A Critical Look at the Bathtub Curve

Georgia-Ann Klutke, Peter C. Kiessler, and M. A. Wortman

***Abstract*—This paper addresses some of the fundamental assumptions underlying the bathtub curve. It is shown to be unlikely that any practical hazard function is decreasing near zero. Great care should be taken in interpreting the hazard function, particularly in applying quality-control practices, such as burn-in or environmental-stress-screeing to manufactured products.**

***Index Terms*—Bathtub curve, burn-in, hazard function, infant mortality, mixture of distributions.**

ACRONYMS[1]

Cdf	cumulative distribution function
ESS	environmental stress screening
hzf	hazard function
IFR	increasing hazard (failure) rate
pdf	probability density function
Sf	survivor function.

NOTATION

$f(t)$	pdf of time to failure
$t_1, t_2, \ldots$	$t_n \in [0, \infty)$: inflection points of f
$h(t)$	hzf of time to failure
A	compact parameter space for mixture of distributions
α	element of A
p	probability measure on A
$F(t)$, $F_\alpha(t)$	Cdfs of time to failure
$\overline{F}(t)$, $\overline{F_\alpha}(t)$	Sfs of time to failure.

I. INTRODUCTION

A FUNDAMENTAL tenet of reliability theory is that the hzf displays a "bathtub shape."

The origins of this curve are unclear; it appears in actuarial life-table analysis as long ago as 1693 [6]. The bathtub curve is described in nearly every standard reliability text, e.g., [2], [3], [7], [13], [15]–[19]. The curve represents the idea that the operation of a population of devices can be viewed as comprised of 3 distinct periods:

- an "early failure" (burn-in) period, where the hzf decreases over time,
- a "random failure" (useful life) period, where the hzf is constant over time,
- a "wear-out" period, where the hzf increases over time.

The bathtub curve occupies a place of considerable importance in reliability practice, particularly in justifying burn-in strategies for improving system reliability. This paper exposes some of the limitations of the traditional bathtub curve, and shows that a bathtub-shaped hzf cannot rigorously agree with a simple bi-modal lifetime density. Thus, the value of the bathtub curve in characterizing infant mortalities is questionable. While the useful life and wearout intervals are not examined, the realism of these segments of the curve might also be questioned. Some simple analyses are offered that examine the foundations for the traditional bathtub curve for manufactured products; its indiscriminate use is discouraged here.

While most reliability texts mention the bathtub curve, there is considerable disagreement on its applicability.

- Reference [16] describes it as a "typical hazard rate" shape;
- [19] claims that "a few products show a decreasing failure rate in the early life and an increasing failure rate in later life";
- [11] asserts that the bathtub curve describes "only 10% to 15%" of applications;
- [18] states that "the bathtub curve can model the reliability characteristics of a generic piece-part type, but not of an assembly, a circuit, or a system.";
- [8], [12] represent the hzf as the sum (superposition) of a decreasing hzf, a constant hzf, and an increasing hzf.

Interestingly, none of the standard references cited here provide compelling *empirical* evidence in support of the bathtub curve for manufactured products. Indeed, there have been several efforts to dislodge the bathtub curve from its place of importance. References [20], [21], and later, [9], advocated a "roller-coaster" curve for electronic components and provide both philosophical and analytic justification for such behavior. These papers point to the limitations of the traditional bathtub curves to adequately-model early life failures.

The presence of decreasing hazard near time zero has been explained in several ways; two are listed here, although other explanations can certainly arise.

1) The "physics of failure" explanation [14] postulates that individual devices improve with age in their early life. While this explanation might be viable in certain situations (e.g., biological systems, curing of materials), [14] points out that thermodynamic considerations make it an unconvincing argument for manufactured devices.

2) References [9], [10] (see also [4], [5], [14]) explain that a manufactured component belongs to a population at risk of

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[1]The singular and plural of an acronym are always spelled the same.

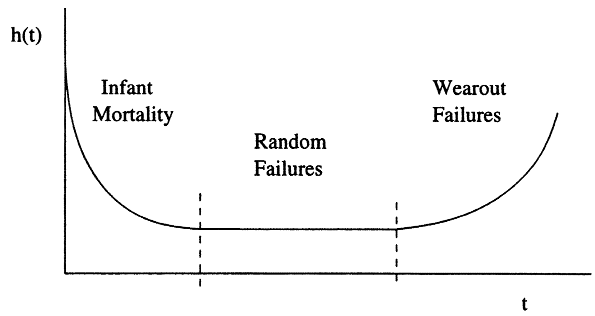

Fig. 1. The hzf over time (the classical bathtub curve).

failure, and this population is comprised of sub-populations exhibiting differing hazards. While each of these subpopulations can exhibit nondecreasing hzfs, the population as a whole might exhibit a decreasing hzf. In a manufacturing context, the subpopulations might represent component parts from various suppliers, each of which operates a stable manufacturing process.[2]

Section II begins with the assertion that it is physically unreasonable that any *homogeneous* population of devices should exhibit decreasing hazard, and then investigates the relationship between a convex hzf and infant mortalities; a mixture of such populations cannot have a hzf that exhibits the classical bathtub shape in the region near zero.

Section III builds a model for lifetime distributions that are mixtures of distributions with increasing hazard, and then investigates the hzf of this mixture and proves that, under reasonable conditions on the underlying mixture Cdfs, the hzf is always increasing in the neighborhood of zero, and thus cannot exhibit the classical bathtub shape. Also examined is the practical situation of hzfs for mixtures of Weibull distributions; several examples illustrate that the hzf can take on several different shapes.

Section IV comments on quality control issues related to the hzf.

II. What Does the hzf Imply About Early Failures?

The initial decreasing hazard region of the classical bathtub curve shown in Fig. 1 is often supposed to model "infant mortalities" due to design or manufacturing defects that cannot be completely eliminated, resulting in a subpopulation of so-called "weak sisters" [2]. The "weak sister" explanation of infant mortality is not analytically consistent with the bathtub curve. Consider the case where early failures are characterized by a population pdf that has at least 2 modes [10], such as indicated in Fig. 2. This section shows that a multimodal pdf cannot have a convex hzf: the classical bathtub curve is not appropriate in this scenario.

Let $f(t)$ be a bimodal pdf of device lifetime and $F(t)$ its Cdf. Let $f(t)$ be twice differentiable, and $f(0) = 0$. When f is bimodal, there exist (at least) 3 points where f changes direction; let $t_1, t_2, \ldots, t_n, n \geq 3$, be such that $f'(t_i) = 0$. The basic idea is that, if a subpopulation of early failing devices exists, then the corresponding lifetime pdf should be, at least, bimodal: have at least 3 stationary points. One of the 2 stationary points nearest zero, is an "infant mortality mode."

[2]This paper does not address "reliability growth": improvements of a manufacturing process over time.

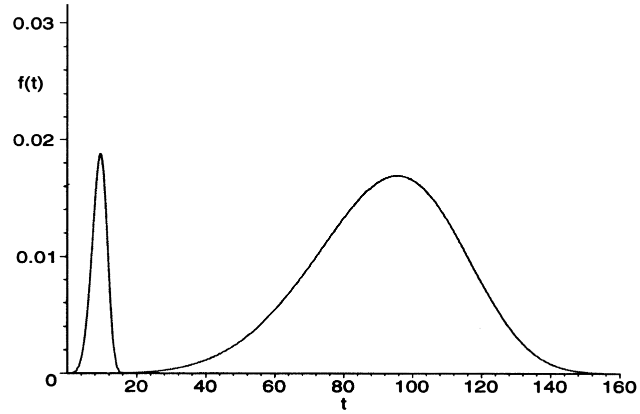

Fig. 2. The pdf for a population with early failures.

The $h(t)$ is

$$h(t) = \frac{f(t)}{\overline{F}(t)}, \qquad t \geq 0; \tag{1}$$

thus

$$f'(t) = h'(t) \cdot \overline{F}(t) + h(t) \cdot \overline{F}'(t). \tag{2}$$

But, because $\overline{F}'(t) = -f(t)$, then

$$f'(t) = h'(t) \cdot \overline{F}(t) - h(t) \cdot f(t) \tag{3}$$

and

$$h'(t_i) \cdot \overline{F}(t_i) = h(t_i) \cdot f(t_i) \tag{4}$$

and therefore, $h'(t_i) = h^2(t_i)$.

Now let $h(t)$ follow the bathtub shape of Fig. 1: $h(t)$ is convex and positive. It follows that $h^2(t)$ must also be convex and positive, and $h'(t)$ must be monotone nondecreasing. Because $h(t)$ follows a bathtub shape, $h^2(t)$ and $h'(t)$ cannot possibly intersect in the decreasing hazard rate region [where $h'(t)$ is negative] of the bathtub curve. Hence the corresponding $f(t)$ can have no stationary points in the "burn-in region." Thus the bimodal pdf as a representation of a mixture of subpopulations does not yield a decreasing hzf during the early life interval, and the bathtub curve does not accommodate this characterization of early failures.

III. Some Remarks About the hzf and Convexity

This section proves a general result that characterizes mixtures of distributions whose hzfs increase in a neighborhood of 0. In particular, a sufficient condition is given for the hzf of a mixture of such distributions to be increasing in a neighborhood of 0.

Let $\{F_\alpha, \alpha \in A\}$ be a family of sufficiently smooth (continuous second time-derivatives) distributions. Define the mixture distribution by

$$F(t) = \int_A F_\alpha(t) \cdot p(d\alpha). \tag{5}$$

Observe that if $\overline{F}$ is strictly concave at any point t_0, then its hzf h is strictly increasing at t_0. To see this, note that

$$h'(t_0) = \left(-\frac{\overline{F}'(t_0)}{\overline{F}(t_0)}\right)' = h^2(t_0) - \frac{\overline{F}''(t_0)}{\overline{F}(t_0)}. \tag{6}$$

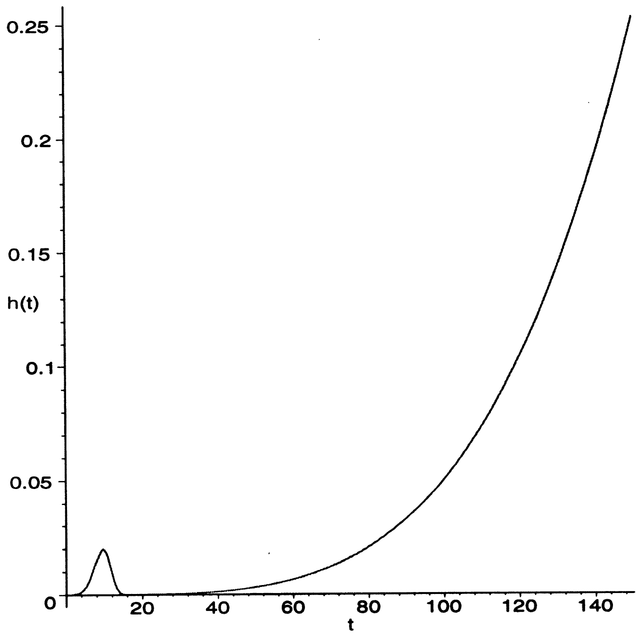

Fig. 3. The hzf, $\lambda = 0.1$, $\beta_1 = 5$, $\eta_1 = 10$, $\beta_2 = 5$, $\eta_2 = 100$.

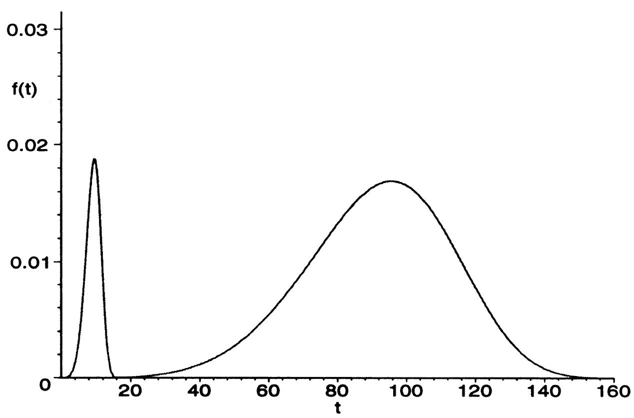

Fig. 4. The pdf, $\lambda = 0.1$, $\beta_1 = 5$, $\eta_1 = 10$, $\beta_2 = 5$, $\eta_2 = 100$.

Then $f'(t_0) > 0 \iff \overline{F}''(t_0) < 0 \Rightarrow h'(t_0) > 0$.

Moreover, if each $\overline{F}_\alpha$ is concave in a neighborhood of zero, and if at least 1 of the $\overline{F}_\alpha$ is strictly concave in this neighborhood, then $\overline{F}$ is strictly concave in this neighborhood.

To summarize, a sufficient condition for the mixture of distributions with concave Sfs in a neighborhood of 0 to have an increasing hzf at 0 is that at least 1 of the distributions has a *strictly* concave Sf: such mixtures cannot follow the classical bathtub shape.

The rest of this section illustrates this observation with mixtures of Weibull distributions with increasing hazard rates. Reference [1] considers the hzf of an inverse Gaussian–Weibull mixture and obtains results consistent with this paper. The Weibull distribution is important in reliability theory, both for empirical reasons (it can be used to model failure-time data with an increasing or decreasing hzf) and for its relationship to certain extreme value distributions [17, p. 86]. For Weibull distributions with increasing hzfs, the Appendix shows that there is a neighborhood around 0 on which the Sf is strictly concave. It follows that, although a mixture of such Weibulls might not be IFR, the corresponding hzf is increasing in the neighborhood of 0, which does not agree with the traditional bathtub curve.

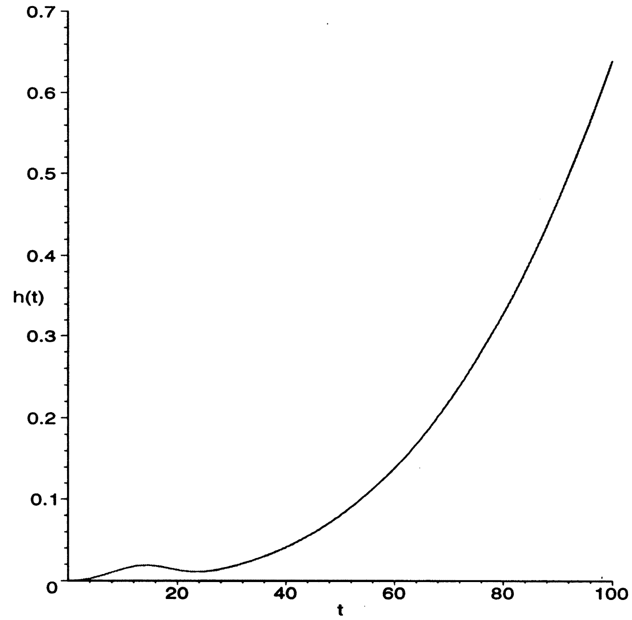

Fig. 5. The hzf, $\lambda = 0.2$, $\beta_1 = 3$, $\eta_1 = 15$, $\beta_2 = 4$, $\eta_2 = 50$.

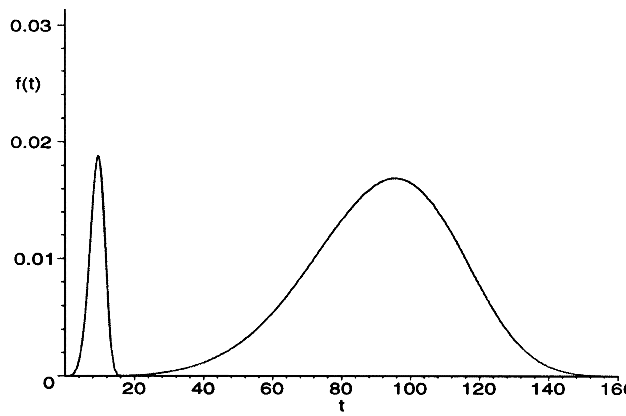

Fig. 6. The pdf, $\lambda = 0.2$, $\beta_1 = 3$, $\eta_1 = 15$, $\beta_2 = 4$, $\eta_2 = 50$.

Consider mixtures of the form

$$F(t) = \lambda \cdot \left(1 - \exp\left[-\left(\frac{t}{\eta_1}\right)^{\beta_1}\right]\right) + (1-\lambda) \cdot \left(1 - \exp\left[-\left(\frac{t}{\eta_2}\right)^{\beta_2}\right]\right), \quad t > 0, \qquad (7)$$

with $\beta_1, \beta_2 > 1$. In Figs. 3–10, the caption reports the Weibull shape parameters β_1, β_2 and scale parameters η_1, η_2 as well as the mixing parameter λ.

Figs. 3 and 4 show the hzf and pdf for the infant mortality distribution in Fig. 2. The hook in the hazard-rate curve corresponds to the infant mortality mode in the pdf.

Figs. 5 and 6 show the hzf and pdf for a mixture of 2 populations that are somewhat more dispersed than in the previous

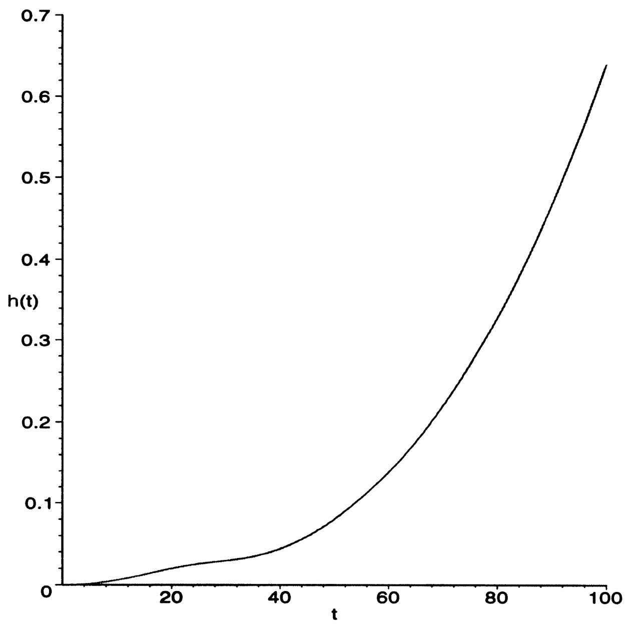

Fig. 7. The hzf, $\lambda = 0.3$, $\beta_1 = 3$, $\eta_1 = 30$, $\beta_2 = 4$, $\eta_2 = 50$.

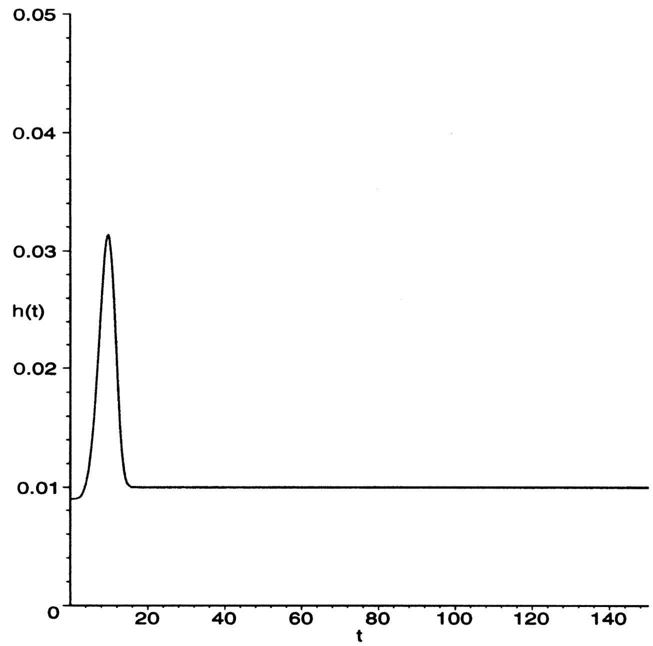

Fig. 9. The hzf, $\lambda = 0.1$, $\beta_1 = 5$, $\eta_1 = 10$, $\beta_2 = 1$, $\eta_2 = 100$.

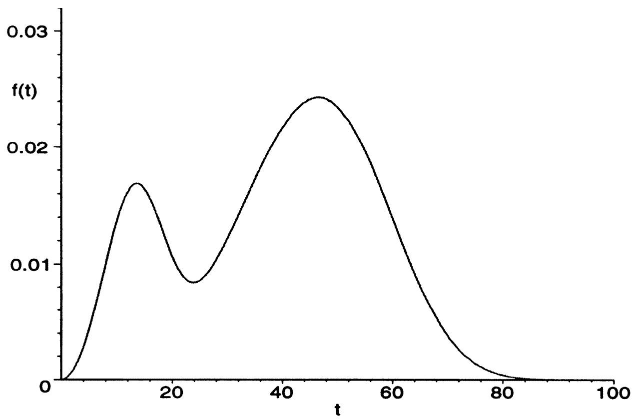

Fig. 8. The pdf, $\lambda = 0.3$, $\beta_1 = 3$, $\eta_1 = 30$, $\beta_2 = 4$, $\eta_2 = 50$.

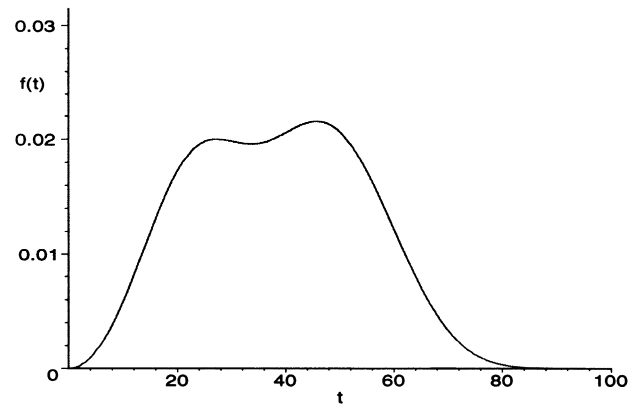

Fig. 10. The pdf, $\lambda = 0.1$, $\beta_1 = 5$, $\eta_1 = 10$, $\beta_2 = 1$, $\eta_2 = 1000$.

example. The hzf has a much less noticeable hook and resembles the classical bathtub curve much less closely, even though 2 distinct modes are visible in the pdf.

Figs. 7 and 8 show the hzf and pdf for a mixture of 2 populations that are both relatively dispersed and whose means are similar. While the populations remain somewhat distinct on the pdf curve, their differences are almost imperceptible on the hzf curve.

Figs. 9 and 10 show the hzf and pdf for a mixture consisting of an exponential distribution and an IFR Weibull distribution. The hzf curve has almost no similarity with the classical bathtub shape.

IV. Implications for Quality Control and Screening Procedures

Using only the hzf when studying the early-life characteristics of a population formed by a mixture of subpopulations can be quite misleading. In particular, one should not use the bathtub curve to characterize early-life behavior of such populations. Instead, practitioners should pay particular attention to the population lifetime pdf whenever it is believed that a population mixture might include subpopulations of so-called "weak devices" or "freaks" that lead to early failures in the total population.

The initial decreasing hzf period of the bathtub curve is often cited as justification for using certain quality-control practices, such as burn-in or ESS. Burn-in and ESS are an "art," based as much on engineering judgment as on statistical principles [17, p. 521]. They are expensive techniques and their effects on surviving devices are not well understood. Reference [14] exposes some common misconceptions about burn-in, and presents compelling arguments against the bathtub curve as well. This paper hopes to encourage reliability practitioners to examine failure-data to determine whether or not the assumed benefits of these techniques are realized. As stated in [17, p. 522], remember that improved reliability through continuous improvement in product design and manufacturing techniques is far more effective in reducing quality problems.

APPENDIX

Consider the case: the F_αs of Section III have Weibull distributions with increasing hzfs: A is 2-dimensional and

$$F_{(\beta,\eta)}(t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^\beta\right], \qquad t > 0,\ \beta > 1.$$

A neighborhood around 0 is derived in which such a distribution has an increasing hzf.

The Sf for a Weibull distribution with parameters β and η is

$$\overline{F}(t) = \exp\left[-\left(\frac{t}{\eta}\right)^\beta\right].$$

Taking two derivatives -

$$\begin{aligned}\overline{F}''(t) &= \left[-\frac{\beta\cdot(\beta-1)}{\eta^2}\cdot\left(\frac{t}{\eta}\right)^{\beta-2} + \frac{\beta^2}{\eta^2}\cdot\left(\frac{t}{\eta}\right)^{2\beta-2}\right]\cdot\overline{F}(t)\\ &= \left[-(\beta-1)+\beta\cdot\left(\frac{t}{\eta}\right)^\beta\right]\cdot\frac{\beta}{\eta^2}\cdot\left(\frac{t}{\eta}\right)^{\beta-2}\cdot\overline{F}(t).\end{aligned}$$

Because

$$\frac{\beta}{\eta^2}\cdot\left(\frac{t}{\eta}\right)^{\beta-2}\cdot\overline{F}(t) > 0,$$

then $\overline{F}''(t) < 0$ if and only if

$$\beta\cdot\left(\frac{t}{\eta}\right)^\beta < \beta - 1.$$

For $\beta > 1$, solving for t gives: $\overline{F}(t)$ is strictly concave when

$$t < \eta\cdot\left(\frac{\beta-1}{\beta}\right)^{1/\beta}.$$

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