Inequalities

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1 Definitions

Definition 1.1. A function is said to be *convex* (or *concave up*) if the line segment between any two points on the graph of the function lies above the graph. A function is said to be *concave* (or *concave down*) if the line segment between any two points on the graph of the function lies below the graph. Formally, a function $f : I \to \mathbb{R}$ is concave up if for any two points $x_1, x_2 \in I$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

The sign of the inequality is flipped if f is concave down. An easy way to check if f is convex in an interval is by taking its second derivative, and checking that it is nonnegative for the interval.

Definition 1.2. A sequence $a_1, a_2, \cdots, a_n$ is said to *majorize* the sequence $b_1, b_2, \cdots, b_n$ if

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$$

for all $k = 1, 2, \cdots, n$, and is written as $(a_1, a_2, \cdots, a_n) \succ (b_1, b_2, \cdots, b_n)$.

Definition 1.3. A shorthand way of writing inequalities is by using summation notation. Define the *cyclic sum* of $f(a, b, c)$ by

$$\sum_{cyc} f(a, b, c) = f(a, b, c) + f(b, c, a) + f(c, a, b)$$

and define the *symmetric sum* of $f(a, b, c)$ by

$$\sum_{sym} f(a, b, c) = f(a, b, c) + f(a, c, b) + f(b, a, c) + f(b, c, a) + f(c, a, b) + f(c, b, a).$$

Cyclic products and *symmetric products* are defined analogously.

Definition 1.4. The *gradient* of a function f is defined as

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}.$$

2 The Standard Dozen

As listed in the beginning of Thomas Mildorf's *Olympiad Inequalities*[1], the following twelve inequalities are the most famous in Olympiad mathematics.

[1] `http://www.artofproblemsolving.com/Resources/Papers/MildorfInequalities.pdf`

Theorem 2.1 (Jensen). *Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, \cdots, x_n \in I$ and any nonnegative reals $\omega_1, \omega_2, \cdots, \omega_n$,*

$$\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n) \geq (\omega_1 + \omega_2 + \cdots + \omega_n) f\left(\frac{\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n}{\omega_1 + \omega_2 + \cdots + \omega_n}\right).$$

Theorem 2.2 (Weighted Power Mean). *If $x_1, x_2, \cdots, x_n$ are nonnegative reals with a positive sum, then*

$$f(r) = \left(\frac{\omega_1 x_1^r + \omega_2 x_2^r + \cdots + \omega_n x_n^r}{\omega_1 + \omega_2 + \cdots + \omega_n}\right)^{\frac{1}{r}}$$

is a non-decreasing function of r, with the convention that $r = 0$ is the weighted gometric mean. f is strictly increasing unless all the x_i are equal except possible for $r \in (-\omega, 0]$, where if some x_i is zero f is identically 0. In particular, $f(1) \geq f(0) \geq f(-1)$ gives the AM-GM-HM inequality.

Theorem 2.3 (Hölder). *Let $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots$; $z_1, z_2, \cdots, z_n$ be sequences of nonnegative real numbers, and let $\lambda_a, \lambda_b, \cdots, \lambda_z$ be positive reals which sum to 1. Then*

$$(a_1 + a_2 + \cdots + a_n)^{\lambda_a}(b_1 + b_2 + \cdots + b_n)^{\lambda_b} \cdots (z_1 + z_2 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \cdots z_n^{\lambda_z}.$$

Theorem 2.4 (Rearrangement). *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two nondecreasing sequences of real numbers. Then, for any permutation π of $\{1, 2, \cdots, n\}$, we have*

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{\pi(1)} + a_2 b_{\pi(2)} + \cdots + a_n b_{\pi(n)} \geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1.$$

Theorem 2.5 (Chebyshev). *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two nondecreasing sequences of real numbers. Then*

$$\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \frac{b_1 + b_2 + \cdots + b_n}{n} \geq \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}.$$

Theorem 2.6 (Schur). *Let a, b, c be nonnegative reals and $r > 0$. Then*

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0.

Theorem 2.7 (Newton). *Let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers. Define the symmetric polynomials $s_0, s_1, \cdots, s_n$ by $(x + x_1)(x + x_2) \cdots (x + x_n) = s_n x^n + \cdots + s_1 x + s_0$, and define the symmetric averages by $d_i = \frac{s_i}{\binom{n}{i}}$. Then*

$$d_i^2 \geq d_{i+1} d_{i-1}.$$

Theorem 2.8 (Maclaurin). *Let d_i be defined as above. Then*

$$d_1 \geq \sqrt{d_2} \geq \sqrt[3]{d_3} \geq \cdots \geq \sqrt[n]{d_n}.$$

Theorem 2.9 (Majorization). *Let $f : I \to \mathbb{R}$ be a convex function on I and suppose that the sequence $x_1, x_2, \cdots, x_n$ majorizes the sequence $y_1, y_2, \cdots, y_n$, where $x_i, y_i \in I$. Then*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq f(y_1) + f(y_2) + \cdots + f(y_n).$$

Theorem 2.10 (Popoviciu). *Let $f : I \to \mathbb{R}$ be a convex function on I and let $x, y, z \in I$. Then for any*

positive reals p, q, r,

$$pf(x)+qf(y)+rf(z)+(p+q+r)f\left(\frac{px+qy+rz}{p+q+r}\right) \geq (p+q)f\left(\frac{px+qy}{p+q}\right)+(q+r)f\left(\frac{qy+rz}{q+r}\right)+(r+p)f\left(\frac{rz+px}{r+p}\right).$$

Theorem 2.11 (Bernoulli). *For all $r \geq 1$ and $x \geq -1$,*

$$(1+x)^r \geq 1+xr.$$

Theorem 2.12 (Muirhead). *Suppose the sequence $a_1, a_2, \cdots, a_n$ majorizes the sequence $b_1, b_2, \cdots, b_n$. Then for any positive reals $x_1, x_2, \cdots, x_n$,*

$$\sum_{sym} x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}.$$

3 Monotonicity

Monotonicity of functions can be used to prove that the left side increases faster than the right side, so that the left side is always greater than the right side.

Theorem 3.1. *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions, where $I = [a, b]$, such that $f(a) = g(a)$, f and g are differentiable on I, and $f'(x) > g'(x)$ for all $x \in I$. Then $f(x) > g(x)$ for all $x \in I$.*

Example 3.1. Let x, y, and z be nonnegative real numbers such that $x + y + z = 1$. Prove the inequality

$$0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}.$$

Solution. WLOG let $0 \leq x \leq y \leq z \leq 1$. Hence $x \leq \frac{1}{3}$, and so $xy + yz + zx - 2xyz = (1 - 3x)yz + xy + zx + xyz \geq 0$, which is the left inequality.

Now we prove the right inequality. By AM-GM, $yz \leq \left(\frac{y+z}{2}\right)^2 = \left(\frac{1-x}{2}\right)^2$. Thus

$$xy+yz+zx-2xyz = x(y+z)+yz(1-2x) \leq x(1-x)+\left(\frac{1-x}{2}\right)^2(1-2x) = \frac{-2x^3+x^2+1}{4}.$$

Let $f(x) = \frac{-2x^3+x^2+1}{4}$. Differentiating f, we see that

$$f'(x) = \frac{-3x^2+x}{2} = \frac{3x}{2}\left(\frac{1}{3}-x\right) \geq 0,$$

implying that f is increasing on $[0, \frac{1}{3}]$. Therefore, $f(x) \leq f\left(\frac{1}{3}\right) = \frac{7}{27}$, as desired. $\square$

4 *pqr* Method

The *pqr* method is a method of solving inequalities by making substitutions and simplifying the inequality. Let x, y, and z be positive reals and define the variables p, q, and r by $p = x + y + z$, $q = xy + yz + zx$, $r = xyz$. A list of relationships between the variables x, y, z, and p, q, r are given below, where $\sum$ and $\prod$ are all taken to be cyclic.

1. $\sum x^2 = p^2 - 2q$
2. $\sum x^3 = p(p^2 - 3q) + 3r$
3. $\sum x^2y^2 = q^2 - 2pr$
4. $\sum x^4 = (p^2 - 2q)^2 - 2(q^2 - 2pr)$
5. $\prod(x + y) = pq - r$
6. $\sum(x + y)(y + z) = p^2 + q$
7. $\sum(x + y)^2(y + z) = (p^2 + q)^2 - 4p(pq - r)$
8. $\sum xy(x + y) = pq - 3r$
9. $\prod(1 + x) = 1 + p + q + r$
10. $\sum(1 + x)(1 + y) = 3 + 2p + q$
11. $\sum(1 + x)^2(1 + y)^2 = (3 + 2p + q)^2 - 2(3 + p)(1 + p + q + r)$
12. $\sum x^2(y + z) = pq - 3r$
13. $\sum x^3y^3 = q^3 - 3pqr - 3r^2$
14. $\sum xy(x^2 + y^2) = p^2q - 2q^2 - pr$
15. $\prod(1 + x^2) = p^2 + q^2 + r^2 - 2pr - 2q + 1$
16. $\prod(1 + x^3) = p^3 + q^3 + r^3 - 3pqr - 3pq - 3r^2 + 3r + 1$

There are also inequalities that relate p, q, and r to themselves:

1. $p^3 - 4pq + 9r \geq 0$
2. $p^4 - 5p^2q + 4q^2 + 6pr \geq 0$
3. $pq \geq 9r$
4. $p^2 \geq 3q$
5. $p^3 \geq 27r$
6. $q^3 \geq 27r^2$
7. $q^2 \geq 3pr$
8. $2p^3 + 9r \geq 7pq$
9. $p^4 + 3q^2 \geq 4p^2q$
10. $2p^3 + 9r^2 \geq 7pqr$
11. $p^2q + 3pr \geq 4q^2$
12. $q^3 + 9r^2 \geq 4pqr$
13. $pq^2 \geq 2p^2r + 3qr$

The implementation of the *pqr* method is simple, but usually hidden. The transformation from the given inequality into *pqr* notation is probably the trickiest part, and the rest is generally just substituting the above inequalities/identities.

Example 4.1. Let $x, y, z > 0$ such that $x + y + z = 1$. Prove the inequality

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$

Solution. Let $p = x+y+z = 1$, $q = xy+yz+zx$, and $r = xyz$. The desired inequality is $(x+1)(y+1)(z+1) \geq 64xyz$. The left side expands to $1+p+q+r$, and the right side is $64r$. It remains to prove $2+q \geq 63r$. We have that $pq \geq 9r$, that is, $q \geq 9r$. In addition, $p^3 \geq 27r$, so $r \leq \frac{1}{27}$. Now, $2+q \geq 63r \iff 2 \geq 54r$, which is true. $\square$

Example 4.2 (TSTST 2012)**.** Positive real numbers x, y, z satisfy $xyz + xy + yz + zx = x + y + z + 1$. Prove that

$$\frac{1}{3}\left(\sqrt{\frac{1+x^2}{1+x}}+\sqrt{\frac{1+y^2}{1+y}}+\sqrt{\frac{1+z^2}{1+z}}\right) \leq \left(\frac{x+y+z}{3}\right)^{5/8}.$$

Solution. (David Stoner) The given condition implies that $p+1 = q+r$. Note that

$$\begin{aligned}\sum_{cyc}\frac{1+x^2}{1+x} = p + \sum_{cyc}\frac{1-x}{1+x} &= p + \frac{\sum_{cyc}(x-1)(y+1)(z+1)}{p+q+r+1} \\ &= p + \frac{-q-3r+p+3}{2p+2} \\ &= p + \frac{1-r}{p+1} \\ &= \frac{p^2+p-r+1}{p+1} = \frac{p^2+q}{p+1}.\end{aligned}$$

We have that $3q \leq p^2$, so $\sum_{cyc}\frac{1+x^2}{1+x} \leq \frac{4p^2}{3(p+1)}$. By Jensen's Inequality,

$$\frac{1}{3}\sum_{cyc}\sqrt{\frac{1+x^2}{1+x}} \leq \sqrt{\frac{1}{3}\sum_{cyc}\frac{1+x^2}{1+x}} \leq \sqrt{\frac{4p^2}{9(p+1)}}.$$

Now $\frac{4p^2}{9(p+1)} \leq \left(\frac{p}{3}\right)^{5/4} \iff \frac{4^4p^3}{9^4(p+1)^4} \leq \frac{1}{3^5} \iff 27(p+1)^4 \geq 256p^3$, which is true since $3p+3 = p+p+p+3 \geq 4\sqrt[4]{3p^3}$. $\square$

5 *uvw* Method

The *uvw* method is similar to the *pqr* method in that it also involves substitution, but instead we use the substitutions $3u = a+b+c$, $3v^2 = ab+bc+ca$, and $w^3 = abc$. The theorems and proofs of these theorems are outlined in Tejs Knudsen's article[2] on this method.

Theorem 5.1 (Idiot)**.** *If $a, b, c \geq 0$, then $u \geq v \geq w$.*

Theorem 5.2 (Positivity)**.** *$a, b, c \geq 0$ if and only if $u, v^2, w^3 \geq 0$.*

[2] `http://tinyurl.com/uvwmethod`

Theorem 5.3 (*uvw*)**.** *u, v^2, w^3 are real if and only if $u^2 \geq v^2$ and*

$$w^3 \in \left[3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}, 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}\right].$$

Example 5.1. (IMO 2006/3) Determine the least real number M such that the inequality

$$|ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b, and c.

Solution. Since $ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2) = (a-b)(b-c)(a-c)(a+b+c)$, the given inequality is equivalent to

$$(a-b)^2(b-c)^2(c-a)^2(a+b+c)^2 \leq M^2(a^2+b^2+c^2)^4.$$

Making the substitutions $3u = a+b+c$, $3v^2 = ab+bc+ca$, and $w^3 = abc$, the inequality becomes

$$9u^2 \cdot (3^3(-(w^3-(3uv^2-2u^3))^2 + (4u^2-v^2)^3)) \leq M^2(9u^2-6v^2)^4.$$

Since $-(w^3-(3uv^2-2u^3))^2 \leq 0$, it suffices to prove that $12u^2(u^2-v^2)^3 \leq M^2(3u^2-2v^2)^4$. Dividing through by v^8 and making the substitution $t = \frac{u}{v}$, we have $12t(t-1)^3 \leq M^2(3t-2)^4$. Hence, the minimum value of M is attained at the maximum of $f(t) = \frac{12t(t-1)^3}{(3t-2)^4}$.

Differentiating f, we see that $f'(t) = \frac{12(t-1)^2(t+2)}{(3t-2)^5}$, which is positive for $t \in [1, \infty)$. Thus f is increasing in $[1, \infty)$, so in this interval, the maximum of f is

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{12t(t-1)^3}{(3t-2)^4} = \frac{12}{81} = \frac{4}{27}.$$

If t is negative, solving $f'(t) = 0$ shows that $t = -2$ yields a maximum. Now $f(-2) = \frac{3^4}{2^9} > \frac{4}{27}$, so the minimum of M^2 is $\frac{3^4}{2^9}$, implyping that the minimum value of M is $\frac{9}{16\sqrt{2}}$. □

6 ABC Method

The *Abstract Conconcreteness Method*, abbreviated the *ABC method*, is an extremely powerful but limited way of proving polynomial inequalities. The main idea of the method is to express an inequality as a function of abc, $ab+bc+ca$, and $a+b+c$, and then observing where the minimum and maximum values occur, and checking the inequality for the specific case.

Theorem 6.1 (ABC Theorem)**.** *If the function $f(abc, ab+bc+ca, a+b+c)$ is monotonic/convex/concave, then f achieves its maximum and minimum values on $\mathbb{R}$ when $(a-b)(b-c)(c-a) = 0$, and on $\mathbb{R}^+$ when $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.*

This theorem produces a few corollaries:

Corollary 6.2. *Let $f(abc, ab+bc+ca, a+b+c)$ be a linear function/quadratic trinomial with variable abc. Then f achieves its maximum and minimum values on $\mathbb{R}$ iff $(a-b)(b-c)(c-a) = 0$, and on $\mathbb{R}^+$ iff $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.*

Corollary 6.3. *All symmetric three-variable polynomials of degree less than or equal to 5 achieve their maximum and minimum values on $\mathbb{R}$ if and only if $(a-b)(b-c)(c-a) = 0$, and on $\mathbb{R}^+$ if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.*

Corollary 6.4. *All symmetric three-variable polnomials of degree less than or equal to 8 with nonnegative coefficient of $(abc)^2$ in the form $f(abc, ab+bc+ca, a+b+c)$ achieve their maximum on $\mathbb{R}$ iff $(a-b)(b-c)(c-a) = 0$, and on $\mathbb{R}^+$ iff $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.*

Example 6.1. Let $a, b, c > 0$ be real numbers. Prove that

$$\frac{abc}{a^3+b^3+c^3} + \frac{2}{3} \geq \frac{ab+bc+ca}{a^2+b^2+c^2}.$$

Solution. Clearing denominators and bringing all of the terms to one side, we have to prove that

$$abc(a^2+b^2+c^2) + \frac{2}{3}(a^3+b^3+c^3)(a^2+b^2+c^2) - (a^3+b^3+c^3)(ab+bc+ca) \geq 0.$$

Both $a^2+b^2+c^2$ and $a^3+b^3+c^3$ can be written as polynomial expressions of abc, $ab+bc+ca$, and $a+b+c$, and so this function has degree 3 in abc. By Corollary 6.2, the minimum of the left hand side occurs when $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.

For the first case, WLOG let $a = b$. We have to prove that

$$\frac{a^2c}{2a^3+c^3} + \frac{2}{3} \geq \frac{a^2+2ac}{2a^2+c^2} \iff \frac{2(a-b)^4(a+b)}{3(2a^2+b^2)(2a^3+b^3)} \geq 0,$$

which is true. If $abc = 0$, then WLOG let $a = 0$. The inquality becomes

$$\frac{2}{3} \geq \frac{bc}{b^2+c^2} \iff b^2+c^2+3(b-c)^2 \geq 0,$$

which is true. $\square$

7 Sum of Squares (SOS)

Sum of Squares, abbreviated *SOS*, is a brute-force but powerful method of proving 3-variable homogenous inequalities one of whose equality cases is $a = b = c$. This is the technique that the Vietnamese are well-known for on the Inequalities forum on AoPS. The main idea of the method is to transform an inequality into the form

$$S_a(a-b)^2 + S_b(b-c)^2 + S_c(c-a)^2 \geq 0$$

where S_a, S_b, and S_c are functions of a, b, and c.

The steps to expressing an inequality X into SOS form are:

1. Group X into one or more terms that are 0 when $a = b = c$.
2. Write everything as a multiple of $(a-b)$, $(b-c)$, or $(c-a)$.
3. Group everything by $(a-b)$, $(b-c)$, and $(c-a)$, and ensure the coefficients vanish when $a = b = c$.
4. Write everything as a multiple of $(a-b)^2$, $(b-c)^2$, $(c-a)^2$, $(a-b)(b-c)$, $(b-c)(c-a)$, or $(c-a)(a-b)$.
5. Replace $(a-b)(b-c)$ with $\frac{1}{2}((a-c)^2 - (a-b)^2 - (b-c)^2)$.

This exact procedure is reiterated in David Arthur's article[3] on Sum of Squares for the Canada IMO Training Camp, and is much better explained because the author provides an example to go along with it.

[3] `http://tinyurl.com/davidarthursos`

Assuming that it has equality case $a = b = c$, it is possible to write any inequality in SOS form using this procedure. Steps 4 and 5 are optional, as there aren't many inequalities that have that extraneous $(a-b)(b-c)$ term.

Example 7.1. Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\le\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

for all positive real numbers a, b, and c satisfying $a^2+b^2+c^2=1$.

Solution 1. Let $x=a^2$, $y=b^2$, and $z=c^2$. It suffices to show that $\sum_{cyc}\frac{1}{x+3y+3z}\ge\sum_{cyc}\frac{1}{3x+2y+2z}$ after using AM-GM and homogenizing. Now,

$$\begin{aligned}\sum_{cyc}\left(\frac{1}{x+3y+3z}-\frac{1}{3x+2y+2z}\right)&=\sum_{cyc}\frac{(x-y)+(x-z)}{(x+3y+3z)(3x+2y+2z)}\\&=\sum_{cyc}(x-y)\left(\frac{1}{(x+3y+3z)(3x+2y+2z)}-\frac{1}{(y+3x+3z)(3y+2x+2z)}\right)\\&=\sum_{cyc}(x-y)\left(\frac{3(x^2-y^2)+xz-yz}{(x+3y+3z)(3x+2y+2z)(y+3x+3z)(3y+2x+2z)}\right)\\&=\sum_{cyc}(x-y)^2\left(\frac{3x+3y+z}{(x+3y+3z)(3x+2y+2z)(y+3x+3z)(3y+2x+2z)}\right)\ge 0.\end{aligned}$$

□

Solution 2. As above, it suffices to show that $\sum_{cyc}\frac{1}{a^2+3b^2+3c^2}\ge\sum_{cyc}\frac{1}{3a^2+2b^2+2c^2}$. Note that

$$(a^2+3b^2+c^2,3a^2+b^2+3c^2,a^2+3b^2+3c^2)\succ(3a^2+2b^2+2c^2,2a^2+3b^2+2c^2,2a^2+2b^2+3c^2).$$

Since $f(x)=\frac{1}{x}$ is concave up for $x>0$, applying Majorization Inequality gives us the desired inequality. □

Solution 3. By Muirhead's Inequality, we have that $\sum_{cyc}x^3y^3z\ge\sum_{cyc}x^3y^2z^2$. Making the substitution $x=t^{a^2-\frac{1}{7}}$, $y=t^{b^2-\frac{1}{7}}$, and $z=t^{c^2-\frac{1}{7}}$ gives us

$$\sum_{cyc}t^{3a^2+3b^2+c^2-1}\ge\sum_{cyc}t^{3a^2+2b^2+2c^2-1},$$

which implies that

$$\int_0^1\sum_{cyc}t^{3a^2+3b^2+c^2-1}\,dt\ge\int_0^1\sum_{cyc}t^{3a^2+2b^2+2c^2-1}\,dt,$$

implying the desired inequality. □

8 SMV Method

The *Strongly Mixing Variables Method*, or just the *Mixing Variables Method*, abbreviated *SMV*, is a technique based on averaging. For proofs to the following theorems and more example problems, Pham Kim Hung's

article on the SMV Method[4] and Zdravko Cvetkovski's book *Inequalities - Theorems, Techniques, and Selected Problems* are good resources.

The main idea of the technique is proving that substituting the average of two variables in place of the two variables increases/decreases a function, and then after applying this substitution infinitely many times, all of the variables must end up being equal. Written in mathematical terminology,

Lemma 8.1. *Let $(x_1, x_2, \cdots, x_n)$ be a sequence of real numbers. Define a transformation as follows:*

1. *Let $x_i = \min\{x_1, x_2, \cdots, x_n\}$ and $x_j = \min\{x_1, x_2, \cdots, x_n\}$.*
2. *Replace x_i and x_j with $\frac{x_i+x_j}{2}$.*

After infinitely many of the above transformation, each term of sequence $x_1, x_2, \cdots, x_n$ tends to the same limit $x = \frac{x_1+x_2+\cdots+x_n}{n}$.

In fact, this transformation does not necessarily have to be the average $a, b \to \frac{a+b}{2}$. It can be the geometric mean $\sqrt{ab}$, or the quadratic mean $\sqrt{\frac{a^2+b^2}{2}}$, and so on. Let us call such transformations Δ-*transformations.*

Theorem 8.2 (SMV Theorem)**.** *Let $f : I \subset \mathbb{R}^n \to \mathbb{R}$ be a symmetric and continuous function such that $f(a_1, a_2, \cdots, a_n) \geq f(b_1, b_2, \cdots, b_n)$, where the sequence $(b_1, b_2, \cdots, b_n)$ is obtained from the sequence $(a_1, a_2, \cdots, a_n)$ by some Δ-transformation. Then $f(x_1, x_2, \cdots, x_n) \geq f(x, x, \cdots, x)$, where $x = \frac{x_1+x_2+\cdots+x_n}{n}$.*

Example 8.1 (Nesbitt)**.** Let a, b, and c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Solution. Define $f(a,b,c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$. Note that

$$\begin{aligned} f(a,b,c) - f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right) &= \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) - \left(\frac{a+b}{a+b+2c} + \frac{a+b}{a+b+2c} + \frac{c}{a+b}\right) \\ &= \frac{a}{b+c} + \frac{b}{c+a} - \frac{2(a+b)}{a+b+2c} \\ &= \frac{a^3 + a^2c + b^2c + b^3 - 2abc - ab^2 - a^2b}{(b+c)(a+c)(a+b+2c)} \geq 0. \end{aligned}$$

Hence $f(a,b,c) \geq f\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$, and so applying the SMV Theorem, it remains to prove that $f\left(\frac{a+b+c}{3}, \frac{a+b+c}{3}, \frac{a+b+c}{3}\right) \geq \frac{3}{2}$, which is clearly true. □

Example 8.2 (ISL 1993)**.** Let a, b, c, and d be four nonnegative real numbers satisfying $a+b+c+d = 1$. Prove the inequality

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

Solution. WLOG let $a \leq b \leq c \leq d$. Denote $f(a,b,c,d) = abc + bcd + cda + dab - \frac{176}{27}abcd = ac(b+d) + bd\left(a + c - \frac{176}{27}ac\right)$. Note that $a + c \leq \frac{1}{2}(a+b+c+d) = \frac{1}{2}$, and so $a + c \geq \frac{4ac}{a+c} \geq 8ac > \frac{176}{27}ac$, implying that $a + c - \frac{176}{27}ac > 0$. Hence,

$$f(a,b,c,d) \leq f\left(a, \frac{b+d}{2}, c, \frac{b+d}{2}\right).$$

[4] https://www.awesomemath.org/wp-content/uploads/reflections/2006_6/2006_6_mixing.pdf

By the SMV Theorem, we have that $f(a,b,c,d) \le f(a,t,t,t)$, where $t = \frac{b+c+d}{3}$. Now we must prove that $t^2(3a+t) \le \frac{1}{27} + \frac{176}{27}at^3$ given that $a + 3t = 1$. Letting $a = 1 - 3t$, we obtain the polynomial inequality

$$(1-3t)(4t-1)^2(11t+1) \ge 0,$$

which is true. $\square$

9 Dumbassing

To *dumbass* an inequality is to mindlessly brute-force the inequality by first expanding it out into a polynomial expression, and then using Muirhead and AM-GM many, many times and adding everything up to obtain the desired inequality. This "art of dumbassing" is enunciated in Brian Hamrick's article[5] on Chinese Dumbass Notation.

10 Lagrange Multipliers

The *method of Lagrange multipliers* is a technique for finding the maximum and minimum of a function subject to various constraints. This method is highly looked down upon by Olympiad graders, as it involves Multivariable Calculus, and thus it is often referred to as "Lagrange murderpliers" by MOPpers. In addition, in order to receive full points for a Lagrange multipliers solution, one must fully rigorize all of one's steps, such as compactification.

The method is outlined below:

Theorem 10.1. *To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ (assuming that these extreme values exist):*

1. *Find all values of x, y, z, and λ such that*

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z),$$
$$g(x,y,z) = k.$$

2. *Evaluate f at all the points (x,y,z) that result from the previous step. The largest of these values is the maximum value of f; the smallest is the minimum value of f.*

Example 10.1. Let $0 \le a,b,c \le \frac{1}{2}$ be real numbers with $a+b+c=1$. Show that

$$a^3+b^3+c^3+4abc \le \frac{9}{32}.$$

Solution. (Thomas Mildorf) Let $f(a,b,c) = a^3+b^3+c^3+4abc$ and $g(a,b,c) = a+b+c-1$. Because f and g are polynomials, they have continuous first partial derivatives. Moreover, the gradient of g is never zero. Hence, by the theorem of Lagrange multipliers, any extrema occur on the boundary or where $\nabla f = \lambda \nabla g$ for suitable scalars λ. As

$$\nabla f = \langle 3a^2+4bc, 3b^2+4ca, 3c^2+4ab \rangle$$

[5] `http://www.tjhsst.edu/~2010bhamrick/files/dumbassing.pdf`

and $\nabla g = < 1, 1, 1 >$, we have

$$\begin{aligned} \lambda &= 3a^2 + 4bc, \\ &= 3b^2 + 4ca, \\ &= 3c^2 + 4ab, \\ g(a, b, c) &= a + b + c - 1. \end{aligned}$$

We have $3a^2 + 4bc = 3b^2 + 4ca$ or $(a-b)(3a+3b-4c) = (a-b)(3-7c) = 0$ for any permutation of a, b, and c. Hence, without loss of generality, $a = b$. Now, $3a^2 + 4ac = 3c^2 + 4a^2$ and $a^2 - 4ac + 3c^2 = (a-c)(a-3c) = 0$. The interior local extrema therefore occur when $a = b = c$ or when two of $\{a, b, c\}$ are three times as large as the third. Checking, we have $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{7}{27} < \frac{13}{49} = f(\frac{1}{7}, \frac{3}{7}, \frac{3}{7})$. Recalling that $f(a, b, c)$ is symmetric in a, b, and c, the only boundary check we need is $f(\frac{1}{2}, t, \frac{1}{2} - t) \le \frac{9}{32}$ for $0 \le t \le \frac{1}{2}$. We solve

$$\begin{aligned} h(t) &= f\left(\frac{1}{2}, t, \frac{1}{2} - t\right) \\ &= \frac{1}{8} + t^3 + \left(\frac{1}{2} - t\right)^3 + 2t\left(\frac{1}{2} - t\right) \\ &= \frac{1}{4} + \frac{t}{4} - \frac{t^2}{2}. \end{aligned}$$

$h(t)$ is $\frac{1}{4}$ at either endpoint. Its derivative $h'(t) = \frac{1}{4} - t$ is zero only at $t = \frac{1}{4}$. Checking, $h(\frac{1}{4}) = f(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) = \frac{9}{32}$. Since $h(t)$ has a continuous derivative, we are done. (As a further check, we could observe that $h''(t) = -1 < 0$, which guarantees that $h(\frac{1}{4})$ is a local minimum.) $\square$

11 Problems

1. (Iran 1996) Prove that for all positive real numbers a, b, and c,
$$(ab + bc + ca)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \ge \frac{9}{4}.$$
2. (USAMO 2003) Let a, b, and c be positive real numbers. Prove that
$$\frac{(2a+b+c)^2}{2a^2 + (b+c)^2} + \frac{(2b+c+a)^2}{2b^2 + (c+a)^2} + \frac{(2c+a+b)^2}{2c^2 + (a+b)^2} \le 8.$$
3. Let $x > 0$ be a real number. Prove that $x - \frac{x^2}{2} < \ln(x+1)$.
4. (Japan 1997) Show that for all positive reals a, b, and c,
$$\frac{(a+b-c)^2}{(a+b)^2 + c^2} + \frac{(b+c-a)^2}{(b+c)^2 + a^2} + \frac{(c+a-b)^2}{(c+a)^2 + b^2} \ge \frac{3}{5}.$$
5. (IMO 2008) If x, y, and z are real numbers, all different from 1, such that $xyz = 1$, then prove that
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \ge 1.$$

6. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}}+\frac{b}{\sqrt{2c^2+2a^2-b^2}}+\frac{c}{\sqrt{2a^2+2b^2-c^2}}\geq\sqrt{3}.$$

7. (ISL 2009/A2) Let a, b, and c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\leq\frac{3}{16}.$$

8. (ISL 2009/A4) Let a, b, and c be positive real numbers such that $ab+bc+ca\leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\leq\sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)$$

9. (IMC 2006) Compare $\tan(\sin x)$ and $\sin(\tan x)$ for all $x\in(0,\frac{\pi}{2})$.

10. (Gabriel Dospinescu) Prove that for any positive real numbers a, b, and c the following inequality holds:

$$\frac{1}{3a}+\frac{1}{3b}+\frac{1}{3c}+2\left(\frac{1}{2a+b}+\frac{1}{2b+c}+\frac{1}{2c+a}\right)\geq 3\left(\frac{1}{a+2b}+\frac{1}{b+2c}+\frac{1}{c+2a}\right).$$