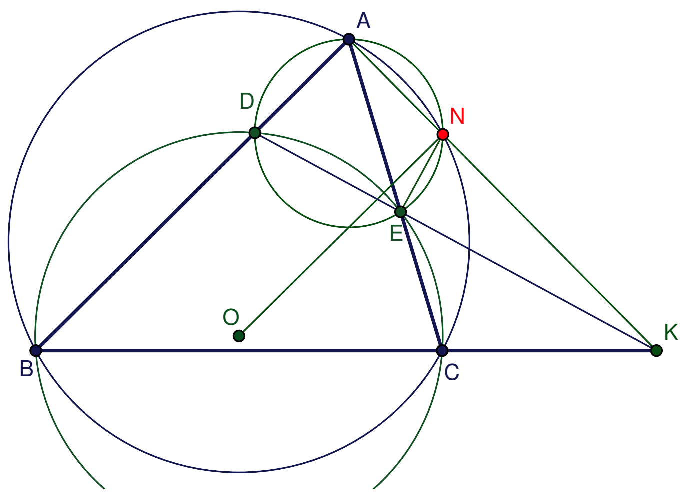

Figure 3: Figure to Lemma 2

so $CKNE$ is cyclic. Moreover,

$$\begin{aligned}
\measuredangle CND &= \measuredangle CNE + \measuredangle END \\
&= \measuredangle CKE + \measuredangle EAD \\
&= \measuredangle BKD + \measuredangle CAB \\
&= -(\measuredangle KDB + \measuredangle DBK) - (\measuredangle ABC + \measuredangle BCA) \\
&= \measuredangle BDK + \measuredangle KBD + \measuredangle CBA + \measuredangle ACB \\
&= (\measuredangle KBD + \measuredangle CBA) + (\measuredangle BDK + \measuredangle ACB) \\
&= (\measuredangle CBD + \measuredangle CBD) + (\measuredangle BDE + \measuredangle ECB) \\
&= 2 \cdot \measuredangle CBD + (\measuredangle BDE + \measuredangle EDB) \\
&= 2 \cdot \measuredangle CBD = \measuredangle COD.
\end{aligned}$$

Hence, $OCND$ is cyclic, so

$$\begin{aligned}
\measuredangle ONK &= \measuredangle ONC + \measuredangle CNK \\
&= \measuredangle ODC + \measuredangle CEK \\
&= (90^\circ - \measuredangle CBD) + \measuredangle CED \\
&= 90^\circ + (\measuredangle CED - \measuredangle CBD) = 90^\circ.
\end{aligned}$$ $\square$

Second Proof of Lemma 2. It is sufficient to prove that $AO^2 - KO^2 = AN^2 - KN^2$.

As in the first proof, we can show that $CKNE$ is cyclic. Let r be the circumradius of the cyclic quadrilateral $BCED$. Then, by the power-of-a-point theorem, we have

$$\begin{aligned}AO^2 - KO^2 &= (AO^2 - r^2) - (KO^2 - r^2)\\ &= (AE \cdot AC) - (KE \cdot KD)\\ &= AN \cdot AK - KN \cdot KA\\ &= AK \cdot (AN - NK)\\ &= (AN + NK) \cdot (AN - NK)\\ &= AN^2 - NK^2 = AN^2 - KN^2. \qquad \square\end{aligned}$$

The proof of property (e) is also rather trivial if one is familiar with basic theorems about the pole-polar-transformation (if not, see [3]).

Lemma 3. *In the configuration Proposition 1, the line OS is perpendicular to AK.*

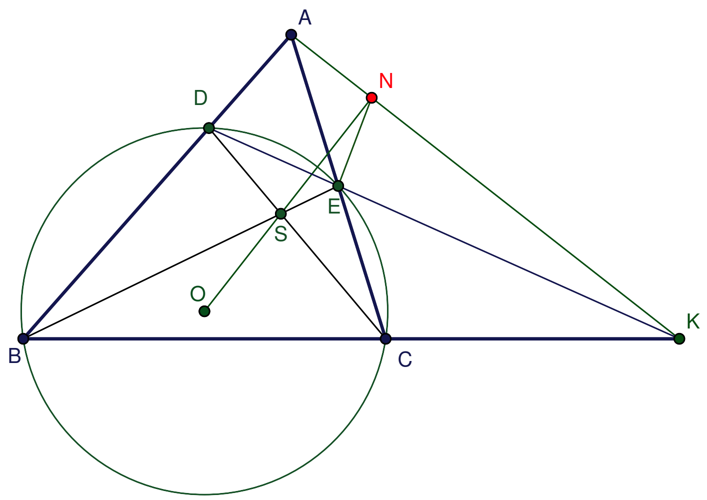

Figure 4: Figure to Lemma 3

Proof. This directly follows from Theorem 4 of [3], since AK is the polar of S with respect to the circumcircle of $BCED$. $\square$

Now the remaining two properties (f) and (g) of Proposition 1 are the statements to be proven in Problem 2. However, for these statements, the given property that $BCED$ is cyclic is not neccessary, in fact, the following generalisation of Problem 2 can be proven:

Lemma 4. *Let ABC be a triangle with circumcircle Γ and let D and E be arbitrary points on AB and AC respectively. Let G and H be the second points of intersection of CD and BE with Γ respectively. The line CD meets the tangent line at B to Γ at M and the tangent line at C to Γ at L. Then MG and LH concur on Γ. Moreover, if BC and DE are not parallel and intersect at K, then the point of concurrence is the point where AK meets Γ apart from A again and if BC and DE are parallel, then the point connecting A with the point of concurrence is parallel to BC.*

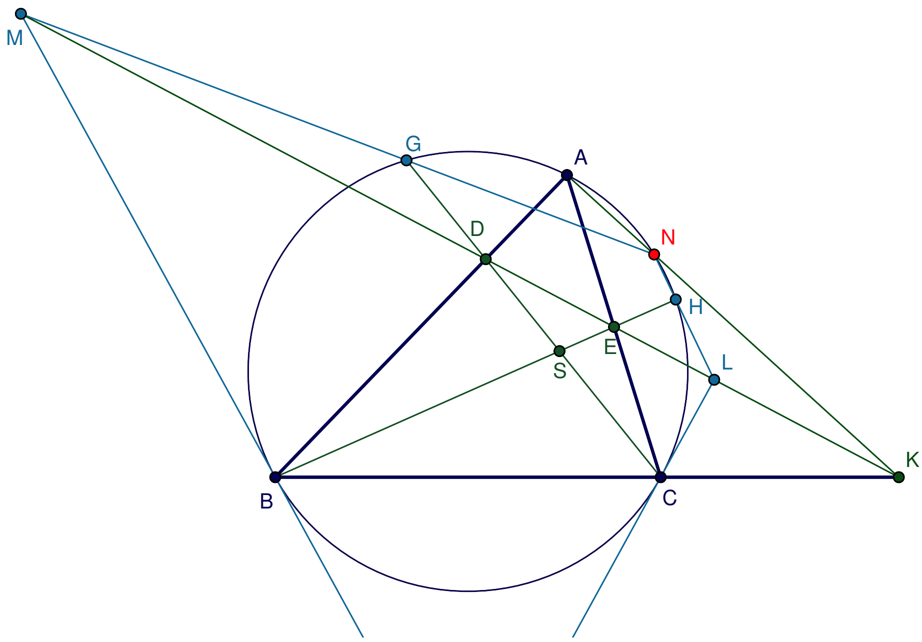

Figure 5: Figure to Lemma 4

Proof. Suppose that BC and DE are not parallel and intersect at K. Let N be the point where AK meets Γ apart from A again. Furthermore, let GN meet the tangent line at B to Γ at M'.

Using Pascal's Theorem on the points B, B, C, G, N and A (the line BB being interpreted as the tangent line at B to Γ), we see that $M' = BB \cap GN, K = BC \cap NA$ and $D = CG \cap AB$ are collinear. It follows that M' is the point where the tangent line to Γ at B meets the line DE since D, E and K are collinear. But this point was defined as M, so $M = M'$ which means that M, G and N are collinear. With the same reasoning, we can show that L, H and N are collinear, so Lemma 4 is proven for BC and DE not being parallel. The case for BC and DE being parallel is left as an exercise to the reader. $\square$

We have seen in the proof of Lemma 2 that the quadrilateral $CKNE$ is cyclic. With the same arguments, we can prove that $BKND$ is cyclic as well. Thus, the point N can also be seen as the point where the circumcircles of the triangles ABC, BKD, ADE and CKE meet. This however, is a famous theorem.

Theorem 1. *Suppose that g_1, g_2, g_3 and g_4 are four lines, no two of them being parallel and no three of them being concurrent. These four lines form four triangles. Then the circumcircles of these four triangles concur at a point M, which is also known as the* Miquel-Point *of the complete quadrilateral formed by g_1, g_2, g_3, g_4.*

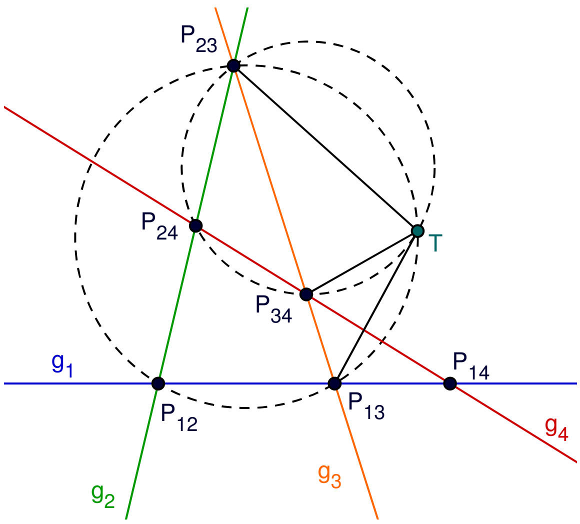

Figure 6: Figure to Theorem 1

Proof. For $1 \leq i < j \leq 4$, let the lines g_i and g_j intersect at P_{ij}. Let M be the second point of intersection of the circumcircles of $\triangle P_{12}P_{13}P_{23}$ and $\triangle P_{23}P_{24}P_{34}$. Then, using directed angles modulo 180°, we have

$$\begin{aligned}
\measuredangle MP_{34}P_{14} &= \measuredangle MP_{34}P_{24} \\
&= \measuredangle MP_{23}P_{24} \\
&= \measuredangle MP_{23}P_{12} \\
&= \measuredangle MP_{13}P_{12} \\
&= \measuredangle MP_{13}P_{14}
\end{aligned}$$

and thus, M lies on the circumcircle of the triangle $P_{13}P_{14}P_{34}$. With the same argument, we can show that M also lies on the circumcircle of the triangle $P_{23}P_{24}P_{34}$. □

In the configuration of Proposition 1, we see that N is the Miquel-Point of the complete quadrilateral formed by the lines AB, AC, BC, DE, which, since $BCED$ is cyclic, we have deduced to lie also on the line AK. It can easily be shown that the converse also holds:

Theorem 2. *Let A, B, C, D, E and K be six distinct points in a plane so that the points A, D, B, the points A, E, C, the points B, C, K and the points D, E, K are collinear respectively. Then the miquel-Point N of the complete quadrilateral formed by the lines AB, AC, BC and DE lies on AK if and only if $BCED$ is a cyclic quadrilateral.*

Proof. Again, we will make use of directed angles modulo 180°.

The points A, K and N are collinear if and only if $\measuredangle ANE = \measuredangle KNE$. However, N lies on the circumcircles of the triangles ADE and KCE, so $\measuredangle ANE = \measuredangle ADE$ and $\measuredangle KNE = \measuredangle KBE$. Hence,

$$\begin{aligned} & \measuredangle ANE = \measuredangle KNE \\ \Leftrightarrow \quad & \measuredangle ADE = \measuredangle KCE \\ \Leftrightarrow \quad & \measuredangle BDE = \measuredangle BCE \end{aligned}$$

and the latter holds if and only if $BCED$ is cyclic. □

References

[1] Mathlinks, *Circle center O passes through the vertices A and C*, `http://www.mathlinks.ro/Forum/viewtopic.php?t=60787`

[2] Mathlinks, *Two lines meet at circle*, `http://www.mathlinks.ro/viewtopic.php?t=195491`

[3] Kin Y. Li, *Pole and Polar*, Mathematical Excalilbur, 2006, Issue 4, `http://www.math.ust.hk/excalibur/v11_n4.pdf`