Inequalities

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1. (IMO '95) Let a, b, c be positive real numbers with $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq 1.$$

2. (MOP '00?) Show that if k is a positive integer and x_1, x_2, $\ldots$, x_n are positive real numbers which sum to 1, then
$$\prod_{i=1}^{n} \frac{1 - x_i^k}{x_i^k} \geq (n^k - 1)^n.$$
(Hint: the case $k = 1$ is equivalent to USAMO 98/3.)

3. (IMO '01) Let a, b, c be positive real numbers. Prove that
$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

4. (USAMO '04) Let a, b, c be positive reals. Prove that
$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

5. (IMO '96 shortlist) Let a, b, c be positive reals with $abc = 1$. Prove that
$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1.$$

6. (APMO '05) Let a, b, c be real numbers with $abc = 8$. Prove that
$$\frac{a^2}{\sqrt{(a^2+1)(b^2+1)}} + \frac{b^2}{\sqrt{(b^2+1)(c^2+1)}} + \frac{c^2}{\sqrt{(c^2+1)(a^2+1)}} \geq \frac{4}{3}.$$

7. (Poland '96?) Let a, b, c be real numbers with $a + b + c = 1$ and $a, b, c \geq -3/4$. Prove that
$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}.$$

8. (Japan, '97) Let a, b, c be positive real numbers. Prove that
$$\frac{(b+c-a)^2}{(b+c)^2 + a^2} + \frac{(c+a-b)^2}{(c+a)^2 + b^2} + \frac{(a+b-c)^2}{(a+b)^2 + c^2} \geq \frac{3}{5}.$$

9. (MOP '02) Let a, b, c be positive real numbers. Prove that
$$\left(\frac{2a}{b+c}\right)^{2/3} + \left(\frac{2b}{c+a}\right)^{2/3} + \left(\frac{2c}{a+b}\right)^{2/3} \geq 3.$$

10. (USAMO '97) Prove that for all positive real numbers a, b, c,
$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}.$$