Inequalities II: Tricks of the Trade

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1. (Titu) Prove that
$$\frac{a}{3b+c}+\frac{b}{3c+d}+\frac{c}{3d+a}+\frac{d}{3a+b}\geq 1$$
for all positive real numbers a, b, c, d.

2. (MOP '00?) Show that if k is a positive integer and x_1, x_2, $\ldots$, x_n are positive real numbers which sum to 1, then
$$\prod_{i=1}^{n}\frac{1-x_i^k}{x_i^k}\geq (n^k-1)^n.$$
(Hint: the case $k=1$ is equivalent to USAMO 98/3.)

3. (IMO '01) Let a, b, c be positive real numbers. Prove that
$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

4. (IMO '96 shortlist) Let a_1, $\ldots$, a_n be nonnegative real numbers, not all zero. Let $A=\sum_{j=1}^n a_j$, $B=\sum_{j=1}^n ja_j$, and let R be the unique positive real root of the equation $x^n-a_1x^{n-1}-\cdots-a_{n-1}x-a_n=0$. Prove that $A^A\leq R^B$.

5. (IMO '00) Let a, b, c be positive real numbers such that $abc=1$. Prove that
$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

6. (MOP '02) Let a, b, c be positive real numbers. Prove that
$$\left(\frac{2a}{b+c}\right)^{2/3}+\left(\frac{2b}{c+a}\right)^{2/3}+\left(\frac{2c}{a+b}\right)^{2/3}\geq 3.$$

7. ("Majorization") Let $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ be two sequences of real numbers such that
$$\begin{array}{rcl} a_1 & \geq & b_1 \\ a_1+a_2 & \geq & b_1+b_2 \\ & \vdots & \\ a_1+\cdots+a_{n-1} & \geq & b_1+\cdots+b_{n-1} \\ a_1+\cdots+a_n & = & b_1+\cdots+b_n. \end{array} \qquad (1)$$
Prove that
$$a_1^2+\cdots+a_n^2\geq b_1^2+\cdots+b_n^2.$$
(More generally, if f is any convex function, then $f(a_1)+\cdots+f(a_n)\geq f(b_1)+\cdots+f(b_n)$. When the inequalities (1) hold, the sequence $(a_1,\ldots,a_n)$ is said to *majorize* the sequence $(b_1,\ldots,b_n)$.)