CS 577: Introduction to Algorithms Fall 2020

Homework 11

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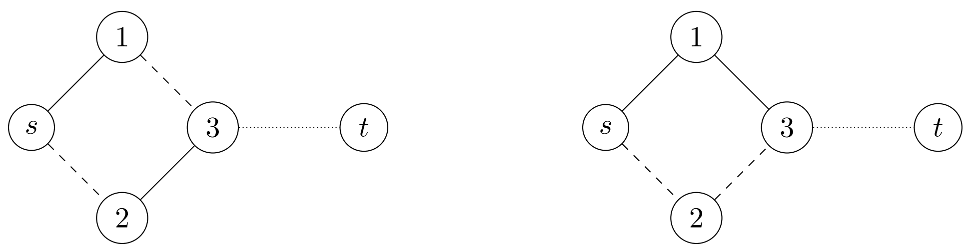
This homework covers NP-completeness of a variety of problem types. **Problems 1 and 3 must be submitted for grading by 11:59pm on 12/7**. Please refer to the homework guidelines on Canvas for detailed instructions.

Warm-up problems

1. You are given a directed graph $G = (V, E)$, two vertices $s, t \in V$, and a partition of the set of edges: $E = \cup_{i=1}^{k} E_i$, where the sets E_i are pairwise disjoint. Your goal is to find a path from s to t that contains exactly one occurrence of an edge of each E_i, or report that none exists.

   Consider the examples below on the same graph $G = (V, E)$, s, and t, but two different partitions, each with $k = 3$. In both examples, the solid edges form E_1, the dashed edges E_2, and the dotted edge E_3.

   In the example on the left there exists a solution, namely the path $s \to 1 \to 3 \to t$ (as well as the path $s \to 2 \to 3 \to t$). In the example on the right there does not exist a solution because every path from s to t has to use two edges from E_1 or two edges from E_2.

   Prove that the decision version of the problem is NP-complete.

2. Recall the Interval Scheduling problem from class. Consider the variant, Multiple Interval Scheduling, in which each job may require the machine to be reserved for multiple time intervals. Show that this variant is NP-hard.

Regular problems

3. You're president of the student government and planning a flashmob dance. You want to surprise the teachers and administrators, so you can't send an email blast to all of the students. You've been procrastinating, so it's the night before the event and you need to get the word out. You can choose at most k "chosen" students to tell about the flashmob tonight. Tomorrow, the chosen students will tell everyone they have a class with. Note that you can't rely on any students other than the original chosen ones to pass on the word; if you tell Alice but not Bob tonight and Alice has a class with Bob tomorrow, Alice will tell Bob about the flashmob, but Bob will not tell any students in his other classes. Is there a way to choose at most k students such that all of the students at your school hear about the flashmob by the

end of the school day tomorrow? If so, what is such a group of at most k students you could tell tonight?

The input is a list of m sets C_c, each of which represents the students enrolled in class c. There are n students and m classes, and each student is enrolled in at least one class. Show that the Flashmob problem is NP-complete.

4. Your friend recently complained to you about a controversal policy in her company: All employees should work from 9am to 9pm, 6 days a week, and go jogging on the non-working day of the week to keep fit. The company has distributed smartwatches to monitor the workout statistics and requires that each employee realizes a total elevation change of at least k ft while jogging. When you start at 100 ft, ascend to 120 ft, and then decend to 90 ft, you achieve a total elevation change of $|100 - 120| + |120 - 90| = 50$ ft.

   Your friend is not a fan of jogging, but does not want to quit the job either. Therefore, she plans to achieve a total elevation change of *exactly* k ft every week. She found a nice jogging park in the city and plans to jog from checkpoint s to checkpoint t. She also downloaded the park map including all jogging checkpoints with their elevations, and the roads connecting the checkpoints. Since everyone in the park is jogging, all roads connecting the checkpoints are one-way. All elevations are nonnegative integers.

   Show that finding a route from s to t with a total elevation change of exactly k is NP-complete.

5. In the unit on network flow we saw that the problem MIN-CUT can be solved in polynomial time. Now let us consider the problem MAX-CUT: Given a network G and a number c, does there exist an st-cut with capacity at least c? Show that MAX-CUT is NP-complete by giving a reduction from NAE-3-SAT.

Challenge problem

6. Show that the following problem is NP-complete: Given positive integers $n, n_1, \ldots, n_k$ in binary representation, decide whether n can be written as a product $\prod_{i=1}^{k} n_i^{e_i}$ where the e_i's are nonnegative integers.

Programming problem

7. SPOJ problem Oil Company (problem code OILCOMP).