Q5

- Paper Title: Iterative Hard Thresholding for Compressed Sensing
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- Venue: University of Southampton
- Year of Publication: 2008

1. Iterative Hard Thresholding algorithm (IHT_s)

$$\mathbf{x} = \boldsymbol{\phi}\mathbf{y} + \mathbf{e}$$

where e is the observation noise and ϕ is the sampling matrix.

Let $y^{[i]}$ be the value of desired vector y during the i^{th} iteration of the algorithm.

Pseudo-Code:

a) $y^{[0]} = 0$

b) Update $y\ as\ y^{[n+1]} = H_s\left(y^{[n]} + \phi^T\left(x - \phi y^{[n]}\right)\right)$

c) $H_s(\boldsymbol{a})$ is the non-linear operator that sets all but the largest (in magnitude) s elements of **a** to zero. If there is no unique such set, a set can be selected either randomly or based on a predefined ordering of the elements

2. Key Theorem (stated as it is for the most part)

a) A matrix ϕ satisfies the Restricted Isometry Condition (RIP) if

$$(1 - \delta_s)||y||_2^2 \leq ||\phi y||^2 \leq (1 + \delta_s)||y||_2^2$$

b) Instead of using the above restricted isometry property, we will use a re-scaled matrix $\phi = \frac{\phi'}{1+\delta_s}$, which satisfies the following non-symmetric isometry property, which is equivalent to the RIP defined above.

$$(1 - \beta_s)||y||_2^2 \leq ||\phi y||_2^2 \leq ||y||_2^2, \quad \beta_s = 1 - \frac{1-\delta_s}{1+\delta_s}$$

for all s-sparse y. We will say that for a matrix ϕ the RIP holds for sparsity s, if β_s< 1.

c) *Theorem:* Given a noisy observation $x = \phi y + e$, were y is an arbitrary vector. Let y_s be an approximation to y with no-more than s non-zero elements for which $||y - y_s||_2$ is minimal. If ϕ has restricted isometry property with $\beta_{3s} < \frac{1}{8}$, then, at iteration k, IHT_s will recover an approximation $y^{[k]}$ satisfying

$$||y - y_s||_2 \leq 2^{-k}||y_s||_2 + 5\epsilon_s^{\sim}$$

where $\epsilon_s^{\sim} = ||y - y_s||_2 + \frac{1}{\sqrt{s}}||y - y_s||_1 + ||e||_2$

d) Furthermore, after at most

$$k^* = \lceil \log_2\left(\frac{||y_s||_2}{\epsilon_s^{\sim}}\right) \rceil$$

iterations, IHT_s estimates y with accuracy

a. $||y - y_{k^*}||_2 \leq 6\left(||y - y_s||_2 + \frac{1}{\sqrt{s}}||y - y_s||_1 + ||e||_2\right)$