1 The LIF Model

The membrane potential as per the leaky integrate-and-fire (LIF) model, the deduced differential equation is

$$\tau_m \frac{du}{dt} = -(u - u_{rest}) + R_m I(t), \tag{1}$$

where

- u is the membrane potential
- $\tau_m = R_m C_m$ is the membrane time constant
- R_m is the membrane resistance
- C_m is the membrane capacitance
- u_{rest} is the resting potential

As the membrane potential reaches the firing treshold ϑ, a spike is generated and the membrane motential is reset to the reset potential u_{reset}:

$$\text{if } u \geq \vartheta : u \leftarrow u_{reset} \tag{2}$$

First assume, at time t=0, the membrane potential takes a value u_{rest} + z [Note: z=0, if the noise is not being assumed as present]. The constant current is supplied and that there is no absolute refractory period (i.e. $I(t) = I, \Delta_{abs} = 0$).

$$\tau_m \frac{du}{dt} = -(u - u_{rest}) + R_m I(t) \tag{3}$$

$$\tau_m \frac{du}{dt} = -(u - u_{rest}) + R_m I \tag{4}$$

$$\frac{du}{dt} + \frac{1}{\tau_m}(u - u_{rest}) = \frac{R_m I(t)}{\tau_m} \tag{5}$$

[Note : This is of the form Linear Differential Standard equation, which is

$$\frac{dy}{dx} + P(x)y = Q(x)] \tag{6}$$

$$\tag{7}$$

Integrating factor is given by

$$e^{\int \frac{1}{\tau_m} dt} = e^{\frac{t}{\tau_m}} \tag{8}$$

Multiplying Integrating factor both the sides to eq. (5) we get

$$e^{\frac{t}{\tau_m}} \frac{du}{dt} + \frac{e^{\frac{t}{\tau_m}}(u - u_{rest})}{\tau_m} = \frac{R_m I}{\tau_m} e^{\frac{t}{\tau_m}} \tag{9}$$

L.H.S of eq.(9) can be simplified as below

$$((u - u_{rest})e^{\frac{t}{\tau_m}})' \tag{10}$$

Subsituting to eq(9) and Integrating both sides w.r.t 't' we get

$$\int ((u - u_{rest})e^{\frac{t}{\tau_m}})' = \int \frac{R_m I}{\tau_m} e^{\frac{t}{\tau_m}} dt \tag{11}$$

$$(u - u_{rest})e^{\frac{1}{\tau_m}} = \int \frac{R_m I}{\tau_m} e^{\frac{t}{\tau_m}} dt \tag{12}$$

$$(u - u_{rest}) = e^{\frac{-1}{\tau_m}} \frac{R_m I}{\tau_m} \left(\tau_m e^{\frac{t}{\tau_m}} + C\right) \tag{13}$$

$$(u - u_{rest}) = R_m I + C \frac{R_m I}{\tau_m} e^{-\frac{t}{\tau_m}} \tag{14}$$

$$(u - u_{rest}) = R_m I \left(1 + \frac{C}{\tau_m} e^{-\frac{t}{\tau_m}}\right) \tag{15}$$

In our case, C is an integration constant. And RC is the characteristic time of the decay. Since this constant act as an initial offset, For our convenience we set, $C = -\tau_m$.

$$(u - u_{rest}) = R_m I\Big(1 - e^{-\frac{t}{\tau_m}}\Big) \tag{16}$$

Applying initial boundary conditions to determine the time, at which the first spike occurs, by solving for $t^{(1)}$, i.e when the potential $(u - u_{rest})$ reaches the threshold ϑ.

$$\vartheta = R_m I\Big(1 - e^{-\frac{t^{(1)}}{\tau_m}}\Big) \tag{17}$$

$$\vartheta = R_m I - R_m I e^{-\frac{t^{(1)}}{\tau_m}} \tag{18}$$

$$R_m I e^{-\frac{t^{(1)}}{\tau_m}} = R_m I - \vartheta \tag{19}$$

$$-e^{\frac{t^{(1)}}{\tau_m}} = \frac{R_m I - \vartheta}{R_m I} \tag{20}$$

$$-\frac{t^{(1)}}{\tau_m} = \ln\left(\frac{R_m I - \vartheta}{R_m I}\right) \tag{21}$$

$$t^{(1)} = -\tau_m \ln\left(\frac{R_m I - \vartheta}{R_m I}\right) \tag{22}$$

$$t^{(1)} = \tau_m \ln\left(\frac{R_m I}{R_m I - \vartheta}\right) \tag{23}$$

With $\Delta_{abs} = 0$, we get a interspike interval T and a spike frequencs F of

$$T = \tau_m \ln\left(\frac{R_m I}{R_m I - \vartheta}\right) \tag{24}$$

$$F = \left[\tau_m \ln\left(\frac{R_m I}{R_m I - \vartheta}\right)\right]^{-1} \tag{25}$$

, where with $\Delta_{abs} > 0$, we get

$$T = \Delta_{abs} + \tau_m \ln\left(\frac{R_m I}{R_m I - \vartheta}\right) \tag{26}$$

$$F = \left[\Delta_{abs} + \tau_m \ln\left(\frac{R_m I}{R_m I - \vartheta}\right)\right]^{-1} \tag{27}$$

2 Verification by simulating in PyNest

In this task we used the given parameters.

R_m	C_m	u_{rest}	u_{reset}	ϑ	Δ_{abs}
0.03	1000	-65	-80	-45	5

Plugging in values of parameters to the deduced solution:

$$T = \Delta_{abs} + \tau_m \ln\left(\frac{R_m I}{R_m I - \vartheta}\right) \tag{28}$$

$$T = \Delta_{abs} + 0.03 \cdot 1000 \cdot \ln\left(\frac{0.03 \cdot 1000}{0.03 \cdot 1000 + 45}\right) \tag{29}$$

$$T = \Delta_{abs} + 30 \cdot \underbrace{\ln\left(\frac{90}{135}\right)}_{<0} \tag{30}$$

Even though the deduced mean spike interval equation correctly informs about the behaviour of the stimulus of neuron i.e Neuron stimulus cycle is equal to Absolute refractory period + membrane decay potential until to reach reset potential. Because of the presence of negative threshold voltage , in order to increase the membrane potential after the absolute refracrory period, the membrane decay potential has to be subtracted from the absolute refractory period which is given by

$$T = \Delta_{abs} - \tau_m \ln \left(\frac{R_m I}{R_m I - \vartheta} \right) \tag{31}$$

$$F = \left[\Delta_{abs} - \tau_m \ln \left(\frac{R_m I}{R_m I - \vartheta} \right) \right]^{-1} \tag{32}$$

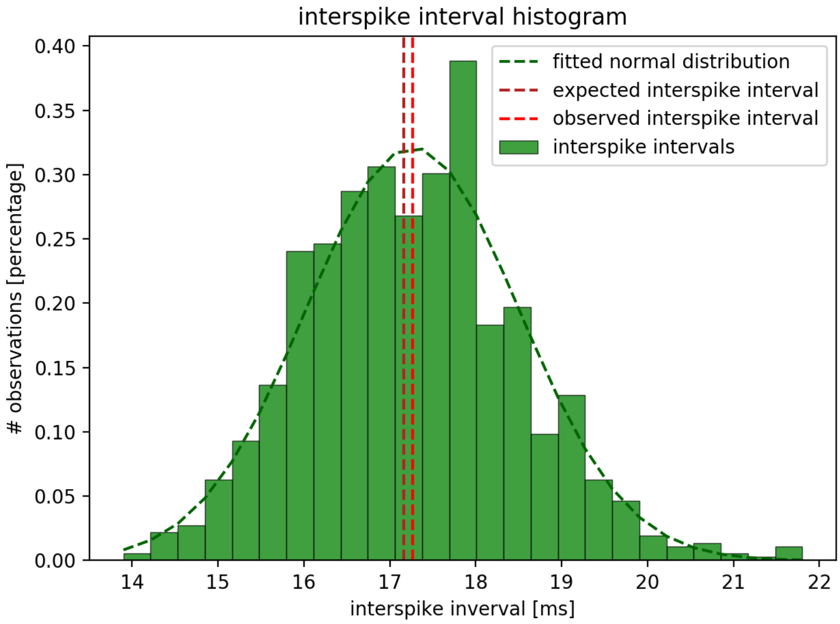

Figure 1: Interspike Histogram