

## 5. TRANSFORMATIONS

- Functions of Random Variables

Ex:  $X$  is a R.V

$|X|$  - errors

$\log X$

$X^2$

$\sin X$

$\alpha X + b$

→ For R.V  $X$ ,  $y = g(x)$  is also a R.V  
we find pdf, cdf ... for  $y$ .

Ex:1  $y = \alpha X + b$

$$\begin{aligned} F_y(y) &= P(Y \leq y) = P(\alpha X + b \leq y) \\ &= P\left(X \leq \frac{y-b}{\alpha}\right) \\ &= F_x\left(\frac{y-b}{\alpha}\right) \end{aligned}$$

Ex:2  $y = |X|$

$$\begin{aligned} F_y(y) &= P(Y \leq y) = P(|X| \leq y) \\ &= P(-y \leq X \leq y) \quad P(a < X < b) \\ F_y(y) &= F_x(y) - F_x(-y) \quad = F(b) - F(a) \end{aligned}$$

On differentiating cdf of  $y$  i.e.,  $F_y(y)$   
we get pdf of  $y$ .

$$\text{Ex: } P(X=x) = \begin{cases} \frac{e^{-\lambda} \cdot \lambda^x}{x!} & x=0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$y = x^2 + 3$$

$$\begin{aligned} P(Y=y) &= P(X^2 + 3 = y) \\ &= P(X = \sqrt{y-3}) \\ &= \frac{e^{-\lambda} \cdot \lambda^{\sqrt{y-3}}}{(\sqrt{y-3})!} \quad y = 3, 4, 7, \dots \end{aligned}$$

$$P(Y=y) = \begin{cases} e^{-\lambda} \cdot \lambda^{\sqrt{y-3}} & y = 3, 4, 7, \dots \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Ex: } P(X=-2) = \frac{1}{5}, \quad P(X=-1) = \frac{1}{6}$$

$$P(X=0) = \frac{1}{5}, \quad P(X=1) = \frac{1}{15}$$

$$P(X=2) = \frac{11}{30}$$

$$y = x^2$$

$$A = \{-2, -1, 0, 1, 2\}$$

↓  
values of x.

$$P(Y=y) = \begin{cases} \frac{1}{5} & (y=0) \\ \frac{1}{6} + \frac{1}{15} = \frac{7}{30} & y=1 \\ \frac{1}{5} + \frac{11}{30} = \frac{17}{30} & y=4 \end{cases}$$

B = {0, 1, 4}  
↓  
values of y.

(Y)	P(Y=y)
0	$\frac{1}{5}$
1	$\frac{7}{30}$
4	$\frac{17}{30}$

$$\begin{aligned}
 E(Y) &= 0 \cdot \frac{1}{5} + 1 \cdot \frac{7}{30} + 4 \cdot \frac{17}{30} \\
 &= \frac{7}{30} + \frac{68}{30} \\
 &= \frac{75}{30} \\
 &= \frac{5}{2}
 \end{aligned}$$

Ex. 16/2)

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$Y = x^2$$

$$\begin{aligned}
 F_Y(y) &= P(Y \leq y) = P(x^2 \leq y) \\
 &= P(x \leq y^{1/2})
 \end{aligned}$$

$$F_x(\sqrt{y}) = \int_0^{\sqrt{y}} f(x) dx$$

$$= \int_0^{\sqrt{y}} 3x^2 dx = \left[ \frac{3x^3}{3} \right]_0^{\sqrt{y}}$$

$$F_x(\sqrt{y}) = y^{3/2}$$

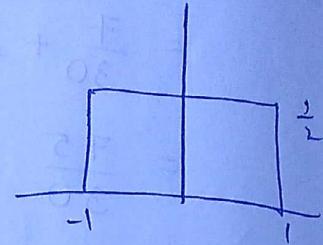
$$\text{cdf } F_Y(y) = \begin{cases} y^{3/2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{pdf } f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{3}{2} y^{1/2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

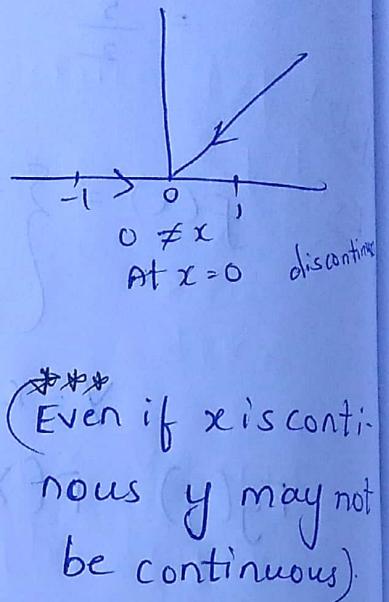
Ex.

$$x \sim U[-1, 1]$$

$$f(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$Y = X^+ = \begin{cases} X & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2} & y = 0 \\ \frac{1}{2} + \frac{y}{2} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

Theorem:

Let  $x$  be R.V of continuous type with pdf  $f(x)$ . Let  $y = g(x)$  be a differentiable function for all  $x$  and  $g'(x) < 0 \quad \forall x$  or  $g'(x) > 0 \quad \forall x$  then  $y = g(x)$  is also R.V of continuous type. Pdf which is given by

$$f(y) = \begin{cases} f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \alpha < y < \beta \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha = \min(g(-\infty), g(\infty))$$

$$\beta = \max(g(-\infty), g(\infty))$$

(it can also be a finite interval)

if  $g'(x) > 0$   $g(x)$  is always strictly increasing  
 if  $g'(x) < 0$   $g(x)$  is strictly decreasing  
 $g(x)$  should be monotonic.

Ex:

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g'(x) = e^x > 0 \quad \forall x$$

so  $g(x)$  is continuous & strictly increasing

$$f(y) = f(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

$$y = e^x$$

$$\log y = x$$

$$g^{-1}(y) = \log y$$

$$f(y) = f(\log y) \left| \frac{d}{dy}(\log y) \right|$$

$$= \begin{cases} 1 \cdot \frac{1}{y} & |1 < y < e| \\ 0 & \text{otherwise} \end{cases}$$

$0 < x < 1$   
 $e^0 < e^x < e^1$   
 $1 < y < e$

Ex:

$$X \sim U(0, 1)$$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y = g(x) = \frac{x}{1+x}$$

$$g(x) = \frac{1+x-1}{1+x}$$

$$g(x) = 1 - \frac{1}{1+x}$$

$$g'(x) = \frac{1}{(1+x)^2} > 0$$

$g(x)$  is strictly increasing

$$y = \frac{x}{1+x}$$

$$y + xy = x$$

$$x(y-1) = -y$$

$$x = \frac{y}{1-y}$$

$$g^{-1}(y) = \frac{y}{1-y}$$

$$f(y) = f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= 1 \cdot \left| \frac{d}{dy} \left( \frac{y}{1-y} \right) \right| \quad \left\{ \begin{array}{l} 0 < x < 1 \\ 0 < \frac{x}{1+x} < \frac{1}{2} \end{array} \right.$$

$$= \left| \frac{d}{dy} \left( \frac{-1+y+1}{1-y} \right) \right| \quad 0 < y < \frac{1}{2}$$

$$= \left| \frac{d}{dy} \left( 1 + \frac{1}{1-y} \right) \right|$$

$$f(y) = \begin{cases} \frac{1}{(1-y)^2} & 0 < y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Ex: } f(x) = \frac{1}{\pi(1+x^2)} \quad -\infty < x < \infty$$

$$y = g(x) = \tan^{-1} x$$

$$g^{-1}(y) = x$$

$$g^{-1}(y) = \tan y$$

$$f(y) = f(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= f(\tan y) \left| \frac{d}{dy} \tan y \right| \quad -\infty < x < \infty \\ -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$= \frac{1}{\pi(1+\tan^2 y)} \cdot |\sec^2 y| \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$= \frac{1}{\pi \cdot (\sec^2 y)} |\sec^2 y| \quad \sec^2 \theta - \tan^2 \theta = 1 \\ \sec^2 \theta = 1 + \tan^2 \theta$$

$$f(y) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < y < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Ex: } f(x) = 5x^4 \quad 0 < x \leq 1$$

$$g(x) = 1-x^2 = y \Rightarrow g^{-1}(x) = \sqrt{1-x}$$

$$x^2 = 1-y$$

$$x = \sqrt{1-y}$$

$$g^{-1}(y) = \sqrt{1-y}$$

$g(x)$  is decreasing  
for the given  
interval.

$$f(y) = f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= f(\sqrt{1-y}) \left| \frac{d}{dy} \sqrt{1-y} \right|$$

$$\begin{aligned}
 &= 5 (\sqrt{1-y})^4 \cdot \left| \frac{-1}{2\sqrt{1-y}} \right| \\
 &= 5 (\sqrt{1-y})^4 \cdot \frac{1}{2\sqrt{1-y}} \\
 f(y) &= \begin{cases} \frac{5}{2} (\sqrt{1-y})^3 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

## Functions of Random Vectors /

### Several Random Variables

Let  $X_1, X_2, \dots, X_n$  be the random variables defined on  $(\Omega, \mathcal{S}, P)$

Ex: Functions  $X_1 - X_2, X_3 + 2X_4 - X_5, X_1, X_2, \dots$

we study the joint distributions of functions

→ Functions are denoted by  $y = g(X_1, X_2, \dots, X_n)$

cdf's :  $P(Y=y) = P(g(X_1, X_2, \dots, X_n) \leq y)$

$$\left\{ \begin{array}{l} \sum P(X_1=x_1, X_2=x_2, \dots, X_n=x_n) \\ \{(x_1, x_2, \dots, x_n) / g(x_1, x_2, \dots, x_n) \leq y\} \end{array} \right. \quad \text{For discrete}$$

$$\left\{ \begin{array}{l} \int f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ \{(x_1, x_2, \dots, x_n) / g(x_1, x_2, \dots, x_n) \leq y\} \end{array} \right. \quad \text{For continuous}$$

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Example on function of Random Variable

$$X \sim U[-1, 1]$$

pdf of  $X$ :  $f(x)$  is continuous

$$g(x) = y$$

$$F_y(y) = D \cdot F = \begin{cases} 0 & y < 0 \\ \frac{y+1}{2} & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

distribution function.

$$g'(x) = \begin{cases} \frac{1}{2} & y = 0 \\ \frac{1}{2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

### Functions of Random Vectors

$x_1, x_2, \dots, x_n$

$$g_1(x_1, x_2, \dots, x_n), g_2(x_1, x_2, \dots, x_n), \dots, g_n(x_1, x_2, \dots, x_n)$$

Ex 21

$x$	-1	0	1	$U =  x $	$V = Y^2$
-2	$y_6$	$y_{12}$	$y_6$	$U = 0, 1$	$V = 1, 4$
1	$y_6$	$y_{12}$	$y_6$	$0, -1, 1$	$1, 4, 1, 4$
2	$y_{12}$	0	$y_{12}$		

$$P(U=0, V=1) = P(X=0, Y=1) = \frac{1}{12}$$

$$P(U=0, V=4) = P(X=0, Y=-2) + P(X=0, Y=2) = \frac{1}{12} + 0 = \frac{1}{12}$$

$$P(U=1, V=1) = P(X=-1, Y=1) + P(X=1, Y=1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\begin{aligned}
 P(U=1, V=4) &= P(X=-1, Y=-2) + P(X=-1, Y=2) \\
 &\quad + P(X=1, Y=-2) + P(X=1, Y=2) \\
 &= \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} \\
 &= \frac{1}{2}
 \end{aligned}$$

$\sqrt{U}$	0	1
1	$\frac{1}{12}$	$\frac{1}{3}$
4	$\frac{1}{12}$	$\frac{1}{2}$

Ex 2

$$f(x, y) = \begin{cases} \frac{1+xy}{4} & |x| < 1, |y| < 1 \\ & -1 < x < 1, -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$U = X^2, V = Y^2$$

$$F_{U,V}(u, v) = P(U \leq u, V \leq v)$$

$$= P(X^2 \leq u, Y^2 \leq v)$$

$$= P(-\sqrt{u} < X < \sqrt{u}, -\sqrt{v} < Y < \sqrt{v})$$

$$= \int_{x=-\sqrt{u}}^{\sqrt{u}} \int_{y=-\sqrt{v}}^{\sqrt{v}} \frac{1+xy}{4} dy dx.$$

$$= \int_{x=-\sqrt{u}}^{\sqrt{u}} \left[ \frac{y}{4} + \frac{x}{4} \cdot \frac{y^2}{2} \right]^{\sqrt{v}} \cdot dx$$

$$= \frac{1}{4} \int_{x=-\sqrt{u}}^{\sqrt{u}} \left[ \sqrt{v} + x \cdot \frac{v}{2} - \left( -\sqrt{v} + \frac{xv}{2} \right) \right] dx$$

$$= \frac{1}{4} \cdot \int_{x=-\sqrt{u}}^{\sqrt{u}} \left( \sqrt{v} + \frac{xv}{2} + \sqrt{v} - \frac{xv}{2} \right) dx$$

$$= \frac{1}{4} \int_{x=-\sqrt{u}}^{\sqrt{u}} 2\sqrt{v} dx$$

$$= \frac{1}{2} \cdot \int_{x=-\sqrt{u}}^{\sqrt{u}} \sqrt{v} dx$$

$$= \frac{1}{2} \cdot [\sqrt{v} \cdot x]_{-\sqrt{u}}^{\sqrt{u}}$$

$$= \frac{1}{2} \cdot \sqrt{v} \cdot (2 \cdot \sqrt{u})$$

$$F_{u,v}(u, v) = \sqrt{v} \cdot \sqrt{u}$$

$$U = X^2 \quad V = Y^2$$

$$|X| \leq 1 \quad |Y| \leq 1$$

$$\frac{\partial^2 F_{u,v}(u, v)}{\partial v \partial u} = \frac{\partial}{\partial u} \left( \frac{\partial F}{\partial v} \right)$$

$$= \frac{\partial}{\partial u} \left( \frac{\sqrt{u} \cdot 1}{2\sqrt{v}} \right)$$

$$= \frac{1}{2\sqrt{v} \cdot 2\sqrt{u}}$$

$$= \frac{1}{4\sqrt{u}\sqrt{v}} \quad 0 \leq u \leq 1 \quad 0 \leq v \leq 1$$

## Theorem:

Let  $(x_1, x_2, \dots, x_n)$  be Random Vector of dimension of  $n$ , which is continuous type with pdf or  $f(x_1, x_2, \dots, x_n)$

$$\text{Let } y_1 = g_1(x_1, x_2, \dots, x_n)$$

$$y_2 = g_2(x_1, x_2, \dots, x_n)$$

$$\vdots \quad \vdots \\ y_n = g_n(x_1, x_2, \dots, x_n)$$

be a mapping from  $R^n$  to  $R^n$

there exists inverse mapping

$$x_1 = h_1(y_1, y_2, \dots, y_n)$$

$$x_2 = h_2(y_1, y_2, \dots, y_n)$$

$$\vdots \quad \vdots \\ x_n = h_n(y_1, y_2, \dots, y_n)$$

$$(x_1, x_2, x_3) \\ \downarrow$$

$$(y_1, y_2, y_3)$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ g_1 & g_2 & g_3 \end{matrix}$$

Conditions:

① Assume both mapping  $(g_i(x))$  & inverse  $(h(y))$  are continuous

② Assume that partial derivatives

$$\frac{\partial x_i}{\partial y_j}, 1 \leq i \leq n, 1 \leq j \leq n \quad (n^2 \text{ partial derivatives})$$

should exist & continuous

③ Assume that Jacobian of inverse mapping  
 $\downarrow$   
 will be formed  
 only for square mat.

$$J = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)}$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}_{n \times n} = \text{function of } y$$

is non zero for  $(y_1, y_2, \dots, y_n)$

then  $(y_1, y_2, \dots, y_n)$  has a joint continuous Distribution function with PDF

$$\omega(y_1, y_2, \dots, y_n) = |J| \cdot f(h_1(y_1, y_2, \dots, y_n), h_2(y_1, y_2, \dots, y_n), \dots, h_n(y_1, y_2, \dots, y_n))$$

Ex  $x_1, x_2, x_3$  be independent R.V's with common exponential distribution

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(h_1, h_2, h_3) = f(h_1) \cdot f(h_2) \cdot f(h_3)$$

$$\left\{ \begin{array}{l} \text{Ex: if } f(x_1, x_2, x_3) = x_1 + x_2^2 + x_3 \\ \text{then } f(h_1, h_2, h_3) = h_1 + h_2^2 + h_3 \end{array} \right\}$$

$$Y_1 = X_1 + X_2 + X_3, \quad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \quad Y_3 = \frac{X_1}{X_1 + X_2}$$

Sol

$$Y_2 = \frac{X_1 + X_2}{Y_1}$$

$$Y_2 = \frac{X_1 + X_2}{Y_1}$$

$$Y_2 Y_1 = Y_1 + Y_2 Y_3 + X_2$$

$$Y_2 = \frac{X_1}{Y_1 Y_3}$$

$$X_2 = Y_2 Y_1 - Y_1 Y_2 Y_3$$

$$\boxed{X_1 = Y_1 Y_2 Y_3}$$

$$\boxed{X_2 = Y_1 Y_2 (1 - Y_3)}$$

$$Y_1 = X_1 + X_2 + X_3$$

$$Y_1 = X_1 Y_2 Y_3 + Y_1 Y_2 (1 - Y_3) + X_3$$

$$X_3 = Y_1 - Y_1 Y_2 Y_3 - Y_1 Y_2 (1 - Y_3)$$

$$= Y_1 - Y_1 Y_2$$

$$\boxed{X_3 = Y_1 (1 - Y_2)}$$

$$X_1 = Y_1 Y_2 Y_3 \quad X_2 = Y_1 Y_2 (1 - Y_3) \quad X_3 = Y_1 (1 - Y_2)$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \frac{\partial X_1}{\partial Y_3} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \frac{\partial X_2}{\partial Y_3} \\ \frac{\partial X_3}{\partial Y_1} & \frac{\partial X_3}{\partial Y_2} & \frac{\partial X_3}{\partial Y_3} \end{vmatrix} = \begin{vmatrix} Y_2 Y_3 & Y_1 Y_3 & Y_1 Y_2 \\ Y_2 (1 - Y_3) & Y_1 (1 - Y_3) & -Y_1 Y_2 \\ 1 - Y_2 & -Y_1 & 0 \end{vmatrix}$$

$$= Y_2 Y_3 (-Y_1^2 Y_2) - Y_1 Y_3 (Y_1 Y_2 (1 - Y_2)) + Y_1 Y_2$$

$$\begin{pmatrix} -Y_1 Y_2 (1 - Y_3) - (1 - Y_2) \\ Y_1 (1 - Y_3) \end{pmatrix}$$

$$= -y_1^2 y_2^2 y_3 - y_1^2 y_2 y_3 + y_1^2 y_2^2 y_3 \\ + y_1 y_2 \left[ -y_1 y_2 + y_1 y_2 y_3 \right. \\ \left. - (y_1 - y_1 y_2)(1 - y_3) \right]$$

$$= -y_1^2 y_2 y_3 - y_1^2 y_2^2 + y_1^2 y_2^2 y_3 + -y_1 y_2 (y_1 - y_1 y_3) \\ - y_1 y_2 + y_1 y_2 y_3$$

$$= -y_1^2 y_2^2 - y_1^2 y_2 + y_1^2 y_2^2 y_3 + y_1^2 y_2^2 - y_1^2 y_2^2 y_3 \\ = -y_1^2 y_2$$

$$J = -y_1^2 y_2$$

$$|J| = y_1^2 y_2$$

Joint pdf of  $\omega(y_1, y_2, y_3) = |J| f(h_1, h_2, h_3)$

$$= y_1^2 y_2 f(h_1) \cdot f(h_2) \cdot f(h_3) \\ = y_1^2 y_2 \cdot f(y_1 y_2 y_3) f(y_1 y_2 (1-y_3)) \\ \cdot f(y_1 (1-y_2)) \\ = y_1^2 y_2 \cdot e^{-y_1 y_2 y_3} \cdot e^{-y_1 y_2 (1-y_3)} \cdot e^{-y_1 (1-y_2)} \\ = y_1^2 y_2 \cdot e^{-y_1} \quad 0 < y_1 < \infty \\ 0 < y_2 < 1 \\ 0 < y_3 < 1$$

$$x_1 > 0 \quad y_1 = x_1 + x_2 + x_3$$

$$x_2 > 0 \quad y_1 > 0$$

$$x_3 > 0 \quad y_2 = \frac{x_1 + x_2}{x_1 + x_2 + x_3}$$

$\hookrightarrow$  fraction  $\Rightarrow 0 < y_2 < 1$

$$y_3 = \frac{x_1}{x_1 + x_2}$$

$\hookrightarrow$  fraction

$0 < y_3 < 1$

solution

$$\omega(y_1, y_2, y_3) = \begin{cases} y_1^2 y_2 e^{-y_3} & \text{for } 0 < y_1 < \infty \\ & 0 < y_2 < 1 \\ & 0 < y_3 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Ex: 2

Let  $x_1, x_2$  be two independent R.V

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_1 = X_1 + X_2$$

$$Y_2 = X_1 - X_2$$

sol

$$Y_1 + Y_2 = 2X_1$$

$$Y_1 - Y_2 = 2X_2$$

$$X_1 = \frac{Y_1 + Y_2}{2} \quad X_2 = \frac{Y_1 - Y_2}{2}$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$|J| = \frac{1}{2}$$

$$\omega(y_1, y_2) = \frac{1}{2} \cdot f(h_1, h_2)$$

$$= \frac{1}{2} f\left(\frac{y_1 + y_2}{2}\right) \cdot f\left(\frac{y_1 - y_2}{2}\right)$$

$$= \frac{1}{2} \cdot 1 \cdot 1$$

$$= \frac{1}{2}$$

$$0 < x_1 < 1, 0 < x_2 < 1$$

$$y_1 = x_1 + x_2, y_2 = x_1 - x_2$$

$$0 < y_1 < 2, -1 < y_2 < 1$$

$$\omega(y_1, y_2) = \begin{cases} \frac{1}{2} & 0 < y_1 < 2 \\ & -1 < y_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(Y_1 = 1, Y_2 = 1)$$

$$2 \cdot 0 = 0$$

boundary & normalizing by  $P(Y_1 = 1, Y_2 = 1)$

$$P(X_1 = 1, X_2 = 1) = \frac{P(Y_1 = 1, Y_2 = 1)}{P(Y_1 = 1, Y_2 = 1)} = \frac{1}{P(Y_1 = 1, Y_2 = 1)}$$

$$(0.5 - 0.5) \cdot \left(\frac{1}{2}\right)^2 \cdot 0 + P(Y_1 = 1, Y_2 = 1)$$

$$= (0.5) \cdot \frac{1}{4} + P(Y_1 = 1, Y_2 = 1)$$

$$= \frac{1}{8} + P(Y_1 = 1, Y_2 = 1)$$

$$= 2 \cdot P(Y_1 = 1, Y_2 = 1)$$

$$= P(Y_1 = 1, Y_2 = 1) \cdot \left(0.5 - \left(\frac{1}{2}\right)^2\right)$$

$$= P(Y_1 = 1, Y_2 = 1) \cdot (0.5 - 0.25)$$

$$= P(Y_1 = 1, Y_2 = 1) \cdot \left(\frac{1}{4}\right)$$

$$= \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}$$

## 4/6/21 Tutorial problems

Bivariate normal distribution

- ① Let  $x$  be the height of father  
 $y$  be the height of son

Assume  $x$  &  $y$  follow bivariate normal distribution.

Assume that  $E(x) = 68$  inches

$$E(y) = 69$$

$$\sigma_x = \sigma_y = 2$$

$$\rho = 0.5$$

Given  $x=80$ , find the Mean & Variance of  $y$ .

Sol

$$\begin{aligned}
 E(Y/x=80) &= \mu_y + \frac{\rho \sigma_y}{\sigma_x} (x - \mu_x) \\
 &= 69 + 0.5 \left( \frac{2}{2} \right) (80 - 68) \\
 &= 69 + \frac{1}{2} (12) \\
 &= 69 + 6 \\
 &= 75
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(Y/x=80) &= (1-\rho^2) \sigma_y^2 \\
 &= (1-(0.5)^2) (2)^2 \\
 &= (4) \left(1 - \frac{1}{4}\right) \\
 &= 4 \left(\frac{3}{4}\right) \\
 &= 3
 \end{aligned}$$

② Measurements taken on 'n' heart attack patients on their cholesterol levels. For each patient measurements were taken 0, 2, 4 days after attack.

Sample mean.

$$\begin{array}{ll} \text{0th} & x_1 \\ \text{2nd} & x_2 \\ \text{4th} & x_3 \end{array} \begin{array}{l} 259.5 \\ 230.8 \\ 221.5 \end{array}$$

Covariance matrix

$$\begin{matrix} & 0 & 2 & 4 \\ 0 & 2276 & 1508 & 813 \\ 2 & 1508 & 2206 & 1349 \\ 4 & 813 & 1349 & 1865 \end{matrix}$$

Covariance matrix is always symmetric

Find Mean and Variance of  $x_1 - x_2$ .

$$E(x_1 - x_2) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 259.5 \\ 230.8 \\ 221.5 \end{pmatrix} \quad \begin{matrix} \text{Coefficient} \\ \text{vector} \end{matrix} \\ = (1, -1, 0)$$

$$\begin{aligned} &= 259.5 - 230.8 \\ &= 28.7 \end{aligned}$$

$$\begin{aligned} \text{Var} &= (1 \ -1 \ 0)_{1 \times 3} \begin{pmatrix} 2276 & 1508 & 813 \\ 1508 & 2206 & 1349 \\ 813 & 1349 & 1865 \end{pmatrix}_{3 \times 3} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}_{3 \times 1} \\ &= (768 \ -698 \ -536) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1466 \end{aligned}$$

$$③ f(x,y) = \begin{cases} \frac{xy}{96} & 0 < x < 4, 0 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

Find  $E(X)$ ,  $E(Y)$ ,  $E(XY)$ ,  $\text{cov}(X, Y)$ ,  $E(2X+3Y)$

sol

$$\begin{aligned} f_x(x) &= \int_0^5 f(x,y) dy = \int_0^5 \frac{xy}{96} dy \\ &= \frac{x}{96} \left[ \frac{y^2}{2} \right]_0^5 \\ &= \frac{x}{192} (25-1) \\ &= \frac{24x}{192} \end{aligned}$$

Marginal  $f_x(x) = \begin{cases} \frac{x}{8} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} E(X) &= \int_0^4 x \cdot \frac{x}{8} dx = \frac{1}{8} \int_0^4 x^2 dx \\ &= \frac{1}{8} \cdot \left[ \frac{x^3}{3} \right]_0^4 = \frac{1}{24} \cdot 64 \end{aligned}$$

$$f_y(y) = \int_0^4 f(x,y) dx = \int_0^4 \frac{xy}{96} dx$$

$$= \frac{y}{96} \cdot \left[ \frac{x^2}{2} \right]_0^4$$

$$= \frac{y}{192} \cdot 16$$

$$f_y(y) = \begin{cases} \frac{y}{12} & \text{if } 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$E(Y) = \int_1^5 y \cdot \frac{y}{12} dy$$

$$= \frac{1}{12} \cdot \left[ \frac{y^3}{3} \right]_1^5$$

$$= \frac{1}{36} \cdot (125 - 1)$$

$$= \frac{124}{36}$$

$$E(Y) = \frac{31}{9}$$

$$E(XY) = \int_{x=0}^4 \int_{y=1}^5 xy \cdot \frac{xy}{96} dy dx$$

$$= \int_{x=0}^4 \int_{y=1}^5 \frac{x^2 y^2}{96} dy dx$$

$$= \int_{x=0}^4 \frac{x^2}{96} \cdot \left[ \frac{y^3}{3} \right]_1^5 dx$$

$$= \int_{x=0}^4 \frac{x^2}{96} \cdot \left(\frac{124}{3}\right) dx$$

$$= - \int_0^4 \frac{x^3}{3} = \frac{124}{3 \times 96} \int_0^4 x^2 dx$$

$$= \frac{124}{3 \times 96} \cdot \left[ \frac{x^3}{3} \right]_0^4$$

$$= \frac{124}{3 \times 96} \times \frac{64}{3}$$

$$E(XY) = \frac{248}{27}$$

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X) \cdot E(Y) \\ &= \frac{248}{27} - \frac{8}{3} \cdot \frac{31}{9} \end{aligned}$$

$$\text{Cov}(X, Y) = 0$$

$$E(2X+3Y) = 2E(X)+3E(Y)$$

$$= 2 \cdot \frac{8}{3} + 3 \cdot \frac{31}{9}$$

$$= \frac{16}{3} + \frac{31}{3}$$

$$E(2X+3Y) = \frac{47}{3}$$

④ Let  $Y$  be a continuous Random Variable.

$$\text{where } f(y) = \begin{cases} \frac{3}{2}y^2 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine density function of  $U = 3-y$ .

$$\begin{aligned} U &= 3-y & U &= g(y) \\ g(y) &= 3-y & g^{-1}(u) &= y \\ g'(y) &= -1 < 0 & g^{-1}(u) &= 3-u \end{aligned}$$

$\therefore$  function is strictly decreasing

$$\begin{aligned} f(u) &= f(g^{-1}(u)) \left| \frac{d}{du} g^{-1}(u) \right| \\ &= f(3-u) \cdot \left| \frac{d}{du} (3-u) \right| \\ &= \frac{3}{2} \cdot (3-u)^2 \cdot |-1| & -1 \leq y \leq 1 \\ & & -1 \leq u \leq 1 \end{aligned}$$

$$f(u) = \begin{cases} \frac{3}{2} (3-u)^2 & 2 \leq u \leq 4 \quad 2 \leq 3-y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$⑤ f(y) = \begin{cases} \frac{1}{2} & 9 \leq y \leq 11 \\ 0 & \text{otherwise} \end{cases}$$

$$U = 2y^2$$

$$g(y) = U = 2y^2 \Rightarrow g'(y) = 4y > 0 \quad \forall [9, 11]$$

$$g^{-1}(u) = y = \sqrt{\frac{u}{2}}$$

$$\begin{aligned}
 f(u) &= f(g^{-1}(u)) \left| \frac{d}{du} \cdot g^{-1}(u) \right| \\
 &= f\left(\sqrt{\frac{u}{2}}\right) \left| \frac{d}{du} \cdot \sqrt{\frac{u}{2}} \right| \\
 &= \frac{1}{2} \cdot \left| \frac{1}{2\sqrt{\frac{u}{2}}} \right| \cdot \frac{1}{\sqrt{2}} \quad 9 \leq y \leq 11 \\
 f(u) &= \begin{cases} \frac{1}{4\sqrt{2}\sqrt{u}} & 162 \leq u \leq 242 \\ 0 & \text{otherwise} \end{cases} \quad 81 \leq y^2 \leq 121 \\
 &\quad 162 \leq u \leq 242
 \end{aligned}$$

$$\textcircled{6} \quad f(x) = \begin{cases} 2^{-x} & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$U = X^4 + 1$$

$$g(x) = u$$

$$g^{-1}(u) = x = (u-1)^{1/4} = \sqrt[4]{u-1}$$

$$x = A = 1, 2, 3$$

$$u = B = 2, 17, 82$$

$$g(u) = P(U=u) = P(X^4 + 1 = u)$$

$$= P(X = \sqrt[4]{u-1})$$

$$= f((u-1)^{1/4})$$

$$\begin{aligned}
 g(u) &= \begin{cases} 2^{-(u-1)^{1/4}} & u = 2, 17, 82, \dots \\ 0 & \text{otherwise} \end{cases} \\
 &\quad u = 2, 17, 82, \dots
 \end{aligned}$$

Q) If  $X$  and  $Y$  are independent gamma distributions

Compute joint density function of  $U=X+Y$

$$V = \frac{X}{X+Y}$$

$$\text{sol} \quad f_{XY} = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{r(\alpha)} \cdot \frac{\lambda e^{-\lambda y} (\lambda y)^{\beta-1}}{r(\beta)}$$

$$U = X+Y \quad V = \frac{X}{X+Y}$$

$$X = UV$$

$$Y = U-X$$

$$= U - UV$$

$$Y = U(1-V)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$$\begin{aligned} |J| &= |-uv - u(1-v)| \\ &= |-uv - u + uv| \\ &= | -u | \end{aligned}$$

$$\begin{aligned} f(u, v) &= |J| f(h_1, h_2) \\ &= u \cdot f(uv, u(1-v)) \\ &\approx u \cdot f(uv) \cdot f(u(1-v)) \end{aligned}$$

$$= \frac{u \cdot \lambda e^{-\lambda u v}}{\Gamma(\alpha)} \cdot \frac{(\lambda u v)^{\alpha-1} \cdot \lambda^{-\alpha} e^{-\lambda(u(1-v))}}{\Gamma(\beta) \cdot (\lambda \cdot u(1-v))^{\beta-1}}$$

$$= \frac{u \cdot \lambda^{\alpha-1} \cdot \lambda^{B-1}}{\Gamma(\alpha) \cdot \Gamma(\beta)}$$

$$= \frac{u \cdot \lambda^2 \cdot (\lambda u v)^{\alpha-1} \cdot (\lambda u)^{B-1} \cdot (1-v)^{B-1} \cdot e^{-\lambda u v - \lambda u + \lambda u v}}{\Gamma(\alpha) \cdot \Gamma(\beta)}$$

$$= \frac{u \cdot \lambda^{\alpha+\beta} \cdot u^{\alpha+\beta-2} \cdot v^{\alpha-1} \cdot (1-v)^{B-1} \cdot e^{-\lambda u}}{\Gamma(\alpha) \cdot \Gamma(\beta)}$$

$$f(u, v) = \frac{\lambda^{\alpha+\beta} \cdot u^{\alpha+\beta-1} \cdot v^{\alpha-1} \cdot (1-v)^{B-1} \cdot e^{-\lambda u}}{\Gamma(\alpha) \cdot \Gamma(\beta)}$$

$0 < u < 1$   
 $0 < v < 1$

H.W

$$\textcircled{1} \quad f(x) = \begin{cases} x^2/81 & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

$$U = \frac{1}{3}(12-x)$$

Determine density function of  $U$ .

$$g(x) = u \quad u = 4 - \frac{x}{3}$$

$$g^{-1}(u) = x \quad \frac{x}{3} = 4 - u$$

$$g^{-1}(u) = 3(4-u)$$

$$g(x) = u = 4 - \frac{x}{3}$$

$$g'(x) = -\frac{1}{3} < 0$$

Function is strictly decreasing

$$f(u) = f(g^{-1}(u)) \left| \frac{d}{du} \cdot g^{-1}(u) \right|$$

$$= f(3(4-u)) \left| \frac{d}{du} \cdot 3(4-u) \right|$$

$$= \frac{3^2(4-u)^2}{81} \cdot |-3| \quad -3 < x < 6$$

$$= \frac{9}{81} \cdot (4-u)^2 \cdot 3 \quad 6 < 12-x < 15$$

$$f(u) = \begin{cases} \frac{1}{3} \cdot (4-u)^2 & 2 < u < 5 \\ 0 & \text{otherwise.} \end{cases} \quad 2 < u < 5$$

②  $x_1, x_2, x_3$  be R.V

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$$

$$y_1 = \frac{x_1 - x_2}{\sqrt{2}}, \quad y_2 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}$$

$$y_3 = \frac{x_1 + x_2 + x_3}{\sqrt{3}}$$

$$f(y_1, y_2, y_3) = ?$$

$$Y_1 = \frac{X_1 - X_2}{\sqrt{2}}, Y_2 = \frac{X_1 + X_2 - 2X_3}{\sqrt{6}}$$

$$X_1 - X_2 = \sqrt{2} Y_1 \rightarrow ① \quad X_1 + X_2 - 2X_3 = \sqrt{6} Y_2 \rightarrow ②$$

$$Y_3 = \frac{X_1 + X_2 + X_3}{\sqrt{3}}$$

$$X_1 + X_2 + X_3 = \sqrt{3} Y_3 \rightarrow ③$$

$$③ - ② \quad 3X_3 = \sqrt{3} Y_3 - \sqrt{6} Y_2$$

$$X_3 = \frac{\sqrt{3} Y_3 - \sqrt{6} Y_2}{3}$$

$$X_3 = \frac{Y_3 - \sqrt{2} Y_2}{\sqrt{3}}$$

Put  $X_3$  in ③

$$X_1 + X_2 + \frac{Y_3 - \sqrt{2} Y_2}{\sqrt{3}} = \sqrt{3} Y_3$$

$$X_1 + X_2 = \sqrt{3} Y_3 - \frac{Y_3 - \sqrt{2} Y_2}{\sqrt{3}}$$

$$= \frac{3Y_3 - Y_3 + \sqrt{2} Y_2}{\sqrt{3}}$$

$$X_1 + X_2 = \frac{2Y_3 + \sqrt{2} Y_2}{\sqrt{3}} \rightarrow ④$$

$$① + ④ \Rightarrow 2X_1 = \sqrt{2} Y_1 + \frac{2Y_3 + \sqrt{2} Y_2}{\sqrt{3}}$$

$$X_1 = \frac{\sqrt{6} Y_1 + 2Y_3 + \sqrt{2} Y_2}{2\sqrt{3}}$$

$$\textcircled{1} \Rightarrow 2X_2 = \frac{2Y_3 + \sqrt{2}Y_2}{\sqrt{3}} - \sqrt{2}Y_1$$

$$2X_2 = \frac{2Y_3 + \sqrt{2}Y_2 - \sqrt{6}Y_1}{\sqrt{3}}$$

$$X_2 = \frac{2Y_3 + \sqrt{2}Y_2 - \sqrt{6}Y_1}{2\sqrt{3}}$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \frac{\partial X_1}{\partial Y_3} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \frac{\partial X_2}{\partial Y_3} \\ \frac{\partial X_3}{\partial Y_1} & \frac{\partial X_3}{\partial Y_2} & \frac{\partial X_3}{\partial Y_3} \end{vmatrix} = (X_1, X_2, X_3) t$$

$$\begin{vmatrix} \frac{\sqrt{6}}{2\sqrt{3}} & \cancel{\frac{2\sqrt{2}}{\sqrt{3}}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{6}}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \left( \frac{\sqrt{6}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{\sqrt{6}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \right) + \frac{1}{\sqrt{3}} \left( \frac{\sqrt{6}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{2\sqrt{3}} \right)$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \left( \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{6}} \right) + \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}} \right)$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{12}}$$

$$= \frac{2\sqrt{2}}{3\sqrt{2}} + \frac{\frac{2}{6}}{\sqrt{2}}$$

$$= \frac{\frac{2}{3} + \frac{1}{3}}{\sqrt{2}}$$

$$J = 1\sqrt{2} - \sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{2} = \sqrt{2}$$

$$|J| = 1$$

$$f(y_1, y_2, y_3) = |J| \cdot f(h_1, h_2, h_3)$$

$$= 1 \cdot f\left(\frac{\sqrt{6}y_1 + 2y_3 + \sqrt{2}y_2}{2\sqrt{3}}\right) \cdot f\left(\frac{2y_3 + \sqrt{2}y_2 - \sqrt{6}y_1}{2\sqrt{3}}\right)$$

$$\cdot f\left(\frac{y_3 - \sqrt{2}y_2}{\sqrt{3}}\right).$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \left(\frac{\sqrt{6}y_1 + 2y_3 + \sqrt{2}y_2}{2\sqrt{3}}\right)^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \left(\frac{2y_3 + \sqrt{2}y_2 - \sqrt{6}y_1}{2\sqrt{3}}\right)^2}$$

$$\cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \left(\frac{y_3 - \sqrt{2}y_2}{\sqrt{3}}\right)^2}$$

$$f(y_1, y_2, y_3) = \begin{cases} \frac{1}{(\sqrt{2\pi})^3} \cdot e^{-\frac{1}{2}(y_1^2 + y_2^2 + y_3^2)} & -\infty < y_1 < \infty \\ & -\infty < y_2 < \infty \\ & -\infty < y_3 < \infty \\ 0 & \text{otherwise} \end{cases}$$