CS 577- Intro to Algorithms

Computational Intractability (Part 2)

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Outline

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Motivation

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- Recognizing infeasible approaches

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- P vs NP problem

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P, NP, and NPC

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P, NP, and NPC

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- NP: decision problems with yes-instances that have polynomial-time verifiable certificates

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- P: decision problems that have polynomial-time algorithms
- NP: decision problems with yes-instances that have polynomial-time verifiable certificates
- Fact: $P \subseteq NP$

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P, NP, and NPC

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- NP: decision problems with yes-instances that have polynomial-time verifiable certificates
- Fact: $\mathrm{P} \subseteq \mathrm{NP}$
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- NP-complete (NPC): hardest problems in NP

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- NP-complete (NPC): hardest problems in NP
- Assume $\mathrm{P} \neq \mathrm{NP}$. If $B \in \mathrm{NPC}$ then $B \notin \mathrm{P}$.

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Today: Satisfiability

NP Decision/Search/Optimization

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- Parameters:
  - $c \in \mathbb{N}$
  - $V : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$ in P
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- Goal:

Decision	Is $S_x \neq \emptyset$?
Search	Find $y \in S_x$ or report that no such y exists.
Optimization	Find $y^* \in S_x$ such that $f(x,y^*) = \min_{y \in S_x}(f(x,y))$ respectively $f(x,y^*) = \max_{y \in S_x}(f(x,y))$

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- Examples: Independent Set, Satisfiability, Graph Coloring, Traveling Salesperson Problem, ...

NP-Hardness and NP-Completeness

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Corollary
Assume $\mathrm{P} \neq \mathrm{NP}$. If B is NP-hard then $B \notin \mathrm{P}$.

Circuit Satisfiability

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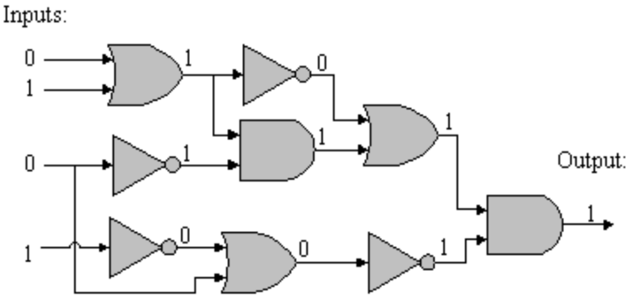

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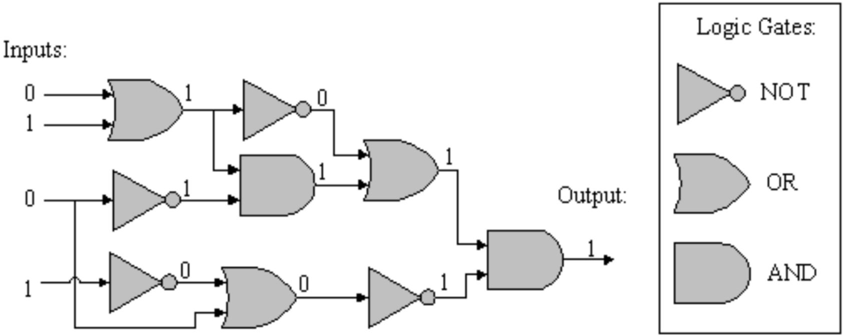

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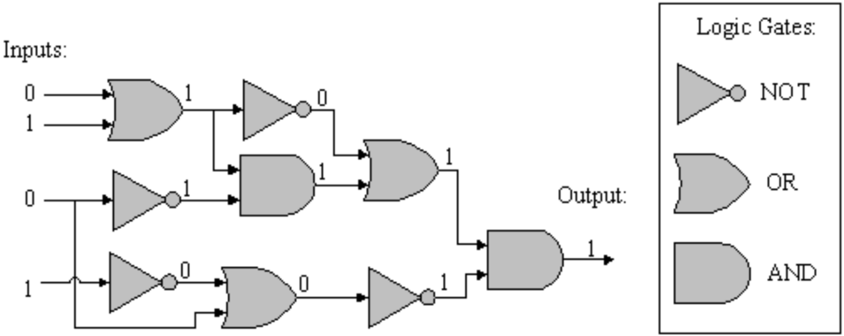

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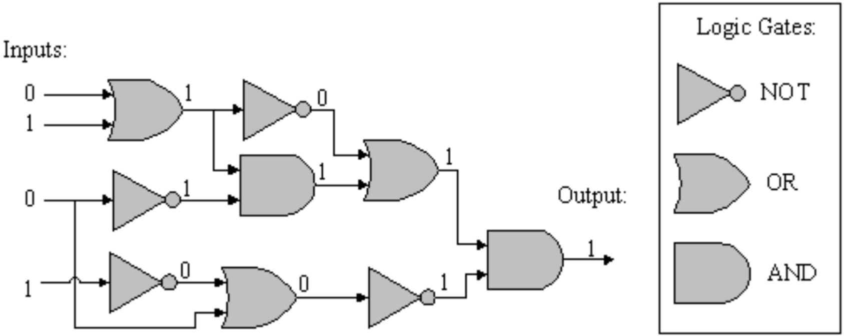

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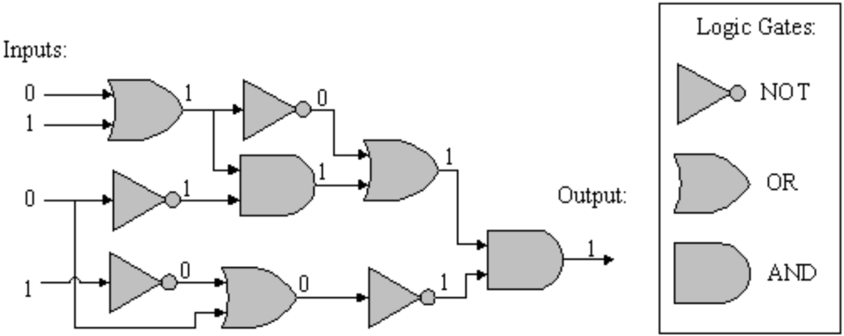

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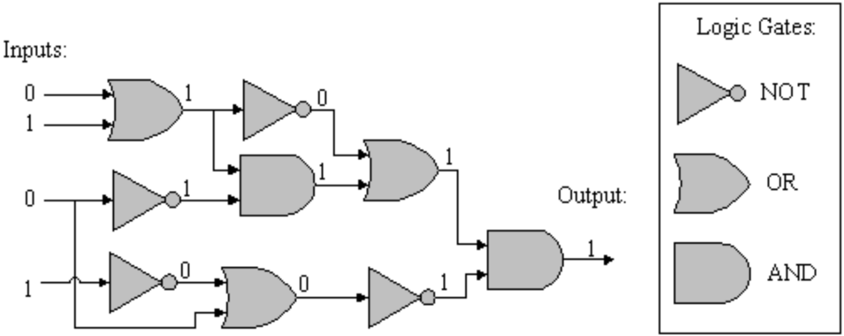

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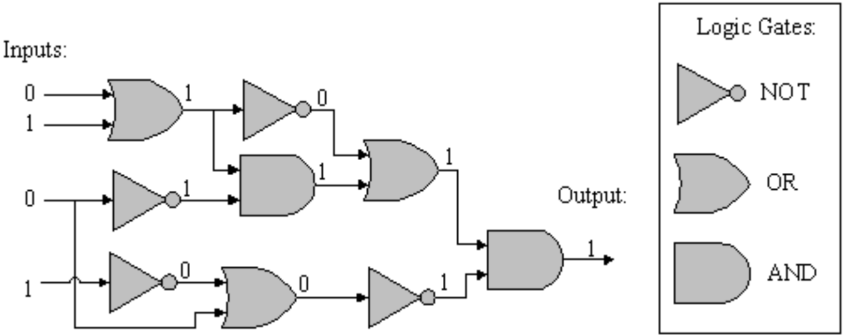

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Lemma

A Boolean circuit C_x on $m \doteq n^c$ inputs such that $C_x(y) = V(x,y)$ for all $y \in \{0,1\}^m$ can be constructed in time $n^{O(1)}$.

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- 3-SAT is NP-hard.
- 2-SAT can be solved in polynomial time.

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- Therefore $A \leq^p B \leq^p C$.
- By transitivity $A \leq^p C$.

3-SAT is NP-Hard

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Mapping reduction Circuit-SAT $\leq^p$ 3-SAT

3-SAT is NP-Hard

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Gadget reduction

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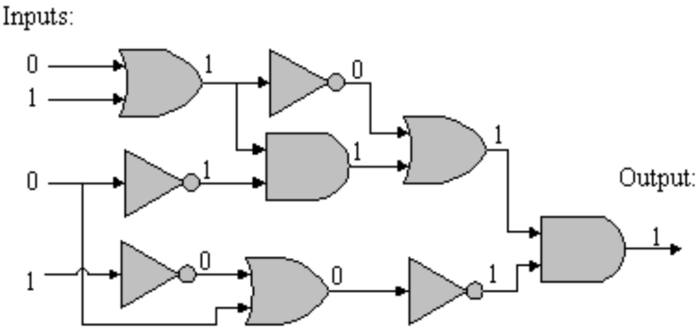
Gadget reduction



Reduction Circuit-SAT $\leq^p$ 3-SAT

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- For each gate g, include clauses with at most 3 literals each that combined force the variable of the gate to the value of the gate on the input.

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    - $g' =$ NOT g

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  - $g' = g_1 \text{ AND } g_2 \rightarrow \begin{cases} \overline{g_1} \Rightarrow \overline{g'} \\ \overline{g_2} \Rightarrow \overline{g'} \\ g_1 \wedge g_2 \Rightarrow g' \end{cases}$

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- Add unit clause consisting of the variable for the output gate.

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  $\Leftrightarrow \varphi$ has a satisfying assignment.

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Polynomial running time

Polynomial-Time Algorithm for 2-SAT

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Reduction to digraph reachability

Polynomial-Time Algorithm for 2-SAT

Reduction to digraph reachability

Construction of the digraph G

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- Introduce a vertex for each variable x_i that occurs in φ, and another one for its negation $\overline{x_i}$.

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- Introduce a vertex for each variable x_i that occurs in φ, and another one for its negation $\overline{x_i}$.
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Claim

φ has a satisfying assignment
$\Leftrightarrow$ for no variable x_i there are paths $x_i \rightsquigarrow \overline{x_i}$ and $\overline{x_i} \rightsquigarrow x_i$ in G.

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Proof

$\Rightarrow$ By contraposition.

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Proof

$\Rightarrow$ By contraposition.

$\Leftarrow$ Pick a literal ℓ such that there is no path $\ell \rightsquigarrow \overline{\ell}$ in G, set ℓ to true, simplify the formula/digraph, and repeat until there are no variables left.