

Q3)

For upper bound observe that dot product of 2 unit vectors can be almost 1 by Cauchy Schwartz inequality $(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2$.

ϕ^i and ψ_j are both unit vectors, hence $|\phi^i \cdot \psi_j| \leq 1$ for any i, j , hence

$\max(|\phi^i \cdot \psi_j|) \leq 1$ and $\mu(\phi, \psi) \leq \sqrt{n}$.

For lower bound, take $g_i = \phi^i$ and express it as linear combination of columns of ψ and take transpose, since all rows of ϕ are normalised to unit magnitude. Hence we expressed a row of ϕ as linear combination of columns of ψ and then took transpose. So g_i is a row vector now.

Let $g_i^t = \sum_{k=1}^n a_k \psi_k$. Since g_i is unit normal $g_i \cdot g_i^t = 1$ and hence $\sum_{k=1}^n a_k^2 = 1$.

ψ is an orthonormal matrix and hence $\psi_i \cdot \psi_j = 1$ for $(i=j)$ and 0 otherwise so

$|g_i \cdot \psi_j| = |a_j|$ as only $a_j(\psi_j^t \cdot \psi_j)$ is the non zero term in the dot product.

So $\mu(g_i, \psi) = \sqrt{n} \max_{j \in (0,1,\dots,n-1)} |a_j|$. The denominator term in the original formula for coherence is one since $\sum_{k=1}^n a_k^2 = 1$.

$\sum_{k=1}^n |a_k|^2 = 1$, hence the min value of $S = \max_{j \in (0,1,\dots,n-1)} |a_j|$ can be $\frac{1}{\sqrt{n}}$ since if the min value is $< \frac{1}{\sqrt{n}}$, then for all j , $|a_j| < \frac{1}{\sqrt{n}}$, this implies $|a_j|^2 < \frac{1}{n}$ and $\sum_{k=1}^n a_k^2 < n * \frac{1}{n} = 1$.

Hence we get $\sum_{k=1}^n a_k^2 < 1$ which is a contradiction to $\sum_{k=1}^n a_k^2 = 1$.

Whereas the min value of $\frac{1}{\sqrt{n}}$ can be achieved by setting $|a_j| = \frac{1}{\sqrt{n}}$ for all j . Then $\sum_{k=1}^n a_k^2 = 1$ also and $S = \max_{j \in (0,1,\dots,n-1)} |a_j| = \frac{1}{\sqrt{n}}$ also.

Hence the minimal value of coherence is obtained when $g_i^t = \frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_k$ and

$$\min \mu(g_i, \psi) = \sqrt{n} * S = \sqrt{n} * \frac{1}{\sqrt{n}} = 1$$

So $\mu(g_i, \psi) \geq 1$

Now $\mu(\phi, \psi) = \max_{i \in (0,1,\dots,n-1)} \mu(g_i, \psi)$ where we represented $g_i = \phi^i$ and all rows of ϕ can be independently chosen.

$\mu(g_i, \psi) \geq 1$ for all i , hence $\mu(\phi, \psi) \geq 1$ and we showed that $\mu(g_i, \psi) = 1$ is possible when $g_i^t = \frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_k$. Hence for all i , choose $g_i^t = \frac{1}{\sqrt{n}} \sum_{k=1}^n \psi_k$, then each $\mu(g_i, \psi) = 1$ and hence

the minimum value of $\mu(\phi, \psi) = \max_{i \in (0,1,\dots,n-1)} \mu(g_i, \psi) = 1$.

Thus the value of coherence $\mu(\phi, \psi)$ must lie in the range $[1, \sqrt{n}]$.