

16/14/21

3. Random Variables

→ Quantifying a phenomena

Tossing 3 coins

$$\Omega = \{ \cancel{\{TTT\}}, \{TTH\}, \{HTT\}, \{HTH\} \\ \{THH\}, \{HTH\}, \{HHT\}, \{HHH\} \}$$

No. of heads can be $X=0, 1, 2 or } 3.$

$$X(\{TTT\}) = 0$$

$$X(\{TTH\}, \{HTT\}, \{HTH\}) = 1$$

$$X(\{THH\}, \{HTH\}, \{HHT\}) = 2$$

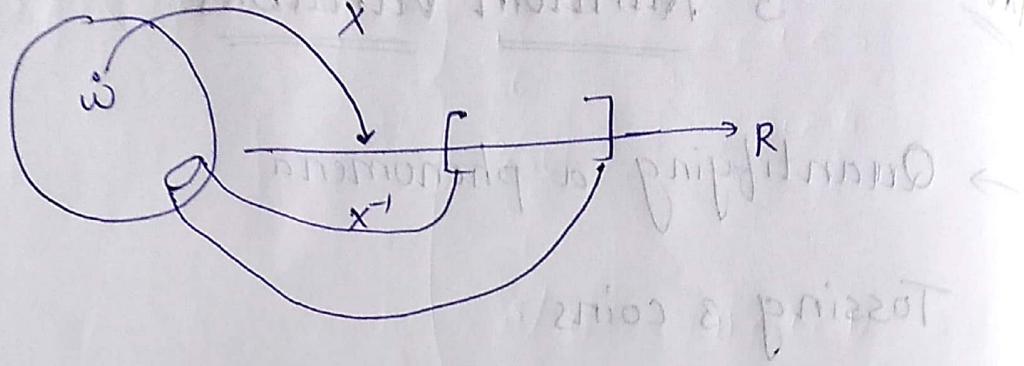
$$X(\{HHH\}) = 3$$

→ Random Variable associates a real number

to each outcome

Defn: Let (Ω, \mathcal{S}) be a sample space. A function X maps Ω into \mathbb{R} is called a Random Variable if the inverse images under X of all Borel sets in \mathbb{R} are events. i.e; $\forall B \in \mathcal{B}$

$$X^{-1}(B) = \{ \omega : X(\omega) \in B \} \in \mathcal{S}$$



$$X^{-1}(x_0) = X^{-1}(-\infty, 0) = \emptyset$$

$$X^{-1}(0 \leq x < 1) = \{TTT\}$$

$$X^{-1}(1 \leq x < 2) = \{HTT, THT, TTH\}$$

$$X^{-1}(2 \leq x < 3) = \{HHT, HTH, THH\}$$

$$X^{-1}(2 \leq x \leq 3) = \{HHH\}$$

Note:

$$X^{-1}(a \leq x \leq b) = \{\text{events i.e. } X(\omega) \in [a, b]\}$$

If X is a R.V $\{X=x\}, \{a < x \leq b\}, \{X < x\},$

$\{a \leq X < b\}, \{a \leq X \leq b\}, \{a \leq X \leq b\}$ are all events

Ex: Let $\Omega = \{H, T\} \quad X(H) = 1, X(T) = 0$

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset & \text{if } x < 0 \\ \{T\} & \text{if } 0 \leq x < 1 \\ \{H, T\} = \Omega & \text{if } 1 \leq x \end{cases}$$

<u>Ex</u>	3 coins are tossed	$X(\omega) = \text{no. of heads}$
	X	0 1 2 3
	$P(X)$	$\frac{1}{8} \quad \frac{3}{8} \quad \frac{3}{8} \quad \frac{1}{8}$

Total = 3

Ex: $\Omega = \{HH, TT, HT, TH\}$

$X(\omega) = \text{no. of heads}$

$$X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 0$$

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset & x < 0 \\ \{TT\} & 0 \leq x < 1 \\ \{HT, TH, TT\} & 1 \leq x < 2 \\ \Omega & x \geq 2 \end{cases}$$

$P(X^{-1}) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$

Def The Random Variable X defined on probability space (Ω, S, P) induces a probability space called Probability distribution function

$$P(X^{-1}(B)) = P\{\omega : X(\omega) \in B\} \quad \forall B \in \mathcal{B}$$

↓
events

Ex: Throwing 2 dice $\Omega = \{(i,j) ; 1 \leq i, j \leq 6\}$

$X \rightarrow \text{sum of the two numbers on the dice}$

Possible values of $X : 2, 3, 4, 5, \dots, 12$

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset & x < 2 \\ \{(1,1)\} & 2 \leq x < 3 \\ \{(1,2), (2,1), (1,1)\} & 3 \leq x < 4 \\ \vdots & \vdots \\ \Omega & x \geq 12 \end{cases}$$

$$P(X \in (-\infty, x)) = \begin{cases} 0 & x < 2 \\ \frac{1}{36} & 2 \leq x < 3 \\ \frac{3}{36} & 3 \leq x < 4 \\ \vdots & \vdots \\ \frac{35}{36} & 11 \leq x < 12 \\ 1 & x \geq 12 \end{cases}$$

19/4/21

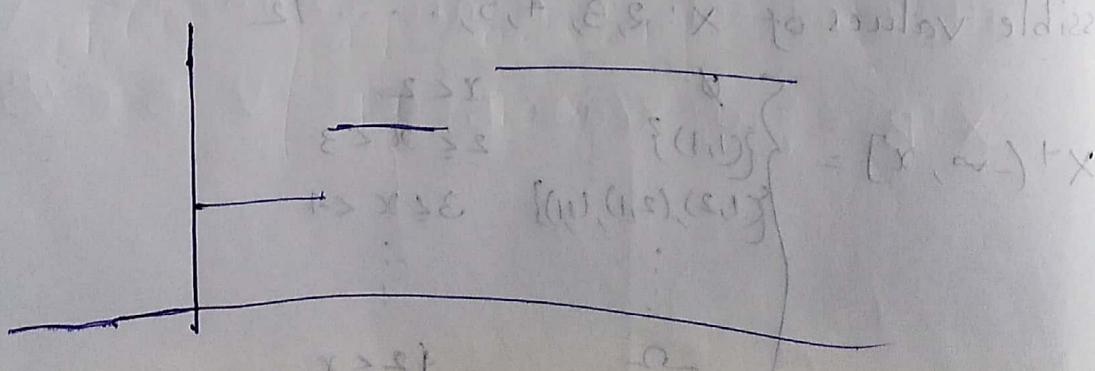
Cumulative distribution Function:

For any random variable X , cdf is the distribution function defined by

$$F(x) = P(X \leq x), x \in \mathbb{R}$$

Properties:

- i) $F(x)$ is always non decreasing
($F(x) \leq F(y)$ if $x \leq y$)
- ii) $\lim_{x \rightarrow -\infty} F(x) = 0$, $\lim_{x \rightarrow \infty} F(x) = 1$
- iii) $F(x)$ is continuous only right continuous.
i.e., $\lim_{x \rightarrow c^+} F(x) = F(c)$



Discrete Random Variable: (X takes finite no. of values or countable no. of values)

For discrete R.V the c.d.f is given by

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

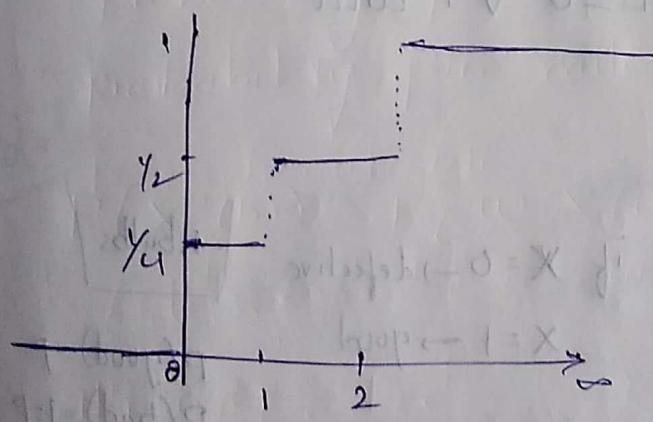
Ex: $x = 0, 1, 2, 3, 4$

$$P(X \leq 2) = P(X=1) + P(X=2) + P(X=0)$$

Ex: Tossing 2 coins. X = no. of heads

$$\begin{array}{c|ccc} X & 0 & 1 & 2 \\ \hline P(X=x) & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$$

$$F(x) = P(X \leq (-\infty, x]) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{3}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



Continuous Random Variable: For continuous R.V

cdf is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du \text{ for } x \in \mathbb{R}$$

& interval probability $P(a \leq x \leq b) = \int_a^b$

Ex: $f(x) = \frac{x}{6}$ \times interval is $[2, 4]$

$$\int_{-\infty}^{\infty} f(x) dx = \int_2^4 \frac{x}{6} dx = \frac{1}{6} \left[\frac{x^2}{2} \right]_2^4 = 1$$

$$cdf F(x \leq 2.5)$$

$$= F(x \leq 2.5) = \int_2^{2.5} \frac{x}{6} dx = \frac{1}{6} \left[\frac{x^2}{2} \right]_2^{2.5}$$

$$= \frac{6 \cdot 2.5 - 4}{12} = \frac{2.25}{12}$$

$$= 0.1875$$

Probability Density Func

Probability Mass Function: (Discrete R.V)

Def. collection of all numbers $\{P_i\}$ is called PMF

if i) $P\{X = x_i\} = P_i \geq 0 \quad \forall i$ and

ii) $\sum_{i=1}^{\infty} P_i = 1$

$$P(X=x) = \begin{cases} 1-p & \text{if } X=0 \rightarrow \text{defective} \\ p & \text{if } X=1 \rightarrow \text{good} \end{cases}$$

$\boxed{n \text{ bulbs}}$

$$p(\text{good}) = p$$
$$p(\text{bad}) = 1-p$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ p & 1 \leq x \end{cases}$$

$$\therefore (x \geq x) q = (x) F$$

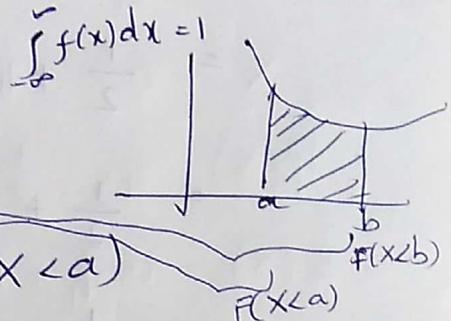
Probability Density Function:

For a continuous random variable X , if there exists non negative function $f(x)$ such

that $F(x) = \int_{-\infty}^x f(t) dt$ is called pdf of X

$$P(a < x < b) = \int_a^b f(t) dt$$

$$= F(x < b) - F(x < a)$$



Ex:

cdf $F(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$ $F'(x) = f(x)$

pdf $f(x) = \begin{cases} 0 & x \leq 0 \text{ & } x \geq 1 \\ 1 & 0 < x < 1 \end{cases}$

20/4/21

$\uparrow \text{cdf}$ $\uparrow \text{pdf}$
 $F(x) = \int f(x) dx$

discrete \rightarrow pmf
 continuous \rightarrow pdf

$$F'(x) = f(x)$$

~~pdf~~ $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$

$$\int f(x) dx = 1$$

$$\int_0^1 x dx + \int_1^2 (2-x) dx$$

$$= \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2$$

$$= \frac{1}{2} + \left(4 - \frac{4}{2} \right) - \left(2 - \frac{1}{2} \right)$$

$$= \frac{1}{2} + 2 - \frac{3}{2}$$

$$= 1$$

$\therefore f(x)$ is a pdf

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x t dt = \frac{x^2}{2} & 0 \leq x < 1 \\ \int_0^1 t dt + \int_1^x (2-x) dx & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

$$c.d.s \quad F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \leq x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$\begin{aligned}
 P\{0.3 < X < 1.5\} &= F(1.5) - F(0.3) \\
 &= 2(1.5) - \frac{(1.5)^2}{2} - 1 - \frac{(0.3)^2}{2} \\
 &= 3 - \frac{2.25}{2} - 1 - \frac{0.09}{2} \\
 &= 2 - \frac{2.34}{2} \\
 &= 2 - 1.17 \\
 &= 0.83
 \end{aligned}$$

Expectation of a Random Variable:

Denoted by $E(X)$ defined as $\mu = \sum_{K=1}^{\infty} x_i P(X=x_i)$
 for discrete case

$$\mu = \int_{-\infty}^{\infty} xf(x)dx \text{ for continuous case.}$$

Ex: Two players A & B
 $X(A) =$

Find expected gain of A.

$$X(H) = -1, X(T) = 1$$

$$P(H) = p \quad P(T) = 1-p$$

$$E(x) = (-1)p + 1(1-p) = 1-2p$$

If $p = \frac{1}{2}$ $E(x) = 0$ (fair game)

Thm If X is a random variable & $Y = g(X)$

$$E(Y) = \sum_{j=1}^{\infty} g(x_j) \cdot P(X=x_j)$$

Ex: Tossing 2 coins $X = \text{no. of heads}$

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$Y = X^2 \quad E(X^2) = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = \frac{3}{2}$$

$$\boxed{E(Y = ax + b) = aE(X) + b}$$

Continuous case:

$$f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

$$\int x f(x) dx = E(X) = \int_1^{\infty} x \cdot \frac{2}{x^3} dx = \int_1^{\infty} \frac{2}{x^2} dx$$
$$= \left(-\frac{2}{x} \right)_1^{\infty} = 0 - (-2) = 2$$

$$E(X^2) = \int_1^{\infty} x^2 \cdot \frac{2}{x^3} dx = \int_1^{\infty} \frac{2}{x} dx = \text{undefined}$$

does not exist.

$$(comparing) \quad a = (x) \quad , \quad \frac{1}{x} = q + b$$

Variance: For any random variable X ,

$E(X-\mu)^2$ is called variance.

$$\sigma^2 = \text{Var}(X) = E((X-\mu)^2)$$

σ is called standard deviation.

$$\begin{aligned} E((X-\mu)^2) &= E(X^2 + \mu^2 - 2\mu X) \\ &= E(X^2) + E(\mu^2) - 2\mu E(X) \\ &= E(X^2) + \mu^2 - 2\mu E(X) \\ &= E(X^2) + \mu^2 - 2\mu^2 \\ \boxed{E((X-\mu)^2)} &= E(X^2) - (E(X))^2 \end{aligned}$$

Ex: Calculate $\text{Var}(X)$, when X is representing the outcome when a fair die is rolled

X	1	2	3	4	5	6
$P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 2 + \frac{9}{6} = 2 + \frac{3}{2} = \frac{7}{2}$$

$\text{Var}(X)$

$$\begin{aligned} E(X^2) &= 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} \\ &= \frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \frac{25}{6} + \frac{36}{6} \\ &= \frac{91}{6} \end{aligned}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{91}{6} - \frac{49}{4} \quad \cancel{\frac{49}{4}} = \frac{35}{12} \quad \sigma = \sqrt{\frac{35}{12}}$$

$$\text{Var}(ax+b) = a^2 \text{Var}(x) + 0 \\ = a^2 \text{Var}(x)$$

$$\text{Var}(x) = E(x-\mu)^2 = E(x^2) - (E(x))^2$$

$$E(c) = c, \text{Var}(c) = 0 \text{ for any } c \in R$$

21/4/21

Moments:

For any positive integer k , & any constant c

$E((x-c)^k)$, is called moment of order k about point c .

* if $c=0$ i.e., $E(x^k)$ is called k^{th} moment

about origin $m_i = E(x^i)$ or μ_i

* if $c=\mu$ $E((x-\mu)^k)$ is called k^{th} moment about mean $[\mu_k = E(x^k)]$ is called central moments

About origin ($c=0$) About Mean ($c=\mu$)

$$\mu'_1 = E(X) = \mu = \text{Mean}$$

$$\mu_1 = E(X-\mu) = E(X) - E(\mu) = 0$$

$$\mu'_2 = E(X^2)$$

$$\mu_2 = E((X-\mu)^2) = V(X) = \sigma^2$$

$$\mu'_3 = E(X^3)$$

$$\mu_3 = E((X-\mu)^3)$$

$$\mu'_4 = E(X^4)$$

$$\mu_4 = E((X-\mu)^4)$$

Ex: $P(X=k) = 1$ & 0 elsewhere Degenerate distribution

cdf $F(x) = \begin{cases} 0 & x < k \\ 1 & x \geq k \end{cases}$ At only 1 point probability = 1

$$E(X) \times E(x) = \sum x_i P(X=x_i) = k \cdot 1 = k$$

$$\mu'_2 = E(X^2) = k^2 \cdot 1 = k^2$$

$$\mu'_l = E(X^l) = k^l \cdot 1 = k^l$$

$$\begin{aligned} \mu_1 &= 0 & \mu_2 &= E((X-\mu)^2) \\ &&&= E(X^2) - (E(X))^2 \\ &&&= k^2 - k^2 \end{aligned}$$

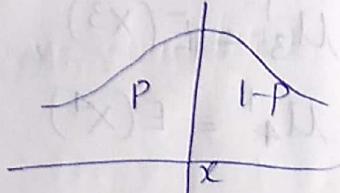
For degenerate distribution $\Rightarrow \text{Var}(X) = 0$

Quantiles:

A number x satisfying

$$P\{X \leq x\} \geq P, \quad P\{X \geq x\} \geq 1-P, \quad P^{\text{th}} \text{ quantile}$$

if $P = \frac{1}{2}$, x is called median



For $F(x)$ $P \leq F(x) \leq P + \{X=x\}$ F is cdf

$$P\{X \geq x\} \geq 1-P$$

$$1 - P\{X \leq x\} \geq 1-P$$

$$-P\{X < x\} \geq -P$$

$$P\{X < x\} \leq P$$

$$P\{X < x\} + P\{X = x\} \leq P + P\{X = x\}$$

For $F(x)$
$$\boxed{P \leq F(x) \leq P + \{X=x\}}$$
 For discrete case

F is cdf

$$P \leq F(x) \leq P + P\{X=x\}$$

For continuous case $P\{X=x\}=0$

$$P \leq F(x) \leq P$$

$$F(x) = P$$

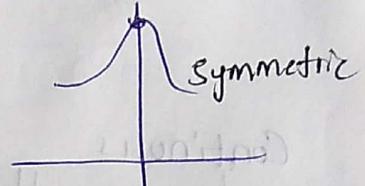
Median: Let X be a R.V. with distribution function $F(x)$. A number x , satisfying

$$\frac{1}{2} \leq F(x) \leq \frac{1}{2} + P\{X=x\} \quad \text{For discrete}$$

$$F(x) = \frac{1}{2} \quad \text{For continuous}$$

Ex ① $P\{X=-2\} = P\{X=0\} = \frac{1}{4}$

$$P\{X=1\} = \frac{1}{3}, P\{X=2\} = \frac{1}{6}$$



Centre is median
for symmetric

$x=0$:

$$P\{X \leq 0\} = F(0) = \frac{1}{2}$$

$$P\{X \geq 0\} = \frac{1}{4} + \frac{1}{3} + \frac{1}{6} = \frac{3}{4} > \frac{1}{2}$$

$$P\{X \geq 0\} \geq \frac{1}{2} = P$$

$$P\{X \geq 0\} = \frac{3}{4} \geq 1 - P$$

For any point
 $0 < x < 1$

$$P\{X \leq x\} \geq \frac{1}{2}$$

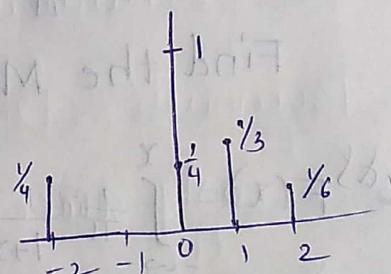
$$P\{X \geq x\} \geq \frac{1}{2}$$

Any $x \in (0, 1)$ is a median

if $P = 0.2$

$$P\{X \leq x\} \geq 0.2 \quad P\{X \geq x\} \geq 1 - 0.2 = 0.8$$

$$P\{X \leq -2\} \geq 0.2 \quad P\{X \geq -2\} \geq 0.8$$



$$F(-2) = \frac{1}{4} = 0.25$$

$$F(0) = \frac{1}{2} = 0.5$$

$$F(1) = \frac{5}{6}$$

$$F(2) = 1$$

$$P\{X \geq x\} \geq 1 - 0.2 = 0.8$$

$$P\{X \geq -2\} \geq 0.8$$

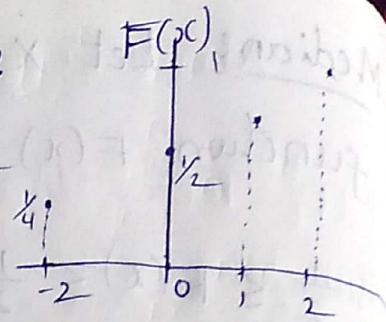
~~Frac
Error~~

$$P \leq F(x) \leq p + P\{X=x\}$$

Let $x = -2$

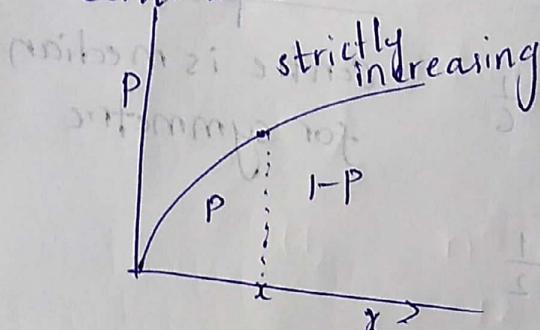
$$0.2 \leq F(x) \leq 0.2 + 0.25$$

$$0.2 \leq F(x) \leq 0.45$$

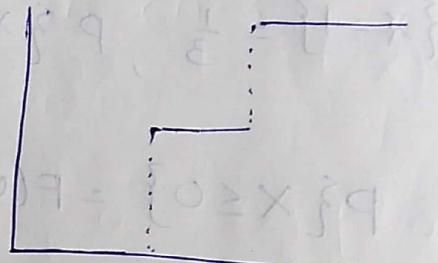


$F(-2) = 0.25$ satisfies the above condition.

Continuous



Discrete



For any given P ,
 x is unique

For any given P ,
there are infinite quantiles.

Ex
②

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad -\infty < x < \infty$$

Find the Median

$$\begin{aligned} F(x) &= \int_{-\infty}^x \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi} (\tan^{-1} x) \Big|_{-\infty}^x \\ &= \frac{1}{\pi} (\tan^{-1} x + \frac{\pi}{2}) \\ &= \frac{1}{\pi} \tan^{-1} x + \frac{1}{2} \end{aligned}$$

For median case $p = \frac{1}{2}$

$x = 0$ is median.

23/4/21

Quartiles: When we divide into 4 parts

they are called quartiles denoted by Q

For 4

Q_1

Q_2

Q_3

Quartiles

For 10

Q_{10}

Q_{20}

Q_{30}

deciles

For 100

Q_{100}

Q_{200}

Q_{300}

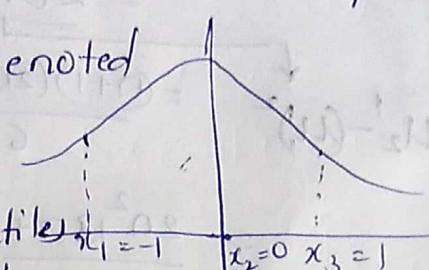
Percentiles

$F(x) = \frac{1}{\pi} [\tan^{-1} x + \frac{\pi}{2}]$

$x_1 = -1$

$x_2 = 0$

$x_3 = 1$



$$F(x) = \frac{1}{4} \text{ lower quartile}$$

$$F(x) = \frac{1}{4} \Rightarrow x = -1 \quad F(x) = \frac{3}{4} \text{ upper quartile}$$

$$F(x) = \frac{3}{4} \Rightarrow x = 1$$

Discrete distribution:

1. Degenerate distribution: $P\{X=k\} = 1$,
 $P=0$ for any other value.

2. Uniform Discrete Distribution: The R.V X is said to have uniform distribution on n points $\{x_1, x_2, \dots, x_n\}$ if its PMF is $P\{X=x_i\} = \frac{1}{n}$, $i=1, 2, \dots, n$

Ex: Tossing coin, $\{H, T\}$ $P(H) = P(T) = \frac{1}{2}$

Throwing dice $\{1, 2, 3, 4, 5, 6\}$ $P(1) = P(2) = \dots = P(6) = \frac{1}{6}$

$$\mu'_1 = E(X) = \sum_{i=1}^n x_i P(X=x_i)$$

$$= \sum_{i=1}^n x_i \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$$

$$\begin{aligned}
 \mu_1 &= 0 & \mu_2' &= E(X-\mu)^2 & \mu_2' &= E(X^2) \\
 && &= E(X^2) - (E(X))^2 & &= \sum_{i=1}^n \frac{1}{n} \cdot x_i^2 \\
 && &= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 & &= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} \\
 && &= \frac{2n^2 + 3n + 1}{6} - \frac{n^2 + 2n + 1}{4} & &= \frac{(n+1)(2n+1)}{6} \\
 && &= \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} & & \\
 \mu_2 &= \frac{n^2 - 1}{12}
 \end{aligned}$$

Note: Using central moments we can find non central moments and viceversa.

→ Moment generating function generates non central moments.

3. Bernoulli's Distribution:

only success or failure.

$X=0 \Rightarrow$ failure, $X=1 \Rightarrow$ success.

Ex: Tossing coin $P(X=1(H)) = \frac{1}{2}$

$$P(X=0(H)) = \frac{1}{2}$$

probability of success is p
failure is $1-p$ or q .

PMF $P\{X=1\} = p \quad P\{X=0\} = 1-p = q$

$$\mu_1' = E(X) = \sum x_i p(x) = 1 \cdot p + 0 \cdot (1-p) = p$$

$$\begin{aligned}M_1 &= 0 \\M'_2 &= E(X^2) = \sum x^2 p(x) \\&= 1^2 \cdot P + 0^2 (1-P) = P\end{aligned}$$

$$\begin{aligned}E(X^k) &= M_k = P \\M_2 &= E(X^2) - (E(X))^2 \\&= P - P^2 = P(1-P) = PQ\end{aligned}$$

A Binomial Distribution: We say X has Binomial distribution if $P(X=k) = \sum_{k=0}^n {}^n C_k \cdot P^k \cdot (1-P)^{n-k}$

* n trials must be independent.

* Success or failure for each trial.

* R.V, $X \rightarrow$ no. of successes

$$P(\text{success}) = P, P(\text{failure}) = 1-P$$

$$\text{pmf} = \sum_{x=0}^n {}^n C_x \cdot P^x q^{n-x} = \sum_{x=0}^n (P+q)^x$$

Ex: 4 coins are tossed $x = 0, 1, 2, 3, 4$ $= 1$

What is probability that $P(2H \& 2T)$

$$n=4$$

Treating head as success $k=2$

$$P\{X=2\} = {}^4 C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = \frac{4 \times 3}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{8}$$

$$P\{X \leq 3\} = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 1 - P(X > 3)$$

$$= 1 - P\{X=4\}$$

$$= 1 - {}^4 C_4 \cdot \left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16}$$

$$= \frac{15}{16}$$

Moments:

$$\begin{aligned}
 E(X) &= \sum_{x=0}^{\infty} x \cdot P(X=x) \\
 &= \sum_{x=0}^{\infty} x \cdot {}^n C_x \cdot P^x \cdot q^{n-x} \\
 &= \sum_{x=0}^{\infty} x \cdot \frac{n!}{(n-x)! x!} \cdot P^x \cdot q^{n-x} \\
 &= \sum_{x=0}^{\infty} x \cdot \frac{n(n-1)!}{x \cdot (x-1)! (n-x)!} \cdot P^x \cdot q^{n-x} \\
 &\quad \boxed{x(n-x) = q^n} \quad \frac{n(n-1)!}{(x-1)! (n-x)!} \cdot P^x \cdot q^{n-x} \\
 &= n \cdot \sum_{x=0}^{\infty} \frac{(n-1)!}{(x-1)! (n-x)!} \cdot P^x \cdot q^{n-x}
 \end{aligned}$$

$$\begin{aligned}
 &\stackrel{q=1-(p+q)}{=} n \cdot P \cdot \sum_{x=0}^{n-1} {}^{n-1} C_{x+1} \cdot P^{x+1} \cdot q^{n-x} \\
 &\stackrel{1-p=q}{=} n \cdot P \cdot \sum_{x=0}^{n-1} (p+q)^{n-1-x}
 \end{aligned}$$

$$\mu_1 = E(X) = np$$

$$\mu_1 = 0$$

$$\begin{aligned}
 \mu_2 &= E(X^2) = \sum x^2 \cdot P(X=x) \\
 &= \sum x^2 \cdot \frac{n!}{(n-x)! x!} \cdot P^x \cdot q^{n-x}
 \end{aligned}$$

$$\begin{aligned}
 E(X^2) &= E(X(x-1) + x) \\
 &= E(X^2 - x + x) \\
 &= E(X(X-1)) + E(X)
 \end{aligned}$$

$$\begin{aligned}
 E(X(X-1)) &= \sum x(x-1) \cdot \frac{n!}{x!(n-x)!} \cdot p^x \cdot q^{n-x} \\
 &= \sum x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} \cdot p^x \cdot q^{n-x} \\
 &= n(n-1)p^2 \cdot \sum \frac{(n-2)!}{(x-2)!(n-x)!} \cdot p^{x-2} \cdot q^{n-x} \\
 &= n(n-1)p^2.
 \end{aligned}$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$\mu_2' = n(n-1)p^2 + np$$

$$\begin{aligned}
 \text{Var} &= E(X^2) - (E(X))^2 \\
 &= n(n-1)p^2 + np - (np)^2 \\
 &= n^2p^2 - np^2 + np - n^2p^2 \\
 &= np(1-p)
 \end{aligned}$$

$$\text{Var} = npq$$

26|4|2)

5) Poisson Distribution: A Random Variable
is said to have poisson distribution if

PMF

$$P\{X=x\} = \frac{e^{-\lambda} \lambda^x}{x!}, \lambda > 0, x=0, 1, 2, \dots$$

$$X \sim P(\lambda)$$

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

→ Independent event

Ex: If no. of accidents on a highway each day
is a poisson distribution with parameter $\lambda = 3$

$P\{\text{there is no accident}\}$

Sol Let X be the no. of accidents

$$P\{X=0\} = \frac{e^{-3} \cdot (3)^0}{0!} = e^{-3} = (2.718)^{-3} = 0.049$$

Moments

$$E(X) = \sum_{x=0}^{\infty} x \cdot P(X=x)$$

$$= \sum x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \sum \frac{x \cdot \lambda^x}{x(x-1)!}$$

$$= e^{-\lambda} \cdot \sum \frac{\lambda \cdot \lambda^{x-1}}{(x-1)!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$E(X) = \lambda$$

$$E(X^2) = E(x(x-1) + x)$$

$$= E(x(x-1)) + E(x) = (\lambda)$$

$$E(x(x-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^2 \lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda}$$

$$= \lambda^2$$

$$E(X^2) = E(x(x-1)) + E(x)$$

$$= \lambda^2 + \lambda$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$\mu_2 = \text{Var}(X) = \lambda$$

For poison distribution Mean & Variance are same
i.e.,

Ex: Suppose that typographical errors on a single page of a book has poison distribution with parameter $\lambda = 1$

$P\{ \text{atleast one error on this page} \}$

$P(X = \text{no. of errors})$

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - \frac{e^{-1} \cdot 1^0}{0!}$$

$$= 1 - \frac{e^{-1} \cdot (1)^0}{1!}$$

$$= 1 - \frac{1}{e}$$

$$= 0.633$$

6) Geometric Distribution:

Ex A coin is tossed until a head appears

4th success T T T H

nth success (n-1) failures 1 success

Def A random variable is said to follow geometric distribution if

$$P\{X=x\} = (1-p)^{x-1} p$$

$$= q^{x-1} \cdot p$$

Ex: If your probability of getting success is 0.2 in voting, what is the probability that you get success on third day?

$$\text{Sol} \quad X: 0, 1, 2, \dots$$

$$P(X=3) = (1-P)^{x-1} \cdot P$$

$$= (0.8)^2 \cdot (0.2)$$

$$= (0.64)(0.2)$$

$$= 0.128.$$

Moments

$$E(X) = \sum_{x=1}^{\infty} x (1-P)^{x-1} P$$

$$= P \sum_{x=1}^{\infty} x (1-P)^{x-1}$$

$$= P \cdot \left(\frac{1}{(1-(1-P))^2} \right)$$

$$= \frac{P}{P^2} = \frac{1}{P}$$

$$\mu_1' = E(X) = \frac{1}{P}$$

$$E(X^2) = E(X(X-1) + X)$$

$$= P \cdot \frac{1}{P^2} = \frac{1}{P}$$

$$E(X(X-1)) = \sum_{x=1}^{\infty} x(x-1) (1-P)^{x-1} \cdot P$$

$$= P \sum_{x=1}^{\infty} x(x-1) (1-P)^{x-1}$$

$$= P \sum_{x=1}^{\infty} x(x-1) (1-P)^{x-2}$$

$$\begin{aligned}
 &= P(1-P) \sum_{x=1}^{\infty} x(x-1) (1-P)^{x-2} \\
 &= P(1-P) \frac{d^2 \sum (1-P)^x}{dP^2} \\
 &= P(1-P) \cdot \frac{d^2}{dP^2} \left(\frac{1-P}{P} \right) \\
 &= P(1-P) \cdot \frac{d}{dP} \left(-\frac{1}{P^2} \right) \\
 &= P(1-P) \cdot \left(\frac{2}{P^3} \right)
 \end{aligned}$$

$$E(X(X-1)) = \frac{2(1-P)}{P^2}$$

$$E(X^2) = \frac{2(1-P)}{P^2} + \frac{1}{P}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{2(1-P)}{P^2} + \frac{1}{P} - \frac{1}{P^2}$$

$$= \frac{1-2P}{P^2} + \frac{1}{P} = \frac{1-2P+P}{P^2}$$

$$= \frac{1-P}{P^2} = \frac{q}{P^2}$$

Continuous Distribution:

i) Uniform distribution: A R.V is said to follow uniform distribution if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Ex: If X is uniformly distributed over $[0, 10]$

calculate

$$f(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

a) $P\{X < 3\}$

b) $P\{X > 7\}$

c) $P\{1 < X < 6\}$

$$P\{X < 3\} = \int_0^3 f(x) dx = \frac{1}{10} \int_0^3 dx = \frac{3}{10}$$

$$P\{X > 7\} = \int_7^{10} f(x) dx = \frac{3}{10}$$

$$P\{1 < X < 6\} = \int_1^6 f(x) dx = \frac{5}{10}$$

$$\begin{aligned} E(X) &= \int_a^b x \cdot f(x) dx = \frac{1}{b-a} \int_a^b x \cdot dx \\ &= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b \\ &= \frac{1}{b-a} \left[\frac{b^2 - a^2}{2} \right] \end{aligned}$$

$$\mu_1 = E(X) = \frac{a+b}{2}, \mu_1 = 0$$

$$\begin{aligned}
 E(X^2) &= \int_a^b x^2 \cdot f(x) dx \\
 &= \frac{1}{b-a} \int_a^b x^2 dx \\
 &= \frac{1}{b-a} \cdot \left[\frac{x^3}{3} \right]_a^b \\
 &= \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} \\
 &= \frac{(b-a)(a^2 + b^2 + ab)}{(b-a) \cdot 3}
 \end{aligned}$$

$$\mu'_2 = E(X^2) = \frac{a^2 + b^2 + ab}{3}$$

$$\begin{aligned}
 \mu_2 = \text{Var}(X) &= E(X^2) - (E(X))^2 \\
 &= \frac{a^2 + b^2 + ab}{3} - \left(\frac{a+b}{2} \right)^2 \\
 &= \frac{a^2 + b^2 + ab}{3} - \frac{a^2 + b^2 + 2ab}{4} \\
 &= \frac{4a^2 + 4b^2 + 4ab - 3a^2 - 3b^2 - 6ab}{12} \\
 &= \frac{a^2 + b^2 - 2ab}{12}
 \end{aligned}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

27/4/21

2) Gamma distribution:

$$\text{Gamma Function } \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0$$

$$\Gamma(\alpha) = (\alpha-1) \Gamma(\alpha-1)$$

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2) \Gamma(\alpha-2)$$

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)(\alpha-3)\dots 1 = (\alpha-1)!$$

$$\Gamma(1) = 1$$

$$\Gamma(\alpha) = (\alpha-1)!$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1 + \frac{1}{2}\right) \\ = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\ = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0$$

$$\text{Substitute } x = \frac{y}{\beta} \Rightarrow dx = \frac{dy}{\beta}$$

$$\Gamma(\alpha) = \int_0^{\infty} \frac{\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y/\beta} dy}{\beta}$$

$$\Gamma(\alpha) = \int_0^{\infty} \frac{y^{\alpha-1} e^{-y/\beta} dy}{\beta^{\alpha}}$$

$$1 = \frac{1}{\Gamma(\alpha) \beta^{\alpha}} \int_0^{\infty} y^{\alpha-1} e^{-y/\beta} dy$$

A R.V is said to follow gamma distribution if its pdf is given by

$$\text{pdf } f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} & 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

denoted by $y \sim r(\alpha, \beta)$

Moments

$$\begin{aligned} \mu'_1 &= E(Y) = \int_0^\infty y \cdot \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^\alpha \cdot e^{-y/\beta} dy \\ &= \frac{\beta^{\alpha+1} \Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty \frac{y^\alpha \cdot e^{-y/\beta}}{\Gamma(\alpha+1) \beta^{\alpha+1}} dy}_1 \\ &= \frac{\beta^{\alpha+1} \Gamma(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \\ &= \frac{\beta (\alpha+1) \Gamma(\alpha)}{\beta^\alpha \Gamma(\alpha)} \end{aligned}$$

$y \sim r(\alpha+1, \beta)$

$$\mu'_1 = \alpha\beta$$

$$\mu'_1 = 0$$

$$\begin{aligned}
 \mu_2' &= E(Y^2) = \int_0^\infty y^2 \cdot \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy \\
 &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha+1} \cdot e^{-y/\beta}}{\Gamma(\alpha+2)} dy \\
 &= \frac{\Gamma(\alpha+2) \beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha+1} \cdot e^{-y/\beta}}{\Gamma(\alpha+2) \beta^{\alpha+2}} dy \\
 &= \frac{\Gamma(\alpha+2) \beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \quad [Y \sim \Gamma(\alpha+2, \beta)] \\
 &= \frac{\alpha(\alpha+1) \beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)}
 \end{aligned}$$

$$\mu_2' = \alpha(\alpha+1)\beta^2$$

$$\begin{aligned}
 \text{Var}(X) &= \mu_2 - E(X)^2 \\
 &= \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 \\
 &= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 \\
 &= \alpha\beta^2
 \end{aligned}$$

Ex: i) Let the time to fix a car follows gamma distribution with mean = 2 & Variance is 2 hours

Sol

$$\text{Mean} = 2, \sigma^2 = 2$$

$$\alpha\beta = 2 \quad \alpha\beta^2 = 2$$

$$\beta = 1$$

$$\alpha = 2$$

$$f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} & 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$f(y) = \begin{cases} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} & 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} y e^{-y} & 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

ii) What is the chance that the waiting time is atmost 4.5 hours

$$P\{Y \leq 4.5\} = \int_0^{4.5} y \cdot e^{-y} dy$$

$$= \left[y \cdot (-e^{-y}) \right]_0^{4.5} + \int_0^{4.5} e^{-y} dy$$

$$= \left[-ye^{-y} \right]_0^{4.5} + \left[-e^{-y} \right]_0^{4.5}$$

$$= -(4.5)e^{-4.5} + [-e^{-4.5} + 1]$$

$$= 1 - e^{-4.5} (1+4.5) = 1 - e^{-4.5} (5.5) \approx 0.9395 \approx 0.94$$

3) Exponential Distribution:

Special case of Gamma distribution

where $\alpha = 1$

$$f(y) = \begin{cases} \frac{e^{-y/\beta}}{\beta} & 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Moments

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = E(X) = \mu_1$$

$$\begin{aligned} &= \int_0^\infty x \cdot \frac{1}{\beta} e^{-x/\beta} dx \\ &= \frac{1}{\beta} \int_0^\infty x e^{-x/\beta} dx \\ &= \frac{1}{\beta} \left(\left[x \cdot \frac{e^{-x/\beta}}{-\frac{1}{\beta}} \right]_0^\infty - \int_0^\infty \frac{1 \cdot e^{-x/\beta}}{-\frac{1}{\beta}} dx \right) \\ &= \frac{1}{\beta} \frac{1}{\beta} \int_0^\infty e^{-x/\beta} dx \\ &= \frac{1}{\beta} \frac{1}{\beta} \frac{1}{-\beta^2} \left(e^{-x/\beta} \right)_0^\infty \\ &= \frac{1}{\beta} \frac{1}{\beta} \frac{1}{\beta^2} (0 - 1) \end{aligned}$$

$$\mu_1 = \beta$$

$$\mu_1 = 0$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \frac{1}{\beta} \int_0^\infty x^2 \cdot e^{-x/\beta} dx$$

$$= \frac{1}{\beta} \left[x^2 \frac{e^{-x/\beta}}{-1/\beta} \right]_0^\infty - \int_0^\infty 2x \left(\frac{e^{-x/\beta}}{-1/\beta} \right) dx$$

$$= \frac{1}{\beta} \frac{1}{(\frac{1}{\beta})} 2 \cdot \int_0^\infty x \cdot e^{-x/\beta} dx$$

$$= 2 \cdot \int_0^\infty x \cdot e^{-x/\beta} dx$$

$$= 2 \cdot \left[\left(x \frac{e^{-x/\beta}}{-1/\beta} \right)_0^\infty - \int_0^\infty \frac{e^{-x/\beta}}{-1/\beta} dx \right]$$

$$= 2 \left[\beta \cdot \frac{(e^{-x/\beta})_0^\infty}{-1/\beta} \right]$$

$$= -2\beta^2 [0-1]$$

$$\mu_2' = 2\beta^2$$

$$\text{Var} = E(X^2) - (E(X))^2 = 2\beta^2 - \beta^2$$

$$\mu_2 = \beta^2$$

Ex: Evaluate $\int_0^\infty x^6 e^{-5x} dx$

$$f(y) = \int_0^\infty \frac{y^{x-1} e^{-y/\beta}}{\Gamma(x) \beta^x} dy = 1$$

$$I = \int_0^\infty x^6 \cdot e^{-5x} dx$$

$$= \left(\frac{1}{5}\right)^7 \cdot \Gamma(7) \quad \alpha = 7 \\ \beta = \frac{1}{5}$$

$$= \frac{1}{5^7} \cdot 6! \quad \Gamma(x) = (x-1)!$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2}{5^7}$$

$$= \frac{144}{55}$$

$$= 0.0092$$

Ex: Let X be amount of time a postal clerk spends on his customer which has exponential distribution with average amount of time is

~~4 minutes~~ 4 minutes i) Find Mean & S.D
ii) Formulate PDF

$$\text{Mean} = \beta = 4$$

$$\text{Variance} = \beta^2 = 16, \text{S.D.} = \sqrt{\text{Var}} = 4$$

$$f(x) = \frac{1}{\beta} e^{-x/\beta} = 0.25 e^{-0.25x}, x > 0$$

iii) Find $f(5)$

$$f(x) = 0.25 e^{-0.25x}$$
$$f(5) = 0.25 e^{-0.25(5)}$$
$$= 0.072$$

iv) $P\{4 < x < 5\}$

$$= F(5) - F(4)$$

$$= \int_4^5 0.25 e^{-0.25x} dx$$

$$= 0.25 \left[\frac{e^{-0.25x}}{-0.25} \right]_4^5$$

$$= \frac{0.25}{-0.25} \left[e^{-0.25x} \right]_4^5$$

$$= \left[e^{-0.25x} \right]_4^5$$

$$= 0.3679 - 0.2865$$

$$= 0.0814$$

28/4/21

4) Beta Distribution: Domain $[0, 1]$

Beta Function $B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx, \alpha, \beta > 0$

$$1 = \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad \text{defines pdf}$$

for Beta Function

$$\text{i.e., } f(x) = \begin{cases} \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha, \beta)} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

we say R.V $X \sim B(\alpha, \beta)$ ~~Bf~~

Relation between beta & gamma

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \Gamma(\alpha) = (\alpha - 1)!$$

$$B(5, 6) = \frac{\Gamma(5) \cdot \Gamma(6)}{\Gamma(5+6)}$$

$$= \frac{4! 5!}{10!}$$

$$\text{Moments: } E(X) = \mu_1 = \int_0^1 x f(x) dx$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^1 x \cdot x^{\alpha-1} (1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha, \beta)} \int_0^1 x^\alpha (1-x)^{\beta-1} dx$$

$$= \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} \int_0^1 \frac{x^\alpha (1-x)^{\beta-1} dx}{B(\alpha+1, \beta)} \xrightarrow{\textcircled{1}} \\ = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)}$$

$$= \frac{\frac{\Gamma(\alpha+1) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$= \frac{\alpha \Gamma(\alpha)}{(\alpha+\beta) \Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$\mu'_1 = \frac{\alpha}{\alpha+\beta}$$

$$\mu_1 = 0$$

$$\mu_1 = E(X^2) - (E(X))^2$$

$$\mu'_2 = \frac{1}{B(\alpha, \beta)} \int_0^1 x^2 \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx$$

$$\begin{aligned}
 &= \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha+1} (1-x)^{\beta-1} dx \\
 &= \frac{B(\alpha+2, \beta)}{B(\alpha, \beta)} \int_0^1 \frac{x^{\alpha+1} (1-x)^{\beta-1}}{B(\alpha+2, \beta)} dx \\
 &= \frac{B(\alpha+2, \beta)}{B(\alpha, \beta)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\Gamma(\alpha+2) \Gamma(\beta)}{\Gamma(\alpha+2+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \\
 &= \frac{(\alpha+1)\alpha \cdot \Gamma(\alpha) \Gamma(\alpha+\beta)}{(\alpha+1)(\alpha+\beta) \Gamma(\alpha+\beta) \Gamma(\alpha)}
 \end{aligned}$$

$$\mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+1+\beta)(\alpha+\beta)}$$

$$\begin{aligned}
 \mu_1 &= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta} \right)^2 \\
 &= \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
 &= \frac{\alpha(\alpha+\beta)(\alpha+1) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\alpha(\alpha^2 + \alpha + \alpha\beta + \beta) - \cancel{\alpha^2(\alpha+\beta+1)}}{(\alpha+\beta)^2(\alpha+\beta+1)} = \cancel{\frac{\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}}
 \end{aligned}$$

$$= \frac{\cancel{x^3} + \cancel{x^2} + \cancel{\alpha^2\beta} + \alpha\beta - \cancel{\alpha^3} - \cancel{\alpha^2\beta} - \cancel{x}}{(\alpha+\beta)^2 (\alpha+\beta+1)}$$

$$= \frac{\cancel{x^2(\alpha+\beta)} + \alpha\beta}{(\cancel{\alpha+\beta})^2 (\alpha+\beta+1)}$$

$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Ex:

$$f(x) = \begin{cases} \frac{1}{12}x^2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

i) Find the parameters α, β

$$\alpha = 3, \beta = 2$$

ii) Find $E(X)$ & Variance

$$E(X) = \frac{\alpha}{\alpha+\beta} = \frac{3}{5}$$

$$\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{6}{25 \times 6} = \frac{1}{25}$$

iii) $P\{0.2 < x < 0.5\}$

$$= \int_{0.2}^{0.5} \frac{1}{12}x^2(1-x)dx$$

$$= \frac{1}{12} \cdot \int_{0.2}^{0.5} (x^2 - x^3) dx$$

$$\begin{aligned}
 &= \frac{1}{12} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_{0.2}^{0.5} \\
 &= \frac{1}{12} \cdot \left[\left(\frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} \right) - \left(\frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} \right) \right] \\
 &= \frac{1}{12} \cdot \left[\frac{4(0.5)^3 - 3(0.5)^4}{12} - \frac{4(0.2)^3 - 3(0.2)^4}{12} \right] \\
 &= \frac{1}{12} \cdot \left[\frac{(0.5)^3 [4 - 1.5]}{12} - \frac{(0.2)^3 [4 - 0.6]}{12} \right] \\
 &= \frac{1}{12} \left[\frac{(0.5)^3 (2.5)}{12} - \frac{(0.2)^3 (3.4)}{12} \right]
 \end{aligned}$$

$$\begin{aligned}
 &\frac{0.3125 - 0.0272}{144} \\
 &= \frac{\cancel{0.2853}}{\cancel{144}} 0.00198
 \end{aligned}$$

5) Normal Distribution / Gaussian Distribution:

Any R.V X is said to have normal distribution if its pdf is given by

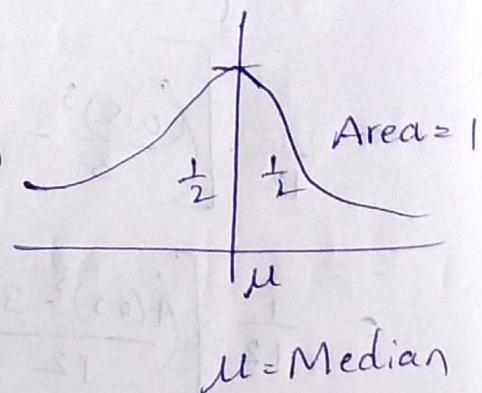
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)} \quad \text{for } -\infty < x < \infty$$

where μ is mean &
 σ is standard deviation

we say $X \sim N(\mu, \sigma^2)$

Properties:

- i) Bell shaped curve
- ii) symmetric about Mean
- iii) Mean & Median
are same
- iv) Curve is peak at Mean



Standard Normal distribution

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $-\infty < x < \infty$

$$\text{Let } z = \frac{x-\mu}{\sigma}$$

$$f(z) = \phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sigma \sqrt{2\pi}} \quad (\because \sigma = 1)$$

Here the R.V $z \sim N(0, 1)$

changing μ = shift the entire curve on R

changing σ will stretch the curve

pdf $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1$

$$\frac{z^2}{2} = t$$

$$z^2 = 2t$$

$$2zdz = 2dt$$

$$zdz = dt$$

$$f(z) = \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$F(-z) = \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$F(z) = F(-z)$$

$$F(z) = 2 \int_0^{\infty} e^{-\frac{z^2}{2}} dz$$

$z dz = dt$

$\sqrt{2t} dz = dt$

$$= 2 \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= 2 \int_0^{\infty} \frac{1}{\sqrt{2t}} e^{-t} dt$$

~~$$= \sqrt{\frac{1}{2}} \int_0^{\infty} \sqrt{t} e^{-t} dt$$~~

$$= 2 \frac{1}{\sqrt{2}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt$$

$$\alpha = \frac{1}{2}$$

$$= \frac{2}{\sqrt{2}} \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2}{\sqrt{2}} \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = 1$$

Moments

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 z \cdot e^{-\frac{z^2}{2}} dz + \int_0^{\infty} z \cdot e^{-\frac{z^2}{2}} dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 e^{-t} \cdot dt + \int_0^{\infty} e^{-t} dt \right] \quad \begin{matrix} \frac{z^2}{2} = t \\ 2z dz = 2t \end{matrix}$$

$$= \frac{1}{\sqrt{2\pi}} \left[(-e^{-t}) \Big|_{-\infty}^0 + (-e^{-t}) \Big|_0^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[(-e^{-\frac{z^2}{2}}) \Big|_{-\infty}^0 + (-e^{-\frac{z^2}{2}}) \Big|_0^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-1 + 0 + (-0 + 1) \right]$$

$$= \frac{1}{\sqrt{2\pi}} (-1 + 1)$$

$$\mu_1 = E(X) = 0$$

30/41

$$\text{Var}(z) = E(z^2) - (E(z))^2$$

$$= E(z^2)$$

$$\mu_2 = \mu_2'$$

$$E(z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot z e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[z \left(-e^{-\frac{z^2}{2}} \right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 1 \cdot \left(-e^{-\frac{z^2}{2}} \right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$

$$= 1$$

$$\text{Var}(z) = 1$$

$$\frac{x-\mu}{\sigma} = z, \quad x \sim N(\mu, \sigma^2)$$

$$x = z\sigma + \mu$$

$$E(x) = E(z\sigma) + E(\mu)$$

$$= \sigma E(z) + \mu$$

$$= \sigma \cdot 0 + \mu$$

$$= \mu$$

$$\text{Var}(X) = \text{Var}(z\sigma) + \text{Var}(\mu)$$

$$= \sigma^2 \text{Var}(z) + 0$$

$$= \sigma^2 \cdot 1$$

$$= \sigma^2$$

Ex:

$$X \sim N(3, 4)$$

$$\mu = 3, \quad \sigma^2 = 4 \Rightarrow \sigma = 2$$

$$P\{2 \leq X \leq 5\} = P\left\{\frac{2-3}{2} \leq \frac{X-\mu}{\sigma} \leq \frac{5-3}{2}\right\}$$

$$Z = \frac{X-\mu}{\sigma}$$

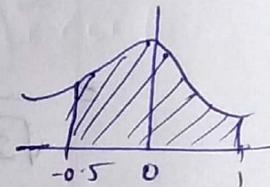
$$= P\left\{-\frac{1}{2} \leq Z \leq 1\right\}$$

$$= P\{-0.5 \leq Z \leq 1\}$$

$$= \phi(1) - \phi(-0.5)$$

$$= 0.8413 - 0.3085$$

$$= 0.5328$$



Ex: $X \sim N(2, 4)$

$$\mu = 2, \quad \sigma^2 = 4 \Rightarrow \sigma = 2$$

i) $P\{X \leq 0\}$

$$= P\left\{\frac{X-\mu}{\sigma} \leq \frac{0-2}{2}\right\}$$

$$= P\{Z \leq -1\}$$

$$= 0.1587$$

$$\text{ii) } P\{|X| \geq 2\}$$

$$= P\{X \geq 2 \text{ or } X \leq -2\}$$

$$= P\left\{\frac{X-\mu}{\sigma} \geq \frac{2-\mu}{\sigma} \text{ or } \frac{X-\mu}{\sigma} \leq \frac{-2-\mu}{\sigma}\right\}$$

$$= P\{Z \geq 0\} + P\{Z \leq -2\}$$

$$= P\{Z \leq -2\} + P\{Z \geq 0\}$$

$$= \phi(-2) + 1 - P\{Z < 0\}$$

$$= 0.0228 + 1 - \frac{1}{2}$$

$$= 0.0228 + 0.5$$

$$= 0.5228$$

Ex:

Scores of an exam are normally distributed with $\mu = 527$ & $\sigma = 112$

What is probability that

$$P\{\text{he/she scores more than 500}\}$$

$$P\{X \geq 500\} = P\left\{\frac{X-\mu}{\sigma} \geq \frac{500-527}{112}\right\}$$

$$= P\left\{\frac{X-\mu}{\sigma} \geq -0.24\right\}$$

$$= P\{Z \geq -0.24\}$$

$$= 1 - P\{Z < -0.24\}$$

$$= 1 - \phi(-0.24)$$

$$= 1 - 0.4052$$

$$= 0.5948$$

Ex:

$$\mu = 4.11 \quad \sigma = 1.37$$

$$\begin{aligned} P\{X < 3\} &= P\left\{\frac{X-\mu}{\sigma} < \frac{3-4.11}{1.37}\right\} \\ &= P\{Z < -0.81\} \\ &= \Phi(-0.81) \\ &= 0.2090 \end{aligned}$$

$P\{X \leq a\}$ & $P\{X < a\}$ For continuous both are
same
 $\hookrightarrow P\{X < a\} + P\{X = a\}$

$P\{X = a\} = 0$ for continuous.

Moment Generating Function:

The function $M_x(t) = E(e^{tx})$ is the MGF of the Random Variable x , if it exists in some neighbourhood of origin

ex:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \cdot \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{(t-\frac{1}{2})x} dx$$

$$= \frac{1}{2} \left[\frac{e^{(t-\frac{1}{2})x}}{t-\frac{1}{2}} \right]_0^{\infty} \quad t < \frac{1}{2}$$

$$= \frac{1}{2} \cdot \left[\frac{0-1}{(t-\frac{1}{2})} \right]$$

$$\text{MGF of PDF} = \frac{1}{2(\frac{1}{2}-t)} = \frac{1}{1-2t}$$

All the time mgf may not exists

since \int or \sum may diverge sometimes

$$M_x(t) = E(e^{tx}) = \frac{1}{1-2t}, \quad t < \frac{1}{2}$$

Thm For given mgf, n th derivative at $t=0$ will give n th order moment about origin

$$M_x(t) = E(e^{tx})$$

$$\frac{d}{dt} M_x(t) = \frac{d}{dt} E(e^{tx})$$

$$= E(x e^{tx})$$

$$\text{At } t=0, \mu'_x(t) = E(x) = \mu_1'$$

$$\mu''_x(t) = E(x^2 e^{tx})$$

$$\text{At } t=0 \quad \mu''_x(t) = E(x^2) = \mu_2'$$

Statement $\mu_x^n = 0$ at $t=0$ will give you n^{th} order moments about origin.

$$M_x(t) = \frac{1}{1-2t}$$

$$M'_x(t) = \frac{+2}{(1-2t)^2}$$

$$\text{At } t=0 \quad \mu'_x(t) = 2$$

$$\mu_1' = E(x) = 2$$

$$\mu_2' = M''_x(t) \Big|_{t=0} = \frac{8}{(1-2t)^3} = 8$$

$$\text{Var} = \mu_2 = 8 - 2^2 = 4$$

Mgf for continuous $E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$
For discrete $E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot P\{X=x\}$

$$\text{Var}(x) = \mu_2 = \mu_1^2 - (\mu_1')^2$$

$$\begin{aligned}\mu_3 &= E((x-\mu)^3) = E(x^3 - 3x^2\mu + 3x\mu^2 - \mu^3) \\ &= E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3\end{aligned}$$

$$E(x) = \mu = \mu_1'$$

$$= \mu_3' - 3\mu_1'\mu_2' + 3(\mu_1')^2\mu_1' - (\mu_1')^3$$

$$= \mu_3' - 3\mu_1'\mu_2' + 3(\mu_1')^3 - (\mu_1')^3$$

$$= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

Binomial Distribution:

$$E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot P\{X=x\}$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot {}^n C_x \cdot p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{\infty} {}^n C_x \cdot (pe^t)^x (1-p)^{n-x}$$

$$= (pe^t + 1-p)^n$$

$$= (pe^t + q)^n$$

Mgf of $X \sim B(n, p)$

$$\mu_x'(t) = n (pe^t + (1-p))^{n-1} \cdot p \cdot e^t$$

$$\mu_x'(t) \Big|_{t=0} = n (p+1-p)^{n-1} \cdot p = np$$

$$\mu_1' = E(x) = np$$

$$M_x(t) = \text{exp} [n(P e^t + (1-P))^{n-1} \cdot P e^t]$$

$$= nP \left[(P e^t + (1-P))^{n-1} \cdot e^t \right]$$

$$M_x''(t) = nP \left[(n-1)(P e^t + (1-P))^{n-2} e^t \cdot P e^t \right] \\ \left. - (P e^t + (1-P))^{n-1} \cdot e^t \right]$$

$$\left. M_x''(t) \right|_{t=0} = nP \left[(n-1) (P + (1-P))^{n-2} \cdot P + (P + (1-P))^{n-1} \right]$$

$$= nP [P(n-1) + 1]$$

$$M_2'(t) = n(n-1)P^2 + nP$$

$$M_2 = \text{Var}(X) = n(n-1)P^2 + nP - n^2 P^2 \\ = np(1-p) \\ = npq$$

Poisson Distribution

$$E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!} \quad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^\lambda$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$E(e^{tx}) = e^{\lambda(e^t - 1)}$$

$$M_x(t) = \frac{e^{\lambda(e^t - 1)}}{(e^t - 1)}$$

$$\begin{aligned} M_x'(t) &= e^{-\lambda} \cdot e^{\lambda e^t} \cdot \lambda e^t \\ &= e^{-\lambda} \cdot \lambda [e^{\lambda e^t} \cdot e^t] \end{aligned}$$

$$M_x'(t) \Big|_{t=0} = e^{-\lambda} \cdot \lambda \cdot e^\lambda = \lambda = E(x) = \mu_1$$

$$M_x''(t) = e^{-\lambda} \cdot \lambda [e^{\lambda e^t} \cdot e^t + \lambda \cdot e^t \cdot e^{\lambda e^t} \cdot e^t]$$

$$\begin{aligned} M_x''(t) \Big|_{t=0} &= e^{-\lambda} \cdot \lambda [e^\lambda + \lambda e^\lambda] \\ &= \lambda + \lambda^2 = \mu_2' \end{aligned}$$

$$Var = \mu_2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Continuous Distribution

Exponential Distribution:

$$\begin{aligned} E(e^{tx}) &= \int_{x=0}^{\infty} e^{tx} \cdot e^{-x/\beta} \cdot dx \\ &= \frac{1}{\beta} \int_0^{\infty} e^{(t - \frac{1}{\beta})x} dx \\ &= \frac{1}{\beta} \cdot \frac{\left[e^{(t - \frac{1}{\beta})x} \right]_0^{\infty}}{(t - \frac{1}{\beta})} \\ &= \frac{1}{\beta} \left[0 - \frac{1}{t - \frac{1}{\beta}} \right] \end{aligned}$$

$$= \frac{1}{\beta \left(\frac{1}{\beta} - t \right)}$$

$$\mu_x'(t) = \frac{1}{\beta \left(\frac{1}{\beta} - t \right)^2}$$

$$\mu_x' \Big|_{t=0} = \frac{1}{\beta \left(\frac{1}{\beta} - 0 \right)^2} = \frac{1}{\beta \cdot \frac{1}{\beta^2}} = \beta = E(X)$$

$$\begin{aligned} \mu_x''(t) &= \frac{1}{\beta} \cdot \frac{2}{\left(\frac{1}{\beta} - t \right)^3} \\ &= \frac{2}{\beta \left(\frac{1}{\beta} - t \right)^3} \end{aligned}$$

$$\mu_x''(t) \Big|_{t=0} = \frac{2}{\beta \left(\frac{1}{\beta} - 0 \right)^3} = \frac{2\beta^2}{\beta^3} = \underline{\underline{\mu_2}}$$

$$\underline{\underline{\mu_2}} = 2\beta^2 - \beta^2$$

$$= \beta^2$$

A15

Moment Generating function of Normal

Distribution:

Let $Z \sim N(0, 1)$

$$M_x(t) = E(e^{tz}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2 - 2tz}{2}\right)} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2 - 2tz + t^2}{2}\right)} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} dz$$

$$= e^{t^2/2}$$

$M_x(t)$ for Standard Normal

$$X = Z\sigma + \mu$$

$$E(e^{tx}) = E\left(e^{t(z\sigma + \mu)}\right)$$

$$= e^{t\mu} E(e^{tz\sigma}) = e^{t\mu} E(e^{t\sigma z})$$

$$= e^{t\mu} \cdot e^{\frac{\sigma^2 t^2}{2}}$$

$$= e^{\left(\frac{\sigma^2 t^2}{2} + t\mu\right)}$$

is the mgf of $X \sim N(\mu, \sigma^2)$

Mgf of $X \sim N(\mu, \sigma^2)$ is $e^{\frac{\sigma^2 t^2}{2} + t\mu}$

$$M_x'(t) = e^{\frac{\sigma^2 t^2}{2} + t\mu} \cdot \left(\frac{\sigma^2 \cdot 2t}{2} + \mu \right)$$

$$= e^{\frac{\sigma^2 t^2}{2} + t\mu} \cdot (t\sigma^2 + \mu)$$

$$M_x'(t) \Big|_{t=0} = \mu = E(X) = \mu_1$$

$$\mu_1 = 0$$

$$M_x'(t) = E(X^t) = e^{\frac{\sigma^2 t^2}{2} + t\mu} \cdot (\sigma^2 t + \mu)$$

$$M_x''(t) = \frac{(\sigma^2 t^2 + \mu t)}{(\sigma^2)} \cdot (\sigma^2) + \frac{(\sigma^2 t^2 + \mu t)}{(\sigma^2 t + \mu)} \cdot (\sigma^2 t + \mu)$$

$$= \frac{(\sigma^2 t^2 + \mu t)}{\sigma^2} + \frac{(\sigma^2 t^2 + \mu t)}{(\sigma^2 t + \mu)^2} \cdot (\sigma^2 t + \mu)$$

$$M_x''(t) \Big|_{t=0} = \sigma^2 + \mu^2 = E(X^2) = \mu_2$$

$$\text{Var}(X) = \mu_2 - (\mu_1)^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

$$P(X > 1000) = P\left(Z > \frac{1000 - \mu}{\sigma}\right)$$

$$= P\left(Z > \frac{1000 - 100}{50}\right)$$

$$= P\left(Z > 1000 \cdot \frac{1}{50}\right)$$

$$= P(Z > 20)$$

Inequalities (unit-6)

1) Markov's Inequality: Suppose X is non-negative random variable then for any $a > 0$, we have $P\{X \geq a\} \leq \frac{E(X)}{a}$.

Proof: Let X be continuous random variable

$$E(X) = \int_{-\infty}^{\infty} xf(x) dx$$

$$= \int_0^a xf(x) dx + \int_a^{\infty} xf(x) dx$$

$$E(X) \geq \int_a^{\infty} xf(x) dx$$

$$E(X) \geq \int_a^{\infty} af(x) dx \quad \because x \geq a$$

$$\frac{E(X)}{a} \geq \int_a^{\infty} f(x) dx$$

$$\frac{E(X)}{a} \geq P\{X \geq a\}$$

2) Chebyshew's Inequality: Suppose X is a random variable then for any with mean μ and variance σ^2 then for any $k > 0$

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Proof:

Since $(x-\mu)^2$ is positive

$$P\{X \geq a\} \leq \frac{E(X)}{a}$$

Let us take $(x-\mu)^2 > 0$ by taking $a=k^2$ in

$$P\{(x-\mu)^2 \geq k^2\} \leq \frac{E((x-\mu)^2)}{k^2} = \frac{\sigma^2}{k^2}$$

$(x-\mu)^2 \geq k^2$, if $|x-\mu| \geq k$

$$P\{|x-\mu| \geq k\} \leq \frac{\sigma^2}{k^2} \quad (\therefore \text{hence proved}).$$

Ex.

Suppose we know that the no. of items produced in a factory during a week is a R.V with mean 500

i) What can be said about probability that this week's production will be atleast

Sol:

$$P\{X \geq 1000\} \leq \frac{500}{1000} \quad (\text{By Markov's inequality})$$

$$P\{X \geq 1000\} \leq \frac{1}{2}$$

$$P\{X < 1000\} = ?$$

$$1 - P\{X \geq 1000\} \geq \frac{1}{2} \quad \text{upper bound}$$

$$1 - P\{X \geq 1000\} \leq 1 - \frac{1}{2} = \frac{1}{2} \quad \text{lower bound}$$

$$P\{X < 1000\} \geq \frac{1}{2}$$

ii) if variance of week's production is 100, what can be said about the probability that the production will be between 400 & 600?

Sol

$$\mu = 500, \sigma^2 = 100$$

$$P\{|x-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

$$P\{400 < x < 600\} = ?$$

$$P\{|x-500| < 100\}$$

$$P\{-100 < x-500 < 100\}$$

$$P\{400 < x < 600\}$$

Using chebyshew's Inequality

$$P\{|x-500| \geq 100\} \leq \frac{100}{(100)^2} = \frac{1}{100}$$

$$1 - P\{|x-500| \geq 100\} \geq 1 - \frac{1}{100}$$

$$P\{|x-500| < 100\} \geq \frac{99}{100}$$

chebyshew's inequality suppose X is a random variable. Then for any $\epsilon > 0$, if μ is the mean of the random variable X and σ^2 is its variance, then for any $k > 0$,

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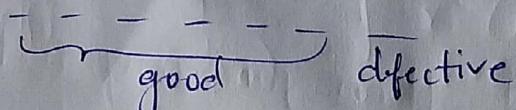
- 1) Consider an experiment which consists of counting α -particles in a one-second interval if we know that on an average 3.2 such α -particles are given, what is the probability that no more than 2 α -particles will appear.

$$\begin{aligned}
 \text{sol} \quad P\{X \leq 2\} &= P\{X = 0\} + P\{X = 1\} + P\{X = 2\} \\
 &= e^{-3.2} + \frac{e^{-3.2} \cdot (3.2)}{1} + \frac{e^{-3.2} \cdot (3.2)^2}{2} \\
 &= e^{-3.2} \left(1 + 3.2 + \frac{(3.2)^2}{2}\right) \\
 &= 0.382
 \end{aligned}$$

- 2) Assume that $P(\text{defective computer component}) = 0.02$

- a) Find $P\{\text{the 1st defect is caused by 7th component}\}$
- b) How many components do you expect to test until one is found to be defective?

sol a)



Follows geometric distribution

$$P\{X=7\} = q^7 \cdot p \quad \text{success - getting defective}$$

$$= (0.98)^6 \cdot (0.02)$$

$$= 0.0177$$

b) $E(X) = \frac{1}{P}$

$$= \frac{1}{0.02}$$

$$= 50$$

$$\sigma^2 = \frac{q}{P^2} = \frac{0.98}{0.02 \times 0.02} = \frac{98}{2 \times 0.02}$$

$$= 49 \times 50$$

3) An item produced by a machine will be defective with $P=0.1$ what is the probability that at most one will be defective in a sample of 3 items.

Sol $P=0.1, n=3$

$$P\{X \leq 1\} = P\{X=0\} + P\{X=1\}$$

$$= {}^3C_0 \cdot (0.1)^0 \cdot (0.9)^3 + {}^3C_1 \cdot (0.1)^1 \cdot (0.9)^2$$

$$= (0.9)^3 + 3(0.1)(0.9)^2$$

$$= 0.972$$

4) Time in hours to finish a job follows beta distribution with $\mu = \frac{1}{3}$ and variance $\frac{2}{63}$ hours. Find the $P\{\text{Job will be finished in } 30 \text{ min}\}$

so) For Beta distribution

$$\mu = \frac{\alpha}{\alpha + \beta} = \frac{1}{3}$$

$$\sigma^2 = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} = \frac{2}{63}$$

$$3\alpha = \alpha + \beta$$

$$2\alpha = \beta$$

$$\beta = 4$$

$$63\alpha\beta = 2(\alpha + \beta)^2(\alpha + \beta + 1)$$

$$63\alpha(2\alpha) = 2(\alpha + 2\alpha)^2(\alpha + 2\alpha + 1)$$

$$126\alpha^2 = \alpha(9\alpha^2)(3\alpha + 1)$$

$$3\alpha + 1 = 7$$

$$3\alpha = 6$$

$$\alpha = 2$$

$$P\{X \leq \frac{1}{2}\} = \frac{1}{B(2,4)} \int_0^{\frac{1}{2}} x^1 \cdot (1-x)^3 dx$$

$$B(2,4) = \frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{1! \cdot 3!}{5!} = \frac{3 \cdot 2}{120} = \frac{1}{20}$$

$$P\{X \leq \frac{1}{2}\} = \frac{1}{\binom{5}{2}} \int_0^{\frac{1}{2}} x^1 \cdot (1-x^3 + 3x^2 - 3x) dx$$

$$= 20 \int_0^{\frac{1}{2}} (x - x^4 + 3x^3 - 3x^2) dx$$

$$\begin{aligned}
 &= 20 \cdot \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{3x^4}{4} + \frac{3x^3}{3} \right]_0^{1.5} \\
 &= 20 \left[\frac{1}{8} - \left(\frac{1}{2}\right)^5 \cdot \frac{1}{5} + 3 \cdot \left(\frac{1}{2}\right)^4 \cdot \frac{1}{4} - 3 \cdot \left(\frac{1}{2}\right)^3 \right] \\
 &= 20 \left[\frac{3}{64} - \frac{1}{32 \times 5} \right] = 20 \left[\frac{15}{64 \times 8} \right] = \frac{13}{16}
 \end{aligned}$$

5) Let waiting time for emails follow exponential distribution if $\beta = 2$, what is chance of waiting atmost 1.1 minutes.

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\beta = 2$$

$$\begin{aligned}
 P\{X \leq 1.1\} &= \int_0^{1.1} \frac{1}{2} \cdot e^{-x/2} dx \\
 &= \frac{1}{2} \left[e^{-x/2} \right]_0^{1.1} \\
 &= \frac{1}{2} \cdot \left[\frac{e^{-x/2}}{-1/2} \right]_0^{1.1} \\
 &= 1 - e^{-1.1/2} \\
 &= 0.423
 \end{aligned}$$

6) Suppose that reaction time x of a randomly selected individual to a certain stimulus has gamma distribution with $\beta = 1$ & $\alpha = 2$

$$\text{a) } P\{3 < x < 5\} \quad \text{b) } P\{x > 4\}$$

sol: pdf $f(x) = \frac{1}{r(\alpha)\beta^\alpha} \int_0^{\infty} y^{\alpha-1} e^{-y/\beta} dy$

$$\begin{aligned} \text{a) } P\{3 < x < 5\} &= \frac{1}{r(2)1^2} \int_3^5 x \cdot e^{-x} dx \\ &= \int_3^5 x \cdot e^{-x} dx \\ &= [x \cdot (-e^{-x})]_3^5 - \int_3^5 (-e^{-x}) dx \\ &= -5e^{-5} + 3e^{-3} + (-e^{-x})_3^5 \\ &= -5e^{-5} + 3e^{-3} \\ &\quad - \cancel{2e^{-5}} + (-e^{-5} + e^{-3}) \\ &= -6e^{-5} + 4e^{-3} \\ &= 0.1587 \end{aligned}$$

$$\begin{aligned} P\{x > 4\} &= 1 - P\{x \leq 4\} \\ &= 1 - \int_0^4 x e^{-x} dx = 1 - 0.908 \\ &= 0.092 \end{aligned}$$

7) Let $X \sim N(2, 4)$ $\mu = 2, \sigma = 2$

a) $P\{1 < X \leq 3\}$

$$= P\left\{\frac{1-2}{2} < Z < \frac{3-2}{2}\right\}$$

$$= P\{-0.5 < Z < 0.5\}$$

$$= \Phi(0.5) - \Phi(-0.5)$$

$$= 0.6915 - 0.3085$$

$$= 0.3830$$

b) $P\{X \leq 3 / X > 1\}$

$$= \frac{P\{1 \leq X \leq 3\}}{P\{X > 1\}}$$

$$= \frac{0.3830}{0.6915}$$

$$P\{X > 1\} = 1 - P\{X \leq 1\}$$

~~0.3830~~

$$= 1 - P\{Z \leq (-0.5)\}$$

$$= 1 - 0.3085$$

$$= 0.6915$$

$$P\{X \leq 3 / X > 1\} = 0.553$$