Assignment

PES1201801482

Linear Algebra and its Applications

1. $y = A + Bx + Cx^2$. (parabola)

passes through $(1,1), (2,-1), (3,1)$

$\rightarrow A + B + C = 1$

$\rightarrow A + 2B + 4C = -1$

$\rightarrow A + 3B + 9C = 1$

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \end{array} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$[A\,b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array}\right] \quad R_3 \rightarrow R_3 - 2R_2 \quad \approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$2C = 4$, $C = 2$

$B + 3C = -2$, $B = -8$

$A + B + C = 1$, $A = 7$.

Equation of the Parabola: $y = 7 - 8x + 2x^2$.

2. LU Decomposition

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 + 5R_1$, $R_4 \Rightarrow R_4 - 5R_1$

$$\approx \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 + 2R_2 \\ R_4 \rightarrow R_4 + 2R_2 \end{array}$$

$$\approx \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix} \quad R_4 \rightarrow R_4 - 3R_3$$

$$\approx \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \therefore LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

3. $T: \mathbb{R}^3 \to \mathbb{R}^3$,

$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$

i) $T = ?$

$T(1\,0\,0) = (1, 0, 1)$
$T(0\,1\,0) = (2, 1, 1)$
$T(0\,0\,1) = (-1, 1, -2)$

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

ii) Basis for 4 subspaces

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_1 \approx \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\approx \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$C(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ Column Space

$C(T^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ Row Space

$x + 2y + -z = 0$
$y + z = 0$ $\quad y = -z$.
$x + 2(-z) - z = 0$, $x = 3z$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} z$$

$N(T) = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ Null Space.

$$T^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array} \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x + z = 0$, $x = -z$
$y - z = 0$, $y = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} z$$

$N(T^T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ Left Null Space.

(iii) Eigen value,

$$|T - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)\left[(1-\lambda)(-2-\lambda) - 1\right] - 2(-1) + (-1)(\lambda-1)$$

$$-\lambda^3 + 3\lambda = 0$$

$$\lambda(\lambda^2 - 3) = 0$$

$\therefore \lambda_1 = -\sqrt{3}, \lambda_2 = 0, \lambda_3 = \sqrt{3}.$

Eigen vectors.

$\lambda_1 = -\sqrt{3}$, ~~A+J~~

$$T + \sqrt{3}I = \begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 1+\sqrt{3} & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1+\sqrt{3} & -1 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1+\sqrt{3} & 2 \\ 0 & 1+\sqrt{3} \end{vmatrix}}$$

$$\frac{x}{3+\sqrt{3}} = \frac{-y}{1+\sqrt{3}} = \frac{z}{4+2\sqrt{3}} = k_1.$$

~~$k_1(-\sqrt{3}+3$~~ $\therefore k_1(3+\sqrt{3}, -1-\sqrt{3}, 4+2\sqrt{3})$

$$= k_1\left(\frac{3-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, 1\right)$$

$\lambda_2 = 0$,

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \qquad \frac{x}{3} = \frac{-y}{1} = \frac{z}{1} = k_2$$

$$= k_2(3, -1, 1)$$

$\lambda_3 = \sqrt{3}$

$$~~A~~ - \sqrt{3}I = \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix}$$

$$\frac{x}{3-\sqrt{3}} = \frac{y}{-1+\sqrt{3}} = \frac{z}{4-2\sqrt{3}} = k_3$$

$$= k_3(3-\sqrt{3}, -1+\sqrt{3}, 4-2\sqrt{3})$$

$$= k_3\left(\frac{\sqrt{3}+3}{2}, \frac{\sqrt{3}+1}{2}, 1\right)$$

(iv) $T = QR$.

$a = (1\ 0\ 1) \quad b = (2\ 1\ 1) \quad c = (-1\ 1\ -2)$

$$q_1 = \frac{a}{\|a\|} = \frac{(1\ 0\ 1)}{\sqrt{2}}$$

$$q_2 = \frac{B}{\|b\|} = \frac{(b - q_1^T b)q_1}{\|b\|} = \frac{1}{\sqrt{6}}(1, 2, -1)$$

$$q_3 = \frac{c}{\|c\|} = \frac{c - (q_2^T c)q_2 - (q_1^T c)q_1}{\|c\|} = (0, 0, 0)$$

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 \\ 0 & 0 & 0 \end{bmatrix}$$

4. $y = c + dx$. , Least square principles

x	-4	1	2	3
y	4	6	10	8

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix} \qquad (A^T A)^{-1} A^T b$$

$$A^T A = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \qquad (A^T A^{-1}) = \frac{1}{116}\begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}$$

$$(A^T A^{-1}) A^T = \frac{1}{116}\begin{bmatrix} 38 & 28 & 26 & -24 \\ -18 & 2 & 6 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} A^T b = \frac{1}{116}\begin{bmatrix} 772 \\ 80 \end{bmatrix} = \hat{x}\begin{bmatrix} c \\ d \end{bmatrix}$$

$$y = \frac{193}{29} + \frac{20}{29}x.$$

5. Projection matrices P, Q, to plane.

$x_1 + x_2 + 3x_3 + 4x_5 = 0$. & orthogonal complement.

$$A = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} \qquad P = \frac{AA^T}{A^TA} = \frac{1}{27}\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \\ 1 & 1 & 3 & 0 & 4 \\ 3 & 3 & 9 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 12 & 0 & 16 \end{bmatrix}$$

$$Q = I - P = \frac{1}{27}\begin{bmatrix} 26 & -1 & -3 & 0 & 4 \\ -1 & 26 & -3 & 0 & -4 \\ -3 & -3 & 18 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & -4 & -12 & 0 & 11 \end{bmatrix}$$

6. (i) $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

For A to be +ve definite, all subdomains must be +ve.

$a^2 - 4 > 0$, $a \in (-\infty, 2) \cup (2, \infty)$

$a > 0$, $a \in (0, \infty)$

$(a+4)(a-2)^2 > 0$ $\quad a \in (-4, \infty)$

$\therefore a \in (2, \infty)$

(ii) $$[x_1 \; x_2 \; x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$

$\Longrightarrow 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

$\therefore a_{11} = a_{22} = a_{33} = 2$

$a_{12} = -1$

$a_{13} = 0$

$a_{23} = -1$ $\qquad \therefore B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

7. $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ $U\Sigma V^T$, SVD = ?

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} = B$$

$|B - \lambda I| = 0$

$(81-\lambda)(9-\lambda) - 27^2 = 0$

$\lambda^2 - 90\lambda = 0$, $\lambda(\lambda - 90) = 0$

$\therefore \lambda = 0, 90 \longrightarrow$ Eigen values

1) $\lambda = 0$, $A = \begin{bmatrix} -9 & -27 \\ -27 & \cancel{3}9 \end{bmatrix}$

$\frac{x}{1} = \frac{y}{3} = k_1(1, 3)$

2) $\lambda = 90$ $A = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}$

~~$9x - 27$~~ $\frac{x}{-3} = \frac{y}{1} = k_2(-3, 1)$

$$V = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

$|(AA)^T - \lambda I| = 0$

$(10-\lambda)(\lambda^2 - 80\lambda) + 800\lambda = 0$

$\lambda^2(\lambda - 90) = 0$

$\lambda = 90, 0$, $\sigma = 3\sqrt{10}, 0$

$$\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$$

$AV = U\Sigma$, $\begin{bmatrix} 0 & \sqrt{10} \\ 0 & -2\sqrt{10} \\ 0 & -2\sqrt{10} \end{bmatrix} = U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$

$$U = \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & -2/3 \end{bmatrix}$$

$$A = U\Sigma V^T = \begin{bmatrix} 0 & 0 & 1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$