CS 577- Intro to Algorithms

Network Flow (Part 3)

Dieter van Melkebeek

November 3, 2020

Outline

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Recap

- Notions: network, flow, cut
- Computational problems: max flow vs min cut

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- Computational problems: max flow vs min cut

Applications of max flow

- Bipartite matching
- Edge-disjoint paths
- Survey design

Recap - notions

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Network

- a digraph (V, E)
- edge capacities $c : E \to [0, \infty)$
- the source $s \in V$, which has indegree 0, and
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Flow

A mapping $f : E \to [0, \infty)$ satisfying

- [capacity constraints] $(\forall e \in E)\, f(e) \leq c(e)$
- [conservation constraints] $(\forall v \in V \setminus \{s, t\})\, f_{\text{in}}(v) = f_{\text{out}}(v)$
  where $f_{\text{in}}(v) \doteq \sum_{u \in V : e \doteq (u,v) \in E} f(e)$ and
  $f_{\text{out}}(v) \doteq \sum_{u \in V : e \doteq (v,u) \in E} f(e)$.

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st-Cut

A partition (S, T) of V such that $s \in S$ and $t \in T$.

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Output: st-cut (S, T) such that $c(S, T) \doteq \sum_{e \in S \times T} c(e)$ is minimized

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Duality

$$\max_{\text{flow } f}(\nu(f)) = \min_{st\text{-cut } (S,T)}(c(S, T))$$

Bipartite Matching

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Definition

A matching M in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that each $v \in V$ appears in at most one $e \in M$.

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$$\text{matching } M \text{ in } G \quad \stackrel{\text{bijection}}{\longleftrightarrow} \quad \text{integral flow } f \text{ in } N$$

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A bipartite graph $G = (V, E)$ with bipartition (L, R) and $|L| = |R| = n$ has a perfect matching iff

$$(\forall A \subseteq L)\,|G(A)| \geq |A|.$$

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$\Leftarrow$: Proof of contrapositive on next slide.

Proof of Marriage Theorem

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- Consider $A \doteq L \cap S$.

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- Consider $A \doteq L \cap S$.

$$|G(A)| \stackrel{(2)}{\leq} |R \cap S| \stackrel{(3)}{=} c(S, T) - |L \cap T| \stackrel{(1)}{<} n - |L \cap T| = |L \cap S| = |A|$$

Edge-Disjoint Paths

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Computational problem

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- Resulting algorithm runs in time $O(nm)$.

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- $c(S, T) = |E \cap S \times T|$
- Removing $E \cap S \times T$ from G ensures no st-path remains.

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Edge connectivity duality

The maximum number of edge-disjoint st-paths equals the minimum number of edges to be removed so no st-path remains.

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- $\max_{\text{flow } f}(\nu(f)) = \ell \leq r \leq \min_{st\text{-cut } (S,T)}(c(S, T))$
- By duality, LHS = RHS so $\ell = r$.

Survey Design

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Computational problem

Input: n customers $i \in [n]$
m products $j \in [m]$
$S_i \subseteq [m]$: products that customer $i \in [n]$ can survey
$c_i \in \mathbb{N}$: max number of surveys for customer $i \in [n]$
$p_j \in \mathbb{N}$: min number of surveys of product $j \in [m]$

Output: set $D \subset [n] \times [m]$ such that

- $(\forall (i,j) \in D)\, j \in S_i$
- $(\forall i \in [n])\, |\{j \in [m] : (i,j) \in D\}| \leq c_i$
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Model

Represent each $(i,j) \in D$ as a unit of flow that passes through customer i and product j.

Survey Design – reduction to max flow

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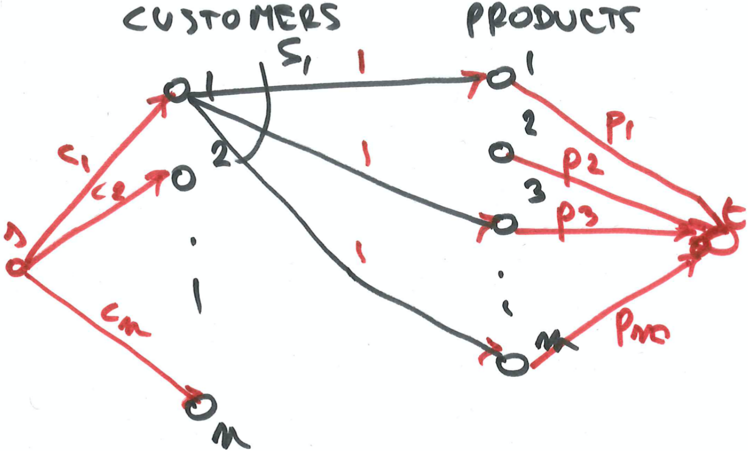

$$S_1 = \{1, 3, m\}$$

Survey Design – analysis

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$$D \quad = \quad \text{middle edges that carry flow}$$

Survey Design – analysis

- Minimal survey D satisfies

$$(\forall j \in [m])\, |\{i \in [m] : (i,j) \in D\}| = p_j.$$

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$$\begin{aligned} \text{minimal survey } D \quad &\overset{\text{bijection}}{\longleftrightarrow} \quad \text{integral flow } f \text{ with } \nu(f) = \sum_{j \in [m]} p_j \\ D \quad &= \quad \text{middle edges that carry flow} \end{aligned}$$

- Resulting algorithm runs in time $O((n+m)(n+m+\sum_{i\in[n]} |S_i|))$.