Module 5 : Numerical Methods - 2.

Numerical methods for Initial value Problems:-

Consider a differential equation of first order and first degree in the form $\frac{dy}{dx} = f(x, y) \rightarrow (1)$ with the initial condition $y(x_0) = y_0$ [means when $x = x_0$, $y = y_0$] $\rightarrow (2)$.

The problem of solving (1) subjected to the condition (2) is called initial value problem. [IVP]

Taylor's Series Method:-

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$.

We have Taylor's series expansion of $y(x)$ about the point x_0 in the form:

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

Here $y'(x_0), y''(x_0) \ldots$ denote the value of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ at x_0.

Problems:

1) Solve $\frac{dy}{dx} = x^2y - 1$ with $y(0) = 1$ using Taylor's series method and find $y(0.1)$ by considering upto fourth degree term.

Sol^n:- Given $\frac{dy}{dx} = x^2y - 1$; $y(0) = 1$; $x_0 = 0$, $y_0 = 1$, $x = 0.1$

The Taylor's series sol^n is

$$y(x) = y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\frac{(x-x_0)^4}{4!}y^{IV}(x_0)$$

$$y(0.1) = y(0)+ 0.1\,y'(0) + \frac{(0.1)^2}{2}y''(0)+\frac{(0.1)^3}{6}y'''(0)+\frac{(0.1)^4}{24}y^{IV}(0)+\cdots \rightarrow (1).$$

$y' = x^2y-1 \;;\; y'(0) = -1$

$y'' = x^2y'+2xy \;;\; y''(0)=0$

$y''' = x^2y''+2xy'+2xy'+2y = x^2y''+4xy'+2y \;;\; y'''(0)= 2.$

$y^{IV} = x^2y'''+2xy''+4xy''+4y'+2y' = x^2y'''+6xy''+6y' \;;\; y^{IV}(0)=-6.$

$(1) \Rightarrow y(0.1)= 1+(0.1)(-1)+\frac{0.01}{2}(0)+\frac{0.001}{6}(2)+\frac{0.0001}{24}(-6)$

$\therefore \boxed{y(0.1)=0.90033}.$

2) Employ Taylor's series method to obtain approximate value of y at $x=0.1$ & 0.2 for differential equation $\frac{dy}{dx}=2y+3e^x$, $y(0)=0$ considering upto fourth degree term.

Sol^n:- Given $\frac{dy}{dx}=2y+3e^x$, $y(0)=0$; $x_0=0$, $y_0=0$.

The Taylor's series sol^n is

$$y(x) = y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\frac{(x-x_0)^4}{4!}y^{IV}(x_0)$$

$y' = 2y+3e^x \;;\; y'(0)=0+3=3.$

$y'' = 2y'+3e^x \;;\; y''(0)=6+3=9.$

$y''' = 2y''+3e^x \;;\; y'''(0)=18+3=21.$

$y^{IV} = 2y'''+3e^x \;;\; y^{IV}(0)=42+3=45.$

$$y(x) = 0+3x+\frac{9}{2}x^2+\frac{7}{2}x^3+\frac{15}{8}x^4+\cdots$$

Put $x=0.1$ & $x=0.2$ we get

$y(0.1)=0.3487$	$y(0.2)=0.8110$

3) Use Taylor's series method to solve $y' = x^2 + y$ in the range $0 \le x \le 0.2$ by taking step size $h = 0.1$ given that $y = 10$ at $x = 0$ initially, Considering the terms upto the fourth degree.

Soln:- Since the step length is 0.1, the problem has to be solved in two stages.

First, We have to find $y(0.1)$ and using this as the initial condition we have to compute $y(0.2)$

The Taylor's series soln is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0) + \cdots$$

I Stage :— Given $y' = x^2 + y$, $x_0 = 0$, $y_0 = 10$.

$$y' = x^2 + y, \quad y'(0) = 0^2 + 10 = 10$$

$$y'' = 2x + y', \quad y''(0) = 10$$

$$y''' = 2 + y'', \quad y'''(0) = 12$$

$$y^{iv} = y''', \quad y^{iv}(0) = 12.$$

Substitute $x_0 = 0$ along with $x = 0.1$ We get

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0) + \frac{(0.1)^3}{3!}y'''(0) + \frac{(0.1)^4}{4!}y^{iv}(0) + \cdots$$

$$\boxed{y(0.1) = 11.052}$$

II Stage :- Now $x_0 = 0.1$, $y_0 = 11.052$.

$$y' = x^2 + y \; ; \quad y'(0.1) = (0.1)^2 + 11.052 = 11.052$$

$$y'' = 2x + y' \; ; \quad y''(0.1) = 11.262.$$

$$y''' = 2 + y'' \; ; \quad y'''(0.1) = 13.262.$$

$$y^{iv} = y''' \; ; \quad y^{iv}(0.1) = 13.262.$$

With $x_0 = 0.1$ & $x = 0.2$ We get

$$\boxed{y(0.2) = 12.2168}.$$

4) Use Taylor's series method to find y at $x=0.1$ and at $x=0.2$ given that $\frac{dy}{dx}=x^2+y^2$ with $y(0)=1$ by considering the terms upto fourth degree.

Sol^n :- Given $\frac{dy}{dx}=x^2+y^2$, $y(0)=1$, $x_0=0$, $y_0=1$.

The Taylor's series sol^n is

$$y(x)=y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\frac{(x-x_0)^4}{4!}y^{IV}(x_0)$$

$y'=x^2+y^2$; $y'(0)=0+1=1$.

$y''=2x+2yy'$; $y''(0)=2$

$y'''=2+2[yy''+(y')^2]$; $y'''(0)=2+2(2+1)=8$.

$y^{IV}=2[yy'''+y''y'+2y'y''] = 2[yy'''+3y'y'']$

$y^{IV}(0)= 2[(1)(8)+3(1)(2)] = 28$.

$\therefore$ $y(0.1)=1.1114$	$y(0.2)=1.2508$.

Modified Euler's Method :-

Consider the initial value problem $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$

The initial approximation for y_1 at $x_1=x_0+h$ is found using Euler's formula is given by

$$y_1^{(0)}=y_0+h\,f(x_0,y_0)$$

The other approximations are found using Modified Euler's formula.

$$y_1^{(1)}=y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(0)})]$$

$$y_1^{(2)}=y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(1)})]$$

$$y_1^{(3)}=y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(2)})]$$

Note:- Euler's formula and Modified Euler's formula jointly called as 'Euler's Predicted and corrected formulae'.

Problems:-

1> Use modified Euler's method to solve the initial value problem $y' = x^2 + y^2$, $y(0) = 0$, find $y(0.1)$.

Solⁿ:- Given $x_0 = 0$, $y_0 = 0$, $f(x,y) = x^2 + y^2$.

$f(x_0, y_0) = 0$, $h = 0.1$, $x_1 = x_0 + h = 0.1$.

To find $y(x_1) = y(0.1)$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 0 + 0.1(0) = 0$.

By Euler's modified formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

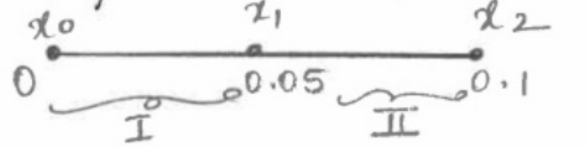
$$y_1^{(1)} = 0 + 0.05\{0 + [(0.1)^2 + 0^2]\} = 0.0005.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(2)} = 0 + (0.05)\{0 + [(0.1)^2 + (0.0005)^2]\} = 0.0005.$$

$$\therefore \boxed{y(0.1) = 0.0005.}$$

2> Determine the value of y when $x = 0.1$ given that $y(0) = 1$ and $y' = x^2 + y$. Using modified Euler's formula. Take $h = 0.05$.



Solⁿ:- Stage I :

$x_0 = 0$, $y_0 = 1$, $f(x,y) = x^2 + y^2$, $h = 0.05$

$f(x_0, y_0) = 1$, $x_1 = x_0 + h = 0.05$.

To find $y(x_1) = y(0.05)$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.05)(1) = 1.05$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + (0.025)\{1 + [(0.05)^2 + 1.05] = 1.0513.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = 1 + (0.025)\{1 + [(0.05)^2 + 1.0513] = 1.0513.$$

$$\therefore \boxed{y(0.05) = 1.0513.}$$

<u>Stage II</u>:- Let $x_0 = 0.05$, $y_0 = 1.0513$, $h = 0.05$.

$f(x, y) = x^2 + y \Rightarrow f(x_0, y_0) = 1.0538$, $x_1 = x_0 + h = 0.05 + 0.05 =$

To find $y(x_1) = y(0.1)$.

By Euler's formula, $y_1^{(0)} = y_0 + h f(x_0, y_0) = 1.0513 + (0.05)(1.0$

$$y_1^{(0)} = 1.104.$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[(f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1.0513 + (0.025)\{1.0538 + [(0.1)^2 + 1.104]\} = 1.1055$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

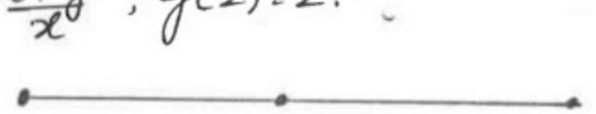
$$y_1^{(2)} = 1.0513 + (0.025)\{1.0538 + [(0.1)^2 + 1.1055]\} = 1.1055$$

$$\therefore \boxed{y(0.1) = 1.1055.}$$

3) Using Euler's predictor and Corrector formula compute $y(2.5)$ with $h = 0.25$ from $\frac{dy}{dx} = \frac{x+y}{x}$, $y(2) = 2$.

<u>Stage I</u>:-

<u>Sol^n</u>:- Given $x_0 = 2$, $y_0 = 2$, $h = 0.25$



$f(x, y) = \frac{x+y}{x} = 1 + \frac{y}{x}$. $f(x_0, y_0) = 1 + \frac{2}{2} = 2$

$x_1 = x_0 + h = 2 + 0.25 = 2.25$.

To find $y(x_1) = y(2.25)$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 2 + (0.25)(2) = 2.5$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(0)})\}$$

$$y_1^{(1)} = 2 + (0.125)\left\{2 + \left(1 + \frac{2.5}{2.25}\right)\right\} = 2.5139 .$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

$$y_1^{(2)} = 2 + (0.125)\left\{2 + \left(1 + \frac{2.5139}{2.25}\right)\right\} = 2.5147 .$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(2)})\}$$

$$y_1^{(3)} = 2 + (0.125)\left\{2 + \left(1 + \frac{2.5147}{2.25}\right)\right\} = 2.5147 .$$

$$\therefore \boxed{y(2.25) = 2.5147}$$

<u>Stage II</u>: Let $x_0 = 2.25$, $y_0 = 2.5147$, $h = 0.25$.

$f(x, y) = 1 + \frac{y}{x} \Rightarrow f(x_0, y_0) = 2.1176$, $x_1 = x_0 + h = 2.25 + 0.25 = 2.5$

To find $y(x_1) = y(2.5)$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 2.5147 + (0.25)(2.117$

$$y_1^{(0)} = 3.0441 .$$

By Modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(0)})\}$$

$$y_1^{(1)} = 2.1176 + (0.125)\left\{2.1176 + \left(1 + \frac{3.0441}{2.5}\right)\right\} = 3.0566$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

$$y_1^{(2)} = 2.1176 + (0.125)\left\{2.1176 + \left(1 + \frac{3.0566}{2.5}\right)\right\} = 3.0572$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(2)})\} = 3.0572 .$$

$$\therefore \boxed{y(2.5) = 3.0572.}$$

4) Given $y' - \sqrt{xy} = 2$, $y(1) = 1$, find $y(1.25)$ using modified Euler's method, take $h = 0.25$.

Soln: Given $x_0 = 1, y_0 = 1, h = 0.25, f(x,y) = 2 + \sqrt{xy}$

$f(x_0, y_0) = 2 + \sqrt{(1)(1)} = 3$. $x_1 = x_0 + h = 1.25$.

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.25)(3) = 1.75$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 1 + (0.125)[3 + (2 + \sqrt{(1.25)(1.75)})] = 1.8099$$

$y_1^{(2)} = 1.8130$,

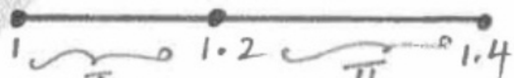
$y_1^{(3)} = 1.8132$,

$y_1^{(4)} = 1.8132$.

$$\therefore \boxed{y(1.25) = 1.8132}$$

5) Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$, find the approximate value of y at $x = 1.4$ by taking step size $h = 0.2$ applying modified Euler's method.

Soln:- Stage I :



$x_0 = 1, y_0 = 2, f(x,y) = 1 + \frac{y}{x}, h = 0.2, x_1 = x_0 + h = 1.2$.

$f(x_0, y_0) = 1 + (2/1) = 3$.

To find $y(x_1) = y(1.2)$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 2 + (0.2)(3) = 2.6$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 2 + (0.1)[3 + (1 + \frac{2.6}{1.2})] = 2.6167$$

$y_1^{(2)} = 2.6181$, $y_1^{(3)} = 2.6182$, $y_1^{(4)} = 2.6182$

$$\therefore \boxed{y(1.2) = 2.6182}$$

Stage II : $x_0 = 1.2$, $y_0 = 2.6182$, $f(x_0, y_0) = 1 + (y_0/x_0) = 3.1818$

$x_1 = x_0 + h = 1.4$, $y(x) = y(1.4)$:

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0) = 2.6182 + (0.2)(3.1818)$

$$y_1^{(0)} = 3.2546.$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 2.6182 + (0.1)\left[3.1818 + \left(1 + \frac{3.2546}{1.4}\right)\right] = 3.2689$$

$$y_1^{(2)} = 3.2699, \ y_1^{(3)} = 3.2699.$$

$$\therefore \boxed{y(1.4) = 3.2699.}$$

Runge-Kutta Method of fourth order :—

Consider the initial value problem $\frac{dy}{dx} = f(x,y)$ where $y(x_0) = y_0$.

We need to find the value of y at $x = x_1 = x_0 + h$.

Compute four Quantities namely k_1, k_2, k_3 & k_4 where

$$k_1 = h f(x_0, y_0) \ ; \ k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), \ k_4 = h f(x_0 + h, y_0 + k_3)$$

Then the value of y at $x = x_1 = x_0 + h$ is given by

$$y_1 = y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Problems:-

1) Applying R-K method of order four and find approximate value of y at $x=0.1$ given $\frac{dy}{dx}=3e^{x}+2y$ with $y(0)=0$ and $h=0.1$.

Sol^n: Here $f(x,y)=3e^{x}+2y$, $x_0=0$, $y_0=0$, $h=0.1$, $x_1=x_0+h=0.1$

We have, $K_1=hf(x_0,y_0)=(0.1)f(0,0)=(0.1)(3e^{0}+0)=0.3$

$$K_2=hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2})=(0.1)f(0.05, 0.15)$$

$$K_2=(0.1)[3.e^{0.05}+2(0.15)]=0.3454.$$

$$K_3=hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})=(0.1)f(0.05, 0.1727)$$

$$K_3=(0.1)[3.e^{0.05}+2(0.1727)]=0.3499$$

$$K_4=hf(x_0+h, y_0+K_3)=(0.1)f(0.1, 0.3499)$$

$$K_4=(0.1)[3.e^{0.1}+2(0.3499)]=0.4015$$

$$\therefore y_1=y(x_1)=y_0+\tfrac{1}{6}(K_1+2K_2+2K_3+K_4)$$

$$y(0.1)=0+\tfrac{1}{6}(0.3+0.6908+0.6998+0.4015)$$

$$\boxed{y(0.1)=0.34868}$$

2) By R-K method of order four solve $y'=3x+y/2$ with $y(0)=1$ and hence find $y(0.2)$ by taking $h=0.1$

Sol^n:- Here $f(x,y)=3x+y/2$, $x_0=0$, $y_0=1$, $h=0.1$, $x_1=x_0+h=0.1$

Stage I: $K_1=hf(x_0,y_0)=0.1f(0,1)=(0.1)(3\times0+\frac{1}{2})=0.05$

$$K_2=hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2})=0.1f(0.05, 1.025)$$

$$=0.1\left(3\times0.05+\frac{1.025}{2}\right)=0.066$$

$$K_3=hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})=0.1f(0.05, 1.033)$$

$$=0.1\left(3\times0.05+\frac{1.033}{2}\right)=0.067$$

$$k_4 = hf(x_0+h, y_0+k_3) = 0.1f(0.1, 1.067)$$
$$= 0.1\left(3\times0.1 + \frac{1.067}{2}\right) = 0.083.$$

$$y(x_1) = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$y(0.1) = 1 + \frac{1}{6}(0.05+0.132+0.134+0.083)$$

$$\boxed{y(0.1) = 1.067}$$

<u>Stage II</u>: $f(x,y) = 3x + y/2$, $x_0 = 0.1$, $y_0 = 1.067$, $h = 0.1$

$x_1 = x_0 + h = 0.1 + 0.1 = 0.2$.

We have $k_1 = hf(x_0, y_0) = 0.1f(0.1, 1.067) = 0.1\left(3\times0.1 + \frac{1.067}{2}\right)$

$k_1 = 0.0834$.

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.1f(0.15, 1.1087)$

$k_2 = 0.1\left(3\times0.15 + \frac{1.1087}{2}\right) = 0.1004$

$k_3 = 0.1\ f(0.15, 1.1172) = 0.1009$.

$k_4 = 0.1\ f(0.2, 1.1679) = 0.1184$.

$\therefore$ $\boxed{y(0.2) = 1.1677}$

3) Given $\frac{dy}{dx} = x+y^2$, $y(0)=1$, estimate the value of $y(0.2)$ with $h=0.1$ using R-K method of order 4.

Sol^n:- <u>Stage I</u>: Here $f(x,y) = x+y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$,

$x_1 = x_0 + h = 0 + 0.1 = 0.1$.

We have $k_1 = h f(x_0, y_0) = 0.1f(0,1) = 0.1[0+1^2] = 0.1$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.1\ f(0.05, 1.05)$$
$$= 0.1\ [0.05+(1.05)^2] = 0.115525$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = 0.1f(0.05, 1.0576)$$
$$= 0.1[0.05+(1.0576)^2] = 0.1169$$

$K_4 = hf(x_0+h, y_0+K_3) = 0.1 f(0.1, 1.1169)$
$= 0.1[0.1 + (1.1169)^2] = 0.1347$.

$\therefore y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$$\boxed{y(0.1) = 1.1165}$$

Stage II: $f(x,y) = x+y^2$, $x_0 = 0.1$, $y_0 = 1.1165$, $h = 0.1$, $x_1 = x_0+h = 0.1+0.1$
$= 0.2$.

We have $K_1 = hf(x_0, y_0) = 0.1 f(0.1, 1.1165) = 0.1[0.1+(1.1165)^2]$

$K_1 = 0.1347$

$K_2 = (0.1) f(0.15, 1.1839) = 0.1552$.

$K_3 = (0.1) f(0.15, 1.1941) = 0.1576$

$K_4 = (0.1) f(0.2, 0.2741) = 0.1823$

$\therefore \boxed{y(0.2) = 1.2736.}$

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4) Solve: $(y^2-x^2)dx = (y^2+x^2)dy$ for $x = 0(0.2)0.4$ given that $y=1$ at $x=0$ initially, by applying R-K method of order 4.

Soln: We have $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$; $x_0 = 0$, $y_0 = 1$, $h = 0.2$.

I stage:- $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$.

$K_1 = hf(x_0, y_0) = 0.2 f(0,1) = 0.2(1) = 0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2 f(0.1, 1.1) = 0.1967$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2 f(0.1, 1.0987) = 0.1967$

$K_4 = hf(x_0+h, y_0+K_3) = 0.2 f(0.2, 1.1967) = 0.1891$

$y(x_1) = \boxed{y(0.2) = 1.19598}$

II stage:- $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0 = 0.2$, $y_0 = 1.196$, $h = 0.2$

$$K_1 = 0.2 f(0.2, 1.196) = 0.1891$$
$$K_2 = 0.2 f(0.3, 1.29055) = 0.1795$$
$$K_3 = 0.2 f(0.3, 1.28575) = 0.1793$$
$$K_4 = 0.2 f(0.4, 1.3753) = 0.1688$$
$$y(x_1) = \boxed{y(0.4) = 1.37525}$$

Milne's Predictor - Corrector formula :-

Consider the initial value problem $y' = \frac{dy}{dx} = f(x,y)$ given $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$ where x_0, x_1, x_2, x_3 are equidistant values of x with step length h.

To compute $y_4 = y(x_4) = y(x_0 + 4h)$; Milne's Predictor and Corrector formulae are as follows:

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \text{ ---- (Predictor formula)}$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \text{ ---- (Corrector formula)}$$

Working Rule :-

Step 1: Construct the table showing the value of y corresponding four equidistant values of x and calculation of $y' = f(x,y)$.

Step 2: Calculate y_4 using the predictor formula.

Step 3: Using this value of y_4, Calculate $y_4' = f(x_4, y_4)$.

Step 4: Apply Corrector formula to get Corrected value of y_4.

Step 5: Use this new value of y_4 to calculate y_4' to apply Corrector formula once again.

Step 6: Continue the procedure till two consecutive value of y_4 are almost same.

Problems:-

1) Apply Milne's method to compute $y(1.4)$ correct to four decimal places given $\frac{dy}{dx} = x^2 + y/2$ and following the data: $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$

Soln:-

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + 2/2 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

We have $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$$y_4^{(P)} = 2 + \frac{4(0.1)}{3}(2(2.3178) - (2.67245) + 2(3.0657))$$

$$\boxed{y_4^{(P)} = 3.0793}.$$

$$y_4' = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0793}{2} = 3.49965.$$

Consider, $y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$$y_4^{(C)} = 2.4649 + \frac{0.1}{3}\left[2.67245 + 4(3.0657) + 3.49965\right]$$

$$\boxed{y_4^{(C)} = 3.0794}$$

Now $y_4' = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0794}{2} = 3.4997.$

Substituting this value of y_4' again in the corrector formula, we obtain $\boxed{y_4^{(C)} = 3.0794}$

$$\therefore \quad \boxed{y_4 = y(1.4) = 3.0794}.$$

2) Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. Compute y at $x = 0.8$ by applying Milne's method.

Sol^n:-

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y'_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.5687$
$x_4 = 0.8$	$y_4 = ?$	

We have the predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$\boxed{y_4^{(P)} = 0.3049}$$

Now $y'_4 = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$.

We have the Corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4)$$

$$\boxed{y_4^{(C)} = 0.3046.}$$

Now $y'_4 = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

Substituting this value of y'_4 again in the Corrector formula, $\boxed{y_4^{(C)} = 0.3046}$.

$$\therefore \boxed{y_4 = y(0.8) = 0.3046.}$$

3) Apply Milne's method to solve the equation $(y^2+1)dy - x^2dx = 0$ at $x=1$ given $y(0)=1$, $y(0.25)=1.0026$, $y(0.5)=1.0206$, $y(0.75)=1.0679$.

Soln: $\frac{dy}{dx} = y' = \frac{x^2}{y^2+1}$:

$y_4^{(P)} = 1.1552$; $y_4' = 0.4284$

$y_4^{(C)} = 1.154$; $y_4' = 0.4289$

$y_4^{(C)} = 1.1541$.

$\therefore y(1) = 1.1541$.