Inequalities: A Brief Introduction

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1 Theorems

Throughout this lecture, we refer to *convex* and *concave* functions. Write I and I' for the intervals $[a, b]$ and (a, b) respectively. A function f is said to be convex on I if and only if $\lambda f(x)+(1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function f that is continuous on I and twice differentiable on I' is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$ (Concave if the inequality is flipped.)

Let $x_1 \geq x_2 \geq \cdots \geq x_n$; $y_1 \geq y_2 \geq \cdots \geq y_n$ be two sequences of real numbers. If $x_1 + \cdots + x_k \geq y_1+\cdots+y_k$ for $k = 1, 2, \ldots, n$ with equality where $k = n$, then the sequence $\{x_i\}$ is said to *majorize* the sequence $\{y_i\}$. An equivalent criterion is that for all real numbers t,

$$|t - x_1| + |t - x_2| + \cdots + |t - x_n| \geq |t - y_1| + |t - y_2| + \cdots + |t - y_n|$$

We use these definitions to introduce some famous inequalities.

Theorem 1 (Jensen) *Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, \ldots, x_n \in I$ and any nonnegative reals $\omega_1, \ldots, \omega_n$ with positive sum,*

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq (\omega_1 + \cdots + \omega_n) f\left(\frac{\omega_1 x_1 + \cdots + \omega_n x_n}{\omega_1 + \cdots + \omega_n}\right)$$

If f is concave, then the inequality is flipped.

Theorem 2 (Weighted Power Mean) *If $x_1, \ldots, x_n$ are nonnegative reals and $\omega_1, \ldots, \omega_n$ are nonnegative reals with a postive sum, then*

$$f(r) := \left(\frac{\omega_1 x_1^r + \cdots + \omega_n x_n^r}{\omega_1 + \cdots + \omega_n}\right)^{\frac{1}{r}}$$

is a non-decreasing function of r, with the convention that $r = 0$ is the weighted geometric mean. f is strictly increasing unless all the x_i are equal except possibly for $r \in (-\infty, 0]$, where if some x_i is zero f is identically 0. In particular, $f(1) \geq f(0) \geq f(-1)$ gives the AM-GM-HM inequality.

Theorem 3 (Hölder) *Let $a_1, \ldots, a_n$; $b_1, \ldots, b_n$; $\cdots$; $z_1, \ldots, z_n$ be sequences of nonnegative real numbers, and let $\lambda_a, \lambda_b, \ldots, \lambda_z$ positive reals which sum to 1. Then*

$$(a_1 + \cdots + a_n)^{\lambda_a}(b_1 + \cdots + b_n)^{\lambda_b} \cdots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_z} b_n^{\lambda_b} \cdots z_n^{\lambda_z}$$

This theorem is customarily identified as Cauchy when there are just two sequences.

Theorem 4 (Rearrangement) *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two nondecreasing sequences of real numbers. Then, for any permutation π of $\{1, 2, \ldots, n\}$, we have*

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{\pi(1)} + a_2 b_{\pi(2)} + \cdots + a_n b_{\pi(n)} \geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1$$

with equality on the left and right holding if and only if the sequence $\pi(1), \ldots, \pi(n)$ is decreasing and increasing respectively.

[1]Thanks to Thomas Mildorf for writing *the* lecture on Inequalities, available online at http://web.mit.edu/ tmildorf/www/Inequalities.pdf, as well as for generously providing me with the .tex file for said lecture so that I did not have to recopy all of the theorems. In addition, many example problems are also taken from his lecture.

Theorem 5 (Chebyshev) *Let $a_1 \le a_2 \le \cdots \le a_n$; $b_1 \le b_2 \le \cdots \le b_n$ be two nondecreasing sequences of real numbers. Then*

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \ge \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \ge \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}$$

Theorem 6 (Schur) *Let a, b, c be nonnegative reals and $r > 0$. Then*

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \ge 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0.

Remark *- This can be improved considerably. (See the problems section.) However, they are not as well known (as of now) as this form of Schur, and so should be proven whenever used on a contest.*

Theorem 7 (Majorization) *Let $f : I \to \mathbb{R}$ be a convex on I and suppose that the sequence $x_1, \ldots, x_n$ majorizes the sequence $y_1, \ldots, y_n$, where $x_i, y_i \in I$. Then*

$$f(x_1) + \cdots + f(x_n) \ge f(y_1) + \cdots + f(y_n)$$

Theorem 8 (Muirhead) *Suppose the sequence $a_1, \ldots, a_n$ majorizes the sequence $b_1, \ldots, b_n$. Then for any positive reals $x_1, \ldots, x_n$,*

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \ge \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

where the sums are taken over all permutations of n variables.

Remark *- Although Muirhead's theorem is a named theorem, it is generally not favorably regarded as part of a formal olympiad solution. Essentially, the majorization criterion guarantees that Muirhead's inequality can be deduced from a suitable application of AM-GM. Hence, whenever possible, you should use Muirhead's inequality only to deduce the correct relationship and then explicitly write all of the necessary applications of AM-GM. For a particular case this is a simple matter.*

Also note the **Trivial Inequality**, which states that $x^2 \ge 0$.

2 Techniques

One valuable technique, known as **dumbassing**, involves clearing all of the dominators and then employing AM-GM, Schur, and sometimes other inequalities. It should be avoided if possible while practicing, but during a contest, if you can find a dumbass solution there is no real reason not to use it.

Another technique is called **homogenization**. Here the idea is that, if all terms have an equal degree (the inequality is homogenous), then one can scale all of the variables until some inhomogenous condition (say, for example, $abc = 1$) holds.

Smoothing is where you show (rigorously!) that the inequality is "least" true (closest to the equality case) when two variables are equal, oftentimes while holding something, such as the sum of the two variables, fixed).

Substitutions can also help to simplify inequalities. In particular, $x = \frac{1}{y}$ can be useful to clear denominators, and $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ can be used if $abc = 1$, or to guarantee that $abc = 1$.

3 Problems

The only way to get better at Inequalities is to do lots of problems. Here are some to start you off (in all cases you may assume that variables are positive reals):

1. $(a+b)(b+c)(c+a) \geq 8abc$.
2. Given $abc = 1$, then $a+b+c \geq 3$.
3. $a^3 \geq 3a - 2$.
4. $(a^2+1)(b^2+1) \geq (a+b)^2$.
5. $a^3+b^3+c^3+3abc \geq a^2b+b^2c+c^2a+ab^2+bc^2+ca^2$.
6. (TMildorf) $a+b+c \leq a^2+b^2+c^2$ given $abc = 1$.
7. $\sum_{sym} a^5b^2c \geq \sum_{sym} a^3b^3c^2$.
8. (Tony Liu) $x^2 + \frac{2}{x} \geq 3$.
9. Given $abc \geq a+b+c$, then $abc \geq 3\sqrt{3}$.
10. (Titu Andreescu) $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$.
11. $(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$.
12. (USAMO 04/5) $(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+c) \geq (a+b+c)^3$.
13. (MOP) Given $a_1 + \ldots a_n = 1$, then $\displaystyle\prod_{i=1}^{n} \frac{1-a_i^k}{a_i^k} \geq (n^k-1)^n$ for any positive integer k.
14. (Aaron Pixton) Given $abc = 1$, then $5 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq (1+a)(1+b)(1+c)$.
15. (TMildorf) Given $a_1 \ldots a_n = 1$, then $\displaystyle\frac{1}{1+a_1} + \ldots + \frac{1}{1+a_n} \leq \frac{a_1 + \ldots + a_n + n}{4}$.