Limits and Derivatives

- **Derivatives**
  - Suppose *f* is a real-valued function and *a* is a point in its domain of definition. The derivative of *f* at *a* [denoted by $f'(a)$] is defined as

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists.
Derivative of $f(x)$ at *a* is denoted by $f'(a)$.

  - Suppose *f* is a real-valued function. The derivative of $f\left\{\text{denoted by } f'(x) \text{ or } \frac{d}{dx}[f(x)]\right\}$ is defined as

$\frac{d}{dx}[f(x)] = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.
This definition of derivative is called the first principle of derivative.

Example: Find the derivative of $f(x) = x^2 + 2x$ using first principle of derivative.

Solution: We know that $f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore f'(x) = \lim_{h \to 0} \frac{(x+h) + 2(x+h) - \left(x^2 + 2x\right)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2hx + 2x + 2h - x^2 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2hx + 2h}{h}$$

$$= \lim_{h \to 0} (h + 2x + 2)$$

$= 0 + 2x + 2 = 2x + 2$

$f'(x) = 2x + 2$

- **Derivatives of Polynomial Functions**

For the functions *u* and *v* (provided u' and v' are defined in a common domain),

  - $(u \pm v)' = u' \pm v'$
  - $(uv)' = u'v + uv'$ (Product rule)
  - $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ (Quotient rule)

- **Derivatives of Trigonometric Functions**
  - $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for any positive integer *n*
  - $\frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\right) = na_n x^{n-1} + \left(n-1\right)a_{n-1}x^{n-1} + \ldots + a_1$
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\cos x) = -\sin x$

- $\frac{d}{dx}(\tan x) = \sec^2 x$

Example: Find the derivative of the function $f(x) = \left(3x^2 + 4x + 1\right) \cdot \tan x$

Solution: We have,