fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
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STUDY PACKAGE

Subject : Mathematics

Topic : Permutation and Combination

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Permutation and Combination

Permutations are arrangements and combinations are selections. In this chapter we discuss the methods of counting of arrangements and selections. The basic results and formulas are as follows:

1. Fundamental Principle of Counting :

(i) Principle of Multiplication: If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then total number of different ways of simultaneous occurrence of both the events in a definite order is $m \times n$.

(ii) Principle of Addition: If an event can occur in 'm' different ways, and another event can occur in 'n' different ways, then exactly one of the events can happen in m + n ways.

Example # 1 There are 8 buses running from Kota to Jaipur and 10 buses running from Jaipur to Delhi. In how many ways a person can travel from Kota to Delhi via Jaipur by bus.

Solution. Let E_1 be the event of travelling from Kota to Jaipur & E_2 be the event of travelling from Jaipur to Delhi by the person.
E_1 can happen in 8 ways and E_2 can happen in 10 ways.
Since both the events E_1 and E_2 are to be happened in order, simultaneously, the number of ways = 8 × 10 = 80.

Example # 2 How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if
(i) No digit is repeated in any number. (ii) Digits can be repeated.

Solution.
(i) Number of two digit numbers = 5 × 4 = 20
Number of three digit numbers = 5 × 4 × 3 = 60
Number of four digit numbers = 5 × 4 × 3 × 2 = 120 Total = 200
(ii) Number of two digit numbers = 5 × 5 = 25
Number of three digit numbers = 5 × 5 × 5 = 125
Number of four digit numbers = 5 × 5 × 5 × 5 = 625 Total = 775

Self Practice Problems :

1. How many 4 digit numbers are there, without repetition of digits, if each number is divisible by 5. **Ans.** 952
2. Using 6 different flags, how many different signals can be made by using atleast three flags, arranging one above the other. **Ans.** 1920

2. Arrangement : If ${}^nP_r$ denotes the number of permutations of n different things, taking r at a time, then

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

NOTE : (i) factorials of negative integers are not defined. (ii) 0 ! = 1 ! = 1 ;
(iii) ${}^nP_n = n! = n.(n-1)!$ (iv) $(2n)! = 2^n.\ n!\ [1.\ 3.\ 5.\ 7 \ldots (2n-1)]$

Example # 3: How many numbers of three digits can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits. How many of these are even.

Solution.: Three places are to be filled with 5 different objects.
∴ Number of ways = ${}^5P_3 = 5 \times 4 \times 3 = 60$
For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^4P_2$ ways.
∴ Number of even numbers = $2 \times {}^4P_2 = 24$.

Example # 4: If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the given word.

Solution.: Number of words beginning with E = ${}^4P_4$ = 24
Number of wards beginning with QE = ${}^3P_3$ = 6
Number of words beginning with QS = 6
Number of words beginning withQT = 6.
Next word is 'QUEST' ∴ its rank is 24 + 6 + 6 + 6 + 1 = 43.

Self Practice Problems :

3. Find the sum of all four digit numbers (without repetition of digits) formed using the digits 1, 2, 3, 4, 5. **Ans.** 399960
4. Find 'n', if ${}^{n-1}P_3 : {}^nP_4 = 1 : 9$. **Ans.** 9
5. Six horses take part in a race. In how many ways can these horses come in the first, second and third place, if a particular horse is among the three winners (Assume No Ties). **Ans.** 60

3. Circular Permutation : The number of circular permutations of n different things taken all at a time is; (n – 1) !. If clockwise & anti–clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

Note: Number of circular permutations of n things when p alike and the rest different taken all at a time distinguishing clockwise and anticlockwise arrangement is $\frac{(n-1)!}{p!}$.

Example # 5: In how many ways can we arrange 6 different flowers in a circle. In how many ways we can form a garland using these flowers.

Solution.: The number of circular arrangements of 6 different flowers = (6 – 1)! = 120
When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the number of ways of forming garland = $\frac{1}{2}(6-1)! = 60$.

Example # 6: In how many ways 6 persons can sit at a round table, if two of them prefer to sit together.

Solution.: Let $P_1, P_2, P_3, P_4, P_5, P_6$ be the persons, where P_1, P_2 want to sit together.
Regard these person as 5 objects. They can be arranged in a circle in (5 – 1)! = 24. Now P_1P_2 can be arranged in 2! ways. Thus the total number of ways = 24 × 2 = 48.

Self Practice Problems : 6. In how many ways the letters of the word 'MONDAY' can be written around a circle if the vowels are to be separated in any arrangement. **Ans.** 72

7. In how many ways we can form a garland using 3 different red flowers, 5 different yellow flowers and 4 different blue flowers, if flowers of same colour must be together. **Ans.** 17280.

4. Selection : If ${}^nC_r$ denotes the number of combinations of n different things taken r at a time, then

$${}^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{{}^nP_r}{r!} \quad \text{where } r \le n\ ;\ n \in N \text{ and } r \in W.$$

NOTE : (i) ${}^nC_r = {}^nC_{n-r}$ (ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (iii) ${}^nC_r = 0$ if $r \notin \{0, 1, 2, 3 \ldots\ldots, n\}$

Example # 7 Fifteen players are selected for a cricket match.

(i) In how many ways the playing 11 can be selected
(ii) In how many ways the playing 11 can be selected including a particular player.
(iii) In how many ways the playing 11 can be selected excluding two particular players.

Solution. (i) 11 players are to be selected from 15
Number of ways = $^{15}C_{11}$ = 1365.
(ii) Since one player is already included, we have to select 10 from the remaining 14
Number of ways = $^{14}C_{10}$ = 1001.
(iii) Since two players are to be excluded, we have to select 11 from the remaining 13.
Number of ways = $^{13}C_{11}$ = 78.

Example # 8 If $^{49}C_{3r-2} = {}^{49}C_{2r+1}$, find 'r'.
Solution. $^nC_r = {}^nC_s$ if either r = s or r + s = n.
Thus 3r – 2 = 2r + 1 $\Rightarrow$ r = 3
or 3r – 2 + 2r + 1 = 49 $\Rightarrow$ 5r – 1 = 49 $\Rightarrow$ r = 10
$\therefore$ r = 3, 10

Example # 9 A regular polygon has 20 sides. How many triangles can be drawn by using the vertices, but not using the sides.
Solution. The first vertex can be selected in 20 ways. The remaining two are to be selected from 17 vertices so that they are not consecutive. This can be done in $^{17}C_2$ – 16 ways.
$\therefore$ The total number of ways = 20 × ($^{17}C_2$ – 16)
But in this method, each selection is repeated thrice.
$\therefore$ Number of triangles = $\dfrac{20 \times (^{17}C_2 - 16)}{3}$ = 800.

Example # 10 10 persons are sitting in a row. In how many ways we can select three of them if adjacent persons are not selected.
Solution. Let $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}$ be the persons sitting in this order.
If three are selected (non consecutive) then 7 are left out.
Let PPPPPPP be the left out & q, q, q be the selected. The number of ways in which these 3 q's can be placed into the 8 positions between the P's (including extremes) is the number ways of required selection. Thus number of ways = $^8C_3$ = 56.

Example # 11 In how many ways we can select 4 letters from the letters of the word MISSISSIPPI.
Solution. M
IIII
SSSS
PP
Number of ways of selecting 4 alike letters = $^2C_1$ = 2.
Number of ways of selecting 3 alike and 1 different letters = $^2C_1 \times {}^3C_1$ = 6
Number of ways of selecting 2 alike and 2 alike letters = $^3C_2$ = 3
Number of ways of selecting 2 alike & 2 different = $^3C_1 \times {}^3C_2$ = 9
Number of ways of selecting 4 different = $^4C_4$ = 1 Total = 21

Self Practice Problems :8. In how many ways 7 persons can be selected from among 5 Indian, 4 British & 2 Chinese, if atleast two are to be selected from each country. **Ans.** 100
9. 10 points lie in a plane, of which 4 points are collinear. Barring these 4 points no three of the 10 points are collinear. How many quadrilaterals can be drawn. **Ans.** 185.
10. In how many ways 5 boys & 5 girls can sit at a round table so that girls & boys sit alternate. **Ans.** 2880
11. In how many ways 4 persons can occupy 10 chairs in a row, if no two sit on adjacent chairs. **Ans.** 840.
12. In how many ways we can select 3 letters of the word PROPORTION. **Ans.** 36

5. The number of permutations of 'n' things, taken all at a time, when 'p' of them are similar & of one type, q of them are similar & of another type, 'r' of them are similar & of a third type & the remaining n – (p + q + r) are all different is $\dfrac{n!}{p!\,q!\,r!}$.

Example # 12 In how many ways we can arrange 3 red flowers, 4 yellow flowers and 5 white flowers in a row. In how many ways this is possible if the white flowers are to be separated in any arrangement (Flowers of same colour are identical).
Solution. Total we have 12 flowers 3 red, 4 yellow and 5 white.
Number of arrangements = $\dfrac{12!}{3!\,4!\,5!}$ = 27720.
For the second part, first arrange 3 red & 4 yellow
This can be done in $\dfrac{7!}{3!\,4!}$ = 35 ways
Now select 5 places from among 8 places (including extremes) & put the white flowers there.
This can be done in $^8C_5$ = 56.
$\therefore$ The number of ways for the 2nd part = 35 × 56 = 1960.

Example # 13 In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative positions of vowels & consonants.
Solution. The consonants in their positions can be arranged in $\dfrac{4!}{2!}$ = 12 ways.
The vowels in their positions can be arranged in $\dfrac{3!}{2!}$ = 3 ways
$\therefore$ Total number of arrangements = 12 × 3 = 26

Self Practice Problems : 13. How many words can be formed using the letters of the word ASSESSMENT if each word begin with A and end with T. **Ans.** 840
14. If all the letters of the word ARRANGE are arranged in all possible ways, in how many of words we will have the A's not together and also the R's not together. **Ans.** 660
15. How many arrangements can be made by taking four letters of the word MISSISSIPPI. **Ans.** 176.

6. **Formation of Groups :** Number of ways in which (m + n + p) different things can be divided into three different groups containing m, n & p things respectively is $\dfrac{(m+n+p)!}{m!\,n!\,p!}$, If m = n = p and the groups have identical qualitative characteristic then the number of groups = $\frac{(3n)!}{n!\,n!\,n!\,3!}$.

However, if 3n things are to be divided equally among three people then the number of ways = $\frac{(3n)!}{(n!)^3}$.

Ex.14. 12 different toys are to be distributed to three children equally. In how many ways this can be done.
Solution. The problem is to divide 12 different things into three different groups.

Number of ways = $\frac{12!}{4!\,4!\,4!}$ = 34650.

Example # 15 In how many ways 10 persons can be divided into 5 pairs.
Solution. We have each group having 2 persons and the qualitative characteristic are same (Since there is no purpose mentioned or names for each pair).

Thus the number of ways = $\frac{10!}{(2!)^5\,5!}$ = 945.

Self Practice Problems : 16. 9 persons enter a lift from ground floor of a building which stops in 10 floors (excluding ground floor). If is known that persons will leave the lift in groups of 2, 3, & 4 in different floors. In how many ways this can happen. **Ans.** 907200

17. In how many ways one can make four equal heaps using a pack of 52 playing cards. **Ans.** $\frac{52!}{(13!)^4\,4!}$

18. In how many ways 11 different books can be parcelled into four packets so that three of the packets contain 3 books each and one of 2 books, if all packets have the same destination. **Ans.** $\frac{11!}{(3!)^4\,2}$

7. Selection of one or more objects

(a) Number of ways in which atleast one object be selected out of 'n' distinct objects is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n - 1$$

(b) Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is

$$(p + 1)\,(q + 1)\,(r + 1) - 1$$

(c) Number of ways in which atleast one object may be selected from 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest n – (p + q + r) are different, is

$$(p + 1)\,(q + 1)\,(r + 1)\,2^{n-(p+q+r)} - 1$$

Example # 16 There are 12 different books on a shelf. In how many ways we can select atleast one of them.
Solution. We may select 1 book, 2 books,........, 12 books.
∴ The number of ways = ${}^{12}C_1 + {}^{12}C_2 + \ldots\ldots + {}^{12}C_{12} = 2^{12} - 1$. = 4095

Example # 17 There are 12 fruits in a basket of which 5 are apples, 4 mangoes and 3 bananas (fruits of same species are identical). How many ways are there to select atleast one fruit.
Solution. Let x be the number of apples being selected
y be the number of mangoes being selected and
z be the number of bananas being selected.
Then x = 0, 1, 2, 3, 4, 5
y = 0, 1, 2, 3, 4
z = 0, 1, 2, 3
Total number of triplets (x, y, z) is 6 × 5 × 4 = 120
Exclude (0, 0, 0) ∴ Number of combinations = 120 – 1 = 119.

Self Practice Problems

19. In a shelf there are 5 physics, 4 chemistry and 3 mathematics books. How many combinations are there if (i) books of same subject are different (ii) books of same subject are identical.
Ans. (i) 4095 (ii) 119

20. From 5 apples, 4 mangoes & 3 bananas in how many ways we can select atleast two fruits of each variety if (i) fruits of same species are identical (ii) fruits of same species are different.
Ans. (i) 24 (ii) 1144

8. Multinomial Theorem: Coefficient of x^r in expansion of $(1 - x)^{-n} = {}^{n+r-1}C_r$ (n ∈ N)

Number of ways in which it is possible to make a selection from m + n + p = N things, where p are alike of one kind, m alike of second kind & n alike of third kind taken r at a time is given by coefficient of x^r in the expansion of

$$(1 + x + x^2 + \ldots\ldots + x^p)\,(1 + x + x^2 + \ldots\ldots + x^m)\,(1 + x + x^2 + \ldots\ldots + x^n).$$

(i) For example the number of ways in which a selection of four letters can be made from the letters of the word **PROPORTION** is given by coefficient of x^4 in

$(1 + x + x^2 + x^3)\,(1 + x + x^2)\,(1 + x + x^2)\,(1 + x)\,(1 + x)\,(1 + x)$.

(ii) **Method of fictious partition :**
Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is; ${}^{n+p-1}C_n$.

Example # 18: Find the number of solutions of the equation x + y + z = 6, where x, y, z ∈ W.
Solution. Number of solutions = coefficient of x^6 in $(1 + x + x^2 + \ldots\ldots\ldots x^6)^3$
= coefficient of x^6 in $(1 - x^7)^3\,(1 - x)^{-3}$
= coefficient of x^6 in $(1 - x)^{-3}$

$$= \begin{pmatrix} 3+6-1 \\ 6 \end{pmatrix} = {}^8C_6 = 28.$$

Example # 19: In a bakery four types of biscuits are available. In how many ways a person can buy 10 biscuits if he decide to take atleast one biscuit of each variety.
Solution. Let x be the number of biscuits the person select from first variety, y from the second, z from the third and w from the fourth variety. Then the number of ways = number of solutions of the equation x + y + z + w = 10.
where x = 1, 2, ...,7
y = 1, 2, ...,7
z = 1, 2, ...,7
w = 1, 2, ...,7

This is equal to = coefficient of x^{10} in $(x + x^2 + \ldots\ldots + x^7)^4$
= coefficient of x^6 in $(1 + x + \ldots\ldots + x^6)^4$
= coefficient of x^6 in $(1 - x^7)^4 (1 - x)^{-4}$
= coefficient x^6 in $(1 - x)^{-4}$ $= \binom{4+6-1}{6} = 84.$

Self Practice Problems:

21. Three distinguishable dice are rolled. In how many ways we can get a total 15. **Ans.** 10.

22. In how many ways we can give 5 apples, 4 mangoes and 3 oranges (fruits of same species are similar) to three persons if each may receive none, one or more. **Ans.** 3150

9. Let N = $p^{a} q^{b} r^{c} \ldots$ where p, q, r...... are distinct primes & a, b, c..... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = $(a + 1)(b + 1)(c + 1)\ldots\ldots$

(b) The sum of these divisors is =
$(p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots\ldots$

(c) Number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1)\ldots & \text{if N is not a perfect square} \\ \frac{1}{2}[(a+1)(b+1)(c+1)\ldots+1] & \text{if N is a perfect square} \end{cases}$$

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Example # 20 Find the number of divisors of 1350. Also find the sum of all divisors.

Solution. $1350 = 2 \times 3^3 \times 5^2$

$\therefore$ Number of divisors = (1+ 1) (3 + 1) (2 + 1) = 24

sum of divisors = $(1 + 2)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2) = 3720$.

Example # 21 In how many ways 8100 can be resolved into product of two factors.

Solution. $8100 = 2^2 \times 3^4 \times 5^2$

Number of ways = $\frac{1}{2}((2 + 1)(4 + 1)(2 + 1) + 1) = 23$

Self Practice Problems :

23. How many divisors of 9000 are even but not divisible by 4. Also find the sum of all such divisors.
Ans. 12, 4056.

24. In how many ways the number 8100 can be written as product of two coprime factors. **Ans.** 4

10. Let there be 'n' types of objects, with each type containing atleast r objects. Then the number of ways of arranging r objects in a row is n^r.

Example # 22 How many 3 digit numbers can be formed by using the digits 0, 1, 2, 3, 4, 5. In how many of these we have atleast one digit repeated.

Solution. We have to fill three places using 6 objects (repeatation allowed), 0 cannot be at 100[th] place.

The number of numbers = 180. $\overset{5}{\square}\overset{6}{\square}\overset{6}{\square}$

Number of numbers in which no digit is repeated = 100 $\overset{5}{\square}\overset{5}{\square}\overset{4}{\square}$

$\therefore$ Number of numbers in which atleast one digit is repeated = 180 – 100 = 80

Example # 23 How many functions can be defined from a set A containing 5 elements to a set B having 3 elements. How many these are surjective functions.

Solution. Image of each element of A can be taken in 3 ways.

$\therefore$ Number of functions from A to B = $3^5 = 243$.

Number of into functions from A to B = $2^5 + 2^5 + 2^5 - 3 = 93$.

$\therefore$ Number of onto functions = 150.

Self Practice Problems : 25. Find the sum of all three digit numbers those can be formed by using the digits. 0, 1, 2, 3, 4.
Ans. 27200.

26. How many functions can be defined from a set A containing 4 elements to a set B containing 5 elements. How many of these are injective functions. **Ans.** 625, 120

27. In how many ways 5 persons can enter into a auditorium having 4 entries. **Ans.** 1024.

11. **Dearrangement :**

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots\ldots\ldots\ldots + (-1)^n \frac{1}{n!}\right)$$

Example # 24 In how many ways we can put 5 writings into 5 corresponding envelopes so that no writing go to the corresponding envelope.

Solution. The problem is the number of dearragements of 5 digits.

This is equal to $5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44.$

Example # 25 Four slip of papers with the numbers 1, 2, 3, 4 written on them are put in a box. They are drawn one by one (without replacement) at random. In how many ways it can happen that the ordinal number of atleast one slip coincide with its own number.

Solution. Total number of ways = 4 ! = 24.

The number of ways in which ordinal number of any slip does not coincide with its own number is the number of dearrangements of 4 objects $= 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$

Thus the required number of ways. = 24 – 9 = 15

Self Practice Problems:

28. In a match column question, Column I contain 10 questions and Column II contain 10 answers written in some arbitrary order. In how many ways a student can answer this question so that exactly 6 of his matchings are correct. **Ans.** 1890

29. In how many ways we can put 5 letters into 5 corresponding envelopes so that atleast one letter go to wrong envelope. **Ans.** 119

