SHORT REVISION

1. Definition : Rectangular array of mn numbers . Unlike determinants it has no value.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{pmatrix}$$

Abbreviated as : $A = \left[a_{ij}\right]$ $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column is called a matrix of order $m \times n$.

2. Special Type Of Matrices :

(a) Row Matrix : $A = [\, a_{11}, a_{12}, \ldots\ldots a_{1n} \,]$ having one row . $(1 \times n)$ matrix.(or row vectors)

(b) Column Matrix : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ having one column. $(m \times 1)$ matrix (or column vectors)

(c) Zero or Null Matrix : $(A = O_{m \times n})$

An $m \times n$ matrix all whose entries are zero .

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(e) Verical Matrix : A matrix of order $m \times n$ is a vertical matrix if $m > n$. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : (Order n) If number of row = number of column $\Rightarrow$ a square matrix.

Note (i) In a square matrix the pair of elements a_{ij} & a_{ji} are called **Conjugate Elements** .

e.g. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

(ii) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots a_{nn}$ are called **Diagonal Elements** . The line along which the diagonal elements lie is called **"Principal or Leading"** diagonal.

The qty $\Sigma\, a_{ii}$ = trace of the matrice written as , i.e. $t_r A$

Square Matrix

Triangular Matrix

Diagonal Matrix denote as $d_{dia}(d_1, d_2, \ldots.., d_n)$ all elements except the leading diagonal are zero

diagonal Matrix Unit or Identity Matrix

Note: Min. number of zeros in a diagonal matrix of order $n = n(n-1)$

"It is to be noted that with square matrix there is a corresponding determinant formed by the elements of A in the same order."

3. Equality Of Matrices :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if ,

(i) both have the same order . **(ii)** $a_{ij} = b_{ij}$ for each pair of i & j.

4. Algebra Of Matrices :

Addition : $A + B = \left[a_{ij} + b_{ij}\right]$ where A & B are of the same type. (same order)

(a) Addition of matrices is commutative.

i.e. $A + B = B + A$ $\quad A = m \times n \quad ; \quad B = m \times n$

(b) Matrix addition is associative .

$(A + B) + C = A + (B + C)$ **Note :** A, B & C are of the same type.

(c) Additive inverse.

If $A + B = \mathbf{O} = B + A$ $\quad A = m \times n$

5. Multiplication Of A Matrix By A Scalar :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

6. Multiplication Of Matrices : (Row by Column)

AB exists if, $\underset{2 \times 3}{A = m \times n}$ & $\underset{3 \times 3}{B = n \times p}$

AB exists, but BA does not $\Rightarrow AB \neq BA$

Note : In the product AB, $\begin{cases} A = \text{pre factor} \\ B = \text{post factor} \end{cases}$

$\underset{1 \times n}{A = (a_1, a_2, \ldots\ldots a_n)}$ & $\underset{n \times 1}{B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}$

$AB = [a_1 b_1 + a_2 b_2 + \ldots\ldots + a_n b_n]$

If $A = \left[a_{ij}\right]$ $m \times n$ & $B = \left[b_{ij}\right]$ $n \times p$ matrix, then $(AB)_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}$

Properties Of Matrix Multiplication :

1. Matrix multiplication is not commutative .

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow AB \neq BA$ (in general)

2. $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow AB = \mathbf{O} \not\Rightarrow A = \mathbf{O}$ or $B = \mathbf{O}$

Note: If A and B are two non- zero matrices such that AB = **O** then A and B are called the divisors of zero. Also if $[AB] = \mathbf{O} \Rightarrow |AB| \Rightarrow |A||B| = 0 \Rightarrow |A| = 0$ or $|B| = 0$ but not the converse.

If A and B are two matrices such that

(i) $AB = BA \Rightarrow$ A and B commute each other

(ii) $AB = -BA \Rightarrow$ A and B anti commute each other

3. Matrix Multiplication Is Associative :

If A, B & C are conformable for the product AB & BC, then

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$

4. Distributivity :

$\left.\begin{aligned} A(B + C) &= AB + AC \\ (A + B)C &= AC + BC \end{aligned}\right]$ Provided A, B & C are conformable for respective products

5. Positive Integral Powers Of A Square Matrix :

For a square matrix A, $A^2 A = (AA)A = A(AA) = A^3$.

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.

6. MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_nx^0$ then we define a matrix polynomial

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots.. + a_nI^n$

where A is the given square matrix. If f (A) is the null matrix then A is called the zero or root of the polynomial f (x).

DEFINITIONS :

(a) **Idempotent Matrix :** A square matrix is idempotent provided $A^2 = A$.

Note that $A^n = A \ \forall \ n \geq 2, n \in N$.

(b) **Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = \mathbf{O}$, $A^{m-1} \neq \mathbf{O}$.

(c) **Periodic Matrix :** A square matrix is which satisfies the relation $A^{K+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(d) **Involutary Matrix :** If $A^2 = I$, the matrix is said to be an involutary matrix.

Note that $A = A^{-1}$ for an involutary matrix.

7. The Transpose Of A Matrix : (Changing rows & columns)

Let A be any matrix . Then, $A = a_{ij}$ of order $m \times n$

$\Rightarrow$ A^T or $A' = [a_{ji}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$

Properties of Transpose : If A^T & B^T denote the transpose of A and B ,

(a) $(A \pm B)^T = A^T \pm B^T$; note that A & B have the same order.

IMP. (b) $(AB)^T = B^T A^T$ A & B are conformable for matrix product AB.

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$ k is a scalar .

General : $(A_1, A_2, \ldots\ldots A_n)^T = A_n^T, \ldots\ldots., A_2^T, A_1^T$ (reversal law for transpose)

8. Symmetric & Skew Symmetric Matrix :

A square matrix $A = \left[a_{ij}\right]$ is said to be ,

symmetric if ,

$a_{ij} = a_{ji} \quad \forall \ i \ \& \ j$ (conjugate elements are equal) **(Note $A = A^T$)**

Note: Max. number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

and skew symmetric if ,

$a_{ij} = -a_{ji} \quad \forall \ i \ \& \ j$ (the pair of conjugate elements are additive inverse of each other)

(Note $A = -A^T$)

Hence If A is skew symmetric, then

$$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \quad \forall \ i$$

Thus the digaonal elements of a skew symmetric matrix are all zero , but not the converse .

Properties Of Symmetric & Skew Matrix :

P – 1 A is symmetric if $A^T = A$

A is skew symmetric if $A^T = -A$

P – 2 $A + A^T$ is a symmetric matrix

$A - A^T$ is a skew symmetric matrix .

Consider $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$A + A^T$ is symmetric . Similarly we can prove that $A - A^T$ is skew symmetric .

P – 3 The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix .

Let $A^T = A$; $B^T = B$ where A & B have the same order .

$\mathbf{(A + B)^T = A + B}$ Similarly we can prove the other

P – 4 If A & B are symmetric matrices then ,

(a) AB + BA is a symmetric matrix

(b) AB – BA is a skew symmetric matrix .

P – 5 Every square matrix can be uniquely expressed as a sum of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\substack{P \\ \text{Symmetric}}} + \underbrace{\frac{1}{2}(A - A^T)}_{\substack{Q \\ \text{Skew Symmetric}}}$$

9. Adjoint Of A Square Matrix :

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is = $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$.

Then (adj A) = $\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

V. Imp. Theorem : $A\,(\text{adj. }A) = (\text{adj. }A).A = |A|\, I_n$, If A be a square matrix of order n.

Note : If A and B are non singular square matrices of same order, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) $\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$

(iii) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

Inverse Of A Matrix (Reciprocal Matrix) :

A square matrix A said to be invertible (non singular) if there exists a matrix B such that,

$$AB = I = BA$$

B is called the inverse (reciprocal) of A and is denoted by A^{-1} . Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA .$$

We have , $A\,.\,(\text{adj } A) = |A|\; I_n$

$A^{-1}\, A\,(\text{adj } A) = A^{-1}\, I_n\, |A|$

$I_n\,(\text{adj } A) = A^{-1}\, |A|\, I_n$ $\therefore$ $A^{-1} = \dfrac{(\text{adj } A)}{|A|}$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Imp. Theorem : If A & B are invertible matrices ofthe same order , then $(AB)^{-1} = B^{-1}A^{-1}$. This is reversal law for inverse.

Note :(i) If A be an invertible matrix , then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(ii) If A is invertible, **(a)** $(A^{-1})^{-1} = A$; **(b)** $(A^k)^{-1} = (A^{-1})^k = A^{-k}$, $k \in N$

(iii) If A is an Orthogonal Matrix. $AA^T = I = A^TA$

(iv) A square matrix is said to be **orthogonal** if , $A^{-1} = A^T$. **(v)** $|A^{-1}| = \dfrac{1}{|A|}$

SYSTEM OF EQUATION & CRITERIAN FOR CONSISTENCY

GAUSS - JORDAN METHOD

$$x + y + z = 6$$
$$x - y + z = 2$$

$2x + y - z = 1$

or $\begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$AX = B \quad \Rightarrow \quad A^{-1} A X = A^{-1} B$

$$X = A^{-1} B = \frac{(\text{adj. } A).B}{|A|}.$$

Note :(1) If $|A| \neq 0$, system is consistent having unique solution

(2) If $|A| \neq 0$ & (adj A) . B $\neq$ **O** (Null matrix), system is consistent having unique non−trivial solution.

(3) If $|A| \neq 0$ & (adj A) . B = **O** (Null matrix), system is consistent having trivial solution .

(4) If $|\mathbf{A}| = 0$, **matrix method fails**

If (adj A) . B = null matrix = **O** → Consistent (Infinite solutions)

If (adj A) . B $\neq$ **O** → Inconsistent (no solution)

EXERCISE-4

Q1. Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that Cb = D. Solve the matrix equation Ax = b.

Q2. Find the value of x and y that satisfy the equations.

$$\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

Q 3. If, $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ calculate the matrix product EF & FE and show that $E^2F + FE^2 = E$.

Q 4. If A is an orthogonal matrix and B = AP where P is a non singular matrix then show that the matrix PB^{-1} is also orthogonal.

Q 5. The matrix, R(t) is defined by $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Show that, $R(s)\,R(t) \equiv R(s+t)$.

Q 6. Prove that the product of two matrices, $\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ & $\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix when θ & ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Q 7. If, $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that the maxtrix A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.

Q.8 For a non zero λ, use induction to prove that : (Only for XII CBSE)

(a) $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$, for every $n \in N$

(b) If, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(aI + bA)^n = a^n I + na^{n-1} b A$, where I is a unit matrix of order 2, $\forall\ n \in N$.

Q9. Find the number of 2×2 matrix satisfying

(i) a_{ij} is 1 or –1 ; (ii) $a_{11}^2 + a_{12}^2 = a_{21}^2 + a_{22}^2 = 2$; (iii) $a_{11}\, a_{21} + a_{12}\, a_{22} = 0$

Q 10. Prove that $(AB)^T = B^T . A^T$, where A & B are conformable for the product AB . Also verify the result for the matrices, $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$.

Q 11 Express the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also Express the matrix as a sum of a symmetric & a skew symmetric matrix.

Q 12. Find the inverse of the matrix :

(i) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$; **(ii)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}$ where w is the cube root of unity.

(iii) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Q 13. Find the matrix A satisfying the matrix equation, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} . A . \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$.

Q 14. A is a square matrix of order n.
l = maximum number of distinct entries if A is a triangular matrix
m = maximum number of distinct entries if A is a diagonal matrix
p = minimum number of zeroes if A is a triangular matrix
If $l + 5 = p + 2m$, find the order of the matrix.

Q 15. If A is an idempotent matrix and I is an identity matrix of the same order, find the value of n, $n \in N$, such that $(A + I)^n = I + 127\,A$.

Q.16 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that value of f and g satisfying the maxtrix equation $A^2 + f A + g I = \mathbf{O}$ are equal to $-t_r(A)$ and determinant of A respectively. Given a, b, c, d are non zero reals and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Q17. Matrices A and B satisfy $AB = B^{-1}$ where $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. Find

(i) without finding B^{-1}, the value of K for which $KA - 2B^{-1} + I = \mathbf{O}$

(ii) Without finding A^{-1}, the matrix X satisfying $A^{-1}XA = B$ (iii) the matrix A, using A^{-1}

Q18. For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2}.

Q19. Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Q 20. Use matrix to solve the following system of equations.

(i) $\begin{matrix} x+y+z=3 \\ x+2y+3z=4 \\ x+4y+9z=6 \end{matrix}$ (ii) $\begin{matrix} x+y+z=6 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$ (iii) $\begin{matrix} x+y+z=3 \\ x+2y+3z=4 \\ 2x+3y+4z=7 \end{matrix}$ (iv) $\begin{matrix} x+y+z=3 \\ x+2y+3z=4 \\ 2x+3y+4z=9 \end{matrix}$

EXERCISE-5

Q1. Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(i) AX = A (ii) XA = I (iii) $XB = \mathbf{O}$ but $BX \neq \mathbf{O}$.

Q 2. If A & B are square matrices of the same order & A is symmetrical, show that $B'AB$ is also symmetrical.

Q 3. Show that, $\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Q.4 If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a, b, c, d not all simultaneously zero) commute, find the value of $\frac{d-b}{a+c-b}$. Also show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha-\beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$

Q 5. If the matrix A is involutary, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I+A)\cdot\frac{1}{2}(I-A)=\mathbf{O}$.

Q 6. Prove that (i) $|\text{adj}\,(\text{adj}\,A)| = |A|^{(n-1)^2}$, where A is a non-singular matrix of order 'n'.

(ii) $\text{adj}\,(\text{adj}\,A) = |A|^{n-2}.\,A$, where $|A|$ denotes the determinant of co-efficient matrix.

Q 7. Find the product of two matrices A & B, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the following system of linear equations,

$x + y + 2z = 1$; $3x + 2y + z = 7$; $2x + y + 3z = 2$.

Q 8. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then, find a non-zero square matrix X of order 2 such that AX = **O**. Is XA = **O**.

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, is it possible to find a square matrix X such that AX = O. Give reasons for it.

Q 9. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$; $B = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$ Where $0 < \beta < \frac{\pi}{2}$ then prove that $BAB = A^{-1}$. Also find the least positive value of α for which $BA^4B = A^{-1}$.

Q 10. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of f(a), where $f(x) = x - x^2$, when bc = 1/4. Hence otherwise evaluate a.

Q 11. If A is a skew symmetric matrix and I + A is non singular, then prove that the matrix $B = (I - A)(I + A)^{-1}$ is an orthogonal matrix. Use this to find a matrix B given $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$.

Q 12. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that F(x). F(y) = F(x + y)

Hence prove that $[F(x)]^{-1} = F(-x)$.

Q 13. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ then solve the following matrix equation.

(a) AX = B – I (b) (B – I)X = IC (c) CX = A

Q 14. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(i) has a unique solution ; (ii) has no solution and (iii) has infinitely many solutions

Q 15. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$ then find the minimum value of $\sum(\cos^x\theta + \sin^x\theta)$, $\theta \in R$.

Q16. Determine the matrices B and C with integral element such that

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = B^3 + C^3$$

Q17. If $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is an orthogonal matrix, find the values of α, β, γ.

Q18. If $A = \begin{bmatrix} k & m \\ l & n \end{bmatrix}$ and $kn \neq lm$; then show that $A^2 - (k + n)A + (kn - lm)\, I = \mathbf{O}$. Hence find A^{-1}.

Q19. Evaluate $\lim\limits_{n\to\infty} \begin{bmatrix} 1 & \frac{x}{n} \\ -\frac{x}{n} & 1 \end{bmatrix}^n$

Q.20 Given matrices $A = \begin{bmatrix} 1 & x & 1 \\ x & 2 & y \\ 1 & y & 3 \end{bmatrix}$; $B = \begin{bmatrix} 3 & -3 & z \\ -3 & 2 & -3 \\ z & -3 & 1 \end{bmatrix}$

Obtain x, y and z if the matrix AB is symmetric.

EXERCISE-6

Q.1 If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, abc = 1 and $A^TA = I$, then find the value of $a^3 + b^3 + c^3$. [JEE 2003, Mains-2 out of 60]

Q.2 If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then α =

(A) ± 3 (B) ± 2 (C) ± 5 (D) 0[JEE 2004 (Screening)]

Q.3 If M is a 3 × 3 matrix, where $M^TM = I$ and det (M) = 1, then prove that det (M – I) = 0.

Q.4 $A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$.

If there is vector matrix X, such that AX = U has infinitely many solution, then prove that BX = V cannot have a unique solution. If afd $\neq$ 0, then prove that BX = V has no solution.

Q.5 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$, then the value of c and d are

(A) –6, –11 (B) 6, 11 (C) –6, 11 (D) 6, – 11

Q.6 If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^TQ^{2005}P$, then x is equal to

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Q7. If f(x) is a quadratic polynomial and a, b, c are three real and distinct numbers satisfying $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 & + & 3a \\ 3b^2 & + & 3b \\ 3c^2 & + & 3c \end{bmatrix}$. Given f(x) cuts the x-axis at A and V is the point of mixima.

If AB is any chord which subtends right angle at V, find curve f(x) and area bounded by chord AB and curve f(x).

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, if U_1, U_2 and U_3 are columns matrices satisfying.

$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and U is 3 x 3 matrix whose columns are U_1, U_2, U_3 then answer the following questions.

Q8. The value of |U| is [JEE 2006]
(A) 3 (B) -3 (C) 3/2 (D) 2

Q9. The sum of the elements of U^{-1} is [JEE 2006]
(A) -1 (B) 0 (C) 1 (D) 3

Q10. The value of $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ is [JEE 2006]
(A) 5 (B) 5/2 (C) 4 (D) 3/2

ANSWER SHEET

EXERCISE-4

Q.1 $x_1 = 1$, $x_2 = -1$, $x_3 = 1$ Q.2 $x = \frac{3}{2}$, $y = 2$ Q.3 $EF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $FE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.9 8 Q.11 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$

Q.12 (i) $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (ii) $\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix}$, (iii) $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$ Q.13 $\frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

Q.14 4 Q.15 n = 7 Q.16 f = – (a + d) ; g = ad – bc

Q.17 (i) K = 2, (ii) X = B, (iii) $A = \frac{1}{4}\begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$ Q.18 $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$ Q.19 $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$

Q.20 (i) x = 2, y = 1, z = 0 ; (ii) x = 1, y = 2, z = 3 ;
(iii) x = 2 + k, y = 1 – 2k, z = k where $k \in R$; (iv) inconsistent, hence no solution

EXERCISE-5

Q.1 (i) $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$ for $a, b \in R$; (ii) X does not exist. ;

(iii) $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ $a, c \in R$ and $3a + c \neq 0$; $3b + d \neq 0$

Q.4 1 Q.7 $x = 2, y = 1, z = -1$ Q.8 $X = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$, where $c, d \in R - \{0\}$, NO

Q.9 $\frac{2\pi}{3}$ Q.10 $f(a) = 1/4$, $a = 1/2$ Q.11 $\frac{1}{13}\begin{bmatrix} -12 & -5 \\ 5 & -12 \end{bmatrix}$

Q.13(a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$, (b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, (c) no solution

Q.14 (i) $a \neq -3$, $b \in R$; (ii) $a = -3$ and $b \neq 1/3$; (iii) $a = -3$, $b = 1/3$

Q.15 2 Q.16 $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Q.17 $\alpha = \pm\frac{1}{\sqrt{2}}$, $\beta = \pm\frac{1}{\sqrt{6}}$, $\gamma = \pm\frac{1}{\sqrt{3}}$ Q.18 $\frac{1}{kn - lm}\begin{bmatrix} n & -m \\ -l & k \end{bmatrix}$

Q.19 $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

Q.20 $\left(-\frac{4\sqrt{2}}{3}, \frac{2}{3}, 2\sqrt{2}\right), \left(\frac{4\sqrt{2}}{3}, \frac{2}{3}, -2\sqrt{2}\right), (3, 3, -1)$

EXERCISE-6

Q.1 4 Q.2 A Q.5 C Q.6 A Q7. $\frac{125}{3}$ sq. units Q.8 A

Q9. B Q10. A