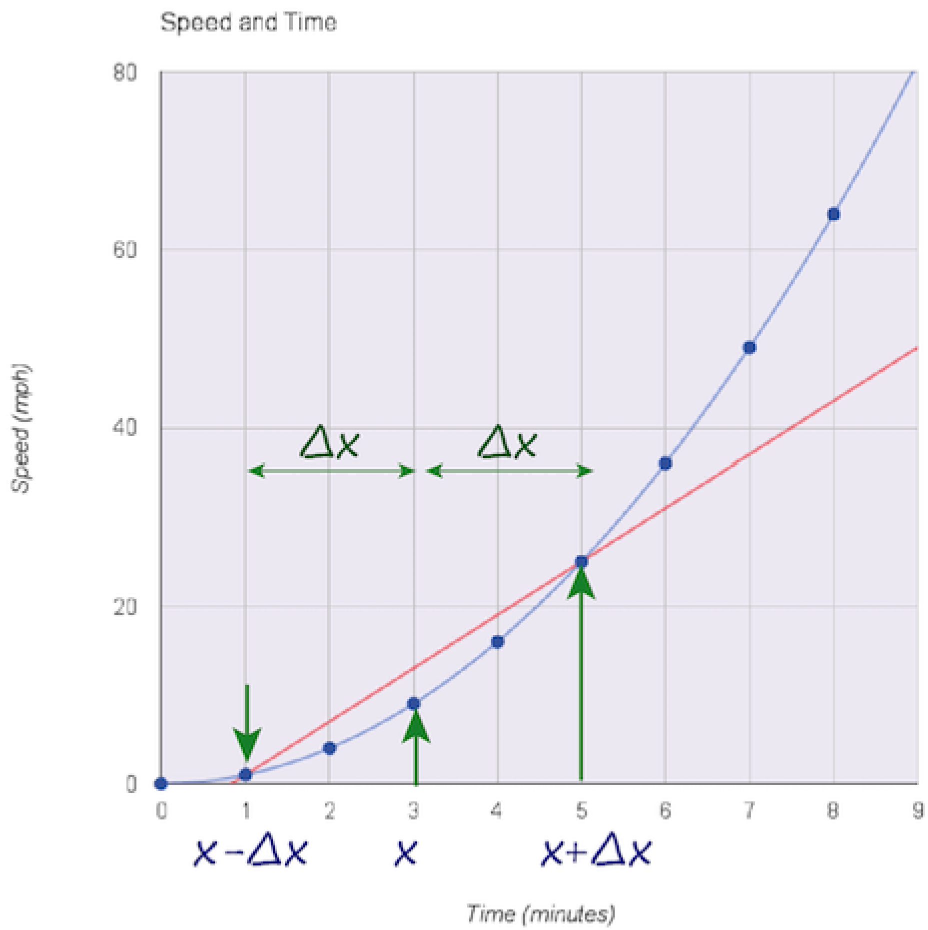
Calculus Not By Hand

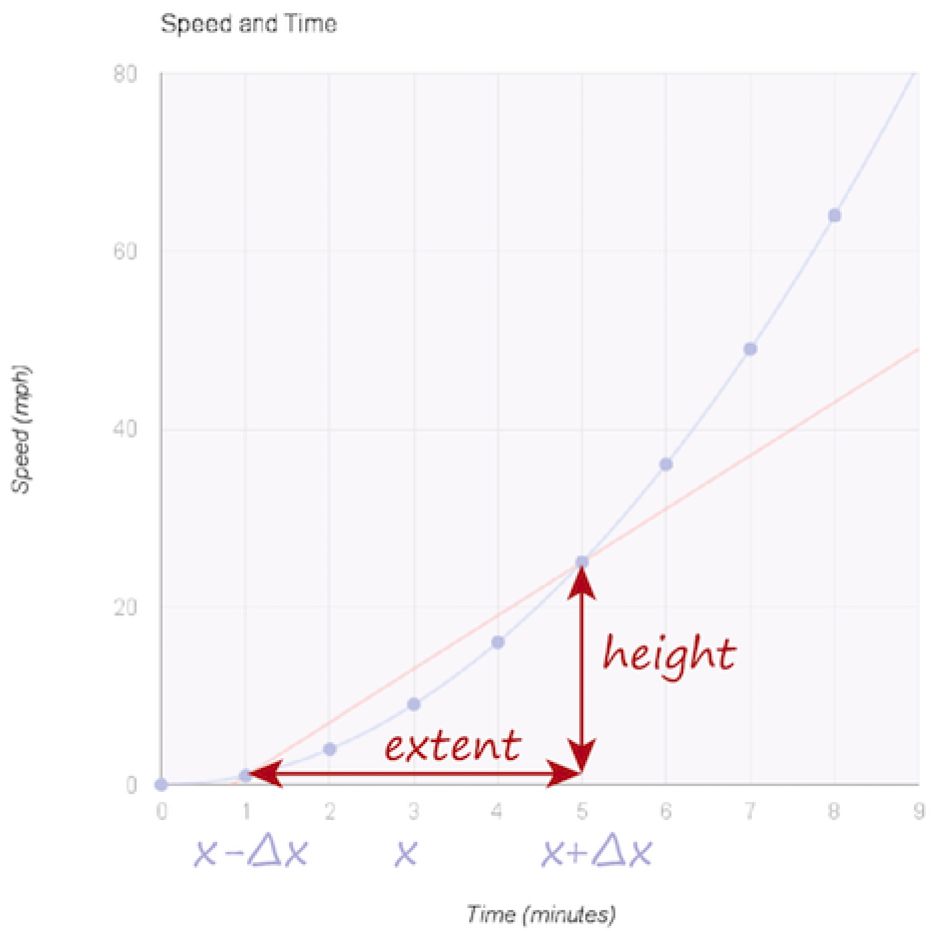
Look at the following graph which has a new line marked on it. It isn't the tangent because it doesn't touch the curve only at a single point. But it does seem to be centered around time 3 minutes in some way.



In fact, there is connection to time 3 minutes. What we've done is chosen a time above and below this point of interest at $t = 3$. Here, we've selected points 2 minutes above and below $t = 3$ minutes. That is, $t = 1$ and $t = 5$ minutes. Using our mathematical notation, we say we have a Δx of 2 minutes. And we have chosen points x-Δx and x+Δx. Remember that symbol Δ just means a "small change", so Δx is a small change in x.

Why have we done this? It will become clear very soon — hang in there just a bit. If we look at the speeds at times x-Δx and x+Δx, and draw a line between those two points, we have something that very roughly has the same slope as

a tangent at the middle point *x*. Have a look again at the diagram above to see that straight line. Sure, it's not going to have exactly the same slope as a true tangent at *x*, but we'll fix this. Let's work out the gradient of this line. We use the same approach as before where the gradient is the height of the incline divided by the extent. The following diagram makes clearer what the height and extent are here.



The height is the difference between the two speeds at x-Δx and x+Δx, that is, 1 and 5 minutes. We know the speeds are $1^2 = 1$ and $5^2 = 25$ mph at these points, so the difference is 24. The extent is the very simple distance between x-Δx and x+Δx, that is, between 1 and 5, which is 4. So we have:

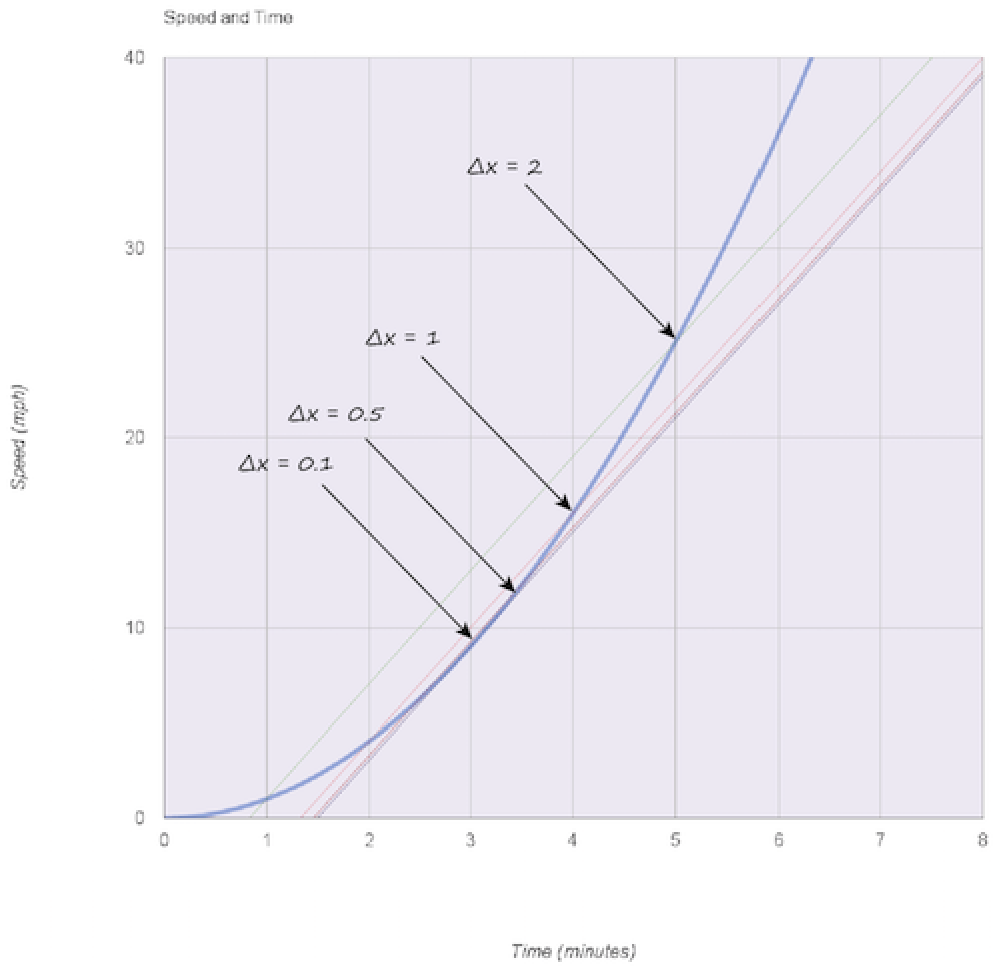
$$\text{gradient} = \frac{\text{height}}{\text{extent}}$$

$$= 24 / 4$$

$$= 6$$

The gradient of the line, which approximates the tangent at $t = 3$ minutes, is 6 mph per min. Let's pause and have a think about what we've done. We first tried to work out the slope of a curved line by hand drawing a tangent. This approach will never be accurate, and we can't do it many many times because, being human, we'll get tired, bored, and make mistakes. The next approach doesn't need us to hand draw a tangent. Instead, we follow a recipe to create a different line which seems to have approximately the right slope. This second approach can be automated by a computer and done many times and very quickly, as no human effort is needed.

That's good but not good enough yet! That second approach is only an approximation. How can we improve it, so it's not an approximation? That's our aim, after all, to be able to work out how things change, the gradient, in a mathematically precise way. This is where the magic happens! We'll see one of the neat tools that mathematicians have developed and had a bit too much fun with! What would happen if we made the extent smaller? Another way of saying that is, what would happen if we made the Δx smaller? The following diagram illustrates several approximations or slope lines resulting from a decreasing Δx.



We've drawn the lines for $\Delta x = 2.0$, $\Delta x = 1.0$, $\Delta x = 0.5$ and $\Delta x = 0.1$. You can see that the lines are getting closer to the point of interest at 3 minutes. You can imagine that as we keep making Δx smaller and smaller, the line gets closer and closer to a true tangent at 3 minutes.

As Δx becomes infinitely small, the line becomes infinitely closer to the true tangent. That's pretty cool! This idea of approximating a solution and improving it by making the deviations smaller and smaller is very powerful. It allows mathematicians to solve problems which are hard to attack directly. It's a bit like creeping up to a solution from the side, instead of running at it head on!