UNIT - 2

Date / /
Page

1-9-21

NUMERICAL ANALYSIS

* Interpolation:

x	:	x_0	x_1	x_2	------	x_n
$y = f(x)$	:	$f(x_0)$	$f(x_1)$	$f(x_2)$	----	$f(x)$

Interpolation is finding $f(x)$ $\forall$ x is between the given range.

eg - if we have $x = 1, 2, 3 \ldots n$.

then finding $f(x = 1.5)$.

* Interpolation with evenly spaced points:

⇒ Finite difference Operators:

$$(x_i, f(x_i)), \quad \boxed{x_i = x_0 + ih}, \quad i = 0, 1, 2 \ldots n$$

① Shift operator E:

$$E(f(x_i)) = f(x_i + h)$$
$$= f(x_{i+1})$$

→

$$E(f(x_0)) = f(x_1) = f(x_0+h)$$

$$E(f(x_1)) = f(x_2) = f(x_0+2h) = f(x_1+h)$$

$$\Rightarrow \quad E^2 f(x_i) = E[E(f(x_i))]$$

$$= E(f(x_i+h)) = \boxed{\begin{aligned} &f(x_i+2h) \\ =\ &f(x_{i+2}) \end{aligned}}$$

$\Rightarrow$ So $E^K f(x_i) = f(x_i + Kh) = f(x_{i+K})$

$\Rightarrow$ If $K = \frac{1}{2}$,

$$E^{1/2}(f(x_i)) = f\left(x_i + \frac{h}{2}\right) = f\left(x_{i+\frac{1}{2}}\right)$$

Date / /
Page

(2) Forward difference Operator (Δ):

$$\boxed{\Delta f(x_i) = f(x_i + h) - f(x_i) = f_{i+1} - f_i}$$

$$f_{(x_{i+1})} - f_{(x_i)}$$

$$\Delta f(x_0) = f(x_1) - f(x_0)$$

$$\Delta f(x_1) = f(x_2) - f(x_1)$$

$$\Rightarrow \Delta^2 f(x_i) = \Delta\left[f(x_{i+1}) - f(x_i)\right]$$

$$= f(x_{i+2}) - f(x_{i+1}) - f(x_{i+1}) + f(x_i)$$

$$= \boxed{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}$$

$$\Rightarrow \Delta^3 f(x_i) = f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)$$

$$\therefore \Delta f(x_i) = f(x_i + h) - f(x_i) = E f(x_i) - f(x_i)$$

$$\Delta f(x_i) = (E - 1) f(x_i)$$

$$\therefore \boxed{\Delta = E - 1}$$

$\rightarrow$

$$\therefore \Delta^n f(x_i) = (E-1)^n f(x_i)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!} \left(f(x_i + n - k)\right)$$

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	$f(x_0)$	$\Delta f_0 = f_1 - f_0$		
			$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	
x_1	$f(x_1)$	$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
			$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	
x_2	$f(x_2)$	$\Delta f_2 = f_3 - f_2$		
x_3	$f(x_3)$			

$\Rightarrow$

Δ

Date / /
Page

③ Backward Difference Operator (∇):

$\Rightarrow \nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1}$

$\Rightarrow \nabla f(x_1) = f(x_1) - f(x_0)$

$\Rightarrow \nabla^2 f(x_i) = \nabla(\nabla f(x_i))$

$= f(x_i) - 2f(x_{i-1}) + f(x_{i-2})$

$\Rightarrow \nabla^3 f(x_i) = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}$

$\Rightarrow \nabla f(x_i) = f(x_i) - f(x_i - h)$

$= \boxed{(1 - E^{-1}) f(x_i)}$

$\therefore \boxed{\nabla = 1 - \frac{1}{E}}$ or $\boxed{E = \frac{1}{1-\nabla}}$

$\Rightarrow \boxed{\nabla^n f(x_i) = (1 - E^{-1})^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k}}$

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	$f(x_0)$			
		$\nabla f_1 = f_1 - f_0$		
x_1	$f(x_1)$		$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	
		$\nabla f_2 = f_2 - f_1$		$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$
x_2	$f(x_2)$		$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$	
		$\nabla f_3 = f_3 - f_2$		
x_3	$f(x_3)$			

(4) Central difference operator (δ):

$\Rightarrow$ $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

$$= \boxed{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}$$

$\Rightarrow$ $\delta\left(f\left(x_i + \frac{h}{2}\right)\right) = \delta f_{i+\frac{1}{2}} = f(x_i + h) - f(x_i)$

$$= \underline{f_{i+1} - f_i}$$

$\delta(f_{1/2}) = f_1 - f_0$

$\delta(f_{3/2}) = f_2 - f_1$

$\Rightarrow$ $\delta^2 f(x_i) = \delta(\delta(f(x_i)))$

$= \boxed{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}$

$\Rightarrow$ $\delta^3 f(x_i) = \boxed{f\left(i+\tfrac{3}{2}\right) - 3f\left(i+\tfrac{1}{2}\right) + 3f\left(i-\tfrac{1}{2}\right) - f\left(i-\tfrac{3}{2}\right)}$

$$\boxed{\delta \equiv E^{1/2} - E^{-1/2}}$$

$\Rightarrow$ $\delta^n f(x_i) = ?$

$\delta^n f(x_i) = (E^{1/2} - E^{-1/2})^n f(x_i)$

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	
x_0	$f(x_0)$			
		$\delta f_{1/2} = f_1 - f_0$		
x_1	$f(x_1)$		$\delta^2 f_1 = \delta f_{3/2} - \delta f_{1/2}$	
		$\delta f_{3/2} = f_2 - f_1$		$\delta^3 f_{3/2} = \delta^2 f_2 - \delta^2 f_1$
x_2	$f(x_2)$		$\delta^2 f_2 = \delta f_{5/2} - \delta f_{3/2}$	
		$\delta f_{5/2} = f_3 - f_2$		
x_3	$f(x_3)$			

Date / /
Page

Relations:

1. $$\boxed{\Delta^n f_i \equiv \nabla^n f_{i+n} \equiv \delta^n f_{i+\left(\frac{n}{2}\right)}}$$

$\therefore \nabla = 1 - E^{-1} = (E - 1)E^{-1} = \Delta E^{-1}$

Proof:

$\Rightarrow \nabla^n f_{i+n} = \Delta^n E^{-n} f_{i+n} = \Delta^n f_i$

$\Rightarrow \delta = (E^{1/2} - E^{-1/2})$

$\therefore \delta = (E - 1) E^{-1/2} = \Delta E^{-1/2}$

$\therefore \delta^n f_{i+\frac{n}{2}} = \Delta^n E^{-n/2} f_{i+\frac{n}{2}} = \boxed{\Delta^n f_i}$

$\Rightarrow$ **Remark:**

let $P_n(x) = a_0 x^n + a_1 x^{n-1} \cdots\cdots a_n$

degree $= n$

$\Delta^k P_n(x) = ?$ $\boxed{k > n}$

$\Delta P_n(x) = \left(P_n(x+h) - P_n(x)\right)$ degree $= n - 1$

$\Delta^2 P_n(x) \Rightarrow n - 2$ degree.

$\therefore \boxed{\Delta^k P_n(x) = 0}$

$\therefore \Delta^k P_n(x) = 0 \quad , k > n$

$= a_0\, n! \quad , k = n$

Same for ∇:

$\nabla^k P_n(x) = 0$ for $k > n$

$= a_0\, n!$ for $k = n$

~~★ Mean~~

⑤ Mean operator (μ):

$$\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[f_{i+1/2} + f_{i-\frac{1}{2}}\right]$$

$$\therefore \boxed{\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)}$$

Date / /
Page ____

eg- $\Delta^3 (1-2x)(1-3x)(1-4x)$

$\Rightarrow \Delta^3(\underbrace{-24}_{a_0}x^3 - - - - - .)$

$\because$ degree $= 3$

& $k = 3$.

$\therefore \; = a_0 \, n!$

$= -24 \times 3!$

$= \boxed{-144}$

Relation:

(2) (i) $\delta = \nabla(1-\nabla)^{-1/2}$ (ii) $\mu = \left[1 - \frac{\delta^2}{4}\right)^{1/2}$

(iii) $\Delta(f_i)^2 = (f_i + f_{i+1})\Delta f_i$

Proof :

$\Delta(f_i^2) = f_{i+1}^2 - f_i^2$.

$\cancel{g(x) = f^2(x)}$

$\cancel{\Delta g(x_i) = (\Delta f(x_i))^2}$

$\Longrightarrow$

Date / /
Page

here, let

$$g(x_i) = [f(x)]^2$$

$$\Delta g(x_i) = \Delta (f(x_i))^2$$

$$= g(x_{i+1}) - g(x_i)$$

$$= (f_{i+1})^2 - (f_i)^2$$

$$\therefore \Delta (f_i^2) = (f_{i+1})^2 - f_i^2$$

$$= (f_{i+1} + f_i)(f_{i+1} - f_i)$$

$$= (f_{i+1} + f_i) \times \Delta f_i$$

Relation :

(3) $$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x)\cdot g(x+h)}$$

Proof :

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{g(x)\,f(x+h) - f(x)\,g(x+h)}{g(x+h)\cdot g(x)}$$

$$= \frac{g(x)\left[f(x+h) - f(x)\right] - f(x)\left[g(x+h) - g(x)\right]}{g(x)\cdot g(x+h)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x)\cdot g(x+h)}$$

$\Rightarrow$

Date / /
Page ______

☆ Newton's Forward Difference Interpolation Formula:

let $y = f(x)$. $\forall$ x are equidistant.

$$\Rightarrow F(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{1!\,(h)}$$

$$+ (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!\,h^2}$$

~~$+ (x-x_0)(x-x_1)\ldots$~~

$$+ \ldots\ldots$$

$$+ (x - x_0)(x - x_1)\ldots(x - x_{n-1})\frac{\Delta^n f_0}{n!\,h^n}$$

n degree

$x_1 = x_0 + h$
$x_2 = x_0 + 2h$
$\vdots$
$x_n = x_0 + n!$

and let $\boxed{x = x_0 + sh}$

$$\therefore (x - x_i) = x_0 + sh - (x_0 + ih)$$

$$= (s - i)h$$

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$\therefore x - x_0 = sh$

$x - x_1 = (s-1)h$

$\Rightarrow f(x) = f(x_0 + sh)$

$$\therefore f(x) = E^s f(x_0)$$
$$= (\Delta+1)^s f(x_0)$$
$$= \left({}^sC_0 + {}^sC_1 \Delta + {}^sC_2 \Delta^2 \dots\right) f(x_0)$$
$$= {}^sC_0 f(x_0) + {}^sC_1 \Delta f_0 + {}^sC_2 \Delta^2 f_0 \dots$$
$$= f(x_0) + \frac{(x-x_0)}{h} \Delta f_0 + \dots$$

$$f(x) - P_n(x) = E_n(f, x)$$

$$E_n(f, x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{n+1}(\xi)$$

error

$0 < \xi < n$

$f(\xi) = f(x_\xi)$

$\Rightarrow$

$x = x_0 + sh$

$$f(x) = f(x_0) + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0$$

$$+ \cdots \frac{s(s-1)(s-2)\cdots(s-n+1)}{n!}\Delta^n f_0$$

NFDI :

1. x's ~~negative~~ are near the beginning of the nodal points.

2. applicable for evenly spaced x's.

$$x_i - x_{i-1} = h, \quad \forall\ i = 1, 2, \ldots n$$

3. has permanence property:

if you add another term, the whole expression remains same (permanent) just one new term is added.

Date / /
Page

Newton's Backward difference Interpolation Formula :

$$\Rightarrow f(x) = f(x_n) + \frac{(x-x_n)}{1!\,h} \nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2} \nabla^2 f(x_n)$$

$$\vdots$$

$$+ \frac{(x-x_n) \cdots (x-x_1)}{n!\,h^n} \nabla^n f(x_n).$$

$x = x_n + sh$, $s < 0$

$$\therefore f(x) = f(x_n) + s \nabla f(x_n) + \frac{s(s+1)}{2!} \nabla^2 f(x_n) \cdots + \frac{s(s+1)\cdots(s+n-1)}{n!} \nabla^n f(x_n)$$

error

$$E_n(f, x) = f(x) - P_n(x)$$

$$= \frac{(x-x_n)(x-x_{n-1}) \cdots (x-x_0)}{(n+1)!} f^{n+1}(\xi)$$

where $0 < \xi < n$

Q) construct forward diff table for x & $f(x)$

Ans)

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
-1	-8			
		11		
0	3		-13	
		-2		26
1	1		13	
		11		
2	12			

Q) " backward " " for x & $f(x)$

Ans)

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
-1	-8			
		-11		
0	3		-13	
		2		-26
1	1		13	
		-11		
2	12			

Ex) Find $f(2.4)$.

x	2	4	6	8	10
$f(x)$	9.68	10.96	12.32	13.76	15.28

Ans.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	9.68		
		1.28	
4	10.96		0.08
		1.36	
6	12.32		0.08
		1.44	
8	13.76		0.08
		1.52	
10	15.28		

$$f(2.4) = 9.68 + \frac{(2.4-2)\ (1.28)}{2} + \frac{(2.4-2)(2.4-4)}{2 \times 2^2} \times 0.08$$

$$f(2.4) = 9.68 + 0.256 - \frac{(0.4)(1.6)\,(\cancel{0.08})^{0.01}}{\cancel{8}}$$

$$\boxed{f(2.4) = 9.9296}$$

Q) Find $f(1.0)$:

⇒

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
0.1	−1.699			
0.3	−1.073			
0.5	−0.375			
0.7	0.443			
0.9	1.429			
1.1	2.631			

Central difference Interpolation (Sterling):

$$\delta f(x_r) = f\left(x_{r+\frac{1}{2}}\right) - f\left(x_{r-\frac{1}{2}}\right)$$

$$\mu f(x_r) = \frac{1}{2}\left[f\left(x_{r+\frac{1}{2}}\right) + f\left(x_{r-\frac{1}{2}}\right)\right]$$

$$\boxed{\mu = \frac{E^{-1/2} + E^{1/2}}{2}}$$

~~$f(x_{r_0})$~~ $f(x_r + kh)$ → for 2.2 in 1, 2, 3 . . .

$x_r = 2$, $k = 0.2$, $h = 1$

x	$f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$	$\delta^4 f(x)$
x_0	$f(x_0)$				
		$\delta f_{1/2}$			
x_1	$f(x_1)$				
$\vdots$					
x_{r-2}	$f(x_{r-2})$				
		$\delta f_{r-3/2}$			
x_{r-1}	$f(x_{r-1})$		$\delta^2 f_{r-1}$		
		$\boxed{\delta f_{r-1/2}}$		$\boxed{\delta^3 f_{r-1/2}}$	
x_r	$f(x_r)$		$\delta^2 f_r$		$\delta^4 f_r$
		$\boxed{\delta f_{r+1/2}}$		$\boxed{\delta^3 f_{r+1/2}}$	
x_{r+1}	$f(x_{r+1})$		$\delta^2 f_{r+1}$		
		$\delta f_{r+3/2}$			

$$\mu \delta f(x_r) = \frac{E^{-1/2} + E^{1/2}}{2} \delta f_r$$

$$= \frac{1}{2} \left[\delta f_{r-1/2} + \delta f_{r+\frac{1}{2}} \right]$$

$$\mu \delta^3 f_r = \frac{1}{2} \left[\delta^3 f_{r-\frac{1}{2}} + \delta^3 f_{r+\frac{1}{2}} \right]$$

e^x

x	$f(x)$	δ	δ^2	δ^3	δ^4
0.1	1.1052				
		0.7169			
0.6	1.8221		0.4652		
		1.1821		0.3015	
1.1	3.0042 ~~4.9530~~		0.7667		~~[illegible]~~ 0.1962
		1.9488		0.4977	
1.6	4.9530		1.2644		
		3.2132			
2.1	8.1662				

$f(1.3) = ?$

Date / /
Page

$$\Rightarrow f(x_r + kh) = \left[1 + k\mu\delta + \frac{k^2}{2!}\delta^2 + \frac{k(k-1)(k+1)}{3!}\mu\delta^3 + \frac{k^2(k^2-1)}{4!}\delta^4 \ldots\ldots\right] \times f(x_r)$$

Ans) here $x_r = 1.1$, $h = 0.5$, $k = \frac{x - x_r}{h} = 0.4$

$$f(1.3) = 3.0042 + (0.4)\left[\frac{1}{2}(1.1821 + 1.9488)\right] + \frac{(0.4)^2}{2} \times 0.7667 + \frac{(0.4)(0.4-1)(0.4+1)}{3!}\left[\frac{1}{2}(0.3015 + 0.4977)\right] + \frac{(0.4)^2((0.4)^2-1)}{4!}(0.1962)$$

$$\boxed{f(1.3) = 3.6682}$$

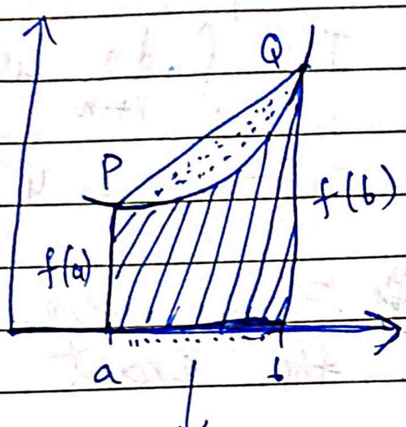
$$\Rightarrow f(x_r + kh) = \left[1 + k\mu\delta + \frac{k^2}{2!}\delta^2 + \frac{k(k-1)(k+1)}{3!}\mu\delta^3 + \frac{k^2(k^2-1)}{4!}\delta^4 \ldots\ldots\right] \times f(x_r)$$

NUMERICAL INTEGRATION:

$$I = \int_a^b f(x)\,dx$$

$$\simeq \boxed{\frac{b-a}{2}\left[f(a) + f(b)\right]}$$

Trapezoidal Rule.



$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$\vdots$

$b = x_n = x_0 + nh.$

$$\therefore \int_a^b f(x)\,dx$$

$$= \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx \cdots \int_{x_{n-1}}^{x_n} f(x)\,dx$$

By Trapezoidal Rule, $= \frac{h}{2}\left[\{f(x_0) + f(x_1)\} + \{f(x_1) + f(x_2)\} \cdots \{f(x_{n-1}) + f(x_n)\}\right]$

Composite Trapezoidal Rule. $\boxed{= \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) \cdots f(x_{n-1})\} + f(x_n)\right]}$

Date / /
Page

Q) Find the approximate value of
$I = \int_0^1 \frac{dx}{1+x}$ using the trapezoidal rule.
with 2, 4 and 8 equal sub intervals.
~~Ans~~ Find the absolute errors using the exact solution.

Ans) Exact $sol^n = \int_0^1 \frac{1}{1+x}\, dx$

$= \left[\log(1+x)\right]_0^1$

$= \boxed{\log 2} = \boxed{0.6931}$

(i) $N = 2$

$h = \frac{1-0}{2} = \frac{1}{2}$, Nodes $= 0, 0.5, 1$

$\Rightarrow \frac{1/2}{2}\left(f(0) + 2f(0.5) + f(1)\right)$

$= \frac{1}{4}\left(1 + \frac{4}{3} + \frac{1}{2}\right) = \frac{17}{24} = \boxed{0.7083}$

$\Rightarrow$

$\cancel{0.8 + 0.667 + 0.571}$

$\cancel{1.538}$

(ii) $N = 4$,

$h = \frac{1-0}{4}$, Nodes $= 0, 0.25, 0.5, 0.75, 1$

$\Rightarrow \frac{1/4}{2}\left[1 + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) + \frac{1}{2}\right]$

$\Rightarrow \frac{1}{8}\left[1.5 + 2(2.038)\right]$

$\Rightarrow \frac{5.576}{8} \Rightarrow \boxed{0.6977}$

(iii) $N = 8$

$h = \frac{1-0}{8}$, Nodes, $0, 0.125, 0.25, 0.375, 0.5$
$0.625, 0.75, 0.875, 1.$

$\Rightarrow \frac{1}{16}\left[1 + 2\left[\frac{8}{9} + \frac{4}{5} + \frac{8}{11} + \frac{2}{3} + \frac{8}{13} + \frac{4}{7} + \frac{8}{15}\right] + \frac{1}{2}\right]$

$\Rightarrow \frac{1}{16}\left[1.5 + 2(4.8028)\right]$

$\Rightarrow \boxed{0.6941}$

$|I_{N=2} - I| = |0.7083 - 0.6931| = \boxed{0.0152}$

$|I_{N=4} - I| = |0.6970 - 0.6931| = \boxed{0.0039}$

$|I_{N=8} - I| = |0.6941 - 0.6931| = \boxed{0.001}$

Date / /
Page ______

Q) Evaluate $I = \int_1^2 \frac{dx}{5+3x}$ with 4 and 8 sub intervals using trapezoidal rule. Find errors.

Ans) $I = \int_1^2 \frac{1}{5+3x}\, dx$

$5x = t$
$5dx = \frac{dt}{3}$

$= \frac{1}{3} \left| \log(5+3x) \right|_1^2$

$= \frac{1}{3} \left[\log 11 - \log 8 \right]$

$= \frac{1}{3} (2.397895 - 2.079441)$

$= \boxed{0.106151}$

$N = 2$

$h = \frac{2-1}{2} = \boxed{\frac{1}{2}}$, Nodes, $(1, 1.5, 2)$

$\therefore I_{N=2} = \frac{1}{4} \left[0.125 + 2(0.1052) + 0.0909 \right]$

$= \frac{1}{4} \left[0.125 + 0.2104 + 0.0909 \right]$

$$\boxed{I_{N=2} = 0.106575}$$

$$I_{N=4} = \frac{1}{8}\left[0.125 + 2\left(0.1142 + 0.1052 + 0.0975\right) + 0.0909\right]$$

$$= \frac{1}{8}\left[0.125 + 0.6338 + 0.0909\right]$$

$$\boxed{= 0.10627}$$

$$\boxed{error = 0.00012}$$

$$I_{N=8} = \frac{1}{16}\left[0.125 + 0.0909 + 2\left(0.119 + 0.114 + 0.109 + 0.1052 + 0.101 + 0.0975 + 0.0945\right)\right]$$

$$= \frac{1}{16}\left[0.2159 + 2\left(0.7398\right)\right]$$

$$= \boxed{0.10618}$$

$$error = \left|1.06151 - 1.0618\right|$$

$$= \cancel{0.000183} = (0.00003)$$

Date / /
Page

Order = P

polynomial of degree $\leq$ P

$\Rightarrow$ Trapezoidal rule $\rightarrow$ 1

$\Rightarrow$ Composite Trapezoidal rule $\rightarrow$ 1

$\Rightarrow \boxed{P = 1}$

(2 dimensions)

★ Simpson's $1/3^{rd}$ rule: $\longrightarrow$ 3

p = 3

$\therefore f(x) = 1, x, x^2, x^3$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2}\Big\{ f(x_0) + \frac{1}{h}(x - x_0)\,\Delta f(x_0) + \frac{1}{2h^2}(x - x_0)(x - x_1)\,(\Delta^2 f(x_0))$$

$$= (x_2 - x_0)\,f(x_0) + \left(\frac{1}{h}\left[\frac{1}{2}(x - x_0)^2\right]_{x_0}^{x_2} \Delta f(x_0)\right)$$

$$+ \text{I}$$

$$= 2h\,f(x_0) + \frac{1}{h} \times 2h^2\,\Delta f(x_0) + \frac{h}{3}\Delta^2 f(x_0)$$

$$= \frac{h}{3}\Big\{ 6h\,f(x_0) + 6\,(f(x_1) - f(x_0)) + \big[f(x_0) - 2f(x_1) + f(x_2)\big]\Big]$$

$$\Rightarrow \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$\cancel{\Rightarrow \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]}$$

★ Composite Simpson's 1/3rd Rule:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_{2N}} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots \int_{x_{2N-2}}^{x_{2N}} f(x)\,dx.$$

$$\Rightarrow \frac{h}{3}\left\{\{f(x_0) + 4f(x_1) + f(x_2)\} + \{f(x_2) + 4f(x_3) + f(x_4)\} + \cdots \{f(x_{2N-2}) + 4f(x_{2N-1}) + f(x_{2N})\}\right\}$$

$$\Rightarrow \frac{h}{3}\left\{f(x_0) + 4\{f(x_1) + f(x_3) \cdots f(2N-1)\} + 2\{f(x_2) + f(x_4) \cdots f(x_{2N-2})\} + f(x_{2N})\right\}$$

Both Simpson's rules are of order 3.

eg – $f(x) = x^3$. by Simpson

$$\text{error}(f, x) = \int_a^b x^3\,dx - \left(\frac{b-a}{6}\right)\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right]$$

where $a = x_0$, $\frac{a+b}{2} = x_1$, $b = x_2$

$$= \frac{b^4 - a^4}{4} - \frac{b^4 - a^4}{4} = \boxed{0}$$

---

eg – $I = \int_0^1 \frac{dx}{1+x}$ using Simpson's $\frac{1}{3}$rd rule with 2, 4, 8 equal subintervals.

Find errors. ($\log 2 = 0.693147$)

Ans) i) $2N = 2 \Rightarrow$

x	0	0.5	1
$f(x)$	1.0	0.667	0.5

$h = 0.5$

$$I_2 = \frac{h}{3}\left[f(0) + 4f(0.5) + f(1)\right]$$

$$I_2 = \frac{1}{6}\left[1 + 4 \times 0.667 + 0.5\right]$$

$$I_2 = \boxed{0.674}$$

$$\text{error} = 0.693147 - 0.674$$
$$= \boxed{0.019147}$$

$h = 1/4$

(ii) $2N = 4 \Rightarrow$

x	0.25	0.75
$f(x)$	0.8	0.571429

$$I_4 = \frac{h}{3}\left[f_0 + 4\left(f(0.25) + f(0.75)\right) + 2\left(f(0.5)\right) + f(1) \right]$$

$$I_4 = \frac{1}{12}\left[1 + 4(0.8 + 0.571429) + 2(0.667) + 0.5 \right]$$

$$I_4 = \boxed{0.69324}$$

$$\text{error} = |0.693147 - 0.69324|$$
$$= \boxed{0.000177}$$

$h = 1/8$

(iii) $2N = 8 \Rightarrow$

x	0.125	0.375	0.625	0.875
$f(x)$	8/9	8/11	8/13	8/15

$$I_8 = \frac{1}{24}\left[1.5 + 4\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right) + 2(0.8 + 0.667 + 0.57142) \right]$$

$$= \frac{1}{24}\left[1.5 + 11.6595 + 4.07684 \right]$$

$$= 0.693180$$

$\therefore$ error $= 0.693147 - 0.693180 = \boxed{0.000033}$

ERROR :

$$I = \int_a^b w(x)\, f(x)\, dx = \sum_{K=0}^{n} \lambda_K\, f(x_K)$$

error $\underline{R(f)} = \int_a^b w(x)\, f(x)\, dx - \sum_{K=0}^{n} \lambda_K\, f(x_K)$

Order $\longrightarrow$ P

$$\therefore\ R(1) = 0$$
$$R(x) = 0$$
$$R(x^2) = 0$$
$$\vdots$$
$$R(x^P) = 0$$

but for $R(x^{P+1}) = c$ (some error).

$$c = \int_a^b w(x)\, x^{P+1}\, dx - \sum_{K=0}^{n} \lambda_K\, x_K^{P+1}$$

$$\Longrightarrow$$

$$\boxed{R(f) = \frac{c}{(P+1)!} \times f^{(P+1)}(\xi)} \quad , \quad a < \xi < b$$

⇒ Error term in Trapezoidal Rule:

Order = 1 $\quad f(x) = 1, x \longrightarrow R(f) = 0.$

$$c = \int_a^b x^2 \, dx - \frac{b-a}{2}\left(b^2 + a^2\right)$$

$$c = \frac{1}{6}(b-a)^3$$

by formula, $R(f) = \frac{c}{2!} \cdot f''(\xi)$

$a < \xi < b$

$$\therefore \left|R(f)\right| \le \frac{(b-a)^3}{12} M_2 \quad , \quad M_2 = \max \left|f''(x)\right|$$

$a \le x \le b$

⇒ For composite Trapezoidal Rule:

$$R(f) = \frac{h^3}{12}\left[f''(\xi_1) + f''(\xi_2) \cdots + f''(\xi_N)\right]$$

$\forall \; h = \frac{b-a}{N}$

$$\therefore \left|R(f)\right| \le \frac{N h^3}{12} M_2 \longrightarrow \boxed{Nh = b-a}$$

$\Rightarrow$ Error in Simpson's $\frac{1}{3}$rd rule:

order $= 3$

$$c = \int_a^b x^4\,dx - \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]$$

$$\boxed{c = \frac{(b-a)^5}{120}}$$

$$R(f) = \frac{c}{4!} f^{(4)}(\xi) = \frac{-(b-a)^5}{2880} f^{(4)}(\xi)$$

$a < \xi < b$

$$|R(f)| \le \frac{(b-a)^5}{2880} \times M_4$$

, M_4 is max^m value of $|f^{(4)}x|$ $\forall\ a \le x \le b$

$\Rightarrow$

In ~~composite~~ composite Simpson's $1/3^{rd}$ rule:

$R(f) =$

Solution of Non-Linear Equations

⇒ **Simple root:** Say α is root of $f(x)$.

then, ~~$f(\alpha)=0$~~ $f(\alpha)=0$ & let $f'(\alpha) \neq 0$,

$$\text{then } \boxed{\begin{array}{c} f(x) = (x-\alpha)\, g(x) \\ \text{here, } g(\alpha) \neq 0. \end{array}}$$

$\therefore \alpha$ is a simple root.

eg → $f(x) = x^3 + x - 2 = 0$

$f(x) = (x-1)(x^2+x+2) = (x-1)\, g(x)$, $g(1) \neq 0$.

$\therefore$ 1 is a simple root.

⇒ **Multiple root:** α is a multiple root, of multiplicity m, of $f(x)=0$ if.

$f(\alpha)=0$, $f'(\alpha)=0$ .... $f^{(m-1)}(\alpha)=0$

and $f^{m}(\alpha) \neq 0$

$$\therefore \boxed{f(x) = (x-\alpha)^{M}\, g(x), \quad g(\alpha) \neq 0}$$

~~eg $x^3 - 3x^2 + 2 \Rightarrow (x-2)^2 (x+1)$~~

eg → $x^3 - 3x^2 + 4 = 0$

$f(x) = (x-2)^2 (x+1)$, $g(2) \neq 0$

∴ 2 is multiple root with multiplicity 2

& −1 is a simple root.

Direct method:

⇒ • They give exact values of all roots in a finite number of steps.

eg - for quadratic eq^n,

$ax^2+bx+c=0$, $x = \frac{1}{2a}\left[-b \pm \sqrt{b^2-4ac}\right]$

but after degree 4, we don't have a direct method. Also for transcendental equations.

So, we use Iterative method.

Date / /
Page ____

Iterative method:

⇒ **Criterion to terminate Iteration procedure:**

(1) $\left| f(x_k) \right| \leq \epsilon$ $\quad$ $f(x) = 0$

$f(x_k) = 0$

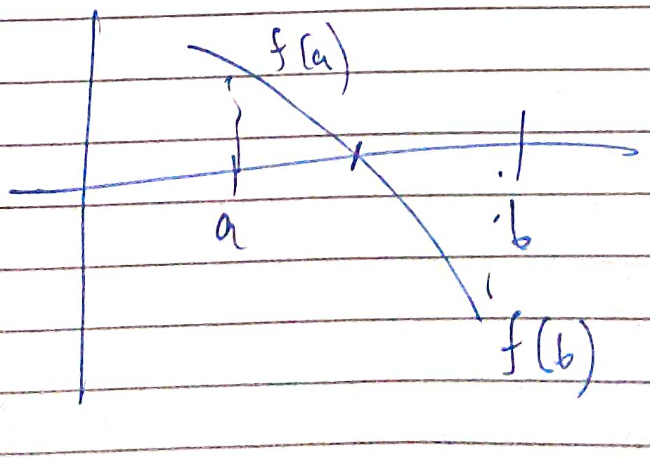
$\boxed{\text{let } \epsilon = 0.00005}$ → error you can allow

(2) $\left| x_{k+1} - x_k \right| \leq \epsilon$

⇒ **Initial approximation for an iterative method:**

(1) Bolzano's theorem (Intermediate Value theorem)

If the function $f(x)$ is continous in $[a, b]$ and if $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one real root of $f(x) = 0$ between a and b.



⇒ Method of Bisection : (Iterative method).

~~Ide~~

This method is based on Bolzano's theorem.

$$\boxed{\text{Let } f(x) \text{ has a root in } [a,b],\ f \text{ is continuous in } [a,b], \text{ and } f(a)\cdot f(b) < 0 \text{ then a real root is b/w } a \,\&\, b}$$

(a) (b) $\because f(a).f(b) < 0$

(1) $x_1 = \dfrac{a+b}{2}$, if $f(x_1) = 0$, x_1 is root of $f(x) = 0$

else,
either $f(x_1) \cdot f(a) < 0$ or $f(x_1).f(b) < 0$.

For instance, let's say $f(x_1).f(a) < 0$.

then,

$a_1 = a$ & $b_1 = x_1$

$\therefore$ (2) $x_2 = \dfrac{a_1 + b_1}{2}$, if $f(x_2) = 0$, x_2 is root

else either $f(x_2).f(a_1) < 0$ or $f(x_2).f(b_1) < 0$

Date / /
Page

and so on. - - - - - - - - - -

Ex-1) Find by bisection method, a real root of

$$2x - \log_{10} x = 7$$

Ans) $f(x) = 2x - \log_{10} x - 7 = 0$

~~$n \quad a_n \quad b_n \quad x_{n+1} = \frac{a_n + b_n}{2} \quad f(x_{n+1})$~~

here, $f(1) = -5$
$f(2) = -3.3$
$f(3) = -1.477$
$f(4) = 0.3979.$

$\therefore \boxed{f(3) \cdot f(4) < 0}$

$\therefore$

n	a_n	b_n	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_n)$
0	3	4	3.5	−0.5441
1	3.5	4	3.75	−0.0740
2	3.75	4	3.875	0.1617
3	3.75	3.875	3.8125	0.0438

n	a_n	b_n	x_{n+1}	$f(x_{n+1})$
4	3.75	3.8125	3.7813	-0.0151
5	3.7813	3.8125	3.7969	0.0143
6	3.7813	3.7969	3.7891	-0.0004
7	3.7891	3.7969	3.7930	0.0070
8	3.7891	3.7930	3.7910	0.0033
9	3.7891	3.7910	3.79	

$$\boxed{\alpha = 3.79}$$

Date / /
Page

⇒ Chord Methods :

we approximate the curve $f(x) = 0$ in a sufficiently small interval which

⇒ Method of False Position (Regula - Falsi):

Start: Find the interval in which the root lies.

Let root of $f(x)$ lies in (x_{K-1}, x_K),

i.e, $f_{K-1} \cdot f_K < 0$.

Then $P(x_{K-1}, f_{K-1})$ and $Q(x_K, f_K)$ are

points on the curve $f(x)$.

Draw a straight line joining P and Q (approximation of the curve in that interval)

eq^n of PQ:

$$\boxed{\frac{y - f_K}{f_{K-1} - f_K} = \frac{x - x_K}{x_{K-1} - x_K}}$$

Next approximation :

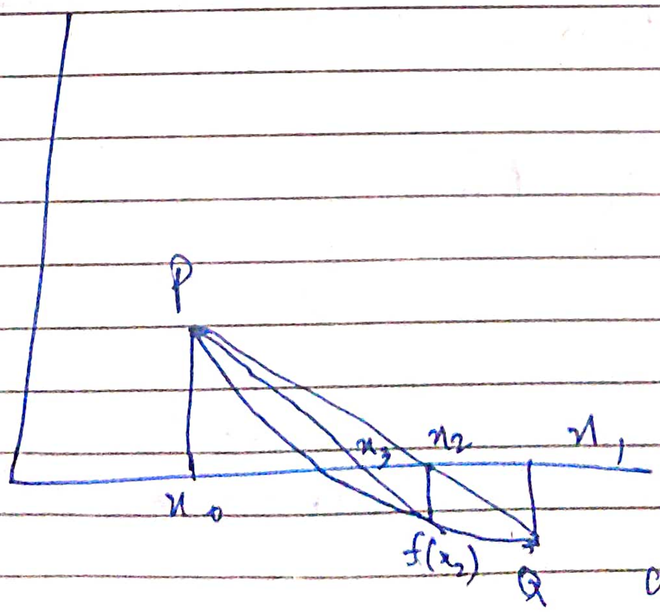
where PQ and X axis intersect
set $Y=0$ and solve for x,

$$x = x_K - \left(\frac{x_{K-1} - x_K}{f_{K-1} - f_K}\right) f_K$$

$$x = x_K - \left(\frac{x_K - x_{K-1}}{f_K - f_{K-1}}\right) f_K$$

$$x_{K+1} = x_K - \left(\frac{x_K - x_{K-1}}{f_K - f_{K-1}}\right) f_K$$

$$\boxed{x_{K+1} = \frac{x_{K-1} f_K - x_K f_{K-1}}{f_K - f_{K-1}}} \quad K = 1, 2 \ldots$$



and so on.

Date / /
Page

starting with initial interval (x_0, x_1) in which root lies,

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

Now if $f(x_0) \cdot f(x_2) < 0$, then root lies b/w (x_0, x_2) otherwise in (x_2, x_1). The iteration is continued using the interval in which the root lies, until the required accuracy criterion is satisfied.

Date / /
Page

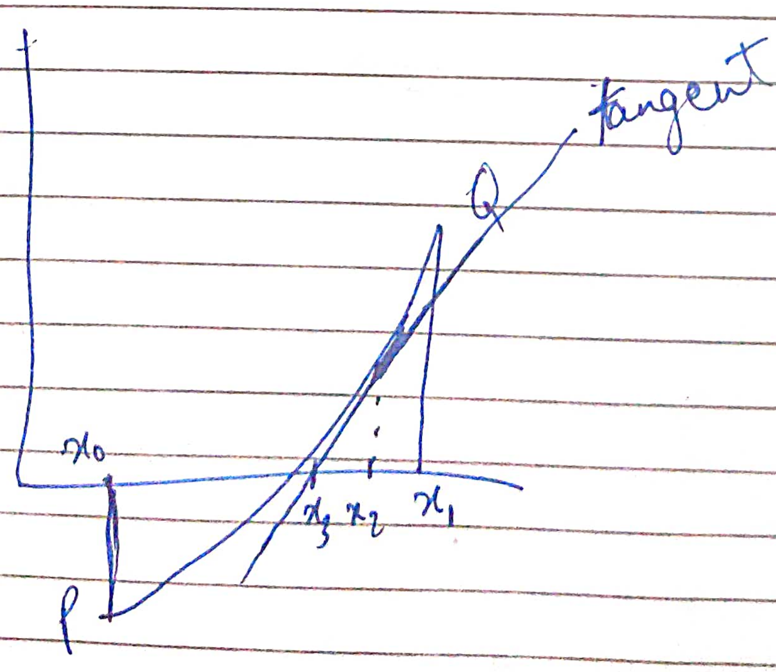
★ Newton - Raphson Method (~~Tag~~ Tangent method):

let $x_0 \to$ initial approximation ~~to~~ the root of $f(x) = 0$

$P(x_0, f_0)$ is a pt on the curve & draw the tangent to the curve at P.
Approximate the curve in the neighbourhood of the root by the tangent to the curve at P.

Next approximation : intersection of tangent ~~to~~ x - axis.

⇒ Another chord method in which we approximate the curve near a root, by a straight line.



- Repeat ~~&~~ until req^r accuracy is obtained

eq^n of tangent to the curve $y=f(x)$ at P is given by.

$$y - f_0 = (x - x_0)\, f'(x_0)$$

$\to$ slope at $f(x_0, f_0)$

Set $y=0$, solve for x,

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \quad , \quad f'(x_0) \neq 0$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \quad , \quad f'(x_k) \neq 0$$

Date / /
Page

Regula - Falsi

Q) $x^3 - 3x + 1 = 0$ solve for root correct to 3 decimal places using regula - falsi method.

Ans)

x	0	1	2	3
$f(x)$	1	-1	3	19

So root b/w (0,1) & (1,2)

we choose (0,1)

(1) $x_0 = 0$, $x_1 = 1$, $f_0 = f(x_0) = f(0) = 1$
$f_1 = f(x_1) = f(1) = -1$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{-1(1)}{-1-1}$$

$$= (0.5)$$

$\therefore f(x_2) = -0.325$

(2) $x_0 = 0$, $x_2 = 0.5$

$f(0) \cdot f(0.5) < 0$

$$\therefore x_3 = \frac{x_0 f_2 - x_2 f_0}{f_2 - f_0} = 0.36364,$$

$f(x_3) = 0.04283.$

$\Longrightarrow$

Date / /
Page

(3) $f(0) \, f(0.36364) < 0$

$\therefore x_4 = \dfrac{x_0 f_3 - x_3 f_0}{f_3 - f_0} = 0.34870$

$f(4) = -0.00370$

(4) $f(0) \cdot f(0.34870) < 0$

$x_5 = \dfrac{x_0 f_4 - x_4 f_0}{f_4 - f_0} = 0.34741$

$f(5) = -0.00030$

(5) $x_6 = 0.347306$

$|x_6 - x_5| = 0.0001 < 0.0005$

correct to 3 decimal places.

$\therefore \boxed{\alpha = 0.347}$

$\Longrightarrow$

Date / /
Page

Q) Perform 4 iterations of Newton - Raphson method to find a root of

$f(x) = x^3 - 5x + 1 = 0$

Ans) $f(0) = 1$, $f(1) = -3$

$\therefore (0,1) \rightarrow$ interval

$\therefore x_{k+1} = x_k - \cancel{\frac{f(x_k)}{f'(x_k)}} \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5}$

$x_{k+1} = \frac{2x_k^3 - 1}{3x_k^2 - 5}$

$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = 0.176471$, $\forall\ x_0 = 0.5$
$\downarrow$
approximated

$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = 0.201568$

$x_3 = 0.201640$

$x_4 = 0.201640$

$\boxed{\text{So, } \alpha = 0.201640}$

Date / /
Page

* Error of Approximation :

error

$$\boxed{\epsilon_k = x_k - \alpha} \quad k = 0, 1, 2 \dots$$

$$|\epsilon_k| \to 0 \text{ as } k \to \infty$$

Order : p or has the rate of convergence p.

If p is the largest positive real no, for which $\exists$ a finite constant c, such that,

$$|\epsilon_{k+1}| \leq c \cdot |\epsilon_k|^p$$

For Bisection : $|\epsilon_{k+1}| = |x_{k+1} - \alpha|$

$$\leq \frac{1}{2} |x_k - \alpha|$$

$$\leq \frac{1}{2} |\epsilon_k|$$

$p = 1$ (Rate of convergence).

Regula Falsi : $p = 1$ linear (Rate of convergence)

Newton Raphson:

$$|\epsilon_{k+1}| \leq c \cdot |\epsilon_k|^2$$

$p = 2$. (Rate of convergence)