$Max^m \rightarrow$ LUB and present in range "[" "]".

$Min^m \rightarrow$ GLB and present in range "[" "]"

Supremum $\rightarrow$ LUB
"(" ")"

Infimium $\rightarrow$ GLB
"(" ")"

If "[", "]" the
max^m = Supremum
min^m = Infimium.

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REAL ANALYSIS, NUMERICAL ANALYSIS AND CALCULUS

☆ REAL ANALYSIS:

☆ **LUB AXIOM:** (Least Upper bound Axiom OR Completeness property)

$T = \{ x : x \in Q, \ x > 0 \wedge x^2 < 2 \}$

Every non-empty subset S of $\mathbb{R}$ that is bounded above has a LUB. That is, sup S exists and is a real no.

Sequence → Infinite terms
but finite distinct terms.

Range or Range Set } — The distinct terms of the sequence.

→ Bounded above sequence:

$$S_n \le K \;,\; \forall\, n \in N$$

→ Bounded below sequence:

$$S_n \ge K \;\forall\; n \in N$$

→ **Bounded Sequence:**

Evidently, a sequence is bounded iff it's range is bounded.

Also, bounds of range are the bounds of sequence.

★ **Convergence of a sequence:** (ε = epsilon (will be given))

Def^n

A sequence $\{S_n\}$ is said to converge to a real no. l (or to have real no. l as it's limit) if for each $\varepsilon > 0$, $\exists$ a +ve integer m

Such that,

$|S_n - l| \le \varepsilon$, $\forall\ n \ge m$

$S_n \rightarrow l$ as $n \rightarrow \infty$

$$\lim_{n \to \infty} S_n = l$$

for $(l-\varepsilon, l+\varepsilon)$ only a finite terms will lie outside.

$(m-1)$ terms can lie outside.

Cause $m, m+1 \ldots \infty$ terms will lie inside $(l-\varepsilon, l+\varepsilon)$.

---

eg- $S_n = \frac{n}{2^n}$ $n \in N$

$S_n = \frac{1}{2}, \frac{2}{2^2}, \frac{3}{2^3}, \frac{4}{2^4} \ldots)$

$S_6 = 0.09375$

let $\boxed{\varepsilon = 0.1}$.

$\therefore N > 5 \quad \left| \frac{N}{2^N} - 0 \right| < 0.1$

here $m = 5$, $l = 0$, $\varepsilon = 0.1$

for $\varepsilon = 0.01$, $\quad \underset{m}{N} > 9 \quad \left| \frac{N}{2^N} - 0 \right| < 0.01$

for $\varepsilon = 0.00001$, $\quad \underset{m}{N} > 22 \quad \left| \frac{N}{2^N} - 0 \right| < 0.00001$

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Theorem ⇒ Every convergent sequence is bounded.

Let $\lim_{n\to\infty} S_n = l$

let $\varepsilon > 0$ be a given no. so that there exist a +ve integer, such that

$|S_n - l| < \varepsilon$, $\forall n \geq m$

$$\boxed{l - \varepsilon < S_n < l + \varepsilon, \forall n \geq m}$$

let, $g = \min\{l-\varepsilon, S_1, S_2 \cdots S_{m-1}\}$

$G = \max\{l+\varepsilon, S_1, S_2 \cdots S_{m-1}\}$

Thus, we have $g \leq S_n \leq G$, $\forall n \in N$.

Thus, $\{S_n\}$ is a bounded sequence.

$\Rightarrow$ The converse of the above theorem may not be ~~be~~ true.
(every bounded sequence is not convergent.

$\Rightarrow$ A sequence cannot converge to more than one limit.

$\Longrightarrow$ Every convergent sequence is bounded and has a unique limit.

Limit Points of a Sequence:

$\xi \rightarrow$ ita

A real no. ξ is said to be a limit point of a sequence $\{S_n\}$, if every neighbourhood of ξ contains an infinite no. of members of the sequence.

i.e, given any +ve no. ε, however small, $S_n \in (\xi - \varepsilon, \xi + \varepsilon)$ for an infinite no. of values of n.

$|S_n - \xi| < \varepsilon$ for infinitely many values of n.

Intuitively, S_n is arbitrarily close to ξ for an infinite no. of values of n.

In a set, a limit pt. is also called cluster pt. or pt. of condensation.

A no. ξ is not a limit pt. of sequence if there exists a no. $\varepsilon > 0$, such that, $S_n \in (\xi - \varepsilon, \xi + \varepsilon)$, has atmost a finite sequence.

A limit pt of the range set of a sequence is also a limit pt of the sequence. Converse may not be true

eg-① $S_n = 1, \forall n \in N$ $\{1, 1, 1, 1, \dots\}$

limit pt of sequence = ? = $\{1\}$

" " " range = ? = X

eg-② $S_n = \frac{1}{n}, \forall n \in N$ $\{1, \frac{1}{2}, \frac{1}{3} \dots\}$

limit pt of sequence = 0

limit pt of range = 0.

eg-③ $S_n = 1 + (-1)^n, n \in N$ $\{0, 2, 0, 2, 0 \dots\}$

limit pt of sequence = $\{0, 2\}$

limit pt of range = X

eg-④ $S_n = (-1)^n \left(1+\frac{1}{n}\right), n \in N$

limit pt of sequence $= \{-1, 1\}$

limit pt of range $= \{-1, 1\}$

The sequence $\Rightarrow \left\{-2, \left(1+\frac{1}{2}\right), -1\left(1+\frac{1}{3}\right), \left(1+\frac{1}{4}\right) \right\}$

So it's going to -1 and 1.

★ **Bolzano - Weierstrass Theorem:**

Every bounded sequence has a limit point.

The converse of the theorem? Not true.

★ Convergent Sequence:

⇒ Every bounded sequence with a unique limit point is convergent.

⇒ A necessary and sufficient cond^n for the convergance sequence is that it is bounded and has a unique limit point. (converse also true).

★ Non-convergent Sequence:

(a) Bounded – A bounded sequence which does not converge and has atleast 2 limit points, is said to oscillate.

(b) Unbounded – limit pt → $-\infty, \infty$.

⇒

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Prove: Show that

$$\lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2$$

$$\lim_{n\to\infty} \frac{3}{\sqrt{n}} + \frac{2\sqrt{n}}{\sqrt{n}}$$

$$= \boxed{2}$$

Th: if $\{a_n\}$ converges to L and $c \in R$, then the sequence $\{ca_n\}$ converges to cL.

OR

$$\boxed{\lim_{n\to\infty} ca_n = c \cdot \lim_{n\to\infty} a_n}$$

Th: $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

Th: $\{a_n\}$ converges to L

$\{b_n\}$ converges to M

and $a_n \leq b_n$, $\forall\, n \geq m$,

then $\boxed{L \leq M}$

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* Sandwich theorem (Squeeze theorem):

if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are three sequences, such that,

(i) $a_n \le b_n \le c_n \quad \forall \; n \in N$

& (ii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = l$.

then $\boxed{\lim\limits_{n\to\infty} b_n = l}$

Proof: Let $\varepsilon > 0$ be given, $\{a_n\}$, $\{c_n\}$ converge to l. Therefore, $\exists$ a +ve integer m_1, m_2, such that,

$|a_n - l| < \varepsilon \quad \forall \; n \ge m_1$ ① $|c_n - l| < \varepsilon \quad \forall n \ge m_2$ ②

~~and~~ ~~$m_2 \ge m_1$~~

and let $m = \max(m_1, m_2)$.

then for $n \ge m$, we have from ① & ②,

$$l - \varepsilon < a_n \le b_n \le c_n < l + \varepsilon .$$

$$\Rightarrow l - \varepsilon < b_n < l + \varepsilon \quad \forall \; n \ge m$$

$$\boxed{|b_n - l| < \varepsilon} \quad \forall \; n \ge m$$

$$\boxed{\text{Hence, } \lim_{n\to\infty} b_n = l}$$

eg- $\{b_n\} = \left\{\frac{\sin(n)}{n}\right\}, n \in \mathbb{N}$ (converges or diverges)

$$-1 \le \sin(n) \le 1, \forall n \in \mathbb{N}$$

$$\therefore -\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}, \forall n \in \mathbb{N}$$

$$\therefore \boxed{\{a_n\} = -\frac{1}{n}} \quad \boxed{\{c_n\} = \frac{1}{n}} \quad \forall n \in \mathbb{N}$$

$$\therefore a_n \le b_n \le c_n, \forall n \in \mathbb{N}$$

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$

$$\therefore \boxed{\lim_{n\to\infty} b_n = 0}$$

$\therefore b_n$ converges at 0.

★ MONOTONIC SEQUENCES:

A sequence $\{S_n\}$ is said to be monotonic inc. if $S_{n+1} \geq S_n \;\forall\, n \in \mathbb{N}$ and monotonic dec. if $S_{n+1} \leq S_n \;\forall\, n \in \mathbb{N}$.

It is said to be monotonic if it is either monotonic inc. or monotonic dec.

⇒ Strictly inc → $S_{n+1} > S_n \;\forall\, n \in \mathbb{N}$

Strictly dec → $S_{n+1} < S_n \;\forall\, n \in \mathbb{N}$

→ Monotonic sequences can either be converging or diverging but not oscillating.

Th A necessary and sufficient ~~cond~~ condⁿ for the convergence of a monotonic sequence is that it is bounded.

Corr 1 A monotonic increasing bounded above sequences converges to it's least upper bound, and a monotonic dec. bounded ~~above~~ below to the GLB.

⟹

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Corr 2. Every monotonic inc sequence which is not bounded above, diverges to $(+\infty)$.

Corr 3: - - - - - - - dec - - - - - - - - - - - - - - below, - - - - - - - $(-\infty)$.

Subsequences:

If $\{s_n\} = \{s_1, s_2, s_3 \ldots\}$ be a sequence, then any infinite succession of it's terms, picked out in any way (but preserving the original order) is called a subsequence of $\{s_n\}$, or, in other words,

eg - (1) $\{s_2, s_4, s_6 \ldots s_{2n} \ldots\}$ is a subsequence of s_n.

(2) $\{s_1, s_4, s_9, \ldots\}$

Th A sequence S_n converges to s iff its every subsequence converges to s. Similarly

$\lim S_n = \infty$ or $-\infty$

iff every subsequence of $\{S_n\}$ tends to $\infty, -\infty$.

Th If ξ is a limit point of ~~a~~ sequence $\{S_n\}$, then there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ which converges to ξ.

i.e, $\boxed{\lim_{k \to \infty} S_{n_k} = \xi}$

Q) $\left\{ \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2^9}, \frac{1}{2^{16}}, \frac{1}{2^{25}}, \dots, \left(\frac{1}{2}\right)^{n^2}, \dots \right\}$

converges or diverges?

Ans) Converges to 0.

The sequence in the question is a subsequence of

$$s_n = \left(\frac{1}{2}\right)^n$$

& $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n \longrightarrow \underline{\underline{0}}.$

$\therefore$ converges to 0.

Q) Show that the sequence $\{b_n\}$ ∀

$$\{b_n\} = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots \frac{1}{(2n)^2}\right]$$

converges to 0.

$\Rightarrow$

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Ans) we know that, $\frac{1}{(n+1)^2} \geq \frac{1}{(2n)^2}$ | $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$

$\frac{1}{(n+2)^2} \geq \frac{1}{(2n)^2}$ | $\frac{1}{(n+2)^2} \leq \frac{1}{n^2}$

$\vdots$ | $\vdots$

$\frac{1}{(2n)^2} \geq \frac{1}{(2n)^2}$

$$\therefore \frac{n}{(2n)^2} \leq b_n \leq \frac{n}{n^2}$$

$$\boxed{\therefore \frac{1}{4n} \leq b_n \leq \frac{1}{n}}$$

$\{a_n\} = \frac{1}{4n}$ $\quad$ $\{c_n\} = \frac{1}{n}$

$\lim_{n\to\infty} a_n = 0$ $\quad$ $\lim_{n\to\infty} c_n = 0$

$$\boxed{\therefore \lim_{n\to\infty} b_n = 0}$$

$\therefore b_n$ converges to 0.

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Questions:

Q1) Show that $\lim_{n \to \infty} \frac{n-1}{n+1} = 1$ (from def^n).

Ans) $\left| \frac{n-1}{n+1} - 1 \right| = \left| \frac{-2}{n+1} \right| = \frac{2}{n+1}$

$\frac{2}{n+1} < \varepsilon$ if $n+1 > \frac{2}{\varepsilon}$

i.e, if $\boxed{n > N, \ \forall \ N = \frac{2}{\varepsilon} - 1}$

Given $\varepsilon > 0$, $\left| \frac{n-1}{n+1} - 1 \right| = \frac{2}{n+1} < \varepsilon$ if $\boxed{n > \frac{2}{\varepsilon} - 1}$

m

$\therefore \frac{n-1}{n+1}$ converges to 1.

$\therefore \lim_{n \to \infty} \frac{n-1}{n+1} = 1.$ ✓

$\Rightarrow$

(Q2) By defⁿ, show that

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$$

we know that

Ans $\left(\sqrt{n+1}-\sqrt{n}\right)=\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{2\sqrt{n}}\right)$

given $\varepsilon>0$, $\frac{1}{2\sqrt{n}}<\varepsilon$ if $\frac{1}{4n}<\varepsilon^2$

, i.e, if $\boxed{n>\frac{1}{4\varepsilon^2}}$ ⟵ M

$$\boxed{\left|\sqrt{n+1}-\sqrt{n}-0\right|<\varepsilon,\ \forall\ n>\frac{1}{4\varepsilon^2}}$$

$\therefore$ $\sqrt{n+1}-\sqrt{n}$ converges to 0.

$$\therefore \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$$

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Q3) $\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4} \ldots\ldots\}$

Does this converge?

Does this have a limit point.

Ans) Does not converge cause not bounded

but limit point $= 0$.

Q4) $\lim_{n \to \infty} \frac{2 - \cos n}{n+3} = ?$

$$-1 \leq \cos n \leq 1$$

$$\cancel{1 \geq -\cos n \geq -1}$$

$$2-1 \leq 2 - \cos n \leq 2+1$$

$$1 \leq 2 - \cos n \leq 3$$

$$\underbrace{\frac{1}{n+3}}_{a_n} \leq \underbrace{\frac{2-\cos n}{n+3}}_{b_n} \leq \underbrace{\frac{3}{n+3}}_{c_n}$$

$\lim_{n \to \infty} a_n = 0$ $\lim_{n \to \infty} c_n = 0$

$$\boxed{\Rightarrow \lim_{n \to \infty} b_n = 0}$$

S_n

Q5) $\left(\sqrt{30}, \sqrt{30+\sqrt{30}}, \sqrt{30+\sqrt{30+\sqrt{30}}} \cdots\right)$.

Does it converge ?

Ans Recursive $\longrightarrow$ $a_{n+1} = \sqrt{30 + a_n} \quad \forall \; n \geq 1$

$a_1 = \sqrt{30}$

$a_1 = \sqrt{30} < 6$

$a_2 = \sqrt{30+\sqrt{30}} < 6$

$\therefore$ converges to 6 .

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Infinite Series:

If $U_1, U_2, U_3 \ldots U_n \ldots$ be an infinite sequence of real nos., then,

$$U_1 + U_2 + U_3 \ldots\ldots U_n + \ldots\ldots \infty$$

is called an infinite series, denoted by $\boxed{\Sigma U_n}$ and the sum of it's first n terms is denoted by $\boxed{S_n}$ (sequence of partial sums)

Convergence, divergence and Oscillation of a Series:

Consider the infinite series $\Sigma U_n = U_1 + U_2 \ldots U_n \ldots \infty$

and let sum of first n terms be $S_n = U_1 + U_2 \ldots U_n$

Clearly, S_n is a func. of n and as n increases indefinitely, 3 possibilities:

1. for $S_n \to$ finite limit, convergent.
2. for $S_n \to \pm\infty$ as $n \to \infty$, divergent.
3. if $S_n \to$ no unique limit as $n \to \infty$, oscillatory.

eg-① Examine the convergence of series:

$$1+2+3.......+n.....\infty$$

Ans) $S_n = 1+2+3.......n = \frac{(n+1)n}{2}$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty} n(n+1) \longrightarrow \infty$$

$\therefore$ S_n ~~this series~~ is divergent.

$\therefore$ The series ΣU_n is divergent.

eg-② Examine the convergence of the series:

$$5-4-1+5-4-1+5-4-1.......\infty$$

Ans) $S_n = 5-4-1+5-4-1+......n$ terms

$= 5, 1, 0$ $\boxed{\begin{array}{l} n = 3m+1 \\ 3m+2 \\ 3m \end{array}}$ respectively.

$\therefore$ Oscillatory.

Geometric Series: $1 + r + r^2 \ldots \infty$

⇒ Show that the series $1 + r + r^2 \ldots \infty$

(1) ~~[illegible]~~ converges if $|r| < 1$

(2) diverges if $r \geq 1$ and

(3) oscillates if $r \leq -1$

Proof: (1) if $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$ such that

$$\boxed{S_n = \frac{1}{1-r}}$$ S_n converges.

(2) when $r > 1$, $\lim_{n \to \infty} r^n$ diverges to ∞.

now $S_n = \frac{r^n - 1}{r - 1} = \frac{r^n}{r-1} - \frac{1}{r-1}$

∴ S_n diverges ~~to~~ ∞.

The series is divergent.

(3) when $r < -1$, let $r = -\rho$, so that $\rho > 1$

then $r^n = (-1)^n (\rho)^n$

$$S_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n \rho^n}{1+\rho}$$

as $\lim_{n\to\infty} \rho^n \to \infty$,

$\lim_{n\to\infty} S_n \to \infty$ or $-\infty$ ~~...~~ if n is even or odd respectively.

$\therefore$ Oscillating.

---

eg- $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots \infty$

$S_n = \frac{1}{2} + \frac{1}{2^2} \dots \frac{1}{2^n}$

$$S_n = \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}$$

$$S_n = \left(1 - \frac{1}{2^n}\right)$$

$$\therefore \boxed{\lim_{n\to\infty} S_n = 1}$$

$\therefore$ series converges to 1.

~~Necessary And Sufficient condⁿ for convergence:~~

Necessary condⁿ for convergence:

A necessary condⁿ for convergence of an infinite series ΣU_n is $\lim_{n\to\infty} U_n = 0$

Let $S_n = U_1 + U_2 \ldots\ldots U_n$ $\{S_n\}$

As ΣU_n converges, the sequence $\{S_n\}$ also converges.

Let $\lim_{n\to\infty} S_n = S$

Now, $U_n = S_n - S_{n-1}$, $n > 1$

$$\therefore \boxed{\lim_{n\to\infty} U_n} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = \boxed{0}$$

$\Rightarrow$ Converse is not true.

, i.e, if $\lim_{n\to\infty} U_n = 0$, then series can be either convergent or divergent.

(1) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots \infty$

~~converges at~~

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2}$

$$\boxed{S_{2^n} > \frac{1}{2}(n-2)}$$

$\therefore$ this series diverges

and is called Harmonic series

eg- $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$ - - - - - - - - .

Ans) $U_n = \frac{n}{n+1} \Rightarrow U_n = \frac{1}{1+\frac{1}{n}}$

$$\lim_{n \to \infty} U_n = 1 \neq 0$$

$\therefore \Sigma U_n$ is not converging

☆ General property of Series:

① The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite no. of it's terms.

② If a series, in which all the terms are +ve, is convergent, then

③ The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite no.

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☆ Series of positive terms:

An infinite series in which all the terms after some particular terms are positive is a +ve term series.

eg→ $-7-3+1+5+9+13+17 \cdots$

~~☆ Integral Test:~~

☆ P Test (Test for comparison):

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} \cdots}$$

$$\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots$$

is convergent (P test)

$$\sum \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \cdots$$

is divergent (by p test)

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* Comparison Tests:

1. If two positive term series ΣU_n and ΣV_n be such that

(i) ΣV_n converges

(ii) $\Sigma U_n \leq V_n \quad \forall \, n \in N$

then ΣU_n converges.

$\Rightarrow$ eg - $\Sigma U_n = \Sigma \frac{1}{2^n + n}$

let $\Sigma V_n = \Sigma \frac{1}{2^n}$

(1) ΣV_n converges because $|r| < 1$ (geometric series)

as $2^n + n > 2^n$

then $\frac{1}{2^n+n} < \frac{1}{2^n}$

$\therefore$ (2) $U_n \leq V_n$ for $n \in N$

$\boxed{\therefore \Sigma U_n \text{ is convergent}}$

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$\boxed{\begin{array}{l} \text{2. If two positive term series } \Sigma U_n \text{ and } \Sigma V_n \text{ be such that} \\ \text{(i) } \Sigma V_n \text{ diverges.} \\ \text{(ii) } U_n \geq V_n \ \forall \, n \in N \\ \text{then } \Sigma U_n \text{ is also divergent.} \end{array}}$

eg - $\Sigma U_n = \Sigma \frac{1}{n+1}$

(1) $\Sigma V_n = \Sigma \frac{1}{2n} \longrightarrow$ diverges

$\Rightarrow n+1 < n+n$

$\therefore \frac{1}{n+1} > \frac{1}{2n} \quad \forall n$

$\therefore$ (2) $U_n \geq V_n \quad , \forall \, n \in N$

$\therefore \boxed{\Sigma U_n \text{ converges.}}$

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D'Alemberts Ratio test :

★ ~~LIMIT FORM : (comparison test)~~

In a positive series ΣU_n,

$$\text{if } \underset{n\to\infty}{Lt} \frac{U_{n+1}}{U_n} = \lambda \text{, then}$$

the series converges if $\lambda < 1$
& diverges if $\lambda > 1$

The test fails if $\lambda = 1$

eg - $\Sigma \left(\frac{n^2-1}{n^2+1}\right) x^n$, $x > 0$

let $U_n = \frac{n^2-1}{n^2+1} \cdot x^n$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = x$$ Solve urself.

$\therefore$ by D'Alembart's ration test, the series converges if $x < 1$
& diverges if $x > 1$.

★ LIMIT FORM: (comparison test).

If two positive term series ΣU_n and ΣV_n be such that,

① $\lim_{n\to\infty} \frac{U_n}{V_n} = K\ (\neq 0)$ (finite)

the ΣU_n and ΣV_n converges or diverges together.

---

eg- $\Sigma U_n = \Sigma \sin \frac{1}{n}$

$U_n = \sin\left(\frac{1}{n}\right)$

Let $V_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \boxed{1}$$

ΣV_n diverges

$\therefore \Sigma U_n$ diverges

$\Rightarrow$ Ratio test fails at $\lambda = 1$.

$$\Sigma U_n = \Sigma \frac{1}{n^p}$$

$$\text{So } \lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \lim_{n \to \infty} \frac{n^p}{(n+1)^p}$$

$$= \lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{n}\right)^p} = \frac{1}{\cancel{}} = \lambda$$

So $\lambda = 1$ for any p.

But we know that from p test, ΣU_n converges for $p > 1$ and ΣU_n diverges for $p \leq 1$.

$\Rightarrow$

☆ Cauchy's Root Test:

In a +ve series ΣU_n, if $\lim_{n \to \infty} (U_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

The test fails when $\lambda = 1$

eg- $\Sigma (\log n)^{-2n}$

$$\lim_{n \to \infty} (U_n)^{1/n} = \lim_{n \to \infty} (\log n)^{-\frac{2n}{n}}$$

$$= \lim_{n \to \infty} \left(\frac{1}{\log n}\right)^2 = \underline{\underline{0}}$$

∴ By Cauchy's root test, the given series converges.

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* **Alternating Series**: A series in which the terms are alternatingly +ve and -ve.

eg → $u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots \infty$

* **Leibnitz Series**: An alternating series $u_1 - u_2 + u_3 \cdots$ converges if (i) each term is numerically less than it's preceeding term.

(ii) $\lim_{n\to\infty} u_n = 0$

eg → $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \infty$ converges or not?

Ans) $u_n = \frac{1}{\sqrt{n}}$

(1) $\boxed{u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}}} < 0$ ✓

(2) $\boxed{\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0}$ ✓

and is an alternating series,

∴ By Leibnitz rule, the given series converges.

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☆ Series of positive and negative terms : (general case)

The +ve series & alternating series are special type of these series.

☆ Absolutely convergent :

If the series $U_1 + U_2 + U_3 \cdots\cdots U_n \cdots \infty$ be such that the series $|U_1| + |U_2| + |U_3| \cdots |U_n| \cdots \infty$ is convergent, then the original series $\sum U_n$ is said to be absolutely convergent.

eg → $(-1)^n \left(\frac{1}{n^2}\right)$ is absolutely convergent.

☆ Conditionally convergent :

~~If $\sum U_n$~~

If $\sum |U_n|$ is divergent but $\sum U_n$ is convergent, then $\sum U_n$ is said to be conditionally convergent.

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eg → $\Sigma U_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots \infty$ is convergent.

(by Leibnitz rule)

$\Sigma |U_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots \infty$ is divergent.

$\therefore 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots \infty$ is conditionally convergent

Q1) Test whether the following series is absolutely convergent or not?

$$\Sigma \frac{(-1)^{n-1}}{2n-1}$$

Ans) $\Sigma U_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots \infty$

→ convergent by Leibnitz rule.

$\Sigma |U_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \cdots$

→ divergent by limit form

$$\lim_{n \to \infty} \frac{|U_n|}{|V_n|} = \frac{1}{2} \ (\neq 0)$$

$V = \boxed{V_n = \frac{1}{n}}$

$\therefore \Sigma U_n$ is conditionally convergent.

✩ POWER SERIES:

A series of the form $a_0 + a_1 x + a_2 x^2 \cdots a_n x^n \cdots \infty$ where the a's are independent of x, is called a power series in x.

The series may converge or diverge for different ~~val~~ values of x.

✩ Interval of convergence:

In the power series, $\Sigma U_n = a_n x^n$.

We use ratio test,

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) x$$

$$\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n} \cdot x\right| = |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= l\,|x|$$

$$\forall\ l = \left|\frac{a_{n+1}}{a_n}\right|$$

$\therefore\ \Sigma U_n$ converges if $l|x| < 1$ as absolute convergence $\Downarrow$ convergent.

$\Rightarrow$

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$\Rightarrow |x| < \frac{1}{d}$

OR

$$\boxed{\frac{-1}{d} < x < \frac{1}{d}}$$ (Interval of convergence)

If the series is +ve,

$\left(x > \frac{1}{d} \text{ and } x < \frac{-1}{d}\right)$ will give interval of divergence.

$\Rightarrow$ cant find interval of divergence for non +ve series.

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Q) $\frac{1}{1-x} + \frac{1}{2(1-x)^2} \cdots\cdots \frac{1}{n(1-x)^n} \cdots\cdots \infty$

$$U_n = \frac{1}{n(1-x)^n}$$

Ans) $\lim_{n\to\infty} \frac{\frac{1}{(n+1)(1-x)^{n+1}}}{\frac{1}{n(1-x)^n}} \Rightarrow \lim_{n\to\infty} \frac{n}{n+1} \times \frac{1}{(1-x)}$

$\therefore \left|\frac{1}{1-x}\right| < 1$

$\frac{1}{x-1} < 1$ $\qquad$ $\frac{1}{1-x} < 1$

$x - 1 > 1$ $\qquad$ $1 - x > 1$

$\boxed{x > 2}$ $\qquad$ $\boxed{x < 0}$

~~$|1-x| > 1$~~

~~$-1 > 1-x > 1$~~

~~$-2 > -x > 0$~~

~~$0 < x < 2$~~

For $x = 0$ & $x = 2$

$x=0$ → harmonic → divergent

$x=2$ → alternating → By Leibnitz rule → convergent.

$$\therefore \boxed{x < 0 \ \& \ x \geq 2}$$

☆ Convergence of Exponential series:

⟹ The series is $1 + \frac{x}{1!} + \frac{x^2}{2!} \cdots \frac{x^n}{n!} \cdots \infty$

⟹ Let $U_n = \frac{x^n}{n!}$, $U_{n+1} = \frac{x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{n+1} = x \lim \frac{1}{n+1} = \boxed{0} < 1$$

∴ The exponential series converges for any value of x.

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Convergence of Logarithmic series:

$$\Rightarrow x - \frac{x^2}{2} + \frac{x^3}{3} + \dots (-1)^{n-1}\frac{x^n}{n} \dots \infty$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(-1)^n \frac{x^{n+1}}{n+1}}{(-1)^{n-1}\frac{x^n}{n}} \right| \Rightarrow \lim_{n\to\infty} \left| (-1)\, x\, \frac{n}{n+1} \right|$$

$$\Rightarrow |x|$$

$|x| < 1 \rightarrow$ converges

$\boxed{-1 < x < 1}$ Interval of convergence

for $x = -1$ $\downarrow$ -ve harmonic $\downarrow$ diverges

for $x = 1$ $\downarrow$ alternating $\downarrow$ by Leibnitz $\downarrow$ converges

$$\therefore \boxed{-1 < x \leq 1}$$

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Questions:

P1) Converges or diverges ?

$$\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$$

WRONG

Ans) $\frac{1}{4}, \frac{8}{35} \dots \infty$

$$\lim_{n\to\infty} \frac{n^3}{n^5+3} \Rightarrow \lim \frac{1}{n^2+\frac{3}{n^3}} = \boxed{0}$$

~~and every preceding term > successor.~~

~~∴ By Leibnitz rule,~~ $\boxed{\text{converging.}}$

Can also use limit form ∀ $V_n = \frac{1}{n^2}$

RIGHT

or use comparison test ∀ $V_n = \frac{1}{n^2}$

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P2) $\sum \frac{3^n}{4^n+4}$

Ans) By cauchy Root Test,

$$\lim_{n\to\infty} \frac{(3^n)^{1/n}}{(4^n \cancel{+4})^{1/n}} = \frac{3}{4} < 1$$

~~$\lim_{n\to\infty}$~~

$$\& \left(\frac{3^n}{4^n+4}\right) < \frac{3^n}{4^n}$$

$\therefore$ Convergent

or

$\left(\frac{3}{4}\right)^n$ is ~~Geom~~ Geometric series with $r = \frac{3}{4}$

$\therefore$ convergent.

P5) Converges or diverges.

$$\sum \frac{n!\,(n+1)!}{(3n)!}$$

Ans) $\lim_{n\to\infty} \dfrac{\dfrac{(n+1)!\,(n+2)!}{(3(n+1))!}}{\dfrac{n!\,(n+1)!}{(3n)!}}$

$\Rightarrow \lim_{n\to\infty} \dfrac{\cancel{(n+1)}\,(n+2)}{3\,(3n+1)(3n+2)\,\cancel{(n+1)}}$

$\Rightarrow \boxed{0} < 1$

$\therefore$ convergent.

By D'Alembert's Ratio Test

P4) $\sum_{n=0}^{\infty} \frac{n^3 \cdot x^{3n}}{n^4+1}$

$\Rightarrow 0, \frac{x^3}{2}, \frac{8x^6}{17}, \ldots \infty$

$\lim_{n\to\infty} \frac{\frac{(n+1)^3 \cdot x^{3(n+1)}}{(n+1)^4+1}}{\frac{n^3 \cdot x^{3n}}{n^4+1}} = \lim_{n\to\infty} \frac{(n+1)^3 (n^4+1) x^3}{n^3 ((n+1)^4+1)}$

$= |x^3| < 1$

when $x=-1$, $x=1$ $\therefore -1 < x^3 < 1$

$\frac{n^3(-1)^{3n}}{n^4+1}$ → converging

$\frac{n^3}{n^4+1}$ → diverging

$\boxed{-1 \leq x < 1}$

P5) converges absolutely or ~~and~~ conditionally or diverges.

$$\sum (-1)^n \frac{1}{\sqrt{n^2+1}}$$

Ans) $\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{5}}, \frac{-1}{\sqrt{9}} \dots \infty$

By Liebnitz test, converging

~~for $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{8}}, \dots \infty$~~

$$\cancel{\lim_{n\to\infty} \frac{\frac{1}{\sqrt{(n+1)^2-1}}}{\frac{1}{\sqrt{n^2-1}}} \Rightarrow \lim_{n\to\infty} \frac{\sqrt{n^2-1}}{\sqrt{(n+1)^2-1}} \Rightarrow \lim_{n\to\infty} \frac{n\sqrt{1-1/n^2}}{n\sqrt{\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}}} \Rightarrow \boxed{\frac{1}{1}}}$$

for $|U_n|$, $\sum \frac{1}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{1}{(n)} \rightarrow$ diverging

By limit form ∴ conditionally convergent.

Sequence and Series of Functions:

⇒ Sequence of Real nos: $f : N \to R$

⇒ Sequence of functions : $E \subset R$ $E = [a, b]$

for each $n \in N$ let $f_n : E \to R$ be a function.

then $\{f_n\}$ is a sequence of fns on E to R.

$\underbrace{E}_{\text{domain}}$

$x \in E$, then $\{f_n(x)\}$ is a sequence of real nos, for some $x \in E$, $\{f_n(x)\}$ may converge, for other $x \in E$, it may not.

If $\{f_n(x)\}$ converges $\forall$ $x \in R$, then $f : E \to R$

$f(x) = \lim_{n \to \infty} f_n(x)$, $x \in E$

$\{f_n\}$ converges ~~to~~ f point-wise,

$f_n \to f$ pointwise $= f_n(x) \to f(x)$, $\forall$ $x \in E$

$\cancel{\epsilon}$ $\epsilon > 0$

eg- $E = [0,1]$, $f_n(x) = x^n$, $0 \le x \le 1$

Ans) $x=0$, $f_n(0) = 0$, $f_1(0) = 0$ .....

for $x=0$, $f_n(x)$ converges to ~~f(x)~~ 0.

So, here $\boxed{\begin{aligned} f(x) &= 0 , \quad 0 \le x < 1 \\ &= 1 , \quad x = 1 \end{aligned}}$

eg) $\forall n \in N$ let $f_n : R \to R$ $f_n(x) = \frac{x}{n}$, $x \in R$

$\Rightarrow$ will converge to 0 for all values of R.

$\Rightarrow$

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Uniform Convergence:

Let $f: E \to R$ be a sequence of functions and $f: E \to R$ be a fn, we say that $f_n \to f$ uniformly if

$\forall\ \epsilon > 0\ \exists\ n_0 \in N$, such that $\forall\ n \geq n_0$

$$|f_n(x) - f(x)| < \epsilon, \quad \forall\ x \in E$$

Th: If $f_n \to f$ uniformly, then $f_n \to f$ pointwise

$f_n \to f$ pointwise $\equiv f_n(x) \to f(x),\ \forall\ x \in E$

$\equiv \forall\ x \in E,\ \forall\ \epsilon > 0,\ \exists\ n_0 \in N$, such that,

$$\underbrace{n \geq n_0}_{n_0(x)} \Rightarrow |f_n(x) - f(x)| < \epsilon$$

Let $f_n \to f$ uniformly,

$$\{|f_n(x) - f(x)| : x \in E\}$$

Summary

$\Rightarrow$ $\{f_n\} \to f$ pointwise on $E \subseteq \mathbb{R}$

$\forall x \in E$, $f(x) = \lim_{n \to \infty} f_n(x)$

$\Rightarrow$ $\{f_n\} \to f$ uniformly

$\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$

$|f_n(x) - f(x)| < \epsilon$, $\forall x \in E$

Last class of Unit 1
missed