Q1

Since matrix $\boldsymbol{A}$ $(\boldsymbol{m} \times \boldsymbol{n})$ follows RIP of order s with RIC δ_s, for any s-sparse vector y,

$$(1-\delta_s)\|y\|_2^2 \leq \|Ay\|_2^2 \leq (1+\delta_s)\|y\|_2^2$$

Let $\tau \subset \{1,2, \ldots n\}$ $be\ a\ s-sized\ index\ set\ (i.e.\ |\tau| = s)$. Then we have,

$$(1-\delta_s)\|y_\tau\|_2^2 \leq \|A_\tau y_\tau\|_2^2 \leq (1+\delta_s)\|y_\tau\|_2^2 \quad \text{where}$$

y_τ , A_τ $are\ subvector\ and\ submatrix\ (selection\ of\ columns)\ of\ y\ and\ A\ respectively.$

Note that in the above equation y_τ can be any general s-size vector since we can then fill zeros and make a y to be used with the first equation.

$A_\tau\ is\ m \times s\ matrix$. Note that $A_t\ has\ all\ non-zero\ eigenvalues\ (to\ satisfy\ left\ inequality)$.

We have for all non-zero y_τ, $1-\delta_s \leq \frac{\|A_\tau y_\tau\|_2^2}{\|y_\tau\|_2^2} \leq 1+\delta_s$.

Claim1: *The nonzero singular values of A are the square roots of the nonzero eigenvalues of A*A.*

Proof: $Considering\ USV^*\ as\ the\ SVD\ of\ A_\tau$:

$$A_\tau^* A_\tau = (VS^*U^*)(USV^*) = V(S^*S)V^*$$

$which\ is\ the\ eigendecomposition. S^*S\ contains\ \sigma_1^2, \ldots \sigma_s^2. Thus\ eigen\ values\ of\ A_\tau^* A_\tau\ are$ square of magnitudes of singular values of A_τ.

For every eigenvector v of A_τ, $\frac{\|A_\tau v\|_2}{\|v\|_2} = |\sigma_v|$, with $\sigma_v\ being\ the\ eigenvalue$ which implies (using the claim)

$$\max(|\sigma(A_\tau)|) = \max\sqrt{\lambda(A_\tau^* A_\tau)} \ \leq \sqrt{1+\delta_s}$$

$$\min(|\sigma(A_\tau)|) = \min\sqrt{\lambda(A_\tau^* A_\tau)} \ \geq \sqrt{1-\delta_s}$$

Here $\lambda\ denotes\ the\ non-zero\ eigenvalues\ of\ A_\tau^* A_\tau$.

Therefore $\delta_s \geq \max\big(\lambda(A_\tau^* A)\big) - 1\ and\ \delta_s \geq -\min\big(\lambda(A_\tau^* A)\big) + 1$. Since $\delta_s\ is\ the\ smallest\ such\ number\ and\ all\ \delta's\ satifying\ the\ above\ follow\ the\ rip\ condition^{\#}$,

$$\delta_s = \max\Big(\max\big(\lambda(A_\tau^* A)\big) - 1, -\min\big(\lambda(A_\tau^* A)\big) + 1\Big).$$

Proof :

Let $\delta_s \geq \max(\lambda(A_\tau^* A)) - 1 \; and \; \delta_s \geq -\min(\lambda(A_\tau^* A)) + 1$. Then,

$1 + \delta_s \geq \lambda_i(A_\tau^* A) \geq 1 - \delta_s \; for \; all \; i.$ By Claim1, all $\sigma(A) \; are \; in \; the \; range \; \left(\sqrt{1-\delta_s}, \sqrt{1+\delta_s}\,\right) \; yielding \; the \; RIP \; condition.$

This is because maximum/minimum of $\|Av\|_2$ for a unit vector v occurs at max/min absolute value eigenvalue.

$$\|Av\|_2^2 = (Av)^*(Av) = v^*(A^*A)v = v^*V(S^*S)V^*v = Q^*(S^*S)Q = \sum q_i^2\sigma_i^2 \leq \sigma_{max}^2 \sum q_i^2$$
$$= \sigma_{max}^2 \; , since \; \sum q_i^2 = 1 \; as \; \|Q\|_2^2 = Q^*Q = \; v^*(VV^*)v = v^*v = 1$$

Similiarly for minimum.

Consider $M = \; A_\tau^* A_\tau - I_\tau$ which has then eigenvalues in the range $[-\delta_s, \delta_s]$ with at least one eigenvalue with absolute value δ_s. (since δ_s is achieving atleast one bound (either the left or the right) both of which

Also max $element \; of \; M \leq \mu$ since each element in M is a dot product between two distinct columns in A_τ or zero.

Claim2: $\|Mv\|_2 \leq \mu(s-1)\|v\|_2$

Proof:

$$\|Mv\|_2^2 = \sum_{r=1}^{r=s}\Big(\sum_{c=1,c\neq r}^{c=s} M_{rc}v_c\Big)^2 \leq \sum_{r=1}^{r=s}\mu^2\Big(\sum_{c=1,c\neq r}^{c=s}|v_c\,|\Big)^2 \leq$$
$$\sum_{r=1}^{r=s}\mu^2(s-1)\sum_{c=1,c\neq r}^{c=s}(|v_c\,|)^2$$
$$= \mu^2(s-1)^2\|v\|_2^2$$

where $\Big(\sum_{c=1,c\neq r}^{c=s}|v_c|\Big)^2 \leq (s-1) * \sum_{c=1,c\neq r}^{c=s}|v_c|^2 \; by \; Cauchy - Schwartz \; inequality.$

If $v \; is \; a \; eigenvector \; \|Mv\|_2 = |\lambda|\|v\|_2.$

$$\mu(s-1) \; is \; thus \; greater \; than \; all \; absolute \; eigenvalues \; of \; M.$$

We have thus $\delta_s \leq \mu(s-1)$.