Sequence and Series

Harmonic Progression $H.P.$

If the reciprocals of the terms of a sequence form an arithmetic progression, then the sequence is said to be in harmonic progression.

- ***n*th term of an H.P.**

If x and y are the first two terms of an H.P., then the nth term of the H.P. is

$$a_n = \frac{1}{\frac{1}{x} + (n-1)\left(\frac{1}{y} - \frac{1}{x}\right)}$$

- **Ways of choosing the terms of an H.P.**

Three terms in an H.P can be taken as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

Four terms in H.P can be taken as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

- **Harmonic Mean**

Harmonic mean, H, between any two non–zero numbers x and y is

$$H = \frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{2xy}{x+y}$$

- **Relation between A.M., G.M. and H.M. of two distinct real numbers**

Let x any y be two distinct positive numbers.

If A, G, and H be their A.M., G.M. and H.M., respectively, then

$$A = \frac{x+y}{2}, G = \sqrt{xy} \text{ and } H = \frac{2xy}{x+y}$$

$$AH = \left(\frac{x+y}{2}\right)\left(\frac{2xy}{x+y}\right)$$
$$\Rightarrow AH = xy$$
$$\Rightarrow AH = G^2$$

Hence, A.M., G.M. and H.M are in G.P.

Also, A.M. ≥ G.M. ≥ H.M.

- **n^{th} term of an AP**

  The n^{th} term (a_n) of an AP with first term a and common difference d is given by $a_n = a + (n-1)d$.
  Here, a_n is called the general term of the AP.

- **n^{th} term from the end of an AP**

  The n^{th} term from the end of an AP with last term l and common difference d is given by $l - (n-1)d$.

  Example:
  Find the 12th term of the AP 5, 9, 13 …
  Solution:
  Here, $a = 5$, $d = 9 - 5 = 4$, $n = 12$

  $a_{12} = a + (n-1)d$
  $= 5 + 12 - 1\ 4$
  $= 5 + 11 \times 4$
  $= 5 + 44$
  $= 49$

  - **Sum of *n* terms of an AP**
    - The sum of the first n terms of an AP is given by

      , where a is the first term and d is the common difference.

  - If there are only n terms in an AP, then $S_n = \frac{n}{2}[a + l]$, where $l = a_n$ is the last term.

Example :

Find the value of 2 + 10 + 18 + …. + 802.

Solution:

2, 10, 18… 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be n terms in the series. Then,

$a_n = 802$
$\Rightarrow a + (n - 1)\, d = 802$
$\Rightarrow 2 + (n - 1)\, 8 = 802$
$\Rightarrow 8(n - 1) = 800$
$\Rightarrow n - 1 = 100$
$\Rightarrow n = 101$

Thus, required sum = $\frac{n}{2}(a + l) = \frac{101}{2}(2 + 802) = 40602$

- **Properties of an Arithmetic progression**
  - If a constant is added or subtracted or multiplied to each term of an A.P. then the resulting sequence is also an A.P.
  - If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

- **Arithmetic mean**
  - For any two numbers a and b, we can insert a number A between them such that a, A, b is an A.P. Such a number i.e., A is called the arithmetic mean $A.M$ of numbers a and b and it is given by $A = \frac{a+b}{2}$.
  - For any two given numbers a and b, we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

Let $A_1, A_2 \ldots A_n$ be n numbers between a and b such that $a, A_1, A_2 \ldots A_n, b$ is an A.P.

Here, common difference (d) is given by $\frac{b-a}{n+1}$.

Example:
Insert three numbers between –2 and 18 such that the resulting sequence is an A.P.
Solution:
Let A_1, A_2, and A_3 be three numbers between – 2 and 18 such that – 2, A_1, A_2, A_3, 18 are in an A.P.
Here, $a = -2$, $b = 18$, $n = 5$
$\therefore 18 = -2 + 5 - 1\ d$
$\Rightarrow 20 = 4\ d$
$\Rightarrow d = 5$
Thus, $A_1 = a + d = -2 + 5 = 3$
$A_2 = a + 2d = -2 + 10 = 8$
$A_3 = a + 3d = -2 + 15 = 13$
Hence, the required three numbers between –2 and 18 are 3, 8, and 13.

- **Geometric Progression:** A sequence is said to be a geometric progression $G.P.$ if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by r.

- In standard form, the G.P. is written as a, ar, ar^2 ... where, a is the first term and r is the common ratio.

- **General Term of a G.P.:** The n^{th} term $orgeneralterm$ of a G.P. is given by $a_n = ar^{n-1}$

Example: Find the number of terms in G.P. 5, 20, 80 ... 5120.
Solution: Let the number of terms be n.
Here $a = 5$, $r = 4$ and $t_n = 5120$
n^{th} term of G.P. = ar^{n-1}
$\therefore 5(4)^{n-1} = 5120$
$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$
$\Rightarrow (2)^{2n-2} = (2)^{10}$
$\Rightarrow 2n - 2 = 10$
$\Rightarrow 2n = 12$
$\therefore n = 6$

- **Sum of n Term of a G.P.:** The sum of n terms (S_n) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, \text{ if } r < 1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, \text{ if } r > 1 \\ na, \text{ if } r = 1 \end{cases}$$

Example: Find the sum of the series 1 + 3 + 9 + 27 + ... to 10 terms.
Solution: The sequence 1, 3, 9, 27, ... is a G.P.
Here, $a = 1$, $r = 3$.
Sum of n terms of G.P. = $\frac{a(r^n-1)}{r-1}$ $[r > 1]$
$S_{10} = 1 + 3 + 9 + 27 + \ldots$ to 10 terms

$= \frac{1 \times \left[(3)^{10} - 1\right]}{(3-1)}$

$= \frac{59049-1}{2}$

$= \frac{59048}{2}$

$= 29524$

- **Geometric Mean:** For any two positive numbers a and b, we can insert a number G between them such that a, G, b is a G.P. Such a number i.e., G is called a geometric mean $G.M.$ and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then $G_1, G_2, \ldots, G_n$ are given by $G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$
Where, r is calculated from the relation $b = ar^{n+1}$, that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

Example: Insert three geometric means between 2 and 162.
Solution:
Let G_1, G_2, G_3 be 3 G.M.'s between 2 and 162.
Therefor 2, G_1, G_2, G_3, 162 are in G.P.
Let r be the common ratio of G.P.
Here, $a = 2$, $b = 162$ and $n = 3$

$r = \left(\frac{162}{2}\right)^{\frac{1}{3+1}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$

$G_1 = ar = 2 \times 3 = 6$
$G_2 = ar^2 = 2 \times 3^2 = 2 \times 9 = 18$
$G_3 = ar^3 = 2 \times 3^3 = 2 \times 27 = 54$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let A and G be the respective A.M. and G.M. of two given positive real numbers a and b. Accordingly, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$

- **Sequence:** A sequence is an arrangement of numbers in definite order according to some rule.
  Also, we define a sequence as a function whose domain is the set of natural numbers or some subset of the type {1, 2, 3… k}.
  - A sequence containing finite number of terms is called a finite sequence.
  - sequence containing infinite number of terms is called an infinite sequence.

- A general sequence can be written as

  $a_1, a_2, a_3 \ldots a_{n-1}, a_n, \ldots$
  Here, $a_1, a_2 \ldots$ etc. are called the terms of the sequence and a_n is called the general term or n^{th} of the sequence.

- **Fibonacci sequence:** An arrangement of numbers such as 1, 2, 4, 6, 10 … has no visible pattern. However, the sequence is generated by the recurrence relation given by
  $a_1 = 1, a_2 = 2, a_3 = 4$
  $a_n = a_{n-2} + a_{n-1}, n > 3$
  This sequence is called the Fibonacci sequence.

- Let $a_1, a_2, \ldots a_n, \ldots$ be a given sequence. Accordingly, the sum of this sequence is given by the expression $a_1 + a_2 + \ldots + a_n + \ldots$
  This is called the series associated with the given sequence.
  The series is finite or infinite according as the given sequence.
  A series is usually represented in a compact form using sigma notation $\sum$.
  This means the series $a_1 + a_2 + \ldots + a_{n-1} + a_n \ldots$ can be written as $\sum_{k=1}^{n} a_k$.

- **Sum of n-terms of some special series:**
  - Sum of first n natural numbers
    $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
  - Sum of squares of the first n natural numbers
    $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
  - Sum of cubes of the first n natural numbers
    $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: Find the sum of n terms of the series whose n^{th} term is $n(n+1)(n-2)$.

Solution: It is given that

$$a_n = n(n+1)(n-2)$$
$$= n(n^2 + n - 2n - 2)$$
$$= n(n^2 - n - 2)$$
$$= n^3 - n^2 - 2n$$

Thus, the sum of *n* terms is given by

$$S_n = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3} - 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)-2(2n+1)-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n-4n-2-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2-n-14}{6}\right]$$

$$= \frac{n(n+1)(3n^2-n-14)}{12}$$

$$= \frac{n(n+1)(3n^2-7n+6n-14)}{12}$$

$$= \frac{n(n+1)[n(3n-7)+2(3n-7)]}{12}$$

$$= \frac{n(n+1)(n+2)(3n-7)}{12}$$