CS 577- Intro to Algorithms

Divide and Conquer

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Outline

Paradigm

1. Break up given instance into smaller ones.
2. Recursively solve those.
3. Combine their solutions into one for the given instance.

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  - Counting inversions

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- Common pattern
  - Sorting (Mergesort)
  - Counting inversions
- Lower bound for sorting

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Problem (specification, method)

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  Output: a^b

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  Input: (a, b) with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$

  Output: a^b
- Example: Sorting

  Input: An array $A[1, \dots, n]$ of length $n \geq 1$.

  Output: Sort(A), i.e., a copy of A sorted from the smallest to the largest element

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- Precise high-level description in words and/or pseudocode.
- Examples for Sorting: Selection Sort, Insertion Sort, Bubblesort, Quicksort, Merge Sort, ...

Program (code, implementation)

- Precise detailed description in a particular programming language

Mergesort in words

Mergesort in pseudocode

Input: $A[1, \dots, n]$
Output: $\mathrm{Sort}(A)$

procedure $\textsc{Merge-Sort}(A)$
    if $n = 1$ **then**
        return A
    else
        $m \leftarrow \lfloor n/2 \rfloor$
        $L \leftarrow \textsc{Merge-Sort}(A[1, \dots, m])$
        $R \leftarrow \textsc{Merge-Sort}(A[m+1, \dots, n])$
        return $\textsc{Merge}(L, R)$

where Merge

Input: sorted arrays $L[1, \dots, n]$ and $R[1, \dots, m]$ with $n, m \geq 1$.

Output: $\mathrm{Sort}(LR)$

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On every valid input, the algorithm produces a correct output.

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For every integer $n \geq 1$, $P(n)$ holds, where
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- Base case: $n = 1$
- Induction step:
  - For every integer $n \geq 1$, $P(n) \Rightarrow P(n+1)$
  - For every integer $n \geq 1$, $P(1) \wedge P(2) \dots P(n) \Rightarrow P(n+1)$

Merging Sorted Arrays

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- PowerPoint presentation
- Correctness
- Running time: $O(n + m)$

Mergesort – running time

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- Recurrence relation

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- Recurrence relation
- Recursion tree

Counting Inversions

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$A = [3, 5, 4, 7, 3, 1]$

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Input: array $A[1, \ldots, n]$ of integers with $n \geq 1$

Output: $\mathrm{Inv}(A) \doteq$ number of inversions in A

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Bounds on $\text{Inv}(A)$

Count in words

Count in pseudocode

Input: $A[1, \ldots, n]$
Output: $\mathrm{Sort}(A)$

procedure $\textsc{Merge-Sort}(A)$
    if $n = 1$ **then**
        return A
    else
        $m \leftarrow \lfloor n/2 \rfloor$
        $L \leftarrow \textsc{Merge-Sort}(A[1, \ldots, m])$
        $R \leftarrow \textsc{Merge-Sort}(A[m+1, \ldots, n])$
        return $\textsc{Merge}(L, R)$

where Count-Cross

Input: sorted arrays $L[1, \ldots, n]$ and $R[1, \ldots, m]$ with $n, m \geq 1$.

Output: $\mathrm{Inv}(LR)$

Count – running time

Improved Count

Input: $A[1 \cdots n]$, an array of length $n \geq 1$
Output: (Inv(A),Sort(A))

procedure COUNT-AND-SORT(A)
    if $n = 1$ **then**
        return $(0, A)$
    else
        $m \leftarrow \lfloor n/2 \rfloor$
        $(c_L, L) \leftarrow$ COUNT-AND-SORT($A[1, \ldots, m]$)
        $(c_R, R) \leftarrow$ COUNT-AND-SORT($A[m+1, \ldots, n]$)
        $c_{\text{cross}} \leftarrow$ COUNT-CROSS(L, R)
        $c \leftarrow c_L + c_R + c_{\text{cross}}$
        $B \leftarrow$ MERGE(L, R)
        return $(c_L + c_R + c_{\text{cross}}, B)$

Improved Count

Input: $A[1 \cdots n]$, an array of length $n \geq 1$
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        $c_{\text{cross}} \leftarrow \textsc{Count-Cross}(L, R)$
        $c \leftarrow c_L + c_R + c_{\text{cross}}$
        $B \leftarrow \textsc{Merge}(L, R)$
        return $(c_L + c_R + c_{\text{cross}}, B)$

Running time: $O(n \log n)$

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Every comparison-based sorting algorithm takes $\Omega(n \log n)$ comparisons on arrays of length n.

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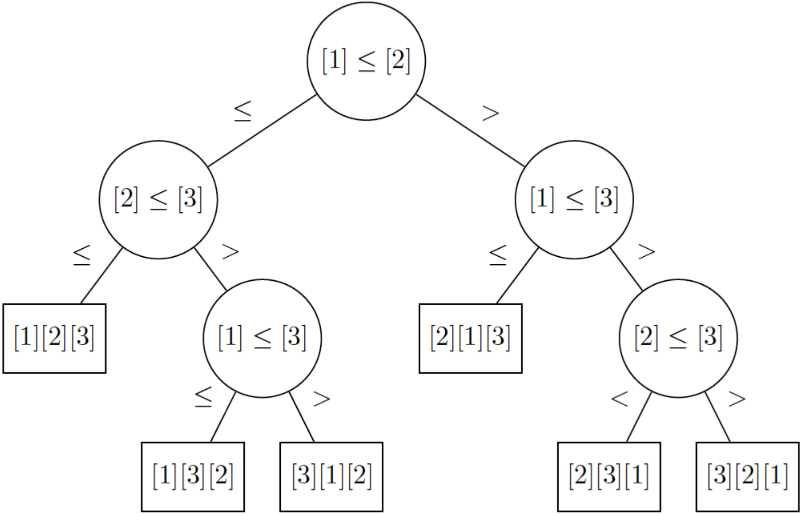
Proof

- Every such algorithm can be modeled as a decision tree.

Decision Tree



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Proof

- Every such algorithm for a given n can be modeled as a binary decision tree T.
- Depth d of T is the maximum number c of comparisons that A makes on arrays of length n.
- Number ℓ of leaves is at least $n! \doteq 1 \cdot 2 \cdot \ldots \cdot n$.
- $\ell \geq 2^d$

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- $(n/2)^{n/2} \leq n! \leq n^n$

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- $\log(n!) = \Theta(n \log n)$