CS 577- Intro to Algorithms

Randomness (Part 2)

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Outline

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Make a random choice when there are many good choices but it is hard to deterministically find one.

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- Balanced search tree: time $O(\log |S|)$, space $O(|S|)$
- Hash table: expected time $O(1)$, space $O(|S|)$

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  - Chaining
  - Open addressing (linear, quadratic, double)

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Two distinct elements $x, x' \in S$ with $h(x) = h(x')$.

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For any two distinct $x, x' \in U$,

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when $h : U \to [m]$ is picked uniformly at random.

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Corollary

For any fixed $x \in S$, the expected number of collisions with x equals $\frac{|S|-1}{m}$.

Closest Pair of Points in the Plane

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- Divide & Conquer: $O(n \log n)$
- Randomized: $O(n)$ in expectation

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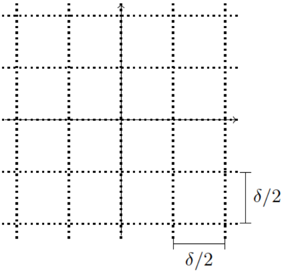

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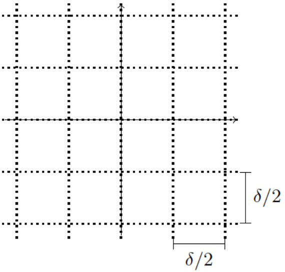

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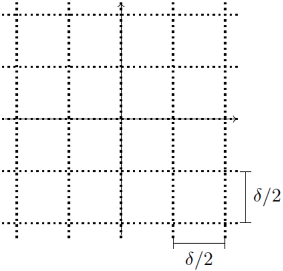

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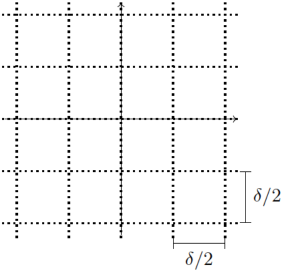

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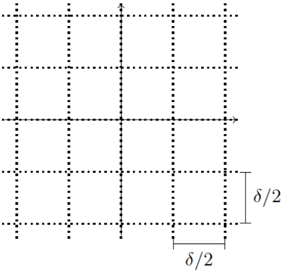

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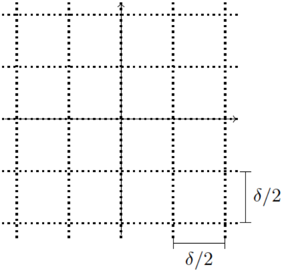

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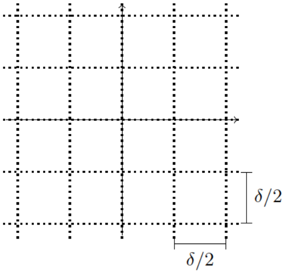

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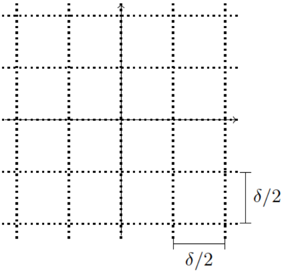

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  - If so, update δ and rehash.

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- If we consider points in random order, i-th point is involve in closest pair among first i points with probability at most $2/i$.
- Total expected time for rehashing then becomes $O(\sum_{i=1}^{n} \frac{2}{i} \cdot i) = O(n)$.

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- $p_2 = x_1 \cdot x_1 - x_2 \cdot x_2$
- $p_3 = (x_1 + x_2) \cdot (x_3 + x_4) \cdot \dots \cdot (x_{2n-1} + x_{2n})$

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- Probability of false positive is at most d/m, where $d = \max(\deg(p_1), \deg(p_2))$.

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- Probability of false positive is at most d/m, where $d = \max(\deg(p_1), \deg(p_2))$.
- Yields polynomial-time algorithm.