σ - algebra

POST-MIDSEM

12/02/19

Random Vectors :

$(\Omega, \mathcal{F}, P)$

Let $X_1, X_2, \ldots, X_r$ be r discrete random variables on $(\Omega, \mathcal{F}, P)$

Define $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix}$ is an r-dimensional vector

For an $\omega \in \Omega$

$$\bar{X}(\omega) = \begin{bmatrix} X_1(\omega) \\ X_2(\omega) \\ \vdots \\ X_r(\omega) \end{bmatrix} \in \mathbb{R}^r$$

Suppose $X_1(\omega) = x_1$; $X_2(\omega) = x_2, \ldots, X_r(\omega) = x_r$

$$\bar{X}(\omega) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} \in \mathbb{R}^r$$

Definition:

Let $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{bmatrix}$ be a vector where X_i is a random variable on $(\Omega, \mathcal{F}, P)$. For every $\bar{x} \in \mathbb{R}^r$ the set

$$\{\omega \in \Omega \mid \bar{X}(\omega) = \bar{x}\} \in \mathcal{F}$$

Then $X : \Omega \longrightarrow \mathbb{R}^r$ is called as ~~an~~ a r-dimensional random vector

We are interested in $\text{Prob}(\bar{X} = \bar{x})$

Let $\bar{X} = \begin{bmatrix} X_1(\omega) \\ X_2(\omega) \\ \vdots \\ X_r(\omega) \end{bmatrix}$ be a discrete random vector.

If ~~or~~ $\bar{x}$ denotes the value assumed by the random vector $\bar{X}$, then

$$\{\bar{x} : P(\bar{X}(\omega) = \bar{x}) > 0\}$$ is finite or countably infinite

<u>Definition:</u> The discrete density / discrete joint p.m.f. of the random vector $\bar{X}$ is defined as

$$f(x_1, x_2, \ldots, x_r) = \text{Prob}(X_1 = x_1, X_2 = x_2, \ldots X_r = x_r)$$

where $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}$ and $\bar{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_r \end{bmatrix}$

In the vector notation

$$f(\bar{x}) = P(\bar{X} = \bar{x}) \quad \forall \ \bar{x} \in \mathbb{R}^r$$

For a subset $A \subseteq \mathbb{R}^r$

$$P(\bar{X} \in A) = \sum_{\bar{x} \in A} f(\bar{x})$$

Definition: A function f is called as joint pmf if

i) $f(\bar{x}) \geq 0 \quad \forall\, x \in \mathbb{R}^n$

Range of the random vector

ii) $\{\bar{x} : f(\bar{x}) \neq 0\}$ is finite or countably infinite. We denote the elements of this set as $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots$

iii) $\sum_i f(x_i) = 1$

Ex:

X_1 \ X_2	1	2	3	4	
1	1/4	1/8	1/16	1/16	1/2
2	1/16	1/16	1/4	1/8	1/2
	5/16	3/16	5/16	3/16	

Not independent

$P(X_1=1, X_2=1) = 1/4$

$P(X_1=1) \cdot P(X_2=1) = \frac{1}{2} \cdot \frac{5}{16} = \frac{5}{32}$

$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad P(\bar{X} = \bar{x}) = P(X_1 = x_1 ; X_2 = x_2)$

$= f(x_1, x_2)$

$= f(\bar{x})$

This table is a discrete joint pmf

$P(X_1 \geq X_2) = \frac{1}{4} + \frac{1}{16} + \frac{1}{16} = P(X_1 = 1, X_2 = 1) + P(X_1 = 2, X_2 = 1) + P(X_1 = 2, X_2 = 2)$

$P(X_1 = 1) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$

$= \sum_{x_2} f(1, x_2) = \frac{1}{2}$

$P(X_1 = 2) = \sum_{x_2} f(2, x_2) = 1/2$

$\text{Prob}(X_1 = x_1)$

$$\boxed{f_{X_1}(x_1) = \sum_{x_2} f(x_1, x_2) \quad \text{for } x_1 \in R_{x_1}}$$

marginal pmf of X_1

Independent random variables

Let $X_1, X_2, \ldots X_r$ be r discrete random variables with pmfs $f_1, f_2, \ldots f_r$ respectively. The random variables $X_1, \ldots, X_r$ are called mutually independent if their joint pmf f is given by

$$f(x_1, x_2, \ldots, x_r) = f_1(x_1) \cdot f_2(x_2) \cdots \cdots f_r(x_r)$$

Notation:

$$P(x_1, x_2, \ldots, x_r) = \text{Prob}(X_1 = x_1, X_2 = x_2, \ldots, X_r = x_r)$$

Define $A_1 \subseteq \Omega$ s.t. $A_1 = [X_1 = x_1] \Rightarrow A_1 = \{\omega : X_1(\omega) = x_1\}$

$$= f(x_1, x_2, \ldots, x_r)$$

$$= \text{Prob}(A_1 \cap A_2 \cap \ldots \cap A_r)$$

$$= \text{Prob}(A_1) \cdot \text{Prob}(A_2) \cdot \text{Prob}(A_3) \ldots \text{Prob}(A_r)$$
$$= \text{Prob}(X_1 = x_1) \ldots \ldots \text{Prob}(X_r = x_r)$$
$$= f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) \ldots f_r(x_r)$$

Ex: Let X_1 & X_2 be two independent random variables each with geometric distribution with parameters p. Find the distribution $\min(X_1, X_2)$

Solⁿ: $Prob(\min(X_1, X_2) \geq z)$

$= Prob(X_1 \geq z ; X_2 \geq z)$ — independence of X_1 & X_2

$= Prob(X_1 \geq z) \cdot Prob(X_2 \geq z)$

$= (1-p)^z \cdot (1-p)^z$

$= (1-p)^{2z} = [1-(1-(1-p)^2)]^z$

$\min(X_1, X_2) \sim$ geometric $(1-(1-p)^2)$

5/3/19

Sum of independent r.v. (Discrete case)

Let X & Y be two independent r.v.

Let $x_1, x_2, \ldots$ be the distinct values taken by X

Interested in the event $\{X+Y = z\}$ $\forall z$

$$\{X+Y=z\} = \bigcup_i \{X = x_i, Y = z - x_i\}$$

← This union is disjoint.

$$P(X+Y=z) = P\left(\bigcup_i \{X = x_i, Y = z - x_i\}\right)$$

$$= \sum_i P(X = x_i, Y = z - x_i)$$

$$= \sum_i P(X = x_i)\, P(Y = z - x_i)$$

$$\boxed{f_{X+Y}(z) = \sum_{x} f_X(x)\, f_Y(z-x)}$$

Expectation:

Let $\bar{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ be a given discrete random vector with

joint pmf $f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$

Let $h(x_1, x_2, \dots, x_n)$ be a function of $(x_1, x_2, \dots, x_n)$.

$$E\left(h(x_1, x_2, \dots, x_n)\right) = \sum_{x_1, x_2, \dots, x_n} h(x_1, x_2, \dots, x_n)\, f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$$

for $n=2$, $h(x_1, x_2) = x_1 + x_2$

$$E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2)\, f_{X_1, X_2}(x_1, x_2)$$

for $n=2$, $h(x_1, x_2) = x_1$

$$E(x_1) = \sum_{x_1}\sum_{x_2} x_1 \, f_{X_1,X_2}(x_1, x_2)$$

$$= \sum_{x_1} x_1 \boxed{\sum_{X_2} f_{X_1,X_2}(x_1, x_2)}$$

↳ marginal pmf of X_1

$$\boxed{E(x_1) = \sum_{x_1} x_1 f_{X_1}(x_1).}$$

In general, take $h(x_1, x_2, \ldots, x_n) = x_i$, for $1 \le i \le n$

$$\boxed{E(X_i) = \sum_{x_i} x_i f_{X_i}(x_i)}$$

Let X and Y be independent discrete random variables with joint pmf $f_{X,Y}(x,y)$

$$E(XY) = \sum_{x}\sum_{y} xy \, f_{X,Y}(x,y)$$

$$E(XY) = \sum_{x}\sum_{y} xy \, f_X(x) \cdot f_Y(y)$$

$$E(XY) = \sum_{x} x f_X(x) \sum_{y} y f_Y(y)$$

$$\boxed{E(XY) = E(X) \cdot E(Y)}$$

11/3/19

X and Y are independent discrete R.V.s with MGF pmf of $f_{x,y}(x,y)$

M.G.F. of $X+Y$

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right)$$

$$= E\left(e^{tX + tY}\right)$$

Independence

$$= E\left(e^{tX}\right) \cdot E\left(e^{tY}\right)$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Generalizing this result.

$X_1, X_2, \ldots, X_n$ are mutually independent.

$$M_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

Ex: Let X_i be bernoulli with probability 'p'. Let $Y = X_1 + X_2 + \cdots + X_n$ find $M_Y(t)$?

$$M_Y(t) = M_{\sum_{i=1} X_i}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \prod_{i=1}^{n} \left((1-p) + pe^t\right)$$

$$= \left((1-p) + pe^t\right)^n$$

$\therefore$ Binomial (n, p) = sum of n independent Bernoulli(p)

Note:

If X & Y are independent $E(XY) = E(X)E(Y)$

Ex: Let (X, Y) be discrete r.v. with range $R(X,Y) = \langle (0,1), (0,-1), (1,0), (-1,0) \rangle$ with each outcome equally likely.

$P(x,y) = 1/4$ for $(x,y) \in R(x,y)$

$= 0$ elsewhere

$E(X) = 0$
$E(Y) = 0$

Now $XY = 0 \Rightarrow E(XY) = 0 = E(X) . E(Y)$

But we prove X & Y are not independent

$P(X=0) = 1/2$
$P(Y=0) = 1/2$

$P(X=0, Y=0) = 0 \neq P(X=0) . P(Y=0) = \frac{1}{2} . \frac{1}{2} = \frac{1}{4}$

Remember

$E(X+Y) = E(X) + E(Y)$
$E(\alpha X) = \alpha E(X)$
$E(X+t) = E(X) + t$

<u>Sum of Variances</u>

Let X and Y be discrete random variables.

$\mathrm{var}(X+Y) = E(X+Y - E(X+Y))^2$

$= E[(X - E(X)) + (Y - E(Y))]^2$

$= E[X - E(X)]^2 + E[Y - E(Y)]^2 + 2\underbrace{E[(X - E(X)).(Y - E(Y))]}_{= \mathrm{cov}(X,Y)}$

$\sigma^2 = E(X^2) - \mu^2$
$E((X-\mu)^2) = E(X^2) - (E(X))^2$

$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y)$

Note:

$$\text{cov}(X,Y) = E[(X-E(X))(Y-E(Y))]$$
$$= E[XY - XE(Y) - YE(X) + E(X).E(Y)]$$
$$= E(XY) - E(X).E(Y) - E(Y).E(X) + E(X).E(Y).$$

$$\boxed{\text{cov}(X,Y) = E(XY) - E(X)E(Y)}$$

If X Y are independent $\Rightarrow \text{cov}(X,Y) = 0$
but converse is NOT true

Corollary:
If X and Y are independent then $\text{cov}(X,Y) = 0$
but converse is NOT true

$\text{cov}(X,Y) = 0 \not\Rightarrow$ X & Y are independent. If X and Y are independent r.v.s. $\boxed{\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)}$

$$\text{var}(aX) = E(aX - E(aX))^2 = a^2\, E[(X) - E(X)]^2$$

$$\boxed{\text{var}(aX) = a^2\,\text{var}(X)}$$

var(c) where c is a constant = 0

$$= E(c - E(c))^2 = E(c-c)^2 = E(0)^2 = 0$$

Let $X_1, X_2, \ldots, X_n$ be independent random variables. with variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$

$$\text{var}(X_1 + X_2 + \ldots + X_n) = \text{var}(X_1) + \text{var}(X_2) + \ldots + \text{var}(X_n)$$
$$= \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$$

Let $X_1, X_2, \ldots, X_n$ be iid (independently, identically, with mean μ and variance σ^2

$$E(\bar{X}) = E\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = E\left(\frac{X_1}{n}\right) + E\left(\frac{X_2}{n}\right) + \ldots + E\left(\frac{X_n}{n}\right)$$

$$= \frac{1}{n} E(X_1) + \ldots + \frac{1}{n} E(X_n)$$

$$= \frac{\mu}{n} + \frac{\mu}{n} + \ldots + \frac{\mu}{n}$$

$$= n \cdot \frac{\mu}{n} = \mu = E(\bar{X})$$

$$Var(\bar{X}) = Var\left(\frac{X_1 + \ldots + X_n}{n}\right) = Var\left(\frac{X_1}{n}\right) + \ldots + Var\left(\frac{X_n}{n}\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} var(X_i)\right)$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\boxed{var(\bar{X}) = \frac{\sigma^2}{n}}$$

Correlation coefficient:

Let X, Y be two discrete r.v.s. Then correlation coefficient $\rho(X,Y)$ is defined as:

$$\rho(X, Y) = \rho = \rho_{X,Y} = \frac{cov(X, Y)}{\sqrt{var(X) \cdot var(Y)}}$$

$$= \frac{cov(X, Y)}{SD(X)\, SD(Y)}$$

X & Y are independent $\Rightarrow$ $\boxed{\rho_{X,Y} = 0}$

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Theorem : Schwartz Inequality

Let X and Y be random variables with finite second order moments

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

Furthermore, equality holds when $P(Y=0)=1$ or

$P(X=aY)=1$ for some $a \in \mathbb{R}$

<u>Proof:</u>

Easy to see that when

$P(Y=0)=1$ or $P(X=aY)=1$

Equality holds!

For Inequality,

for any $\lambda \in \mathbb{R}$

$$0 < E(X-\lambda Y)^2 = \lambda^2 E(Y^2) - 2\lambda E(XY) + E(X^2)$$

Since the above expression is a quadratic in λ, and $E(Y^2) =$, the min. value of this quadratic expression is achieved.

$$2\lambda E(Y^2) - 2E(XY) = 0$$

$$\Rightarrow \lambda = \frac{E(XY)}{E(Y^2)}$$

Then minimum value at the part $\frac{E(XY)}{E(Y^2)}$ is given by:

$$\left(\frac{E(XY)}{E(Y^2)}\right)^2 E(Y^2) - 2\left[\frac{E(XY)}{E(Y^2)}\right] E(XY) + E(X^2) \geq 0$$

$$\Rightarrow -\frac{[E(XY)]^2}{E(Y^2)} + E(X^2) \geq 0$$

$$\Rightarrow [E(XY)]^2 \leq E(X^2)\, E(Y^2)$$

Importance of Schwartz Inequality:

Apply this Schwartz inequality to 2 random variable.

$X - E(X)$ and $Y - E(Y)$

$$\left(E\left[(X - E(X)).(Y - E(Y))\right]\right)^2 \leq E(X - E(X))^2\, E(Y - E(Y))^2$$

$$\Rightarrow [\operatorname{cov}(X, Y)]^2 \leq \operatorname{Var}(X)\operatorname{var}(Y)$$

or $$\frac{[\operatorname{cov}(X, Y)]^2}{\operatorname{var}(X) \cdot \operatorname{var}(Y)} \leq 1$$

or $$\left|\frac{\operatorname{cov}(X, Y)}{S.D(X) \cdot S.D(Y)}\right| \leq 1$$

or $$\boxed{-1 \leq \rho_{X,Y} \leq 1}$$

$$\boxed{\rho(X, Y) = \pm 1 \iff P(X = aY) = 1}$$

Check! If $\operatorname{cov}(X, Y) = 0$, we cannot say anything except that X & Y are NOT linearly related.

Recall:

If X is a non negative r.v., with finite expectation, then, for $t>0$, $P(X \geq t) \leq \frac{E(X)}{t}$

Chebyshev's Inequality

$$\boxed{P(|X-\mu| > t) \leq \frac{\sigma^2}{t^2}}$$

Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables.

Let $\mu = E(X_1)$, $\sigma^2 = Var(X_1)$

$$E\left(\frac{S_n}{n}\right) = E\left(\frac{X_1 + \cdots + X_n}{n}\right) = \mu$$

$$Var\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$$

using chebyshev's Inequality $\Rightarrow$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \delta\right) \leq \left(\frac{Var(S_n/n)}{\delta^2}\right) = \frac{\sigma^2}{n\delta^2}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \delta\right) \leq \frac{\sigma^2}{n\delta^2}$$

In particular

$$\boxed{\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \delta\right) = 0}$$ WLLN (Weak Law of Large Number)

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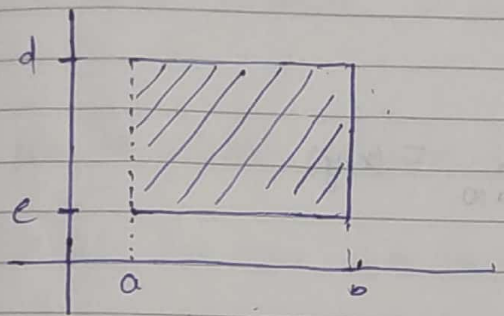
Joint continuous random variables.

X and Y are continuous r.v.s on the same probability space.

$$F(x, y) = \text{Prob}(X \le x, Y \le y) \quad [\text{Joint CDF}]$$

$$-\infty < x, y < \infty$$

Rectangle $R = \{(x,y) \mid a < x \le b,\ c < y \le d\}$



$$P((X,Y) \in R)$$

$$= P(a < X \le b,\ c < Y \le d)$$

$$= F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

Marginal distributions.

$$F_X(x) = P(X \le x) = F(x, \infty) \quad \text{marginal CDF of } X$$

$$= \lim_{y \to \infty} F(x,y)$$

$$F_Y(y) = P(Y \le y) = F(\infty, y) \quad \text{marginal CDF of } Y$$

$$= \lim_{x \to \infty} F(x,y)$$

If there exists a non-negative function $f(x,y)$ over $\mathbb{R}^2$

$$F(X,Y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du$$

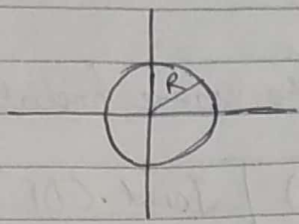
then $f(x,y)$ = joint pdf of (X,Y)

$$\boxed{P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1 \quad \text{and} \quad \boxed{\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)}$$

12/3/19

Example:



$$f(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & (x,y) \in \text{circle} \quad x^2+y^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

$$P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy$$

$$= \frac{\text{Area of } A}{\pi R^2}$$

$$F_X(x) = F(x, \infty) = \lim_{y \to \infty} F(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \qquad \text{(Marginal Pdf of X)}$$

$$= \int_{-\sqrt{R^2+x^2}}^{\sqrt{R^2+x^2}} \frac{1}{\pi R^2}\, dy$$

$$f_X(x) = \frac{2\sqrt{R^2-x^2}}{\pi R^2}, \qquad -R < x < R$$

$$= 0, \qquad \text{otherwise}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx \qquad \text{(Marginal pdf of Y)}$$

$$f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2}, \qquad -R < y < R$$

$$= 0, \qquad \text{other wise}$$

Independence:

X and Y are called independent r.v.s. iff

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

(joint pdf is the product of marginal pdfs)

→ In the above example X and Y are NOT independent

Easy way to generate examples:

Let X & Y be two independent continuous r.v.s with pdfs

f

Ex: $X \sim N(0,1)$
$Y \sim N(0,1)$

$\cancel{\phi_X(y)}$ $\phi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\quad -\infty < x < \infty$

$\phi_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ $\quad -\infty < y < \infty$

$f_{XY}(x,y) = \phi_X(x)\, \phi_Y(y)$

$= \frac{1}{\sqrt{2\pi}} e^{-(x^2+y^2)/2}$ $\quad -\infty < x,y < \infty$

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

18/03/

Ex: Let X and Y have joint density

$$f(x,y) = c e^{-(x^2 - xy + y^2)/2} \quad , \quad -\infty < x, y < \infty$$

$$\Rightarrow 0 \quad , \quad \text{o/w}$$

Marginal of X

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= c\int_{-\infty}^{\infty} e^{-(x^2-xy+y^2)/2}\,dy$$

$$= c\int e^{-\left[(y-x/2)^2 + \frac{3}{4}x^2\right]/2}\,dy$$

$$= c e^{\frac{-3x^2}{8}} \int_{-\infty}^{\infty} e^{-(y-x/2)^2/2}\,dy$$

$$f_X(x) = c e^{\frac{-3x^2}{8}} \int_{-\infty}^{\infty} e^{-u^2/2}\,du$$

$$f_X(x) = \sqrt{2\pi}\, c e^{-3x^2/8} = \sqrt{2\pi}\, c\, e^{-\frac{x^2}{2\cdot\frac{4}{3}}}$$

$$\Rightarrow \boxed{c = \frac{\sqrt{3}}{4\pi}}$$

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18/09/19

Distributions of sums and quotients

Let X and Y be r.v.s. with joint pmf $f(x,y)$

Define $Z = g(X, Y)$

For a fixed $z \in \mathbb{R}$ we are interested in the event $(Z \leq z)$

By $A_z \subseteq \mathbb{R}^2$ define

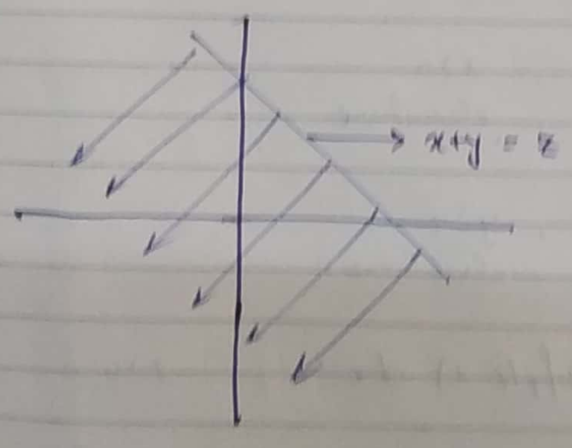
$$A_z = \{(x,y) \mid g(x,y) \leq z\}$$

$$F_Z(z) = P(Z \leq z) = P((X, Y) \in A_z)$$

$$= \iint_{A_z} f(x,y)\, dx\, dy$$

Distribution of sums ($g(X,Y) = X+Y$)

$$A_z = \{(x,y) \mid x+y \leq z\}$$



$$F_Z(z) = \iint_{A_z} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-x} f(x,y)\, dy \right] dx$$

Substitute, $y = v - x$ in inner integral.

$$F_Z(z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z} f(x, v-x)\, dv \right] dx$$

$$= \int_{-\infty}^{z} \underbrace{\left(\int_{-\infty}^{\infty} f(x, v-x)\, dx \right)}\, dv$$

Thus, the pdf of $X+Y$ is given by (if X & Y are independent)

$$\boxed{f_{X+Y}(z) = \int_{-\infty}^{z} f_X(x)\, f_Y(z-x)\, dx} \qquad -\infty < z < \infty$$

The pdf of $X+Y$ is given by (if X & Y are non-negative ind. r.v.s)

$$f_{X+Y}(z) = \int_{0}^{z} f_X(x)\, f_Y(z-x)\, dx \qquad 0 < z < \infty$$
$$= 0 \qquad \text{elsewhere}$$

Eg: $X, Y \sim \exp(\lambda)$ are independent.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$
$$= 0, \quad \text{elsewhere}$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$
$$= 0, \quad \text{elsewhere}$$

$$f_{X+Y}(z) = \int_{0}^{z} f_X(x)\, f_Y(z-x)\, dx, \quad z \geq 0$$
$$= 0, \quad \text{elsewhere}$$

For $z > 0$

$$f_{X+Y}(z) = \int_{0}^{z} \lambda e^{-\lambda x}\, \lambda e^{-\lambda(z-x)}\, dx$$

$$f_{X+Y}(z) = \lambda^2 e^{-\lambda z} \cdot z$$

Thus $X+Y \sim \text{gamma}(2, \lambda)$.

Recall:

$$\Gamma(\alpha, \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

Ex:

X, Y are iid $U(0,1)$

$$f_{X+Y}(z) = \int_0^z f_X(x)\, f_Y(z-x)\, dx$$

$f_X(x)\, f_Y(z-x)$ takes only values zero or 1

$f_X(x)\, f_Y(z-x)$ takes value 1 when $0 \le x \le 1$ and $0 \le z-x \le 1$

If $0 \le z \le 1$, then the integrand has value 1 on the set $0 \le x \le z$

$$f_{X+Y}(z) = z \qquad 0 \le z \le 1$$

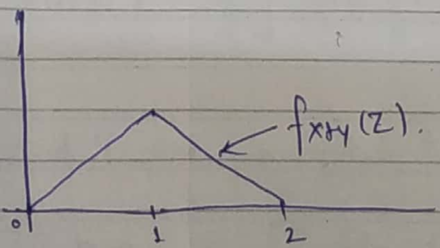
If $1 \le z \le 2$, then the integrand has value 1 on the set

$$z-1 \le x \le 1$$

$$f_{X+Y}(z) = 2-z \qquad , 1 < z \le 2$$

Thus,

$$f_{X+Y}(z) = \begin{cases} 0 & z < 0 \\ z & 0 \le z \le 1 \\ 2-z & 1 < z \le 2 \\ 0 & z > 2 \end{cases}$$



Ex: Let $X \sim \Gamma(\alpha_1, \lambda)$ and $Y \sim \Gamma(\alpha_2, \lambda)$ be independent r.v.s.
Then
$$X + Y \sim \Gamma(\alpha_1 + \alpha_2, \lambda)$$

$$f_X(x) = \frac{\lambda^{\alpha_1} x^{\alpha_1 - 1} e^{-\lambda x}}{\Gamma(\alpha_1)}, \quad x > 0$$

$$f_Y(y) = \frac{\lambda^{\alpha_2} y^{\alpha_2 - 1} e^{-\lambda y}}{\Gamma(\alpha_2)}, \quad y > 0.$$

For $z > 0$

$$f_{X+Y}(z) = \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)} \int_0^z e^{-\lambda z} x^{\alpha_1 - 1} (z - x)^{\alpha_2 - 1} \, dx$$

Ex: Let X, Y be independent r.v.s. with
$X \sim N(\mu_1, \sigma_1^2)$
$Y \sim N(\mu_2, \sigma_2^2)$

Then what is the pdf of $X+Y$?

(Another approach is MGF)

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right)$$

$$= E\left(e^{tX} \cdot e^{tY}\right)$$

since X and Y are independent

$$= E\left(e^{tX}\right) \cdot E\left(e^{tY}\right)$$

$$= M_X(t) \cdot M_Y(t)$$

$$= e^{\mu_1 t + \sigma_1^2 t^2/2} \cdot e^{\mu_2 t + \sigma_2^2 t^2/2}$$

$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)t^2/2}$$

$$X+Y \sim N\left(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2\right)$$

If X and Y are continuous r.v.s with joint pdf $f_{XY}(x,y)$

$$E(\phi(x,y)) = \iint \phi(x,y)\, f_{XY}(x,y)\, dx\, dy$$

If X and Y are independent

$$E(\phi(x,y)) = \iint \phi(x,y)\, f_X(x) \cdot f_Y(y)\, dx\, dy$$

If X and Y are independent and $\phi(x,y) = \phi_1(x) \cdot \phi_2(y)$.

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$$E(\phi(x,y)) = \iint \phi_1(x) \cdot \phi_2(y)\, f_X(x)\, f_Y(y)\, dx\, dy$$
$$= \int \phi_1(x) f_X(x)\, dx \int \phi_2(y) f_Y(y)\, dy$$
$$= E(\phi_1(x)) \cdot E(\phi_2(x)).$$

Generalizing the results:

1. $X_i \sim \exp(\lambda)$ are iid for $i = 1, 2, \ldots n$

$$\sum_{i=1}^{n} X_i \sim \text{gamma}(n, \lambda)$$

2. $X_i \sim \text{gamma}(\alpha_i, \lambda)$ are independent for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} X_i \sim \text{gamma}\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$$

3. $X_i \sim \text{normal}(\mu_i, \sigma_i^2)$ are independent, for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} X_i \sim \text{normal}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Distribution on quotients:

Let X and Y be two continuous r.v.s. with joint pdf $f_{XY}(x,y)$ or $f(x,y)$

What is density for $Z = Y/X$

$$A_z = \{ (x,y) \mid y/x \le z \}$$

If $x < 0$, then $y/x \le z \Leftrightarrow y \ge xz$

$$A_z = \{ (x,y) \mid x<0 \text{ and } y \ge xz \}$$

$$\cup \{ (x,y) \mid x>0 \text{ and } y \le xz \}$$

$$F_{Y/X}(z) = \iint f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{0} \left[\int_{xz}^{\infty} \right.$$

substitute $y = xv$ ($dy = x\,dv$) in the inner integral

$$F_{Y/X}(z) = \int_{-\infty}^{0} \left(\int_{z}^{-\infty} x\, f(x, xv)\, dv \right) dx$$

$$+ \int_{0}^{\infty} \left(\int_{-\infty}^{z} x\, f(x, xv)\, dv \right) dx$$

$$F_{Y/X}(z) = \int_{-\infty}^{z} \boxed{\left(\int_{-\infty}^{\infty} |x|\, f(x, xv)\, dx \right)} dv.$$

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x|\, f(x, xv)\, dx$$

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For X and Y ; non negative and independent.

$$f_{Y/X}(z) = \int_0^\infty x \, f_X(x) \, f_Y(xv) \, dx \qquad 0 < x < \infty.$$

Ex: Let X and Y are independent r.v.s. with densities $\Gamma(\alpha_1, \lambda)$ and $\Gamma(\alpha_2, \lambda)$ respectively.

Prove:

$$f_{Y/X}(z) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)} \cdot \frac{z^{\alpha_2 - 1}}{(\quad)^{\alpha_1+\alpha_2}} , \quad 0 < z < \infty$$

$$= 0 \qquad , \text{elsewhere}$$

Recall

$$f_X(x) = \frac{\lambda^{\alpha_1} x^{\alpha_1 - 1} e^{-\lambda x}}{\Gamma(\alpha_1)} , \quad x > 0$$

$$f_Y(y) = \frac{\lambda^{\alpha_2} y^{\alpha_2 - 1} e^{-\lambda y}}{\Gamma(\alpha_2)} , \quad y > 0$$

$$f_{Y/X}(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)} \cdot z^{\alpha_2 - 1} \int$$

$Y = X^2 \Rightarrow G_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$g_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

Ex: Let X and Y be independent r.v.s, with density $N(0, \sigma^2)$.

Find the density of Y^2/X^2

$X^2, Y^2 \sim \Gamma(1/2, 1/2\sigma^2)$

$$f_{Y^2/X^2}(z) = \frac{1}{\pi(z+1)\sqrt{z}} \quad , 0 < z < \infty$$

$$= 0 \quad , \text{elsewhere.}$$

Conditional densities.

Let (X, Y) be a discrete random vector

conditional probability.

$$P(Y=y \mid X=x) = \frac{P(X=x,\ Y=y)}{P(X=x)}$$

$$= \frac{f(x,y)}{f_X(x)}$$

where $f(x,y)$ is joint pmf of (X,Y) and $f_X(x)$ is the marginal pmf of X.

Definition: Let X and Y be continuous r.v.s. with joint pdf $f(x,y)$. Then the conditional density of Y given X, denoted as $f_{Y|X}$

$$P(a \leq Y \leq b \mid X=x) = \int_a^b f_{Y|X}(y|x)\, dy$$

Also, observe
$$\boxed{f(x,y) = f_X(x)\, f_{Y|X}(y|x)}$$

If X and Y are independent,

$$f_{Y|X}(y|x) = f_Y(y)$$

__Example:__

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-(x^2-xy+y^2)/2} \qquad -\infty < x,y < \infty$$

$X \sim N(0, \frac{4}{3})$ [we have done this before]

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{\sqrt{3}}{4\pi} e^{-(x^2-xy+y^2)/2}}{}$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-(y-x/2)^2/2}$$

Thus,

$f_{Y|X}(y,x)$ is $N(\frac{X}{2}, 1)$.

$\text{Prob}(0 \le Y \le 2 \mid X=0) = \Phi(2) - \Phi(0)$.

where $\Phi(x)$ is CDF of $N(0,1)$

$\text{Prob}(0 \le Y \le 2 \mid X=2) = 2\Phi(1) - 1$

__Ex:__

Let $X \sim U[0,1]$ and the r.v. $Y \sim U[0,X]$
Find joint density of X, Y and marginal density of Y

$$f_X(x) = \begin{cases} 1 & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} 1/x & , 0 < y < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$f(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & , 0 < y < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_y^1 \frac{1}{x}\, dx$$

$$f_Y(y) = \begin{cases} -\log y & , \quad 0 < y < 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Baye's Rule

$$f_{(X|Y)}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{f_X(x)\, f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(x)\, f_{Y|X}(y|x)\, dx}$$

19/3/19

Bivariate Normal Density

Random vector (X_1, X_2) is said to follow bivariate normal density if the joint pdf is given by

$$f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2\sqrt{1-\rho^2}} \left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right] \right\}$$

$-\infty < x_1 < \infty$

$-\infty < x_2 < \infty$

$\rho \rightarrow$ corelation coefficient

Probability computation:

$$P(a_1 \le X_1 \le b_1,\ a_2 \le X_2 \le b_2)$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2)\, dx_2\, dx_1$$

The range of parameters:

$$-\infty < \mu_1 < \infty$$
$$-\infty < \mu_2 < \infty$$
$$\sigma_1 > 0,\ \sigma_2 > 0$$
$$-1 < \rho < 1$$

The marginal densities

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1,x_2)\, dx_2$$
$$= \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}$$

$$-\infty < X_1 < \infty$$

Marginal density of X_1 is $N(\mu_1, \sigma_1^2)$

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f(x_1,x_2)\, dx_1$$
$$= \frac{1}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$$-\infty < X_2 < \infty.$$

Marginal density of X_2 is $N(\mu_2, \sigma_2^2)$

$E(X_1) = \mu_1$, $var(X_1) = \sigma_1^2$

$E(X_2) = \mu_2$, $var(X_2) = \sigma_2^2$

$$\operatorname{cov}(X_1, X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1-\mu_1)(x_2-\mu_2)\, f(x_1,x_2)\, dx_2\, dx_1$$

$$=$$

Then,

$$\rho = \frac{\operatorname{cov}(X_1, X_2)}{\sigma_1 \cdot \sigma_2}$$

In general, independence $\Rightarrow$ uncorrelatedness and converse is NOT true.

In case of bivariate normal density, if X_1 & X_2 are uncorrelated ($\rho = 0$) $\Rightarrow$ X_1 & X_2 are independent. [only for normal]

(joint = product of marginals)

The conditional densities.

$$f_{X_1|X_2} = \frac{f(x_1,x_2)}{f_{X_2}(x_2)}$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\sigma_1^2(1-\rho^2)}\left\{x_1 - \left[\mu_1 + \rho(\sigma_1/\sigma_2)(x-\mu_2)\right]\right\}^2\right]$$

$$f_{X_2|X_1} = \frac{f(x_1,x_2)}{f_{X_1}(x_1)}$$

$$= \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\sigma_2^2(1-\rho^2)}\left\{x_1 - \left[\mu_2 + \rho(\sigma_2/\sigma_1)(x-\mu_1)\right]\right\}^2\right]$$

Thus, conditional densities $X_2|X_1$ and $X_1|X_2$ are both normal densities.

$$\boxed{X_2 | X_1 = x_1 \sim N\left(\mu_2 + \rho(\sigma_2/\sigma_1)(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right)}$$

$$\boxed{X_1 | X_2 = x_2 \sim N\left(\mu_1 + \rho(\sigma_1/\sigma_2)(x_2-\mu_2),\ \sigma_1^2(1-\rho^2)\right)}$$

Ex: (X_1, X_2) is a bivariate normal r.v. with parameters $\mu_1 = 0.2$, $\mu_2 = 1100$, $\sigma_1^2 = 0.02$, $\sigma_2^2 = 525$, $\rho = 0.9$

Compute $E(X_2|x_1)$.

$$E(X_2|x_1) = \mu_2 + \rho(\sigma_2/\sigma_1)(x_1-\mu_1)$$

$$\boxed{E(X_2|x_1) = 145.8x_1 + 1070.84}$$

$P(X_2 \geq 1080 | X_1 = 1)$

$$var(X_2|X_1=1) = \sigma_2^2(1-\rho^2) = 525(1-\rho^2) = 525(1-0.81)$$
$$= 99.75$$

Note that $Y = X_2|x_1=1$ is a Normal density with mean μ_Y and variance σ_Y^2 given by.

$$\mu_Y = 1216.64$$
$$\sigma_Y^2 = 99.75.$$

$$P(X_2 \geq 1080 | X_1 = 1) = P(Y \geq 1080)$$
$$= P\left(\frac{Y-\mu_Y}{\sigma_Y} \geq \frac{1080-\mu_Y}{\sigma_Y}\right)$$
$$= P\left(Z \geq \frac{1080-1216.64}{\sqrt{99.75}}\right)$$
$$= P \; Z \geq -1$$

25/3/19

Sampling Distributions :

1) Standard Normal $N(0,1)$

2) χ^2 distribution with n degree of freedom (Chi-squared)

Let $X_1, X_2, \dots, X_n$ be iid $N(0, \sigma^2)$

$$\frac{X_1 + X_2 + X_3 + \dots + X_n}{n} \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$\frac{X_1 + X_2 + \dots + X_n}{\sigma\sqrt{n}} = \left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) \Big/ (\sigma/\sqrt{n})$$

$$= \left\{\frac{N(\mu, \sigma^2) - \mu}{\sigma}\right\}$$

$N(0,1)$

In particular $\frac{X_1}{\sigma} \sim N(0,1)$

$$\frac{X_1^2}{\sigma^2} \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

From the additivity property of $\Gamma(\alpha_i, \beta)$,

We know then,

$$\frac{X_1^2}{\sigma^2} + \frac{X_2^2}{\sigma^2} + \dots + \frac{X_n^2}{\sigma^2} \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

$\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ is χ^2 density with n degree of freedom

Note:

χ^2 is sum of squares of 'n' independent standard normal r.v.s.

[n : (degree of freedom) dof]

Theo:

Let $Y_1, Y_2, \ldots, Y_m$ be independent χ^2 random variables with dofs $k_1, k_2, \ldots, k_m$ respectively.

$$\boxed{Y_1 + Y_2 + \ldots + Y_m \sim \chi^2_{k_1+k_2+\ldots+k_m}}$$

Ratio of χ^2 densities

Let $Y_1 \sim \chi^2_{k_1}$
$Y_2 \sim \chi^2_{k_2}$

and Y_1 and Y_2 are independent.

Then the random variable defined by the ratio $\dfrac{Y_1/k_1}{Y_2/k_2}$ is called as F distributed random variable with (k_1, k_2) dofs

Particular case: $k = 1$ $=$ $F(1, k_2)$

$$F(1, k_2) = \frac{\text{square of std. normal}}{\left(\text{sum of square of std. normal variables divided by dof}\right)}$$

t- distribution

Let X be std. Normal r.v. and Y be a χ^2 r.v. with n dof and X and Y are independent of each other

$$X \sim N(0,1) \quad \text{and } X, Y \text{ are independent.}$$
$$Y \sim \chi^2_n$$

Then $\frac{X}{\sqrt{Y/n}}$ is said to follow t- distribution with n dof:

$$\frac{X}{\sqrt{Y/n}} \sim t_n$$

Clearly $t_n^2 \sim F(1, n)$

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be iid with μ and variance σ^2

Objective: To study the distribution of $\sum_{i=1}^{n} X_i = S_n$

Notice:

$$E(S_n) = n\mu$$
$$Var(S_n) = n\sigma^2$$

Suppose X_i has density F, then $f_{S_n}(x) = \sum_y f_{S_{n-1}}(y) f(x-y)$

if X_i's are discrete.

$$f_{S_n}(x) = \int_{-\infty}^{\infty} f_{S_{n-1}}(y) f(x-y)\, dy$$ if X_is are continuous

* Note: $\boxed{S_n^* = \frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}} \sim N(0,1)$

Let $S_n = X_1 + X_2 + \dots + X_n$

Then

$$\lim_{n \to \infty} P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right] = \Phi(x), \quad -\infty < x < \infty$$

where $\Phi(x)$ is CDF of $N(0,1)$

Note that

$$P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right] = F_{S_n^*}(x)$$

$$= \text{CDF of } S_n^*$$

CLT states.

$$\lim_{n \to \infty} F_{S_n^*}(x) = \Phi(x) \qquad -\infty < x < \infty$$

Observe:

$$P(S_n \le x) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le \frac{x - n\mu}{\sigma\sqrt{n}}\right)$$

$$\simeq \Phi\left(\frac{x - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{x - E(S_n)}{\sqrt{Var(S_n)}}\right)$$

Generally for $n \ge 25$, these approximation are really, very good for any other distribution.

Example:

Let $X_i \sim \exp(1)$ for $i = 1, 2, \dots, n$

$S_n = X_1 + \dots + X_n$

$$\mu = \frac{1}{\lambda} = 1, \quad \sigma^2 = \frac{1}{\lambda^2} = 1$$

~~Prob (Sn ≤ x)~~

$$\text{Prob}(S_n \le x) \simeq \Phi\left(\frac{x - n}{\sqrt{n}}\right) \quad -\infty < x < \infty$$

Example:

Suppose life of a bulb after it is installed follows exponential distribution with mean = 10 days. As soon as the bulb runs out, another bulb with same characteristics is installed. What is the probability that 50 bulbs are required in one year.

$$\mu = \frac{1}{\lambda} = 10 \Rightarrow \boxed{\lambda = \frac{1}{10}}, \quad \sigma^2 = \frac{1}{\lambda^2}$$

$$\Rightarrow X_i \sim \exp(1/10)$$

$$S_{50} = \sum_{i=1}^{50} X_i$$

Probability of interest:

$$P(S_{50} \leq 365) \approx \Phi\left(\frac{x - E(S_n)}{\sqrt{Var(S_n)}}\right)$$

$$\simeq \Phi\left(\frac{365 - 500}{\sqrt{5000}}\right) = \Phi(-1.91)$$

$$\boxed{P(S_{50} \leq 365) = 0.028.}$$

$$\left({}^{r+x-1}C_x \right) p^r (1-p)^x$$

	μ	σ^2	mgf.	
Bernoulli(p)	p	$p(1-p)$	$(1-p) + pe^t$	$p^x(1-p)^{1-x}$ $\quad x = 0, 1$
Binomial (n,p)	np	$np(1-p)$	$[(1-p) + pe^t]^n$	${}^nC_x\, p^x (1-p)^{n-x}$
Geometric (p)	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$	$p(1-p)^x$
Negative Binomial (r,p)	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$	$p^r \left({}^{r+x-1}C_x \right) (1-p)^x$
Poisson(λ)	λ	λ	$e^{\lambda(e^t-1)}$	$\frac{e^{-\lambda}\lambda^x}{x!}$
Exponential(λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$	$\lambda e^{-\lambda x}$
Gamma (α, λ)	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$; $\int_0^\infty x^{\alpha-1} \cdot e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(a-b)^2}{12}$	$\frac{e^{bt} - e^{at}}{(b-a)t}$	$\frac{1}{b-a}$
Normal (μ, σ^2)	μ	σ^2	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$n(2 - \log 5) = 2$