Relations and Functions

- A relation *R* from a set *A* to a set *B* is said to be a **function** if for every *a* in *A*, there is a unique *b* in *B* such that (*a*, *b*) ∈*R*.
- If *R* is a function from *A* to *B* and (*a*, *b*) ∈ *R*, then *b* is called the **image** of *a* under the relation *R* and *a* is called the **preimage** of *b* under *R*.
- For a function *R* from set *A* to set *B*, set *A* is the **domain** of the function; the images of the elements in set *A* or the second elements in the ordered pairs form the **range**,while the whole of set *B* is the **codomain** of the function.

For example, in relation $f=\{(-1,3)(0,2),(1,3),(2,6),(3,11)\}$ since each element in *A* has a unique image, therefore *f* is a function.

Each image in *B* is 2 more than the square of pre-image.

Hence, the formula for *f* is $f(x)=x^2+2$ Or $f:x\rightarrow x^2+2$

Domain = {–1, 0, 1, 2, 3}

Co-domain = {2, 3, 6, 11, 13}

Range = {2, 6, 3, 11}

- **Cartesian product of two sets:** Two non-empty sets P and Q are given. The Cartesian product $\mathrm{P} \times \mathrm{Q}$ is the set of all ordered pairs of elements from P and Q, i.e.,

P × Q = $\{p, q$: p ∈ P and q ∈ **Q}**

Example: If P = {x, y} and Q = {-1, 1, 0}, then $\mathrm{P} \times \mathrm{Q} = \{x,-1, x,1, x,0, y,-1, y,1, y,0\}$

If either P or Q is a null set, then P × Q will also be a null set, i.e., $\mathrm{P} \times \mathrm{Q} = \Phi$.

In general, if A is any set, then A × $\Phi = \Phi$.

- **Property of Cartesian product of two sets:**
  - If nA = p, nB = q, then n$A \times B$ = pq.
  - If A and B are non-empty sets and either A or B is an infinite set, then so is the case with A × B
  - .A × A × A = $\{a, b, c$: a, b, c ∈ A}. Here, (a, b, c) is called an ordered triplet.
  - A × $B \cap C = A \times B \cap A \times C$
  - A × $B \cup C = A \times B \cup A \times C$

- Two ordered pairs are equal if and only if the corresponding first elements are equal and the second elements are also equal. In other words, if (*a*, *b*) = (*x*, *y*), then *a* = *x* and *b* = *y*.

Example: Show that there does not exist *x*, *y* ∈ R if (*x* – *y* + 1, 4*x* – 2*y* – 6) = (*y* – *x* – 4, 7*x* – 5*y*

$-2)$.

Solution: It is given that

$(x-y+1, 4x-2y-6) = (y-x-4, 7x-5y-2)$.

$\Rightarrow x-y+1 = y-x-4$ and $4x-2y-6 = 7x-5y-2$

$\Rightarrow 2x-2y+5 = 0$ … 1

And $-3x+3y-4 = 0$ … 2

Now,

$\frac{2}{-3} = -\frac{2}{3}, \frac{-2}{3} = -\frac{2}{3}$ and $\frac{5}{-4} = -\frac{5}{4}$

Since $\frac{2}{-3} = \frac{-2}{3} \neq \frac{5}{-4}$, equations 1 and 2 have no solutions. This shows that there does not exist $x, y \in$ R if $(x-y+1, 4x-2y-6) = (y-x-4, 7x-5y-2)$.

In general, for any two sets A and B, A × B ≠ B × A.

- **Relation:** A relation *R* from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element *x* and the second element *y* of the ordered pairs (*x*, *y*) in A × B.
- The image of an element *x* under a relation *R* is *y*, where (*x*, *y*) ∈ R
- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation *R*. Range ⊆Co-domain

Example: In the relation X from **W** to **R**, given by X = {(*x*, *y*): $y = 2x+1$; $x \in$ W, $y \in$ R}, we obtain X = $\{0,1, 1,3, 2,5, 3,7 \ldots\}$. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 …}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 …}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

If $nA = p$ and $nB = q$, then $nA \times B = pq$ and the total number of relations is 2^{pq}.

- **Real-valued Function:** A function having either R $realnumbers$ or one of its subsets as its range is called a real-valued function. Further, if its domain is also either R or *a* subset of R, it is called a real function.

Types of functions:

  - **Identity function:** The function *f*: R → R defined by $y = f(x) = x$, for each $x \in$ R, is called the identity function.

Here, R is the domain and range of *f*.

- **Constant function:** The function f: R → R defined by $y = f(x) = x$, for each $x \in$ R, where c is a constant, is a constant function.

Here, the domain of f is R and its range is $\{c\}$.

- **Polynomial function:** A function f: R → R is said to be a polynomial function if for each $x \in$ R, $y = f(x) = a^0 + a_1 x +$ ___ $+ a_n x^n$a where n is a non-negative integer and $a_0, a_1, \ldots, a_n \in$ R .
- **Rational function:** The functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$, are called rational functions.
- **Modulus function:** The function f: R → R^+ defined by $f(x) = |x|$, for each $x \in$ R, is called the modulus function.

In other words, $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

- **Signum function:** The function f: R → R defined by

$$f(x) = \begin{Bmatrix} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{Bmatrix}$$

is called the signum function. Its domain is R and its range is the set {–1, 0, 1}.

- **Greatest Integer function**: The function f: R → R defined by $f(x) = [x]$, $x \in$ R, assuming the value of the greatest integer less than or equal to x, is called the greatest integer function.

Example:

$$-2.7$$

= –3,

$$2.7$$

= 2,

$$2$$

= 2

- **Linear function:** The function f defined by $f(x) = mx + c$, for $x \in$ R, where m and c are constants, is called the linear function. Here, R is the domain and range of f.

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define

- Addition of Functions

  $(f+g)$: X → R by $(f+g)\ (x) = f(x) + g(x)$, $x \in$ X

- Subtraction of Functions

  $(f - g)$: X → R by $(f - g)\ (x) = f(x) - g(x)$, $x \in$ X

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(f+g)\ (x) = f(x) + g(x)$
$= (2x - 3) + (x^2 + 3x + 2)$
$= x^2 + 5x - 1$

$(f - g) = f(x) - g(x)$
$= (2x - 3) - (x^2 + 3x + 2)$
$= -x^2 - x - 5$

- **Multiplication of real functions:** For functions f: X → R and g: X → R, we define

  - Multiplication of two real functions

    (fg): X → R by $(fg)(x) = f(x).\ g(x)$ $x \in$ X

  - Multiplication of a function by a scalar

    af: X → R by $(a\,f)\ (x) = af(x)$ $x \in$ X and a is a real number

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(fg)\ (x) = f(x) \times g(x)$
$= (2x - 3) \times (x^2 + 3x + 2)$
$= 2x^3 + 3x^2 - 5x - 6$

$(2f)(x) = 2.f(x)$
$= 2 \times (2x - 3)$
$= 4x - 6$

- **Addition and Subtraction of functions:** For functions f: X → R and g: X → R, we define

  - Addition of Functions

    $(f+g)$: X → R by $(f+g)\ (x) = f(x) + g(x)$, $x \in$ X

  - Subtraction of Functions

    $(f - g)$: X → R by $(f - g)\ (x) = f(x) - g(x)$, $x \in$ X

Example: Let $f(x) = 2x - 3$ and $g(x) = x^2 + 3x + 2$ be two real functions, then

$(f+g)\ (x) = f(x) + g(x)$

$= (2x - 3) + (x^2 + 3x + 2)$

$$= x^2 + 5x - 1$$

$$(f - g) = f(x) - g(x)$$

$$= (2x - 3) - (x^2 + 3x + 2)$$

$$= - x^2 - x - 5$$