CS 577: Introduction to Algorithms Fall 2020

Introduction

Instructor: Dieter van Melkebeek

In this lecture we give a brief overview of the topics to be covered in the course, discuss some administrative issues, and get a taste of the kinds of arguments we will be using later on by looking at the *Stable Marriage Problem*.

1 Course Overview

See the slides and schedule on Canvas.

2 Administrativia

See the slides, syllabus, and guidelines on Canvas.

3 Stable Marriage Problem

For the rest of the lecture, we discuss the *Stable Marriage Problem*. The notes here are given for completeness and clarity, but the powerpoint presentation from lecture may also be helpful, and can be found on Canvas.

The problem is, given a set of n boys and n girls, each of whom has a ranked preference list of all members of the opposite sex (with no ties), how should we pair them off?

The first question is, what do we want out of a pairing? A good criterion is that there be no *rogue couples*. We say that if a boy and a girl who are not paired prefer each other to the ones with whom they are paired, they are a rogue couple. Since the members of a rogue couple have an incentive to break up with their assigned partners, we call a pairing with one or more rogue couples *unstable*.

3.1 First Attempt: Let Them Figure It Out

One way to go about finding stable marriages is to start with an arbitrary set of marriages (in which each boy is married to exactly one girl, and each girl is married to exactly one boy), and the rogue couples swap around as they want. That is, we repeatedly find a girl and a boy who are not married to each other, but prefer one another to their current marriages, and swap the marriages.

This approach is obviously *correct*, in the sense that, whenever it terminates, it will always leave stable marriages. However, this approach does *not* always terminate.

Challenge 1. *Exhibit an example where this approach does not terminate. Hint: Pick* $n = 3$.

3.2 Traditional Marriage Algorithm

Let's try out the "traditional" marriage algorithm (TMA). We'll see that there is always a stable pairing, and that the traditional marriage algorithm always finds one.

Algorithm 1 Traditional Marriage Algorithm

For each day that some boy gets a "no", do:
  —Morning
    Each girl stands on her balcony.
    Each boy proposes under the balcony of the best girl whom he has not yet crossed off.
  —Afternoon (for girls with at least one suitor)
    To today's best suitor: "Maybe, come back tomorrow."
    To any others: "No, I will never marry you."
  —Evening
    Any rejected boy crosses the girl off his list.
Finally, each girl marries the boy to whom she last said "maybe".

For this algorithm, it's not quite so clear that it either terminates or even yields a stable matching. Maybe it will run for a while, and find that some boy has crossed off every girl on his list. Or perhaps some girl is never propositioned, and thus never says "maybe" to some boy. These are all issues that need to be addressed.

To address these problems, we will argue that the TMA is always well-defined, and is guaranteed to terminate. Then, we will show that it always produces a (particular) stable pairing (and *a fortiori*, that such a pairing always exists). First, some lemmas:

Lemma 1 (Improvement Lemma)**.** *If a girl ever tells a boy "maybe," then she will always have a suitor that she likes at least as well (or a husband).*

Proof. The proof is by induction: we show that, for every girl A, and at every time step t after the first time that A says "maybe", that A will tell some boy "maybe" at time step t, and this boy will be at least as good as the boy she said "maybe" to at time step $t-1$.

Base case: For the base case, the statement of the lemma gives us that at some time t_0, A has a boy B (Bob) on a string. At time t_0+1 (the first value of t in our induction statement), B proposes to A again, so A will say "maybe" to some boy, and this boy will be at least as good as B.

Inductive case: We are at time step $t+1$, and we know that at time step t, A said maybe to some boy D (Dave) who is at least as good for her as B. Then D will return to propose to A at time step $t+1$, and so A will say "maybe" to some boy who is at least as good as D.

□

Lemma 2. *No boy can be rejected by all the girls.*

Proof. By contradiction. Suppose some boy b is rejected by all the girls. At that point, every girl must have a boy on a string (by Lemma 1). Moreover, all these boys are different (girls don't share balconies) and different from b. That is, the n girls have n different suitors, none of whom is b. But then there must be at least $n+1$ boys, which is a contradiction! □

Now we show that the TMA terminates:

Theorem 1. *The TMA terminates in at most n^2 days.*

Proof. A "master list" of all n boys' lists starts with a total of n^2 girls on it (one entry per boy per girl). Each day that at least one boy gets a "no," at least one girl gets crossed off the master list. Therefore, the number of days is bounded by the original size of the master list. □

We can say more: since no list ever becomes empty (Lemma 2), the number of days is bounded by $n(n-1) \leq n^2$.

Challenge 2. *Assuming that we count the first and the last days of the TMA, the tight upper bound is $n(n-2)+2$. Argue this by proving that the expression always is an upper bound, and exhitibing for each n an example list of preferences on which the TMA requires that many days to complete.*

Note the following:

Corollary 1. *Each girl will marry her absolute favorite of the boys who visit her during the TMA.*

Since no boy gets rejected by all girls, when TMA terminates, every boy must be at the balcony of a different girl, so the result produced by TMA is a valid pairing. Finally, we show that this pairing is stable (which implies that a stable pairing must exist).

Theorem 2. *Let T be the pairing produced by the TMA. T is stable.*

Proof. Let B (Bob) and A (Alice) be any couple in T. Suppose B prefers C (Carol) to A. In this case, we can argue that C prefers her husband (D, Dave) to B.

During the TMA, B proposed to C before he proposed to A. Therefore, C rejected B for someone she liked better. By Lemma 1, the person she married, D, was also someone she liked better than B.

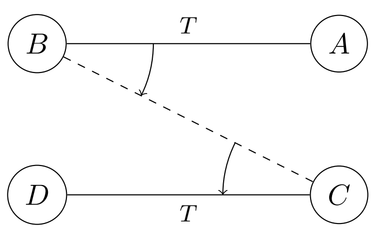

Figure 1: If TMA pairs C with D and A with B, but B prefers C over A, then it must be the case that C prefers D over B.

□

3.3 Moving First

Who does better in the TMA, boys or girls? We'll show that in a very strong sense, the boys are better off. In fact, the boys get their best imaginable pairings, and the girls get their worst!

We say that a boy's *optimal girl* is the highest ranked girl for whom there is some stable pairing in which the boy marries her. A boy's *pessimal girl* is the lowest ranked girl for whom there is a stable pairing in which the boy marries her. We say that a pairing is *male-optimal* if every boy marries his optimal girl, and *male-pessimal* if every boy marries his pessimal girl.

Theorem 3 (The Naked Mathematical Truth!). *The Traditional Marriage Algorithm always produces a male-optimal, female-pessimal pairing.*

Proof.

Male-optimality: Let T be the pairing produced by the TMA. Suppose, by way of contradiction, that some boy gets rejected by his optimal girl during the TMA. Let t be the earliest time at which this happens. In particular, at time t, some boy B (Bob) gets rejected by his optimal girl A (Alice) because she says "maybe" to a preferred boy D (Dave). Since B is the first boy to be rejected by his optimal girl, D has not yet been rejected by his optimal girl. Therefore, D likes A at least as much as his optimal girl. On the other hand, since A is B's optimal girl, there must exist some stable pairing S in which they are married. Let C (Carol) be D's wife in S. Figure 2 shows the picture as it involves B, D, A, and C.

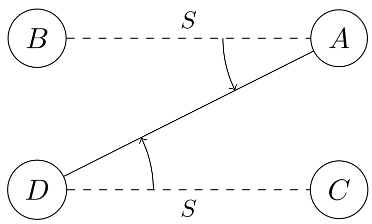

Figure 2: D and A are a rogue couple under S, so S can't be stable!

As the picture shows, we've already reached a contradiction to the stability of S, since under our assumptions D and A would prefer each other to their pairings in S. Therefore, the TMA pairing must be male-optimal.

Female-pessimality: We now know that the TMA pairing is male-optimal. Now suppose that it is not female-pessimal. That is, there is a stable pairing S where some girl A (Alice) does worse than in T. Let B (Bob) be her husband in S, and let D (Dave) be her husband in T. By assumption, A likes D better than B. We also know that D likes A better than his wife in S, C (Carol), since A is his optimal girl due to male optimality of T. Therefore, D and A are a rogue couple under S and so S cannot be stable (see Figure 3). This is a contradiction, and the theorem is proved.

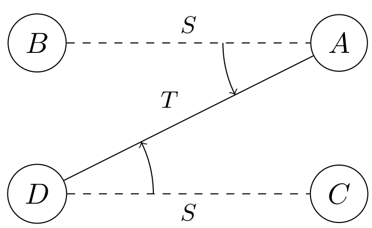

Figure 3: A and D are a rogue couple under S, so S can't be stable!

$\square$