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STUDY PACKAGE

Subject : Mathematics

Topic : COMPLEX NUMBER

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Complex Numbers

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1. The complex number system

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of complex numbers is introduced.

We can consider a complex number as having the form a + bi where a and b are real number and i, which is called the imaginary unit, has the property that $i^2 = -1$.

It is denoted by z i.e. z = a + ib. 'a' is called as real part of z which is denoted by (Re z) and 'b' is called as imaginary part of z which is denoted by (Im z).

Any complex number is :

(i) Purely real, if b = 0 ; (ii) Purely imaginary, if a = 0

(iii) Imaginary, if $b \neq 0$.

NOTE : (a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary but not imaginary.

(c) $i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non - negative.

(e) is z = a + ib, then a – ib is called complex conjugate of z and written as $\bar{z} = a - ib$

Self Practice Problems

1. Write the following as complex number

(i) $\sqrt{-16}$ (ii) $\sqrt{x}$, (x > 0)

(iii) $-b + \sqrt{-4ac}$, (a, c> 0)

Ans. (i) $0 + i\sqrt{16}$ (ii) $\sqrt{x} + 0i$ (iii) $-b + i\sqrt{4ac}$

2. Write the following as complex number

(i) $\sqrt{x}$ (x < 0) (ii) roots of $x^2 - (2\cos\theta)x + 1 = 0$

2. Algebraic Operations:

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by – 1 when it occurs.

1. Addition $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d) i$
2. Subtraction $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d) i$
3. Multiplication $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad+ bc)i$
4. Division $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-bi}{c-di} = \dfrac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$

$$= \frac{ac+bd+(bc-ad)i}{c^2-d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i$$

Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. z > 0, 4 + 2i < 2 + 4 i are meaningless.

In real numbers if $a^2 + b^2 = 0$ then a = 0 = b however in complex numbers,
$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Example : Find multiplicative inverse of 3 + 2i.

Solution Let z be the multiplicative inverse of 3 + 2i. then

$\Rightarrow$ z . (3 + 2i) = 1

$$\Rightarrow \quad z = \frac{1}{3+2i} = \frac{3-2i}{(3+2i)(3-2i)}$$

$$\Rightarrow \quad z = \frac{3}{13} - \frac{2}{13}\,i$$

$$\left(\frac{3}{13} - \frac{2}{13}i\right)$$ **Ans.**

Self Practice Problem

1. Simplify $i^{n+100} + i^{n+50} + i^{n+48} + i^{n+46}$, $n \in I$.

Ans. **0**

3. Equality In Complex Number:

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively

i.e. $z_1 = z_2$ $\Rightarrow$ $Re(z_1) = Re(z_2)$ and $I_m(z_1) = I_m(z_2)$.

Example: Find the value of x and y for which $(2 + 3i)\ x^2 - (3 - 2i)\ y = 2x - 3y + 5i$ where $x, y \in R$.

Solution $(z + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$

$\Rightarrow \quad 2x^2 - 3y = 2x - 3y$

$\Rightarrow \quad x^2 - x = 0$

$\Rightarrow \quad x = 0, 1 \quad$ and $\quad 3x^2 + 2y = 5$

$\Rightarrow \quad$ if $x = 0, y = \dfrac{5}{2}$ and if $x = 1, y = 1$

$\therefore \quad x = 0,\ y = \dfrac{5}{2}$ and $x = 1,\ y = 1$

are two solutions of the given equation which can also be represented as $\left(0, \dfrac{5}{2}\right)$ & (1, 1)

$\left(0, \dfrac{5}{2}\right)$, (1, 1) **Ans.**

Example: Find the value of expression $x^4 - 4x^3 + 3x^2 - 2x + 1$ when $x = 1 + i$ is a factor of expression.

Solution. $x = 1 + i$

$\Rightarrow \quad x - 1 = i$

$\Rightarrow \quad (x - 1)^2 = -1$

$\Rightarrow \quad x^2 - 2x + 2 = 0$

Now $\quad x^4 - 4x^3 + 3x^2 - 2x + 1$

$= (x^2 - 2x + 2)\ (x^2 - 3x - 3) - 4x + 7$

$\therefore \quad$ when $x = 1 + i \quad$ i.e. $\quad x^2 - 2x + 2 = 0$

$x^4 - 4x^3 + 3x^2 - 2x + 1 = 0 - 4\ (1 + i) + 7$

$= -4 + 7 - 4i$

$= 3 - 4i$ **Ans.**

Example: Solve for z if $z^2 + |z| = 0$

Solution. Let $z = x + iy$

$\Rightarrow \quad (x + iy)^2 + \sqrt{x^2 + y^2} = 0$

$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ and $2xy = 0$

$\Rightarrow \quad x = 0$ or $y = 0$

when $x = 0 \quad -y^2 + |\ y\ | = 0$

$\Rightarrow \quad y = 0, 1, -1$

$\Rightarrow \quad z = 0, i, -i$

when $y = 0 \quad x^2 + |\ x\ | = 0$

$\Rightarrow \quad x = 0 \quad \Rightarrow \quad z = 0 \quad$ **Ans.** $\quad z = 0,\ z = i,\ z = -i$

Example: Find square root of $9 + 40i$

Solution. Let $(x + iy)^2 = 9 + 40i$

$\therefore \quad x^2 - y^2 = 9 \quad$ ....(i)

and $\quad xy = 20 \quad$ ....(ii)

squing (i) and adding with 4 times the square of (ii)

we get $x^4 + y^4 - 2x^2\ y^2 + 4x^2\ y^2 = 81 + 1600$

$\Rightarrow \quad (x^2 + y^2)^2 = 168$

$\Rightarrow \quad x^2 + y^2 = 4 \quad$ ....(iii)

from (i) + (iii) we get $\quad x^2 = 25 \quad \Rightarrow \quad x = \pm 5$

and $y = 16 \quad \Rightarrow \quad y = \pm 4$

from equation (ii) we can see that

x & y are of same sign

$\therefore \quad x + iy = +(5 + 4i)$ or $= (5 + 4i)$

$\therefore \quad$ sq. roots of a + 40i = $\pm$ (5 + 4i) $\quad$ **Ans.** $\quad \pm$ (5 + 4i)

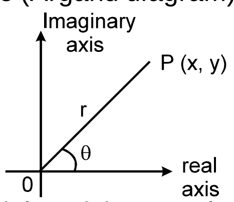
Self Practice Problem

1. Solve for z : $\bar{z} = i\ z^2$ $\quad$ **Ans.** $\pm \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i,\ 0,\ i$

4. Representation Of A Complex Number:

(a) Cartesian Form (Geometric Representation) :

Every complex number $z = x + i\ y$ can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number which is denoted by $|\,z\,|$ & θ is called the argument or amplitude.

$|\,z\,| = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis)

NOTE : (i) Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2\,n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

(iii) By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0\,i$ the argument is not defined and this is the only complex number which is only given by its modulus.

(b) Trignometric/Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

NOTE : $\cos\theta + i\sin\theta$ is also written as CiS θ or $e^{i\theta}$.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$ are known as Euler's identities.

(c) **Euler's Representation :**

$z = re^{i\theta}$; $|z| = r$; $\arg z = \theta$; $\bar{z} = re^{-i\theta}$

(d) **Vectorial Representation :**

Every complex number can be considered as if it is the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

Example: Express the complex number $z = -1 + \sqrt{2}\,i$ in polar form.

Solution. $z = -1 + i\sqrt{2}$

$|z| = \sqrt{(-1)^2 + \left(\sqrt{2}\right)^2} = \sqrt{1+2} = \sqrt{3}$

$\text{Arg } z = \pi - \tan^{-1}\left(\dfrac{\sqrt{2}}{1}\right) = \pi - \tan^{-1}\sqrt{2} = \theta$ (say)

$\therefore \quad z = \sqrt{3}\,(\cos\theta + i\sin\theta)$ where $\theta = \pi - \tan^{-1}\sqrt{2}$

Self Practice Problems

1. Find the principal argument and |z|

$$z = \frac{-1(9+i)}{2-i}$$

Ans. $-\tan^{-1}\dfrac{17}{11}, \sqrt{\dfrac{8^2}{5}}$

2. Find the |z| and principal argument of the complex number $z = 6(\cos 310^\circ - i\sin 310^\circ)$

Ans. 6, 50°

5. Modulus of a Complex Number :

If $z = a + ib$, then it's modulus is denoted and defined by $|z| = \sqrt{a^2 + b^2}$. Infact |z| is the distance of z from origin. Hence $|z_1 - z_2|$ is the distance between the points represented by z_1 and z_2.

Properties of modulus

(i) $|z_1 z_2| = |z_1| \,.\, |z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (provided $z_2 \ne 0$)

(iii) $|z_1 + z_2| \le |z_1| + |z_2|$ (iv) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

(Equality in (iii) and (iv) holds if and only if origin, z_1 and z_2 are collinear with z_1 and z_2 on the same side of origin).

Example: If $|z - 5 - 7i| = 9$, then find the greatest and least values of $|z - 2 - 3i|$.

Solution. We have $9 = |z - (5 + 7i)|$ = distance between z and 5 + 7i.
Thus locus of z is the circle of radius 9 and centre at 5 + 7i. For such a z (on the circle), we have to find its greatest and least distance as from 2 + 3i, which obviously 14 and 4.

Example: Find the minimum value of $|1 + z| + |1 - z|$.

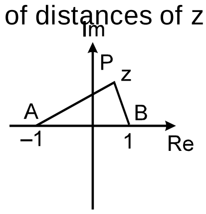
Solution $|1 + z| + |1 - z| \ge |1 + z + 1 - z|$ (triangle inequality)

$\Rightarrow \quad |1 + z| + |1 - z| \ge 2$

$\therefore \quad$ minimum value of $(|1 + z| + |1 - z|) = 2$

Geometrically $|z + 1| + |1 - 2| = |z + 1| + |z - 1|$ which represents sum of distances of z from 1 and -1

it can be seen easily that minimu (PA + PB) = AB = 2



Ans. $2^{1/4} e^{1\left(\frac{\pi}{8} + n\pi\right)}$

Example: $\left|z-\frac{2}{z}\right| = 1$ then find the maximum and minimum value of |z|

Solution. $\left|z-\frac{2}{z}\right| = 1$ $\quad \left|\,|z| - \left|\frac{2}{z}\right|\,\right| \le \left|z-\frac{2}{2}\right| \le |z| + \left|-\frac{2}{z}\right|$

Let | z | = r

$\Rightarrow \quad \left|r-\frac{2}{r}\right| \le 1 \le r + \frac{2}{r}$

$r + \frac{2}{r} \ge 1 \quad \Rightarrow \quad r \in R^+$ ....(i)

and $\left|r-\frac{2}{r}\right| \le 1 \Rightarrow \quad -1 \le r - \frac{2}{r} \le 1$

$\Rightarrow \quad r \in (1, 2)$ ....(ii)

$\therefore$ from (i) and (ii) $r \in (1, 2)$

Ans. $r \in (1, 2)$

Self Practice Problem

1. |z – 3| < 1 and |z – 4i| > M then find the positive real value of M for which these exist at least one complex number z satisfy both the equation.
Ans. $M \in (0, 6)$

6. Agrument of a Complex Number :

Argument of a non-zero complex number P(z) is denoted and defined by arg(z) = angle which OP makes with the positive direction of real axis.
If OP = |z| = r and arg(z) = θ, then obviously z = r(cosθ + isinθ), called the polar form of z. In what follows, 'argument of z' would mean principal argument of z(i.e. argument lying in $(-\pi, \pi]$ unless the context requires otherwise. Thus argument of a complex number z = a + ib = r(cosθ + isinθ) is the value of θ satisfying rcosθ = a and rsinθ = b.

Thus the argument of z = θ, $\pi - \theta$, $-\pi + \theta$, $-\theta$, $\theta = \tan^{-1}\left|\frac{b}{a}\right|$, according as z = a + ib lies in I, II, III or IVth quadrant.

Properties of arguments

(i) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.

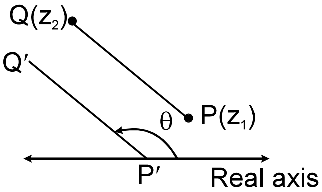
(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.

(iii) $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.

(iv) $\arg(z) = 0 \iff$ z is real, for any complex number $z \neq 0$

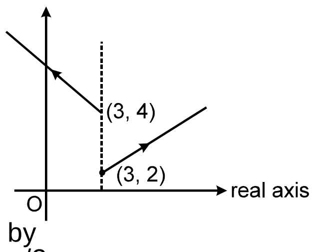
(v) $\arg(z) = \pm\pi/2 \iff$ z is purely imaginary, for any complex number $z \neq 0$

(vi) $\arg(z_2 - z_1)$ = angle of the line segment
P′Q′ || PQ, where P′ lies on real axis, with the real axis.



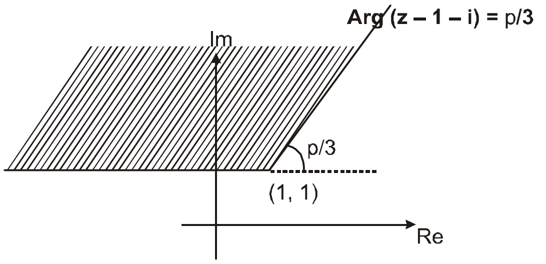
Example: Solve for z, which satisfy Arg (z – 3 – 2i) = $\frac{\pi}{6}$ and Arg (z – 3 – 4i) = $\frac{2\pi}{3}$.

Solution From the figure, it is clear that there is no z, which satisfy both ray

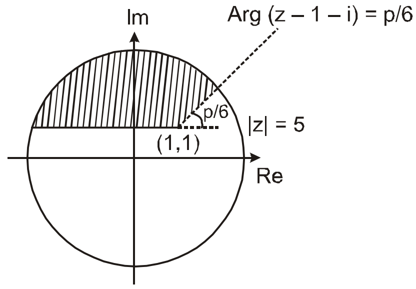


Example: Sketch the region given by
(i) Arg (z – 1 – i) $\ge \pi/3$
(ii) |z| = $\le$ 5 & Arg (z – i – 1) > $\pi/3$

Solution (i)



(ii)



Self Practice Problems

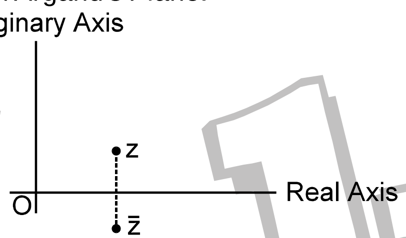
1. Sketch the region given by
(i) $|\text{Arg}(z - i - 2)| < \pi/4$ (ii) $\text{Arg}(z + 1 - i) \le \pi/6$

2. Consider the region $|z - 15i| \le 10$. Find the point in the region which has
(i) max $|z|$ (ii) min $|z|$
(iii) max arg (z) (iv) min arg (z)

7. Conjugate of a complex Number

Conjugate of a complex number z = a + b is denoted and defined by $\bar{z} = a - ib$.

In a complex number if we replace i by – i, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.



Properties of conjugate

(i) $|z| = |\bar{z}|$ (ii) $z\bar{z} = |z|^2$

(iii) $\overline{(z_1 + z_2)} = (\bar{z}_1) + (\bar{z}_2)$ (iv) $\overline{(z_1 - z_2)} = (\bar{z}_1) - (\bar{z}_2)$

(v) $\overline{(z_1 z_2)} = \bar{z}_1\, \bar{z}_2$ (vi) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(\bar{z}_1)}{(\bar{z}_2)}$ $(z_2 \ne 0)$

(vii) $|z_1 + z_2|^2 = (z_1 + z_2)\,\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

(viii) $\overline{(\bar{z}_1)} = z$ (ix) If w = f(z), then $\bar{w} = f(\bar{z})$

(x) $\arg(z) + \arg(\bar{z}) = 0$

Example: If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$

Solution. $\text{Re}\left(\dfrac{z-1}{z+1}\right) = 0$

$\Rightarrow \dfrac{z-1}{z+1} + \overline{\left(\dfrac{z-1}{z+1}\right)} = 0$ $\Rightarrow \dfrac{z-1}{z+1} + \dfrac{\bar{z}-1}{\bar{z}+1} = 0$

$\Rightarrow z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1 = 0$

$\Rightarrow z\bar{z} = 1$ $\Rightarrow |z|^2 = 1$

$\Rightarrow |z| = 1$ **Hence proved**

Self Practice Problem

1. If $\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unmodulus and z_2 is not unimodulus then find $|z_1|$.

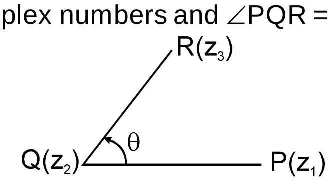
Ans. $|z_1| = 2$

8. Rotation theorem

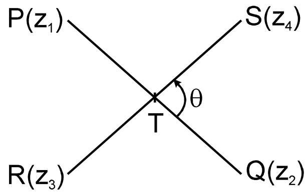
(i) If $P(z_1)$ and $Q(z_z)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1 e^{i\theta}$ where $\theta = \angle POQ$

(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then

$$\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$$

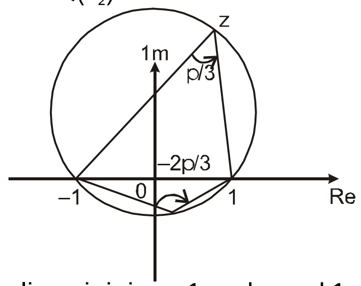


(iii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers and $\angle STQ = \theta$, then



$$\frac{z_3 - z_2}{z_1 - z_2} = \left|\frac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$$

Example: If $\arg\left(\frac{z-1}{z+i}\right) = \frac{\pi}{3}$ then interrupter the locus.



Solution $\arg\left(\frac{z-1}{z+i}\right) = \frac{\pi}{3}$

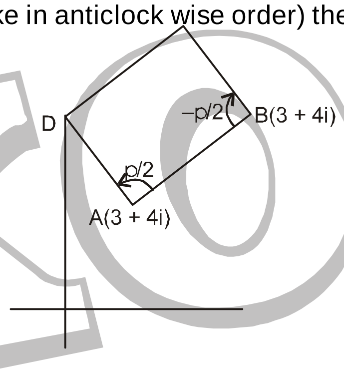
$\Rightarrow \quad \arg\left(\frac{1-z}{-1-z}\right) = \frac{\pi}{3}$

Here $\arg\left(\frac{1-z}{-1-z}\right)$ represents the angle between lines joining –1 and z and 1 + z. As this angle is constant, the locus of z will be a of a circle segment. (angle in a segment is count). It can be seen that locus is not the complete side as in the major are $\arg\left(\frac{1-z}{-1-z}\right)$ will be equal to $-\frac{2\pi}{3}$.

Now try to geometrically find out radius and centre of this circle.

centre $\equiv \left(0, \frac{1}{\sqrt{3}}\right)$ Radius $\equiv \frac{2}{\sqrt{3}}$ **Ans.**

Example: If A(z + 3i) and B(3 + 4i) are two vertices of a square ABCD (take in anticlock wise order) then find C and D.



Solution. Let affix of C and D are $z_3 + z_4$ respectively

Considering $\angle DAB = 90^o$ + AD = AB

we get $\frac{z_4 - (2+3i)}{AD} = \frac{(3+4i)-(2+3i)}{AB} \quad e^{\frac{i\pi}{2}}$

$\Rightarrow \quad z_4 - (2 + 3i) = (1 + i)\, i$

$\Rightarrow \quad z_4 = 2 + 3i + i - 1 = 1 + zi$

and $\frac{z_3 - (3+4i)}{CB} = \frac{(z+3i)-(3-4i)}{AB} \quad e^{-\frac{i\pi}{2}}$

$\Rightarrow \quad z_3 = 3 + 4i - (1 + i)(-i)$

$z_3 = 3 + 4i + i - 1 = z + 5i$

Self Practice Problems

1. z_1, z_2, z_3, z_4 are the vertices of a square taken in anticlockwise order then prove that

$2z_2 = (1 + i)\, z_1 + (1 - i)\, z_3$

Ans. $(1 + i)\, z_1 + (1 - i) z_3$

2. Check that z_1z_2 and z_3z_4 are parallel or, not

where, $z_1 = 1 + i \qquad z_3 = 4 + 2i$

$z_2 = 2 - i \qquad z_4 = 1 - i$

Ans. Hence, z_1z_2 and z_3z_4 are not parallel.

3. P is a point on the argand diagram on the circle with OP as diameter "two point Q and R are taken such that $\angle POQ = \angle QOR$

If O is the origin and P, Q, R are represented by complex z_1, z_2, z_3 respectively then show that

$z_2^2 \cos 2\theta = z_1z_3\cos^2\theta$

Ans. $z_1z_3 \cos^2\theta$

9. Demoivre's Theorem:

Case I

Statement :

If n is any integer then

(i) $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

(ii) $(\cos\theta_1 + i \sin\theta_1)(\cos\theta_2) + i \sin\theta_2)(\cos\theta_3 + i \sin\theta_2)(\cos\theta_3 + i \sin\theta_3) \ldots..(\cos\theta_n + i \sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\ldots \theta_n) + i \sin(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots. + \theta_n)$

Case II

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i \sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i \sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where k = 0, 1, 2, 3, ……, q – 1

NOTE : Continued product of the roots of a complex quantity should be determined using theory of equations.

10. Cube Root Of Unity :

(i) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :

$\cos 0 + i \sin 0$; $\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.

(v) The following factorisation should be remembered :
($a, b, c \in R$ & ω is the cube root of unity)
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $a^2 + ab + b^2 = (a - bw)(a - bw^2)$
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Example: Find the value of $\omega^{192} + \omega^{194}$

Solution. $\omega^{192} + \omega^{194}$
$= 1 + \omega^2 \quad = -\omega$
Ans. $-\omega$

Example: If $1, \omega, \omega^2$ are cube roots of unity prove
(i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$
(ii) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$
(iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = 9$
(iv) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ .... to 2n factors $= 2^{2n}$

Solution. (i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2)$
$= (-2\omega)(-2\omega^2)$
$= 4$

Self Practice Problem

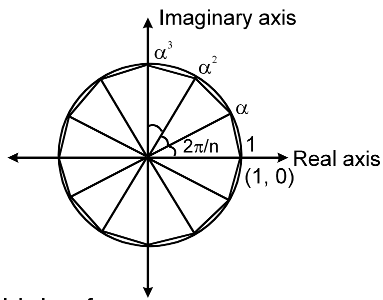
1. Find $\sum_{r=0}^{10}(1 + \omega^r + \omega^{2r})$

Ans. 12

11. n^{th} Roots of Unity :

If $1, \alpha_1, \alpha_2, \alpha_3 .... \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$ &



(ii) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2) ...... (1 - \alpha_{n-1}) = n$ &
$(1 + \alpha_1)(1 + \alpha_2) .... (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1. \alpha_1 . \alpha_2 . \alpha_3 .... \alpha_{n-1} = 1$ or -1 according as n is odd or even.

Example: Find the roots of the equation $z^6 + 64 = 0$ where real part is positive.

Solution. $z^6 = -64$
$z^6 = z^6 . e^{+ i(2n+1)\pi}$ $\quad x \in z$

$\Rightarrow \quad z = z\ e^{i(2n+1)\frac{\pi}{6}}$

$\therefore \quad z = 2\ e^{i\frac{\pi}{6}}, 2e^{i\frac{\pi}{2}}, z e^{i\frac{\pi}{2}}, z e^{i\frac{5\pi}{6}} = e^{i\frac{7\pi}{6}}, z e^{i\frac{3\pi}{2}}, z e^{i\frac{11\pi}{2}}$

$\therefore$ roots with +ve real part are $= e^{\frac{i\pi}{6}} + e^{i\frac{11\pi}{6}}$

$2e^{i\left(-\frac{\pi}{6}\right)}$ **Ans.**

Example: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Solution. $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right)$

$= \sum_{k=0}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$- \sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) + 1

$= 0 - 0 + 1 = 1$

i **Ans.**

Self Practice Problems

1. Resolve $z^7 - 1$ into linear and quadratic factor with real coefficient.

Ans. $(z - 1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right).\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right).\left(z^2 - 2\cos\frac{6\pi}{7}z + 1\right)$

2. Find the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$.

Ans. $-\frac{1}{2}$

12. The Sum Of The Following Series Should Be Remembered :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta$.

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta$.

NOTE : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

13. Logarithm Of A Complex Quantity :

(i) $\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}$, $n \in I$.

Example: Find the value of

(i)	$\log(1 + \sqrt{3}\, i)$	**Ans.**	$\log 2 + i(2n\pi + \frac{\pi}{3})$
(ii)	$\log(-1)$	**Ans.**	$i\pi$
(iii)	z^i	**Ans.**	$\cos(\ln 2) + i\sin(\ln 2) = e^{i(\ln 2)}$
(iv)	i^i	**Ans.**	$e^{-(4n+1).\frac{\pi}{2}}$
(v)	$\lvert(1 + i)^i\rvert$	**Ans.**	$e^{-(8n+1).\frac{\pi}{4}}$
(vi)	$\arg((1 + i)^i)$	**Ans.**	$\frac{1}{2}\ell n(2)$.

Solution. (i) $\log(1 + \sqrt{3}\, i) = \log\left(2\, e^{i\left(\frac{\pi}{3} + 2n\pi\right)}\right)$

$= \log 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$

(iii) $2^i = e^{i\ell n\, 2} = \cos(\ell n\, 2)\cos(\ell n\, 2) + i\sin(\ell n\, 2)$]

Self Practice Problem

1. Find the real part of cos (1 + i)

Ans. $\frac{1-e^2}{2ei}$

14. Geometrical Properties :

Distance formula :

If z_1 and z_2 are affixies of the two points ↓ P and Q respectively then distance between P + Q is given by $|z_1 - z_2|$.

Section formula

If z_1 and z_2 are affixes of the two points P and Q respectively and point C devides the line joining P and Q internally in the ratio m : n then affix z of C is given by

$$z = \frac{mz_2 + nz_1}{m+n}$$

If C devides PQ in the ratio m : n externally then

$$z = \frac{mz_2 - nz_1}{m-n}$$

(b) If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where a + b + c = 0 and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(1) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the Δ ABC = $\frac{z_1 + z_2 + z_3}{3}$:

(ii) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \text{ or } \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(iii) Incentre of the Δ ABC = $(az_1 + bz_2 + cz_3) \div (a + b + c)$.

(iv) Circumcentre of the Δ ABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(2) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the x– axis.

(3) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(4) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(5) $z = z_1(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$ This is also the condition for three complex numbers to be collinear. The above equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

NOTE : If we replace z by $ze^{i\theta}$ and $\bar{z}$ by $\bar{z}e^{-i\theta}$ then we get equation of a straight line which. Passes through the foot of the perpendicular from origin to given straight line and makes an angle θ with the given straightl line.

(7) The equation of circle having centre z_0 & radius ρ is :
$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form
$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + k = 0$, k is real. Centre is $-\alpha$ & radius = $\sqrt{\alpha\bar{\alpha} - k}$.
Circle will be real if $\alpha\bar{\alpha} - k \geq 0$..

(8) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is
$\arg\frac{z - z_2}{z - z_1} = \pm\frac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(9) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number
$\frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear

points z_1, z_2 & z_3 can be taken as $\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real

$\Rightarrow \quad \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$.

(10) Arg$\left(\frac{z-z_1}{z-z_2}\right) = \theta$ represent (i) a line segment if $\theta = \pi$

(ii) Pair of ray if $\theta = 0$ (iii) a part of circle, if $0 < \theta < \pi$.

(11) Area of triangle formed by the points z_1, z_2 & z_3 is $\left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$

(12) Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\frac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

(13) (i) Complex slope of a line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\omega = -\frac{\alpha}{\bar{\alpha}}$.

(ii) Complex slope of a line joining by the points z_1 & z_2 is $\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(iii) Complex slope of a line making θ angle with real axis = $e^{2i\theta}$

(14) ω_1 & ω_2 are the compelx slopes of two lines.

(i) If lines are parallel then $\omega_1 = \omega_2$

(ii) If lines are perpendicular then $\omega_1 + \omega_2 = 0$

(15) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(16) If $|z - z_0| = \left|\frac{\bar{\alpha} z + \alpha\bar{z} + r}{2|\alpha|}\right|$ then locus of z is parabola whose focus is z_0 and directrix is the line $\bar{\alpha} z_0 + \bar{\alpha} z_0 + r = 0$

(17) If $\left|\frac{z-z_1}{z-z_2}\right| = k \neq 1, 0$, then locus of z is circle.

(18) If $\big|\,|z - z_1| - |z - z_2|\,\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

Match the following columns :

	Column - I		Column - II
(i)	If $\lvert z - 3+2i \rvert - \lvert z + i \rvert = 0$, then locus of z represents	(i)	circle
(ii)	If arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, then locus of z represents...	(ii)	Straight line
(iii)	if $\lvert z - 8 - 2i \rvert + \lvert z - 5 - 6i \rvert = 5$ then locus of z represents	(iii)	Ellipse
(iv)	If arg $\left(\frac{z-3+4i}{z+2-5i}\right) = \frac{5\pi}{6}$, then locus of z represents	(iv)	Hyperbola
(v)	If $\lvert z - 1 \rvert + \lvert z + i \rvert = 10$ then locus of z represents	(v)	Major Arc
(vi)	$\lvert z - 3 + i \rvert - \lvert z + 2 - i \rvert = 1$ then locus of z represents	(vi)	Minor arc
(vii)	$\lvert z - 3i \rvert = 25$	(vii)	Perpendicular bisector of a line segment
(viii)	arg $\left(\frac{z-3+5i}{z+i}\right) = \pi$	(viii)	Line segment

Ans.

I	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
II	(vii)	(v)	(viii)	(vi)	(iii)	(iv)	(i)	(ii)

15. (a) Reflection points for a straight line :

Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(b) Inverse points w.r.t. a circle :

Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if:
(i) the point O, P, Q are collinear and P, Q are on the same side of O.
(ii) OP. OQ = ρ^2.

Note : that the two points z_1 & z_2 will be the inverse points w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

16. Ptolemy's Theorem:

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

Example: If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

(i) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(iii) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

Solution. Let $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$,
$z_3 = \cos\gamma + i\sin\gamma$.

$\therefore$ $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$
$= 0 + i\,.\,0 = 0$ (1)

(i) Also $\dfrac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$\dfrac{1}{z_2} = \cos\beta - i\sin\beta$, $\dfrac{1}{z_3} = \cos\gamma - \sin\gamma$

$\therefore$ $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ (2)
$= 0 - i\,.\,0 = 0$

Now $z_1^2 + z_2^2 + z_3^3 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 - 2z_1z_2z_3\left(\dfrac{1}{z_3} + \dfrac{1}{z_1} + \dfrac{1}{z_2}\right)$

$= 0 - 2z_1z_2z_3\,.\,0 = 0$, using (1) and (2)

or $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

or $\cos 2\alpha + i\sin 2\alpha)^2 + \cos 2\beta + i\sin 2\beta + \cos 2\gamma + i\sin 2\gamma = 0 + i.0$

Equation real and imaginary parts on both sides, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ and $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(ii) $z_1^3 + z_2^3 + z_3^3 = (z_1 + z_2)^3 - 3z_1z_2(z_1 + z_2) + z_3^3$
$= (-z_3)^3 - 3z_1z_2(-z_3) + z_3^3$, using (1)
$= 3z_1z_2z_3$

$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$
$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

or $\cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$
$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)$

Equation imaginary parts on both sides, $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Alternative method

Let $C \equiv \cos\alpha + \cos\beta + \cos\gamma = 0$
$S \equiv \sin\alpha + \sin\beta + \sin\gamma = 0$
$C + iS = e^{i\alpha} + e^{i\beta} + e^{i\gamma} = 0$ (1)
$C - iS = e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma} = 0$ (2)

From (1) $\Rightarrow$ $(e^{-i\alpha})^2 + (e^{-i\beta})^2 + (e^{-i\gamma})^2 = (e^{i\alpha})(e^{i\beta}) + (e^{i\beta})(e^{i\gamma}) + (e^{i\gamma})(e^{i\alpha})$
$\Rightarrow$ $e^{i2\alpha} + e^{i2\beta} + e^{i2\gamma} = e^{i\alpha}e^{i\beta}e^{i\gamma}(e^{-2\gamma} + e^{-i\alpha} + e^{i\beta})$
$\Rightarrow$ $e^{i(2\alpha)} + e^{i2\beta} + e^{i2\gamma} = 0$ (from 2)

Comparing the real and imaginary parts we
$\cos 2\alpha + \cos 2\beta + \cos 2\gamma - \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$
Also from (1) $(e^{i\alpha})^3 + (e^{i\beta})^3 + (e^{i\gamma})^3 = 3e^{i\alpha}\,e^{i\beta}\,e^{i\gamma}$
$\Rightarrow$ $e^{i3\alpha} + e^{i3\beta} + e^{i3\gamma} = 3e^{i(\alpha+\beta+\gamma)}$
Comparing the real and imaginary parts we obtain the results.

Example: If z_1 and z_2 are two complex numbers and c > 0, then prove that

$|z_1 + z_2|^2 \le (I + C)\,|z_1|^2 + (I + C^{-1})\,|z_2|^2$

Solution. We have to prove :

$|z_1 + z_2|^2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$

i.e. $|z_1|^3 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_2 z_2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^3$

or $z_1\bar{z}_2 + \bar{z}_2 z_2 \le c|z_1|^2 + c^{-1}|z_2|^2$ or $c|z_1|^2 + \dfrac{1}{c}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_2 z_2 \ge 0$

(using Re $(z_1\bar{z}_2) \le |z_1\bar{z}_2|$)

or $\left(\sqrt{c}\,|z_1| - \dfrac{1}{\sqrt{c}}\,|z_2|\right)^2 \ge 0$ which is always true.

Example: If $\theta, \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5, and $z^4 \cos\theta_1 + z^3 \cos\theta_2 + z^3 \cos\theta_3. + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \dfrac{3}{4}$

Solution. Given that

$\cos\theta_1 \,.\, z^4 + \cos\theta_2 \,.\, z^3 + \cos\theta_3 \,.\, z^2 + \cos\theta_4 \,.\, z + \cos\theta_5 = 2\sqrt{3}$

or $|\cos\theta_1 \,.\, z^4 + \cos\theta_2 \,.\, z^3 + \cos\theta_3 \,.\, z^2 + \cos\theta_4 \,.\, z + \cos\theta_5| = 2\sqrt{3}$

$2\sqrt{3} \le |\cos\theta_1 \,.\, z^4| + |\cos\theta_2 \,.\, z^3| + |\cos\theta_3 \,.\, z^2| + \cos\theta_4 \,.\, z| + |\cos\theta_5|$

$\because$ $\theta i \in [\pi/6, \pi/3]$

$\therefore$ $\dfrac{1}{2} \le \cos\theta_i \le \dfrac{\sqrt{3}}{2}$

$2\sqrt{3} \le \dfrac{\sqrt{3}}{2}|z|^4 + \dfrac{\sqrt{3}}{2}|z|^3 + \dfrac{\sqrt{3}}{2}|z|^2 + \dfrac{\sqrt{3}}{2}|z| + \dfrac{\sqrt{3}}{2}$

$3 \le |z|^4 + |z|^3 + |z|^2 + |z|$

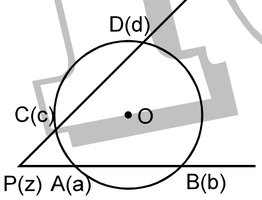
$3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \ldots\ldots\ldots\infty$

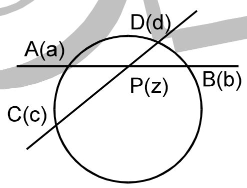
$3 < \dfrac{|z|}{1-|z|}$ $3 - e\,|z| < |z|$

$4|z| > 3$ $\therefore$ $|z| > \dfrac{3}{4}$

Example: Two different non parallel lines cut the circle |z| = r in point a, b, c, d respectively. Prove that these lines meet in the point z given by $z = \dfrac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$

Solution. Since point P, A, B are collinear





$\therefore$ $\begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0 \Rightarrow$ $z\left(\bar{a} - \bar{b}\right) - \bar{z}\,(a - b) + \left(a\bar{b} - a\bar{b}\right) = 0$ (i)

Similarlym, since points P, C, D are collinear

$\therefore$ $z\left(\bar{a} - \bar{b}\right)(c - d) - z\left(\bar{c} - \bar{d}\right)(a - b) = \left(c\bar{d} - \bar{c}d\right)(a - b) - \left(a\bar{b} - \bar{a}b\right)(c - d)$ (iii)

$\because$ $z\bar{z} = r^2 = k$ (say) $\therefore$ $\bar{a} = \dfrac{k}{a}$, $\bar{b} = \dfrac{k}{b}$, $\bar{c} = \dfrac{k}{c}$ etc.

From equation (iii) we get

$z\left(\dfrac{k}{a} - \dfrac{k}{b}\right)(c - d) - z\left(\dfrac{k}{c} - \dfrac{k}{d}\right)(a - b) = \left(\dfrac{ck}{d} - \dfrac{kd}{c}\right)(a - b) - \left(\dfrac{ak}{b} - \dfrac{bk}{a}\right)(c - d)$

$\therefore$ $z = \dfrac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$

Short Revision

1. DEFINITION :

Complex numbers are definited as expressions of the form a + ib where a, b $\in$ R & i = $\sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z).

EVERY COMPLEX NUMBER CAN BE REGARDED AS

Purely real	Purely imaginary	Imaginary
if b = 0	if a = 0	if b $\neq$ 0

Note :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complete Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is both purely real as well as purely imaginary but not imaginary.

(c) i = $\sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

2. CONJUGATE COMPLEX :

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z}$ = a − ib.

Note that :

(i) $z + \bar{z} = 2\,\mathrm{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\mathrm{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

(iv) If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

3. ALGEBRAIC OPERATIONS :

The algebraic operations on complex numbers are similiar to those on real numbers treating *i* as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. z > 0, 4 + 2i < 2 + 4 i are meaningless .

However in real numbers if $a^2 + b^2 = 0$ then a = 0 = b but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

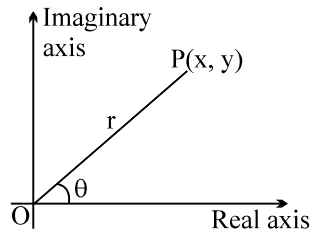
4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometric Representation) :

Every complex number z = x + i y can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



length OP is called modulus of the complex number denoted by $|z|$ & θ is called the argument or amplitude .

eg. $|z| = \sqrt{x^2 + y^2}$ &

$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis)

NOTE :(i) $|z|$ is always non negative . Unlike real numbers $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$ is **not correct**

(ii) Argument of a complex number is a many valued function . If θ is the argument of a complex number then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(iii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) By specifying the modulus & argument a complex number is defined completely. For the complex number 0 + 0 i the argument is not defined and this is the only complex number which is given by its modulus.

(vi) There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

(b) Trignometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note: $\cos\theta + i\sin\theta$ is also written as CiS θ.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$ are known as Euler's identities.

(c) Exponential Representation :

$z = re^{i\theta}$; $|z| = r$; $\arg z = \theta$; $\bar{z} = re^{-i\theta}$

6. IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :

If $z, z_1, z_2 \in C$ then ;

(a) $z + \bar{z} = 2\,\text{Re}(z)$; $z - \bar{z} = 2i\,\text{Im}(z)$; $\overline{(\bar{z})} = z$; $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$;

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ $\quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$

(b) $|z| \geq 0$; $|z| \geq \text{Re}(z)$; $|z| \geq \text{Im}(z)$; $|z| = |\bar{z}| = |-z|$; $z\bar{z} = |z|^2$;

$|z_1 z_2| = |z_1| \cdot |z_2|$; $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$, $|z^n| = |z|^n$;

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

$\big|\,|z_1| - |z_2|\,\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[TRIANGLE INEQUALITY]**

(c) **(i)** $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi.$ $\quad k \in I$

(ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi$; $\quad k \in I$

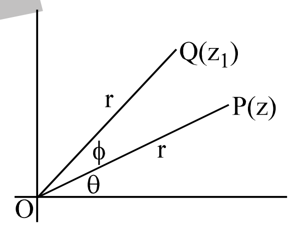
(iii) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$.

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(7) VECTORIAL REPRESENTATION OF A COMPLEX :

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then, $\overrightarrow{OP} = z$ & $|\overrightarrow{OP}| = |z|$.

NOTE :



(i) If $\overrightarrow{OP} = z = re^{i\theta}$ then $\overrightarrow{OQ} = z_1 = re^{i(\theta+\phi)} = z \cdot e^{i\phi}$. If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$

(ii) If A, B, C & D are four points representing the complex numbers z_1, z_2, z_3 & z_4 then

AB $||$ CD if $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely real ; $\quad$ AB $\perp$ CD if $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary]

(iii) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(a) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ $\quad$ **(b)** $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

8. DEMOIVRE'S THEOREM : Statement :

$\cos n\theta + i\sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$ ¥ $n \in Q$. The theorem is very useful in determining the roots of any complex quantity

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

9. CUBE ROOT OF UNITY :

(i) The cube roots of unity are 1, $\dfrac{-1 + i\sqrt{3}}{2}$, $\dfrac{-1 - i\sqrt{3}}{2}$.

(ii) If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$. In general $1 + w^r + w^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are :

$\cos 0 + i\sin 0$; $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$, $\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.

(v) The following factorisation should be remembered :

($a, b, c \in R$ & ω is the cube root of unity)

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