fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM+ rjr e/;e eu dj ';keA
i#"k flg ldYi dj] lgr foifr vud] ^cuk^ u NkM+ /;; dk] j?kcj jk[k VdAA
jfpr% ekuo /keZ iz.krk
lnx# Jh j.kNkMnklth egkjkt

STUDY PACKAGE

Subject : Mathematics

Topic : DETERMINANTS & MATRICES

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Determinant

1. Definition:

Let us consider the equations $a_1x + b_1y = 0$, $a_2x + b_2y = 0$

$$\Rightarrow \quad -\frac{a_1}{b_1} = \frac{y}{x} = -\frac{a_2}{b_2} \quad \Rightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} \quad \Rightarrow \quad a_1b_2 - a_2b_1 = 0$$

we express this eliminant as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of order two.

Its value is given by: $D = a_1b_2 - a_2b_1$

2. Expansion of Determinant:

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three.

Its value can be found as:

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$ **OR**

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \ldots \text{ \& so on.}$$

In this manner we can expand a determinant in 6 ways using elements of ; R_1, R_2, R_3 or C_1, C_2, C_3.

3. Minors:

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands. For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order two will have "4 minors" & a determinant of order three will have "9 minors".

4. Cofactor:

Cofactor of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$; Where i & j denotes the row & column in which the particular element lies.

Note that the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as:

$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

OR $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ & so on.

5. Transpose of a Determinant:

The transpose of a determinant is a determinant obtained after interchanging the rows & columns.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Rightarrow \quad D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6. Symmetric, Skew-Symmetric, Asymmetric Determinants:

(i) A determinant is symmetric if it is identical to its transpose. Its i^{th} row is identical to its i^{th} column i.e. $a_{ij} = a_{ji}$ for all values of 'i' and 'j'

(ii) A determinant is skew-symmetric if it is identical to its transpose having sign of each element inverted i.e. $a_{ij} = -a_{ji}$ for all values of 'i' and 'j'. A skew-symmetric determinant has all elements zero in its principal diagonal.

(iii) A determinant is asymmetric if it is neither symmetric nor skew-symmetric.

7. Properties of Determinants:

(i) The value of a determinant remains unaltered, if the rows & columns are inter changed,

i.e. $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

(ii) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \& \quad D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{Then } D' = -D.$$

NOTE : A skew-symmetric deteminant of odd order has value zero.

(iii) If a determinant has all the elements zero in any row or column then its value is zero,

i.e. $D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(iv) If a determinant has any two rows (or columns) identical, then its value is zero,

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(v) If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number, i.e.

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = KD$

(vi) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(vii) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

i.e. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then $D' = D$

Example : Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution. Let $R_1 \to R_1 + R_2 + R_3$ $\Rightarrow$ $\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (a + b + c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$= (a + b + c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$

$= (a + b + c)\,((b - c)(a - b) - (c - a)^2)$
$= (a + b + c)\,(ab + bc - ca - b^2 - c^2 + 2ca - a^2)$
$= (a + b + c)\,(ab + bc + ca - a^2 - b^2 - c^2)$
$= 3abc - a^3 - b^3 - c^3$

Example : Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Solution. Given detereminant is equal to $= \dfrac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$

$= \dfrac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$

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$$= (a - b)(b - c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a - b) (b - c) [ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 - a^2b - a^2c - ab^2 - abc - b^3 - b^2c]$
$= (a - b) (b - c) [c(ab + bc + ca) - a(ab + bc + ca)]$
$= (a - b) (b - c) (c - a) (ab + bc + ca)$ Use of factor theorem.

USE OF FACTOR THEOREM TO FIND THE VALUE OF DETERMINANT

If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

Example : Prove that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca)$ by using factor theorem.

Solution. Let $a = b$

$\Rightarrow$ $D = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ac & ab \end{vmatrix} = 0$ Hence $(a - b)$ is a factor of determinant

Similarly, let $b = c$, $D = 0$
$c = a$, $D = 0$

Hence, $(a - b)(b - c)(c - a)$ is factor of determinant. But the given determinant is of fifth order so

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a - b)(b - c)(c - a)(\lambda(a^2 + b^2 + c^2) + \mu(ab + bc + ca))$$

Since this is an identity so in order to find the values of λ and μ. Let
$a = 0, b = 1, c = -1$
$-2 = (2)\ (2\lambda - \mu)$
$(2\lambda - \mu) = -1.$ ....(i)

Let $a = 1, b = 2, c = 0$

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = (-1)\ 2\ (-1)\ (5\lambda + 2\mu)$$

$\Rightarrow$ $5\lambda + 2\mu = 2$ ....(ii)

from (i) and (ii) $\lambda = 0$ and $\mu = 1$

Hence $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca).$

Self Practice Problems

1. Find the value of $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$. **Ans.** 0

2. Simplify $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$. **Ans.** 0

3. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)^3$.

4. Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a - b)(b - c)(c - a)$ by using factor theorem .

8. Multiplication Of Two Determinants:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1+b_1\ell_2 & a_1m_1+b_1m_2 \\ a_2\ell_1+b_2\ell_2 & a_2m_1+b_2m_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1\ell_1+b_1\ell_2+c_1\ell_3 & a_1m_1+b_1m_2+c_1m_3 & a_1n_1+b_1n_2+c_1n_3 \\ a_2\ell_1+b_2\ell_2+c_2\ell_3 & a_2m_1+b_2m_2+c_2m_3 & a_2n_1+b_2n_2+c_2n_3 \\ a_3\ell_1+b_3\ell_2+c_3\ell_3 & a_3m_1+b_3m_2+c_3m_3 & a_3n_1+b_3n_2+c_3n_3 \end{vmatrix}$$

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We have multiplied here rows by rows but we can also multiply rows by columns, columns by rows and columns by columns.

If $\Delta = |a_{ij}|$ is a detereminant of order n, then the value of the determinant $|A_{ij}| = \Delta^{n-1}$. This is also known as power cofactor formula.

Example : Find the value of $\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \times \begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix}$ and prove that it is equal to $\begin{vmatrix} 1 & 8 \\ -6 & 12 \end{vmatrix}$.

Solution. $\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \times \begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix}$

$= \begin{vmatrix} 1\times 3 - 2\times 1 & 1\times 0 + 2\times 4 \\ -1\times 3 + 3\times(-1) & -1\times 0 + 3\times 4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 8 \\ -6 & 12 \end{vmatrix}$ $\qquad = 60$

Example : Prove that $\begin{vmatrix} a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3 \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 \\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3 \end{vmatrix} = 0$

Solution. Given determinant can be splitted into product of two determinants

i.e. $\begin{vmatrix} a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3 \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 \\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Example : Prove that $\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$

$= 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$.

Solution. $\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 \end{vmatrix}$

$= \begin{vmatrix} {a_1}^2+{b_1}^2-2a_1b_1 & {a_1}^2+{b_2}^2-2a_1b_2 & {a_1}^2+{b_3}^2-2a_1b_3 \\ {a_2}^2+{b_1}^2-2a_2b_1 & {a_2}^2+{b_2}^2-2a_2b_2 & {a_2}^2+{b_3}^2-2a_2b_3 \\ {a_3}^2+{b_1}^2-2a_3b_1 & {a_3}^2+{b_2}^2-2a_3b_2 & {a_3}^2+{b_3}^2-2a_3b_3 \end{vmatrix}$

$= \begin{vmatrix} {a_1}^2 & 1 & -2a_1 \\ {a_2}^2 & 1 & -2a_2 \\ {a_3}^2 & 1 & -2a_3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ {b_1}^2 & {b_2}^2 & {b_3}^2 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= 2 \begin{vmatrix} 1 & {a_1}^2 & a_1 \\ 1 & {a_2}^2 & a_2 \\ 1 & {a_3}^2 & a_3 \end{vmatrix} \times \begin{vmatrix} 1 & {b_1}^2 & b_1 \\ 1 & {b_2}^2 & b_2 \\ 1 & {b_3}^2 & b_3 \end{vmatrix}$

$= 2(a_1-a_2)(a_2-a_3)(a_3-a_1)(b_1-b_2)(b_2-b_3)(b_3-b_1)$

Example : Prove that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$

Solution. $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$= \begin{vmatrix} \cos A\cos P+\sin A\sin P & \cos A\cos Q+\sin A\sin Q & \cos A\cos R+\sin A\sin R \\ \cos B\cos P+\sin B\sin P & \cos B\cos Q+\sin B\sin Q & \cos B\cos R+\sin B\sin R \\ \cos C\cos P+\sin C\sin P & \cos C\cos Q+\sin C\sin Q & \cos C\cos R+\sin C\sin R \end{vmatrix}$

$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \cos Q & \cos R \\ \sin P & \sin Q & \sin R \\ 0 & 0 & 0 \end{vmatrix}$ $\qquad = 0 \times 0 = 0.$

Self Practice Problems

1. Find the value of Δ

$$\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$

Ans. $(3abc - a^3 - b^3 - c^3)^2$

2. If A, B, C are real numbers then find the value of $\Delta = \begin{vmatrix} 1 & \cos(B-A) & \cos(C-A) \\ \cos(A-B) & 1 & \cos(C-B) \\ \cos(A-C) & \cos(B-C) & 1 \end{vmatrix}$. **Ans.** 0

9. Summation of Determinants

Let $\Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants indepedent of r, then

$$\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Here function of r can be the elements of only one row or column. None of the elements other then that row or column should be dependent on r. If more than one column or row have elements dependent on r then first expand the determinant and then find the summation.

Example : Evaluate $\sum_{r=1}^{n} \begin{vmatrix} 2r-1 & {}^{n}C_r & 2^r \\ x & \cos^2\theta & y \\ n^2 & 2^n - 1 & 2^{n+1} - 2 \end{vmatrix}$

Solution : $\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} (2r-1) & \sum_{r=1}^{n} {}^{n}C_r & \sum_{r=1}^{n} 2^r \\ x & \cos^2\theta & y \\ n^2 & 2^n - 1 & 2^{n+1} - 2 \end{vmatrix}$

$$= \begin{vmatrix} n^2 & 2^n - 1 & 2^{n+1} - 2 \\ x & \cos^2\theta & y \\ n^2 & 2^n - 1 & 2^{n+1} - 2 \end{vmatrix} = 0$$

Example : $D_r = \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

evaluate $\sum_{r=2}^{n} D_r$

Solution : $\sum_{r=2}^{n} D_r = \sum_{r=2}^{n} \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} {}^{n-2}C_0 + {}^{n-2}C_1 + \ldots + {}^{n-2}C_{n-2} & {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2} & {}^{n-2}C_2 + {}^{n-2}C_3 + \ldots + {}^{n-2}C_{n-2} \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2^{n-2} & 2^{n-2} - 1 & 2^{n-2} - 1 - n \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$C_1 \to C_1 - 2 \times C_2$ $= \begin{vmatrix} 2^{n-2} - 2^{n-1} + 2 & 2^{n-2} - 1 & 2^{n-2} - 1 - n \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$= (-1)\begin{vmatrix} 2^{n-2}-2^{n-1}+2 & 2^{n-2}-1-n \\ 1 & 1 \end{vmatrix}$

$= 2^{n-1} - n - 3$

Example : If $\Delta_r = \begin{vmatrix} r-1 & 1 & 0 \\ 2 & r & 3+r \\ r+1 & -1 & -2 \end{vmatrix}$, find $\sum_{r=1}^{n} \Delta_r$

Solution. On expansion of determinent, we get

$D_r = (r-1)(3-r) + 7 + r^2 + 4r = 8r + 4 \quad \Rightarrow \quad \sum_{r=1}^{n} \Delta_r = 4n(n+2)$

Self Practice Problem

1. Evaluate $\sum_{r=1}^{n} D_r \begin{vmatrix} r-1 & x & 6 \\ (r-1)^2 & y & 4n-2 \\ (r-1)^3 & z & 3n^2-3n \end{vmatrix}$ **Ans.** 0

10. Integration of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants independent of x. Hence

$$\int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Note : If more than one row or one column are function of x then first expand the determinant and then integrate it.

Example : If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then find $\int_0^{\pi/2} f(x)\,dx$

Solution. Here $f(x) = \cos x\,(4\cos^2 x - 1) - 2\cos x$
$= 4\cos^3 x - 3\cos x = \cos 3x$

so $\int_0^{\pi/2} \cos 3x\,dx = \left.\frac{\sin 3x}{3}\right]_0^{\pi/2} = -\frac{1}{3}$

Example : If $\Delta = \begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ x & x^2 & x^3 \end{vmatrix}$, then find $\int_0^1 \Delta(x)\,dx$

Solution. $\int_0^1 \Delta(x)\,dx = \begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ \int_0^1 x\,dx & \int_0^1 x^2\,dx & \int_0^1 x^3\,dx \end{vmatrix}$

$= \begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{vmatrix} = \frac{1}{12}\begin{vmatrix} \alpha^2-1 & \beta^2-2 & \gamma^2-3 \\ 6 & 4 & 3 \\ 6 & 4 & 3 \end{vmatrix} = 0$

11. Differentiation of Determinant:

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

then $\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$

Example : If $f(x) = \begin{vmatrix} 3 & 2 & 1 \\ 6x^2 & 2x^3 & x^4 \\ 1 & a & a^2 \end{vmatrix}$, then find the value of $f''(a)$.

Solution. $f'(x) = \begin{vmatrix} 3 & 2 & 1 \\ 12x & 6x^2 & 4x^3 \\ 1 & a & a^2 \end{vmatrix}$

$f''(x) = \begin{vmatrix} 3 & 2 & 1 \\ 12 & 12x & 12x^2 \\ 1 & a & a^2 \end{vmatrix} \Rightarrow f''(a) = 12 \begin{vmatrix} 3 & 2 & 1 \\ 1 & a & a^2 \\ 1 & a & a^2 \end{vmatrix} = 0.$

Example : Let α be a repeated root of quadratic equation f(x) = 0 and A(x), B(x) and C(x) be polynomial of degree 3, 4 and 5 respectively, then show that

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$ divisible by f(x).

Solution. Let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\Rightarrow g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

Since $g(\alpha) = g'(\alpha) = 0$

$\Rightarrow$ $g(x) = (x - \alpha)^2\, h(x)$ i.e. α is the repeated root of g(x) and h(x) is any polynomial expression of degree 3. Also f(x) = 0 have repeated root α. So g(x) is divisible by f(x).

Example : Prove that F depends only on x_1, x_2 and x_3

$$F = \begin{vmatrix} 1 & 1 & 1 \\ x_1 + a_1 & x_2 + a_1 & x_3 + a_1 \\ x_1^2 + b_1x_1 + b_2 & x_2^2 + b_1x_2 + b_2 & x_3^2 + b_1x_3 + b_2 \end{vmatrix}$$

and simplify F.

Solution : $\dfrac{dF}{da_1} = \begin{vmatrix} 0 & 0 & 0 \\ x_1 + a_1 & x_2 + a_1 & x_3 + a_1 \\ x_1^2 + b_1x_1 + b_2 & x_2^2 + b_1x_2 + b_2 & x_3^2 + b_1x_3 + b_2 \end{vmatrix}$

$+ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ x_1^2 + b_1x_1 + b_2 & x_2^2 + b_1x_2 + b_2 & x_3^2 + b_1x_3 + b_2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ x_1 + a_1 & x_2 + a_1 & x_3 + a_1 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Hence F is independent of a_1.

Similarly $\dfrac{dF}{db_1} = \dfrac{dF}{db_2} = 0$.

Hence F is independent of b_1 and b_2 also.
So F is dependent only on x_1, x_2, x_3

Put $a_1 = 0$, $b_1 = 0$, $b_2 = 0$ $\Rightarrow$ $F = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$

$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$.

Example : If $\begin{vmatrix} e^x & \sin x \\ \cos x & \ell n(1+x) \end{vmatrix} = A + Bx + Cx^2 + ...$, then find the value of A and B.

Solution : Put x = 0 in

$\begin{vmatrix} e^x & \sin x \\ \cos x & \ell n(1+x) \end{vmatrix} = A + Bx + Cx^2 + ...$

$\Rightarrow \quad \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = A \qquad\qquad A = 0.$

Differentiating the given determinant w.r.t x, we get

$$\begin{vmatrix} e^x & \cos x \\ \cos x & \ell n(1+x) \end{vmatrix} + \begin{vmatrix} e^x & \sin x \\ -\sin x & \dfrac{1}{1+x} \end{vmatrix} = B + 2\,C\,x + ...$$

Put x = 0, we get

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad B = -1 + 1 = 0$

$\therefore \quad A = 0,\ B = 0$

Self Practice Problem

1. If $\begin{vmatrix} x & x-1 & x \\ -2x & x+1 & 1 \\ x+1 & 1 & x \end{vmatrix} = ax^3 + bx^2 + cx + d$. Find

(i) d **Ans.** [– 1]
(ii) a + b + c + d **Ans.** [– 5]
(iii) b **Ans.** [– 4]

12. Cramer's Rule: System of Linear Equations

(i) Two Variables

(a) Consistent Equations: Definite & unique solution. [intersecting lines]
(b) Inconsistent Equation: No solution. [Parallel line]
(c) Dependent equation: Infinite solutions. [Identical lines]

Let $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ then:

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent &

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow$ Given equations are dependent

(ii) Three Variables

Let, $a_1x + b_1y + c_1z = d_1$............ (I)
$a_2x + b_2y + c_2z = d_2$............ (II)
$a_3x + b_3y + c_3z = d_3$............ (III)

Then, $x = \dfrac{D_1}{D},\ Y = \dfrac{D_2}{D},\ Z = \dfrac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

(iii) Consistency of a system of Equations

(a) If $D \neq 0$ and alteast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations have either infinite solutions or no solution.

Cofactors of D
- All zeros
  - All the cofactors of D_1, D_2 and D_3 are zero, then system will have infinite solutions
  - Cofactors of atleast one of D_1, D_2 and D_3 is non zero, then system will have no solution.
- Atleast one non zero, then system will have infinite solutions

(Refer **Example** & **Self Practice Problem** with*)

(d) If D = 0 but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(e) If a given system of linear equations have Only Zero Solution for all its variables then the given equations are said to have TRIVIAL SOLUTION.

(iv) Three equation in two variables :

If x and y are not zero, then condition for $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ &

$a_3x + b_3y + c_3 = 0$ to be consistent in x and y is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Example: Find the nature of solution for the given system of equations.

$x + 2y + 3z = 1$
$2x + 3y + 4z = 3$
$3x + 4y + 5z = 0$

Solution. Let $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$D = \begin{vmatrix} -1 & -1 & 3 \\ -1 & -1 & 4 \\ -1 & -1 & 5 \end{vmatrix} = 0 \quad D = 0$

Now, $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix}$

$C_3 \to C_3 - C_2$

$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 0 & 4 & 1 \end{vmatrix}$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$D_1 = \begin{vmatrix} -2 & -1 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 5$

$D = 0$ But $D_1 \neq 0$ Hence no solution

***Example :** Solve the following system of equations

$x + y + z = 1$
$2x + 2y + 2z = 3$
$3x + 3y + 3z = 4$

Solution. $\therefore$ $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 0$

$D_1 = 0$, $D_2 = 0$, $D_3 = 0$

$\because$ Let $z = t$
$x + y = 1 - t$
$2x + 2y = 3 - 2t$

Since both the lines are parallel hence no value of x and y Hence there is no solution of the given equation.

***Example :** Solve the following system of equations

$x + y + z = 2$
$2x + 2y + 2z = 4$
$3x + 3y + 3z = 6$

Solution. $\therefore$ $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 0$

$D_1 = 0$, $D_2 = 0$, $D_3 = 0$

All the cofactors of D, D_1, D_2 and D_3 are all zeros, hence the system will have infinite solutions.

Let $z = t_1$, $y = t_2$ $\Rightarrow$ $x = 2 - t_1 - t_2$
where $t_1, t_2 \in R$.

Example : Consider the following system of equations

$x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

Find values of λ and μ if such that sets of equation have

(i) unique solution (ii) infinite solution
(iii) no solution

Solution. $x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}$

Here for $\lambda = 3$ second and third rows are identical hence $D = 0$ for $\lambda = 3$.

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix}$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix}$$

If $\lambda = 3$ then $D_1 = D_2 = D_3 = 0$ for $\mu = 10$

(i) For unique solution $D \neq 0$ i.e. $\lambda \neq 3$

(ii) For infinite solutions
$D = 0 \Rightarrow \lambda = 3$
$D_1 = D_2 = D_3 = 0 \Rightarrow \mu = 10.$

(iii) For no solution
$D = 0 \Rightarrow \lambda = 3$
Atleast one of D_1, D_2 or D_3 is non zero $\Rightarrow \mu \neq 10.$

Self Practice Problems

***1.** Solve the following system of equations
$x + 2y + 3z = 1$
$2x + 3y + 4z = 2$
$3x + 4y + 5z = 3$
Ans. $x = 1 + t$ $y = -2t$ $z = t$ where $t \in R$

2. Solve the following system of equations
$x + 2y + 3z = 0$
$2x + 3y + 4z = 0$
$x - y - z = 0$ **Ans.** $x = 0, y = 0, z = 0$

3. Solve: $(b + c)(y + z) - ax = b - c$, $(c + a)(z + x) - by = c - a$, $(a + b)(x + y) - cz = a - b$ where $a + b + c \neq 0$.
Ans. $x = \frac{c-b}{a+b+c}$, $y = \frac{a-c}{a+b+c}$, $z = \frac{b-a}{a+b+c}$

4. Let $2x + 3y + 4 = 0$; $3x + 5y + 6 = 0$, $2x^2 + 6xy + 5y^2 + 8x + 12y + 1 + t = 0$, if the system of equations in x and y are consistent then find the value of t. **Ans.** $t = 7$

13. Application of Determinants:

Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is:

$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ If $D = 0$ then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines:
$a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$