Memory and Learning in Neural Networks: A Statistical Physics Approach

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Intro

Abstract

Learning and associative memory are understood as emergent phenomena resulting from interactions between a complex network of neurons.

How does computational phenomena like learning and memory emerge from simple interactions between neurons in brain and other nervous systems?

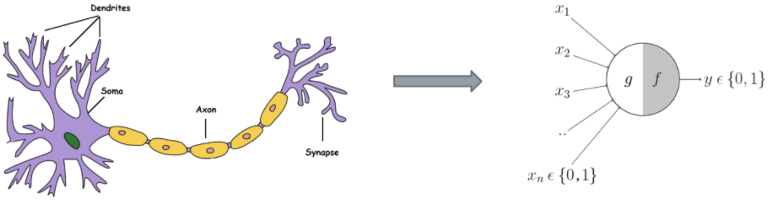
How does the structure of the network affect its function?

To tackle this question, we study the **Hopfield Model of associative memory on regular and Watts Strogatz (WS) networks.**

We obtain estimates of memory capacity on a regular and WS network through numerical simulations. Further, we study how changing the probability of rewiring and local connectivity in WS network affects the performance of network.

McCulloch-Pitts Neurons

- McCulloch and Pitts [6] proposed binary model of neurons where the state of any neuron i is $s_i = 1$ (firing) or $s_i = -1$ (inactive).



- This ideal model makes it possible to systematically study large networks of interacting neurons called neural networks. These act as toy system to study the emergent computational behaviour occurring in biological nervous systems.

For a network of N such neurons, a given neuron i fires only when the sum of their inputs from their neighbouring neurons, weighted by the synaptic efficacy J_{ij} exceeds the threshold potential U_i.

$$s_i(t+1) = \Theta\left(\sum_{j=1}^{N} J_{ij} s_j(t) - U_i\right) \tag{1}$$

- $J_{ij} > 0$ would imply excitatory neruons, and $J_{ij} < 0$ implies inhibitory neurons.

- McCullough and Pitts showed that a dynamical network of such neurons can act like a universal computer and can perform any operation a Turing Machine can.

- With the right values of J_{ij} and U_i one can implement any logical operation. For example if $J_{ij} = 1$ for two presynaptic neurons and $U_i = 1$, the output of neuron i will be an OR function.

Associative Memory: An Illustration

I find it rea*l* amazin* t*at y*u ar*
abl* to re*d t*is tex* despit* th* fac*
*hat more t*an t*ent* perc*n* of t**
char*cte*s a*e mis*ing.
his mean t*at you* brai* i* abl* ** fill
in missin* info*matio*.

From Lecture A1, Neuronal Dynamics - Computational Neuroscience Video Lectures by Wulfram Gerstner (YouTube)

- Associative memory is the ability to learn and remember relationships between objects. When presented with partial information, a subject recovers the complete information of associated items stored in its memory.
- Hebb's rule provides the biological basis for associative memory. Based on experiments of classical conditioning in psychology, Donald Hebb (1949) [4] suggested that in the presence of external stimulus, neuronal networks learn by "neuroplasticity" - modifications to synaptic strengths.
- Hebb's hypothesis was the positive correlations between firing of any two neurons increases their synaptic strength. Memory is thus encoded by synaptic strength matrix J_{ij} and if one is trying to memorize a pattern, encoded as a word of N bits $\{\xi_i = +1, i = 1, N\}$, the synapse between any neuron i and j is modified as

$$J_{ij}^{\text{new}} = \lambda J_{ij}^{\text{old}} + \varepsilon \xi_i \xi_j$$

Hopfield Model: Background

- Consider a network of N McCullough-Pitts neurons. Any configuration of the system can be described as combination of N binary bits and the state space will be given by:

$$\Omega_n = \left\{ \vec{s} = (s_1, \ldots, s_n) : s_i \in \{-1, 1\} \right\} \tag{2}$$

- The system can be described by a set of generalized coordinates $X = X_1, X_2, ...X_n$. Associative memory occurs when we consider a system that has that the system has fixed/stable points $X_a, X_b, ..., X_n$ in the phase space such that if the system begins at any point $X_i = X_a + \delta$, it ultimately converges to $X_i \approx X_a$.
- Hopfield's key insight: any such system with dynamics in phase space such that it attracts to stable points of the system can be used to model associative memory.

- We wish to store an arbitary set of patterns ξ^μ where $\mu = 1, \dots, M$ where $\{\xi^1, \dots, \xi^M\} \subset \Omega_n$ and M is the total number of memory states.
- If we choose an appropriate prescription of J_{ij} so that it leads to the desired dynamics, then network is said to correctly represent a pattern ξ^μ if condition

$$s_i(t) = s_i(t+1) = \xi_i^\mu \tag{3}$$

holds for all neurons $1 \leqslant i \leqslant n$.

- The patterns have no correlation between them and are orthogonal such that

$$\frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu \xi_i^\nu = \delta_{\mu\nu} \tag{4}$$

- We assume no self connections $J_{ii} = 0$ and the threshold value $U_i = 0$ for all the neurons. Then, Hopfield uses the storage algorithm given by Hebb's rule:

$$J_{ij} = \frac{1}{N} \sum_{\mu}^{M} \xi_i^{\mu} \xi_j^{\mu} \tag{5}$$

  This is equivalent to the the Hebbian rule when $\epsilon = 1$ and $\lambda = 1$.
- The dynamical equation for any neuron i is, under the assumption that $U_i = 0$ for all the neurons:

$$S_i(t+1) = \Theta(h_i(t)) \tag{6}$$

  where $\sum_{i \neq j}^{N} J_{ij} S_j(t) \equiv h_i(t)$ are the local fields acting on the neurons, determined by the synaptic strengths.

Energy Function

If we assume symmetric synapes $J_{ij} = J_{ji}$, then at zero temperature there exists of a energy function:

$$E = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j \tag{7}$$

for which the change in energy ΔE due to a variation in the state Δs_i is given by

$$\Delta E = -\Delta s_i \sum_{i \neq j} J_{ij} s_j$$

Thus, this energy function monotonically decreases for any changes in s.

This energy function is isomorphic to the Ising Model Hamiltonian, specifically since the J_{ij} are quenched it is equivalent to the Hamiltonian of a spin glass system called the Sherrington-Kirkpatrick (S-K) Hamiltonian [8]

- A spin glass is a magnetic system with the presence of both ferromagnetic and antiferromagnetic interactions that were quenched randomly in place.

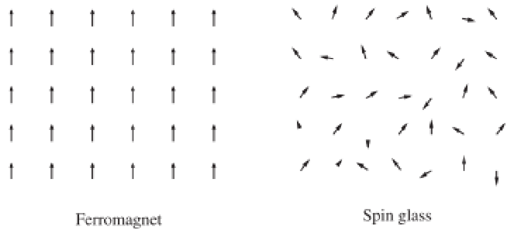


Figure 1: Figure adapted from Dotsenko (1985) [3])

- The couplings are not fixed but instead depend on the pairwise interactions (J_{ij}). The couplings are defined by a probability distribution, often gaussian.

- The interaction is not just between pairs, but is long range - that is the sum is evaluated over all possible pairs of spins.

$$H = -\frac{1}{2}\sum_{i \neq j}^{N} J_{ij}\sigma_j\sigma_j \tag{8}$$

This is the S-K Hamiltonian which is exactly analogous to the Hopfield Energy function where the couplings are decided by the memory states.

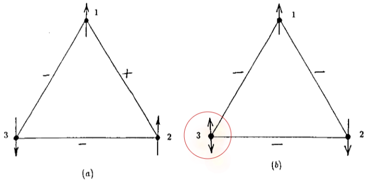
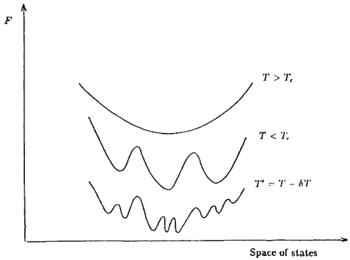


Figure 2: Figure adapted from Dotsenko (1985) [3])

- Frustration occurs due to an impossibility of satisfying all interactions at the same time to achieve an absolute minimum of the free energy. It leads to non degenerate ground states.

Spin Glasses: Energy Landscape

- These two features are the most essential features of spin glasses - *long range disorder*, and *frustration*



- As we go below critical temperature, each valley is divided into many new smaller ones separated by infinite barriers of the free energy. This leads to fragmentation of energy state space further and further as we keep decreasing temperature, with each of the valleys having metastable states as well.

- Thus, the energy landscape of a spin glass system is composed of multiple free lying minima having metastable states. This is why the S-K Hamiltonian works as the multiple energy minima correspond to the stored memory states.

Analogy with Spin Glasses

Electron spins $\Longleftrightarrow$ McCullough-Pitts Neuron states

S-K Hamiltonian $\Longleftrightarrow$ Hopfield's Energy Function

Random FM/AFM couplings $\Longleftrightarrow$ Couplings based on stored memory states

Multiple free energy minima at low temperature "glassy phase" $\Longleftrightarrow$ Stable point attractors corresponding to memory states

- Further, we dynamics only at zero temperature and do not consider effects of thermal noise.

- The **order parameter is the overlap** m_μ between the final state after evolution and a given pattern ξ^μ:

$$m_\mu = \frac{1}{N} \sum_i^N \langle S_i \rangle \, \xi_i^\mu \tag{9}$$

- If a pattern ξ^μ is stable, it implies that $S_i(t) = \xi_i^\mu$ for all neurons i after equilibrium.

Stability

- Thus, at a stable point $\xi_i^\mu = \Theta(h_i^\mu)$ where:

$$h_i^\mu \equiv \sum_{j\neq i}^{N} J_{ij}\xi_j^\mu = \xi_i^\mu + \frac{1}{N}\sum_{\nu\neq\mu}^{P}\xi_i^\nu \sum_{j\neq i}^{N}\xi_j^\nu \xi_j^\mu$$

thus

$$h_i^\mu = \xi_i^\mu + \eta_i^\mu$$

- The second term η_i^μ here is noise caused the "interference" between different patterns. The patterns are stable only when it is small.
- The characteristic value of this noise is given by:

$$<< \eta^2 >>= \frac{1}{N^2}\sum_{\nu,\gamma}^{M}\sum_{j,l}^{N} << \xi_i^\nu \xi_i^\gamma \xi_j^\nu \xi_j^\mu \xi_l^\gamma \xi_l^\mu >>= \frac{M}{N} \equiv \alpha$$

(only nonzero contribution is when $\nu = \gamma$ and $j = l$ since patterns are orthogonal)
Here α **is defined as the memory capacity of a Hopfield Network**.

Memory Capacity and Spin-Glass Solution

- Amit et al [1] solved the S-K Hamiltonian for the Hopfield Model for M states using the replica symmetry method. If memory states increase such that $M = \alpha N$: the system is shown to exhibit three states:

- **Perfect Recall**: $T < T_c$ is the ferromagnetic phase where there are $2M$ degenerate stable states corresponding to patterns.

- **Nearly Perfect Recall with Spurious States**: $T_c < T < T_m$ the system exists in the spin glass state where metastable (spurious) states appear corresponding to mixtures of patterns.

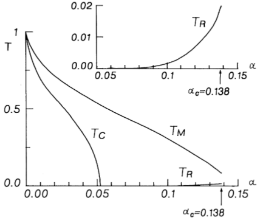


Figure 3: Phase diagram of Hopfield Model, adapted from [1]

- **Catastrophic Forgetting**: The model is useful for associative memory upto $\alpha_c = 0.138$. Above this the system goes to paramagentic state and number of errors increase drastically and the system collapses and no longer returns to any of the expected stable states.

Spurious States

- Amit et al showed that the total memory capacity of a Hopfield Network of N neurons is 0.138N. Beyond this convergence to a memory state doesn't occur.
- As the total number of stored memory states increase, the noise will increase as there will be more interference between the patterns.
- The equation

$$\xi_i^\mu = \Theta(h_i^\mu)$$

  doesn't have $h_i^\mu = \xi_i^{(\mu)}$ as the only solution but instead

$$h_i^\mu = \xi_i^{(\mu_1)} + \xi_i^{(\mu_2)} + \xi_i^{(\mu_3)} + \dots$$

  becomes a valid solution as the interference between the patterns increases.
- The system converges to a superposition of odd number of states. Such final states are called **spurious states**.

Hopfield Model: Methodology and Results

We have implemented a code in Python that simulates a Hopfield Network for a given, arbitrary number of neurons (N) and number of memory states (M) based on user input.

The M memory states are created randomly having bits $+1$ or -1 such that they have no correlations with each other. This generates the *MemoryMatrix* which has M rows corresponding to the memory states, each an array of N bits.

Further a *target* memory state is chosen, superimposing noise on which we get the initial state. Thus, if the associative memory is functioning, the initial state with noise will converge to the desired target state. The noise is introduced by flipping **25 percent** of bits from the target state **randomly**.

Results: 16 Neurons

The code is run for a very small system of 16 neurons to test its working. We can store upto two memory states in this network. The two stored states are

$$M_0 = [1, -1, 1, -1, -1, -1, 1, -1, 1, 1, 1, -1, 1, -1, 1, -1.]$$

$$M_1 = [1, -1, 1, -1, 1, -1, -1, 1, -1, 1, -1, 1, 1, 1, 1, 1.]$$

The chosen target state is M_1. Then the network initializes in an initial state where 25% that is 4 bits from this state are flipped. The network is fully connected as expected, and the initial state is shown below.

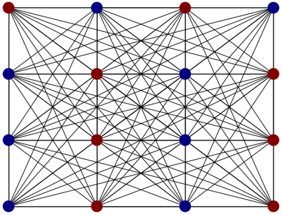


Figure 4: Initial State

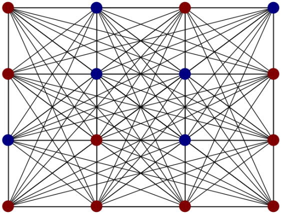


Figure 5: Final State

- The convergence towards the target pattern and the difference between the final state and the memory states is quantified in terms of the **hamming distance** from the given memory state, defined by

$$h(M) = \frac{|S_f - M|}{2} \tag{10}$$

- The hamming distance measures the number of bits that are wrong with respect to any given pattern. We perform an iteration wise analysis by calculating the hamming distance from each pattern after every iteration.

Results: N = 256, M = 10, $M_{max} \approx 35$

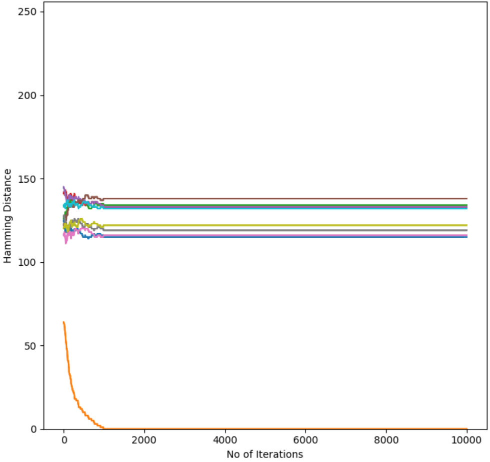


Figure 6: At 10 memory states, the system converges to the desired pattern.

Results: N = 256, M = 50, $M_{max} \approx 35$

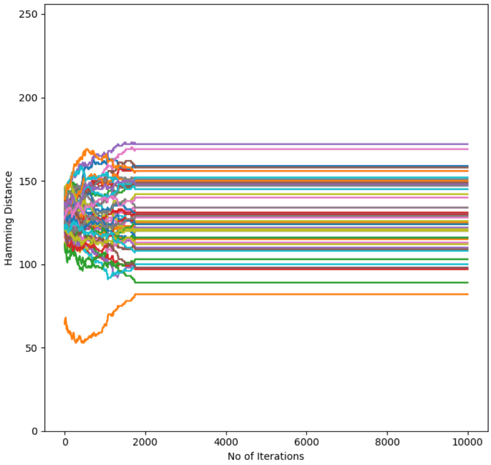


Figure 7: At 50 memory states, the system is beyond $M_{max} \approx 35$ and convergence does not occur

Results: N = 256, M = 33, $M_{max} \approx 35$

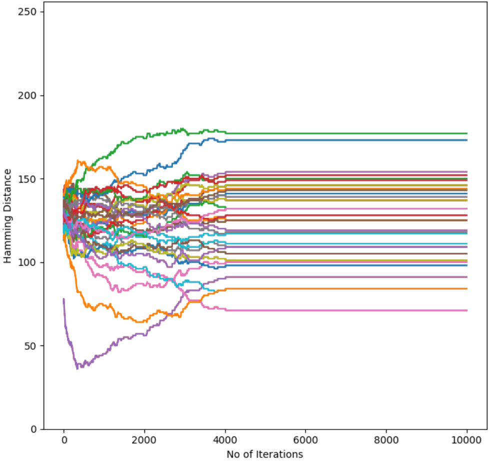


Figure 8: Very close to critical point, at 33 memory states, we see the occurrence of spurious states.

1. We explore the phase diagram in terms of the order parameter *m* and control parameter α for various sizes of networks. We probe the region of α from 0 to 0.2, hence the maximum value of stored memory state is $m_f = 0.2 * N$.
2. To negate the effect of a particular initial state, we implement **ensemble averaging** over a random ensemble of initial states.
3. For a given initial state in the ensemble, we start by an initial value of 2 stored memory states, and keep updating m = m + step until m reaches m_f.
4. Finally we find the overlap of the final state with the targeted state for each of these values of m, and this generates our graph for *m* vs *alpha* for this ensemble.
5. In the end, we take the average of all the columns of the matrix so that at each point in the graph we have the average of the overlap value over the ensemble.

Analysis of Memory Capacity

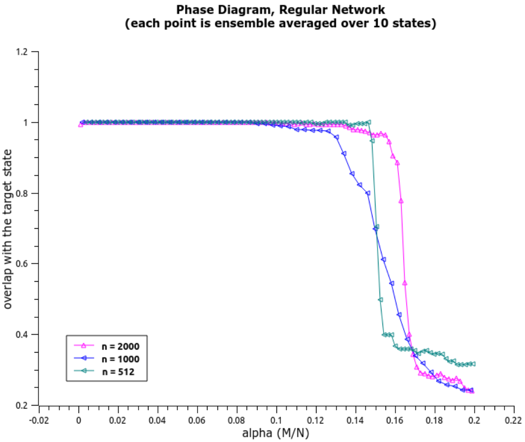


Figure 9: For N = 512, 1000, 2000 - the system collapses around $\alpha_C \approx 0.14$. Further, before transition, over a range the systems show a slight decrease in overlap. This is the region where metastable states occurs due to interference between patterns

Hopfield Model on WS Network

- Unlike magnetic systems, the structure of wiring in brain is far from homogeneous. It is sparse, that is not every neuron is connected to each other.

- With advances in non invasive brain imaging technology (like MRI, DTI amongst others), it is now possible to probe the structure of networks in brain.

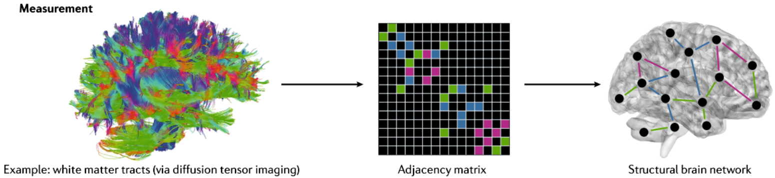


Figure 10: The experimental measurement of brain network structure from DTI imaging is discretized into an adjacency matrix which then results in the structural brain network. Adapted from [5]

Complex Networks

- For example, the nervous system of a nematode worm Caenorhabditis elegans is completely mapped using electron microscopy by [10]. The wirings in brain are far from random like those in spin glasses or regular like that in an ising lattice and instead follows evolutionary favourable organizing principles.
- It is known that structure of brain network optimizes between a metabolic penalty for the formation of long distance connections and the functional benefits of globally integrated topology for efficient information processing.
- To study how the collective function of the brain and neural systems depends on the structure, we use tools from graph theory and generative network models for complex networks.

- Watts and Strogatz [9] proposed an algorithm to model such systems by starting with a regular lattice and rewiring it to various degree for introducing randomness or disorder. The Watts-Strogatz algorithm is described as:

- Start with a regular ring lattice of N nodes, with each node being connected to its K nearest neighbours, with $K/2$ neighbours at each side.
- Then, for each node $i = 0, 1, \ldots N$, take the edge connecting to its $K/2$ neighbours and rewire it to a randomly chosen node z in the lattice with a probability of rewiring p. Self connections and duplicate edges are avoided. This introduces $pNK/2$ long range connections or shortcuts between neighbourhoods, and the parameter p can be varied from $p = 0$ (regular) to $p = 1$ (random).

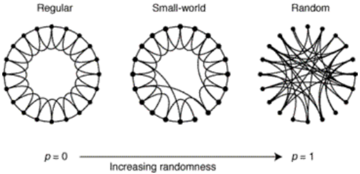


Figure 11: By varying the parameter p, one can interpolate between regular and random networks. Adapted from [9].

To characterize the structural properties of these networks, we introduce:

- **Clustering Coefficient**: For a given network with probability of rewiring p, the clustering coefficient $C(p)$ measures the degree of cliquishness. Say a node i has k_i edges that connect it to k_i other nodes, then:

$$C_i = \frac{2E_i}{k_i(k_i - 1)}$$

for the whole network:

$$C(p) = \frac{1}{N} \cdot \sum_{i=0}^{N} C_i$$

- **Characteristic Path Length**: the characteristic path length $L(p)$ is the number of edges in the shortest path between two nodes, averaged over all possible pairs of nodes.

$$L(p) = \frac{1}{N(N-1)} \cdot \sum_{i \neq j} d(n_i, n_j) \tag{11}$$

Choosing $n \gg k \gg \ln(n)$ ensures that the network remains fully connected.

- Watts and Storgatz [9] showed that as we move from a regular network $p = 0$ towards a random network $p = 1$, the characteristic path length $L(p)$ drops rapidly in comparison to the average clustering coefficient $C(p)$.

- Over a broad range of p, $L(p)$ is almost as small as L_{random} which is the low average path length observed in a random network yet $C(p) \gg C_{\text{random}}$, making the average clustering coefficient higher than in a random network.
- This reduction in characteristic path length is because of the introduction of long range shortcuts between the ordered lattice. Even at a very small value of p, $L(p)$ drops drastically.
- Such networks with very low characteristic path lengths yet average high clustering coefficient are called **small world networks**.

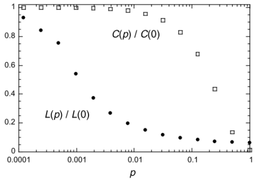


Figure 12: The ratio of average path length $L(p)/L(0)$ drops exponentially while $C(p)/C(0)$ doesn't decrease as quickly, giving the small world regime. Adapted from [9].

- WS analyze real world networks and show that the neural network of the nematode worm C. Elegans has small world characteristics. The characteristic path length observed is 2.65 compared to 2.25 for a random network, yet the average clustering coefficient is high at 0.28 compared to 0.05 for an equivalent random network.

- Further, Shefi et al [7] studied invertebrate neurons cultured in-vitro and study the growth process from isolated neurons as they regenerate and interconnect to form a fully grown 2D network.
- Using optical imaging, the morphology was mapped onto a graph - taking the neurons, the synapses, and synapse-like connections between the neurites of the same neuron, to be the vertices.
- They find that these networks show small world characteristics.

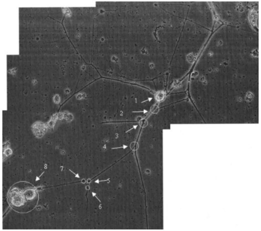


Figure 13: Adapted from [7]

- To extend our description of Hopfield Model for any arbitrary network, we need to take into account that the network isn't fully connected as assumed in the initial model. The structure of the network is encoded in the adjacency matrix c for any graph g such that:

$$c_{ij} = c_{ji} = \begin{cases} 0 & \text{if } \{i, j\} \notin G \\ 1 & \text{if } \{i, j\} \in G \end{cases}$$

- Then, the Hebbian learning rule is modified to nullify the weights for the edges that are not connected:

$$J_{ij}^{g} = \frac{1}{N} \sum_{\mu}^{M} \xi^{\mu} \xi_{j}^{\mu} \cdot c_{ij} \tag{12}$$

- the Hamiltonian is modified to

$$H^{g} = -\frac{1}{2} \sum_{j \neq i}^{N} J_{ij}^{g} S_i S_j \tag{13}$$

Hopfield Model on WS Network:

Methodology and Results

- We use the package *NetworkX* to generate a Watts Strogatz graph for a given N, K and p. Then the training and evolution is undergone as in earlier but with taking into account the adjacency matrix and the updated Hebbian rule.
- On the basis of memory capacity, we can estimate the working region of the associative memory. Then, within this region, we study how the network performance varies with the rewiring probability p. We plot the overlap of the final state after evolution with the randomly chosen *target* state from total memory states M which is defined by the user input.
- We establish that convergence happens for states before the critical memory capacity and determine the number of iterations required for a small world network to converge.
- The initial state is found by introducing noise to a randomly chosen target state from the sample of memory states. We do this in two ways - randomly or sequentially.

Results: Network Generation

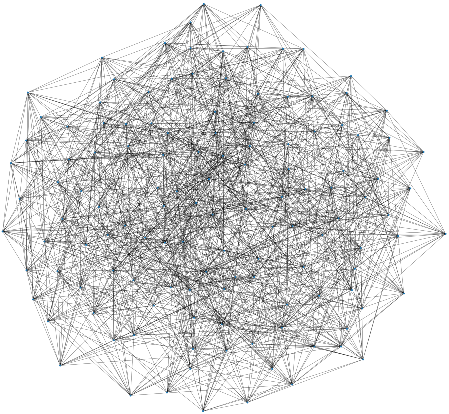
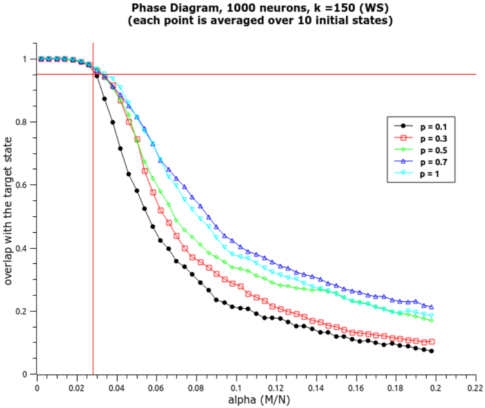


Figure 14: A Watts Strogatz Graph of 128 neurons, with k = 20 and p = 0.5 generated using NetworkX. Blue dots represent nodes and the black lines are edges.

Noise, Ensemble Averaging

- For random noise, we pick up randomly 25% of bits from the target state and flip them. For sequential noise, we choose a starting point in the array of N bits and flip total of $0.25N$ in sequence with it. This allows us to study how the structure of the network plays a role in this error correction.
- Ensemble Averaging: We generate an ensemble of initial states each having either sequential or random noise introduced. This characterizes to capture the behaviour for all the points near a minima in the state space.
- Initially, we choose a dilute connectivity of $k = 0.15N$ for the network. Further, we vary this to understand the effects of further more or less dilution on the performance of the network.
- These experiments allow us to characterize response of the system for all of the state space - for all the points near a minima by ensemble averaging over different initial states near a minima or memory state, and for all different memory states stored in the network.
- With these, we can compare how a small world network performs compared to a random (p=1) or a regular network (as studied earlier).

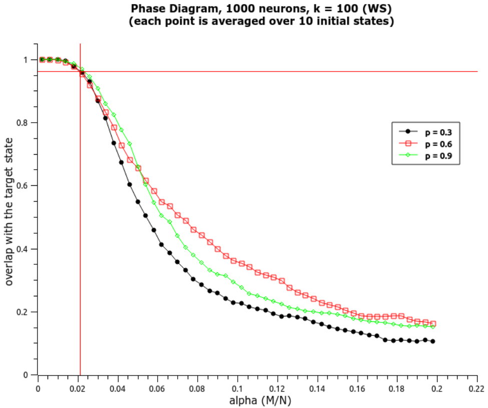
Results: Phase Diagram, N = 1000, k= 150



For N = 1000 neurons with $k = 150$, there is a maximum capacity $\alpha_c^{150} \approx 0.028$ at which the value of overlap is $m = 0.95$ (marked by the red line in the graph).

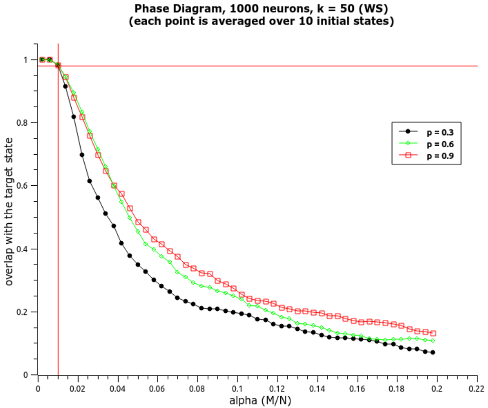
Increasing the rewiring probability reduces the slope of the curve. The shortcut connections in the system provide stability to it and it does not collapse suddenly under increase in stored memory states.

Results: Phase Diagram, N = 1000, k= 100



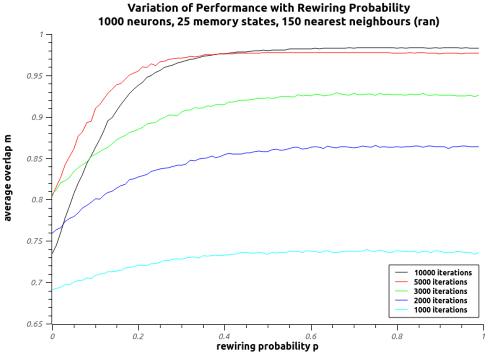
At $k = 100$, we find that the maximum capacity decreases to $\alpha_c^{100} \approx 0.020$ at which the value of overlap is $m = 0.96$ (marked by the red line in graph).

Results: Phase Diagram, N = 1000, k= 50



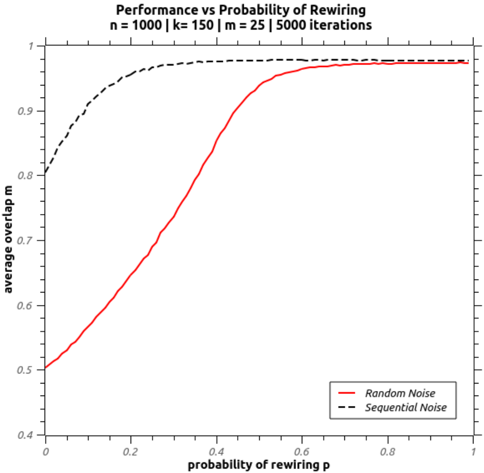
At $k = 50$, we find that the maximum capacity decreases a lot to $\alpha_c^{50} \approx 0.01$ at which the value of overlap is $m = 0.97$ (marked by the red line in graph).

Results: Variation of Performance with Rewiring Probability

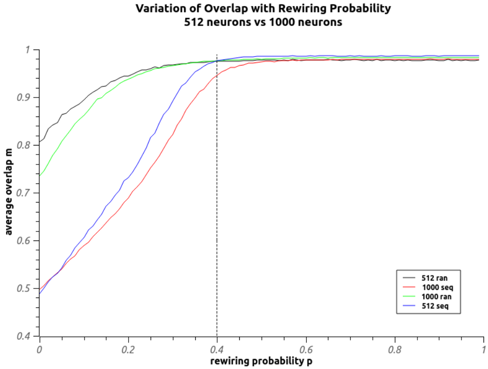


The performance in the case of random errors is way better than that in sequential errors. This is because of the formation of high error domains within the network in sequential noise which do not get corrected easily.

Results: Random Noise vs Seq Noise



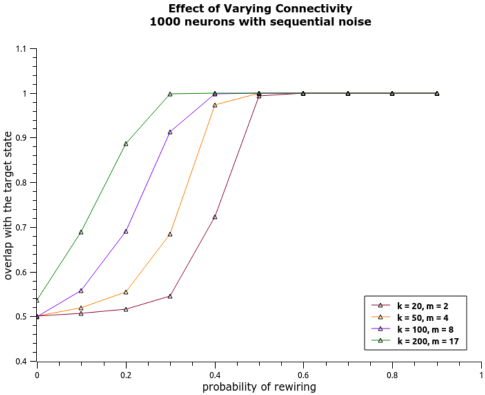
Results: Optimal Rewiring Probability



Finally, irrespective of the size of the system or the nature of the noise, Hopfield Model on a WS network attains this optimal performance well below $p = 1$.

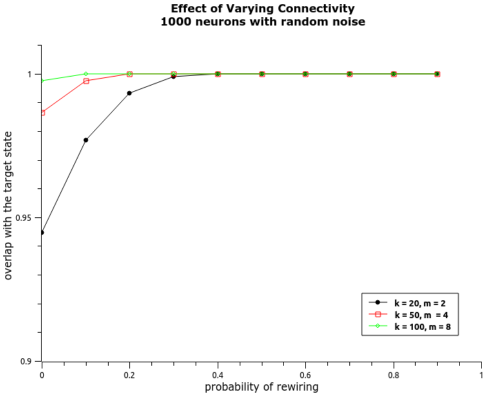
Small world networks obtain equivalent performance as that of random networks with a fraction of connections, and hence are more favourable. This explains the tendency of real world biological networks to show small world characters tics as there is metabolic cost for increasing connections.

Results: Variation of Performance with Connectivity



As the local connectivity is increased, the system attains maximum performance at lower values of rewiring probability p. Thus, having higher local connectivity reduces the need for long range connections to an extent because errors can be resolved due to the inputs from neighbouring neurons.

Results: Variation of Performance with Connectivity



For random noise, changing the value of local connectivity doesn't make much of a difference. All the systems attain optimal performance around $p = 0.4$ irrespective of the local connectivity. However, networks with higher value of local connectivity perform better in the range $p = 0$ to 0.4.

Conclusion

Summary

- The zero temperature phase diagram of regular Hopfield Networks is found to have a critical point around $\alpha_c = 0.14$ in agreement with literature.
- Further, we find that increasing the local connectivity k increased the maximum memory capacity of Hopfield Model on WS networks, and find numerical estimates of the memory capacity based on our simulations.
- We find that small world networks show optimal performance (around $p \approx 0.4$ for $N = 1000$, $k = 150$ and $m = 25$) equivalent to random networks ($p = 1$) at a fraction of wiring length, thus justifying the experimental evidence for existence of small world characteristics in real world biological networks.
- Finally, we study the effect of varying the connectivity k and how it changes the optimal performance of the network. We find that moderately locally connected (moderate K) networks with sparse long range connections (low p) seem to be the most efficient as they have the maximum possible memory capacity, and perform as well as random networks with fraction of the wiring length.

Summary of Findings

Details of the system	Nature of simulations	Results
Hopfield Model on a Regular Network		
$N = 1000$	Zero temeprature phase diagram, probing in the range of $0 < \alpha < 0.2$	$\alpha_c \approx 0.14$
$N = 512$		
$N = 256$		
Hopfield Model on a Watts-Strogatz Network		
$N = 1000, m = 25, k = 150$	Overlap vs p, random and seq noise ensemble of 1000	$p_o \approx 0.4 - 0.5$
$N = 512, m = 12, k = 75$		
$N = 1000, k = 150$	Zero temeprature phase diagram p = 0.3, 0.6 and 0.9, ensemble of 10	$\alpha_c \approx 0.028$
$N = 1000, k = 50$		$\alpha_c \approx 0.1$
$N = 1000, k = 100$		$\alpha_c \approx 0.2$
$N = 1000, k = 200$		$\alpha_c \approx 0.037$
$N = 1000, k = 20, M = 2$	Variation of optimum performance with change in local connectivity k	$p_o \approx 0.6$
$N = 1000, k = 50, M = 3$		$p_o \approx 0.5$
$N = 1000, k = 100, M = 8$		$p_o \approx 0.35$
$N = 1000, k = 200, M = 17$		$p_o \approx 0.3$

Figure 15: Summary of simulations and numerical results. p_o denotes the optimum rewiring probability for which the maximum performance is obtained

Contributions

- We developed an algorithm that can implement Hopfield Network on any given network configuration, and find out the performance as a function of network parameters in the case of WS networks using ensemble averaging over multiple random initial states with random and sequential noise.
- We devise an algorithm that finds the memory capacity of Hopfield Network on any given network structure.
- Although Bohland et al[2] had studied the performance of Hopfield Model on WS networks, our results agree with them for their specific configuration. We further extend their study by studying performance under various configurations of local connectivity, and do a more thorough study based on estimates of memory capacity.
- All the code is original, open source and can be found at: github.com/yashgurbani/watts-hopfield

Scope for Future Work

- The work can be extended to incorporate recent work on Dreaming neural networks where the network operates in awake mode for learning and sleep mode for unlearning consolidating mechanism to remove spurious states. (Fachechi A et al., Neural Networks 112 (2019): 24-40)

- Although we introduce noise randomly or sequentially in the initial state to characterize network's response, we derive our results at zero temperature in this work. A more complete study will take into account the effect of temperature and stochastic noise.

- Although our code can work on any arbitrary network, we have found results primarily for WS networks because of limitations imposed by computational time. This work can be generalized to more types of generative netwroks, and on the connectome of C. Elegans.

Questions?

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