CS 577- Intro to Algorithms

Computational Intractability

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Outline

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Motivation

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- Recognizing infeasible approaches

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- P vs NP problem

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Topics

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- Classes P and NP

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- Classes P and NP
- NP-hardness and NP-completeness

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Input: graph G

Output: independent set S of G such that $|S|$ is maximized

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Satisfiability

Input: Boolean formula φ

Output: satisfying assignment of φ, or report that none exists

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Output: $I \subseteq [n]$ such that $\sum_{i \in I} a_i = t$, or report impossible

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- [in case of optimization problem]
  Objective $f(x,y)$ can be evaluated in time polynomial in n.

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- Parameters:
  - $c \in \mathbb{N}$
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- Goal:

Decision	Is $S_x \neq \emptyset$?
Search	Find $y \in S_x$ or report that no such y exists.
Optimization	Find $y^* \in S_x$ such that $f(x,y^*) = \min_{y \in S_x}(f(x,y))$ respectively $f(x,y^*) = \max_{y \in S_x}(f(x,y))$

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Assuming P $\neq$ NP, the "hardest" problems in NP cannot be solved in polynomial time (but some problems in NP can).

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Corollary

Assume $\mathrm{P} \neq \mathrm{NP}$. If B is NP-hard then $B \notin \mathrm{P}$.

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- By transitivity, $A \leq^p$ Independent Set.

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- Recognizing infeasible approaches.
- Convincing your boss

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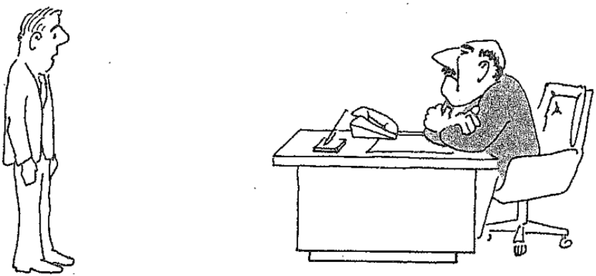
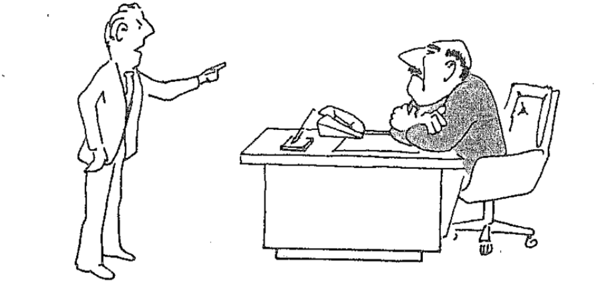
Motivation for Proving NP-Hardness



"I can't find an efficient algorithm, I guess I'm just too dumb."

Motivation for Proving NP-Hardness



“I can’t find an efficient algorithm, because no such algorithm is possible!”

Motivation for Proving NP-Hardness

"I can't find an efficient algorithm, but neither can all these famous people."

Prevalence of NP-completeness

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- Considered strong evidence that $\mathrm{P} \neq \mathrm{NP}$.
- In fact, almost all of the known problems in NP that are not known to be in P, have been shown to be NP-hard.
- Notorious exceptions include: graph isomorphism, factoring integers.