Inequalities

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"Destroy the inequality today, and it will appear again tomorrow."
–Ralph Waldo Emerson

Here are a few inequalities I like, presented in no particular order.

1. (Tournament of Towns 1997) Let $a, b, c > 0$ satisfy $abc = 1$. Prove that
$$\frac{1}{1+a+b} + \frac{1}{1+a+c} + \frac{1}{1+b+c} \leq 1.$$

2. (APMO 2004) For $a, b, c > 0$, prove that
$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

3. Show that for all real numbers a, b, c,
$$a^6+b^6+c^6-3a^2b^2c^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$
When does equality hold?

4. For $a, b, c > 0$, prove that
$$\sqrt{(a^2b+b^2c+c^2a)(b^2a+c^2b+a^2c)} \geq abc + \sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}.$$

5. (TST 2000) For $a, b, c \geq 0$, prove that
$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

6. For $a, b, c > 0$, prove that
$$\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \leq \frac{3\sqrt{2}}{2}.$$

7. (USAMO 2001) Let $a, b, c \geq 0$ satisfy $a^2+b^2+c^2+abc = 4$. Prove that
$$0 \leq ab+bc+ca-abc \leq 2.$$

8. Let $a, b, c > 1$ satisfy
$$\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1} = 1.$$
Prove that
$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1.$$

9. (Japan 1997) For $a, b, c > 0$, prove that
$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

10. (Vietnam 2002) Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 9$. Prove that

$$2(x + y + z) - xyz \leq 10.$$

11. Let $a, b, c > 0$ satisfy $a + b + c + abc = 4$. Prove that

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{a+c}} + \frac{c}{\sqrt{a+b}} \geq \frac{\sqrt{2}}{2} \cdot (a + b + c).$$

12. For x, y, z not all positive, prove that

$$\frac{16}{9}(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1.$$

13. (Vietnam 1998) Let $x_1, \ldots, x_n > 0$ be positive numbers ($n > 2$) satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \ldots x_n}}{n-1} \geq 1998.$$

14. Let $a, b, c, d > 0$ satisfy $abcd = 1$. Prove that

$$\frac{1}{(a+1)^3} + \frac{1}{(b+1)^3} + \frac{1}{(c+1)^3} + \frac{1}{(d+1)^3} \geq \frac{1}{2}.$$

15. For $a, b, c \geq 0$ such that $ab + bc + ca = 1$, prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{5}{2}.$$

16. (Iran 1996) For $x, y, z \geq 0$, prove that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

17. (Romania 2006) Let $a, b, c > 0$ satisfy $a + b + c = 3$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

18. (Bulgaria 1997) Let $a, b, c > 0$ satisfy $abc = 1$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$