CS 577- Intro to Algorithms

Randomness

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December 8, 2020

Motivation

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What

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What

- Flipping coins / bits

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- Flipping coins / bits
- Picking a uniform element from some finite set

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Inherently needed for:

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- Sampling

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Useful for solving deterministic problems

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Useful for solving deterministic problems

- Today: selection and sorting

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- Reproducibility
- Need for randomness

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  - $\Pr[A_2] = 1/2$ if n is even

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- For any nonnegative random variable X and $a \in (0, \infty)$

$$\Pr[X \geq a] \leq E[X]/a.$$

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- The probability that a randomized algorithm on a given input runs for more than twice its expected time is at most 50%.

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Proof

$$\begin{aligned}
E[X] &\doteq \sum_{\omega \in \Omega} \Pr(\omega) \cdot X(\omega) \\
&= \sum_{\omega \in \Omega : X(\omega) < a} \Pr(\omega) \cdot X(\omega) + \sum_{\omega \in \Omega : X(\omega) \geq a} \Pr(\omega) \cdot X(\omega) \\
&\geq 0 + \sum_{\omega \in \Omega : X(\omega) \geq a} \Pr(\omega) \cdot a \\
&= a \cdot \Pr[X \geq a]
\end{aligned}$$

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Problem

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Output: kth element of $\mathrm{Sort}(A)$

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Schema

```
procedure SELECT(A, k)
    if n = 1 then return A[1]
    pick a pivot p from A
    (L, R) ← SPLIT(A, p)
    if k ≤ |L| then
        return SELECT(L,k)
    else if k > n − |R| then
        return SELECT(R,k − (n − |R|))
    else
        return p
```

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Analysis of expected running time

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- For any integer $i \geq 0$

$$X_i \doteq \begin{cases} \text{size of subarray at } i\text{th level of recursion} & \text{if exists} \\ 0 & \text{otherwise.} \end{cases}$$

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- Corollary: $E[X_i] \leq (\frac{3}{4})^i \cdot n$

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- Corollary: $E[X_i] \leq (\frac{3}{4})^i \cdot n$
- $E[T] \leq c \cdot \sum_{i=0}^{\infty} (\frac{3}{4})^i \cdot n = 4cn = O(n)$

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Proof of Shrinkage Lemma

$$
\begin{aligned}
E[X_{i+1}] &= \textstyle\sum_\ell \Pr[X_i = \ell] \cdot E[X_{i+1} \,|\, X_i = \ell] \\
&\leq \textstyle\sum_\ell \Pr[X_i = \ell] \cdot \frac{3}{4} \cdot \ell = \frac{3}{4} \cdot E[X_i]
\end{aligned}
$$

Shrinkage Lemma – figure

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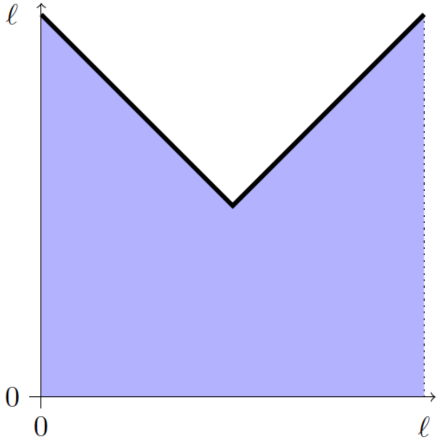

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Pr[number of levels exceeds $s \doteq \lceil \log_{\frac{4}{3}}(n) \rceil + \Delta] \leq \left(\frac{3}{4}\right)^{\Delta}$.

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- Number of levels exceeds s iff $X_s > 0$.

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Proof

- Number of levels exceeds s iff $X_s > 0$.
- Markov's inequality shows

$$\Pr[X_s > 0] = \Pr[X_s \geq 1] \leq E[X_s] \leq \left(\frac{3}{4}\right)^{s} \cdot n \leq \left(\frac{3}{4}\right)^{\Delta}.$$

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Corollary

$E[$ number of levels $] = O(\log n)$

Sorting

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Quicksort

- Pick of pivot p uniformly at random
- Construct L_p, M_p, and R_p
- Recursively sort L_p and R_p
- Return concatenation $\mathrm{Sort}(L_p)\ M_p\ \mathrm{Sort}(R_p)$

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Analysis

- Amount of work per level of recursion is bounded by cn for some constant c.
- Number of levels of recursion exceeds s iff $X_s^{(i)} > 0$ for at least one $i \in [n]$.

Sorting

Quicksort

- Pick of pivot p uniformly at random
- Construct L_p, M_p, and R_p
- Recursively sort L_p and R_p
- Return concatenation $\mathrm{Sort}(L_p)\ M_p\ \mathrm{Sort}(R_p)$

Analysis

- Amount of work per level of recursion is bounded by cn for some constant c.
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- $\Pr[$ number of levels exceeds $s \doteq \lceil 2\log_{\frac{4}{3}}(n)\rceil + \Delta'] \leq \left(\frac{3}{4}\right)^{\Delta'}$

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