Physical World & Measurements

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Chapter 1

The Physical World

1.1 Introduction

Theory

Physics is the fundamental science that seeks to understand the laws governing the universe. It explains how matter and energy interact, from the tiniest subatomic particles to the vastness of galaxies. By observing natural phenomena, conducting experiments, and formulating theories, physics helps us unravel the mysteries of nature and develop technologies that shape our daily lives.

The influence of physics is evident in every aspect of our surroundings. Simple activities such as walking, cycling, or driving involve forces, friction, and motion described by Newton's laws. The operation of electrical appliances, from smartphones to refrigerators, relies on principles of electromagnetism. Even the way we perceive sound and light is governed by the wave properties of these phenomena.

Example

Some clear examples of physics in daily life include:

- **Gravity:** When an object falls, it accelerates towards the Earth due to gravitational attraction, as explained by Newton's law of universal gravitation.
- **Friction:** Rubbing hands together generates heat due to friction, a force that resists motion between two surfaces.
- **Magnetism:** A compass needle aligns itself with Earth's magnetic field, helping with navigation.
- **Buoyancy:** Objects float or sink in water depending on their density relative to the fluid, as described by Archimedes' principle.
- **Optics:** The bending of a straw in a glass of water is due to the refraction of light, which changes speed when transitioning between different media.
- **Thermodynamics:** Ice melts when exposed to heat because of energy transfer, demonstrating concepts of heat and temperature regulation.

Physics not only explains natural occurrences but also enables technological advancements that improve our quality of life. From the mechanics of a simple pendulum to the complex principles behind satellites and space exploration, physics plays a vital role in shaping the modern world.

1.2 Scope and Importance of Physics

Theory

Physics is one of the most fundamental scientific disciplines, influencing nearly every aspect of modern life. Its scope extends from understanding the smallest particles in quantum mechanics to studying galaxies in astrophysics.

The importance of physics can be observed in several domains:

1.2.1 Role of Physics in Science and Technology

Physics serves as the foundation of many scientific disciplines, including chemistry, biology, and engineering. Advances in physics drive technological innovations, such as:

Example

Key Contributions of Physics to Technology:

- **Electricity and Electronics:** The principles of electromagnetism enable the functioning of electrical devices, from household appliances to computers and smartphones.
- **Medical Applications:** Physics plays a vital role in healthcare through imaging technologies like X-rays, MRIs, and ultrasound, as well as radiation therapies for cancer treatment.
- **Space Exploration:** Rockets, satellites, and telescopes rely on Newton's laws of motion and gravitation to explore outer space.
- **Renewable Energy:** Solar panels, wind turbines, and nuclear reactors operate based on principles of thermodynamics and energy conversion.

1.2.2 Interdisciplinary Applications of Physics

Physics is deeply connected with other fields, leading to breakthroughs such as:

Example

Physics in Other Scientific Fields:

- **Biophysics:** Understanding molecular structures and biological processes through the principles of physics.
- **Materials Science:** Developing new materials with unique properties, such as superconductors and nanotechnology.
- **Artificial Intelligence and Computing:** Quantum computing and semiconductor technology are based on the physics of electrons and information processing.

1.2.3 Impact of Physics on Daily Life

Physics governs everyday experiences, from the way we see and hear to how vehicles move. Understanding physics enhances problem-solving skills and critical thinking, making it essential for technological progress and scientific discovery.

1.2.4 Role of Physics in an 11th/12th Student's Life

Physics is a crucial subject for students preparing for board exams, competitive entrance tests, and Olympiads. A strong foundation in physics enhances problem-solving skills and analytical thinking, which are essential for academic success and future careers in science and engineering.

Physics in Board Exams

Exam Tip

Physics is a core subject in 11th and 12th-grade curricula, requiring conceptual understanding and numerical proficiency. Topics such as mechanics, electromagnetism, and optics are frequently tested in board exams, demanding both theoretical knowledge and problem-solving techniques.

Physics in Entrance Exams

Competitive exams like [JEE: Mains/Advanced], [NEET:], and other engineering and medical entrance tests place a significant emphasis on physics. Success in these exams requires:

Exam Tip

Preparation Strategies for Entrance Exams:

- Strong conceptual understanding of fundamental principles.
- Speed and accuracy in solving numerical problems.
- Application of multiple concepts in complex problem scenarios.

Physics in Olympiads

Physics Olympiads, such as the National Standard Examination in Physics (NSEP) and the International Physics Olympiad (IPhO), test students on advanced problem-solving and experimental physics. Preparing for Olympiads helps in:

Exam Tip

Key Benefits of Physics Olympiad Preparation:

- Deepening conceptual clarity.
- Developing logical reasoning and mathematical modeling skills.
- Enhancing problem-solving abilities beyond standard curriculum levels.

Physics not only helps students excel in academics but also lays the foundation for careers in engineering, research, and applied sciences. Mastering physics during school years prepares students for higher education and future challenges in scientific fields.

1.3 Scientific Methods in Physics

Theory

The scientific method is a systematic approach used by physicists to explore natural phenomena, develop theories, and validate principles. It consists of observation, hypothesis formation, experimentation, analysis, and theory development.

1.3.1 Historical Development of the Scientific Method

Historical Insight

The scientific method has evolved over centuries:

- **Aristotle (384–322 BCE):** Relied on logical reasoning without experimentation.
- **Galileo Galilei (1564–1642):** Worked using experimentation and mathematical analysis.
- **Isaac Newton (1643–1727):** Formulated the laws of motion and universal gravitation.
- **Albert Einstein (1879–1955):** Developed relativity through thought experiments and mathematical formulations.

1.3.2 Observation

Science begins with careful observation of natural events.

Example

Example: Ancient astronomers observed planetary motion, leading to Kepler's laws of planetary motion.

1.3.3 Hypothesis Formation

Based on observations, scientists propose a hypothesis—an educated guess explaining a phenomenon.

Example

Example: Newton hypothesized that an invisible force (gravity) governs planetary motion.

1.3.4 Experimentation and Testing

Hypotheses are tested through controlled experiments. Accurate data collection and analysis determine validity.

Example

Example: Galileo's experiments with inclined planes confirmed the laws of motion.

1.3.5 Analysis and Interpretation

Experimental results are analyzed using mathematics and logic. A hypothesis gains credibility if confirmed through multiple tests.

1.3.6 Development of Theories and Laws

A well-tested hypothesis evolves into a theory (e.g., Einstein's theory of relativity). If a theory consistently predicts and explains phenomena, it may become a law (e.g., Newton's laws of motion).

1.3.7 Variations in Scientific Methods

Some discoveries arise purely from theoretical predictions before experimental validation.

Example

Example: Einstein's general theory of relativity was mathematically formulated before being confirmed by experiments (e.g., the bending of light by gravity observed in 1919).

1.3.8 Real-World Applications of the Scientific Method

The scientific method is crucial for technological advancements.

Practical Application

Applications in Technology:

- Discovery of electromagnetism led to the invention of electric generators and motors.
- Understanding semiconductors enabled the development of transistors and modern electronics.
- Quantum mechanics innovations resulted in lasers and quantum computing.
- Vaccines were developed through rigorous scientific testing and validation.

1.3.9 The Scientific Method in Everyday Life

Beyond laboratories, the scientific method is used in everyday decision-making.

Practical Application

Examples in Daily Life:

- Testing which detergent removes stains better by experimenting with different brands.
- Checking if a light bulb is faulty by swapping it with another one.
- Determining the fastest route to school by trying different paths and recording travel time.

Physics thrives on observation, experimentation, and reasoning. By applying the scientific method, physicists continue to uncover new laws that shape our understanding of the universe.

1.4 Fundamental Forces in Nature

Theory

All physical interactions in the universe arise from four fundamental forces governing phenomena from subatomic to cosmic scales.

Exam Tip

Expect 1-2 questions on:

- Force comparisons (Table 1.1)
- Gravitation vs Electromagnetism
- Nuclear force applications

1.4.1 The Four Fundamental Forces

1. **Gravitational Force**

Key Formula

$$F = G\frac{m_1 m_2}{r^2} \quad \text{(Attractive only)}$$

G	$6.674 \times 10^{-11}\text{Nm}^2/\text{kg}^2$
Range	Infinite
Relative Strength	10^{-38}

Einstein's General Relativity describes gravity as spacetime curvature. Governs:

- Planetary orbits [JEE: Kepler's Laws]
- Galaxy formation
- Everyday weight ($W = mg$)

2. **Electromagnetic Force**

Key Formula

$$F = k\frac{q_1 q_2}{r^2} \quad \text{(Attractive/Repulsive)}$$

k	$8.988 \times 10^9 \text{Nm}^2/\text{C}^2$
Range	Infinite
Relative Strength	10^{-2}

Governs:

- Atomic structure [NEET: 2023]
- Light propagation ($c = \lambda\nu$)
- All electronic devices

3. **Strong Nuclear Force**

- Strongest force ($\sim 100\times$ EM force)
- Range: 10^{-15} m (nuclear scale)
- Binds quarks via gluons
- Overcomes proton EM repulsion

Example

- Nuclear power plants
- Stellar nucleosynthesis
- PET scan technology

4. **Weak Nuclear Force**

- Responsible for β-decay: $n \to p + e^- + \bar{\nu}_e$
- Range: 10^{-18} m
- Mediated by W/Z bosons
- Essential for solar fusion

Force	Strength	Range	Mediator	Key Applications
Strong	1	10^{-15} m	Gluons	Nuclear power, QCD
EM	10^{-2}	∞	Photons	Electronics, optics
Weak	10^{-13}	10^{-18} m	W/Z bosons	β-decay, solar fusion
Gravity	10^{-38}	∞	Graviton*	Astronomy, GPS

Table 1.1: Properties of Fundamental Forces [JEE: Memorize for JEE]

1.4.2 Comparative Analysis

Common Mistake

Students often:

- Confuse EM and weak force ranges
- Underestimate gravitational effects at cosmic scales
- Forget strong force doesn't act on electrons

1.4.3 Technological Applications

- **Medical Imaging**
  - MRI (EM)
  - PET (Weak)
- **Energy Production**
  - Nuclear reactors (Strong)
  - Solar cells (EM)
- **Communication**
  - Radio (EM)
  - GPS (Gravity)
- **Materials Science**
  - Superconductors (EM)
  - Neutron stars (Strong)

Exam Tip

For force comparison questions:

1. Identify interaction type
2. Check distance scale
3. Compare relative strengths
4. Consider charge/mass involvement

1.5 Mathematical Tools in Physics

Mathematics is the language of physics. It provides precise methods for formulating laws, solving equations, and analyzing physical phenomena. The key mathematical tools used in physics include:

1.5.1 Algebra and Trigonometry

Algebra and trigonometry are fundamental for solving equations and analyzing relationships between physical quantities. These concepts are used in:

- Solving equations of motion, such as

$$s = ut + \frac{1}{2}at^2$$

- Analyzing wave motion using trigonometric functions like sine and cosine.
- Resolving forces into components using vector decomposition.
- Understanding periodic motion and oscillations.

1.5.2 Calculus (Differentiation and Integration)

Calculus plays a crucial role in understanding change and accumulation, making it essential for physics:

- **Differentiation:** Used to determine instantaneous quantities such as:

$$v = \frac{ds}{dt}, \quad a = \frac{dv}{dt}$$

- **Integration:** Helps calculate areas under curves, total work done by a force, and accumulated quantities, such as displacement from velocity.

Examples:

- Calculating the rate of heat transfer using differential equations.
- Finding electric potential from the electric field using integration.

1.5.3 Vector Analysis

Vectors are essential in physics because many quantities, such as force, velocity, and acceleration, have both magnitude and direction. They form the basis of numerous physical theories, from classical mechanics to electromagnetism and quantum mechanics. Some key operations include:

- **Vector addition and subtraction:** Used in mechanics to determine resultant forces and displacements.
- **Dot product:** Helps find work done:

$$W = \mathbf{F} \cdot \mathbf{d}$$

  and determines the angle between two vectors.
- **Cross product:** Used in torque calculations:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

  and to find the direction of magnetic force in electromagnetism.

Vectors are fundamental in almost every area of physics, from motion and forces to electric and magnetic fields, making them an indispensable mathematical tool.

1.5.4 Importance of Mathematical Tools in Physics

Physics relies on mathematical tools for:

- **Precise Quantification:** Mathematical models express physical laws in exact terms.
- **Prediction and Analysis:** Equations derived using calculus and algebra help predict motion, forces, and energy interactions.
- **Simplifying Complex Problems:** Vectors and calculus allow for efficient problem-solving in mechanics, electromagnetism, and quantum physics.
- **Scientific and Engineering Applications:** From designing bridges to developing electrical circuits, mathematical physics is crucial in real-world problem-solving.

Key Takeaways

- Mathematics provides the foundation for all physical laws and theories.
- Algebra, trigonometry, and calculus are essential for solving physics problems.
- Vector analysis is crucial for understanding forces, motion, and fields.
- Mastering mathematical tools is vital for competitive exams and research.

1.6 Importance of Graphs in Physics

Graphs play a crucial role in physics by providing a visual representation of relationships between different physical quantities. They help in understanding trends, analyzing data, and solving complex problems efficiently.

1.6.1 Types of Graphs in Physics

Different types of graphs are used in physics, each serving a specific purpose:

- **Linear Graphs:** Represent direct proportionality, such as Ohm's Law:
$$V = IR$$
or uniform motion:
$$s = vt$$
- **Parabolic Graphs:** Appear in quadratic relationships, such as projectile motion:
$$y = ut + \frac{1}{2}gt^2$$
- **Exponential and Logarithmic Graphs:** Used in decay processes (radioactive decay, capacitor discharge) and wave phenomena.
- **Trigonometric Graphs:** Describe oscillatory motion, such as simple harmonic motion:
$$x = A\cos(\omega t)$$
- **Bar and Pie Charts:** Used in statistical physics and experimental data analysis.

1.6.2 Why Graphs Are Important in Physics

- **Understanding Relationships:** Graphs visually depict how one physical quantity depends on another, making abstract concepts easier to grasp.
- **Predicting Trends:** From velocity-time graphs to energy distributions, graphs help predict future values.
- **Data Analysis:** Experimental data is often plotted on graphs to identify patterns, errors, and correlations.
- **Problem Solving:** Many physics problems can be solved using graphical methods, such as finding acceleration from a velocity-time graph.
- **Comparing Theoretical and Experimental Data:** Graphs allow scientists to check if experimental results align with theoretical predictions.

Key Takeaways

- Graphs provide a clear and intuitive way to visualize relationships in physics.
- Different types of graphs (linear, parabolic, exponential, trigonometric) are used for specific physics applications.
- They help in predicting trends, solving problems, and validating theoretical models.
- Graph interpretation is a crucial skill for exams, experiments, and research.

Chapter 2

Units and Dimensions

2.1 Introduction to Measurement

Measurement is the cornerstone of physics, enabling us to quantify physical phenomena and compare them with standard references. It provides a systematic way to describe the universe in numerical terms, forming the basis for scientific analysis and technological advancements.

2.1.1 Components of Measurement

A measurement consists of two essential components:

- **Numerical Value** – Represents the magnitude of the quantity being measured.
- **Unit** – Provides a reference standard for the measurement.

2.1.2 Example

For instance, if the length of a table is given as:

$$\textbf{Length} = 2 \text{ meters} \quad (2\,\text{m})$$

- The number **2** represents the **magnitude**.
- The word **meter (m)** represents the **unit of measurement**.

Without a unit, the numerical value alone is meaningless.

Why is Measurement Important?

- Measurements help quantify physical properties and compare them objectively.
- They ensure precision and accuracy in scientific experiments.
- Standardized measurements allow global consistency in research and engineering.

2.1.3 International System of Units (SI)

Precise and accurate measurements are fundamental to experimental physics, engineering, and various scientific disciplines. To ensure consistency and global standardization, scientists rely on the **International System of Units (SI)**, which provides universally accepted definitions for fundamental and derived physical quantities.

2.2 Fundamental and Derived Quantities

Physical quantities are classified into two types: **fundamental quantities** and **derived quantities**.

2.2.1 Fundamental Quantities

Fundamental quantities are the basic physical quantities that do not depend on any other quantity for their definition. These serve as the foundation for all physical measurements.

The International System of Units (SI) defines **seven** fundamental quantities, each with a corresponding unit and symbol. The **units** of **fundamental quantities** are called **fundamental units**:

Quantity	**Symbol**	**SI Unit**
Length	L	meter (m)
Mass	M	kilogram (kg)
Time	T	second (s)
Electric Current	I	ampere (A)
Temperature	T or θ	kelvin (K)
Amount of Substance	n	mole (mol)
Luminous Intensity	I_v	candela (cd)

Table 2.1: Fundamental Quantities and Their SI Units

Note: SI previously classified *plane angle* and *solid angle* as **supplementary units**, but they are now considered *dimensionless derived quantities*. Their SI units are:

Quantity	**Symbol**	**SI Unit**
Plane Angle	θ	radian (rad)
Solid Angle	Ω	steradian (sr)

Table 2.2: SI Units for Plane and Solid Angles

2.2.2 Derived Quantities

Derived quantities are obtained from fundamental quantities through mathematical operations such as multiplication or division. Since they depend on fundamental quantities, their units are derived by combining the units of fundamental quantities.

Some commonly used derived quantities are:

Quantity	**Symbol**	**Formula**	**SI Unit**
Area	A	$L \times L$	m^2
Volume	V	$L \times L \times L$	m^3
Velocity	v	$\frac{\text{displacement}}{\text{time}}$	$\mathrm{m\,s^{-1}}$
Acceleration	a	$\frac{\text{velocity}}{\text{time}}$	$\mathrm{m\,s^{-2}}$
Force	F	$m \times a$	$\mathrm{N\ (kg\,m\,s^{-2})}$
Work	W	$F \times d$	$\mathrm{J\ (kg\,m^2\,s^{-2})}$
Power	P	$\frac{W}{t}$	$\mathrm{W\ (J\,s^{-1})}$
Pressure	P	$\frac{F}{A}$	$\mathrm{Pa\ (N\,m^{-2})}$
Density	ρ	$\frac{m}{V}$	$\mathrm{kg\,m^{-3}}$

Table 2.3: Derived Quantities and Their SI Units

2.3 Systems of Units

A system of units is a standardized set of units used to measure physical quantities. Physical quantities are classified into two types:

- **Fundamental Quantities:** Basic physical quantities that do not depend on other quantities. Their units are called **fundamental units** (e.g., m, kg, s).
- **Derived Quantities:** Obtained by combining fundamental quantities through mathematical relationships. Their units are called **derived units** (e.g., N, J, W).

Different systems of units have been developed over time for convenience:

- **CGS System:** Uses centimeter (cm), gram (g), and second (s) as fundamental units.
- **MKS System:** Uses meter (m), kilogram (kg), and second (s) as fundamental units.
- **SI System:** The internationally accepted system based on seven fundamental units.

2.4 Establishment and Definition of Units

The measurement of physical quantities requires standardized units that remain consistent worldwide. International organizations define and maintain these units.

2.4.1 Who Decides the Units?

The International System of Units (SI) is regulated by:

- **General Conference on Weights and Measures (CGPM)** – Established in 1875 under the Metre Convention.
- **International Committee for Weights and Measures (CIPM)** – Oversees SI unit implementation.
- **International Bureau of Weights and Measures (BIPM)** – Ensures global uniformity.

2.4.2 How Are Units Defined?

Earlier, units were defined based on physical objects or Earth's properties. Modern definitions rely on **fundamental physical constants** for accuracy and reproducibility:

- **Meter:** Defined using the speed of light in vacuum.
- **Second:** Defined based on the cesium-133 atomic transition.
- **Kilogram:** Defined using the Planck constant.

These definitions ensure precision across time and space.

2.4.3 Definition of SI Units

Note to students: Do not memorize these definitions. Instead, understand the concept and move on.

The SI system consists of seven fundamental units, precisely defined using physical constants.

Meter (m) The distance light travels in vacuum in $\frac{1}{299792458}$ seconds.

Kilogram (kg) Defined using the Planck constant:

$$h = 6.626\,070\,15 \times 10^{-34}\,\mathrm{J\,s}$$

Second (s) Defined using atomic transitions in cesium-133:

$$1\ \mathrm{s} = 9.192\,631\,770 \times 10^{9}\,\text{oscillations}$$

Ampere (A) Defined using the elementary charge:

$$e = 1.602\,176\,634 \times 10^{-19}\,\mathrm{C}$$

Kelvin (K) Defined using the Boltzmann constant:

$$k_B = 1.380\,649 \times 10^{-23}\,\mathrm{J\,K^{-1}}$$

Mole (mol) The number of atoms in 12 grams of carbon-12:

$$N_A = 6.022\,140\,76 \times 10^{23}\,\mathrm{mol^{-1}}$$

Candela (cd) Defined based on luminous efficacy at 540 THz.

2.5 Dimensional Formulae and Applications

2.5.1 What are Dimensions?

In physics, **dimensions** represent the fundamental nature of a physical quantity. The **dimension of a physical quantity** refers to the power to which fundamental quantities (Mass, Length, Time, etc.) are raised to express that quantity. Every measurable quantity in physics can be expressed in terms of basic dimensions such as:

- Mass $[M]$
- Length $[L]$
- Time $[T]$
- Electric Current $[A]$
- Temperature $[\Theta]$
- Amount of Substance [mol]
- Luminous Intensity [cd]

Thus, any physical quantity can be expressed in terms of these fundamental dimensions.
For example:

- Distance has the dimension of length $[L]$.
- Speed is the ratio of distance to time, so it has the dimension $[LT^{-1}]$.
- Force, given by Newton's second law ($F = ma$), has the dimension $[MLT^{-2}]$.

2.5.2 Dimensional Formulae

A **dimensional formula** expresses a derived physical quantity in terms of fundamental dimensions.

The general form is:

$$[M^a L^b T^c A^d \Theta^e \mathrm{mol}^f \mathrm{cd}^g]$$

where a, b, c, d, e, f, g are the respective powers of mass, length, time, electric current, temperature, amount of substance, and luminous intensity.

Note: A dimensional formula does not provide numerical constants (like $\frac{1}{2}$ or 2π). It only expresses dependency on fundamental quantities.

Some important dimensional formulae are:

$$\textbf{Velocity} : \boxed{[L^1T^{-1}]}$$
$$\textbf{Acceleration} : \boxed{[L^1T^{-2}]}$$
$$\textbf{Force} : \boxed{[M^1L^1T^{-2}]}$$
$$\textbf{Work / Energy} : \boxed{[M^1L^2T^{-2}]}$$
$$\textbf{Power} : \boxed{[M^1L^2T^{-3}]}$$
$$\textbf{Pressure} : \boxed{[M^1L^{-1}T^{-2}]}$$

2.5.3 Applications of Dimensional Analysis

Dimensional analysis is widely used in physics and engineering for various purposes:

- **Checking the correctness of equations:** According to the **principle of homogeneity**, all terms in a valid equation must have the same dimensions. Example: In the kinematic equation,
$$s = ut + \frac{1}{2}at^2$$
all terms have the dimension of length $[L]$, confirming the equation is dimensionally correct.

- **Deriving relationships between physical quantities:** When the exact formula is unknown, dimensional analysis helps estimate it. Example: The time period (T) of a simple pendulum depends on length (L) and acceleration due to gravity (g). Using dimensional analysis:
$$T \propto \sqrt{\frac{L}{g}}$$

- **Converting units between different systems:** Example: To convert force from CGS (dyne) to SI (newton), we use:
$$1\ \text{N} = 10^5\ \text{dyne}$$

2.6 Dimensional Analysis

2.6.1 Introduction

Dimensional analysis is a mathematical technique used to study the relationships between physical quantities by analyzing their dimensions. It is based on the principle of homogeneity, which states that any physically meaningful equation must have the same dimensions on both sides.

This method is useful for:

- Verifying the correctness of equations.
- Deriving relations between physical quantities.
- Converting units between different measurement systems.
- Establishing the form of physical laws when exact equations are unknown.

2.6.2 Checking Correctness of Equations

Theory

Dimensional Homogeneity An equation is dimensionally correct if all terms on both sides have the same dimensions. This ensures consistency with fundamental physical principles. The process of checking an equation's correctness using dimensions is based on the principle of **dimensional homogeneity**, which states that:

All terms in a physically valid equation must have the same dimensions.

Steps to Check an Equation

To verify an equation, follow these steps:

1. Identify the physical quantities in the equation.
2. Write their dimensional formulas in terms of fundamental units.
3. Substitute the dimensions into the equation.
4. Check if both sides of the equation have the same dimensions.

Example: Verifying the Kinematic Equation

Consider the kinematic equation:

$$s = ut + \frac{1}{2}at^2 \tag{2.1}$$

where:

- s (displacement) has the dimension $[L]$,
- u (initial velocity) has the dimension $[LT^{-1}]$,
- t (time) has the dimension $[T]$,
- a (acceleration) has the dimension $[LT^{-2}]$.

Checking dimensional consistency:

$$\begin{aligned} \text{LHS: } [s] &= [L] \\ \text{RHS: } [ut] &= [LT^{-1}] \times [T] = [L] \\ [at^2] &= [LT^{-2}] \times [T^2] = [L] \\ \Rightarrow [L] &= [L] + [L] \end{aligned}$$

Since both sides have the same dimensions, the equation is **dimensionally correct**.

Example: Checking Newton's Second Law

Newton's Second Law states:

$$F = ma \tag{2.2}$$

where:

- F (force) has the SI unit Newton (N), which is defined as $kg \cdot m/s^2$ with the dimensional formula $[MLT^{-2}]$.
- m (mass) has the dimension $[M]$.
- a (acceleration) has the dimension $[LT^{-2}]$.

Checking dimensions:

$$\begin{aligned} \text{LHS: } [F] &= [MLT^{-2}] \\ \text{RHS: } [m] \times [a] &= [M] \times [LT^{-2}] = [MLT^{-2}] \end{aligned}$$

Since both sides are identical, the equation is **dimensionally valid**.

Common Mistake

- **Dimensional analysis only checks consistency, not accuracy.** It cannot verify numerical constants like $\frac{1}{2}$ or g.
- **Each term in the equation must have the same dimensions.** If even one term differs, the equation is incorrect.
- **Trigonometric, exponential, and logarithmic functions must have dimensionless arguments.** For example, $\sin(\theta)$ is valid only if θ is dimensionless.
- **Dimensional analysis cannot distinguish between scalars and vectors.** It ensures unit consistency but does not account for vector directions.

Key Takeaways

Dimensional analysis is a powerful tool to check the correctness of physical equations. While it ensures unit consistency, it does not confirm the accuracy of numerical coefficients. Despite its limitations, it remains a valuable method for verifying and understanding physical relationships.deriving equations in physics.

2.6.3 Deriving Relations Between Physical Quantities

Dimensional analysis is a powerful tool that helps in determining the form of a physical relation when the exact equation is unknown. It is particularly useful in cases where a physical quantity depends on multiple factors, but the explicit formula is not given.

Basic Idea

Theory

Key Idea of Dimensional Analysis:

- Identify the variables on which the unknown quantity depends.
- Express the unknown quantity as a product of these variables raised to some unknown powers.
- Use dimensional consistency to determine the powers.

To understand this method, let's derive the formula for the time period of a simple pendulum.

Example: Time Period of a Simple Pendulum

The time period T (the time taken for one complete oscillation) of a simple pendulum is known to depend on:

- The length L of the pendulum.
- The acceleration due to gravity g.

Since we do not know the exact relationship, we assume a general proportionality:

$$T \propto L^a g^b$$

where a and b are unknown exponents to be determined using dimensional analysis.

Expressing Dimensions

Theory

Dimensional Forms of Variables:

- Time period T has the dimension: $[T]$.
- Length L has the dimension: $[L]$.
- Acceleration due to gravity g has the dimension: $[LT^{-2}]$.

Substituting these in the assumed proportionality equation:

$$[T] = [L]^a [LT^{-2}]^b$$

Expanding the powers:

$$[T] = [L^a][L^bT^{-2b}]$$

$$[T] = [L^{a+b}T^{-2b}]$$

Equating Powers of Fundamental Quantities

For the equation to be dimensionally consistent, the dimensions on the left-hand side (LHS) and the right-hand side (RHS) must be the same. This gives two equations by comparing the powers of L and T:

Example

Equating Powers of L and T:

- Equating the power of L:

$$a + b = 0$$

- Equating the power of T:

$$-2b = 1$$

Solving for a and b

From the second equation:

$$b = -\frac{1}{2}$$

Substituting this value in the first equation:

$$a - \frac{1}{2} = 0$$

$$a = \frac{1}{2}$$

Final Relationship

Thus, we obtain:

$$T \propto L^{\frac{1}{2}}g^{-\frac{1}{2}}$$

$$T \propto \sqrt{\frac{L}{g}}$$

Since proportionality implies the presence of a dimensionless constant k, we write:

$$T = k\sqrt{\frac{L}{g}}$$

Conclusion

Key Formula

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Here, $k = 2\pi$ is a dimensionless constant that cannot be determined by dimensional analysis. Such constants usually arise from experimental results or deeper theoretical derivations.

Key Points to Remember

Exam Tip

Important Notes:

- Dimensional analysis can only determine the form of the equation (how variables are related), not the exact numerical constants.
- The method relies on the principle of dimensional homogeneity.
- It is useful in deriving unknown relationships and checking consistency of physical formulas.

2.6.4 Converting Units

Dimensional analysis is a useful tool for converting physical quantities from one system of units to another. It ensures consistency in equations and calculations when different unit systems are used.

Method of Unit Conversion

Theory

Steps for Unit Conversion:

1. Express the given quantity in terms of its fundamental units.
2. Use the appropriate conversion factors between the given and required units.
3. Ensure that unit cancellations result in the desired unit.

Example: Converting Force from CGS to SI System

Force in SI units is measured in **newton (N)**, while in the CGS system, it is measured in **dyne**.

Example

Step 1: Express Force in Fundamental Units

- In the SI system:
$$1\ \mathrm{N} = 1\ \mathrm{kg} \cdot \mathrm{m/s}^2$$
- In the CGS system:
$$1\ \mathrm{dyne} = 1\ \mathrm{g} \cdot \mathrm{cm/s}^2$$

Step 2: Express the Relationship Between SI and CGS Units

- $1\ \mathrm{kg} = 10^3\ \mathrm{g}$
- $1\ \mathrm{m} = 10^2\ \mathrm{cm}$

Step 3: Convert Newton to Dyne

$$1\ \mathrm{N} = (1\ \mathrm{kg}) \cdot (1\ \mathrm{m/s}^2)$$
$$= (10^3\ \mathrm{g}) \cdot (10^2\ \mathrm{cm/s}^2)$$
$$= 10^5\ \mathrm{g} \cdot \mathrm{cm/s}^2$$

Since $1\ \mathrm{dyne} = 1\ \mathrm{g} \cdot \mathrm{cm/s}^2$, we get:

$$1\ \mathrm{N} = 10^5\ \mathrm{dyne}$$

Example: Converting Energy from CGS (erg) to SI (joule)

Example

Step 1: Define Energy in Fundamental Units

- In SI: $1\ \text{J} = 1\ \text{N} \cdot \text{m} = 1\ \text{kg} \cdot \text{m}^2/\text{s}^2$.
- In CGS: $1\ \text{erg} = 1\ \text{dyne} \cdot \text{cm} = 1\ \text{g} \cdot \text{cm}^2/\text{s}^2$.

Step 2: Express SI Units in Terms of CGS

- $1\ \text{kg} = 10^3\ \text{g}$
- $1\ \text{m} = 10^2\ \text{cm}$
- $1\ \text{N} = 10^5\ \text{dyne}$

Step 3: Convert Joules to Ergs

$$1\ \text{J} = 1\ \text{N} \cdot 1\ \text{m}$$

$$= (10^5\ \text{dyne}) \cdot (10^2\ \text{cm})$$

$$= 10^7\ \text{erg}$$

Thus,

$$1\ \text{J} = 10^7\ \text{erg}$$

Key Takeaways

Exam Tip

Important Points:

- Dimensional analysis ensures correct unit conversion.
- Always express quantities in fundamental units before converting.
- Remember commonly used conversion factors to simplify calculations.

2.6.5 Limitations of Dimensional Analysis

Theory

Limitations of Dimensional Analysis: Dimensional analysis is a powerful tool but has several limitations, which must be kept in mind when using it.

1. **Cannot determine numerical constants:** Dimensional analysis cannot predict

the exact numerical values of dimensionless constants in an equation. For example, in the formula for the time period of a simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Dimensional analysis can only derive $T \propto \sqrt{L/g}$ but cannot determine the factor 2π. Such constants must be determined experimentally.

2. **Cannot distinguish between scalar and vector quantities:** Dimensional analysis only considers the magnitude of a quantity and does not account for directional properties. For example, both work and torque have the same dimensions:

$$[W] = [T] = [ML^2T^{-2}]$$

However, work is a scalar, while torque is a vector. Dimensional analysis does not differentiate between them.

3. **Not applicable to equations involving logarithmic, exponential, or trigonometric functions:** Functions such as $\sin x$, $\ln x$, and e^x must be dimensionless. However, dimensional analysis cannot be applied directly to derive equations involving such terms. For example, the equation for damped harmonic motion:

$$x = Ae^{-bt/m}\cos(\omega t + \phi)$$

contains an exponential term $e^{-bt/m}$ which is dimensionless, but dimensional analysis cannot predict its presence.

4. **Limited applicability in cases with multiple variables:** When a physical quantity depends on more than three variables, dimensional analysis alone may not be sufficient to derive the complete relation. Additional experimental data or theoretical derivation is often required.

5. **Cannot differentiate between additive and multiplicative constants:** Dimensional analysis does not determine whether terms in an equation should be added or multiplied. For example, in the equation of motion:

$$s = ut + \frac{1}{2}at^2$$

Dimensional analysis only verifies that each term has the same dimension but does not justify why the acceleration term appears with a coefficient $\frac{1}{2}$.

Conclusion:

Common Mistake

Despite its limitations, dimensional analysis remains a valuable tool for verifying equations, deriving relations, and converting units. However, it should always be complemented with experimental data and theoretical understanding.

2.7 Errors in Measurement

Measurement errors are inevitable and arise due to various factors. Errors can be classified into different types based on their origin and behavior.

2.7.1 Types of Errors

Theory

Errors in measurement can be broadly classified into three categories: systematic errors, random errors, and gross errors.

- **Systematic Errors:** These errors occur consistently in the same direction, either always higher or always lower than the true value. They arise due to faulty instruments, calibration issues, improper measurement techniques, or external influences. Systematic errors can be minimized through careful calibration and improved measurement techniques.

  Example

  Examples:

  - A weighing scale showing a constant error due to incorrect zero adjustment.
  - A thermometer consistently reading 2 °C higher due to improper calibration.

- **Random Errors:** These errors occur unpredictably due to small fluctuations in experimental conditions. They can arise from factors such as variations in environmental conditions, observer estimation, or limitations in instrument precision. Unlike systematic errors, random errors can be reduced by taking multiple measurements and averaging the results.

  Example

  Examples:

  - Slight variations in repeated time measurements using a stopwatch.
  - Fluctuations in readings due to slight hand tremors while measuring length.

- **Gross Errors:** These errors occur due to human mistakes such as incorrect readings, miscalculations, or data recording errors. Gross errors can be minimized through careful observation, double-checking readings, and following proper procedures.

Example

Examples:

- Misreading a scale and noting 3.5 cm instead of 5.3 cm.
- Writing 120 g instead of 210 g in an experiment.

2.7.2 Quantifying Errors

Errors in measurement can be quantified using different methods.

Absolute Error

Key Formula

The absolute error is the difference between the measured value and the true value:

$$\Delta X = |X_{\text{measured}} - X_{\text{true}}|$$

where X_{measured} is the observed value and X_{true} is the actual value.

Relative Error

Key Formula

The relative error is the ratio of the absolute error to the true value:

$$\text{Relative Error} = \frac{\Delta X}{X_{\text{true}}}$$

It gives a sense of how significant the error is in comparison to the true value.

Percentage Error

Key Formula

The percentage error expresses the relative error as a percentage:

$$\text{Percentage Error} = \left(\frac{\Delta X}{X_{\text{true}}} \right) \times 100\%$$

It is useful for comparing errors in different measurements.

Example

Example: Suppose the true value of a mass is 50.0 g, but a scale gives a reading of 48.5 g.

- **Absolute Error:**
$$\Delta X = |48.5 - 50.0| = 1.5 \text{ g}$$
- **Relative Error:**
$$\frac{1.5}{50.0} = 0.03$$
- **Percentage Error:**
$$0.03 \times 100 = 3\%$$

2.7.3 Mean of Measurements and Mean Absolute Error

When multiple measurements of a physical quantity are taken, slight variations may occur due to random errors. To obtain a more reliable value, we use the mean (average) of all measurements.

Mean (Average) Value

Key Formula

If a quantity is measured n times and the individual measurements are $X_1, X_2, X_3, \ldots, X_n$, the mean value X_{mean} is given by:

$$X_{\text{mean}} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

Example

Example: Suppose a length is measured five times as:

$$5.1 \text{ cm}, 5.0 \text{ cm}, 5.2 \text{ cm}, 5.1 \text{ cm}, 5.3 \text{ cm}$$

The mean value is:

$$X_{\text{mean}} = \frac{5.1 + 5.0 + 5.2 + 5.1 + 5.3}{5} = \frac{25.7}{5} = 5.14 \text{ cm}$$

Mean Absolute Error

Key Formula

The mean absolute error (ΔX_{mean}) represents the average deviation of individual measurements from the mean value. It is calculated as:

$$\Delta X_{\text{mean}} = \frac{|\Delta X_1| + |\Delta X_2| + \cdots + |\Delta X_n|}{n}$$

where $\Delta X_i = |X_i - X_{\text{mean}}|$ is the absolute error for each measurement.

Example

Example: Using the previous example where $X_{\text{mean}} = 5.14$ cm, we compute absolute errors:

$$\Delta X_1 = |5.1 - 5.14| = 0.04 \text{ cm}$$
$$\Delta X_2 = |5.0 - 5.14| = 0.14 \text{ cm}$$
$$\Delta X_3 = |5.2 - 5.14| = 0.06 \text{ cm}$$
$$\Delta X_4 = |5.1 - 5.14| = 0.04 \text{ cm}$$
$$\Delta X_5 = |5.3 - 5.14| = 0.16 \text{ cm}$$

Mean absolute error:

$$\Delta X_{\text{mean}} = \frac{0.04 + 0.14 + 0.06 + 0.04 + 0.16}{5} = \frac{0.44}{5} = 0.088 \text{ cm}$$

Exam Tip

Final Measurement Representation: The final measured value is reported as:

$$X = X_{\text{mean}} \pm \Delta X_{\text{mean}}$$

For our example:

$$X = (5.14 \pm 0.088) \text{ cm}$$

This means the true value is expected to lie within the range 5.052 cm to 5.228 cm.

2.7.4 Minimizing Errors

To reduce measurement errors, the following precautions should be taken:

Theory

- Calibrate instruments properly before taking measurements.
- Use high-precision instruments to reduce random errors.
- Take multiple readings and compute the average for greater accuracy.
- Avoid parallax errors by taking readings from the correct angle.
- Maintain stable environmental conditions to prevent external influences.

Exam Tip

Conclusion: Errors in measurement are unavoidable but can be minimized through careful experimental design and improved techniques. Understanding different types of errors and their quantification helps improve accuracy and reliability in scientific measurements.

Algebra of Errors

Errors in measurements follow certain mathematical rules when combining different quantities. Understanding these rules helps in estimating the uncertainties in derived results.

1. Absolute Error

Key Formula

The absolute error in a measurement is given by:

$$\Delta X = |X_{\text{measured}} - X_{\text{true}}|$$

For multiple measurements, the mean absolute error is:

$$\Delta \bar{X} = \frac{\sum |\Delta X_i|}{n}$$

where ΔX_i is the absolute error in each measurement, and n is the total number of measurements.

2. Error in Sum and Difference

Key Formula

If two measured quantities A and B have absolute errors ΔA and ΔB, then:

$$\Delta(A \pm B) = \Delta A + \Delta B$$

Example

Example: If $A = 5.0 \pm 0.2$ and $B = 3.0 \pm 0.1$, then:

$$(A + B) = (5.0 + 3.0) \pm (0.2 + 0.1) = 8.0 \pm 0.3$$

3. Error in Product and Quotient

Key Formula

For two measured quantities A and B, the relative error in multiplication and division is given by:

$$\frac{\Delta(A \cdot B)}{A \cdot B} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

$$\frac{\Delta(A/B)}{A/B} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

Example

Example: If $A = 4.0 \pm 0.2$ and $B = 2.0 \pm 0.1$:

$$\frac{\Delta(A \cdot B)}{A \cdot B} = \frac{0.2}{4.0} + \frac{0.1}{2.0} = 0.05 + 0.05 = 0.10$$

Thus, $A \cdot B = 8.0 \pm 0.8$.

4. Error in Power Functions

Key Formula

If a quantity Q depends on a measured quantity A as:

$$Q = A^n$$

then the relative error in Q is:

$$\frac{\Delta Q}{Q} = |n| \frac{\Delta A}{A}$$

Example

Example: If $A = 3.0 \pm 0.1$ and $Q = A^2$, then:

$$\frac{\Delta Q}{Q} = 2 \times \frac{0.1}{3.0} = 0.067$$

Thus, $Q = 9.0 \pm 0.6$.

5. Error in Logarithmic and Exponential Functions

Key Formula

For a function $Q = \ln A$:

$$\Delta Q = \frac{\Delta A}{A}$$

For an exponential function $Q = e^A$:

$$\frac{\Delta Q}{Q} = \Delta A$$

Importance of Algebra of Errors

Theory

- Helps in estimating uncertainties in experimental results.
- Useful in designing experiments with minimal errors.
- Essential in physics, engineering, and data analysis.

2.8 Significant Figures

2.8.1 Definition

Significant figures in a measurement include all the digits known with certainty plus one last digit that is estimated. They indicate the precision of a measurement.

Theory

For example, in the measurement 4.326 m:

- $4, 3$, and 2 are known with certainty.
- 6 is an estimated digit.
- The number has 4 significant figures.

2.8.2 Rules for Identifying Significant Figures

Exam Tip

1. **Nonzero digits** are always significant. Example: 23.4 (3 significant figures).
2. **Zeros between nonzero digits** are significant. Example: 2007 (4 significant figures).
3. **Leading zeros** (zeros before the first nonzero digit) are not significant. Example: 0.0045 (2 significant figures).
4. **Trailing zeros** (zeros at the end) are significant if there is a decimal point. Example: 45.00 (4 significant figures), but 4500 (2 significant figures unless written as 4.500×10^3).
5. **Exact numbers** (like defined constants, counts of objects) have infinite significant figures. Example: 1 dozen = 12 (infinite significant figures).

2.8.3 Rounding Off Significant Figures

Exam Tip

When rounding a number, follow these rules:

1. If the first digit to be dropped is less than 5, leave the preceding digit unchanged. Example: 3.642 rounded to 3 significant figures is 3.64.
2. If the first digit to be dropped is 5 or greater, increase the preceding digit by 1. Example: 3.648 rounded to 3 significant figures is 3.65.

2.8.4 Rounding Off Significant Figures

Key Formula

When rounding a number, follow these rules:

1. If the first digit to be dropped is less than 5, leave the preceding digit unchanged.

   Exam Tip

   Example: 3.642 rounded to 3 significant figures is 3.64.

2. If the first digit to be dropped is 5 or greater, increase the preceding digit by 1.

   Exam Tip

   Example: 3.648 rounded to 3 significant figures is 3.65.

2.8.5 Operations with Significant Figures

Addition and Subtraction

Key Formula

The result should have the same number of decimal places as the least precise measurement.

Example

Example:

$$12.34 + 0.6 = 12.94 \quad \text{(rounded to 12.9)}$$

Multiplication and Division

Key Formula

The result should have the same number of significant figures as the measurement with the fewest significant figures.

Example

Example:

$$(2.34) \times (1.2) = 2.808 \quad \text{(rounded to 2.8 since 1.2 has 2 significant figures)}$$

2.8.6 Scientific Notation and Significant Figures

Theory

Scientific notation helps express significant figures clearly.

Example

Example:

$$0.000345 = 3.45 \times 10^{-4} \quad \text{(3 significant figures)}$$

2.8.7 Practice Problems

1. Identify the number of significant figures in:
    - 0.00560
    - 6.02×10^{23}
    - 4500
2. Perform calculations with proper significant figures:
    - $3.678 + 2.1$
    - $(5.67) \times (0.034)$

2.8.8 Answers to Practice Problems on Significant Figures

Identifying Significant Figures

- 0.00560 has **3 significant figures**
    - Leading zeros are not significant.
    - The digits **5, 6**, and the trailing **0** (after the decimal) are significant.
- 6.02×10^{23} has **3 significant figures**
    - Only the digits in the coefficient (6.02) matter in scientific notation.
- 4500
    - If written as 4500, it has **2 significant figures**.
    - If written as 4500.0, it has **4 significant figures**.
    - If written in scientific notation as 4.50×10^3, it has **3 significant figures**.

Performing Calculations with Significant Figures

Addition

$$3.678 + 2.1 = 5.778$$

- The least precise number (2.1) has **1 decimal place**.
- The final answer should be rounded to **1 decimal place**:

$$5.8$$

Multiplication

$$(5.67) \times (0.034) = 0.19278$$

- 5.67 has **3 significant figures**, and 0.034 has **2 significant figures**.
- The final answer should have **2 significant figures**:

$$0.19$$

Note: Subtraction is similar to Addition and Division is similar to Multiplication for rules regarding calculations with significant figures.

2.9 Illustrative Problems

1. **Convert 5 meters into centimeters.**

   Solution: Since 1 meter = 100 cm, we have:

   $$5 \text{ m} = 5 \times 100 = 500 \text{ cm}$$

   Final Answer: 500 cm

2. **Convert 2 hours into seconds.**

   Solution: Since 1 hour = 3600 seconds:

   $$2 \text{ hours} = 2 \times 3600 = 7200 \text{ s}$$

   Final Answer: 7200 s

3. **Check the dimensional consistency of the equation:** $v^2 = u^2 + 2as$**.**

   Solution: Dimensions of velocity $[v] = [LT^{-1}]$

   $$[v^2] = [L^2T^{-2}], \quad [u^2] = [L^2T^{-2}]$$
   $$[a \cdot s] = [LT^{-2} \cdot L] = [L^2T^{-2}]$$

   Since both sides have the same dimensions, the equation is dimensionally correct.

4. **Derive the dimensional formula of Power.**

   Solution: Power $P = \frac{\text{Work}}{\text{Time}}$, Since Work $W = \text{Force} \times \text{Displacement}$,

   $$W = M^1L^1T^{-2} \times L^1 = M^1L^2T^{-2}$$

   Dividing by time:

   $$P = M^1L^2T^{-3}$$

   Final Answer: $\mathbf{M^1L^2T^{-3}}$

5. **Find the mean value of measurements: 10.2 cm, 10.5 cm, and 10.3 cm.**

   Solution: The mean value is given by:

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3} = \frac{10.2 + 10.5 + 10.3}{3} = 10.33 \text{ cm}$$

   Final Answer: 10.33 cm

6. **Find the percentage error in area if the errors in length and breadth are** ± 0.2 **cm and** ± 0.1 **cm, respectively.**

   Solution:

$$\frac{\Delta A}{A} = \frac{\Delta L}{L} + \frac{\Delta B}{B}$$

   Assuming $L = 10$ cm and $B = 5$ cm:

$$\frac{\Delta A}{A} = \frac{0.2}{10} + \frac{0.1}{5} = 0.02 + 0.02 = 0.04$$

   Percentage error:

$$0.04 \times 100 = 4\%$$

   Final Answer: 4%

7. **Find the absolute and relative error for a true value of 50 and an observed value of 48.**

   Solution: Absolute error:

$$\Delta x = |\text{true value} - \text{measured value}| = |50 - 48| = 2$$

   Relative error:

$$\frac{\Delta x}{\text{true value}} = \frac{2}{50} = 0.04$$

   Final Answer: Absolute error = **2**, Relative error = **0.04**

8. **Express 0.004567 in 3 significant figures.**

   Solution: The first three significant digits are 4, 5, and 7. Rounding off:

$$0.00457$$

   Final Answer: 0.00457

9. **Perform the following operation with correct significant figures:** $(4.56 \times 10^2) + (3.4 \times 10^3)$

   Solution: Converting both values to the same power of ten:

$$4.56 \times 10^2 = 0.456 \times 10^3$$

   Adding:

$$(0.456 + 3.4) \times 10^3 = 3.856 \times 10^3$$

   Since 3.4 has 2 significant figures, the answer should be rounded to 2 significant figures:

$$3.9 \times 10^3$$

   Final Answer: $\mathbf{3.9 \times 10^3}$

10. **Find the number of significant figures in 0.0002050.**

    Solution: The significant figures are 2, 0, 5, and the final 0 (since it is after a decimal). Total significant figures = 4. **Final Answer: 4**

11. **Round off 3.4789 to 3 significant figures.**

    Solution: The first three significant figures are 3, 4, and 7. Rounding off:

    $$3.48$$

    Final Answer: 3.48

12. **Multiply using correct significant figures:** $(3.42) \times (2.1)$

    Solution:

    $$3.42 \times 2.1 = 7.182$$

    Since 2.1 has the least significant figures (2), the answer is rounded to 2 significant figures:

    $$7.2$$

    Final Answer: 7.2

Chapter 3

Mathematical Methods in Physics

3.1 Introduction

Mathematics is the language of physics, providing the tools to decode nature's laws with precision and clarity. From describing motion to unraveling the fabric of space-time, mathematical methods are essential for analyzing and predicting physical phenomena.

For students preparing for entrance exams, mastering fundamental mathematical techniques is crucial. However, for those eager to explore physics beyond exams—delving into modern theories and research—advanced mathematical concepts unlock deeper insights into the universe.

Exam Tip

If your primary goal is to excel in entrance exams, you can safely focus on the material up to Section 4. Everything beyond that is intended for those interested in the more theoretical aspects of physics.

3.2 Algebra, Geometry, and Trigonometry

Mathematics provides essential tools for physics, helping describe motion, forces, waves, and other physical phenomena. This section covers the fundamental algebraic, geometric, and trigonometric techniques needed in physics.

3.2.1 Algebra

Algebra provides essential tools for manipulating equations and expressions, which form the foundation of physics. Many physical laws, such as Newton's equations of motion and energy conservation, rely on algebraic manipulation to derive meaningful results.

Equations and Inequalities

Equations describe fundamental laws of physics, while inequalities define constraints, such as physical limits in thermodynamics and mechanics.

- **Linear equations:** $ax + b = 0$, used in force balance and kinematics.
- **Quadratic equations:** $ax^2 + bx + c = 0$, frequently encountered in projectile motion and energy calculations.
- **Simultaneous equations:** Used to solve multiple interacting variables, e.g., Kirchhoff's laws in electrical circuits.
- **Inequalities:** Appear in concepts like the work-energy theorem, where kinetic energy is always non-negative.

Example

Example: A ball is thrown vertically with an initial height of $h = 5$ m and an initial velocity of 20 m/s. When will it hit the ground? The height as a function of time is given by:

$$h = 5 + 20t - 5t^2$$

Solution: To find when the ball reaches the ground, set $h = 0$:

$$0 = 5 + 20t - 5t^2$$

Rearrange the equation:

$$5t^2 - 20t - 5 = 0$$

Dividing by 5:

$$t^2 - 4t - 1 = 0$$

Using the quadratic formula:

$$t = \frac{4 \pm \sqrt{16 + 4}}{2} = 2 \pm \sqrt{5}$$

Since time cannot be negative, we take the positive root:

$$t \approx 4.24 \text{ s}$$

Thus, the ball reaches the ground after approximately 4.24 seconds.

Polynomials and Factorization

Polynomials appear frequently in physics, particularly in wave equations, oscillatory motion, and approximations in mechanics.

- **Polynomial expressions:** Terms of different degrees, such as kinetic energy $KE = \frac{1}{2}mv^2$.
- **Factorization techniques:** Useful for simplifying equations, e.g., solving motion equations with conserved energy.
- **Binomial theorem:** Expansions such as $(1+x)^n$ are used for small-angle approximations in mechanics and relativity.

Example

Example: Approximate $(1+x)^3$ using the binomial theorem up to the second-order term.

$$(1+x)^3 \approx 1 + 3x + 3x^2$$

This is particularly useful in physics when x is very small.

3.2.2 Geometry

Coordinate Geometry

Coordinate geometry provides a structured way to describe motion, trajectories, and interactions in physics.

- **Cartesian coordinates** (x, y, z) represent positions in space.
- **Distance formula:**

  Key Formula

  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Equation of a straight line:**

  Key Formula

  $$y = mx + c$$

Example

Example: A projectile is launched from $(1, 2)$ and follows a straight line to $(3, 4)$. Find the midpoint of its trajectory.
Solution: Using the midpoint formula:

$$M = \left(\frac{1+3}{2}, \frac{2+4}{2}\right) = (2, 3)$$

This gives the average position of the projectile in flight.

Vectors in Geometry

Vectors describe forces, velocity, acceleration, and electric/magnetic fields.

- **Vector representation:** Expressed as $\mathbf{A} = (A_x, A_y, A_z)$.
- **Magnitude of a vector:**

Key Formula

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Example

Example: A force vector $\mathbf{F} = (3, 4, 0)$ N acts on an object. Find its magnitude.
Solution:

$$|\mathbf{F}| = \sqrt{3^2 + 4^2 + 0^2} = 5 \text{ N}$$

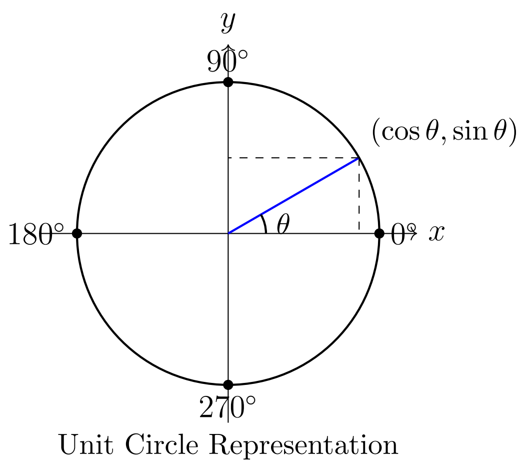
Thus, the total force acting on the object is 5 N.

3.2.3 Trigonometry

Trigonometric functions play a crucial role in physics, especially in wave mechanics, oscillatory motion, and rotational dynamics. This section provides a fundamental understanding of trigonometry using the unit circle, function graphs, and essential formulas.

3.2.4 Unit Circle Representation

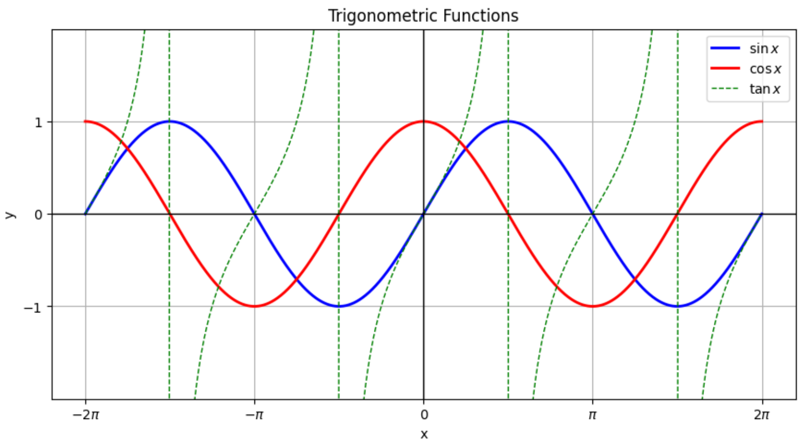
The unit circle is a fundamental tool for understanding trigonometric functions. It provides a geometric representation of sine and cosine values for different angles.



Unit Circle Representation

3.2.5 Trigonometric Function Graphs

The periodic nature of sine, cosine, and tangent functions plays a key role in physics, especially in wave mechanics and oscillatory motion. Below are their graphical representations:



3.2.6 Trigonometric Formulas and Values

Fundamental Identities

Key Formula

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1\\
1+\tan^2\theta &= \sec^2\theta\\
1+\cot^2\theta &= \csc^2\theta
\end{aligned}
$$

Angle Sum and Difference Formulas

Key Formula

$$
\begin{aligned}
\sin(A\pm B) &= \sin A\cos B \pm \cos A\sin B\\
\cos(A\pm B) &= \cos A\cos B \mp \sin A\sin B\\
\tan(A\pm B) &= \frac{\tan A \pm \tan B}{1\mp \tan A\tan B}
\end{aligned}
$$

Double Angle Formulas

Key Formula

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Small Angle Approximations (for $\theta \ll 1$ in radians)

Key Formula

$$\sin\theta \approx \theta$$
$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$
$$\tan\theta \approx \theta$$

Standard Trigonometric Values Table

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Trigonometric formulas and values are essential in physics for solving problems related to waves, oscillations, forces, and rotational motion. A strong grasp of these concepts helps in analyzing simple harmonic motion, AC circuits, and wave propagation.

3.3 Vectors and Their Applications

3.3.1 Introduction to Vectors

In physics, some quantities require both **magnitude** and **direction** for complete representation. These quantities are called **vectors**, whereas those that have only magnitude are called **scalars**.

Common vector quantities:

- **Displacement** – Directed distance between two points.

- **Velocity** – Speed with a specific direction.
- **Acceleration** – Rate of velocity change in a given direction.
- **Force** – A push or pull in a particular direction.
- **Momentum** – Product of mass and velocity.

3.3.2 Representation of Vectors

Vectors can be represented in multiple ways depending on the context and visualization needs.

3.3.3 Graphical Representation

A vector is visually represented as an arrow:

- The **length** of the arrow corresponds to the **magnitude** of the vector.
- The **direction** of the arrow represents the **vector's direction**.

For example, a displacement vector of 5 meters eastward can be drawn as an arrow of proportional length pointing to the right.

3.3.4 Mathematical Representation

Vectors are denoted using:

- **Boldface notation**: $\vec{A}$ or $\vec{v}$ (commonly used in textbooks).
- **Arrow notation**: $\vec{A}$, $\vec{v}$ (used in handwritten work and physics contexts).
- **Component form**:
$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
(in Cartesian coordinates).

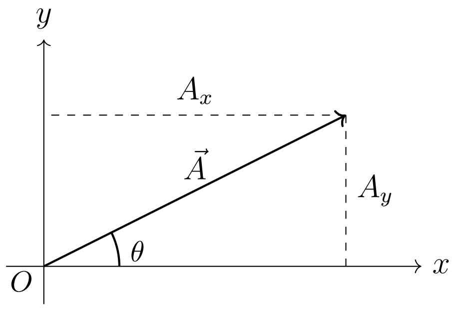
Theory

Position Vector: The position vector represents the location of a point relative to the origin.
For a point $P(x, y, z)$, the position vector is:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

3.3.5 Graphical Representation of a Vector

Below is a graphical depiction of a vector $\vec{A}$ in a two-dimensional coordinate system:



Key Formula

Magnitude of a vector:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

The direction of the vector is given by:

$$\tan\theta = \frac{A_y}{A_x}$$

3.3.6 Resolution of Vectors

Breaking a vector into its components along predefined axes is called **resolution of vectors**. This helps in simplifying vector operations, especially in mechanics.

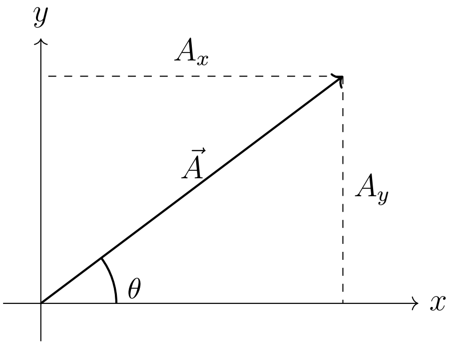
3.3.7 Resolution in Two Dimensions

A vector $\vec{A}$ in a two-dimensional plane can be resolved into two perpendicular components:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

where:

- $A_x = A\cos\theta$ is the component along the x-axis.
- $A_y = A\sin\theta$ is the component along the y-axis.
- θ is the angle the vector makes with the positive x-axis.



3.3.8 Resolution in Three Dimensions

In three-dimensional space, a vector $\vec{A}$ can be resolved into three mutually perpendicular components:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

where:

- $A_x = A\cos\alpha$, $A_y = A\cos\beta$, and $A_z = A\cos\gamma$
- α, β, γ are the angles made by the vector with the x-, y-, and z-axes, respectively.

Key Formula

Magnitude of a vector in three dimensions:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Exam Tip

The direction cosines satisfy the relation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

This is useful in problems involving angles between vectors and coordinate axes.

Example

Example: Find the magnitude of the force vector $\vec{F} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$.
Solution: Using the above formula:

$$|\vec{F}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

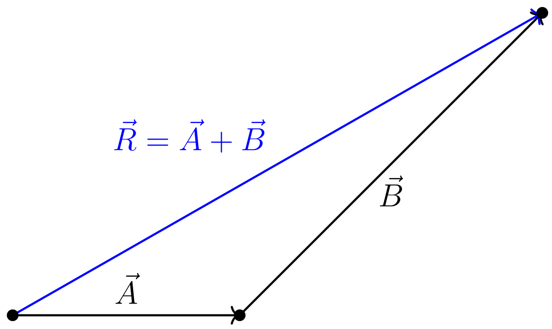
3.3.9 Vector Addition and Subtraction

Vectors can be combined using the following methods:

- **Triangle Law:** If two vectors are placed head-to-tail, their sum is the third side of the triangle.
- **Parallelogram Law:** If two vectors form adjacent sides of a parallelogram, their sum is represented by the diagonal.

Triangle Law of Vector Addition

The triangle law states that if two vectors $\vec{A}$ and $\vec{B}$ are arranged such that the tail of $\vec{B}$ is at the head of $\vec{A}$, then their resultant $\vec{R}$ is the vector drawn from the tail of $\vec{A}$ to the head of $\vec{B}$.



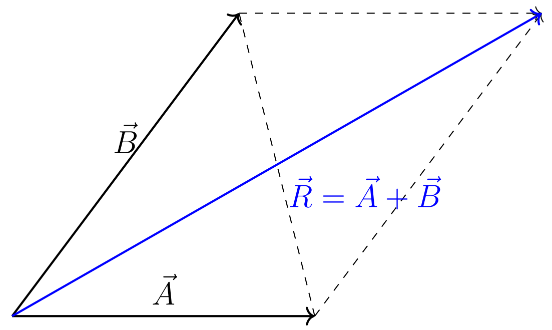
Theory

Mathematical Expression: If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$, then the resultant is:

$$\vec{R} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath}$$

Parallelogram Law of Vector Addition

The parallelogram law states that if two vectors originate from the same point and form adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram.



Key Formula

Magnitude of the Resultant Vector:

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \tag{3.1}$$

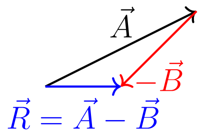
where θ is the angle between the two vectors.

Vector Subtraction

Vector subtraction $\vec{A} - \vec{B}$ is equivalent to adding $\vec{A}$ to the negative of $\vec{B}$:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Graphically, reversing the direction of $\vec{B}$ and applying vector addition gives the result.



Exam Tip

Tip: Vector subtraction follows the same rules as vector addition but with reversed direction. Always check whether you need to subtract components when working with vector equations.

3.3.10 Scalar and Vector Products

Vectors can be multiplied in different ways, each yielding different results. The types of vector multiplication include:

- Multiplication of a vector by a scalar (scaling)
- Dot product (scalar product)
- Cross product (vector product)
- Scalar triple product
- Vector triple product

Multiplication of a Vector by a Scalar

A vector can be multiplied by a scalar to scale its magnitude without changing its direction (unless multiplied by a negative scalar, which reverses direction).

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

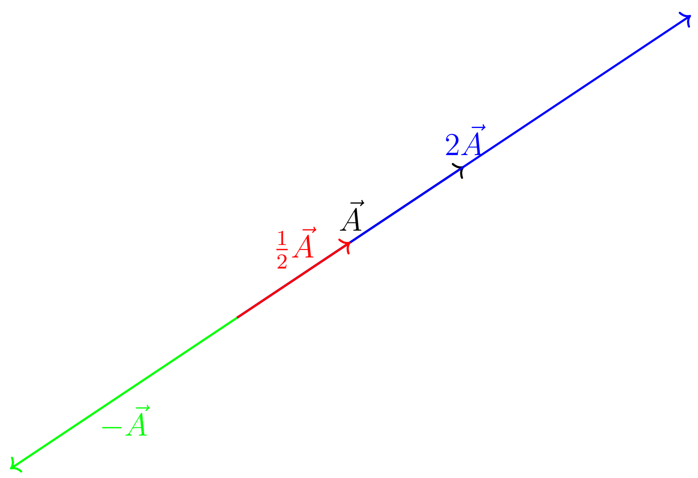
Multiplying by a scalar λ:

$$\lambda\vec{A} = \lambda A_x\hat{i} + \lambda A_y\hat{j} + \lambda A_z\hat{k}$$

Theory

Geometric Interpretation:

- If $|\lambda| > 1$, the vector is stretched.
- If $0 < |\lambda| < 1$, the vector is compressed.
- If $\lambda < 0$, the direction is reversed.



—

Dot Product (Scalar Product)

Key Formula

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. The dot product gives a scalar quantity.

Theory

Component Form: If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, then

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Theory

Properties:

- Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- If $\vec{A} \perp \vec{B}$, then $\vec{A} \cdot \vec{B} = 0$

—

Cross Product (Vector Product)

Key Formula

The **cross product** of two vectors $\vec{A}$ and $\vec{B}$ is:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \hat{n}$$

where:

- $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of the vectors,
- θ is the angle between the two vectors,
- $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Exam Tip

Right-Hand Thumb Rule:

- Point your right-hand fingers in the direction of $\vec{A}$.
- Curl your fingers towards $\vec{B}$ through the smaller angle.
- Your thumb points in the direction of $\vec{A} \times \vec{B}$.

Key Formula

Component Form:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example

Special Cases of Cross Product:

- **Perpendicular Vectors:** If $\theta = 90°$, then $\sin 90° = 1$, so $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\hat{n}$.
- **Parallel Vectors:** If $\theta = 0°$ or $180°$, then $\sin\theta = 0$, so $\vec{A} \times \vec{B} = 0$.

—

Scalar Triple Product

Key Formula

The scalar triple product of three vectors $\vec{A}, \vec{B}, \vec{C}$ is:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

—

Vector Triple Product

Key Formula

The vector triple product is given by:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

—

3.3.11 Applications of Vectors

Theory

Vectors play a crucial role in various fields:

- **Mechanics:** Motion, force, and momentum.
- **Electromagnetism:** Electric and magnetic fields.
- **Fluid Dynamics:** Velocity fields in fluids.
- **Computer Graphics:** 3D transformations.

3.3.12 Illustrative Problems

Problem 1: Magnitude of a Vector

Example

Problem: Find the magnitude of the vector $\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$.
Solution: The magnitude of a vector $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ is given by:

Key Formula

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Substituting the given values:

$$|\vec{A}| = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}.$$

—

Problem 2: Addition of Two Vectors

Example

Problem: Given two vectors $\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{B} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$, find their sum $\vec{A} + \vec{B}$.
Solution: The sum of two vectors is obtained by adding corresponding components:

$$\vec{A} + \vec{B} = (2+1)\hat{\imath} + (3+(-2))\hat{\jmath} + (1+4)\hat{k}$$

$$= 3\hat{\imath} + 1\hat{\jmath} + 5\hat{k}.$$

—

Problem 3: Dot Product of Two Vectors

Example

Problem: Compute the dot product of $\vec{A} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$ and $\vec{B} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$.
Solution: The dot product formula is:

Key Formula

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Substituting values:

$$\vec{A} \cdot \vec{B} = (3 \times 1) + (4 \times (-2)) + (-1 \times 1)$$

$$= 3 - 8 - 1 = -6.$$

Exam Tip

Quick Trick:

- If $\vec{A} \perp \vec{B}$, then $\vec{A} \cdot \vec{B} = 0$.
- If $\vec{A} \parallel \vec{B}$, then $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$.

—

Problem 4: Cross Product of Two Vectors

Example

Problem: Given $\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $\vec{B} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, find $\vec{A} \times \vec{B}$.
Solution: Using determinant expansion:

Key Formula

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}$$

Expanding:

$$\vec{A} \times \vec{B} = \hat{\imath}(3 \times 1 - 4 \times (-2)) - \hat{\jmath}(2 \times 1 - 4 \times 1) + \hat{k}(2 \times (-2) - 3 \times 1)$$

$$= \hat{\imath}(3 + 8) - \hat{\jmath}(2 - 4) + \hat{k}(-4 - 3)$$

$$= 11\hat{\imath} + 2\hat{\jmath} - 7\hat{k}.$$

—

Problem 5: Scalar Triple Product

Example

Problem: Compute the scalar triple product of $\vec{A} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$, $\vec{B} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, and $\vec{C} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$.
Solution: Compute $\vec{B} \times \vec{C}$ first:

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

Expanding:

$$\vec{B} \times \vec{C} = \hat{\imath}((-2)(-2) - (1)(1)) - \hat{\jmath}((1)(-2) - (1)(3)) + \hat{k}((1)(1) - (-2)(3))$$

$$= \hat{\imath}(4 - 1) - \hat{\jmath}(-2 - 3) + \hat{k}(1 + 6)$$

$$= 3\hat{\imath} + 5\hat{\jmath} + 7\hat{k}.$$

Now, compute $\vec{A} \cdot (\vec{B} \times \vec{C})$:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2 \times 3) + (1 \times 5) + (3 \times 7)$$

$$= 6 + 5 + 21 = 32.$$

Thus, the volume of the parallelepiped is $|32| = 32$ cubic units.

3.4 Differentiation and Integration in Physics

Theory

Calculus plays a crucial role in physics, helping us analyze motion, forces, and energy. The key applications include:

- **Differentiation:** Used to find velocity, acceleration, and rate of change in physical systems.
- **Integration:** Used to determine displacement, work, and accumulated quantities.
- **Solving Equations of Motion:** Helps derive kinematic equations and laws in mechanics.

3.4.1 Differentiation in Physics

Differentiation is the mathematical process of finding the rate at which a quantity changes. In physics, it is widely used in kinematics, dynamics, and electromagnetism.

Theory

The derivative of a function $f(x)$ with respect to x is given by:

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{3.2}$$

Velocity and Acceleration

In kinematics, differentiation helps us find velocity and acceleration:

- **Velocity:** The velocity $v(t)$ is the derivative of displacement $s(t)$:

$$v(t) = \frac{ds}{dt}. \tag{3.3}$$

- **Acceleration:** The acceleration $a(t)$ is the derivative of velocity:

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \tag{3.4}$$

Example: Motion Under Gravity

Problem: A particle moves along a straight line with displacement given by $s(t) = 5t^2 + 3t + 2$. Find its velocity and acceleration at any time t.

Solution: Differentiating $s(t)$ to get velocity:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(5t^2 + 3t + 2) = 10t + 3.$$

Differentiating again to get acceleration:

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(10t + 3) = 10.$$

Thus, the acceleration is constant at 10 m/s^2.

Other Applications of Differentiation in Physics

- **Newton's Second Law:** Force is the rate of change of momentum:

$$F = \frac{dp}{dt}. \tag{3.5}$$

- **Optics:** The rate of change of lens power with respect to curvature.

- **Electric Circuits:** The rate of change of charge gives current:

$$I = \frac{dQ}{dt}. \tag{3.6}$$

- **Radioactive Decay:** The decay rate of a substance follows:

$$\frac{dN}{dt} = -\lambda N. \tag{3.7}$$

Understanding Differentiation

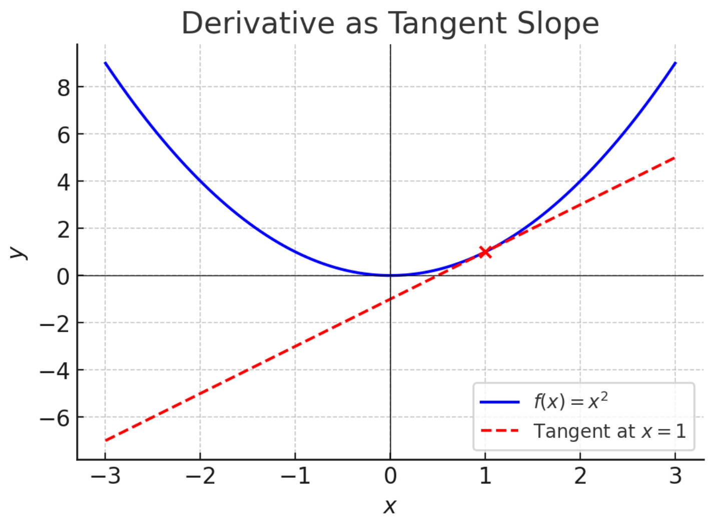
Differentiation measures how a function changes with respect to a variable. It helps determine rates of change, slopes of curves, and instantaneous values.

Theory

For a function $y = f(x)$, the derivative $\frac{dy}{dx}$ represents the rate of change of y with respect to x. It is defined as:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{3.8}$$

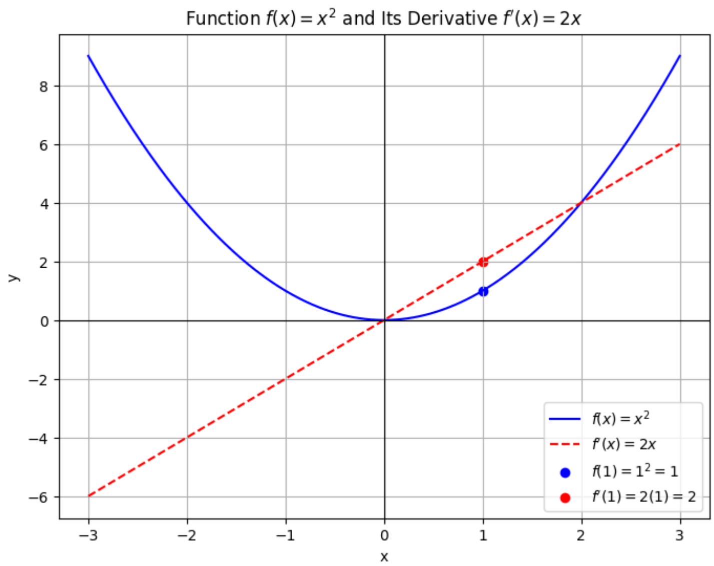
Geometric Interpretation: The derivative at a point is the slope of the tangent line to the function at that point.



In the graph above, the function $y = f(x)$ has a tangent at $x = a$, and its slope is given by $\frac{dy}{dx}$ at that point.

Graphical Understanding: Let's consider a function $f(x) = x^2$:

- The function is increasing for $x > 0$, meaning $\frac{dy}{dx} > 0$.
- The function is decreasing for $x < 0$, meaning $\frac{dy}{dx} < 0$.
- At $x = 0$, $\frac{dy}{dx} = 0$, indicating a minimum point.



Basic Differentiation Rules: Here are some fundamental differentiation rules used in physics:

- **Constant Rule:** $\frac{d}{dx}(C) = 0$, where C is a constant.

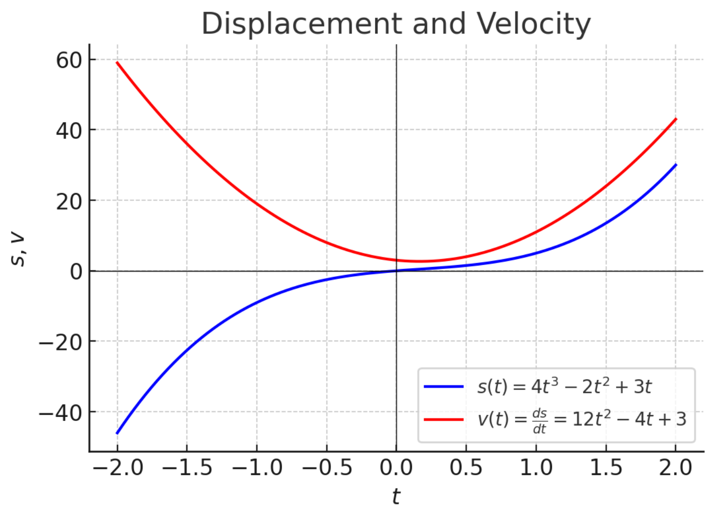
- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$.
- **Sum Rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.
- **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$.
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$.
- **Chain Rule:** $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$.

Example: Velocity from Displacement Given $s(t) = 4t^3 - 2t^2 + 3t$, find the velocity function.

Solution: Differentiate with respect to t:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(4t^3 - 2t^2 + 3t) = 12t^2 - 4t + 3.$$

Graph: The displacement and velocity functions are plotted below:



This shows how velocity changes over time.

3.4.2 Integration in Physics

Integration is the reverse process of differentiation. It is widely used in physics to determine quantities such as displacement from velocity, work done from force, and electric potential from an electric field.

3.4.3 Definition of Integration

Theory

Integration is the process of summing infinitely small quantities to find a total effect. If $F(x)$ is the integral of $f(x)$, then:

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of integration.

3.4.4 Applications of Integration in Physics

Integration is used in physics to solve various problems, such as:

- **Displacement from velocity:** $s = \int v\,dt$
- **Work done by a force:** $W = \int F\,dx$
- **Electric potential from an electric field:** $V = -\int E\,dx$
- **Finding center of mass and moment of inertia**

3.4.5 Illustrative Problems

Problem 1: Displacement from Velocity

Example

Problem: A particle moves with a velocity given by $v(t) = 3t^2 + 2t$. Find its displacement from $t = 0$ to $t = 4$.
Solution: Displacement is given by:

$$s = \int v(t)\,dt = \int (3t^2 + 2t)\,dt$$

Evaluating the integral:

$$s = \left(\frac{3t^3}{3} + \frac{2t^2}{2}\right)\Big|_0^4$$

$$s = (t^3 + t^2)\Big|_0^4 = (4^3 + 4^2) - (0^3 + 0^2) = (64 + 16) - 0 = 80$$

Thus, the displacement is **80** units.

Problem 2: Work Done by a Variable Force

Example

Problem: A force $F(x) = 5x^2$ acts on an object. Find the work done in moving it from $x = 1$ to $x = 3$.
Solution: Work done is given by:

$$W = \int_{x_1}^{x_2} F(x)\,dx$$

$$W = \int_{1}^{3} 5x^2\,dx$$

Evaluating the integral:

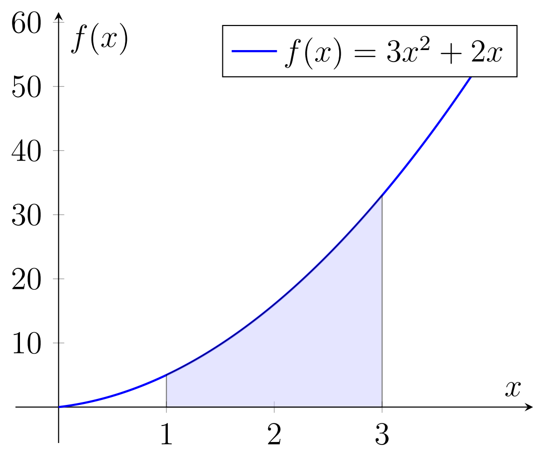
$$W = \left(\frac{5x^3}{3}\right)\Big|_1^3$$

$$W = \frac{5(3^3)}{3} - \frac{5(1^3)}{3} = \frac{5(27)}{3} - \frac{5(1)}{3}$$

$$W = \frac{135}{3} - \frac{5}{3} = \frac{130}{3} \approx 43.33$$

Thus, the work done is **43.33** units.

Graphical Representation of Integration



Exam Tip

The shaded area under the curve represents the integral, which gives the total displacement in this case.

3.4.6 Integration Laws and Formulas

Integration follows certain fundamental rules that help in evaluating integrals efficiently.

Key Formula

Basic Integration Formulas

$$\int a\,dx = ax + C, \qquad \text{(Constant Rule)}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad (n \neq -1)$$

$$\int e^x\,dx = e^x + C,$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \qquad (a > 0, a \neq 1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C, \qquad \text{(Log Rule)}$$

Key Formula

Trigonometric Integrals

$$\int \sin x\,dx = -\cos x + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int \sec^2 x\,dx = \tan x + C,$$

$$\int \csc^2 x\,dx = -\cot x + C,$$

$$\int \sec x \tan x\,dx = \sec x + C,$$

$$\int \csc x \cot x\,dx = -\csc x + C.$$

Key Formula

Exponential and Logarithmic Integrals

$$\int e^x\,dx = e^x + C,$$

$$\int \ln x\,dx = x\ln x - x + C.$$

Integration by Parts

Theory

Integration by Parts Formula: If $u(x)$ and $v(x)$ are differentiable functions, then:

$$\int u\,dv = uv - \int v\,du.$$

The function u is chosen using the **ILATE** rule:

- **I**: Inverse Trigonometric (e.g., $\tan^{-1} x$)
- **L**: Logarithmic (e.g., $\ln x$)
- **A**: Algebraic (e.g., x, x^2)
- **T**: Trigonometric (e.g., $\sin x, \cos x$)
- **E**: Exponential (e.g., $e^x, 2^x$)

The function highest in this order is chosen as u.

Example

Example: Evaluate $\int xe^x\,dx$.
Solution: Using the ILATE rule, let:

$$u = x, \quad dv = e^x dx.$$

Then, differentiating u and integrating dv:

$$du = dx, \quad v = e^x.$$

Using the formula:

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

Since $\int e^x dx = e^x$, we get:

$$\int xe^x\,dx = xe^x - e^x + C.$$

3.5 Exercises

3.5.1 Basic Level

1. Solve for x:
$$2x^2 - 5x + 3 = 0.$$
2. Solve for x:
$$3x^2 + 7x - 10 = 0.$$

3. Find the equation of a line passing through $(2,3)$ and perpendicular to $y = \frac{1}{2}x + 1$.

4. Find the equation of a line that passes through the points $(-3,5)$ and $(2,-4)$.

5. Convert 60° into radians.

6. Convert $\frac{\pi}{3}$ radians into degrees.

7. If θ is small, show that $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$.

8. Find the unit vector in the direction of $\mathbf{A} = 3\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$.

9. Compute the dot product of $\mathbf{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\mathbf{B} = -\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$.

10. Find the magnitude of the vector $\mathbf{A} = 4\hat{\imath} - 3\hat{\jmath} + 12\hat{k}$.

11. Differentiate $f(x) = 3x^2 + 5x - 7$.

12. Find $\frac{d}{dx}(5x^3 - 4x^2 + x - 8)$.

13. Evaluate the integral:
$$\int_0^2 (3x+1)\,dx.$$

14. Compute:
$$\int (5x^2 + 3x - 4)dx.$$

15. Find the sine rule for a triangle.

16. Find the area of a right-angled triangle with base 5 cm and height 12 cm.

17. Compute the cross product of $\mathbf{A} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\mathbf{B} = -\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

18. Find the determinant of the matrix
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

19. Compute:
$$\int x^3 dx.$$

20. Compute:
$$\frac{d}{dx}(\sin x \cos x).$$

3.5.2 Intermediate Level

1. Solve for x:
$$x^4 - 5x^2 + 4 = 0.$$
2. Solve for x:
$$x^3 - 4x + 1 = 0.$$
3. Find the equation of a circle passing through $(1, 2)$ and having a center at $(3, 4)$.
4. Find the focus, directrix, and eccentricity of the parabola $y^2 = 8x$.
5. Find the value of $\sin 75^\circ$ using trigonometric identities.
6. Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
7. Compute the cross product of $\mathbf{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{B} = -\hat{i} + \hat{j} + 2\hat{k}$.
8. Compute:
$$\frac{d}{dx}(x^2 e^x).$$
9. Evaluate the integral:
$$\int \frac{x}{(x^2+1)}\, dx.$$
10. Find the angle between the vectors $\mathbf{A} = (1, 2, 3)$ and $\mathbf{B} = (4, 5, 6)$.
11. Compute:
$$\int_1^3 \frac{dx}{x^2+1}.$$
12. Solve the differential equation:
$$\frac{dy}{dx} = x^2 + y.$$
13. Find the Taylor series expansion for e^x up to x^3.
14. Prove that:
$$\cos^4\theta - \sin^4\theta = \cos 2\theta.$$
15. Compute the work done by a force $\mathbf{F} = (3x^2, -2y, 4z)$ when a particle moves from $(1, 0, 0)$ to $(2, 1, 1)$.
16. Find the volume of the parallelepiped formed by $\mathbf{A} = (1, 2, 3)$, $\mathbf{B} = (4, 5, 6)$, and $\mathbf{C} = (7, 8, 9)$.
17. Evaluate:
$$\int_0^\pi x \sin x\, dx.$$
18. Compute the divergence of $\mathbf{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.
19. Solve the equation:
$$e^x - 3x = 0.$$
20. If $f(x) = \ln(x+1)$, compute $f''(x)$.

3.5.3 Answers

1. $x = 1, x = \frac{3}{2}$.
2. $x = 1, x = -\frac{5}{3}$.
3. $y = -2x + 7$.
4. $y = -\frac{9}{5}x + \frac{2}{5}$.
5. $\frac{\pi}{3}$ radians.
6. 60°.
7. Small angle approximations: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ for $\theta \to 0$.
8. $\frac{3}{13}\hat{\imath} - \frac{4}{13}\hat{\jmath} + \frac{12}{13}\hat{k}$.
9. $-2 + 12 + 2 = 12$.
10. $\sqrt{4^2 + (-3)^2 + 12^2} = \sqrt{169} = 13$.
11. $6x + 5$.
12. $15x^2 - 8x + 1$.
13. $\left[\frac{3}{2}x^2 + x\right]_0^2 = 7$.
14. $\frac{5}{3}x^3 + \frac{3}{2}x^2 - 4x + C$.
15. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
16. $\frac{1}{2} \times 5 \times 12 = 30$ cm^2.
17. $(7\hat{\imath} - 5\hat{\jmath} + 3\hat{k})$.
18. $(1 \times 4 - 2 \times 3) = -2$.
19. $\frac{x^4}{4} + C$.
20. $\cos 2x$.

1. $x^2 = 1, x = \pm 1$.
2. Roots can be found using numerical approximation.
3. $(x-3)^2 + (y-4)^2 = r^2$, where $r = \sqrt{5}$.
4. Focus at $(2, 0)$, directrix: $x = -2$, eccentricity $= 1$.
5. $\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$.
6. Identity proof.
7. $(-7\hat{\imath} - 5\hat{\jmath} - \hat{k})$.

8. $2xe^x + x^2e^x$.
9. $\frac{1}{2}\ln|x^2+1|$.
10. 12.93°.
11. $\tan^{-1} 3 - \tan^{-1} 1$.
12. General solution $y = -x^3/3 + Ce^x$.
13. $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.
14. $2\cos^2\theta - 1$.
15. 3.
16. Volume $= 0$.
17. Integration gives π.
18. $\nabla \cdot \mathbf{F} = 2x + 2y + 2z$.
19. Approximate root using numerical methods.
20. $\frac{1}{(x+1)^2}$.

3.6 Differential Equations

Many physical systems are governed by differential equations. Understanding these is crucial for modeling:

- Simple harmonic motion
- Newton's laws and force equations
- Heat and wave equations

3.6.1 Types of Differential Equations

A differential equation is an equation involving derivatives of a function. Based on order and linearity, differential equations are classified as follows:

- **Ordinary Differential Equations (ODEs)**: Contain derivatives with respect to only one independent variable.
- **Partial Differential Equations (PDEs)**: Involve derivatives with respect to multiple independent variables.
- **Linear vs. Nonlinear Differential Equations**: A linear equation has terms involving the dependent variable and its derivatives only to the first power. Nonlinear equations have terms with higher powers or products of dependent variables and their derivatives.
- **Homogeneous vs. Non-homogeneous Equations**: A differential equation is homogeneous if all terms involve the dependent variable or its derivatives. If an additional function appears, it is non-homogeneous.

First-Order Differential Equations

A first-order differential equation has the form:

$$\frac{dy}{dx} = f(x, y).$$

Theory

Separable Differential Equations: A separable differential equation can be written as:

$$\frac{dy}{dx} = g(x)h(y).$$

It is solved by separating variables:

$$\frac{dy}{h(y)} = g(x)dx.$$

Theory

Linear First-Order Differential Equations: A linear equation is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The integrating factor is:

$$I(x) = e^{\int P(x)dx}.$$

Multiplying by $I(x)$, the solution follows as:

$$y(x) = \frac{1}{I(x)}\left(\int I(x)Q(x)dx + C\right).$$

3.6.2 Solving Differential Equations

Differential equations can be solved using various methods depending on their type and structure. Below are the major techniques:

Separation of Variables

A differential equation is separable if it can be written as:

$$\frac{dy}{dx} = g(x)h(y).$$

Rearranging:

$$\frac{dy}{h(y)} = g(x)dx.$$

Integrating both sides:

$$\int \frac{dy}{h(y)} = \int g(x)dx.$$

Example

Example: Solve $\frac{dy}{dx} = xy$.
Solution: Rearrange the equation:

$$\frac{dy}{y} = xdx.$$

Integrating both sides:

$$\ln|y| = \frac{x^2}{2} + C.$$

Exponentiating:

$$y = Ce^{x^2/2}.$$

Linear First-Order Equations

A first-order linear equation has the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The integrating factor is:

$$I(x) = e^{\int P(x)dx}.$$

Multiplying by $I(x)$, we get:

$$y(x) = \frac{1}{I(x)} \left(\int I(x)Q(x)dx + C \right).$$

Example

Example: Solve $\frac{dy}{dx} + 2y = e^{-x}$.
Solution: Here, $P(x) = 2$, so the integrating factor is:

$$I(x) = e^{\int 2dx} = e^{2x}.$$

Multiplying by e^{2x}:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{x}.$$

Integrating both sides:

$$ye^{2x} = \int e^{x}dx = e^{x} + C.$$

Solving for y:

$$y = e^{-2x}(e^{x} + C) = e^{-x} + Ce^{-2x}.$$

Homogeneous Differential Equations

A first-order equation of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

is called homogeneous. We use the substitution:

$$y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Example

Example: Solve $\frac{dy}{dx} = \frac{x+y}{x}$.
Solution: Substituting $y = vx$:

$$\frac{d(vx)}{dx} = \frac{x+vx}{x}.$$

Simplifying:

$$v + x\frac{dv}{dx} = 1 + v.$$

Canceling v:

$$x\frac{dv}{dx} = 1.$$

Integrating:

$$v = \ln|x| + C.$$

Substituting $v = y/x$:

$$y = x\ln|x| + Cx.$$

Exact Differential Equations

An equation of the form:

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The solution is given by finding a function $\Psi(x, y)$ such that:

$$\frac{\partial \Psi}{\partial x} = M, \quad \frac{\partial \Psi}{\partial y} = N.$$

Example

Example: Solve $(2x + y)dx + (x + 3y^2)dy = 0.$
Solution: Checking exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2x + y) = 1,$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x + 3y^2) = 1.$$

Since both are equal, the equation is exact. We solve:

$$\Psi_x = 2x + y \quad \Rightarrow \quad \Psi = x^2 + xy + f(y).$$

Differentiating with respect to y:

$$\Psi_y = x + f'(y) = x + 3y^2.$$

Solving for $f(y)$:

$$f'(y) = 3y^2 \quad \Rightarrow \quad f(y) = y^3.$$

Thus, the solution is:

$$\Psi(x, y) = x^2 + xy + y^3 = C.$$

3.7 Probability

Classical physics assumes that the behavior of physical systems can be precisely determined if the initial conditions are known. However, at microscopic scales, this deterministic view breaks down. In quantum mechanics, probability is not just a tool for dealing with uncertainty but an inherent feature of reality. Physical quantities such as the position and momentum of a particle cannot be known with absolute certainty at the same time, as described by Heisenberg's Uncertainty Principle. Instead, quantum systems are described by wave functions, whose squared magnitudes provide probability distributions of possible outcomes. This probabilistic nature extends beyond quantum mechanics into areas like statistical mechanics, thermodynamics, and even chaos theory, where complex systems exhibit unpredictable behavior despite deterministic laws.

Some basic concepts in probability includes:

- Basics of probability theory
- Probability in quantum mechanics
- Uncertainty principle and probabilistic wave functions

3.7.1 Basics of Probability Theory

Probability theory provides a mathematical framework to quantify uncertainty. In physics, it is used to describe random processes and statistical behavior.

Fundamental Concepts

- Probability Space: Defined by a sample space S, events $E \subseteq S$, and a probability function $P(E)$ satisfying $0 \leq P(E) \leq 1$ and $P(S) = 1$. - Conditional Probability: The probability of event A given B has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- Bayes' Theorem: Describes the probability of an event based on prior knowledge:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Applications in Physics

- Thermodynamics: Statistical mechanics relies on probability to describe particle distributions. - Radioactive Decay: The decay probability per unit time is constant for a given isotope. - Error Analysis: Measurement uncertainties are modeled using probability distributions.

3.7.2 Probability in Quantum Mechanics

Quantum mechanics departs from classical determinism by incorporating probability fundamentally.

Wave Function and Probability Amplitude

The state of a quantum system is represented by a wave function $\psi(x,t)$, whose squared modulus gives the probability density:

$$P(x) = |\psi(x,t)|^2.$$

Born's Rule

The probability of finding a particle at position x is:

$$P(x) = \int_a^b |\psi(x)|^2 dx.$$

Quantum Superposition and Measurement

- A quantum state can exist in a superposition of multiple states. - Measurement collapses the wave function into a definite state.

3.7.3 Uncertainty Principle and Probabilistic Wave Functions

Heisenberg's uncertainty principle states that certain pairs of physical properties cannot be simultaneously known with arbitrary precision.

Mathematical Formulation

The uncertainty in position Δx and momentum Δp satisfies:

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

Similarly, energy and time uncertainties obey:

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

Implications

- Fundamental Limits: Precision in measurement is inherently limited by quantum mechanics.
- Quantum Tunneling: A particle has a nonzero probability of passing through a potential barrier.
- Wave-Particle Duality: Probability governs both wave-like and particle-like behaviors.

These probabilistic principles are essential in understanding quantum behavior, atomic structure, and modern physics applications.

3.7.4 Brownian Motion and Random Walk

Introduction

Many physical phenomena exhibit random behavior, where a system evolves unpredictably over time. Two fundamental models that describe such behavior are **Brownian motion** and **random walks**. These concepts are crucial in statistical mechanics, diffusion processes, and financial mathematics.

Random Walk

A **random walk** is a mathematical model where a particle moves in discrete steps in a given space (1D, 2D, or 3D), with each step being independent of the previous ones.

1D Random Walk Model:

- Consider a particle starting at $x = 0$ on a number line.
- At each step, it moves **right** $(+1)$ or **left** (-1) with equal probability $P = 0.5$.
- After N steps, the displacement X_N is given by:

$$X_N = \sum_{i=1}^{N} s_i$$

  where s_i is either $+1$ or -1 randomly.

Key Results of Random Walk:

- The **mean displacement** after N steps:

$$\langle X_N \rangle = 0$$

  This means the expected position remains the starting point.

- The **mean squared displacement** (MSD):

$$\langle X_N^2 \rangle = N$$

  This implies that the root-mean-square displacement is proportional to $\sqrt{N}$, meaning the distance traveled grows **sub-linearly** with the number of steps.

Applications of Random Walk:

- **Diffusion of molecules** in a fluid.
- **Stock market fluctuations** in finance.
- **DNA and polymer chain dynamics** in biophysics.

Brownian Motion

Brownian motion describes the continuous, random motion of particles suspended in a fluid due to collisions with surrounding molecules. This was first observed by **Robert Brown** in 1827 while studying pollen grains in water.

Mathematical Description:

- The position $x(t)$ of a particle in Brownian motion follows the **diffusion equation**:

  $$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}$$

  where $P(x,t)$ is the probability of finding the particle at position x at time t, and D is the diffusion coefficient.

- The **mean squared displacement (MSD)** for Brownian motion is:

  $$\langle x^2(t)\rangle = 2Dt$$

  This shows that the particle's displacement grows **linearly** with time, unlike the discrete-step random walk.

Einstein's Explanation (1905): Einstein provided a theoretical explanation for Brownian motion by linking it to **molecular collisions** and the **diffusion coefficient**:

$$D = \frac{k_B T}{6\pi\eta r}$$

where:

- k_B is the **Boltzmann constant**,
- T is the **temperature**,
- η is the **viscosity of the fluid**,
- r is the **radius of the particle**.

This equation established a connection between microscopic molecular motion and macroscopic observations.

Applications of Brownian Motion:

- **Proof of molecular existence** (Einstein's work confirmed the atomic nature of matter).
- **Stock price modeling** in financial mathematics (Black-Scholes model).
- **Diffusion of pollutants** in the atmosphere.
- **Motion of small particles in biological systems**, such as inside cells.

Feature	Random Walk	Brownian Motion
Type	Discrete steps	Continuous motion
Step Size	Fixed (e.g., ± 1)	Varies due to random collisions
Mean Displacement	0	0
Mean Squared Displacement	$\langle x^2 \rangle = N$	$\langle x^2 \rangle = 2Dt$
Application	Diffusion, finance, polymers	Fluid dynamics, atomic motion

Table 3.1: Comparison of Random Walk and Brownian Motion

Comparison Between Random Walk and Brownian Motion

This comparison highlights how Brownian motion can be considered as the **continuous limit** of a random walk as step sizes become infinitesimally small.

3.8 Complex Numbers

3.8.1 Introduction

In mathematics and physics, complex numbers extend the real number system by introducing an imaginary unit i, defined as:

$$i^2 = -1.$$

This extension is crucial in many areas of physics, particularly in wave mechanics, quantum physics, electromagnetism, and control systems.

Historical Insight

The concept of imaginary numbers was first introduced by Gerolamo Cardano in the 16th century while solving cubic equations. However, it was Caspar Wessel and later Carl Friedrich Gauss who gave them a formal geometric interpretation.

3.8.2 Definition and Representation

A complex number z is expressed as:

$$z = a + bi,$$

where:

- a is the **real part**, denoted as $\mathrm{Re}(z)$.
- b is the **imaginary part**, denoted as $\mathrm{Im}(z)$.

Geometric Interpretation: A complex number can be visualized in the **Argand plane**, where:

- The x-axis represents the real part.
- The y-axis represents the imaginary part.

This allows operations like addition and multiplication to be interpreted geometrically.

Example

Example: Consider $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$.
Addition:

$$z_1 + z_2 = (3 + 1) + (4 - 2)i = 4 + 2i.$$

Multiplication:

$$z_1 z_2 = (3 + 4i)(1 - 2i).$$

Expanding using distributive property:

$$= 3 - 6i + 4i - 8i^2.$$

Since $i^2 = -1$, we get:

$$= 3 - 6i + 4i + 8 = 11 - 2i.$$

3.8.3 Modulus and Argument

The modulus (absolute value) of $z = a + bi$ is given by:

$$|z| = \sqrt{a^2 + b^2}.$$

The argument θ (angle with the positive real axis) is:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right).$$

Key Formula

The polar form of a complex number is:

$$z = re^{i\theta}, \quad \text{where } r = |z| \text{ and } \theta = \arg(z).$$

Example

Example: Find the modulus and argument of $z = 1 + i$.
Solution:

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Thus, the polar form is:

$$z = \sqrt{2}e^{i\pi/4}.$$

3.8.4 Complex Conjugate

The complex conjugate of $z = a + bi$ is:

$$\bar{z} = a - bi.$$

It satisfies the property:

$$z\bar{z} = |z|^2.$$

Example

Example: Find the product of $z = 3 + 4i$ and its conjugate.
Solution:

$$\bar{z} = 3 - 4i.$$

$$z\bar{z} = (3 + 4i)(3 - 4i) = 3^2 - (4i)^2 = 9 + 16 = 25.$$

Notice that $|z|^2 = 25$, confirming the property.

3.8.5 Euler's Formula and De Moivre's Theorem

Theory

Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Theory

De Moivre's Theorem: For any integer n,

$$(re^{i\theta})^n = r^n e^{in\theta}.$$

Expanding in trigonometric form:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example

Example: Compute $(1+i)^5$ using De Moivre's theorem.
Solution: Convert to polar form:

$$|z| = \sqrt{2}, \quad \theta = \frac{\pi}{4}.$$

Applying De Moivre's theorem:

$$(1+i)^5 = (\sqrt{2})^5 e^{i5\pi/4}.$$

$$= 4\sqrt{2}e^{i5\pi/4}.$$

Converting back to rectangular form:

$$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).$$

Since $\cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$,

$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right).$$

$$= -4 - 4i.$$

3.8.6 Roots of Complex Numbers

The nth roots of a complex number $z = re^{i\theta}$ are given by:

$$z_k = r^{1/n}e^{i(\theta+2\pi k)/n}, \quad k = 0, 1, 2, \ldots, n-1.$$

Example

Example: Find the cube roots of unity.
Solution: The equation $z^3 = 1$ can be written as:

$$z = e^{i2\pi k/3}, \quad k = 0, 1, 2.$$

The roots are:

$$1, \quad \omega = e^{i2\pi/3}, \quad \omega^2 = e^{i4\pi/3}.$$

These satisfy:

$$1 + \omega + \omega^2 = 0.$$

3.9 Optional Exercise

1. Solve for x:

$$x^3 - 6x^2 + 11x - 6 = 0.$$

2. Find the equation of a circle centered at $(2, -3)$ with radius 4.

3. Find the sine and cosine of $45°$.

4. If $\tan\theta = \frac{3}{4}$, find $\sin\theta$ and $\cos\theta$.

5. Prove that:
$$\sin^2\theta + \cos^2\theta = 1.$$

6. Compute the magnitude of the vector $\mathbf{A} = 3\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$.

7. Find the dot product of $\mathbf{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\mathbf{B} = -\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$.

8. Compute the cross product of $\mathbf{A} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\mathbf{B} = -\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

9. Find the angle between the vectors $\mathbf{A} = (1, 2, 3)$ and $\mathbf{B} = (4, 5, 6)$.

10. Determine if the vectors $(1, 2, 3)$ and $(2, 4, 6)$ are linearly dependent.

11. Differentiate $f(x) = 3x^2 + 5x - 7$.

12. Find $\frac{d}{dx}(5x^3 - 4x^2 + x - 8)$.

13. Compute:
$$\int_0^2 (3x + 1)\, dx.$$

14. Evaluate:
$$\int (5x^2 + 3x - 4)dx.$$

15. Solve the differential equation:
$$\frac{dy}{dx} = x^2 + y.$$

16. Find the general solution of
$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

17. A box contains 3 red, 5 blue, and 2 green balls. What is the probability of picking a blue ball?

18. If a fair coin is tossed 4 times, what is the probability of getting exactly 2 heads?

19. A bag contains 10 balls numbered 1 to 10. What is the probability of drawing a number that is a multiple of 3?

20. Solve for z if
$$z^2 + 4z + 13 = 0.$$

21. Find the modulus and argument of $z = 3 + 4i$.

3.9.1 Answers

1. $x = 1, 2, 3.$
2. $(x-2)^2 + (y+3)^2 = 16.$
3. $\sin 45° = \frac{\sqrt{2}}{2}, \quad \cos 45° = \frac{\sqrt{2}}{2}.$
4. $\sin\theta = \frac{3}{5}, \quad \cos\theta = \frac{4}{5}.$
5. Identity holds for all values of θ.
6. $|\mathbf{A}| = \sqrt{9+16+144} = \sqrt{169} = 13.$
7. $\mathbf{A}\cdot\mathbf{B} = 2(-1) + (-3)(4) + (1)(2) = -2 - 12 + 2 = -12.$
8. $\mathbf{A}\times\mathbf{B} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & 3 \end{bmatrix} = (7\hat{i} - 2\hat{j} + 3\hat{k}).$
9. $\theta = \cos^{-1}\left(\frac{32}{\sqrt{14}\sqrt{77}}\right).$
10. Yes, because one is a scalar multiple of the other.
11. $f'(x) = 6x + 5.$
12. $\frac{d}{dx}(5x^3 - 4x^2 + x - 8) = 15x^2 - 8x + 1.$
13. $\int_0^2 (3x+1)dx = [\frac{3}{2}x^2 + x]_0^2 = \left(\frac{3}{2}(4) + 2\right) - (0) = 8.$
14. $\int (5x^2 + 3x - 4)dx = \frac{5}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$
15. $y = Ce^x - \frac{x^3}{3} - x^2 - x - 1.$
16. $y = C_1e^x + C_2e^{2x}.$
17. $P(\text{blue}) = \frac{5}{10} = 0.5.$
18. $P(\text{2 heads}) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = 6 \times \frac{1}{16} = \frac{6}{16} = \frac{3}{8}.$
19. $P(\text{multiple of 3}) = \frac{3}{10}.$
20. $z = -2 \pm 3i.$
21. $|z| = \sqrt{3^2 + 4^2} = 5, \quad \arg(z) = \tan^{-1}\frac{4}{3}.$