DIGITAL ASSINGMENT-2

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19BEC0740
G2

SIGNALS AND SYSTEMS

1) Suppose $g(t) = x(t)\cos t$ and the fourier transform of the $g(t)$ is $G(j\omega) = \begin{cases} 1, & |\omega| \le 2 \\ 0, & \text{otherwise} \end{cases}$

a) determine $x(t)$

b) specify the fourier transform $X_1(j\omega)$ of a signal $x_1(t)$ such that

$$g(t) = x_1(t)\cos\left(\tfrac{2}{3}t\right)$$

Sol:- a) $w(t) = \cos t$

Apply fourier transform

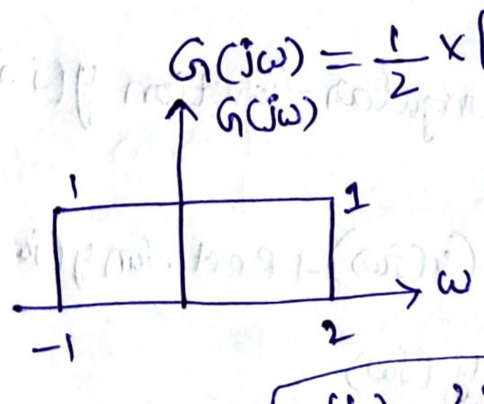
$$W(j\omega) = \pi\,[\delta(\omega-1) + \delta(\omega+1)]$$

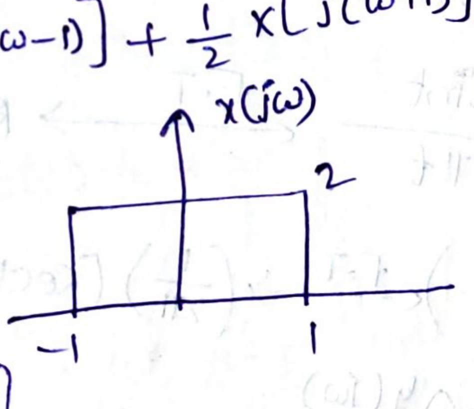
$$g(t) = x(t)\cos t$$

Apply fourier transform

$$G(j\omega) = \frac{1}{2\pi}[X(j\omega) + W(j\omega)]$$

$$G(j\omega) = \frac{1}{2}X[j(\omega-1)] + \frac{1}{2}X[j(\omega+1)]$$





$$\therefore \boxed{x(t) = \frac{2\sin t}{\pi t}}$$

b) $\left(\frac{2\sin t}{\pi t}\right)\cos t = x_1(t)\cos\left(\frac{2}{3}t\right)$

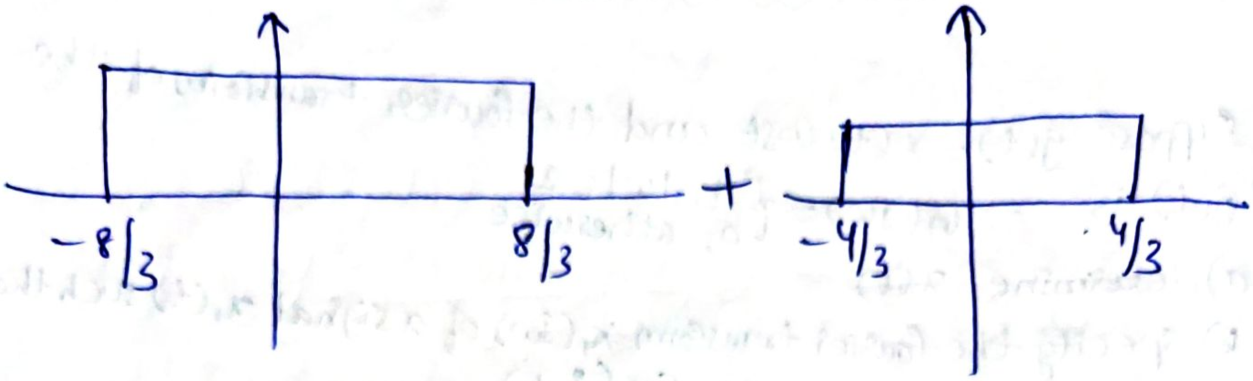
$$x_1(t) = \frac{2\sin t\cos t}{\pi t\cos\left(\frac{2}{3}t\right)} = \frac{\sin 2t}{\pi t\cos\left(\frac{2}{3}t\right)}$$

$$= \frac{2\sin 2t}{\pi t}\left[e^{-2/3t} + e^{2/3t}\right]$$

$X_1(j\omega)$

Apply ~~a~~ fourier transform

$$\left[\frac{2\sin 2t}{\pi t} e^{-2/3 t} + \frac{2\sin 2t}{\pi t} e^{2/3 t}\right]$$



2) a) Determine the fourier transform of given signal

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

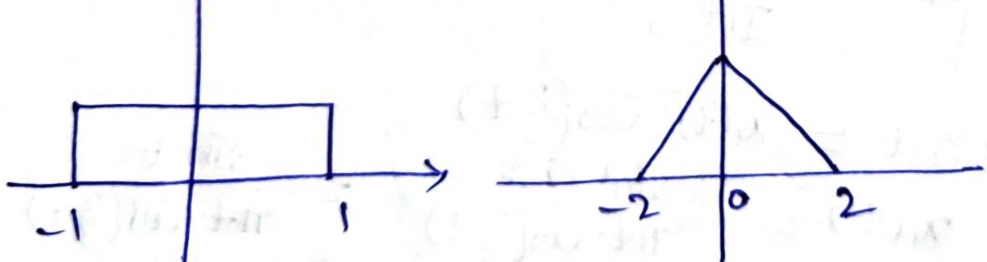
b) use parseval's relation and result of previous part to determine the numerical value of $A = \int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$

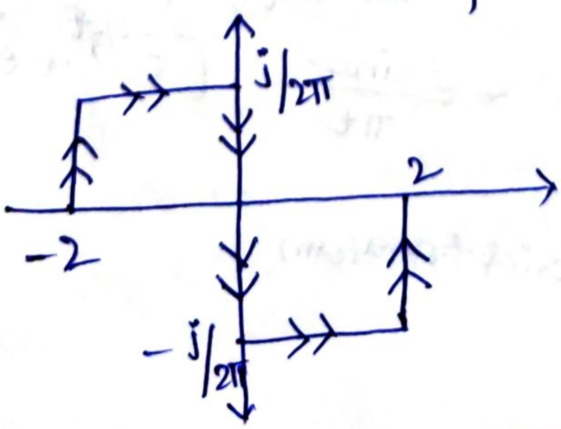
Sol:-

a) fourier transform of $x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$

$$\frac{\sin t}{\pi t} \xleftrightarrow{F.T} \text{Rectangular function } y(j\omega)$$

$$\left(\frac{\sin t}{\pi t}\right) \xleftrightarrow{F.T} \left(\frac{1}{2\pi}\right)\left[\text{Rect.fun}(y(j\omega)) + \text{Rect.fun } y(j\omega)\right]$$





$$t\left(\frac{\sin t}{\pi t}\right)^2 \xleftrightarrow{F.T} X(j\omega) = j\frac{d}{d\omega}[x_1(j\omega)]$$

$$x(j\omega) = \begin{cases} j/2\pi & -2 \le \omega < 0 \\ -j/2\pi & 0 \le \omega < 2 \\ 0 & \text{otherwise} \end{cases}$$

b) Parseval's theorem

$$\int_{-\infty}^{\infty} t^2\left(\frac{\sin t}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |x(j\omega)|^2 d\omega$$

$$= \left(\frac{1}{2\pi^3}\right)$$

3) Compute the fourier transform of each of the following signals

a) $[e^{-\alpha t}\cos\omega_0 t]\,u(t)$, $\alpha > 0$

b) $e^{-3|t|}\sin 2t$

c) $\left(\frac{\sin \pi t}{\pi t}\right)\left(\frac{\sin 2\pi t}{\pi t}\right)$

Sol:- a) $[e^{-\alpha t}\cos\omega_0 t]\,u(t)$, $\alpha > 0$.

$[e^{-\alpha t}\cos\omega_0 t]\,u(t)$

$$= \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t)$$

$$\therefore x(j\omega) = \frac{1}{2(a - j\omega_0 + j\omega)} - \frac{1}{2(a - j\omega_0 + j\omega)}$$

b) $e^{-3|t|}\sin 2t$

$x(t) = e^{-3t}\sin 2t\, u(t) + e^{3t}\sin 2t\, u(-t)$

$x_1(t) = e^{-3t}\sin 2t\, u(t)$

Apply fourier transform

$$x_1(j\omega) = \frac{(1/2j)}{3-2j+j\omega} - \frac{(1/2j)}{3+2j+j\omega}$$

$$x_2(t) = e^{3t}\sin(2t)\,u(-t) = -x_1(-t)$$

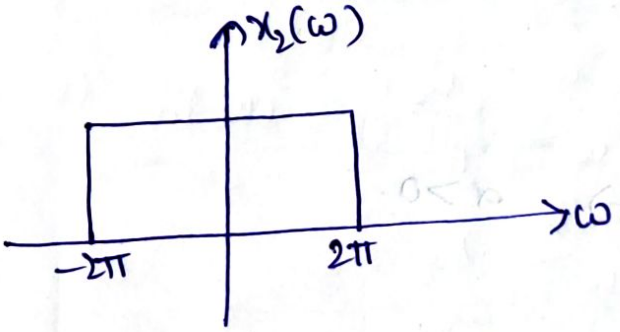
$$x_2(j\omega) = -x_1(-j\omega)$$

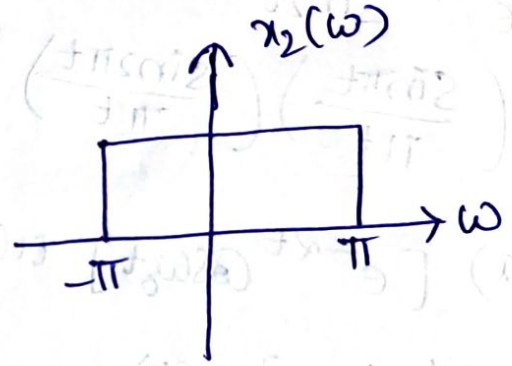
$$x_2(j\omega) = \frac{(1/2j)}{3-2j-j\omega} - \frac{(1/2j)}{3+2j-j\omega}$$

$$x(j\omega) = x_1(j\omega) + x_2(j\omega)$$

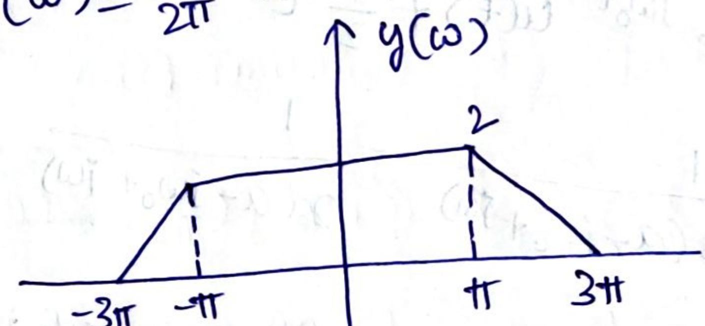
$$\therefore \boxed{x(j\omega) = \frac{3j}{9+(\omega+2)^2} - \frac{3j}{9+(\omega-2)^2}}$$

c) $\left(\frac{\sin \pi t}{\pi t}\right)\left(\frac{\sin 2\pi t}{\pi t}\right)$





$$y(\omega) = \frac{1}{2\pi}\left(x_1(t) * x_2(t)\right)$$



$$y(\omega) = \begin{cases} -\omega/\pi + 3 & \pi \le \omega \le 3\pi \\ 2 & |\omega| = \pi \\ \omega/\pi + 3 & -3\pi \le \omega \le \pi \end{cases}$$

4) Find the Impulse response of a system with the frequency Response

$$H(j\omega) = \frac{(\sin^2(3\omega))\cos\omega}{\omega^2}$$

Sol:-

$$H(j\omega) = \frac{(\sin^2(3\omega))\cos\omega}{\omega^2}$$

$$= \frac{1}{\omega^2}\left(\frac{1-\cos(6\omega)}{2}\right)(\cos\omega)$$

$$= \frac{1}{2\omega^2}\left(\cos\omega - \cos(\omega)\cos(6\omega)\right)$$

$$= \frac{1}{2\omega^2}\left(\cos(\omega) - \frac{1}{2}\cos(7\omega) - \frac{1}{2}\cos(5\omega)\right)$$

$$= \frac{1}{2\omega^2}\cos(\omega) - \frac{1}{4\omega^2}\cos(7\omega) - \frac{1}{4\omega^2}\cos(5\omega)$$

$$H(j\omega) = \frac{1}{2\omega^2}\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) - \frac{1}{4\omega^2}\left(\frac{e^{j7\omega}+e^{-j7\omega}}{2}\right)$$
$$- \frac{1}{4\omega^2}\left(\frac{e^{j5\omega}+e^{-j5\omega}}{2}\right)$$

$$\operatorname{sgn}(t) \longleftrightarrow 2/j\omega$$

$$\operatorname{sgn}(t) * \operatorname{sgn}(t) \longleftrightarrow \left(\frac{2}{j\omega}\right)\left(\frac{2}{j\omega}\right)$$

$$\left[-\sqrt{\frac{\pi}{2}} + \operatorname{sgn}(t) \longleftrightarrow \frac{1}{\omega^2}\right]$$

$$H(j\omega) = \frac{1}{4\omega^2}(e^{j\omega}+e^{-j\omega}) - \frac{1}{8\omega^2}(e^{j7\omega}+e^{-j7\omega})$$
$$- \frac{1}{8\omega^2}(e^{j5\omega}+e^{-j5\omega})$$

By using fourier transform

$$h(t) = \left[\begin{aligned} &\frac{1}{4}\left(-\sqrt{\frac{\pi}{2}}\,(t-1)\,\mathrm{sgn}(t-1) - \sqrt{\frac{\pi}{2}}\,(t+1)\,\mathrm{sgn}(t+1)\right) \\ &-\frac{1}{8}\left(-\sqrt{\frac{\pi}{2}}\,(t-7)\,\mathrm{sgn}(t-7) - \sqrt{\frac{\pi}{2}}\,(t+7)\,\mathrm{sgn}(t+7)\right) \\ &-\frac{1}{8}\left(-\sqrt{\frac{\pi}{2}}\,(t-5)\,\mathrm{sgn}(t-5) - \sqrt{\frac{\pi}{2}}\,(t+5)\,\mathrm{sgn}(t+5)\right) \end{aligned}\right]$$

$$h(t) = \left[\begin{aligned} &\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[(t-1)\,\mathrm{sgn}(t-1) + (t+1)\,\mathrm{sgn}(t+1)\right] + \\ &\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[(t-7)\,\mathrm{sgn}(t-7) + (t+7)\,\mathrm{sgn}(t+7)\right] + \\ &\frac{1}{4}\sqrt{\frac{\pi}{2}}\left[(t-5)\,\mathrm{sgn}(t-5) + (t+5)\,\mathrm{sgn}(t+5)\right] \end{aligned}\right]$$

5) Consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/4)}{(k\pi/4)} \delta\left(t - k\frac{\pi}{4}\right)$$

a) Determine $g(t)$ such that $x(t) = \left(\frac{\sin t}{\pi t}\right) g(t)$

b) use the multiplication property of the fourier transform to argue that $X(j\omega)$ is periodic. Specify $X(j\omega)$ over one period.

Sol:-

a) we way write

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/4)}{(k\pi/4)} \delta(t - k\pi/4)$$

$$= \frac{\sin t}{\pi t} \sum_{k=-\infty}^{\infty} x\,\delta(t - k\pi/4)$$

Therefore $g(t) = \sum_{k=-\infty}^{\infty} x\,\delta(t - k\pi/4)$

b) $G(j\omega) = \frac{\pi\, \delta x}{x/4} \sum_{k=-\infty}^{\infty} \left(\omega - \frac{2\pi k}{x/4}\right)$

$$= \delta x \sum_{k=-\infty}^{\infty} \delta(\omega - 8k)$$

$$x(j\omega) = \frac{1}{2\pi} \left\{ FT\left\{\frac{\sin t}{xt}\right\} \cdot G(j\omega) \right\}$$

if fourier transform $\left\{\frac{\sin t}{xt}\right\}$ by $A(j\omega)$

$$x(j\omega) = \left(\frac{1}{2\pi}\right)\left[A(j\omega)\, \delta x \sum_{k=-\infty}^{\infty} \delta(\omega - 8k)\right]$$

$\therefore$ $X(j\omega)$ may is be viewed as replication of $4A(j\omega)$ every 8 rad/sec

$$A(j\omega) = \begin{cases} 1, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

6) The following facts are given for a particular signal $x[n]$ with fourier $X(e^{j\omega})$

a) $x[n] = 0$; for $n > 0$

b) $x[0] > 0$

c) $Im\{X(e^{j\omega})\} = \sin(\omega) - \sin(2\omega)$

d) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 = 3$

Determine $x[n]$

Sol:- The fourier transform of the signal $x[n]$ is $X(e^{j\omega})$

from Conjugation and Conjugate Symmetry

If $x[n]$ is real valued function then its transform $X(e^{j\omega})$ is Conjugate symmetric

$$odd\{x[n]\} \xleftrightarrow{F.T} j\,Im[X(e^{j\omega})]$$

Consider the following fact about function $x[n]$

$$\rightarrow Im[X(e^{j\omega})] = \sin\omega - \sin 2\omega$$

$$j\,Im[X(e^{j\omega})] = j\sin\omega - j\sin 2\omega$$

$$j\,Im[X(e^{j\omega})] = j\left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right) - j\left(\frac{e^{j2\omega} - e^{-j2\omega}}{2j}\right)$$

$$j\,Im[X(e^{j\omega})] = \frac{1}{2}\left(e^{j\omega} - e^{-j\omega} - e^{2j\omega} + e^{-2j\omega}\right)$$

Oddpart of the function $x[n]$,

$$odd\{x[n]\} = F^{-1}\{j\,Im\{X(e^{j\omega})\}\}$$

$$= \frac{1}{2}\left(\delta(n+1) - \delta(n-1) - \delta(n+2) + \delta(n-2)\right)$$

$$odd\{x[n]\} = \frac{x[n] - x[-n]}{2}$$

Given, $x[n] = 0$ for $n > 0$

$$\therefore x[n] = 2\,odd\{x[n]\}$$

$$x[n] = \delta[n+1] - \delta[n+2] \text{ for } n < 0$$

Using Parseval's theorem

$$x[0] = \frac{1}{2}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$= \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$= \sum_{n=-\infty}^{-1} \left[|x[n]|^2 + |x[0]|^2\right] + \sum_{n=0}^{\infty} |x[n]|^2$$

Using the condition $x[n]=0$ for $n>0$

$$\therefore \sum_{n=0}^{\infty} |x[n]|^2 = 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{-1} |x[n]|^2 + |x[0]|^2$$

$$\left(\because \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 3\right)$$

$$3 = |x[0]|^2 + \sum_{n=-\infty}^{-1} |x[n]|^2$$

$$3 = |x[0]|^2 + (1)^2 + (-1)^2$$

$$3 = |x[0]|^2 + 2$$

$$x[0] = \pm 1$$

$\therefore$ Consider $x[0] = 0$ as $x[0] > 0$

$x[n] = \delta[n]$ for $n = 0$

function for $x[n]$

$$x[n] = \delta[n+1] - \delta[n+2] \quad \text{for } n<0$$
$$= \delta[n] \quad \text{for } n=0$$

∴ This function $x[n]$ is the sum of three impulsive functions

The signal $x[n]$ is ∴ $\boxed{\delta(n) + \delta[n+1] - \delta[n-2]}$

7) Consider a system consisting of the cascade of two LTI system with frequency responses

$$H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \quad \text{and} \quad H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}$$

a) find the difference equation describing the overall systems

b) Determine the impulsive response of the overall system.

Sol:-

a)
$$H_0(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

$$= \left(\frac{2 - e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}\right)$$

$$= \frac{(2 - e^{-j\omega})}{(1 + \frac{1}{2}e^{-j\omega})(1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega})}$$

$$H(e^{j\omega}) = \frac{(2 - e^{-j\omega})}{(1 \cancel{- \frac{1}{2}e^{-j\omega}} \cancel{+ \frac{1}{4}e^{-j2\omega}} \cancel{+ \frac{1}{2}e^{-j\omega}} \cancel{- \frac{1}{4}e^{-j2\omega}} + \frac{1}{8}e^{-j3\omega})}$$

$$H(e^{j\omega}) = \frac{(2 - e^{-j\omega})}{(1 + \frac{1}{8}e^{-j3\omega})}$$

Substitute $\frac{Y(e^{j\omega})}{X(e^{j\omega})}$ for $H(e^{j\omega})$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2-e^{-j\omega}}{1+\frac{1}{8}e^{-j3\omega}}$$

$$Y(e^{j\omega})\left(1+\frac{1}{8}e^{-j3\omega}\right) = (2-e^{-j\omega})X(e^{j\omega})$$

$$\left(Y(e^{j\omega})+\frac{1}{8}e^{-j3\omega}Y(e^{j\omega})\right) = \left(2X(e^{j\omega})-e^{-j\omega}X(e^{j\omega})\right)$$

Apply inverse fourier transform

$$y[n]=\frac{1}{8}y[n-3] = 2x[n]-x[n-1]$$

Thus, the difference equation of overall system is

$$\boxed{y[n]+\frac{1}{8}y[n-3] = 2x[n]-x[n-1]}$$

b)
$$H(e^{j\omega}) = \left(\frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}}\right)$$

$$H(e^{j\omega}) = \left(\frac{2e^{j\omega}-1}{e^{j\omega}+\frac{1}{2}}\right)\left(\frac{e^{j2\omega}}{e^{j2\omega}-\frac{1}{2}e^{j\omega}+\frac{1}{4}}\right)$$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{(2e^{j\omega}-1)e^{j\omega}}{(e^{j\omega}+\frac{1}{2})\left(e^{j\omega}-\left(\frac{1}{4}+j\frac{\sqrt{3}}{4}\right)\right)\left(e^{j\omega}-\left(\frac{1}{4}-j\frac{\sqrt{3}}{4}\right)\right)}$$

By using partial fractions.

$$H(e^{j\omega}) = \left(\frac{A}{e^{j\omega}+\frac{1}{2}}\right)+\left(\frac{B}{e^{j\omega}-\frac{1}{2}\left(\frac{1}{2}+\frac{j\sqrt{3}}{2}\right)}\right)+\left(\frac{C}{e^{j\omega}-\frac{1}{2}\left[\frac{1}{2}-\frac{j\sqrt{3}}{4}\right]}\right)$$

By simplify

$A = 4/3 \quad B = 0.333 + 0.577j \quad C = 0.333 - 0.577j$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \left(\frac{4/3}{e^{j\omega}+1/2}\right) + \left(\frac{0.333+0.577j}{e^{j\omega} - \frac{1}{2}\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}\right)$$
$$+ \left(\frac{0.333-0.577j}{e^{j\omega} - \frac{1}{2}\left(-\frac{1}{2}-j\frac{\sqrt{3}}{4}\right)}\right)$$

$$\left[\frac{1}{2} + j\frac{\sqrt{3}}{2} = 60^\circ\right]$$

$$H(e^{j\omega}) = \left(\frac{4/3}{1+\frac{1}{2}e^{-j\omega}}\right) + \left(\frac{0.333+0.577j}{1-\frac{1}{2}e^{-j\omega}(e^{j60^\circ})}\right)$$
$$+ \left(\frac{0.333-0.577j}{1-\frac{1}{2}e^{-j\omega}(e^{-j60^\circ})}\right)$$

$$\left[0.333+0.577j = \frac{1+j\sqrt{3}}{3}\right]$$

$$H(e^{j\omega}) = \left(\frac{4/3}{1+\frac{1}{2}e^{-j\omega}}\right) + \left(\frac{1+j\sqrt{3}}{3}\right)\left(\frac{1}{1-1/2\, e^{j\omega}(e^{j60^\circ})}\right)$$
$$+ \left(\frac{1-j\sqrt{3}}{3}\right)\left(\frac{1}{1-1/2\, e^{-j\omega}(e^{-j60^\circ})}\right)$$

Inverse fourier transform

$$h(n) = \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n] + \left(\frac{1+j\sqrt{3}}{3}\right)\left(\frac{1}{2}e^{j60^\circ}\right)^n u[n]$$
$$+ \left(\frac{1-j\sqrt{3}}{3}\right)\left(\frac{1}{2}e^{-j60^\circ}\right)^n u[n]$$