Calculus Used in Deep Learning

Intuition, Theory, and Worked Examples

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Why Calculus?

- Deep learning models learn by **minimizing a loss function**
- Learning requires knowing:
  - How output changes with weights
  - Which direction reduces the error
- Calculus provides:
  - Derivatives
  - Gradients
  - Optimization methods

Core idea:

$$\text{Learning} = \text{Gradient-based optimization}$$

Derivative: Intuition

Definition

The derivative measures the rate of change of a function with respect to its input. A derivative tells us how much the output changes when the input changes.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- Large derivative $\rightarrow$ output changes rapidly
- Small derivative $\rightarrow$ output changes slowly

Single Neuron Model

A neuron computes:

$$z = wx + b$$

where:

- w – weight
- x – input
- b – bias

Derivative of Neuron Output

Derivative of z with respect to w:

$$\frac{\partial z}{\partial w} = x$$

Interpretation:

- Large input $\rightarrow$ strong weight update
- Small input $\rightarrow$ weak learning signal

Partial Derivatives: Multiple Inputs and Weights

Consider a neuron with multiple inputs:

$$z = w_1 x_1 + w_2 x_2. \tag{1}$$

The partial derivatives of z with respect to each weight are given by:

$$\frac{\partial z}{\partial w_1} = x_1, \qquad \frac{\partial z}{\partial w_2} = x_2. \tag{2}$$

Interpretation: Each weight is updated independently and the magnitude of the update is proportional to its corresponding input feature.

Loss Function

Loss quantifies prediction error.
Mean Squared Error (MSE):

$$\mathcal{L} = \frac{1}{2}(y - \hat{y})^2$$

- y – true label
- $\hat{y}$ – predicted output

Derivative of Loss

Derivative of loss w.r.t. prediction:

$$\frac{\partial \mathcal{L}}{\partial \hat{y}} = \hat{y} - y$$

Example:

$$y = 1, \quad \hat{y} = 0.6$$

$$\frac{\partial \mathcal{L}}{\partial \hat{y}} = -0.4$$

Meaning: Prediction must increase.

Chain Rule

If:

$$y = f(g(x))$$

that is, y depends on g and g depends on x then x, then

$$\frac{dy}{dx} = \frac{dy}{dg} \cdot \frac{dg}{dx}$$

Interpretation (Intuition): Think of the flow

$$x \longrightarrow g(x) \longrightarrow y$$

A small change in x affects $g(x)$, and that change in $g(x)$ affects y. Therefore, the total rate of change is the product of the two rates:

$$\frac{dy}{dx} = \frac{dy}{dg} \cdot \frac{dg}{dx}.$$

Backpropagation = repeated chain rule

Examples

i)

$$y = \sin(x^2).$$

Outer function: $f(u) = \sin u$, Inner function: $g(x) = x^2$.
Then, by the chain rule,

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x.$$

ii)

$$y = e^{3x} \quad \Rightarrow \quad \frac{dy}{dx} = 3e^{3x}.$$

iii)

$$y = e^{x^2} \quad \Rightarrow \quad \frac{dy}{dx} = 2x\, e^{x^2}.$$

iv)

$$y = e^{e^x} \quad \Rightarrow \quad \frac{dy}{dx} = e^{e^x} \cdot e^x.$$

Backpropagation Example

Neuron with activation:

$$z = wx + b, \quad a = \sigma(z)$$

Loss:

$$\mathcal{L}(a)$$

Forward pass:

$$z = wx + b, \qquad a = \sigma(z), \qquad L = L(a)$$

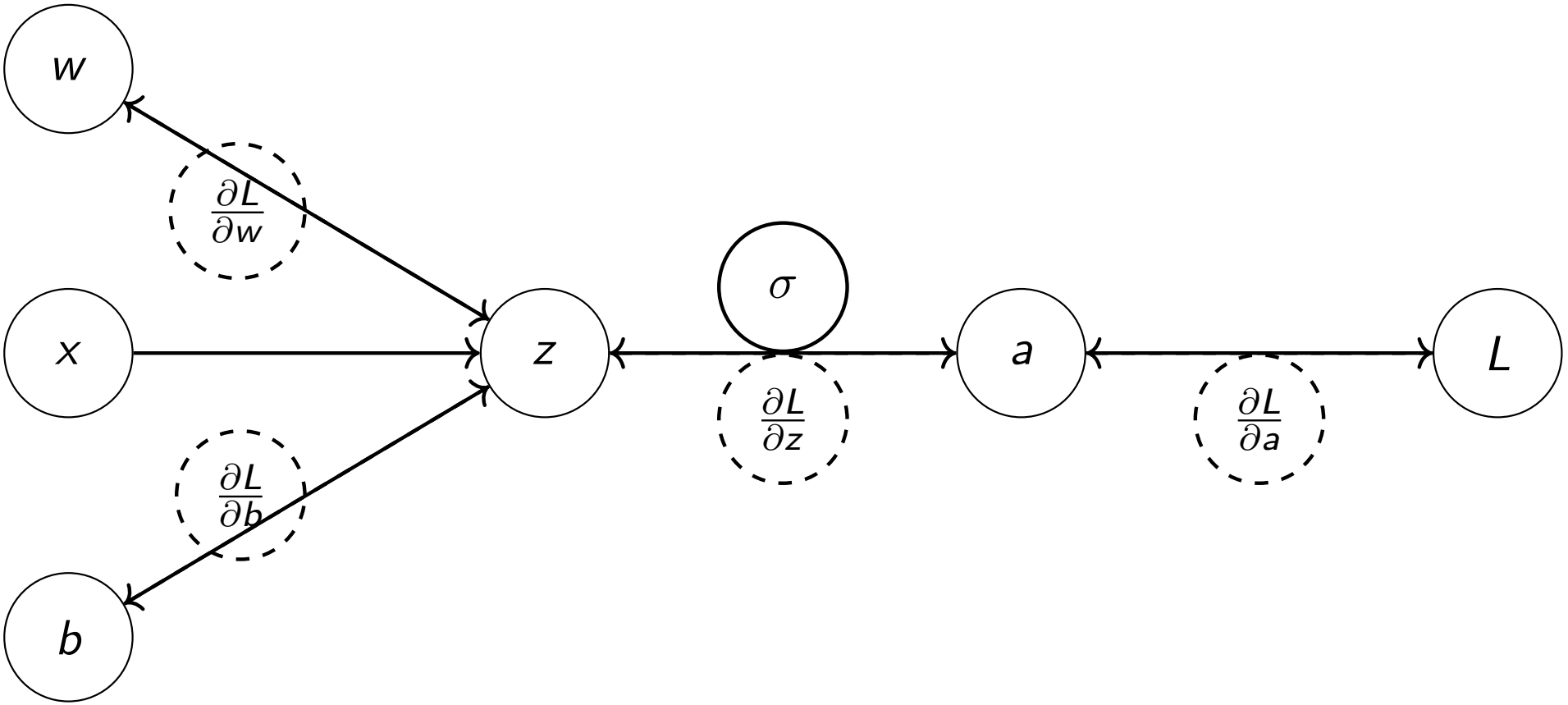
Backward Pass (Chain Rule):

$$w,\ b \longrightarrow z \longrightarrow a \longrightarrow L$$

Gradient:

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{\partial \mathcal{L}}{\partial a} \cdot \frac{\partial a}{\partial z} \cdot \frac{\partial z}{\partial w}$$

Computation Graph



Gradient and Gradient Descent

For weights:

$$\boldsymbol{w} = [w_1, w_2, \ldots, w_n]$$

Gradient:

$$\nabla_{\boldsymbol{w}} \mathcal{L} = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial w_1} \\ \frac{\partial \mathcal{L}}{\partial w_2} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial w_n} \end{bmatrix}$$

Direction of steepest loss increase.

Gradient Descent Update Rule

$$\boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \eta \nabla_{\boldsymbol{w}} \mathcal{L}$$

- η – learning rate
- Move opposite to gradient

Why Taylor Series?

- Approximates complex functions using polynomials
- Enables local linear or quadratic analysis
- Forms the basis of optimization methods
- Widely used in deep learning theory

Key idea: Approximate a function near a point using derivatives.

Taylor Series: Definition

Let $f(x)$ be a function with derivatives of all orders at $x = a$. The Taylor series expansion of $f(x)$ about $x = a$ is:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Example: Taylor Series of $f(x) = e^x$

Derivatives of $f(x) = e^x$:

$$f'(x) = e^x, \quad f''(x) = e^x, \quad \ldots$$

At $x = 0$:

$$f^{(n)}(0) = 1 \quad \forall n$$

Taylor Expansion of e^x

Substitute into the Maclaurin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Polynomial approximation:

- First order: $1 + x$
- Second order: $1 + x + \frac{x^2}{2}$

First-Order Taylor Approximation

Linear approximation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Used in:

- Gradient descent
- Local sensitivity analysis

Second-Order Taylor Approximation

Quadratic approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Used in:

- Newton's method
- Curvature analysis

Numerical Example: Problem Statement

Approximate the function

$$f(x) = \ln(1 + x)$$

around $x = 0$ using a Taylor series, and estimate the value at:

$$x = 0.1$$

Approach:

- Compute derivatives at $x = 0$
- Use first- and second-order Taylor approximations

Step 1: Compute Derivatives

Given:

$$f(x) = \ln(1 + x)$$

Derivatives:

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}$$

At $x = 0$:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1$$

Step 2: First-Order Taylor Approximation

The first-order Taylor expansion about $x = 0$ is:

$$f(x) \approx f(0) + f'(0)x$$

Substitute values:

$$\ln(1 + x) \approx x$$

Numerical estimate at $x = 0.1$:

$$\ln(1.1) \approx 0.1$$

Step 3: Second-Order Taylor Approximation

The second-order Taylor expansion is:

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$$

Substitute values:

$$\ln(1+x) \approx x - \frac{x^2}{2}$$

Numerical estimate at $x = 0.1$:

$$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} = 0.095$$

Step 4: True Value and Error

True value:

$$\ln(1.1) \approx 0.09531$$

- First-order approximation: 0.10000
- Second-order approximation: 0.09500

Observation:

- Second-order Taylor approximation is much more accurate
- Higher-order terms reduce approximation error

Deep Learning Interpretation

- First-order Taylor $\rightarrow$ Gradient Descent
- Second-order Taylor $\rightarrow$ Newton / curvature-aware methods
- Deep learning mostly relies on first-order methods due to efficiency

Key Insight:

$$\text{Optimization} = \text{Local Taylor Approximation of Loss Function}$$

Activation Functions

Function	Expression	Derivative
Sigmoid	$\sigma(x)$	$\sigma(x)(1-\sigma(x))$
ReLU	$\max(0,x)$	$1_{x>0}$
Tanh	$\tanh(x)$	$1-\tanh^2(x)$

Summary

- Derivatives measure sensitivity
- Chain rule enables backpropagation
- Gradients guide learning
- Gradient descent minimizes loss

Without calculus, deep learning cannot learn.