Brilliant Preparatory Section, Sitamarhi

XI Maths Advanced Study Material

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1. MATRICES AND DETERMINANTS

1.1 Matrix Algebra

1.1.1 Introduction

The term ‘matrix’ was first introduced by Sylvester in 1850. He defined a matrix to be an arrangement of terms. In 1858 Cayley outlined a matrix algebra defining addition, multiplication, scalar multiplication and inverses. Knowledge of matrix is very useful and important as it has a wider application in almost every field of Mathematics. Economists are using matrices for social accounting, input – output tables and in the study of inter-industry economics. Matrices are also used in the study of communication theory, network analysis in electrical engineering.

For example let us consider the marks scored by a student in different subjects and in different terminal examinations. They are exhibited in a tabular form as given below.

	Tamil	English	Maths	Science	Social Science
Test 1	70	81	88	83	64
Test 2	68	76	93	81	70
Test 3	80	86	100	98	78

The above statement of marks can also be re-recorded as follows :

$$\begin{array}{ll}
\begin{array}{l}\text{First row}\\ \text{Second row}\\ \text{Third row}\end{array} & \begin{bmatrix}70 & 81 & 88 & 83 & 64\\ 68 & 76 & 93 & 81 & 70\\ 80 & 86 & 100 & 98 & 78\end{bmatrix}\\
 & \begin{array}{ccccc}\begin{array}{c}\text{First}\\ \text{Column}\end{array} & \begin{array}{c}\text{second}\\ \text{Column}\end{array} & \begin{array}{c}\text{Third}\\ \text{Column}\end{array} & \begin{array}{c}\text{Fourth}\\ \text{Column}\end{array} & \begin{array}{c}\text{Fifth}\\ \text{Column}\end{array}\end{array}
\end{array}$$

This representation gives the following informations.

(i) The elements along the first, second, and third rows represent the test marks of the different subjects.

(ii) The elements along the first, second, third, fourth and fifth columns represent the subject marks in the different tests.

The purpose of matrices is to provide a kind of mathematical shorthand to help the study of problems represented by the entries. The matrices may represent transformations of co-ordinate spaces or systems of simultaneous linear equations.

1.1.2 Definitions:

A matrix is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket or parenthesis. The entries or elements may be any kind of numbers (real or complex), polynomials or other expressions. Matrices are denoted by the capital letters like A, B, C…

Here are some examples of Matrices.

$$A = \begin{array}{cl} \left[\begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{array}\right] & \begin{array}{l} \textbf{First Row} \\ \textbf{Second Row} \\ \textbf{Third Row} \end{array} \\ \begin{array}{cc} \begin{array}{c}\textbf{First}\\ \textbf{Column}\end{array} & \begin{array}{c}\textbf{Second}\\ \textbf{Column}\end{array} \end{array} & \end{array} \qquad B = \begin{array}{cl} \left[\begin{array}{ccc} 1 & -4 & 2 \\ 6 & 9 & 4 \\ 3 & -2 & 6 \end{array}\right] & \begin{array}{l} \textbf{First row } (\mathbf{R_1}) \\ \textbf{Second row } (\mathbf{R_2}) \\ \textbf{Third row } (\mathbf{R_3}) \end{array} \\ \begin{array}{ccc} \begin{array}{c}\textbf{First}\\ \textbf{Column}\\ \mathbf{C_1}\end{array} & \begin{array}{c}\textbf{Second}\\ \textbf{Column}\\ \mathbf{C_2}\end{array} & \begin{array}{c}\textbf{Third}\\ \textbf{Column}\\ \mathbf{C_3}\end{array} \end{array} & \end{array}$$

Note : In a matrix, rows are counted from top to bottom and the columns are counted from left to right.

i.e. (i) The horizontal arrangements are known as rows.

(ii) The vertical arrangements are known as columns.

To identify an entry or an element of a matrix two suffixes are used. The first suffix denotes the row and the second suffix denotes the column in which the element occurs.

From the above example the elements of A are $a_{11} = 1$, $a_{12} = 4$, $a_{21} = 2$, $a_{22} = 5$, $a_{31} = 3$ and $a_{32} = 6$

Order or size of a matrix

The order or size of a matrix is the number of rows and the number of columns that are present in a matrix.

In the above examples order of A is 3 × 2, (to be read as 3-by-2) and order of B is 3 × 3, (to be read as 3-by-3).

In general a matrix A of order $m \times n$ can be represented as follows :

$$A = \begin{array}{cl} \left[\begin{array}{cccccc} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] & \begin{array}{l} \\ \\ \\ \to i^{\text{th}} \text{ row} \\ \\ \\ \end{array} \\ \begin{array}{cccccc} \phantom{a_{11}} & \phantom{a_{12}} & & \begin{array}{c} j^{\text{th}} \\ \text{column} \end{array} & & \phantom{a_{1n}} \end{array} & \end{array}$$

This can be symbolically written as A = $[a_{ij}]_{m \times n}$.

The element a_{ij} belongs to i^{th} row and the j^{th} column. i being the row index and j being the column index. The above matrix A is an $m \times n$ or m-by-n matrix. The expression $m \times n$ is the order or size or dimension of the matrix.

Example 1.1: Construct a 3×2 matrix whose entries are given by $a_{ij} = i - 2j$

Solution: The general 3×2 matrix is of the form

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ where } i = 1, 2, 3 \text{ (rows)}, \quad j = 1, 2 \text{ (columns)}$$

It is given that $a_{ij} = i - 2j$

$a_{11} = 1 - 2 = -1 \quad a_{12} = 1 - 4 = -3$

$a_{21} = 2 - 2 = 0 \quad a_{22} = 2 - 4 = -2$

$a_{31} = 3 - 2 = 1 \quad a_{32} = 3 - 4 = -1$

$\therefore$ The required matrix is $A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{bmatrix}$

1.1.3 Types of matrices

(1) Row matrix: A matrix having only one row is called a row matrix or a row vector.

Examples (i) $A = [a_{ij}]_{1 \times 3} = [1 \quad -7 \quad 4]$ is a row matrix of order 1×3.

(ii) $B = [b_{ij}]_{1 \times 2} = [5 \quad 8]$ is a row matrix of order 1×2

(iii) $C = [c_{ij}]_{1 \times 1} = [100]$ is a row matrix of order 1×1

(2) Column matrix:

A matrix having only one column is called a column matrix or a column vector.

Examples (i) $A = [a_{ij}]_{3 \times 1} = \begin{bmatrix} 1 \\ -7 \\ 4 \end{bmatrix}$ is a column matrix of order 3×1

(ii) $B = [b_{ij}]_{2 \times 1} = \begin{bmatrix} 25 \\ 30 \end{bmatrix}$ is a column matrix of order 2×1

(iii) $C = [c_{ij}]_{1 \times 1} = [68]$ is a column matrix of order 1×1

Note : Any matrix of order 1×1 can be treated as either a row matrix or a column matrix.

(3) Square matrix

A square matrix is a matrix in which the number of rows and the number of columns are equal. A matrix of order $n \times n$ is also known as a square matrix of order n.

In a square matrix A of order $n \times n$, the elements a_{11}, a_{22}, a_{33} … a_{nn} are called principal diagonal or leading diagonal or main diagonal elements.

$A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ is a square matrix of order 2

$B = [b_{ij}]_{3 \times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a square matrix of order 3.

Note: In general the number of elements in a square matrix of order n is n^2. We can easily verify this statement from the above two examples.

(4) Diagonal Matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$

In a diagonal matrix all the entries except the entries along the main diagonal are zero.

For example $A = [a_{ij}]_{3 \times 3} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is a diagonal matrix.

(5) Triangular matrix: A square matrix in which all the entries above the main diagonal are zero is called a lower triangular matrix. If all the entries below the main diagonal are zero, it is called an upper triangular matrix.

$A = \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is an upper triangular matrix and $B = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -5 & 7 \end{bmatrix}$ is a lower triangular matrix.

(6) Scalar matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be scalar matrix if $a_{ij} = \begin{cases} a & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

i.e. A scalar matrix is a diagonal matrix in which all the entries along the main diagonal are equal.

$A = [a_{ij}]_{2\times 2} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\quad$ $B = [b_{ij}]_{3\times 3} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$ are examples for scalar matrices.

(7) Identity matrix or unit matrix:

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be an identity matrix if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

i.e. An identity matrix or a unit matrix is a scalar matrix in which entries along the main diagonal are equal to 1. We represent the identity matrix of order n as I_n

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices.

(8) Zero matrix or null matrix or void matrix

A matrix $A = [a_{ij}]_{m \times n}$ is said to be a zero matrix or null matrix if all the entries are zero, and is denoted by O i.e. $a_{ij} = 0$ for all the values of i, j

$[0 \quad 0]$, $\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are examples of zero matrices.

(9) Equality of Matrices:

Two matrices A and B are said to be equal if

(i) both the matrices A and B are of the same order or size.

(ii) the corresponding entries in both the matrices A and B are equal.

i.e. the matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are equal if $m = p$, $n = q$ and $a_{ij} = b_{ij}$ for every i and j.

Example 1.2 :

If $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix}$ then find the values of x, y, z, w.

Solution:

Since the two matrices are equal, their corresponding entries are also equal.

$\therefore x = 4 \quad y = 3 \quad z = 1 \quad w = 5$

(10) Transpose of a matrix:

The matrix obtained from the given matrix A by interchanging its rows into columns and its columns into rows is called the transpose of A and it is denoted by A′ or A^T.

If $A = \begin{bmatrix} 4 & -3 \\ 2 & 0 \\ 1 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \end{bmatrix}$

Note that if A is of order $m \times n$ then A^T is order $n \times m$.

(11) Multiplication of a matrix by a scalar

Let A be any matrix. Let k be any non-zero scalar. The matrix kA is obtained by multiplying all the entries of matrix A by the non zero scalar k.

i.e. $A = [a_{ij}]_{m \times n} \Rightarrow kA = [ka_{ij}]_{m \times n}$

This is called scalar multiplication of a matrix.

Note: If a matrix A is of order $m \times n$ then the matrix kA is also of the same order $m \times n$

For example If $A = \begin{bmatrix} 1 & 7 & 2 \\ -6 & 3 & 9 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 1 & 7 & 2 \\ -6 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 14 & 4 \\ -12 & 6 & 18 \end{bmatrix}$

(12) Negative of a matrix:

Let A be any matrix. The negative of a matrix A is – A and is obtained by changing the sign of all the entries of matrix A.

i.e. $A = [a_{ij}]_{m \times n} \Rightarrow -A = [-a_{ij}]_{m \times n}$

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $-A = \begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

1.1.4 Operations on matrices

(1) Addition and subtraction

Two matrices A and B can be added provided both the matrices are of the same order and their sum A + B is obtained by adding the corresponding entries of both the matrices A and B

i.e. $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Similarly $A - B = A + (-B) = [a_{ij}]_{m \times n} + [-b_{ij}]_{m \times n}$

$= [a_{ij} - b_{ij}]_{m \times n}$

Note:

(1) The matrices A + B and A – B have same order equal to the order of A or B.

(2) Subtraction is treated as negative addition.

(3) The additive inverse of matrix A is – A.

i.e. A + (– A) = (– A) + A = O = zero matrix

For example, if $A = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix}$

then $A + B = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -7 \\ 3 & 1 \\ -8 & 5 \end{bmatrix} = \begin{bmatrix} 7+4 & 2-7 \\ 8+3 & 6+1 \\ 9-8 & -6+5 \end{bmatrix} = \begin{bmatrix} 11 & -5 \\ 11 & 7 \\ 1 & -1 \end{bmatrix}$ and

$A - B = A + (-B) = \begin{bmatrix} 7 & 2 \\ 8 & 6 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} -4 & 7 \\ -3 & -1 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 7-4 & 2+7 \\ 8-3 & 6-1 \\ 9+8 & -6-5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 5 & 5 \\ 17 & -11 \end{bmatrix}$

(2) Matrix multiplication:

Two matrices A and B are said to be conformable for multiplication if the number of columns of the first matrix A is equal to the number of rows of the second matrix B. The product matrix 'AB' is acquired by multiplying every row of matrix A with the corresponding elements of every column of matrix B element-wise and add the results. This procedure is known as row-by-column multiplication rule.

Let A be a matrix of order $m \times n$ and B be a matrix of order $n \times p$ then the product matrix AB will be of order $m \times p$

i.e. order of A is $m \times n$, order of B is $n \times p$

Then the order of AB is $m \times p = \begin{pmatrix} \text{number of rows} \\ \text{of matrix A} \end{pmatrix} \times \begin{pmatrix} \text{number of columns} \\ \text{of matrix B} \end{pmatrix}$

The following example describes the method of obtaining the product matrix AB

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 3 & 6 \end{bmatrix}_{2 \times 3}$ $B = \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}_{3 \times 3}$

It is to be noted that the number of columns of the first matrix A is equal to the number of rows of the second matrix B.

$\therefore$ Matrices A and B are conformable, i.e. the product matrix AB can be found.

$$AB = \begin{bmatrix} 2 & 1 & 4 \\ 7 & 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & 4 & 3 \\ 3 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 6 \\ 3 \\ 7 \end{matrix} & \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 2 & 1 & 4 \end{matrix} \begin{matrix} 3 \\ 5 \\ 1 \end{matrix} \\ \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 6 \\ 3 \\ 7 \end{matrix} & \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 7 & 3 & 6 \end{matrix} \begin{matrix} 3 \\ 5 \\ 1 \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} (2)(6) + (1)(3) + (4)(7) & (2)(4) + (1)(2) + (4)(3) & (2)(3) + (1)(5) + (4)(1) \\ (7)(6) + (3)(3) + (6)(7) & (7)(4) + (3)(2) + (6)(3) & (7)(3) + (3)(5) + (6)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 3 + 28 & 8 + 2 + 12 & 6 + 5 + 4 \\ 42 + 9 + 42 & 28 + 6 + 18 & 21 + 15 + 6 \end{bmatrix} \qquad \therefore\ AB = \begin{bmatrix} 43 & 22 & 15 \\ 93 & 52 & 42 \end{bmatrix}$$

It is to be noticed that order of AB is 2×3, which is the number of rows of first matrix A 'by' the number of columns of the second matrix B.

Note : (i) If AB = AC, it is not necessarily true that B = C. (i.e.) the equal matrices in the identity cannot be cancelled as in algebra.

(ii) AB = O does not necessarily imply A = O or B = O

For example, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq O$

but $AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

(iii) If A is a square matrix then A.A is also a square matrix of the same order. AA is denoted by A^2. Similarly $A^2A = AAA = A^3$

If I is a unit matrix, then $I = I^2 = I^3 = \ldots = I^n$.

1.1.5 Algebraic properties of matrices:

(1) Matrix addition is commutative:

If A and B are any two matrices of the same order then A + B = B + A. This property is known as commutative property of matrix addition.

(2) Matrix addition is associative:

i.e. If A, B and C are any three matrices of the same order then A+(B + C) = (A+B)+C. This property is known as associative property of matrix addition.

(3) Additive identity:

Let A be any matrix then A + O = O + A = A. This property is known as identity property of matrix addition.

The zero matrix O is known as the identity element with respect to matrix addition.

(4) Additive inverse:

Let A be any matrix then A + (– A) = (– A) + A = O. This property is known as inverse property with respect to matrix addition.

The negative of matrix A i.e. – A is the inverse of A with respect to matrix addition.

(5) In general, matrix multiplication is not commutative i.e. $AB \neq BA$

(6) Matrix multiplication is associative i.e. A(BC) = (AB)C

(7) Matrix multiplication is distributive over addition

i.e. (i) A(B + C) = AB + AC (ii) (A + B)C = AC + BC

(8) AI = IA = A where I is the unit matrix or identity matrix. This is known as identity property of matrix multiplication.

Example 1.3: If $A = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix}$ $C = \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix}$

Prove that (i) $AB \neq BA$ (ii) A(BC) = (AB)C

(iii) A(B + C) = AB + AC (iv) AI = IA = A

Solution:

(i)
$$AB = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} (1)(1)+(8)(7) & (1)(3)+(8)(4) \\ (4)(1)+(3)(7) & (4)(3)+(3)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1+56 & 3+32 \\ 4+21 & 12+12 \end{bmatrix} = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} \quad \ldots (1)$$

$$BA = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} (1)(1)+(3)(4) & (1)(8)+(3)(3) \\ (7)(1)+(4)(4) & (7)(8)+(4)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1+12 & 8+9 \\ 7+16 & 56+12 \end{bmatrix} = \begin{bmatrix} 13 & 17 \\ 23 & 68 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2) we have $AB \neq BA$

(ii)
$$(AB)C = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix} \quad \ldots \text{from (1)}$$

$$= \begin{bmatrix} (57)(-4)+(35)(3) & (57)(6)+(35)(-5) \\ (25)(-4)+(24)(3) & (25)(6)+(24)(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -228+105 & 342-175 \\ -100+72 & 150-120 \end{bmatrix}$$

$$\therefore\ (AB)C = \begin{bmatrix} -123 & 167 \\ -28 & 30 \end{bmatrix} \qquad \ldots (3)$$

$$BC = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-4)+(3)(3) & (1)(6)+(3)(-5) \\ (7)(-4)+(4)(3) & (7)(6)+(4)(-5) \end{bmatrix} = \begin{bmatrix} -4+9 & 6-15 \\ -28+12 & 42-20 \end{bmatrix}$$

$$BC = \begin{bmatrix} 5 & -9 \\ -16 & 22 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & -9 \\ -16 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(5)+(8)(-16) & (1)(-9)+(8)(22) \\ (4)(5)+(3)(-16) & (4)(-9)+(3)(22) \end{bmatrix} = \begin{bmatrix} 5-128 & -9+176 \\ 20-48 & -36+66 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} -123 & 167 \\ -28 & 30 \end{bmatrix} \qquad \ldots (4)$$

From (3) and (4) we have, (AB)C = A(BC)

(iii) $$B + C = \begin{bmatrix} 1 & 3 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1-4 & 3+6 \\ 7+3 & 4-5 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 10 & -1 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -3 & 9 \\ 10 & -1 \end{bmatrix} = \begin{bmatrix} -3+80 & 9-8 \\ -12+30 & 36-3 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 77 & 1 \\ 18 & 33 \end{bmatrix} \qquad \ldots (5)$$

$$AB = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} \ldots \text{from (1)}$$

$$AC = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -4 & 6 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -4+24 & 6-40 \\ -16+9 & 24-15 \end{bmatrix} = \begin{bmatrix} 20 & -34 \\ -7 & 9 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 57 & 35 \\ 25 & 24 \end{bmatrix} + \begin{bmatrix} 20 & -34 \\ -7 & 9 \end{bmatrix} = \begin{bmatrix} 57+20 & 35-34 \\ 25-7 & 24+9 \end{bmatrix}$$

$$= \begin{bmatrix} 77 & 1 \\ 18 & 33 \end{bmatrix} \qquad \ldots (6)$$

From equations (5) and (6) we have A(B + C) = AB + AC

(iv) Since order of A is 2 × 2, take $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$AI = \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 8(0) & 1(0) + 8(1) \\ 4(1) + 3(0) & 4(0) + 3(1) \end{bmatrix} = \begin{bmatrix} 1 + 0 & 0 + 8 \\ 4 + 0 & 0 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = A \qquad \ldots (7)$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1(1) + 0(4) & 1(8) + 0(3) \\ 0(1) + 1(4) & 0(8) + 1(3) \end{bmatrix} = \begin{bmatrix} 1 + 0 & 8 + 0 \\ 0 + 4 & 0 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 \\ 4 & 3 \end{bmatrix} = A \qquad \ldots (8)$$

∴ From(7) and (8) AI = IA = A

Example 1.4: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ find $A^2 - 7A - 2I$

Solution: $A^2 = AA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 + 12 & 6 + 15 \\ 8 + 20 & 12 + 25 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} \qquad \ldots (1)$$

$$-7A = -7\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -14 & -21 \\ -28 & -35 \end{bmatrix} \qquad \ldots (2)$$

$$-2I = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad \ldots (3)$$

(1) + (2) + (3) gives $A^2 - 7A - 2I = A^2 + (-7A) + (-2I)$

$$= \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} + \begin{bmatrix} -14 & -21 \\ -28 & -35 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

i.e. $A^2 - 7A - 2I = \begin{bmatrix} 16 - 14 - 2 & 21 - 21 + 0 \\ 28 - 28 + 0 & 37 - 35 - 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Example 1.5: If $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix}$,

show that $(A + B)^2 \neq A^2 + 2AB + B^2$

Solution: $A + B = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1+5 & 4+0 \\ 0+3 & 3+9 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 12 \end{bmatrix}$

$$(A + B)^2 = (A + B)\,(A + B) = \begin{bmatrix} 6 & 4 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 3 & 12 \end{bmatrix} = \begin{bmatrix} 36+12 & 24+48 \\ 18+36 & 12+144 \end{bmatrix}$$

$$(A + B)^2 = \begin{bmatrix} 48 & 72 \\ 54 & 156 \end{bmatrix} \qquad \dots (1)$$

$$A^2 = A.A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & 4+12 \\ 0+0 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ 0 & 9 \end{bmatrix}$$

$$B^2 = B.B = \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 25+0 & 0+0 \\ 15+27 & 0+81 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 42 & 81 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 5+12 & 0+36 \\ 0+9 & 0+27 \end{bmatrix} = \begin{bmatrix} 17 & 36 \\ 9 & 27 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} 17 & 36 \\ 9 & 27 \end{bmatrix} = \begin{bmatrix} 34 & 72 \\ 18 & 54 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 16 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 34 & 72 \\ 18 & 54 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 42 & 81 \end{bmatrix} = \begin{bmatrix} 1+34+25 & 16+72+0 \\ 0+18+42 & 9+54+81 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 60 & 88 \\ 60 & 144 \end{bmatrix} \qquad \dots (2)$$

From (1) and (2) we have

$(A + B)^2 \neq A^2 + 2AB + B^2$

Example 1.6: Find the value of x if $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = O$

Solution: $[2x - 9 \quad 4x + 0] \begin{bmatrix} x \\ 3 \end{bmatrix} = O$ (Multiplying on first two matrices)

$$\Rightarrow \quad [(2x - 9)x + 4x(3)] = O \quad \Rightarrow \quad [2x^2 - 9x + 12x] = O$$

$$\Rightarrow \quad [2x^2 + 3x] = O$$

i.e. $2x^2 + 3x = 0 \Rightarrow x(2x + 3) = 0$

Hence we have $x = 0$, $\quad x = \dfrac{-3}{2}$

Example 1.7: Solve: $X + 2Y = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$; $X - Y = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

Solution: Given $X + 2Y = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$ … (1)

$$X - Y = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \quad \ldots (2)$$

(1) – (2) $\Rightarrow$ $(X + 2Y) - (X - Y) = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

$$3Y = \begin{bmatrix} 3 & 6 \\ -6 & 12 \end{bmatrix} \Rightarrow Y = \frac{1}{3}\begin{bmatrix} 3 & 6 \\ -6 & 12 \end{bmatrix}$$

$$\Rightarrow \quad Y = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$$

Substituting matrix Y in equation (1) we have

$$X + 2\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$$

$$\Rightarrow \quad X + \begin{bmatrix} 2 & 4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 4 & 6 \\ -8 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$$

EXERCISE 1.1

(1) Construct a 3×3 matrix whose elements are (i) $a_{ij} = i + j$ (ii) $a_{ij} = i \times j$

(2) Find the values of x, y, z if $\begin{bmatrix} x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$

(3) If $\begin{bmatrix} 2x & 3x - y \\ 2x + z & 3y - w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$ find x, y, z, w

(4) If $A = \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix}$ find each of the following

(i) $-2A + (B + C)$ (ii) $A - (3B - C)$ (iii) $A + (B + C)$ (iv) $(A + B) + C$
(v) $A + B$ (vi) $B + A$ (vii) AB (viii) BA

(5) Given $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 2 & 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

verify the following results:

(i) $AB \neq BA$ (ii) $(AB)C = A(BC)$ (iii) $A(B + C) = AB + AC$

(6) Solve : $2X + Y + \begin{bmatrix} -2 & 1 & 3 \\ 5 & -7 & 3 \\ 4 & 5 & 4 \end{bmatrix} = O$; $X - Y = \begin{bmatrix} 4 & 7 & 0 \\ -1 & 2 & -6 \\ -2 & 8 & -5 \end{bmatrix}$

(7) If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = O$ where I is the unit matrix of order 2.

(8) If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ find k so that $A^2 = kA - 2I$

(9) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A - 5I = O$

(10) Solve for x if $\begin{bmatrix} x^2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2x & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 7 \end{bmatrix}$

(11) Solve for x if $[x \quad 2 \quad -1] \begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = [0]$

(12) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$ verify the following:

(i) $(A + B)^2 = A^2 + AB + BA + B^2$ (ii) $(A - B)^2 \neq A^2 - 2AB + B^2$

(iii) $(A + B)^2 \neq A^2 + 2AB + B^2$ (iv) $(A - B)^2 = A^2 - AB - BA + B^2$

(v) $A^2 - B^2 \neq (A + B)(A - B)$

(13) Find matrix C if $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$ and $5C + 2B = A$

(14) If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, find x and y.

1.2 Determinants

1.2.1 Introduction:

The term determinant was first introduced by Gauss in 1801 while discussing quadratic forms. He used the term because the determinant determines the properties of the quadratic forms. We know that the area of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)\right] \qquad \ldots (1)$$

Similarly the condition for a second degree equation in x and y to represent a pair of straight lines is $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \qquad \ldots (2)$

To minimize the difficulty in remembering these type of expressions, Mathematicians developed the idea of representing the expression in determinant form.

The above expression (1) can be represented in the form $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Similarly the second expression (2) can be expressed as $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Again if we eliminate x, y, z from the three equations

$a_1x + b_1y + c_1 z = 0 \quad ; \quad a_2x + b_2y + c_2z = 0 \quad ; \quad a_3x + b_3y + c_3z = 0,$

we obtain $a_1(b_2\, c_3 - b_3\, c_2) - b_1\,(a_2\, c_3 - a_3\, c_2) + c_1\,(a_2\, b_3 - a_3\, b_2) = 0$

This can be written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. Thus a determinant is a particular type of expression written in a special concise form. Note that the quantities are arranged in the form of a square between two vertical lines. This arrangement is called a determinant.

Difference between a matrix and a determinant

(i) A matrix cannot be reduced to a number. That means a matrix is a structure alone and is not having any value. But a determinant can be reduced to a number.

(ii) The number of rows may not be equal to the number of columns in a matrix. In a determinant the number of rows is always equal to the number of columns.

(iii) On interchanging the rows and columns, a different matrix is formed. In a determinant interchanging the rows and columns does not alter the value of the determinant.

1.2.2 Definitions:

To every square matrix A of order n with entries as real or complex numbers, we can associate a number called determinant of matrix A and it is denoted by $|\,A\,|$ or $\det(A)$ or Δ.

Thus determinant formed by the elements of A is said to be the determinant of matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|\,A\,| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}a_{12}$

To evaluate the determinant of order 3 or above we define minors and cofactors.

Minors:

Let $|\,A\,| = \left|[a_{ij}]\right|$ be a determinant of order n. The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij}.

Cofactors:

The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j}\,M_{ij}$.

The minors and cofactors of a_{11}, a_{12}, a_{13} of a third order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ are as follows:

(i) Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}\,a_{23}$.

Cofactor of a_{11} is $A_{11} = (-1)^{1+1}\;M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}\,a_{23}$

(ii) Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\,a_{33} - a_{31}a_{23}$

Cofactor of a_{12} is $A_{12} = (-1)^{1+2}\,M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\,(a_{21}a_{33} - a_{23}\,a_{31})$

(iii) Minor of a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$

Cofactor of a_{13} is $A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{31}\, a_{22}$

Note: A determinant can be expanded using any row or column as given below:

Let $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Delta = a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}$ or $a_{11}\, M_{11} - a_{12}\, M_{12} + a_{13}\, M_{13}$ (expanding by R_1)

$\Delta = a_{11}\, A_{11} + a_{21} A_{21} + a_{31}\, A_{31}$ or $a_{11}\, M_{11} - a_{21}\, M_{21} + a_{31}\, M_{31}$ (expanding by C_1)

$\Delta = a_{21}\, A_{21} + a_{22}\, A_{22} + a_{23}\, A_{23}$ or $-\, a_{21}\, M_{21} + a_{22}\, M_{22} - a_{23}\, M_{23}$ (expanding by R_2)

Example 1.8:

Find the minor and cofactor of each element of the determinant $\begin{vmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{vmatrix}$

Solution: Minor of 3 is $M_{11} = \begin{vmatrix} -1 & 2 \\ -2 & 6 \end{vmatrix} = -6 + 4 = -2$

Minor of 4 is $M_{12} = \begin{vmatrix} 0 & 2 \\ 5 & 6 \end{vmatrix} = 0 - 10 = -10$

Minor of 1 is $M_{13} = \begin{vmatrix} 0 & -1 \\ 5 & -2 \end{vmatrix} = 0 + 5 = 5$

Minor of 0 is $M_{21} = \begin{vmatrix} 4 & 1 \\ -2 & 6 \end{vmatrix} = 24 + 2 = 26$

Minor of -1 is $M_{22} = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} = 18 - 5 = 13$

Minor of 2 is $M_{23} = \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = -6 - 20 = -26$

Minor of 5 is M_{31} = $\begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix}$ = 8 + 1 = 9

Minor of – 2 is M_{32} = $\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$ = 6 – 0 = 6

Minor of 6 is M_{33} = $\begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix}$ = – 3 – 0 = – 3

Cofactor of 3 is A_{11} = $(-1)^{1+1} M_{11} = M_{11} = -2$

Cofactor of 4 is A_{12} = $(-1)^{1+2} M_{12} = -M_{12} = 10$

Cofactor of 1 is A_{13} = $(-1)^{1+3} M_{13} = M_{13} = 5$

Cofactor of 0 is A_{21} = $(-1)^{2+1} M_{21} = -M_{21} = -26$

Cofactor of – 1 is A_{22} = $(-1)^{2+2} M_{22} = M_{22} = 13$

Cofactor of 2 is A_{23} = $(-1)^{2+3} M_{23} = -M_{23} = 26$

Cofactor of 5 is A_{31} = $(-1)^{3+1} M_{31} = M_{31} = 9$

Cofactor of – 2 is A_{32} = $(-1)^{3+2} M_{32} = -M_{32} = -6$

Cofactor of 6 is A_{33} = $(-1)^{3+3} M_{33} = M_{33} = -3$

Singular and non-singular matrices:

A square matrix A is said to be singular if | A | = 0

A square matrix A is said to be non-singular matrix, if | A | ≠ 0.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a singular matrix.

$$\because \ |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35)$$

$$= -3 + 12 - 9 = 0$$

$B = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 6 & 3 \\ 4 & 8 & 9 \end{bmatrix}$ is a non-singular matrix.

$$\because \ |B| = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 6 & 3 \\ 4 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 6 & 3 \\ 8 & 9 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} + 5\begin{vmatrix} 2 & 6 \\ 4 & 8 \end{vmatrix}$$

$= 1(54 - 24) - 7(18 - 12) + 5\,(16 - 24)$

$= 1(30) - 7(6) + 5(-8)$

$= -52 \neq 0$

$\therefore$ The matrix B is a non-singular matrix.

1.2.3 Properties of Determinants

There are many properties of determinants, which are very much useful in solving problems. The following properties are true for determinants of any order. But here we are going to prove the properties only for the determinant of order 3.

Property 1:

The value of a determinant is unaltered by interchanging its rows and columns.

Proof:

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Expanding Δ by the first row we get,

$$\Delta = a_1(b_2\,c_3 - b_3\,c_2) - b_1(a_2\,c_3 - a_3\,c_2) + c_1\,(a_2b_3 - a_3\,b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \qquad \ldots (1)$$

Let us interchange the rows and columns of Δ. Thus we get a new determinant.

$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Since determinant can be expanded by any row or any column we get

$$\Delta_1 = a_1(b_2c_3 - c_2b_3) - b_1\,(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \qquad \ldots (2)$$

From equations (1) and (2) we have $\Delta = \Delta_1$ Hence the result.

Property 2:

If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered.

Proof:

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1 (a_2b_3 - a_3 b_2)$$

$$\Delta = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 \qquad \ldots (1)$$

Let Δ_1 be the determinant obtained from Δ by interchanging the first and second rows. i.e. R_1 and R_2.

$$\Delta_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now we have to show that $\Delta_1 = -\Delta$.

Expanding Δ_1 by R_2, we have,

$$\Delta_1 = -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -[a_1b_2c_3 - a_1b_3c_2 + a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1] \qquad \ldots (2)$$

From (1) and (2) we get $\Delta_1 = -\Delta$.

Similarly we can prove the result by interchanging any two columns.

Corollary:

The sign of a determinant changes or does not change according as there is an odd or even number of interchanges among its rows (columns).

Property 3:

If two rows (columns) of a determinant are identical then the value of the determinant is zero.

Proof:

Let Δ be the value of the determinant. Assume that the first two rows are identical. By interchanging R_1 and R_2 we obtain $-\Delta$ (by property2). Since R_1 and R_2 are identical even after the interchange we get the same Δ.

i.e. $\Delta = -\Delta \Rightarrow 2\Delta = 0$ i.e. $\Delta = 0$

Property 4:

If every element in a row (or column) of a determinant is multiplied by a constant "k" then the value of the determinant is multiplied by k.

Proof:

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let Δ_1 be the determinant obtained by multiplying the elements of the first row by 'k' then $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Expanding along R_1 we get,

$$\Delta_1 = ka_1 (b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$$

$$= k[a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1]$$

$$\Delta_1 = k\Delta.$$ Hence the result.

Note:

(1) Let A be any square matrix of order n. Then kA is also a square matrix which is obtained by multiplying every entry of the matrix A with the scalar k. But the determinant k |A| means every entry in a row (or a column) is multiplied by the scalar k.

(2) Let A be any square matrix of order n then $|kA| = k^n|A|$.

Deduction from properties (3) and (4)

If two rows (columns) of a determinant are proportional i.e. one row (column) is a scalar multiple of other row (column) then its value is zero.

Property 5:

If every element in any row (column) can be expressed as the sum of two quantities then given determinant can be expressed as the sum of two determinants of the same order with the elements of the remaining rows (columns) of both being the same.

Proof: Let $\Delta = \begin{vmatrix} \alpha_1 + x_1 & \beta_1 + y_1 & \gamma_1 + z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding Δ along the first row, we get

$$\Delta = (\alpha_1 + x_1)\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - (\beta_1 + y_1)\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (\gamma_1 + z_1)\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \left\{\alpha_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \beta_1\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \gamma_1\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right\}$$

$$+ \left\{x_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - y_1\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + z_1\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right\}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence the result.

Note: If we wish to add (or merge) two determinants of the same order we add corresponding entries of a particular row (column) provided the other entries in rows (columns) are the same.

Property 6:

A determinant is unaltered when to each element of any row (column) is added to those of several other rows (columns) multiplied respectively by constant factors.

i.e. A determinant is unaltered when to each element of any row (column) is added by the equimultiples of any parallel row (column).

Proof:

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let Δ_1 be a determinant obtained when to the elements of C_1 of Δ are added to those of second column and third column multiplied respectively by l and m.

$$\Delta_1 = \begin{vmatrix} a_1 + lb_1 + mc_1 & b_1 & c_1 \\ a_2 + lb_2 + mc_2 & b_2 & c_2 \\ a_3 + lb_3 + nc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} lb_1 & b_1 & c_1 \\ lb_2 & b_2 & c_2 \\ lb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mc_1 & b_1 & c_1 \\ mc_2 & b_2 & c_2 \\ mc_3 & b_3 & c_3 \end{vmatrix} \quad \text{(by property 5)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 + 0 \quad \left[\because \begin{array}{l} C_1 \text{ is proportional to } C_2 \text{ in the second det.} \\ C_1 \text{ is proportional to } C_3 \text{ in the third det.} \end{array}\right]$$

Therefore $\Delta_1 = \Delta$. Hence the result.

Note:

(1) Multiplying or dividing all entries of any one row (column) by the same scalar is equivalent to multiplying or dividing the determinant by the same scalar.

(2) If all the entries above or below the principal diagonal are zero (upper triangular, lower triangular) then the value of the determinant is equal to the product of the entries of the principal diagonal.

For example, let us consider

$$|A| = \begin{vmatrix} 3 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 3(5-0) - 2(0-0) + 7(0-0) = 15$$

The value of the determinant A is 15.

The product of the entries of the principal diagonal is $3 \times 5 \times 1 = 15$.

Example 1.9: Solve $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

Solution: Since all the entries below the principal diagonal are zero, the value of the determinant is $(x-1)(x-2)(x-3)$

$\therefore\ (x-1)(x-2)(x-3) = 0 \quad \Rightarrow \quad x = 1,\ \ x = 2,\ \ x = 3$

Example 1.10: Solve for x if $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

Solution : $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad (x^2 - 35) + (1 - 2) = 0 \ \Rightarrow\ x^2 - 35 - 1 = 0 \ \Rightarrow\ x^2 - 36 = 0$

$\Rightarrow \quad x^2 = 36 \quad \Rightarrow \quad x = \pm 6$

Example 1.11: Solve for x if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$

Solution:

$(0)\begin{vmatrix} 2 & x \\ 3 & x \end{vmatrix} - 1\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} + (0)\begin{vmatrix} x & 2 \\ 1 & 3 \end{vmatrix} = 0 \qquad \Rightarrow \qquad 0 - 1[x^2 - x] + 0 = 0$

$-x^2 + x = 0$ i.e. $x(1-x) = 0 \quad \Rightarrow \quad x = 0,\ x = 1$

Example 1.12: Evaluate (i) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (ii) $\begin{vmatrix} x+2a & x+3a & x+4a \\ x+3a & x+4a & x+5a \\ x+4a & x+5a & x+6a \end{vmatrix}$

Solution:

(i) Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} \quad C_3 \to C_3 + C_2$

$= 0$ [$\because$ C_1 is proportional to C_3]

(ii) Let $\Delta = \begin{vmatrix} x+2a & x+3a & x+4a \\ x+3a & x+4a & x+5a \\ x+4a & x+5a & x+6a \end{vmatrix} = \begin{vmatrix} x+2a & a & 2a \\ x+3a & a & 2a \\ x+4a & a & 2a \end{vmatrix}$ $\begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$

$= 0$ [$\because$ C_2 is proportional to C_3]

Example 1.13: Prove that $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix} = 0$

Solution: $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix} = \begin{vmatrix} 2x & x & y \\ 2y & y & z \\ 2z & z & x \end{vmatrix} + \begin{vmatrix} y & x & y \\ z & y & z \\ x & z & x \end{vmatrix}$

$= 0 + 0$ $\left[\because \begin{matrix} C_1 \text{ is proportional to } C_2 \text{ in the first det.} \\ C_1 \text{ is identical to } C_3 \text{ in the second det.} \end{matrix}\right]$

$= 0$

Example 1.14: Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$ $\begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$

$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$ Take $(a-b)$ and $(b-c)$ from R_1 and R_2 respectively.

$= (a-b)(b-c)\left[(1)(b+c) - (1)(a+b)\right] = (a-b)(b-c)(c-a)$

Example 1.15: Prove that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Solution: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix}$ $\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$= xy$ [$\because$ upper diagonal matrix]

Example 1.16: Prove that $\begin{vmatrix} 1/a^2 & bc & b+c \\ 1/b^2 & ca & c+a \\ 1/c^2 & ab & a+b \end{vmatrix} = 0$

$$\begin{vmatrix} 1/a^2 & bc & b+c \\ 1/b^2 & ca & c+a \\ 1/c^2 & ab & a+b \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1/a & abc & a(b+c) \\ 1/b & abc & b(c+a) \\ 1/c & abc & c(a+b) \end{vmatrix} \quad \text{Multiply } R_1, R_2, R_3 \text{ by } a, b, c \text{ respectively}$$

$$= \frac{abc}{abc}\begin{vmatrix} 1/a & 1 & a(b+c) \\ 1/b & 1 & b(c+a) \\ 1/c & 1 & c(a+b) \end{vmatrix} \quad \text{Take } abc \text{ from } C_2$$

$$= \frac{1}{abc}\begin{vmatrix} bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \\ ab & 1 & c(a+b) \end{vmatrix} \quad \text{Multiply } C_1 \text{ by } abc$$

$$= \frac{1}{abc}\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \quad C_3 \to C_3 + C_1$$

$$= \frac{(ab+bc+ca)}{abc}\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} \quad \text{Take } (ab+bc+ca) \text{ from } C_3$$

$$= \frac{(ab+bc+ca)}{abc}(0) \quad [\because C_2 \text{ is identical to } C_3]$$

$$= 0$$

Example 1.17: Prove that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

Solution: Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

Multiply R_1, R_2 and R_3 by a, b and c respectively

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ab \\ ca^2b^2 & abc & ca+bc \end{vmatrix}$$

$$= \frac{(abc)^2}{abc} \begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ca+bc \end{vmatrix} \quad \text{Take } abc \text{ from } C_1 \text{ and } C_2$$

$$= abc \begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \quad C_3 \to C_3 + C_1$$

$$= abc\,(ab+bc+ca) \begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} \quad \text{Take } (ab+bc+ca) \text{ from } C_3$$

$$= abc\,(ab+bc+ca)\ (0) \qquad [\because C_2 \text{ is identical to } C_3]$$

$$= 0$$

Example 1.18 : Prove that $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)\,(b+c)\,(c+a)$

Solution:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = \begin{vmatrix} a+b & a+b & -(a+b) \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix} \quad \begin{matrix} R_1 \to R_1 + R_2 \\ R_2 \to R_2 + R_3 \end{matrix}$$

$$= (a+b)\,(b+c) \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} \quad \begin{matrix} \text{Take } (a+b), (b+c) \\ \text{from } R_1 \text{ and } R_2 \\ \text{respectively} \end{matrix}$$

$$= (a+b)\,(b+c) \begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} \quad R_1 \to R_1 + R_2$$

$$= (a+b)\,(b+c) \times (-2) \begin{vmatrix} -1 & 1 \\ -b & a+b+c \end{vmatrix}$$

$$= (a+b)\,(b+c) \times (-2)\,\left[-(a+b+c)+b\right]$$

$$= (a+b)\,(b+c) \times (-2)\,[-a-c]$$

$$\Delta = 2(a+b)\,(b+c)\,(c+a)$$

Example 1.19: Prove that $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix} = \lambda^2\,(a^2+b^2+c^2+\lambda)$

Solution: Let $\Delta = \begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$

Multiply R_1, R_2 and R_3,by a, b and c respectively

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(a^2+\lambda) & a^2b & a^2c \\ ab^2 & b(b^2+\lambda) & b^2c \\ ac^2 & bc^2 & c(c^2+\lambda) \end{vmatrix}$$

Take a, b and c from C_1, C_2 and C_3 respectively

$$\Delta = \frac{abc}{abc} \begin{vmatrix} a^2+\lambda & a^2 & a^2 \\ b^2 & b^2+\lambda & b^2 \\ c^2 & c^2 & c^2+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2+\lambda & a^2+b^2+c^2+\lambda & a^2+b^2+c^2+\lambda \\ b^2 & b^2+\lambda & b^2 \\ c^2 & c^2 & c^2+\lambda \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$= (a^2+b^2+c^2+\lambda) \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+\lambda & b^2 \\ c^2 & c^2 & c^2+\lambda \end{vmatrix}$$

$$= (a^2+b^2+c^2+\lambda) \begin{vmatrix} 1 & 0 & 0 \\ b^2 & \lambda & 0 \\ c^2 & 0 & \lambda \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a^2+b^2+c^2+\lambda) \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}$$

$$\therefore \begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix} = \lambda^2(a^2+b^2+c^2+\lambda)$$

EXERCISE 1.2

(1) Find the value of the determinant $\begin{vmatrix} 2 & 6 & 4 \\ -5 & -15 & -10 \\ 1 & 3 & 2 \end{vmatrix}$ without usual expansion.

(2) Identify the singular and non-singular matrix

(i) $\begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{bmatrix}$

(3) Solve (i) $\begin{vmatrix} 2 & x & 4 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -3$ (ii) $\begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 4 & 4 & x \end{vmatrix} = -1$

(4) Evaluate (i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & ab & c(a+b) \\ 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \end{vmatrix}$

(5) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(6) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

where a, b, c are non zero real numbers and hence evaluate the

value of $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$

(7) Prove that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(8) If x, y, z are all different and $\begin{vmatrix} x & x^2 & 1-x^3 \\ y & y^2 & 1-y^3 \\ z & z^2 & 1-z^3 \end{vmatrix} = 0$

then show that $xyz = 1$

(9) Prove that (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

(ii) $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)\ (x-z)^2$

(10) Prove that

(i) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ (ii) $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(iii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

(iv) $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

1.2.4 Factor method

Application of Remainder theorem to determinants

Theorem:

If each element of a determinant (Δ) is polynomial in x and if Δ vanishes for $x = a$ then $(x - a)$ is a factor of Δ.

Proof:

Since the elements of Δ are polynomial in x, on expansion Δ will be a polynomial function in x. (say $p(x)$). For $x = a$, $\Delta = 0$

i.e. $p(x) = 0$ when $x = a$, i.e. $p(a) = 0$

$\therefore$ By Remainder theorem $(x - a)$ is a factor of $p(x)$.

i.e. $(x - a)$ is a factor of Δ.

Note:

(1) This theorem is very much useful when we have to obtain the value of the determinant in 'factors' form. Thus, for example if on putting $a = b$ in the determinant Δ any two of its rows or columns become identical then $\Delta = 0$ and hence by the above theorem $a - b$ will be a factor of Δ.

(2) If r rows (column) are identical in a determinant of order n ($n \geq r$) when we put $x = a$, then $(x - a)^{r-1}$ is a factor of Δ.

(3) $(x + a)$ is a factor of the polynomial $f(x)$ if and only if $x = -a$ is a root of the equation $f(x) = 0$.

Remark: In this section we deal certain problems with symmetric and cyclic properties.

Example 1.20: Prove that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)\,(b-c)\,(c-a)\,(a+b+c)$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$. Put $a = b$, $\Delta = \begin{vmatrix} 1 & b & b^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$ [$\because$ R_1 is identical to R_2]

$\therefore$ $(a-b)$ is a factor of Δ.

Similarly we observe that Δ is symmetric in a, b, c, by putting $b = c$, $c = a$, we get $\Delta = 0$. Hence $(b-c)$ and $(c-a)$ are also factors of Δ. $\therefore$ The product $(a-b)\,(b-c)\,(c-a)$ is a factor of Δ. The degree of this product is 3. The product of leading diagonal elements is $1 . b . c^3$. The degree of this product is 4.

$\therefore$ By cyclic and symmetric properties, the remaining symmetric factor of first degree must be $k(a+b+c)$, where k is any non-zero constant.

Thus $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)\,(b-c)\,(c-a)\,k(a+b+c)$

To find the value of k, give suitable values for a, b, c so that both sides do not become zero. Take $a = 0$, $b = 1$, $c = 2$.

$\therefore$ $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 8 \end{vmatrix} = k(3)\,(-1)\,(-1)\,(2) \Rightarrow k = 1$

$\therefore$ $\Delta = (a-b)\,(b-c)\,(c-a)\,(a+b+c)$

Note: An important note regarding the remaining symmetric factor in the factorisation of cyclic and symmetric expression in a, b and c

If m is the difference between the degree of the product of the factors (found by the substitution) and the degree of the product of the leading diagonal elements and if

(1) m is zero then the other symmetric factor is a constant (k)

(2) m is one then the other symmetric factor of degree 1 is $k(a+b+c)$

(3) m is two then the other symmetric factor of degree 2 is $k(a^2+b^2+c^2)+l\,(ab+bc+ca)$

Example 1.21:

Prove by factor method $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ Put $a = b$ $\quad \Delta = \begin{vmatrix} 1 & b^2 & b^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = 0 \quad [\because R_1 \equiv R_2]$

$\therefore$ $(a-b)$ is a factor of Δ.

By symmetry on putting $b = c$ and $c = a$ we can easily show that Δ becomes zero and therefore $(b-c)$ and $(c-a)$ are also factors of Δ.

This means the product $(a-b)(b-c)(c-a)$ is a factor of Δ. The degree of this product is 3. The degree of the product of leading diagonal elements b^2c^3 is 5.

$\therefore$ The other factor is $k(a^2+b^2+c^2)+l(ab+bc+ca)$

$$\therefore \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \left[k(a^2+b^2+c^2)+l(ab+bc+ca)\right](a-b)(b-c)(c-a)$$

To determine k and l give suitable values for a, b and c so that both sides do not become zero. Take $a = 0$, $b = 1$ and $c = 2$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = [k(5)+l(2)](-1)(-1)(2)$$

$\Rightarrow 4 = (5k+2l)\,2 \quad \Rightarrow \quad 5k+2l = 2$ … (1)

Again put $a = 0$, $b = -1$ and $c = 1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2)+l(-1)](+1)(-2)(1)$$

$\Rightarrow 2 = (2k-l)(-2) \;\Rightarrow\; 2k-l = -1$ … (2)

On solving (1) and (2) we get $k = 0$ and $l = 1$

$\therefore\ \Delta = (ab + bc + ca)\,(a - b)\,(b - c)\,(c - a)$

$= (a - b)\,(b - c)\,(c - a)\,(ab + bc + ca)$

Example 1.22: Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a + b + c)^3$

Solution:

Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ Put $a = 0$ we get

$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ b^2 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0$ [$\because$ C_2 is porportional to C_3]

$\therefore\ (a - 0) = a$ is a factor of Δ.

Similarly on putting $b = 0$, $c = 0$, we see that the value of Δ is zero.

$\therefore\ a, b, c$ are factors of Δ. Put $a + b + c = 0$, we have

$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = 0$

Since three columns are identical, $(a + b + c)^2$ is a factor of Δ.

$\therefore\ abc\,(a + b + c)^2$ is a factor of Δ and is of degree 5. The product of the leading diagonal elements $(b + c)^2\,(c + a)^2\,(a + b)^2$ is of degree 6.

$\therefore$ The other factor of Δ must be $k(a + b + c)$.

$\therefore\ \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\,abc\,(a + b + c)^3$

Take the values $a = 1$, $b = 1$ and $c = 1$

$\therefore\ \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1)\,(1)\,(1)\,(3)^3 \Rightarrow 54 = 27k \Rightarrow k = 2$

$\therefore\ \Delta = 2abc\,(a + b + c)^3$

Example 1.23: Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2 (x+2a)$

Solution:

Let $\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$ Put $x = a$ $\therefore$ $\Delta = \begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix} = 0$

Since all the three rows are identical $(x-a)^2$ is a factor of Δ.

Put $x = -2a$.

$$\Delta = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix} = \begin{vmatrix} 0 & a & a \\ 0 & -2a & a \\ 0 & a & -2a \end{vmatrix} = 0 \quad [C_1 \to C_1 + C_2 + C_3]$$

$(x+2a)$ is a factor of Δ.

$\therefore$ $(x-a)^2 (x+2a)$ is a factor of Δ and is of degree 3. The degree of the product of leading diagonal element is also 3. Therefore the other factor must be k.

$$\therefore \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = k(x-a)^2 (x+2a).$$

Equate x^3 term on both sides, $1 = k$ $\therefore$ $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2 (x+2a)$

Example 1.24: Using factor method, prove $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2 (x+9)$

Solution: Let $\Delta = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix}$

Put $x = 1$, $\Delta = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$

Since all the three rows are identical, $(x-1)^2$ is a factor of Δ.

Put $x = -9$ in Δ, then $\Delta = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0$ $[\because C_1 \to C_1 + C_2 + C_3]$

$\therefore$ $(x + 9)$ is a factor of Δ.

The product $(x - 1)^2 (x + 9)$ is a factor of Δ and is of degree 3. The degree of the product of leading diagonal elements $(x + 1)(x + 2)(x + 4)$ is also 3.

$\therefore$ The remaining factor must be a constant "k"

$\therefore$ $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k(x - 1)^2 (x + 9)$. Equating x^3 term on both sides we get $k = 1$

Thus $\Delta = (x - 1)^2 (x + 9)$

EXERCISE 1.3

(1) Using factor method show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

(2) Prove by factor method $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

(3) Solve using factor method $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$

(4) Factorise $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

(5) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a + b + c)(a - b)(b - c)(c - a)$

1.2.5 Product of determinants

Rule for multiplication of two determinants is the same as the rule for multiplication of two matrices.

While multiplying two matrices "row-by-column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant. Therefore we can also adopt the following procedures for multiplication of two determinants.

(1) Row-by-row multiplication rule
(2) Column-by-column multiplication rule
(3) Column-by-row multiplication rule

Note: The determinant of the product matrix is equal to the product of the individual determinant values of the square matrices of same order.

i.e. Let A and B be two square matrices of the same order.

We have | AB | = | A | | B |

This statement is verified in the following example.

Example 1.25: If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ are two square matrices then show that | AB | = | A | | B |

Solution:

Given that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \ldots (1)$$

$$|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$|A|\ |B| = 1 \times 1 = 1 \qquad \ldots (2)$$

From (1) and (2) | AB | = | A | | B |

Example 1.26: Show that $\begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2 = \begin{vmatrix} b^2 + c^2 & ab & ac \\ ab & c^2 + a^2 & bc \\ ac & bc & a^2 + b^2 \end{vmatrix}$

Solution: L.H.S. $= \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2 = \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix} \begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}$

$$= \begin{vmatrix} o + c^2 + b^2 & o + o + ab & o + ac + o \\ o + o + ab & c^2 + o + a^2 & bc + o + o \\ o + ac + o & bc + o + o & b^2 + a^2 + 0 \end{vmatrix}$$

$$= \begin{vmatrix} c^2+b^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & b^2+a^2 \end{vmatrix} = \text{R.H.S.}$$

Example 1.27: Prove that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1{}^2+a_2{}^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1{}^2+b_2{}^2 \end{vmatrix}$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \left(\begin{array}{l}\text{Interchange rows and} \\ \text{columns of the first determinant}\end{array}\right)$$

$$= \begin{vmatrix} a_1{}^2+a_2{}^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1{}^2+b_2{}^2 \end{vmatrix}$$

Example 1.28: Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

Solution:

$$\text{R.H.S.} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}; \begin{array}{l}\text{Interchanging } R_2 \text{ and } R_3 \\ \text{in the } 2^{\text{nd}} \text{ determinant}\end{array}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ba \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \text{L.H.S.}$$

1.2.6 Relation between a determinant and its co-factor determinant

Consider $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let $A_1, B_1, C_1 \ldots \ldots$ be the co-factors of $a_1, b_1, c_1 \ldots \ldots$ in Δ

$\therefore$ The cofactor determinant is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Let Δ be expanded by R_1 $\therefore \Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$\Rightarrow \quad \Delta = a_1$ (co-factor of a_1) + b_1 (co-factor of b_1) + c_1 (co-factor of c_1)

$\Rightarrow \quad \Delta = a_1A_1 + b_1 B_1 + c_1 C_1$

i.e. **The sum of the products of the elements of any row of a determinant with the corresponding row of co-factor determinant is equal to the value of the determinant.**

Similarly $\Delta = a_2A_2 + b_2B_2 + c_2C_2 \qquad \Delta = a_3A_3 + b_3B_3 + c_3C_3$

Now let us consider the sum of the product of first row elements with the corresponding second row elements of co-factor determinant i.e. let us consider the expression

$a_1A_2 + b_1B_2 + c_1C_2$

$$= -a_1 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= 0$$

$\therefore$ The expression $a_1A_2 + b_1B_2 + c_1C_2 = 0$

Thus we have

$a_1A_3 + b_1B_3 + c_1C_3 = 0$; $a_2A_1 + b_2B_1 + c_2C_1 = 0$; $a_2A_3 + b_2B_3 + c_2C_3 = 0$

$a_3A_1 + b_3B_1 + c_3C_1 = 0$; $a_3A_2 + b_3B_2 + c_3C_2 = 0$

i.e. The sum of the products of the elements of any row of a determinant with any other row of co-factor determinant is equal to 0

Note: Instead of rows, if we take columns we get the same results.

$\therefore \Delta = a_1A_1 + a_2A_2 + a_3A_3$

$\Delta = b_1B_1 + b_2B_2 + b_3B_3$

$\Delta = c_1C_1 + c_2C_2 + c_3C_3$

Thus the above results can be put in a tabular column as shown below.

Row-wise

	R_1	R_2	R_3
r_1	Δ	0	0
r_2	0	Δ	0
r_3	0	0	Δ

Column-wise

	C_1	C_2	C_3
c_1	Δ	0	0
c_2	0	Δ	0
c_3	0	0	Δ

Where r_i's c_i's are i^{th} row and i^{th} column of the original determinant R_i's, C_i's are i^{th} row and i^{th} column respectively of the corresponding co-factor determinant.

Example 1.29: If A_1, B_1, C_1 are the co-factors of a_1, b_1, c_1 in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$

Solution: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_1C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

i.e. $\Delta \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \quad \Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$

EXERCISE 1.4

(1) Show that $\begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2a^2 & -a^2 & -a^2 \\ -a^2 & -1 & a^2-2a \\ -a^2 & a^2-2a & -1 \end{vmatrix}$

(2) Show that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & 1 & 2a \\ b^2 & 1 & 2b \\ c^2 & 1 & 2c \end{vmatrix} = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

2. VECTOR ALGEBRA

2.1 Introduction:

The development of the concept of vectors was influenced by the works of the German Mathematician H.G. Grassmann (1809 – 1877) and the Irish mathematician W.R. Hamilton (1805 – 1865). It is interesting to note that both were linguists, being specialists in Sanskrit literature. While Hamilton occupied high positions, Grassman was a secondary school teacher.

The best features of Quaternion Calculus and Cartesian Geometry were united, largely through the efforts of the American Mathematician J.B. Gibbs (1839 – 1903) and Q. Heariside (1850 – 1925) of England and new subject called Vector Algebra was created. The term vectors was due to Hamilton and it was derived from the Latin word ‘to carry’. The theory of vectors was also based on Grassman’s theory of extension.

It was soon realised that vectors would be the ideal tools for the fruitful study of many ideas in geometry and physics. Vector algebra is widely used in the study of certain type of problems in Geometry, Mechanics, Engineering and other branches of Applied Mathematics.

Physical quantities are divided into two categories – scalar quantities and vector quantities.

Definitions:

Scalar : A quantity having only magnitude is called a scalar. It is not related to any fixed direction in space.

Examples : mass, volume, density, work, temperature, distance, area, real numbers etc.

To represent a scalar quantity, we assign a real number to it, which gives its magnitude in terms of a certain basic unit of a quantity. Throughout this chapter, by scalars we shall mean real numbers. Normally, scalars are denoted by a, b, c…

Vector : A quantity having both magnitude and direction is called a vector.

Examples : displacement, velocity, acceleration, momentum, force, moment of a force, weight etc.

Representation of vectors:

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end denotes the direction of the vector.

A vector denoted by $\vec{a} = \overrightarrow{AB}$ is determined by two points A, B such that the magnitude of the vector is the length of the

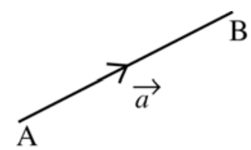


Fig. 2. 1

line segment AB and its direction is that from A to B. The point A is called initial point of the vector $\overrightarrow{AB}$ and B is called the terminal point. Vectors are generally denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ … (read as vector *a*, vector *b*, vector *c*, …)

Magnitude of a vector

The modulus or magnitude of a vector $\vec{a} = \overrightarrow{AB}$ is a positive number which is a measure of its length and is denoted by $\left|\vec{a}\right| = \left|\overrightarrow{AB}\right| = AB$ The modulus of $\vec{a}$ is also written as ***'a'***

$$\text{Thus } \left|\vec{a}\right| = a \ ; \quad \left|\vec{b}\right| = b \ ; \quad \left|\vec{c}\right| = c$$

$$\left|\overrightarrow{AB}\right| = AB \ ; \quad \left|\overrightarrow{CD}\right| = CD \ ; \quad \left|\overrightarrow{PQ}\right| = PQ$$

Caution: The two end points A and B are not interchangeable.

Note: Every vector $\overrightarrow{AB}$ has three characteristics:

Length : The length of $\overrightarrow{AB}$ will be denoted by $\left|\overrightarrow{AB}\right|$ or AB.

Support : The line of unlimited length of which AB is a segment is called the support of the vector $\overrightarrow{AB}$,

Sense : The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ is from B to A. Thus the sense of a directed line segment is from its initial point to the terminal point.

Equality of vectors:

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have the

(i) same magnitude (ii) same direction

In fig (2.2) AB || CD and AB = CD

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are in the same direction

Hence $\overrightarrow{AB} = \overrightarrow{CD}$ or $\vec{a} = \vec{b}$

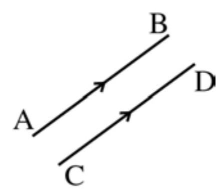


Fig. 2. 2

2.2 Types of Vectors

Zero or Null Vector:

A vector whose initial and terminal points are coincident is called a zero or null or a void vector. The zero vector is denoted by $\vec{O}$

Vectors other than the null vector are called proper vectors.

Unit vector:

A vector whose modulus is unity, is called a unit vector.

The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ (read as '*a* cap'). Thus $|\hat{a}| = 1$

The unit vectors parallel to $\vec{a}$ are $\pm \hat{a}$

Result: $\vec{a} = |\vec{a}|\ \hat{a}$ [i.e. any vector = (its modulus) × (unit vector in that direction)]

$$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|} \ ; \ \left(\vec{a} \neq \vec{O}\right)$$

In general $\boxed{\text{unit vector in any direction} = \dfrac{\text{vector in that direction}}{\text{modulus of the vector}}}$

Like and unlike vectors:

Vectors are said to be **like** when they have the same sense of direction and **unlike** when they have opposite directions.

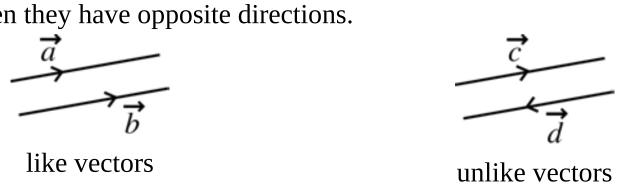


Fig. 2. 3

Co-initial vectors:

Vectors having the same initial point are called co-initial vectors.

Co-terminus vectors:

Vectors having the same terminal point are called co-terminus vectors.

Collinear or Parallel vectors:

Vectors are said to be collinear or parallel if they have the same line of action or have the lines of action parallel to one another.

Coplanar vectors:

Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.

Negative vector:

The vector which has the same magnitude as that of $\vec{a}$ but opposite direction is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus if $\overrightarrow{AB} = \vec{a}$ then $\overrightarrow{BA} = -\vec{a}$.

Reciprocal of a vector:

Let $\vec{a}$ be a non-zero vector. The vector which has the same direction as that of $\vec{a}$ but has magnitude reciprocal to that of $\vec{a}$ is called the reciprocal of $\vec{a}$ and is written as $\left(\vec{a}\right)^{-1}$ where $\left|\left(\vec{a}\right)^{-1}\right| = \dfrac{1}{a}$

Free and localised vector:

When we are at liberty to choose the origin of the vector at any point, then it is said to be a free vector. But when it is restricted to a certain specified point, then the vector is said to be localised vector.

2.3 Operations on vectors:

2.3.1 Addition of vectors:

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$ Join OB.

Then $\overrightarrow{OB}$ represents the addition (sum) of the vectors $\vec{a}$ and $\vec{b}$.

This is written as $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

Thus $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

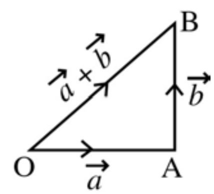


Fig. 2. 4

This is known as the triangle law of addition of vectors which states that, if two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order.

Applying the triangle law of addition of vectors in ΔABC, we have

$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

$$\Rightarrow \quad \overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$$

$$\Rightarrow \quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

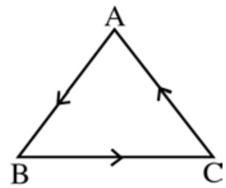


Fig. 2. 5

Thus the sum of the vectors representing the sides of a triangle taken in order is the null vector.

Parallelogram law of addition of vectors:

If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram which is co-initial with the given vectors.

Symbolically we have $\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$

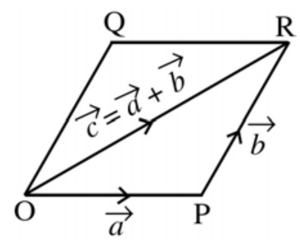


Fig. 2. 6

Thus if the vectors are represented by two adjacent sides of a parallelogram, the diagonal of the parallelogram will represent the sum of the vectors.

By repeated use of the triangle law we can find the sum of any number of vectors.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{c}$, $\overrightarrow{CD} = \vec{d}$, $\overrightarrow{DE} = \vec{e}$ be any five vectors as shown in the fig (2.7). We observe from the figure that each new vector is drawn from the terminal point of its previous one.

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$$

Thus the line joining the initial point of the first vector to the terminal point of the last vector is the sum of all the vectors. This is called the polygon law of addition of vectors.

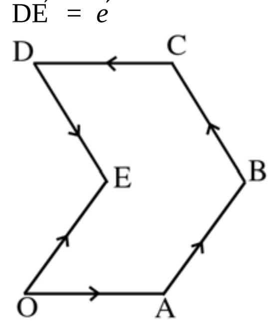


Fig. 2. 7

Note : It should be noted that the magnitude of $\vec{a} + \vec{b}$ is not equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$.

2.3.2 Subtraction of vectors:

If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and is denoted by $\vec{a} - \vec{b}$.

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right)$$

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

Then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

To subtract $\vec{b}$ from $\vec{a}$, produce BA to B′ such that AB = AB′. $\therefore$ $\overrightarrow{AB'} = -\overrightarrow{AB} = -\vec{b}$

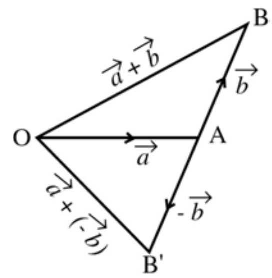


Fig. 2. 8

Now by the triangle law of addition

$$\overrightarrow{OB'} = \overrightarrow{OA} + \overrightarrow{AB'} = \vec{a} + \left(-\vec{b}\right) = \vec{a} - \vec{b}$$

Properties of addition of vectors:

Theorem 2.1:

Vector addition is commutative i.e., if $\vec{a}$ and $\vec{b}$ are any two vectors then $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$

In ΔOAB, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

(by triangle law of add.)

$\Rightarrow$ $\vec{a} + \vec{b} = \overrightarrow{OB}$ … (1)

Complete the parallelogram OABC

$\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$; $\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$

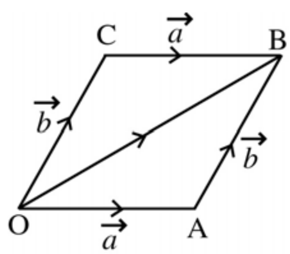


Fig. 2. 9

In ΔOCB, we have $\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$ i.e. $\Rightarrow$ $\vec{b} + \vec{a} = \overrightarrow{OB}$ … (2)

From (1) and (2) we have $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$\therefore$ Vector addition is commutative.

Theorem 2.2:

Vector addition is associative

i.e. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$$\left(\vec{a}+\vec{b}\right)+\vec{c} = \vec{a}+\left(\vec{b}+\vec{c}\right)$$

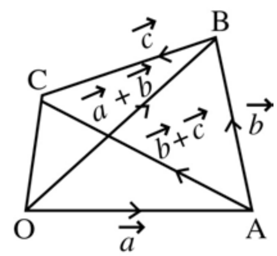


Fig. 2. 10

Proof :

Let $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{AB} = \vec{b}$; $\overrightarrow{BC} = \vec{c}$

Join O and B ; O and C ; A and C

In ΔOAB, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\Rightarrow$ $\vec{a}+\vec{b} = \overrightarrow{OB}$ … (1)

In ΔOBC, $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$

$\Rightarrow$ $\left(\vec{a}+\vec{b}\right)+\vec{c} = \overrightarrow{OC}$ … (2) [using (1)]

In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow$ $\vec{b}+\vec{c} = \overrightarrow{AC}$ … (3)

In ΔOAC $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

$\Rightarrow$ $\vec{a}+\left(\vec{b}+\vec{c}\right) = \overrightarrow{OC}$ … (4) [using (3)]

From (2) and (4), we have $\left(\vec{a}+\vec{b}\right)+\vec{c} = \vec{a}+\left(\vec{b}+\vec{c}\right)$

$\therefore$ vector addition is associative.

Theorem 2.3:

For every vector $\vec{a}$, $\vec{a}+\vec{O} = \vec{O}+\vec{a} = \vec{a}$ where $\vec{O}$ is the null vector. [existence of additive identity]

Proof:

Let $\overrightarrow{OA} = \vec{a}$

Then $\vec{a}+\vec{O} = \overrightarrow{OA} + \overrightarrow{AA} = \overrightarrow{OA} = \vec{a}$

$\therefore \vec{a}+\vec{O} = \vec{a}$

Also $\vec{O}+\vec{a} = \overrightarrow{OO} + \overrightarrow{OA} = \overrightarrow{OA} = \vec{a}$

$$\therefore\ \vec{O} + \vec{a} = \vec{a}$$

$$\therefore\ \vec{a} + \vec{O} = \vec{O} + \vec{a} = \vec{a}$$

Theorem 2.4:

For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that

$\vec{a} + \left(-\vec{a}\right) = \vec{O} = \left(-\vec{a}\right) + \vec{a}$ [existence of additive inverse]

Proof: Let $\overrightarrow{OA} = \vec{a}$. Then $\overrightarrow{AO} = -\vec{a}$

$$\therefore\ \vec{a} + \left(-\vec{a}\right) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO} = \vec{O}$$

$$\left(-\vec{a}\right) + \vec{a} = \overrightarrow{AO} + \overrightarrow{OA} = \overrightarrow{AA} = \vec{O}$$

Hence $\vec{a} + \left(-\vec{a}\right) = \left(-\vec{a}\right) + \vec{a} = \vec{O}$

2.3.3 Multiplication of a vector by a scalar

Let m be a scalar and $\vec{a}$ be any vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is $|\, m \,|$ times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as m is positive or negative.

Result : Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if and only if $\vec{a} = m\vec{b}$ for some non-zero scalar m.

For any vector $\vec{a}$ we define the following:

(1) $\vec{a} = \vec{a}$; $(-1)\,\vec{a} = -\vec{a}$; $0\vec{a} = \vec{O}$

Note: If $\vec{a}$ is a vector then $5\vec{a}$ is a vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is same as that of $\vec{a}$. But $-5\vec{a}$ is a vector whose magnitude is 5 times the magnitude of $\vec{a}$ and whose direction is opposite to $\vec{a}$.

Properties of Multiplication of vectors by a scalar

The following are properties of multiplication of vectors by scalars.

For vectors $\vec{a}$, $\vec{b}$ and scalars m, n we have

(i) $m\left(-\vec{a}\right) = (-m)\vec{a} = -\left(m\vec{a}\right)$ (ii) $(-m)\left(-\vec{a}\right) = m\vec{a}$

(iii) $m\left(n\vec{a}\right) = (mn)\vec{a} = n\left(m\vec{a}\right)$ (iv) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

Theorem 2.5 (Without Proof) :

If $\vec{a}$ and $\vec{b}$ are any two vectors and m is a scalar

then $m\left(\vec{a} + \vec{b}\right) = m\vec{a} + m\vec{b}$.

Result : $m\left(\vec{a} - \vec{b}\right) = m\vec{a} - m\vec{b}$

2.4 Position vector

If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector (P.V.) of P with respect to O.

From the diagram $\overrightarrow{OP} = \vec{r}$

Similarly $\overrightarrow{OA}$ is called the position vector (P.V.) of A with respect to O and $\overrightarrow{OB}$ is the P.V. of B with respect to O.

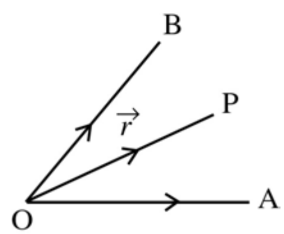


Fig. 2. 11

Theorem 2.6: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the P.Vs of A and B respectively.

Proof: Let O be the origin. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$

In ΔOAB, we have by triangle law of addition

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

i.e. $\overrightarrow{AB}$ = (P.V of B) – (P.V of A)

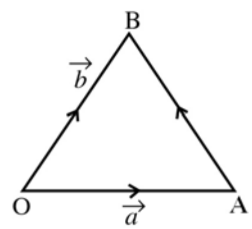


Fig. 2. 12

Note : In $\overrightarrow{AB}$, the point B is the head of the vector and A is the tail of the vector.

$\therefore$ $\overrightarrow{AB}$ = (P.V. of the head) – (P.V. of the tail). Similarly $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

The above rule will be very much useful in doing problems.

Theorem 2.7: [Section Formula – Internal Division]

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB internally in the ratio $m : n$. Then the position vector of P is given by

$$\overrightarrow{OP} = \frac{n\vec{a} + m\vec{b}}{m+n}$$

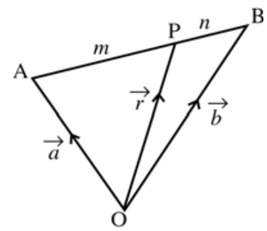


Fig. 2. 13

Proof:

Let O be the origin.

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$

Let the position vector of P with respect to O be $\vec{r}$ i.e. $\overrightarrow{OP} = \vec{r}$

Let P divide AB internally in the ratio $m : n$

Then $\dfrac{AP}{PB} = \dfrac{m}{n} \Rightarrow n\,AP = m\,PB \quad \Rightarrow n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$

$$\Rightarrow n\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = m\left(\overrightarrow{OB} - \overrightarrow{OP}\right) \quad \Rightarrow n\left(\vec{r} - \vec{a}\right) = m\left(\vec{b} - \vec{r}\right)$$

$$\Rightarrow \quad n\vec{r} - n\vec{a} = m\vec{b} - m\vec{r} \qquad \Rightarrow m\vec{r} + n\vec{r} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad (m+n)\,\vec{r} = m\vec{b} + n\vec{a}$$

$$\boxed{\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}}$$

Result (1): If P is the mid point of AB, then it divides AB in the ratio 1 : 1.

$\therefore$ The P.V. of P is $\dfrac{1\,.\,\vec{b} + 1.\,\vec{a}}{1+1} = \dfrac{\vec{a} + \vec{b}}{2}$

$\therefore$ P.V. of the mid point P of AB is $\overrightarrow{OP} = \vec{r} = \dfrac{\vec{a} + \vec{b}}{2}$

Result (2): Condition that three points may be collinear

Proof: Assume that the points A, P and B (whose P.Vs are $\vec{a}$, $\vec{r}$ and $\vec{b}$ respectively) are collinear

We have $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

$$(m+n)\vec{r} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad (m+n)\vec{r} - m\vec{b} - n\vec{a} = 0$$

In this vector equation, sum of the scalar coefficients in the L.H.S. $= (m+n) - m - n = 0$

Thus we have the result, if A, B, C are collinear points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively then there exists scalars x, y, z such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{O}$ and $x + y + z = 0$

Conversely if the scalars x, y, z are such that $x + y + z = 0$ and $x\vec{a} + y\vec{b} + z\vec{c} = \vec{O}$ then the points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear.

Result 3: [Section formula – External division]

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and let P be a point dividing AB externally in the ratio $m : n$. Then the position vector of P is given by

$$\overrightarrow{OP} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

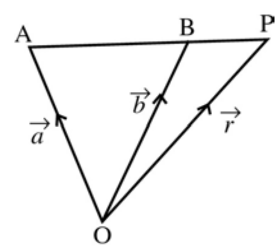


Fig. 2. 14

Proof:

Let O be the origin. A and B are the two points whose position vectors are $\vec{a}$ and $\vec{b}$

Then $\overrightarrow{OA} = \vec{a}$; $\overrightarrow{OB} = \vec{b}$

Let P divide AB externally in the ratio $m : n$. Let the position vector of P with respect to O be $\vec{r}$ i.e. $\overrightarrow{OP} = \vec{r}$

We have $\dfrac{AP}{PB} = \dfrac{m}{n}$ $\Rightarrow$ $n\,AP = m\,PB$

$\Rightarrow$ $n\,\overrightarrow{AP} = -m\,\overrightarrow{PB}$ $\left[\begin{array}{l}\overrightarrow{AP}\ \&\ \overrightarrow{PB} \\ \text{are in the opposite direction}\end{array}\right]$

$\Rightarrow$ $n\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = -m\left(\overrightarrow{OB} - \overrightarrow{OP}\right)$ $\Rightarrow$ $n\left(\vec{r} - \vec{a}\right) = m\left(\vec{r} - \vec{b}\right)$

$\Rightarrow$ $n\vec{r} - n\vec{a} = m\vec{r} - m\vec{b}$ $\Rightarrow$ $m\vec{b} - n\vec{a} = m\vec{r} - n\vec{r}$

$\Rightarrow$ $m\vec{b} - n\vec{a} = (m-n)\vec{r}$

$$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Theorem 2.8: The medians of a triangle are concurrent.

Proof:

Let ABC be a triangle and let D, E, F be the mid points of its sides BC, CA and AB respectively. We have to prove that the medians AD, BE, CF are concurrent.

Let O be the origin and $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of A, B, C respectively.

The position vectors of D, E, F are

$\dfrac{\vec{b} + \vec{c}}{2}$, $\dfrac{\vec{c} + \vec{a}}{2}$, $\dfrac{\vec{a} + \vec{b}}{2}$

Let G_1 be the point on AD dividing it internally in the ratio 2 : 1

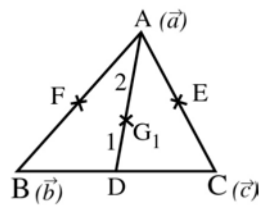


Fig. 2. 15

$\therefore$ P.V. of $G_1 = \dfrac{2\overrightarrow{OD} + 1\overrightarrow{OA}}{2+1}$

$$\overrightarrow{OG_1} = \frac{2\left(\dfrac{\vec{b} + \vec{c}}{2}\right) + 1\,\vec{a}}{3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad (1)$$

Let G_2 be the point on BE dividing it internally in the ratio 2 : 1

$$\therefore \qquad \overrightarrow{OG_2} = \frac{2\overrightarrow{OE} + 1\overrightarrow{OB}}{2+1}$$

$$\overrightarrow{OG_2} = \frac{2\left(\frac{\vec{c} + \vec{a}}{2}\right) + 1.\vec{b}}{3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad (2)$$

Similarly if G_3 divides CF in the ratio 2 : 1 then

$$\overrightarrow{OG_3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad (3)$$

From (1), (2), (3) we find that the position vectors of the three points G_1, G_2, G_3 are one and the same. Hence they are not different points. Let the common point be denoted by G.

Therefore the three medians are concurrent and the point of concurrence is G.

Result:

The point of intersection of the three medians of a triangle is called the centroid of the triangle.

The position vector of the centroid G of ΔABC is $\overrightarrow{OG} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of the vertices A, B, C respectively and O is the origin of reference.

Example 2.1: If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the vectors represented by the three sides of a triangle, taken in order, then prove that $\vec{a} + \vec{b} + \vec{c} = \vec{O}$

Solution:

Let ABC be a triangle such that

$$\overrightarrow{BC} = \vec{a},\ \overrightarrow{CA} = \vec{b} \text{ and } \overrightarrow{AB} = \vec{c}$$

$$\begin{aligned} \vec{a} + \vec{b} + \vec{c} &= \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} \\ &= \overrightarrow{BA} + \overrightarrow{AB} \quad (\because \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}) \\ &= \overrightarrow{BB} = \vec{O} \end{aligned}$$

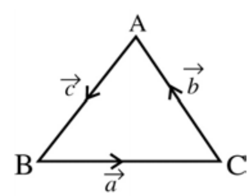


Fig. 2. 16

Example 2.2:

If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, find the vectors determined by the other sides taken in order.

Solution:

Let ABCDEF be a regular hexagon such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$

Since AD || BC such that AD = 2.BC

$\therefore \quad \overrightarrow{AD} = 2\overrightarrow{BC} = 2\vec{b}$

In ΔABC, we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow \quad \overrightarrow{AC} = \vec{a} + \vec{b}$

In ΔACD, $\quad \overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$

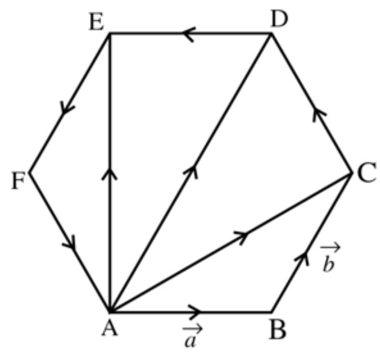


Fig. 2. 17

$$\therefore \quad \overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC} = 2\vec{b} - \left(\vec{a} + \vec{b}\right) = \vec{b} - \vec{a}$$

$$\overrightarrow{DE} = -\overrightarrow{AB} = -\vec{a}$$

$$\overrightarrow{EF} = -\overrightarrow{BC} = -\vec{b}$$

$$\overrightarrow{FA} = -\overrightarrow{CD} = -\left(\vec{b} - \vec{a}\right) = \vec{a} - \vec{b}$$

Example 2.3:

The position vectors of the points A, B, C, D are $\vec{a}$, $\vec{b}$, $2\vec{a} + 3\vec{b}$, $\vec{a} - 2\vec{b}$ respectively. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$

Solution: Given that

$$\overrightarrow{OA} = \vec{a} \quad ; \quad \overrightarrow{OB} = \vec{b} \quad ; \quad \overrightarrow{OC} = 2\vec{a} + 3\vec{b} \quad ; \quad \overrightarrow{OD} = \vec{a} - 2\vec{b}$$

$$\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = \vec{b} - \left(\vec{a} - 2\vec{b}\right) = \vec{b} - \vec{a} + 2\vec{b} = 3\vec{b} - \vec{a}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \left(2\vec{a} + 3\vec{b}\right) - \vec{a}$$

$$= \vec{a} + 3\vec{b}$$

Example 2.4: Find the position vector of the points which divide the join of the points A and B whose P.Vs are $\vec{a} - 2\vec{b}$ and $2\vec{a} - \vec{b}$ internally and externally in the ratio 3 : 2

Solution:

$$\overrightarrow{OA} = \vec{a} - 2\vec{b} \quad ; \quad \overrightarrow{OB} = 2\vec{a} - \vec{b}$$

Let P divide AB internally in the ratio 3 : 2

$$\text{P.V. of P} = \frac{3\overrightarrow{OB} + 2\overrightarrow{OA}}{3+2} = \frac{3\left(2\vec{a} - \vec{b}\right) + 2\left(\vec{a} - 2\vec{b}\right)}{5}$$

$$= \frac{6\vec{a} - 3\vec{b} + 2\vec{a} - 4\vec{b}}{5} = \frac{8\vec{a} - 7\vec{b}}{5} = \frac{8}{5}\vec{a} - \frac{7}{5}\vec{b}$$

Let Q divide AB externally in the ratio 3 : 2

$$\text{P.V. of Q} = \frac{3\overrightarrow{OB} - 2\overrightarrow{OA}}{3-2} = \frac{3\left(2\vec{a} - \vec{b}\right) - 2\left(\vec{a} - 2\vec{b}\right)}{1}$$

$$= 6\vec{a} - 3\vec{b} - 2\vec{a} + 4\vec{b} = 4\vec{a} + \vec{b}$$

Example 2.5: If $\vec{a}$ and $\vec{b}$ are position vectors of points A and B respectively, then find the position vector of points of trisection of AB.

Solution:

Let P and Q be the points of trisection of AB

Let AP = PQ = QB = λ (say)

P divides AB in the ratio 1 : 2

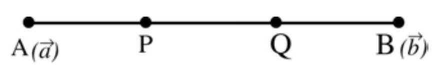


Fig. 2. 18

$$\text{P.V. of P} = \overrightarrow{OP} = \frac{1.\overrightarrow{OB} + 2.\overrightarrow{OA}}{1+2} = \frac{1.\,\vec{b} + 2.\,\vec{a}}{3} = \frac{\vec{b} + 2\vec{a}}{3}$$

Q is the mid-point of PB

$$\text{P.V. of Q} = \frac{\overrightarrow{OP} + \overrightarrow{OB}}{2} = \frac{\dfrac{\vec{b} + 2\vec{a}}{3} + \vec{b}}{2} = \frac{\dfrac{\vec{b} + 2\vec{a} + 3\vec{b}}{3}}{2} = \frac{2\vec{a} + 4\vec{b}}{6}$$

$$= \frac{\vec{a} + 2\vec{b}}{3}$$

Example 2.6: By using vectors, prove that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

Solution:

Let ABCD be a quadrilateral

First we assume that ABCD is a parallelogram

To prove that its diagonals bisect each other

Let O be the origin of reference.

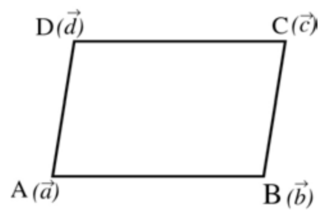


Fig. 2. 19

$\therefore\ \overrightarrow{OA} = \vec{a}\ ,\ \overrightarrow{OB} = \vec{b}\ ,\ \overrightarrow{OC} = \vec{c}\ ,\ \overrightarrow{OD} = \vec{d}$

Since ABCD is a parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$

$$\Rightarrow \quad \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD} \quad \Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$$

$$\Rightarrow \quad \vec{b} + \vec{d} = \vec{a} + \vec{c} \quad \Rightarrow \quad \frac{\vec{b} + \vec{d}}{2} = \frac{\vec{a} + \vec{c}}{2}$$

i.e. P.V. of the mid-point of BD = P.V. of the mid-point of AC. Thus, the point, which bisects AC also, bisects BD. Hence the diagonals of a parallelogram ABCD bisect each other.

Conversely suppose that ABCD is a quadrilateral such that its diagonals bisect each other. To prove that it is a parallelogram.

Let $\vec{a}\ ,\ \vec{b}\ ,\ \vec{c}\ ,\ \vec{d}$ be the position vectors of its vertices A, B, C and D respectively. Since diagonals AC and BD bisect each other.

P.V. of the mid-point of AC = P.V. of the mid-point of BD

$$\Rightarrow \quad \frac{\vec{a} + \vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2} \Rightarrow \vec{a} + \vec{c} = \vec{b} + \vec{d} \qquad \ldots (1)$$

$$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d} \quad \text{i.e.} \quad \overrightarrow{AB} = \overrightarrow{DC}$$

Also (1) $\Rightarrow \quad \vec{d} - \vec{a} = \vec{c} - \vec{b} \quad$ i.e. $\quad \overrightarrow{AD} = \overrightarrow{BC}$

Hence ABCD is a parallelogram.

Example 2.7:

In a triangle ABC if D and E are the midpoints of sides AB and AC respectively, show that $\overrightarrow{BE} + \overrightarrow{DC} = \frac{3}{2}\ \overrightarrow{BC}$

Solution:

For convenience we choose A as the origin.

Let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Since D and E are the mid-points of AB and AC, the position vectors of D and E are $\frac{\vec{b}}{2}$ and $\frac{\vec{c}}{2}$ respectively.

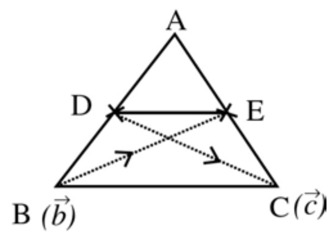


Fig. 2. 20

Now $\quad \overrightarrow{BE}$ = P.V. of E – P.V. of B $= \frac{\vec{c}}{2} - \vec{b}$

$\overrightarrow{DC}$ = P.V. of C – P.V. of D $= \vec{c} - \frac{\vec{b}}{2}$

$$\therefore \overrightarrow{BE} + \overrightarrow{DC} = \frac{\vec{c}}{2} - \vec{b} + \vec{c} - \frac{\vec{b}}{2} = \frac{3}{2}\vec{c} - \frac{3}{2}\vec{b}$$

$$= \frac{3}{2}\left(\vec{c} - \vec{b}\right) = \frac{3}{2}\text{ [P.V. of C – P.V. of B]}$$

$$= \frac{3}{2}\overrightarrow{BC}$$

Example 2.8: Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.

Solution:

Let ABC be a triangle, and let O be the origin of reference. Let D and E be the midpoints of AB and AC respectively.

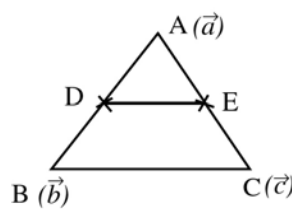


Fig. 2. 21

Let $\quad \overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$

P.V. of D = $\overrightarrow{OD} = \frac{\vec{a} + \vec{b}}{2}$

P.V. of E = $\overrightarrow{OE} = \frac{\vec{a} + \vec{c}}{2}$

Now $\quad \overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \left(\frac{\vec{a} + \vec{c}}{2}\right) - \left(\frac{\vec{a} + \vec{b}}{2}\right)$

$$= \frac{\vec{a} + \vec{c} - \vec{a} - \vec{b}}{2} = \frac{1}{2}\left(\vec{c} - \vec{b}\right) = \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right) = \frac{1}{2}\overrightarrow{BC}$$

$$\therefore \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC} \quad \Rightarrow \quad DE \parallel BC$$

Also $\quad \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC} \quad \Rightarrow \quad \left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{BC}\right| \Rightarrow DE = \frac{1}{2} BC$

Hence DE || BC and DE = $\frac{1}{2}$ BC.

Example 2.9: Using vector method, prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral taken in order form a parallelogram.

Solution:

Let ABCD be a quadrilateral and let P, Q, R, S be the mid-points of the sides AB, BC, CD and DA respectively.

Then the position vectors of P, Q, R, S are

$$\frac{\vec{a} + \vec{b}}{2}, \quad \frac{\vec{b} + \vec{c}}{2}, \quad \frac{\vec{c} + \vec{d}}{2}, \quad \frac{\vec{d} + \vec{a}}{2}$$

respectively.

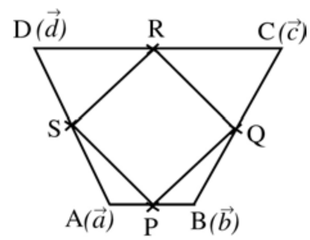


Fig. 2. 22

In order to prove that PQRS is a parallelogram it is sufficient to show that $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\overrightarrow{PS} = \overrightarrow{QR}$

Now $\overrightarrow{PQ}$ = P.V. of Q – P.V. of P = $\left(\frac{\vec{b} + \vec{c}}{2}\right) - \left(\frac{\vec{a} + \vec{b}}{2}\right) = \frac{\vec{c} - \vec{a}}{2}$

$\overrightarrow{SR}$ = P.V. of R – P.V. of S = $\left(\frac{\vec{c} + \vec{d}}{2}\right) - \left(\frac{\vec{d} + \vec{a}}{2}\right) = \frac{\vec{c} - \vec{a}}{2}$

$\therefore \overrightarrow{PQ} = \overrightarrow{SR}$

$\Rightarrow$PQ || SR and PQ = SR

Similarly we can prove that PS = QR and PS || QR

Hence PQRS is a parallelogram.

Example 2.10 :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of three distinct points A, B, C. If there exists scalars l, m, n (not all zero) such that $l\vec{a} + m\vec{b} + n\vec{c} = 0$ and $l + m + n = 0$ then show that A, B and C lie on a line.

Solution:

It is given that l, m, n are not all zero. So, let n be a non-zero scalar.

$$l\vec{a} + m\vec{b} + n\vec{c} = 0 \quad \Rightarrow \quad n\vec{c} = -\left(l\vec{a} + m\vec{b}\right)$$

$$\vec{c} = -\frac{\left(l\vec{a} + m\vec{b}\right)}{n} \quad \Rightarrow \quad \vec{c} = -\frac{\left(l\vec{a} + m\vec{b}\right)}{-(l+m)} = \frac{l\vec{a} + m\vec{b}}{l+m}$$

$\Rightarrow$ The point C divides the line joining A and B in the ratio $m : l$

Hence A, B and C lies on the same line.

Note : $\vec{a}, \vec{b}$ are collinear vectors $\Rightarrow \vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$ for some scalar λ

Collinear points: If A, B, C are three points in a plane such that $\overrightarrow{AB} = \lambda\overrightarrow{BC}$ or $\overrightarrow{AB} = \lambda\overrightarrow{AC}$ (or) $\overrightarrow{BC} = \lambda\overrightarrow{AC}$ for some scalar λ, then A, B, C are collinear.

Example 2.11: Show that the points with position vectors

$$\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - \vec{c} \text{ and } 4\vec{a} - 7\vec{b} + 7\vec{c} \text{ are collinear.}$$

Solution:

Let A, B, C be the points with position vectors

$$\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - \vec{c} \text{ and } 4\vec{a} - 7\vec{b} + 7\vec{c} \text{ respectively.}$$

$$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c},\ \overrightarrow{OB} = -2\vec{a} + 3\vec{b} - \vec{c},\ \overrightarrow{OC} = 4\vec{a} - 7\vec{b} + 7\vec{c}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(-2\vec{a} + 3\vec{b} - \vec{c}\right) - \left(\vec{a} - 2\vec{b} + 3\vec{c}\right)$$

$$= -2\vec{a} + 3\vec{b} - \vec{c} - \vec{a} + 2\vec{b} - 3\vec{c} = -3\vec{a} + 5\vec{b} - 4\vec{c}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(4\vec{a} - 7\vec{b} + 7\vec{c}\right) - \left(-2\vec{a} + 3\vec{b} - \vec{c}\right)$$

$$= 4\vec{a} - 7\vec{b} + 7\vec{c} + 2\vec{a} - 3\vec{b} + \vec{c} = 6\vec{a} - 10\vec{b} + 8\vec{c}$$

Clearly $\overrightarrow{BC} = 6\vec{a} - 10\vec{b} + 8\vec{c} = -2\left(-3\vec{a} + 5\vec{b} - 4\vec{c}\right) = -2\left(\overrightarrow{AB}\right)$

$\Rightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors but B is a point common to them. So $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear vectors. Hence A, B, C are collinear points.

EXERCISE 2.1

(1) If $\vec{a}$ and $\vec{b}$ represent two adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively of a paralleogram ABCD. Find the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

(2) If $\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR}$, show that the points P, Q, R are collinear.

(3) Show that the points with position vectors

$\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} + 2\vec{c}$ and $-8\vec{a} + 13\vec{b}$ are collinear.

(4) Show that the points A, B, C with position vectors $-2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{a} + 2\vec{b} + 3\vec{c}$ and $7\vec{a} - \vec{c}$ respectively, are collinear.

(5) If D is the mid-point of the side BC of a triangle ABC, prove that
$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$

(6) If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{O}$

(7) If ABC and A′B′C′ are two triangles and G, G′ be their corresponding centroids, prove that $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\overrightarrow{GG'}$

(8) Prove that the sum of the vectors directed from the vertices to the mid-points of opposite sides of a triangle is zero

(9) Prove by vector method that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.

(10) Prove by vector method that the internal bisectors of the angles of a triangle are concurrent.

(11) Prove using vectors the mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram.

(12) If ABCD is a quadrilateral and E and F are the mid-points of AC and BD respectively, prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

2.5 Resolution of a Vector

Theorem 2.9 (Without Proof) :

Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors and $\vec{r}$ be a vector coplanar with them. Then $\vec{r}$ can be expressed uniquely as $\vec{r} = l\vec{a} + m\vec{b}$ where *l, m* are scalars.

Note : We call $l\vec{a} + m\vec{b}$ as a linear combination of vectors $\vec{a}$ and $\vec{b}$, where l, m are scalars.

Rectangular resolution of a vector in two dimension

Theorem 2.10 :

If P is a point in a two dimensional plane which has coordinates (x, y) then $\overrightarrow{OP} = x\vec{i} + y\vec{j}$, where $\vec{i}$ and $\vec{j}$ are unit vectors along OX and OY respectively.

Proof:

Let $P(x, y)$ be a point in a plane with reference to OX and OY as co-ordinate axes as shown in the figure.

Draw PL perpendicular to OX.

Then OL = x and LP = y

Let $\vec{i}$, $\vec{j}$ be the unit vectors along OX and OY respectively.

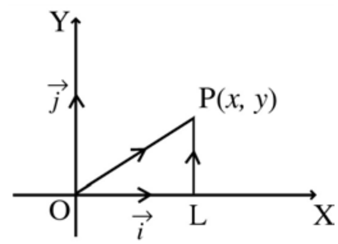


Fig. 2. 23

Then $\overrightarrow{OL} = x\vec{i}$ and $\overrightarrow{LP} = y\vec{j}$

Vectors $\overrightarrow{OL}$ and $\overrightarrow{LP}$ are known as the components of $\overrightarrow{OP}$ along x-axis and y-axis respectively.

Now by triangle law of addition

$$\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = x\vec{i} + y\vec{j} = \vec{r} \quad \text{(say)}$$

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j}$$

Now $$OP^2 = OL^2 + LP^2 = x^2 + y^2$$

$$\Rightarrow \quad OP = \sqrt{x^2 + y^2} \Rightarrow \left|\vec{r}\right| = \sqrt{x^2 + y^2}$$

Thus, if a point P in a plane has coordinates (x, y) then

(i) $\vec{r} = \overrightarrow{OP} = x\vec{i} + y\vec{j}$

(ii) $\left|\vec{r}\right| = \left|\overrightarrow{OP}\right| = \left|x\vec{i} + y\vec{j}\right| = \sqrt{x^2 + y^2}$

(iii) The component of $\overrightarrow{OP}$ along x-axis is a vector $x\vec{i}$ and the component of $\overrightarrow{OP}$ along y-axis is a vector $y\vec{j}$

Components of a vector $\overrightarrow{AB}$ in terms of coordinates of A and B

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in XOY plane. Let $\vec{i}$ and $\vec{j}$ be unit vectors along OX and OY respectively.

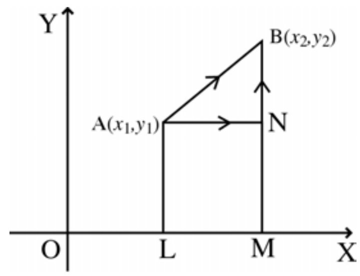


Fig. 2. 24

$AN = x_2 - x_1$, $BN = y_2 - y_1$

$\therefore$ $\overrightarrow{AN} = (x_2 - x_1)\vec{i}$, $\overrightarrow{NB} = (y_2 - y_1)\vec{j}$

Now by triangle law of addition

$$\overrightarrow{AB} = \overrightarrow{AN} + \overrightarrow{NB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$$

Component of $\overrightarrow{AB}$ along x-axis $= (x_2 - x_1)\vec{i}$

Component of $\overrightarrow{AB}$ along y-axis $= (y_2 - y_1)\vec{j}$

$$AB^2 = AN^2 + NB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which gives the distance between A and B.

Addition, Subtraction, Multiplication of a vector by a scalar and equality of vectors in terms of components:

$$\text{Let} \quad \vec{a} = a_1\vec{i} + a_2\vec{j} \quad \text{and} \quad \vec{b} = b_1\vec{i} + b_2\vec{j}$$

We define

(i) $\vec{a} + \vec{b} = \left(a_1\vec{i} + a_2\vec{j}\right) + \left(b_1\vec{i} + b_2\vec{j}\right) = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j}$

(ii) $\vec{a} - \vec{b} = \left(a_1\vec{i} + a_2\vec{j}\right) - \left(b_1\vec{i} + b_2\vec{j}\right) = (a_1 - b_1)\vec{i} + (a_2 - b_2)\vec{j}$

(iii) $m\vec{a} = m\left(a_1\vec{i} + a_2\vec{j}\right) = ma_1\vec{i} + ma_2\vec{j}$ where m is a scalar

(iv) $\vec{a} = \vec{b} \Rightarrow a_1\vec{i} + a_2\vec{j} = b_1\vec{i} + b_2\vec{j} \Rightarrow a_1 = b_1$ and $a_2 = b_2$

Example 2.12: Let O be the origin and P(– 2, 4) be a point in the xy-plane. Express $\overrightarrow{OP}$ in terms of vectors $\vec{i}$ and $\vec{j}$. Also find $\left|\overrightarrow{OP}\right|$

Solution: The position vector of P, $\overrightarrow{OP} = -2\vec{i} + 4\vec{j}$

$$\left|\overrightarrow{OP}\right| = \left|-2\vec{i} + 4\vec{j}\right| = \sqrt{(-2)^2 + (4)^2} = \sqrt{4 + 16} = \sqrt{20}$$
$$= 2\sqrt{5}$$

Example 2.13: Find the components along the coordinates of the position vector of P(– 4, 3)

Solution:

The position vector of P = $\overrightarrow{OP} = -4\vec{i} + 3\vec{j}$

Component of $\overrightarrow{OP}$ along x-axis is $-4\vec{i}$

i.e. component of $\overrightarrow{OP}$ along x-axis is a vector of magnitude 4 and its direction is along the negative direction of x-axis.

Component of $\overrightarrow{OP}$ along y-axis is $3\vec{j}$

i.e. the component of $\overrightarrow{OP}$ along y –axis is a vector of magnitude 3, having its direction along the positive direction of y-axis.

Example 2.14: Express $\overrightarrow{AB}$ in terms of unit vectors $\vec{i}$ and $\vec{j}$, where the points are A(– 6, 3) and B(– 2, – 5). Find also $\left|\overrightarrow{AB}\right|$

Solution:

Given $\quad \overrightarrow{OA} = -6\vec{i} + 3\vec{j}$; $\overrightarrow{OB} = -2\vec{i} - 5\vec{j}$

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(-2\vec{i} - 5\vec{j}\right) - \left(-6\vec{i} + 3\vec{j}\right)$$
$$= 4\vec{i} - 8\vec{j}$$
$$\left|\overrightarrow{AB}\right| = \left|4\vec{i} - 8\vec{j}\right| = \sqrt{(4)^2 + (-8)^2} = \sqrt{16 + 64} = \sqrt{80}$$
$$= 4\sqrt{5}$$

Theorem 2.11 (Without Proof) :

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three given non-coplanar vectors then every vector $\vec{r}$ in space can be uniquely expressed as $\vec{r} = l\vec{a} + m\vec{b} + n\vec{c}$ for some scalars l, m and n

Rectangular Resolution of a vector in three dimension

Theorem 2.12:

If a point P in space has coordinate (x, y, z) then its position vector $\vec{r}$ is $x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along OX, OY and OZ respectively.

Proof:

OX, OY, OZ are three mutually perpendicular axes. $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors along OX, OY, OZ respectively. Let P be any point (x, y, z) in space and let $\overrightarrow{OP} = \vec{r}$

Draw PQ perpendicular to XOY plane and QR perpendicular to OX

Then OR $= x$; RQ $= y$; QP $= z$

$\therefore \overrightarrow{OR} = x\vec{i}$; $\overrightarrow{RQ} = y\vec{j}$; $\overrightarrow{QP} = z\vec{k}$

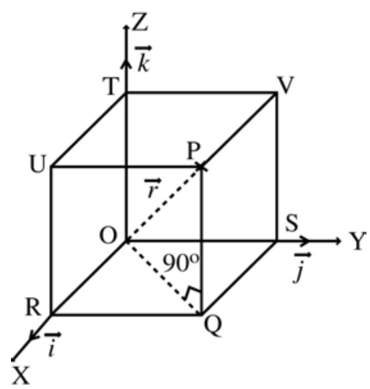


Fig. 2. 25

Now $\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OR} + \overrightarrow{RQ} + \overrightarrow{QP}$

$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Thus if P is a point (x, y, z) and $\vec{r}$ is the position vector of P, then $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

From the right angled triangle OQP, $OP^2 = OQ^2 + QP^2$

From the right angled triangle ORQ, $OQ^2 = OR^2 + RQ^2$

$$\therefore OP^2 = OR^2 + RQ^2 + QP^2 \Rightarrow OP^2 = x^2 + y^2 + z^2$$

$$\Rightarrow OP = \sqrt{x^2 + y^2 + z^2} \Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

2.6 Direction cosines and direction ratios

Let P(x, y, z) be any point in space with reference to a rectangular coordinate system O (XYZ). Let α, β and γ be the angles made by OP with the positive direction of coordinate axes OX, OY, OZ respectively. Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **direction cosines** of $\overrightarrow{OP}$.

In the fig 2.25 $\angle OQP = 90°$; $\angle POZ = \gamma$ $\therefore \angle OPQ = \gamma$ ($\because QP \parallel OZ$)

$$\therefore \cos\gamma = \frac{PQ}{OP} \Rightarrow \cos\gamma = \frac{z}{r} \quad \text{Similarly} \quad \cos\alpha = \frac{x}{r} \text{ and } \cos\beta = \frac{y}{r}$$

$\therefore$ The direction cosines of $\overrightarrow{OP}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$

Result 1: Sum of the squares of direction cosines is unity.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2 = \frac{x^2 + y^2 + z^2}{r^2}$$

$$= \frac{r^2}{r^2} = 1 \qquad [\because r^2 = x^2 + y^2 + z^2]$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Result 2: Sum of the squares of direction sines is 2.

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma)$$

$$= 3 - [\cos^2\alpha + \cos^2\beta + \cos^2\gamma] = 3 - 1 = 2$$

$$\therefore \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Direction ratios:

Any three numbers proportional to direction cosines of a vector are called its direction ratios. (d. r's).

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be any vector

$\Rightarrow$ Direction cosines of $\vec{r}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \cos\alpha = \frac{x}{r}$; $\cos\beta = \frac{y}{r}$; $\cos\gamma = \frac{z}{r}$ where α, β, γ be the angles made by $\vec{r}$ with the coordinate axes OX, OY, OZ respectively

$$\Rightarrow \frac{x}{\cos\alpha} = r, \quad \frac{y}{\cos\beta} = r, \quad \frac{z}{\cos\gamma} = r$$

$$\Rightarrow \frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = r$$

$$\Rightarrow x : y : z = \cos\alpha : \cos\beta : \cos\gamma$$

i.e. the coefficients of i, j, k in the rectangular resolution of a vector are proportional to the direction cosines of that vector.

$\therefore x, y, z$ are the direction ratios of the vector $\vec{r} = x\vec{i} + y\vec{l} + z\vec{k}$

Addition, Subtraction and Multiplication of a vector by a scalar and equality in terms of components:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be any two vectors.

Then

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\vec{i} + (a_2 - b_2)\vec{j} + (a_3 - b_3)\vec{k}$

(iii) $m\vec{a} = m\left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right)$

$= ma_1\vec{i} + ma_2\vec{j} + ma_3\vec{k}$ where m is a scalar

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$

Distance between two points:

Let A (x_1, y_1, z_1) and B(x_2, y_2, z_2) be any two points

Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\right) - \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}\right)$

$= (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

$\therefore$ The distance between A and B is AB $= \left|\overrightarrow{AB}\right|$

$$\left|\overrightarrow{AB}\right| = \left|(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}\right|$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 2.15: Find the magnitude and direction cosines of $2\vec{i} - \vec{j} + 7\vec{k}$

Solution:

Magnitude of $2\vec{i} - \vec{j} + 7\vec{k} = \left|2\vec{i} - \vec{j} + 7\vec{k}\right| = \sqrt{(2)^2 + (-1)^2 + (7)^2}$

$= \sqrt{4 + 1 + 49} = \sqrt{54} = 3\sqrt{6}$

Direction cosines of $2\vec{i} - \vec{j} + 7\vec{k}$ are $\frac{2}{3\sqrt{6}}, -\frac{1}{3\sqrt{6}}, \frac{7}{3\sqrt{6}}$

Example 2.16: Find the unit vector in the direction of $3\vec{i} + 4\vec{j} - 12\vec{k}$

Solution: Let $\vec{a} = 3\vec{i} + 4\vec{j} - 12\vec{k}$

$$\left|\vec{a}\right| = \left|3\vec{i} + 4\vec{j} - 12\vec{k}\right| = \sqrt{(3)^2 + (4)^2 + (-12)^2}$$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

Unit vector in the direction of $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{\left|\vec{a}\right|} = \dfrac{3\vec{i} + 4\vec{j} - 12\vec{k}}{13}$

Example 2.17: Find the sum of the vectors $\vec{i} - \vec{j} + 2\vec{k}$ and $2\vec{i} + 3\vec{j} - 4\vec{k}$ and also find the modulus of the sum.

Solution:

Let $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$$\vec{a} + \vec{b} = \left(\vec{i} - \vec{j} + 2\vec{k}\right) + \left(2\vec{i} + 3\vec{j} - 4\vec{k}\right) = 3\vec{i} + 2\vec{j} - 2\vec{k}$$

$$\left|\vec{a} + \vec{b}\right| = \sqrt{3^2 + 2^2 + (-2)^2} = \sqrt{9 + 4 + 4}$$

$$= \sqrt{17}$$

Example 2.18: If the position vectors of the two points A and B are $\vec{i} + 2\vec{j} - 3\vec{k}$ and $2\vec{i} - 4\vec{j} + \vec{k}$ respectively then find $\left|\overrightarrow{AB}\right|$

Solution:

If O be the origin, then $\overrightarrow{OA} = \vec{i} + 2\vec{j} - 3\vec{k}$, $\overrightarrow{OB} = 2\vec{i} - 4\vec{j} + \vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(2\vec{i} - 4\vec{j} + \vec{k}\right) - \left(\vec{i} + 2\vec{j} - 3\vec{k}\right)$$

$$= \vec{i} - 6\vec{j} + 4\vec{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(1)^2 + (-6)^2 + (4)^2} = \sqrt{53}$$

Example 2.19: Find the unit vectors parallel to the vector $-3\vec{i} + 4\vec{j}$

Solution: Let $\vec{a} = -3\vec{i} + 4\vec{j}$

$$\left|\vec{a}\right| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = 5$$

$$\hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|} = \frac{1}{\left|\vec{a}\right|}\ \vec{a} = \frac{1}{5}\left(-3\vec{i} + 4\vec{j}\right)$$

Unit vectors parallel to $\vec{a}$ are $\pm\ \hat{a} = \pm\left(\frac{-3}{5}\ \vec{i}\ + \frac{4}{5}\vec{j}\right)$

Example 2.20: Find the vectors of magnitude 5 units, which are parallel to the vector $2\vec{i} - \vec{j}$

Solution: Let $\vec{a} = 2\vec{i} - \vec{j}$

$$\left|\vec{a}\right| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|} = \frac{1}{\sqrt{5}}\left(2\vec{i} - \vec{j}\right) = \frac{2}{\sqrt{5}}\ \vec{i} - \frac{1}{\sqrt{5}}\ \vec{j}$$

Vectors of magnitude 5 parallel to $2\vec{i} - \vec{j} = \pm 5\,\hat{a}$

$$= \pm 5\left(\frac{2}{\sqrt{5}}\ \vec{i} - \frac{1}{\sqrt{5}}\ \vec{j}\right) = \pm\left(2\sqrt{5}\ \vec{i} - \sqrt{5}\ \vec{j}\right)$$

Example 2.21: Show that the points whose position vectors $2\vec{i} + 3\vec{j} - 5\vec{k}$, $3\vec{i} + \vec{j} - 2\vec{k}$ and $6\vec{i} - 5\vec{j} + 7\vec{k}$ are collinear.

Solution: Let the points be A, B and C and O be the origin. Then

$$\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 5\vec{k}\ ;\ \overrightarrow{OB} = 3\vec{i} + \vec{j} - 2\vec{k}\ ;\ \overrightarrow{OC} = 6\vec{i} - 5\vec{j} + 7\vec{k}$$

$$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(3\vec{i} + \vec{j} - 2\vec{k}\right) - \left(2\vec{i} + 3\vec{j} - 5\vec{k}\right)$$

$$= \vec{i} - 2\vec{j} + 3\vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(6\vec{i} - 5\vec{j} + 7\vec{k}\right) - \left(2\vec{i} + 3\vec{j} - 5\vec{k}\right)$$

$$\overrightarrow{AC} = 4\vec{i} - 8\vec{j} + 12\vec{k} = 4\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$$

$$= 4\ \overrightarrow{AB}$$

Hence the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. Further they have the common point A.

$\therefore$ The points A, B, C are collinear.

Example 2.22: If the position vectors of A and B are $3\vec{i} - 7\vec{j} - 7\vec{k}$ and $5\vec{i} + 4\vec{j} + 3\vec{k}$, find $\overrightarrow{AB}$ and determine its magnitude and direction cosines.

Solution:

Let O be the origin. Then

$$\overrightarrow{OA} = 3\vec{i} - 7\vec{j} - 7\vec{k},\quad \overrightarrow{OB} = 5\vec{i} + 4\vec{j} + 3\vec{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(5\vec{i} + 4\vec{j} + 3\vec{k}\right) - \left(3\vec{i} - 7\vec{j} - 7\vec{k}\right)$$

$$\overrightarrow{AB} = 2\vec{i} + 11\vec{j} + 10\vec{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(2)^2 + (11)^2 + (10)^2} = 15$$

The direction cosines are $\frac{2}{15}, \frac{11}{15}, \frac{10}{15}$

EXERCISE 2.2

(1) Find the sum of the vectors $4\vec{i} + 5\vec{j} + \vec{k}$, $-2\vec{i} + 4\vec{j} - \vec{k}$ and $3\vec{i} - 4\vec{j} + 5\vec{k}$. Find also the magnitude of the sum.

(2) If $\vec{a} = 3\vec{i} - \vec{j} - 4\vec{k}$, $\vec{b} = -2\vec{i} + 4\vec{j} - 3\vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ find $\left|2\vec{a} - \vec{b} + 3\vec{c}\right|$

(3) The position vectors of the vertices A, B, C of a triangle ABC are respectively

$2\vec{i} + 3\vec{j} + 4\vec{k}$, $-\vec{i} + 2\vec{j} - \vec{k}$ and $3\vec{i} - 5\vec{j} + 6\vec{k}$

Find the vectors determined by the sides and calculate the length of the sides.

(4) Show that the points whose position vectors given by

(i) $-2\vec{i} + 3\vec{j} + 5\vec{k}$, $\vec{i} + 2\vec{j} + 3\vec{k}$, $7\vec{i} - \vec{k}$

(ii) $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $-7\vec{j} + 10\vec{k}$ are collinear.

(5) If the vectors $\vec{a} = 2\vec{i} - 3\vec{j}$ and $\vec{b} = -6\vec{i} + m\vec{j}$ are collinear, find the value of *m*.

(6) Find a unit vector in the direction of $\vec{i} + \sqrt{3}\,\vec{j}$

(7) Find the unit vectors parallel to the sum of $3\vec{i} - 5\vec{j} + 8\vec{k}$ and $-2\vec{j} - 2\vec{k}$

(8) Find the unit vectors parallel to $3\vec{a} - 2\vec{b} + 4\vec{c}$ where $\vec{a} = 3\vec{i} - \vec{j} - 4\vec{k}$, $\vec{b} = -2\vec{i} + 4\vec{j} - 3\vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$

(9) The vertices of a triangle have position vectors $4\vec{i} + 5\vec{j} + 6\vec{k}$, $5\vec{i} + 6\vec{j} + 4\vec{k}$, $6\vec{i} + 4\vec{j} + 5\vec{k}$. Prove that the triangle is equilateral.

(10) Show that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $3\vec{i} - 4\vec{j} - 4\vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ form a right angled triangle.

(11) Prove that the points $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$, $4\vec{i} + 2\vec{j} + 3\vec{k}$ form an equilateral triangle.

(12) If the vertices of a triangle have position vectors $\vec{i} + 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} + \vec{k}$ and $3\vec{i} + \vec{j} + 2\vec{k}$, find the position vector of its centroid.

(13) If the position vectors of P and Q are $\vec{i} + 3\vec{j} - 7\vec{k}$ and $5\vec{i} - 2\vec{j} + 4\vec{k}$, find $\overrightarrow{PQ}$ and determine its direction cosines.

(14) Show that the following vectors are coplanar

(i) $\vec{i} - 2\vec{j} + 3\vec{k}$, $-2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{j} + 2\vec{k}$

(ii) $5\vec{i} + 6\vec{j} + 7\vec{k}$, $7\vec{i} - 8\vec{j} + 9\vec{k}$, $3\vec{i} + 20\vec{j} + 5\vec{k}$

(15) Show that the points given by the vectors $4\vec{i} + 5\vec{j} + \vec{k}$, $-\vec{j} - \vec{k}$, $3\vec{i} + 9\vec{j} + 4\vec{k}$ and $-4\vec{i} + 4\vec{j} + 4\vec{k}$ are coplanar.

(16) Examine whether the vectors $\vec{i} + 3\vec{j} + \vec{k}$, $2\vec{i} - \vec{j} - \vec{k}$ and $7\vec{j} + 5\vec{k}$ are coplanar.

3. ALGEBRA

3.1 Partial Fractions:

Definitions:

Rational Expression: An expression of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x) \neq 0$ are polynomials in x is called a rational expression.

The expressions $\frac{5x - 2}{x^2 + 3x + 2}$, $\frac{3x^2 + 2x - 1}{x^2 + x - 22}$ are examples for rational expressions.

Proper Fraction: A proper fraction is one in which the degree of the numerator is less than the degree of the denominator.

The expressions $\frac{3x + 1}{x^2 + 4x + 3}$, $\frac{7x^2 + 9}{x^3 + x^2 - 5}$ are examples for proper fractions.

Improper Fraction: An improper fraction is a fraction in which the degree of the numerator is greater than or equal to the degree of the denominator.

The expressions $\frac{x^3 + 5x^2 + 4}{x^2 + 2x + 3}$, $\frac{x^2 - x + 1}{x^2 + x + 3}$ are examples for improper fractions.

Partial Fraction:

Consider the sum of $\frac{7}{x - 2}$ and $\frac{5}{x - 1}$

We simplify it as follows:

$$\frac{7}{x-2} + \frac{5}{x-1} = \frac{7(x-1) + 5(x-2)}{(x-2)(x-1)} = \frac{7x - 7 + 5x - 10}{(x-2)(x-1)} = \frac{12x - 17}{(x-2)(x-1)}$$

Conversely the process of writing the given fraction $\frac{12x - 17}{(x-2)(x-1)}$ as $\frac{7}{x-2} + \frac{5}{x-1}$ is known as splitting into partial fractions (or) expressing as partial fractions.

A given proper fraction can be expressed as the sum of other simple fractions corresponding to the factors of the denominator of the given proper fraction. This process is called splitting into Partial Fractions. If the given fraction $\frac{p(x)}{q(x)}$ is improper then convert into sum of a polynomial expression and a proper rational fraction by dividing $p(x)$ by $q(x)$.

Working Rule :

Given the proper fraction $\frac{p(x)}{q(x)}$. Factorise $q(x)$ into prime factors.

Type 1: Linear factors, none of which is repeated.

If a linear factor $ax + b$ is a factor of the denominator $q(x)$ then corresponding to this factor associate a simple fraction $\frac{A}{ax + b}$, where A is a constant ($A \neq 0$).

i.e., When the factors of the denominator of the given fraction are all linear factors none of which is repeated, we write the partial fraction as follows :

$$\frac{x + 3}{(x + 5)(2x + 1)} = \frac{A}{x + 5} + \frac{B}{2x + 1}$$ where A and B are constants to be determined.

Example 3.1: Resolve into partial fractions $\frac{3x + 7}{x^2 - 3x + 2}$

The denominator $x^2 - 3x + 2$ can be factorised into linear factors.

$x^2 - 3x + 2 = x^2 - x - 2x + 2 = x(x - 1) - 2(x - 1) = (x - 1)(x - 2)$

We assume $\frac{3x + 7}{x^2 - 3x + 2} = \frac{A}{x - 1} + \frac{B}{x - 2}$ where A and B are constants.

$$\Rightarrow \quad \frac{3x + 7}{x^2 - 3x + 2} = \frac{A(x - 2) + B(x - 1)}{(x - 1)(x - 2)}$$

$$\Rightarrow \quad 3x + 7 = A(x - 2) + B(x - 1) \qquad \ldots(1)$$

Equating the coefficients of like powers of x, we get

Coefficient of x : $A + B = 3$ … (2)

Constant term : $-2A - B = 7$ … (3)

Solving (2) and (3) we get

$$A = -10$$
$$B = 13$$

$$\therefore \frac{3x + 7}{x^2 - 3x + 2} = \frac{-10}{x - 1} + \frac{13}{x - 2} = \frac{13}{x - 2} - \frac{10}{x - 1}$$

Note: The constants A and B can also be found by successively giving suitable values for x.

To find A, put $x = 1$ in (1)

$$3(1) + 7 = A(1 - 2) + B(0)$$
$$10 = A(-1)$$
$$A = -10$$

To find B, put $x = 2$ in (1)

$$3(2) + 7 = A(0) + B(2 - 1)$$
$$B = 13$$
$$\therefore \frac{3x + 7}{x^2 - 3x + 2} = \frac{-10}{x - 1} + \frac{13}{x - 2}$$
$$\frac{3x + 7}{x^2 - 3x + 2} = \frac{13}{x - 2} - \frac{10}{x - 1}$$

Example: 3.2: Resolve into partial fractions $\dfrac{x + 4}{(x^2 - 4)(x + 1)}$

The denominator $(x^2 - 4)(x + 1)$ can be further factored into linear factors

i.e. $(x^2 - 4)(x + 1) = (x + 2)(x - 2)(x + 1)$

Let $\dfrac{x + 4}{(x^2 - 4)(x + 1)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 2} + \dfrac{C}{x + 1}$, where A, B and C are constants to be determined.

$$\frac{x + 4}{(x^2 - 4)(x + 1)} = \frac{A(x - 2)(x + 1) + B(x + 2)(x + 1) + C(x + 2)(x - 2)}{(x + 2)(x - 2)(x + 1)}$$
$$\Rightarrow \quad x + 4 = A(x - 2)(x + 1) + B(x + 2)(x + 1) + C(x + 2)(x - 2) \quad \ldots (1)$$

To find A, put $x = -2$ in (1)

$$-2 + 4 = A(-2 - 2)(-2 + 1) + B(0) + C(0)$$
$$2 = 4A \Rightarrow \quad A = 1/2$$

To find B, put $x = 2$ in (1), we get $B = 1/2$

To find C, put $x = -1$ in (1), we get $C = -1$

$$\therefore \frac{x + 4}{(x^2 - 4)(x + 1)} = \frac{1/2}{(x + 2)} + \frac{1/2}{(x - 2)} + \frac{(-1)}{x + 1}$$
$$\Rightarrow \quad \frac{x + 4}{(x^2 - 4)(x + 1)} = \frac{1}{2(x + 2)} + \frac{1}{2(x - 2)} - \frac{1}{x + 1}$$

Type 2: Linear factors, some of which are repeated

If a linear factor $ax + b$ occurs n times as a factor of the denominator of the given fraction, then corresponding to these factors associate the sum of n simple fractions,

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \ldots + \frac{A_n}{(ax + b)^n}$$

Where $A_1, A_2, A_3, \ldots A_n$ are constants.

Example 3.3: Resolve into partial fractions $\dfrac{9}{(x-1)(x+2)^2}$

Let $\dfrac{9}{(x-1)(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$

$$\Rightarrow \quad \frac{9}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

$$\Rightarrow \quad 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad \ldots (1)$$

To find A, put $x = 1$ in (1)

We get $9 = A(1+2)^2 \quad \Rightarrow A = 1$

To find C, put $x = -2$ in (1)

We get $9 = C(-2-1) \quad \Rightarrow C = -3$

In (1), equating the coefficient of x^2 on both sides we get

$$A + B = 0$$

$$\Rightarrow \quad 1 + B = 0 \quad \Rightarrow B = -1$$

$$\therefore \frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

Type 3: Quadratic factors, none of which is repeated

If a quadratic factor $ax^2 + bx + c$ which is not factorable into linear factors occurs only once as a factor of the denominator of the given fraction, then corresponding to this factor associate a partial fraction $\dfrac{Ax + B}{ax^2 + bx + c}$ where A and B are constants which are not both zeros.

Consider $\dfrac{2x}{(x+1)(x^2+1)}$

We can write this proper fraction in the form $\dfrac{2x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+1}$

The first factor of the denominator $x + 1$ is of first degree, so we assume its numerator as a constant A. The second factor of the denominator $x^2 + 1$ is of 2^{nd} degree and which is not factorable into linear factors. We assume its numerator as a general first-degree expression $Bx + C$.

Example 3.4: Resolve into partial fractions $\dfrac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)}$

$$\text{Let} \quad \frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} = \frac{Ax + B}{x^2 + x + 6} + \frac{C}{x + 1}$$

$$\Rightarrow \quad \frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} = \frac{(Ax + B)(x + 1) + C(x^2 + x + 6)}{(x^2 + x + 6)(x + 1)}$$

$$\Rightarrow \quad x^2 - 2x - 9 = (Ax + B)(x + 1) + C(x^2 + x + 6) \qquad \ldots (1)$$

To find C put $x = -1$ in (1)

We get $1 + 2 - 9 = C(1 - 1 + 6) \Rightarrow C = -1$

To find B, put $x = 0$ in (1)

We get $-9 = B + 6C$

$-9 = B - 6 \quad \Rightarrow \quad B = -3$

To find A, Put $x = 1$ in (1)

$1 - 2 - 9 = (A - 3)(2) + (-1)(8) \Rightarrow -10 = 2A - 14$

$A = 2$

$$\therefore \quad \frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} = \frac{2x - 3}{x^2 + x + 6} - \frac{1}{x + 1}$$

Example 3.5: Resolve into partial fractions $\dfrac{x^2 + x + 1}{x^2 - 5x + 6}$

Solution:

Here the degree of the numerator is same as the degree of the denominator, i.e. an improper fraction.

$$\text{On division} \quad \frac{x^2 + x + 1}{x^2 - 5x + 6} = 1 + \frac{6x - 5}{x^2 - 5x + 6} \qquad \ldots (1)$$

$$\text{Let} \quad \frac{6x - 5}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$6x - 5 = A(x - 3) + B(x - 2)$$

By putting $x = 2$, $-A = 12 - 5 \Rightarrow A = -7$

By putting $x = 3$, $B = 18 - 5 \Rightarrow B = 13$

$$\therefore \quad \frac{x^2 + x + 1}{x^2 - 5x + 6} = -\frac{7}{x - 2} + \frac{13}{x - 3}$$

$$\therefore (1) \Rightarrow \quad \frac{x^2 + x + 1}{x^2 - 5x + 6} = 1 - \frac{7}{x - 2} + \frac{13}{x - 3}$$

EXERCISE 3.1

Resolve into partial fractions

(1) $\dfrac{1}{(x-1)(x+1)}$ (2) $\dfrac{7x-1}{6-5x+x^2}$ (3) $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$

(4) $\dfrac{1}{(x-1)(x+2)^2}$ (5) $\dfrac{x-2}{(x+2)(x-1)^2}$ (6) $\dfrac{x+1}{(x-2)^2(x+3)}$

(7) $\dfrac{x^2-6x+2}{x^2(x+2)}$ (8) $\dfrac{2x^2-5x-7}{(x-2)^3}$ (9) $\dfrac{x^2-3}{(x+2)(x^2+1)}$

(10) $\dfrac{x+2}{(x+1)(x^2+1)}$ (11) $\dfrac{7x^2-25x+6}{(x^2-2x-1)(3x-2)}$ (12) $\dfrac{x^2+x+1}{x^2+2x+1}$

3.2 Permutations:

Factorial:

The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\underline{\lfloor n}$

$$\text{i.e.}\quad n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$$
$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

Zero Factorial:

We will require zero factorial in the latter sections of this chapter. It does not make any sense to define it as the product of the integers from 1 to zero. So, we define $0! = 1$.

Deduction:

$$\begin{aligned} n! &= 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n \\ &= [1 \times 2 \times 3 \times 4 \times \ldots \times (n-1)]n \\ &= [(n-1)!]\, n \end{aligned}$$

Thus, $n! = n\,[(n-1)!]$

For example,

$$8! = 8(7!)$$

3.2.1 Fundamental Principles of Counting:

In this section we shall discuss two fundamental principles viz., principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations and form the base for permutations and combinations.

Fundamental Principle of Multiplication:

If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

Explanation:

If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

Example 3.6: In a class there are 15 boys and 20 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution :

Here the teacher is to perform two jobs :

(i) Selecting a boy among 15 boys, and

(ii) Selecting a girl among 20 girls

The first of these can be performed in 15 ways and the second in 20 ways.

Therefore by the fundamental principle of multiplication, the required number of ways is $15 \times 20 = 300$.

Fundamental Principle of Addition:

If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Example 3.7: In a class there are 20 boys and 10 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution:

Here the teacher is to perform either of the following two jobs :

(i) selecting a boy among 20 boys. (or)

(ii) Selecting a girl among 10 girls

The first of these can be performed in 20 ways and the second in 10 ways.

Therefore, by fundamental principle of addition either of the two jobs can be performed in $(20 + 10) = 30$ ways.

Hence, the teacher can make the selection of either a boy or a girl in 30 ways.

Example 3.8: A room has 10 doors. In how many ways can a man enter the room through one door and come out through a different door?

Solution:

Clearly, a person can enter the room through any one of the ten doors. So, there are ten ways of entering into the room.

After entering into the room, the man can come out through any one of the remaining 9 doors. So, he can come out through a different door in 9 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door = $10 \times 9 = 90$.

Example 3.9: How many words (with or without meaning) of three distinct letters of the English alphabets are there?

Solution:

Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place.

Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place.

After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of words

$$= 26 \times 25 \times 24 = 15600$$

Example 3.10:

How many three-digit numbers can be formed by using the digits 1, 2, 3, 4, 5.

Solution :

We have to determine the total number of three digit numbers formed by using the digits 1, 2, 3, 4, 5.

Clearly, the repetition of digits is allowed.

A three digit number has three places viz. unit's, ten's and hundred's. Unit's place can be filled by any of the digits 1, 2, 3, 4, 5. So unit's place can be filled in 5 ways.

Similarly, each one of the ten's and hundred's place can be filled in 5 ways.

$\therefore$ Total number of required numbers

$$= \quad 5 \times 5 \times 5 = 125$$

Example 3.11: There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?

Solution:

Here we have to perform 6 jobs of answering 6 multiple choice questions.

Each of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences

$$= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$$

Example 3.12: How many three-digit numbers greater than 600 can be formed by using the digits 4, 5, 6, 7, 8?

Solution:

Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6, 7 or 8 at hundred's place. So, hundred's place can be filled in 3 ways.

Each of the ten's and one's place can be filled in 5 ways.

Hence, total number of required numbers

$$= 3 \times 5 \times 5 = 75$$

Example 3.13: How many numbers divisible by 5 and lying between 5000 and 6000 can be formed from the digits 5, 6, 7, 8 and 9?

Solution:

Clearly, a number between 5000 and 6000 must have 5 at thousand's place. Since the number is divisible by 5 it must have 5 at unit's place.

Now, each of the remaining places (viz. Hundred's and ten's) can be filled in 5 ways.

Hence the total number of required numbers

$$= 1 \times 5 \times 5 \times 1 = 25$$

Example 3.14: How many three digit odd numbers can be formed by using the digits 4, 5, 6, 7, 8, 9 if :

(i) the repetition of digits is not allowed?

(ii) the repetition of digits is allowed?

Solution:

For a number to be odd, we must have 5, 7 or 9 at the unit's place.

So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways.

Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers

$$= 3 \times 5 \times 4 = 60$$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence required number of numbers $= 3 \times 6 \times 6 = 108$

EXERCISE 3.2

1. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent a competition. In how many ways can the teacher make this selection?
2. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other?
3. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles?
4. Twelve students compete in a race. In how many ways first three prizes be given?
5. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-in charge are to be appointed. In how many ways this can be done?
6. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each?
7. How many numbers are there between 500 and 1000 which have exactly one of their digits as 8?
8. How many five-digit number license plates can be made if

   (i) first digit cannot be zero and the repetition of digits is not allowed.

   (ii) the first digit cannot be zero, but the repetition of digits is allowed?
9. How many different numbers of six digits can be formed from the digits 2, 3, 0, 7, 9, 5 when repetition of digits is not allowed?
10. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed?
11. In how many ways can an examinee answer a set of 5 true / false type questions?

12. How many 4-digit numbers are there?
13. How many three – letter words can be formed using *a, b, c, d, e* if :
    (i) repetition is allowed (ii) repetition is not allowed?
14. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there?

3.2.2. Concept of Permutations:

The word permutation means arrangement.

For example, given 3 letters *a, b, c* suppose we arrange them taking 2 at a time.

The various arrangements are *ab, ba, bc, cb, ac, ca*.

Hence the number of arrangements of 3 things taken 2 at a time is 6 and this can be written as $3P2 = 6$.

Definition:

The number of arrangements that can be made out of n things taking r at a time is called the number of permutations of n things taken r at a time.

Notation:

If n and r are positive integers such that $1 \leq r \leq n$, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or nPr.

We use the symbol nPr throughout our discussion. Thus nPr = Total number of permutations of n distinct things taken r at a time.

Note: In permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

Example 3.15: Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking 3 at a time and starting with E.

Solution: The permutations of vowels A, E, I, O, U taking three at a time and starting with E are

EAI, EIA, EIO, EOI, EOU, EUO, EAO, EOA, EIU, EUI, EAU, EUA.

Clearly there are 12 permutations.

Theorem 3.1:

Let r and n be positive integers such that $1 \leq r \leq n$.

Then the number of all permutations of n distinct things taken r at a time is given by $n(n-1)(n-2)\ldots\left(n-r-1\right)$

i.e. $nPr = n(n-1)(n-2)\ldots\left(n-r-1\right)$

Proof:

Let $n\mathrm{P}r$ denote the number of permutations of n things taken r at a time.

Clearly the total number of permutations required is same as the number of possible ways of filling up r blank spaces by n things.

$$\begin{array}{ccccc} \boxed{} & \boxed{} & \boxed{} & \dots & \boxed{} \\ 1 & 2 & 3 & & r \end{array}$$

Let there be r blank spaces arranged in a row

The first place can be filled by any one of the n things in n ways.

If the first place is filled up by any one of the n things, there will be $(n - 1)$ things remaining. Now the second place can be filled up by any one of the $(n - 1)$ remaining things.

Here it can be filled up in $(n - 1)$ ways.

Hence the first two places can be together filled in $n(n - 1)$ ways.

Having filled up these two places, we have $(n - 2)$ things remaining with which we can fill up the third place. So the third place can be filled up by any one of these things in $(n - 2)$ ways.

Hence the first three places can be together filled in $n(n - 1)(n - 2)$ ways.

Proceeding in this way, we find that the total number of ways of filling up the r spaces is

$n(n - 1)(n - 2)\dots$ upto r factors

i.e. $n(n - 1)(n - 2) \dots \left(n - \overline{r-1}\right)$

$\therefore\ n\mathrm{P}r = n(n - 1)(n - 2) \dots \left(n - \overline{r-1}\right) = n(n - 1)(n - 2) \dots (n - r + 1)$

Theorem 3.2:

Let r and n be positive integers such that $1 \le r \le n$. Then $n\mathrm{P}r = \dfrac{n!}{(n-r)!}$

Proof:

$$\begin{aligned} n\mathrm{P}r &= n(n-1)(n-2)\dots\left(n-\overline{r-1}\right) \\ &= \frac{n(n-1)(n-2)\dots\left(n-\overline{r-1}\right)(n-r)\left(n-\overline{r+1}\right)\dots 2.1}{(n-r)\left(n-\overline{r+1}\right)\dots 2.1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Theorem 3.3:

The number of all permutations of n distinct things, taken all at a time is $n!$

Proof: $\quad nPr = n(n-1)(n-2) \ldots \left(n - \overline{r-1}\right)$

By putting $r = n$,

$$\begin{aligned} nPn &= n(n-1)(n-2) \ldots \left(n - \overline{n-1}\right) \\ &= n(n-1)(n-2) \ldots \left(n - \overline{n-1}\right) \\ &= n(n-1)(n-2) \ldots 1 \\ &= n! \\ \therefore\ nPn &= n! \end{aligned}$$

Remark: We have already defined $0! = 1$. This can also be derived as follows.

We know that $\quad nPr = \dfrac{n!}{(n-r)!}$

By putting $r = n$, $\quad nPn = \dfrac{n!}{(n-n)!}$

$$\Rightarrow \quad n! = \frac{n!}{0!} \quad (\because\ nPn = n!)$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1$$

$$0! = 1$$

Example 3.16: Evaluate $8P_3$

Solution:

$$\begin{aligned} 8P_3 &= \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{(8 \times 7 \times 6) \times 5!}{5!} \\ &= 8 \times 7 \times 6 \\ &= 336 \end{aligned}$$

Example 3.17 : If $5Pr = 6P_{r-1}$, find r

Solution: $\quad 5Pr = 6P_{r-1}$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6!}{\left(6 - \overline{r-1}\right)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6 \times 5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6 \times 5!}{\{(7-r)(6-r)\}(5-r)!}$$

$$\Rightarrow \quad 1 = \frac{6}{(7-r)(6-r)}$$

$$\Rightarrow \quad (7-r)(6-r) = 6 \quad \Rightarrow \quad 42 - 7r - 6r + r^2 - 6 = 0$$

$$\Rightarrow \quad r^2 - 13r + 36 = 0 \quad \Rightarrow \quad (r-9)(r-4) = 0$$

$$\Rightarrow \quad r = 9 \text{ or } r = 4$$

$$\Rightarrow \quad r = 4 \qquad (\because \ 5\text{P}r \text{ is meaningful for } r \le 5)$$

Example 3.18:

If $n\text{P}_4 = 360$, find the value of n.

Solution: $n\text{P}_4 = 360 \quad \Rightarrow \quad \dfrac{n\,!}{(n-4)\,!} = 6 \times 5 \times 4 \times 3$

$$\Rightarrow \quad \frac{n\,!}{(n-4)\,!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = \frac{6!}{2!}$$

$$\Rightarrow \quad n! = 6!$$

$$\Rightarrow \quad n = 6$$

Example 3.19:

If $9\text{P}r = 3024$, find r.

Solution: $9\text{P}r = 3024$

$$\Rightarrow \quad = 9 \times 8 \times 7 \times 6 = 9P_4$$

$$\Rightarrow \quad r = 4$$

Example 3.20:

If $(n-1)\text{P}_3 : n\text{P}_4 = 1 : 9$, find n.

Solution:

$$(n-1)\text{P}_3 : n\text{P}_4 = 1 : 9$$

$\Rightarrow (n-1)(n-2)(n-3) : n(n-1)(n-2)(n-3) = 1 : 9$

$\Rightarrow$ i.e. $9(n-1)(n-2)(n-3) = n(n-1)(n-2)(n-3)$

$\Rightarrow n = 9$

Example 3.21: In how many ways can five children stand in a queue?

Solution:

The number of ways in which 5 persons can stand in a queue is same as the number of arrangements of 5 different things taken all at a time.

Hence the required number of ways

$$= \quad {}_5\text{P}_5 = 5! = 120$$

Example 3.22: How many different signals can be made by hoisting 6 differently coloured flags one above the other, when any number of them may be hoisted at one time?

Solution:

The signals can be made by using at a time one or two or three or four or five or six flags.

The total number of signals when *r*-flags are used at a time from 6 flags is equal to the number of arrangements of 6, taking *r* at a time i.e. $6P_r$

Hence, by the fundamental principle of addition, the total number of different signals

$$= {}_6P_1 + {}_6P_2 + {}_6P_3 + {}_6P_4 + {}_6P_5 + {}_6P_6$$

$$= 6 + (6 \times 5) + (6 \times 5 \times 4) + (6 \times 5 \times 4 \times 3) + (6 \times 5 \times 4 \times 3 \times 2) + (6 \times 5 \times 4 \times 3 \times 2 \times 1)$$

$$= 6 + 30 + 120 + 360 + 720 + 720 = 1956$$

Example 3.23: Find the number of different 4-letter words with or without meanings, that can be formed from the letters of the word 'NUMBER'

Solution:

There are 6 letters in the word 'NUMBER'.

So, the number of 4-letter words

= the number of arrangements of 6 letters taken 4 at a time

$= {}_6P_4$

$= 360$

Example 3.24: A family of 4 brothers and 3 sisters is to be arranged in a row, for a photograph. In how many ways can they be seated, if

(i) all the sisters sit together.

(ii) all the sisters are not together.

Solution :

(i) Since the 3 sisters are inseparable, consider them as one single unit.

This together with the 4 brothers make 5 persons who can be arranged among themselves in 5! ways.

In everyone of these permutations, the 3 sisters can be rearranged among themselves in 3! ways.

Hence the total number of arrangements required $= 5! \times 3! = 120 \times 6 = 720$

(ii) The number of arrangements of all the 7 persons without any restriction $=7! = 5040$

Number of arrangements in which all the sisters sit together = 720

$\therefore$ Number of arrangements required $= 5040 - 720 = 4320$

3.2.3 Permutations of objects not all distinct:

The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is $\dfrac{n!}{p!\, q!}$

Example 3.25: How many permutations of the letters of the word 'APPLE' are there?

Solution:

Here there are 5 letters, two of which are of the same kind.

The others are each of its own kind.

$\therefore$ Required number of permutations is $= \dfrac{5!}{2!\ 1!\ 1!\ 1!} = \dfrac{5!}{2!} = \dfrac{120}{2} = 60$

Example 3.26: How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

Solution:

There are 4 odd digits 1, 1, 3, 3 and 4 odd places.

So odd digits can be arranged in odd places in $\dfrac{4!}{2!\ 2!}$ ways.

The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\dfrac{3!}{2!}$ ways.

Hence, the required number of numbers $= \dfrac{4!}{2!\ 2!} \times \dfrac{3!}{2!} = 6 \times 3 = 18$

Example 3.27: How many arrangements can be made with the letters of the word "MATHEMATICS"?

Solution:

There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct.

$\therefore$ required number of arrangements $= \dfrac{11!}{2! \times 2! \times 2!} = 4989600$

3.2.4 Permutations when objects can repeat:

The number of permutations of n different things, taken r at a time, when each may be repeated any number of times in each arrangement, is n^r

Consider the following example:

In how many ways can 2 different balls be distributed among 3 boxes?

Let A and B be the 2 balls. The different ways are

Box 1	Box 2	Box 3
$\boxed{A}$	$\boxed{B}$	$\square$
$\boxed{B}$	$\boxed{A}$	$\square$
$\square$	$\boxed{A}$	$\boxed{B}$
$\square$	$\boxed{B}$	$\boxed{A}$
$\boxed{A}$	$\square$	$\boxed{B}$
$\boxed{B}$	$\square$	$\boxed{A}$
$\square$	$\square$	$\boxed{AB}$
$\boxed{AB}$	$\square$	$\square$
$\square$	$\boxed{AB}$	$\square$

i.e. 9 ways. By formula $n^r = 3^2 = 9$ ways

Example 3.28: In how many ways can 5 different balls be distributed among 3 boxes?

Solution:

There are 5 balls and each ball can be placed in 3 ways.

So the total number of ways = $3^5 = 243$

Example: 3.29: In how many ways can 3 prizes be distributed among 4 boys, when (i) no boy gets more than one prize?

(ii) a boy may get any number of prizes?

(iii) no boy gets all the prizes?

Solution:

(i) The total number of ways is the number of arrangements of 4 taken 3 at a time.

So, the required number of ways = ${}_4P_3 = 4! = 24$

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys.

The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize.

Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away

$= 4 \times 4 \times 4 = 4^3 = 64$

(iii) Since any one of the 4 boys may get all the prizes. So, the number of ways in which a boy get all the 3 prizes = 4.

So, the number of ways in which a boy does not get all the prizes = 64–4=60

3.2.5 Circular Permutations:

We have seen that the number of permutations of n different things taken all together is $n!$, where each permutation is a different arrangement of the n things in a row, or a straight line. These permutations are called linear permutations or simply permutations.

A circular permutation is one in which the things are arranged along a circle. It is also called closed permutation.

Theorem 3.4:

The number of circular permutations of n distinct objects is $(n - 1)!$

Proof:

Let $a_1, \quad a_2, \dots, \quad a_{n-1}, \; a_n$ be n distinct objects.

Let the total number of circular permutations be x.

Consider one of these x permutations as shown in figure.

Clearly this circular permutation provides n near permutations as given below

$a_1, a_2, a_3, \quad \dots, \quad a_{n-1}, \quad a_n$

$a_2, a_3, a_4, \quad \dots, \quad a_n, \quad a_1$

$a_3, a_4, a_5, \quad \dots, \quad a_1, \quad a_2$

$\dots \dots \quad \dots \quad \dots$

$\dots \dots \quad \dots \quad \dots$

$a_n, a_1, a_2, \quad \dots, \quad a_{n-2}, \quad a_{n-1}$

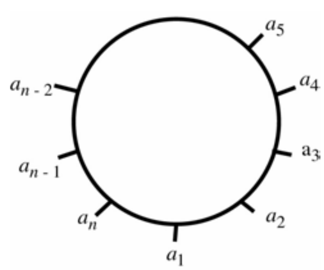


Fig. 3. 1

Thus, each circular permutation gives n linear permutations.

But there are x circular permutations.

So, total number of linear permutations is xn.

But the number of linear permutations of n distinct objects is $n!$.

$$\therefore xn = n!$$

$$\Rightarrow \quad x = \frac{n!}{n}$$

$$x = (n-1)\,!$$

$\therefore$ The total number of circular permutations of n distinct objects is $(n - 1)!$

Note: In the above theorem anti-clockwise and clockwise order of arrangements are considered as distinct permutations.

Difference between clockwise and anti-clockwise arrangements:

Consider the following circular permutations:

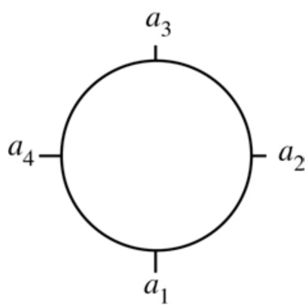


Fig. 3. 2

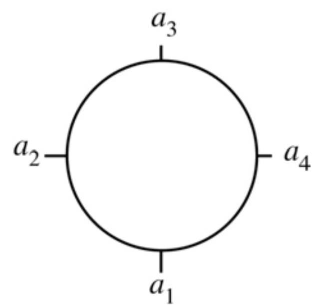


Fig. 3. 3

We observe that in both, the order of the circular arrangement is a_1, a_2, a_3, a_4.

In fig (3.2) the order is anti-clockwise, whereas in fig. (3.3) the order is clockwise.

Thus the number of circular permutation of n things in which clockwise and anti-clockwise arrangements give rise to different permutations is $(n - 1)!$

If there are n things and if the direction is not taken into consideration, the number of circular permutations is $\frac{1}{2}\ (n - 1)!$

Example 3.30:

In how many ways 10 persons may be arranged in a (i) line (ii) circle?

Solution:

(i) The number of ways in which 10 persons can be arranged in a line $= {}_{10}P_{10} = 10!$

(ii) The number of ways in which 10 persons can be arranged in a circle $= (10 - 1)! = 9!$

Example 3.31: In how many ways can 7 identical beads be stung into a ring?

Solution: Since the arrangement is circular either clockwise arrangement or anti-clockwise arrangement may be considered.

$\therefore$ The required number of ways $= \frac{1}{2}\ (7 - 1)! = \frac{6!}{2} = 360$

Example 3.32: In how many ways can 5 gentlemen and 5 ladies sit together at a round table, so that no two ladies may be together?

Solution:

The number of ways in which 5 gentlemen may be arranged is $(5 - 1)! = 4!$

Then the ladies may be arranged among themselves in 5! ways.

Thus the total number of ways $= 4! \times 5! = 24 \times 120 = 2880$

Example 3.33: Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Solution:

Considering 4 particular flowers as one flower, we have five flowers, which can be strung to form a garland in 4! ways.

But 4 particular flowers can be arranged in 4! ways.

$\therefore$ Required number of ways $= 4\,! \times 4\,! = 576$

EXERCISE 3.3

1. Evaluate the following :

   (i) ${}_5P_3$ (ii) ${}_{15}P_3$ (iii) ${}_5P_5$ (iv) ${}_{25}P_{20}$ (v) ${}_9P_5$

2. If ${}_nP_4 = 20\,.\,{}_nP_3$, find n.

3. If ${}_{10}P_r = 5040$, find the value of r.

4. If ${}_{56}P_{(r+6)} : {}_{54}P_{(r+3)} = 30800 : 1$, find r

5. If P_m stands for ${}_mP_m$, then prove that $1 + 1.P_1 + 2.P_2 + 3.P_3 + \dots$

   $+ n.P_n = (n+1)!$

6. Prove that ${}_nP_r = {}_{(n-1)}P_r + r\,.\,{}_{(n-1)}P_{(r-1)}$.

7. Three men have 4 coats, 5 waistcoats and 6 caps. In how many ways can they wear them?

8. How many 4-letter words, with or without meaning, can be formed, out of the letters of the word, ‘LOGARITHMS’, if repetition of letters is not allowed?

9. How many 3-digit numbers are there, with distinct digits, with each digit odd?

10. Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.

11. How many different words can be formed with the letters of the word ‘MISSISSIPPI’?

12. (i) How many different words can be formed with letters of the word ‘HARYANA’?

    (ii) How many of these begin with H and end with N?

13. How many 4-digit numbers are there, when a digit may be repeated any number of times?

14. In how many ways 5 rings of different types can be worn in 4 fingers?

15. In how many ways can 8 students are seated in a (i) line (ii) circle?

16. In how many ways can a garland of 20 similar flowers are made?

3.3 Combinations:

The word combination means selection. Suppose we are asked to make a selection of any two things from three things *a*, *b* and *c*, the different selections are *ab*, *bc*, *ac*.

Here there is no reference to the order in which they are selected.

i.e. *ab* and *ba* denote the same selection. These selections are called combinations.

Definition:

A selection of any *r* things out of *n* things is called a combination of *n* things *r* at a time.

Notation:

The number of all combinations of *n* objects, taken *r* at a time is generally denoted by ${}_nC_r$ or $C(n,r)$ or $\binom{n}{r}$. We use the symbol ${}_nC_r$ throughout our discussion.

$$\text{Thus } {}_nC_r = \begin{cases}\text{Number of ways of selecting} \\ r \text{ objects from } n \text{ objects}\end{cases}$$

Difference between Permutation and Combination:

1. In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

i.e. in a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential.

2. Usually the number of permutation exceeds the number of combinations.
3. Each combination corresponds to many permutations.

Combinations of *n* different things taken *r* at a time:

Theorem 3.5:

The number of all combinations of *n* distinct objects, taken *r* at a time is given by ${}_nC_r = \dfrac{n!}{(n-r)!\,r!}$

Proof: Let the number of combinations of *n* distinct objects, taken *r* at a time be denoted by ${}_nC_r$.

Each of these combinations contains *r* things and all these things are permuted among themselves.

$\therefore$ The number of permutations obtained is $r\,!$

Hence from all the ${}_nC_r$ combinations we get ${}_nC_r \times r!$ permutations.

But this gives all the permutations of n things taken r at a time i.e. ${}_nP_r$.

Hence, $\quad {}_nC_r \,.\, r\,! = {}_nP_r$

$$\therefore\ {}_nC_r = \frac{{}_nP_r}{r!}$$

$$= \frac{n!}{(n-r)!\ r!} \qquad \left(\because\ {}_nP_r = \frac{n!}{(n-r)!}\right)$$

Properties

(1) ${}_nC_n = 1$ $\qquad$ (2) ${}_nC_0 = 1$ $\qquad$ (3) ${}_nC_r = {}_nC_{n-r}$ $\qquad 0 \le r \le n$

Proof:

(1) We know that $\quad {}_nC_r = \dfrac{n!}{(n-r)!\ r!}$

Putting $r = n$, we have $\quad {}_nC_n = \dfrac{n!}{(n-n)!\ n!} = \dfrac{n\,!}{0!\ n!}$

$= 1$

(2) Putting $r = 0$, we have

$$ {}_nC_0 = \frac{n!}{(n-0)\ !\ 0!} = \frac{n\,!}{n!} = 1$$

(3) We have $\quad {}_nC_{n-r} = \dfrac{n!}{(n-r)!\,\left(n-\overline{n-r}\right)!} = \dfrac{n!}{(n-r)!\ r!}$

$= {}_nC_r$

Note: The above property can be restated as follows :

If x and y are non-negative integers such that $x + y = n$, then ${}_nC_x = {}_nC_y$

(4) If n and r are positive integers such that $r \le n$,

then ${}_nC_r + {}_nC_{(r-1)} = {}_{(n+1)}C_r$

Proof: We have

$$ {}_nC_r + {}_nC_{(r-1)} = \frac{n!}{(n-r)!\ r!} + \frac{n!}{\left(n-\overline{r-1}\right)!\ (r-1)!}$$

$$= \frac{n!}{(n-r)!\ r!} + \frac{n!}{(n-r+1)\ !\ (r-1)!}$$

$$= \frac{n!}{(n-r)\ !\ r\{(r-1)!\}} + \frac{n!}{(n-r+1)\ \{(n-r)!\ (r-1)!\}}$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left\{\frac{1}{r} + \frac{1}{n-r+1}\right\}$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left\{\frac{n-r+1+r}{r(n-r+1)}\right\}$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \left\{\frac{n+1}{r(n-r+1)}\right\}$$

$$= \frac{(n+1)\,\{n!\}}{(n-r+1)\,(n-r)!\,r(r-1)!}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!}$$

$$= {}_{(n+1)}C_r$$

(5) If n and r are positive integers such that $1 \leq r \leq n$,

then ${}_nC_r = \frac{n}{r}\ {}_{(n-1)}C_{(r-1)}$

Proof:

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n(n-1)!}{[(n-1)-(r-1)]!\,r\,(r-1)!}$$

$$= \frac{n}{r} \quad \frac{(n-1)!}{[(n-1)-(r-1)]!\,(r-1)!}$$

$$= \frac{n}{r} \quad {}_{(n-1)}C_{(r-1)}$$

(6)If $1 \leq r \leq n$, then $n \,.\, {}_{(n-1)}C_{(r-1)} = (n-r+1) \,.\, {}_nC_{(r-1)}$

Proof:

We have $\quad n \,.\, {}_{(n-1)}C_{(r-1)} = n\left\{\frac{(n-1)!}{[(n-1)-(r-1)]!\,(r-1)!}\right\}$

$$= \frac{n!}{(n-r)!\,(r-1)!}$$

$$= \frac{(n-r+1)n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$= (n - r + 1)\left[\frac{n!}{(n - r + 1)!\,(r - 1)!}\right]$$

$$= (n - r + 1)\left[\frac{n!}{\left(n - r - 1\right)!\,(r - 1)!}\right]$$

$$= (n - r + 1)\,.\,{}_nC_{(r - 1)}$$

(7) For any positive integers x and y,

$${}_nC_x = {}_nC_y \Rightarrow x = y \text{ or } x + y = n$$

Proof: We have ${}_nC_x = {}_nC_y$

$\Rightarrow$ ${}_nC_x = {}_nC_y = {}_nC_{(n-y)}$ $\quad [\because {}_nC_y = {}_nC_{(n-y)}]$

$\Rightarrow x = y$ or $x = n - y$

$\Rightarrow x = y$ or $x + y = n$

Note: If ${}_nC_x = {}_nC_y$ and $x \neq y$, then $x + y = n$

Example 3.34: Evaluate the following :

(i) ${}_6C_3$ (ii) $\sum_{r=1}^{5} {}_5C_r$

Solution:

(i) $${}_6C_3 = \frac{{}_6P_3}{3!} = \frac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$$

(ii) $$\sum_{r=1}^{5} {}_5C_r = {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5$$

$$= 5 + 10 + 10 + 5 + 1 = 31$$

Example 3.35: If ${}_nC_4 = {}_nC_6$, find ${}_{12}C_n$

Solution:

$${}_nC_4 = {}_nC_6 \Rightarrow n = 4 + 6 = 10$$

Now $${}_{12}C_n = {}_{12}C_{10}$$

$$= {}_{12}C_{(12-10)} = {}_{12}C_2 = \frac{12 \times 11}{1 \times 2}$$

$$= 66$$

Example 3.36: If ${}_{15}C_r : {}_{15}C_{(r-1)} = 11 : 5$, find r

Solution:

$$^{15}C_r : {}^{15}C_{(r-1)} = 11 : 5 \Rightarrow \frac{^{15}C_r}{^{15}C_{(r-1)}} = \frac{11}{5}$$

$$\Rightarrow \frac{\dfrac{15!}{r!(15-r)!}}{\dfrac{15!}{(r-1)!\,(15-r+1)!}} = \frac{11}{5}$$

$$\Rightarrow \frac{15!}{r!\,(15-r)!} \times \frac{(r-1)!\,(16-r)!}{15!} = \frac{11}{5}$$

$$\Rightarrow \frac{(r-1)!\,(16-r)\,\{(15-r)!\}}{r(r-1)!\,(15-r)!} = \frac{11}{5}$$

$$\Rightarrow \frac{16-r}{r} = \frac{11}{5}$$

$$\Rightarrow 5(16-r) = 11r \Rightarrow 80 = 16r$$

$$\Rightarrow r = 5$$

Example 3.37: Show that the product of r consecutive integers is divisible by r!

Solution:

Let the r consecutive integers be $n + 1, n + 2, n + 3, \ldots, n + r$

$$\text{Hence their product} = (n+1)\,(n+2)\,(n+3)\ldots(n+r)$$

$$= \frac{1.2.3\ldots n.\,(n+1)\,(n+2)\ldots(n+r)}{1.2.3\ldots n}$$

$$= \frac{(n+r)!}{n!}$$

$$\therefore \frac{\text{their product}}{r!} = \frac{(n+r)!}{n!\,r!}$$

$$= {}^{(n+r)}C_r \text{ which is an integer.}$$

$\therefore$ The product of r consecutive integers is divisible by r!

Example 3.38: Let r and n be positive integers such that $1 \le r \le n$. Then prove the following :

$$\frac{^nC_r}{^nC_{(r-1)}} = \frac{n-r+1}{r}$$

Solution: $$\frac{^nC_r}{^nC_{(r-1)}} = \frac{\dfrac{n!}{r!(n-r)!}}{\dfrac{n!}{(r-1)!\,(n-r+1)!}}$$

$$= \frac{n!}{r!\,(n-r)!} \times \frac{(r-1)!\,(n-r+1)!}{n!}$$

$$= \frac{(r-1)!\,(n-r+1)\,\{(n-r)!\}}{r(r-1)!\,(n-r)!}$$

$$= \frac{n-r+1}{r}$$

Example 3.39 : If ${}_nP_r = {}_nP_{(r+1)}$ and ${}_nC_r = {}_nC_{(r-1)}$, find the values of n and r

Solution:

$${}_nP_r = {}_nP_{(r+1)} \Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow \frac{1}{(n-r)\,(n-r-1)!} = \frac{1}{(n-r-1)!}$$

$$\Rightarrow n-r = 1 \qquad \ldots (1)$$

$${}_nC_r = {}_nC_{(r-1)} \Rightarrow \frac{n!}{r!\,(n-r)!} = \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$\Rightarrow \frac{n!}{r(r-1)!\,(n-r)!} = \frac{n!}{(r-1)!\,(n-r+1)\,\{(n-r)!\}}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{n-r+1}$$

$$\Rightarrow n-r+1 = r$$

$$\Rightarrow n-2r = -1 \qquad \ldots (2)$$

Solving (1) and (2) we get $n = 3$ and $r = 2$

EXERCISE 3.4

1. Evaluate the following:

(i) ${}_{10}C_8$ (ii) ${}_{100}C_{98}$ (iii) ${}_{75}C_{75}$

2. If ${}_nC_{10} = {}_nC_{12}$, find ${}_{23}C_n$

3. If ${}_8C_r - {}_7C_3 = {}_7C_2$, find r

4. If ${}_{16}C_4 = {}_{16}C_{r+2}$, find ${}_rC_2$

5. Find n if (i) $2 \,.\, {}_nC_3 = \frac{20}{3}\ {}_nC_2$ (ii) ${}_nC_{(n-4)} = 70$

6. If ${}_{(n+2)}C_8 : {}_{(n-2)}P_4 = 57 : 16$, find n.

7. If ${}_{28}C_{2r} : {}_{24}C_{(2r-4)} = 225 : 11$, find r.

Practical problems on Combinations

Example 3.40: From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways this can be done?

Solution:

There are 15 players in a group. We have to select 11 players from the group.

$\therefore$ The required number of ways = ${}_{15}C_{11}$

$$ {}_{15}C_{11} \quad = \quad \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4} = 1365 \text{ ways} $$

Example 3.41: How many different teams of 8, consisting of 5 boys and 3 girls can be made from 25 boys and 10 girls?

Solution:

5 boys out of 25 boys can be selected in ${}_{25}C_5$ ways.

3 girls out of 10 girls can be selected in ${}_{10}C_3$ ways.

$\therefore$ The required number of teams = ${}_{25}C_5 \times {}_{10}C_3 = 6375600$

Example 3.42: How many triangles can be formed by joining the vertices of a hexagon?

Solution:

There are 6 vertices of a hexagon.

One triangle is formed by selecting a group of 3 vertices from given 6 vertices.

This can be done in ${}_6C_3$ ways.

$$ \therefore \text{ Number of triangles } = {}_6C_3 = \frac{6!}{3!\ 3!} = 20 $$

Example 3.43:

A class contains 12 boys and 10 girls. From the class 10 students are to be chosen for a competition under the condition that atleast 4 boys and atleast 4 girls must be represented. The 2 girls who won the prizes last year should be included. In how many ways can the selection are made?

Solution:

There are 12 boys and 10 girls. From these we have to select 10 students.

Since two girls who won the prizes last year are to be included in every selection.

So, we have to select 8 students from 12 boys and 8 girls, choosing atleast 4 boys and atleast 2 girls. The selection can be formed by choosing

(i) 6 boys and 2 girls

(ii) 5 boys and 3 girls

(iii) 4 boys and 4 girls

$$\begin{aligned}\therefore \text{ Required number of ways } &= ({}_{12}C_6 \times {}_8C_2) + ({}_{12}C_5 \times {}_8C_3) + ({}_{12}C_4 \times {}_8C_4)\\ &= (924 \times 28) + (792 \times 56) + (495 \times 70)\\ &= 25872 + 44352 + 34650\\ &= 104874\end{aligned}$$

Example 3.44: How many diagonals are there in a polygon?

Solution: A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon.

$$\left.\begin{array}{c}\text{Number of line segments obtained by}\\ \text{joining the vertices of a } n \text{ sided}\\ \text{polygon taken two at a time}\end{array}\right\} = \text{Number of ways of selecting 2 out of } n$$

$$= {}_nC_2 = \frac{n(n-1)}{2}$$

Out of these lines, n lines are the sides of the polygon.

$$\begin{aligned}\therefore \text{ Number of diagonals of the polygon } &= \frac{n(n-1)}{2} - n\\ &= \frac{n(n-3)}{2}\end{aligned}$$

Example 3.45 How many different sections of 4 books can be made from 10 different books, if (i) there is no restriction

(ii) two particular books are always selected;

(iii) two particular books are never selected?

Solution:

(i) The total number of ways of selecting 4 books out of 10 = ${}_{10}C_4 = \frac{10!}{4!\,6!} = 210$

(ii) If two particular books are always selected.

This means two books are selected out of the remaining 8 books

$$\therefore \text{ Required number of ways } = {}_8C_2 = \frac{8!}{2!\,6!} = 28$$

(iii) If two particular books are never selected

This means four books are selected out of the remaining 8 books.

$$\therefore \text{ Required number of ways } = {}_8C_4 = \frac{8!}{4!\,4!} = 70$$

Example 3.46:

In how many ways players for a cricket team can be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers? Assume that the team requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.

Solution:

The selection of team is divided into 4 phases:

(i) Selection of 5 batsmen out of 10. This can be done in ${}_{10}C_5$ ways.

(ii) Selection of 3 all-rounders out of 5. This can be done in ${}_5C_3$ ways.

(iii) Selection of 2 bowlers out of 8. This can be done in ${}_8C_2$ ways.

(iv) Selection of one wicket keeper out of 2. This can be done in ${}_2C_1$ ways.

$\therefore$ The team can be selected in ${}_{10}C_5 \times {}_5C_3 \times {}_8C_2 \times {}_2C_1$ ways

$= 252 \times 10 \times 28 \times 2$ ways

$= 141120$ ways

Example 3.47: Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many

(i) straight lines (ii) triangles can be formed by joining them?

Solution:

(i) Number of straight lines formed joining the 18 points,
taking 2 at a time $= {}_{18}C_2 = 153$

Number of straight lines formed by joining the 5 points,
taking 2 at a time $= {}_5C_2 = 10$

But 5 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines $= 153 - 10 + 1 = 144$

(ii) Number of triangles formed by joining the 18 points,
taken 3 at a time $= {}_{18}C_3 = 816$

Number of triangles formed by joining the 5 points,
taken 3 at a time $= {}_5C_3 = 10$

But 5 collinear points cannot form a triangle when taken 3 at a time.

$\therefore$ Required number of triangles $= 816 - 10 = 806$

EXERCISE 3.5

1. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?

2. In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women?
3. How many triangles can be obtained by joining 12 points, five of, which are collinear?
4. A box contains 5 different red and 6 different white balls. In how many ways 6 balls be selected so that there are atleast two balls of each colour?
5. In how many ways can a cricket team of eleven be chosen out of a batch of 15 players if
   (i) there is no restriction on the selection
   (ii) a particular player is always chosen;
   (iii) a particular player is never chosen?
6. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions.
7. There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the
   (i) the number of straight lines obtained from the pairs of these points
   (ii) number of triangles that can be formed with the vertices as these points.
8. In how many ways can 21 identical books on Tamil and 19 identical books on English be placed in a row on a shelf so that two books on English may not be together?
9. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?

3.4 Mathematical Induction:

Introduction:

The name 'Mathematical induction' in the sense in which we have given here, was first used by the English Mathematician Augustus De-Morgan (1809 – 1871) in his article on 'Induction Mathematics' in 1938. However the originator of the Principle of Induction was Italian Mathematician Francesco Mau Rolycus (1494 – 1575). The Indian Mathematician Bhaskara (1153 A.D) had also used traces of 'Mathematical Induction' in his writings.

"Induction is the process of inferring a general statement from the truth of particular cases".

For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 7 + 3$ and so on.

From these cases one may make a general statement "every even integer except 2 can be expressed as a sum of two prime numbers. There are hundreds of particular cases where this is known to be true. But we cannot conclude that this statement is true unless it is proved. Such a statement inferred from particular cases is called a conjecture. A conjecture remains a conjecture until it is proved or disproved.

Let the conjecture be a statement involving natural numbers. Then a method to prove a general statement after it is known to be true in some particular cases is the principle of mathematical induction.

Mathematical induction is a principle by which one can conclude that a statement is true for all positive integers, after proving certain related propositions.

The Principle of Mathematical Induction:

Corresponding to each positive integer n let there be a statement or proposition $P(n)$.

If (i) $P(1)$ is true,

and (ii) $P(k + 1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all positive integers n.

We shall not prove this principle here, but we shall illustrate it by some examples.

Working rules for using principle of mathematical induction:

Step (1) : Show that the result is true for $n = 1$.

Step (2) : Assume the validity of the result for n equal to some arbitrary but fixed natural number, say k.

Step (3) : Show that the result is also true for $n = k + 1$.

Step (4) : Conclude that the result holds for all natural numbers.

Example 3.48: Prove by mathematical induction $n^2 + n$ is even.

Solution: Let $P(n)$ denote the statement "$n^2 + n$ is even"

Step (1):

$$\text{Put } n = 1$$

$$n^2 + n = 1^2 + 1 = 2, \text{ which is even}$$

$$\therefore P(1) \text{ is true}$$

Step (2):

Let us assume that the statement be true for $n = k$

(i.e.) assume P(k) be true.

(i.e.) assume "$k^2 + k$ is even" be true … (1)

Step (3):

To prove P(k + 1) is true.

(i.e.) to prove $(k + 1)^2 + (k + 1)$ is even

Consider
$$\begin{aligned}(k + 1)^2 + (k + 1) &= k^2 + 2k + 1 + k + 1\\ &= k^2 + 2k + k + 2\\ &= (k^2 + k) + 2(k + 1)\\ &= \text{an even number} + 2(k + 1), \text{ from (1)}\\ &= \text{sum of two even numbers}\\ &= \text{an even number}\end{aligned}$$

$\therefore$ P(k + 1) is true.

Thus if P(k) is true, then P(k + 1) is also true.

Step (4):

$\therefore$ By the principle of Mathematical induction, P(n) is true for all $n \in$ N.

i.e. $n^2 + n$ is even for all $n \in$ N.

Example 3.49: Prove by Mathematical induction $1 + 2 + 3 + \ldots + n = \dfrac{n(n + 1)}{2}$, $n \in$ N

Solution: Let P(n) denote the statement : "$1 + 2 + 3 + \ldots + n = \dfrac{n(n + 1)}{2}$"

Put $n = 1$

P(1) is the statement :
$$1 = \frac{1\,(1 + 1)}{2}$$
$$1 = \frac{1(2)}{2}$$
$$1 = 1$$

$\therefore$ P(1) is true

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true.

(i.e.) assume $1 + 2 + 3 + \ldots + k = \dfrac{k(k + 1)}{2}$ … (1) be true

To prove P(k + 1) is true

(i.e.) to prove $1 + 2 + 3 + \ldots + k + (k + 1) = \dfrac{(k + 1)\,(k + 2)}{2}$ is true,

$$[1 + 2 + 3 + \ldots + k] + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \quad \text{from (1)}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ P(k + 1) is true.

Thus if P(k) is true, then P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all n$\in$N

$$\therefore\ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad \text{for all } n\in N$$

Example 3.50: Prove by Mathematical induction

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for all } n\in N$$

Solution:

Let P(n) denote the statement "$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$"

Put $n = 1$

P(1) is the statement : $1^2 = \frac{1(1+1)[2(1)+1]}{6}$

$$1 = \frac{1(2)(3)}{6}$$

$$1 = 1$$

$\therefore$ P(1) is true.

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true.

(i.e.) $$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \ldots (1)$$

To prove : P(k + 1) is true

(i.e.) to prove: $1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ is true.

$$[1^2 + 2^2 + 3^2 + \ldots + k^2] + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore$ P(k + 1) is true

Thus if P(k) is true, then P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in N$

(i.e.) $$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in N$$

Example 3.51: Prove by Mathematical induction

$$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}, \; n \in N.$$

Solution:

Let P(n) denote the statement "$1.2+2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$"

Put $n = 1$

P(1) is the statement : $$1(1+1) = \frac{1(1+1)(1+2)}{3}$$

$$1(2) = \frac{1(2)(3)}{3}$$

$$2 = \frac{2(3)}{3}$$

$$2 = 2$$

$\therefore$ P(1) is true.

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true

(i.e.) assume $1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$ be true

To prove : P(k + 1) is true

i.e. to prove :

$$1.2 + 2.3 + 3.4 + \ldots, + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Consider $1.2 + 2.3 + 3.4 + \ldots, + k(k+1) + (k+1)(k+2)$

$$= \left[1.2 + 2.3 + \ldots + k(k+1)\right] + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\therefore$ P(k + 1) is true

Thus if P(k) is true, P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in$ N.

$$1.2 + 2.3 + 3.4 + \ldots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$$

Example 3.52: Prove by Mathematical induction $2^{3n} - 1$ is divisible by 7, for all natural numbers n.

Solution:

Let P(n) denote the statement "$2^{3n} - 1$ is divisible by 7"

Put $n = 1$

Then P(1) is the statement : $2^{3(1)} - 1 = 2^3 - 1$

$= 8 - 1$

$= 7$, which is divisible by 7

$\therefore$ P(1) is true

Now assume that the statement be true for $n = k$

(i.e.) assume P(k) be true. (i.e.) "$2^{3k} - 1$ is divisible by 7" be true

Now to prove P(k + 1) is true. (i.e.) to prove $2^{3(k + 1)} - 1$ is divisible by 7

Consider
$$\begin{aligned} 2^{3(k + 1)} - 1 &= 2^{3k + 3} - 1 \\ &= 2^{3k} . 2^3 - 1 = 2^{3k} . 8 - 1 \\ &= 2^{3k} . 8 - 1 + 8 - 8 \qquad \text{(add and subtract 8)} \\ &= (2^{3k} - 1) 8 + 8 - 1 \\ &= (2^{3k} - 1) 8 + 7 = \text{a multiple of } 7 + 7 \\ &= \text{a multiple of } 7 \end{aligned}$$

$\therefore$ $2^{3(k + 1)} - 1$ is divisible by 7

$\therefore$ P(k + 1) is true

Thus if P(k) is true, then P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in$ N

$\therefore$ $2^{3n} - 1$ is divisible by 7 for all natural numbers n.

Example 3.53: Prove by Mathematical induction that $a^n - b^n$ is divisible by (a–b) for all $n \in$ N

Solution: Let P(n) denote the statement "$a^n - b^n$ is divisible by $a - b$".

Put $n = 1$

Then P(1) is the statement : $a^1 - b^1 = a - b$ is divisible by $a - b$

$\therefore$ P(1) is true.

Now assume that the statement be true for $n = k$.

(i.e.) assume P(k) be true. (i.e.) $a^k - b^k$ is divisible by $(a - b)$ be true.

$$\Rightarrow \quad \frac{a^k - b^k}{a - b} = c \text{ (say) where } c \in \text{N}$$

$$\Rightarrow \quad a^k - b^k = c(a - b)$$

$$\Rightarrow \quad a^k = b^k + c(a - b) \qquad \ldots (1)$$

Now to prove P(k + 1) is true. (i.e.) to prove : $a^{k+1} - b^{k+1}$ is divisible by $a - b$

Consider

$$\begin{aligned} a^{k+1} - b^{k+1} &= a^k \,.\, a - b^k\; b \\ &= \left[b^k + c(a - b)\right]\; a - b^k\; b \\ &= b^k a + ac(a - b) - b^k\, b \\ &= b^k(a - b) + ac\,(a - b) \\ &= (a - b)\,(b^k + ac) \text{ is divisible by } (a - b) \end{aligned}$$

$\therefore$ P(k + 1) is true.

By the principle of Mathematical induction, P(n) is true for all $n \in$ N

$\therefore$ $a^n - b^n$ is divisible by $a - b$ for all $n \in$ N

EXERCISE 3.6

Prove the following by the principle of Mathematical Induction.

(1) $(2n + 1)\,(2n - 1)$ is an odd number for all $n \in$ N

(2) $2 + 4 + 6 + 8 + \ldots + 2n = n\,(n + 1)$

(3) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

(4) $1 + 4 + 7 + \ldots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

(5) $4 + 8 + 12 + \ldots + 4n = 2n(n + 1)$

(6) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2\,(n + 1)^2}{4}$

(7) $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

(8) In the arithmetic progression a, $a + d$, $a + 2d$, …
the n^{th} term is $a + (n - 1)d$

(9) $5^{2n} - 1$ is divisible by 24 for all $n \in$ N

(10) $10^{2n-1} + 1$ is divisible by 11.

(11) $n(n + 1)(n + 2)$ is divisible by 6 where n is a natural number.

(12) The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for all $n \in$ N

(13) $7^{2n} + 16n - 1$ is divisible by 64

(14) $2^n > n$ for all $n \in$ N

3.5 Binomial Theorem:

Introduction:

A BINOMIAL is an algebraic expression of two terms which are connected by the operation '+' (or) '–'

For example, $x + 2y$, $x - y$, $x^3 + 4y$, $a + b$ etc.. are binomials.

Expansion of Binomials with positive Integral Index:

We have already learnt how to multiply a binomial by itself. Finding squares and cubes of a binomial by actual multiplication is not difficult.

But the process of finding the expansion of binomials with higher powers such as $(x + a)^{10}$, $(x + a)^{17}$, $(x + a)^{25}$ etc becomes more difficult. Therefore we look for a general formula which will help us in finding the expansion of binomials with higher powers.

We know that

$(x + a)^1 = x + a = {}_1C_0\, x^1 a^0 + {}_1C_1\, x^0 a^1$

$(x + a)^2 = x^2 + 2ax + a^2 = {}_2C_0 x^2 a^0 + {}_2C_1 x^1 a^1 + {}_2C_2 x^0 a^2$

$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3 = {}_3C_0 x^3 a^0 + {}_3C_1 x^2 a^1 + {}_3C_2 x^1 a^2 + {}_3C_3 x^0 a^3$

$(x+a)^4 = x^4 + 4x^3 + 6x^2a^2 + 4xa^3 + a^4 = {}_4C_0 x^4 a^0 + {}_4C_1 x^3 a^1 + {}_4C_2 x^2 a^2 + {}_4C_3 x^1 a^3 + {}_4C_4 x^0 a^4$

For $n = 1, 2, 3, 4$ the expansion of $(x + a)^n$ has been expressed in a very systematic manner in terms of combinatorial coefficients. The above expressions suggest the conjecture that $(x + a)^n$ should be expressible in the form,

$$(x + a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \dots + nC_{n-1}\, x^1 a^{n-1} + nC_n\, x^0 a^n$$

In fact, this conjecture is proved to be true and we establish it by using the principle of mathematical induction.

Theorem 3.6: (Binomial theorem for a Positive Integral Index)

Statement: For any natural number n

$$(x + a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \dots + nC_r\, x^{n-r} a^r + \dots + nC_{n-1}\, x^1 a^{n-1} + nC_n\, x^0 a^n$$

Proof:

We shall prove the theorem by the principle of mathematical induction.

Let P(n) denote the statement :

$$(x + a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \dots + nC_r\, x^{n-r} a^r + \dots + nC_{n-1}\, x^1 a^{n-1} + nC_n\, x^0 a^n$$

Step (1) :

Put $n = 1$

Then P(1) is the statement : $(x + a)^1 = {}_1C_0 x^1 a^0 + {}_1C_1\, x^{1-1} a^1$

$x + a = x + a$

$\therefore$ P(1) is true

Step (2):

Now assume that the statement be true for $n = k$

(i.e.) assume P(k) be true.

$$(x + a)^k = kC_0 x^k a^0 + kC_1 x^{k-1} a^1 + kC_2\, x^{k-2} a^2 + \dots + kC_r\, x^{k-r} a^r + \dots + {}_kC_k\, x^0 a^k$$

be true ... (1)

Step (3):

Now to prove P(k + 1) is true

(i.e.) **To prove:**

$$(x + a)^{K+1} = {}_{(k+1)}C_0 x^{k+1} + {}_{(k+1)}C_1 x^{(k+1)-1} a^1 + {}_{(k+1)}C_2 x^{(k+1)-2} a^2 + \dots + {}_{(k+1)}C_r x^{(k+1)-r} a^r + \dots + {}_{(k+1)}C_{(k+1)}\, a^{k+1}$$

Consider $(x + a)^{k+1} = (x + a)^k (x + a)$

$$= [kC_0 x^k + kC_1 x^{k-1} a^1 + kC_2\, x^{k-2} a^2 + \dots + kC_{(r-1)}\, x^{k-(r-1)}\, a^{(r-1)} + kC_r\, x^{k-r} a^r + \dots + kC_k a^k]\,(x + a)$$

$$= [kC_0x^{k+1} + kC_1x^{k}a^1 + kC_2x^{k-1}a^2 + \dots + kC_{r-1}x^{k-r+2}a^{r-1}$$
$$+ kC_rx^{k-r+1}a^r + \dots + kC_kxa^k]$$
$$+ [kC_0x^ka + kC_1x^{k-1}a^2 + kC_2x^{k-2}a^3 + \dots + kC_{r-1}x^{k-r+1}a^r$$
$$+ kC_rx^{k-r}a^{r+1} + \dots + kC_ka^{k+1}]$$

$$(x+a)^{k+1} = kC_0x^{k+1} + (kC_1 + kC_0)x^k.a + (kC_2 + kC_1)x^{k-1}a^2$$
$$+ \dots + (kC_r + kC_{r-1})x^{k-r+1}a^r + \dots + kC_k a^{k+1} \quad \dots (2)$$

We know that $kC_r + kC_{r-1} = {}_{(k+1)}C_r$

Put $r = 1, 2, 3, \dots$ etc.

$$kC_1 + kC_0 = {}_{(k+1)}C_1$$
$$kC_2 + kC_1 = {}_{(k+1)}C_2$$
$$kC_r + kC_{r-1} = {}_{(k+1)}C_r \quad \text{for } 1 \le r \le k$$
$$kC_0 = 1 = {}_{(k+1)}C_0$$
$$kC_k = 1 = {}_{(k+1)}C_{(k+1)}$$

$\therefore$ (2) becomes

$$(x+a)^{k+1} = {}_{(k+1)}C_0\, x^{k+1} + {}_{(k+1)}C_1\, x^k a + {}_{(k+1)}C_2\, x^{k-1}a^2$$
$$+ \dots + {}_{(k+1)}C_r\, x^{k+1-r}a^r + \dots + {}_{(k+1)}C_{(k+1)}\, a^{k+1}$$

$\therefore$ P(k + 1) is true

Thus if P(k) is true, P(k + 1) is true.

$\therefore$ By the principle of mathematical induction P(n) is true for all n$\in$N

$$(x+a)^n = nC_0\, x^n a^0 + nC_1x^{n-1}a^1 + \dots + nCr\, x^{n-r}a^r + \dots$$
$$+ nC_{n-1}\, x^1 a^{n-1} + nCn\, x^0 a^n \qquad \text{for all } n \in N$$

Some observations:

1. In the expansion

   $(x+a)^n = nC_0\, x^n a^0 + nC_1x^{n-1}a^1 + \dots + nCr\, x^{n-r}a^r + \dots$

   $+ nC_{n-1}\, x^1 a^{n-1} + nCn\, x^0 a^n$, the general term is $nC_r\, x^{n-r} a^r$.

   Since this is nothing but the $(r+1)^{th}$ term, it is denoted by T_{r+1}

   i.e. $\mathbf{T_{r+1} = nC_r\, x^{n-r}\, a^r}$.

2. The $(n+1)^{th}$ term is $T_{n+1} = nC_n\, x^{n-n}\, a^n = nC_n\, a^n$, the last term.

   Thus there are (n + 1) terms in the expansion of $(x + a)^n$

3. The degree of x in each term decreases while that of "a" increases such that the sum of the powers in each term is equal to n.

$$\boxed{\text{We can write } (x+a)^n = \sum_{r=0}^{n} nC_r\, x^{n-r} a^r}$$

4. nC_0, nC_1, nC_2, …, nC_r, … , nC_n are called binomial coefficients. They are also written as C_0, C_1 , C_2, … , C_n.
5. From the relation $nC_r = nC_{n-r}$, we see that the coefficients of terms equidistant from the beginning and the end are equal.
6. The binomial coefficients of the various terms of the expansion of $(x + a)^n$ for $n = 1, 2, 3, \ldots$ form a pattern.

Binomials	**Binomial coefficients**
$(x+a)^o$	1
$(x+a)^1$	1 1
$(x+a)^2$	1 2 1
$(x+a)^3$	1 3 3 1
$(x+a)^4$	1 4 6 4 1
$(x+a)^5$	1 5 10 10 5 1

This arrangement of the binomial coefficients is known as Pascal's triangle after the French mathematician Blaise Pascal (1623 – 1662). The numbers in any row can be obtained by the following rule. The first and last numbers are 1 each. The other numbers are obtained by adding the left and right numbers in the previous row.

1, 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, 4 + 1 = 5, 1

Some Particular Expansions:

In the expansion

$$(x+a)^n = nC_0\, x^n a^0 + nC_1 x^{n-1} a^1 + \ldots + nC_r\, x^{n-r} a^r + \ldots + nC_{n-1}\, x^1 a^{n-1} + nC_n\, x^0 a^n \quad \ldots (1)$$

1. If we put $-a$ in the place of a we get

$$\therefore\ (x-a)^n = nC_0 x^n - nC_1\, x^{n-1} a^1 + nC_2 x^{n-2} a^2 - \ldots + (-1)^r\, nC_r\, x^{n-r} a^r + \ldots + (-1)^n\, nC_n\, a^n$$

We note that the signs of the terms are positive and negative alternatively.

2. If we put 1 in the place of a in (1) we get,

$$(1 + x)^n = 1 + nC_1x + nC_2x^2 + \dots + nC_rx^r + \dots + nC_nx^n \quad \dots (2)$$

3. If we put $-x$ in the place of x in (2) we get

$$(1 - x)^n = 1 - nC_1\, x + nC_2x^2 - \dots + (-1)^r\, nC_rx^r + \dots + (-1)^n\, nC_nx^n$$

Middle Term:

The number of terms in the expansion of $(x + a)^n$ depends upon the index n. The index is either even (or) odd. Let us find the middle terms.

Case (i) : n is even

The number of terms in the expansion is $(n + 1)$, which is odd.

Therefore, there is only one middle term and it is given by $T_{\frac{n}{2}+1}$

Case (ii) : n is odd

The number of terms in the expansion is $(n + 1)$, which is even.

Therefore, there are two middle terms and they are given by $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

Particular Terms:

Sometimes a particular term satisfying certain conditions is required in the binomial expansion of $(x + a)^n$. This can be done by expanding $(x + a)^n$ and then locating the required term. Generally this becomes a tedious task, when the index n is large. In such cases, we begin by evaluating the general term T_{r+1} and then finding the values of r by assuming T_{r+1} to be the required term.

To get the term independent of x, we put the power of x equal to zero and get the value of r for which the term is independent of x. Putting this value of r in T_{r+1}, we get the term independent of x.

***Example 3.54*:**Find the expansion of : (i) $(2x + 3y)^5$ (ii) $\left(2x^2 - \frac{3}{x}\right)^4$

Solution:

(i) $(2x + 3y)^5 = {}_5C_0\, (2x)^5\, (3y)^o + {}_5C_1\, (2x)^4\, (3y)^1 + {}_5C_2\, (2x)^3\, (3y)^2$

$+ {}_5C_3\, (2x)^2\, (3y)^3 + {}_5C_4\, (2x)^1\, (3y)^4 + {}_5C_5\, (2x)^0\, (3y)^5$

$= 1(32)x^5\, (1) + 5(16x^4)\, (3y) + 10(8x^3)\, (9y^2)$

$$+ 10(4x^2)(27y^3) + 5(2x)(81y^4) + (1)(1)(243y^5)$$

$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

(ii) $\left(2x^2 - \frac{3}{x}\right)^4 = {}_4C_0\,(2x^2)^4\left(-\frac{3}{x}\right)^0 + {}_4C_1\,(2x^2)^3\left(-\frac{3}{x}\right)^1$

$$+ {}_4C_2\,(2x^2)^2\left(-\frac{3}{x}\right)^2 + {}_4C_3\,(2x^2)^1\left(-\frac{3}{x}\right)^3 + {}_4C_4\,(2x^2)^0\left(-\frac{3}{x}\right)^4$$

$$= (1)\,16x^8(1) + 4(8x^6)\left(-\frac{3}{x}\right) + 6(4x^4)\left(\frac{9}{x^2}\right) + 4(2x^2)\left(-\frac{27}{x^3}\right)$$

$$+ (1)(1)\left(\frac{81}{x^4}\right)$$

$$= 16x^8 - 96x^5 + 216x^2 - \frac{216}{x} + \frac{81}{x^4}$$

Example 3.55: Using binomial theorem, find the 7^{th} power of 11.

Solution:

$$11^7 = (1 + 10)^7$$

$$= {}_7C_0\,(1)^7\,(10)^0 + {}_7C_1\,(1)^6\,(10)^1 + {}_7C_2(1)^5(10)^2 + {}_7C_3(1)^4(10)^3 + {}_7C_4\,(1)^3(10)^4$$

$$+ {}_7C_5\,(1)^2\,(10)^5 + {}_7C_6\,(1)^1\,(10)^6 + {}_7C_7\,(1)^0\,(10)^7$$

$$= 1 + 70 + \frac{7 \times 6}{1 \times 2}\,10^2 + \frac{7 \times 6 \times 5}{1 \times 2 \times 3}\,10^3 + \frac{7 \times 6 \times 5}{1 \times 2 \times 3}\,10^4 + \frac{7 \times 6}{1 \times 2}\,10^5 + 7(10)^6 + 10^7$$

$$= 1 + 70 + 2100 + 35000 + 350000 + 2100000 + 7000000 + 10000000$$

$$= 19487171$$

Example 3.56: Find the coefficient of x^5 in the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$

Solution:

In the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$, the general term is

$$T_{r+1} = {}_{17}C_r\,x^{17-r}\left(\frac{1}{x^3}\right)^r$$

$$= {}_{17}C_r x^{17-4r}$$

Let T_{r+1} be the term containing x^5

then, $\quad 17 - 4r = 5 \quad \Rightarrow r = 3$

$$\therefore\ T_{r+1} = T_{3+1}$$

$$= {}_{17}C_3\, x^{17-4(3)} = 680x^5$$

$$\therefore \text{ coefficient of } x^5 = 680$$

Example 3.57: Find the constant term in the expansion of $\left(\sqrt{x} - \dfrac{2}{x^2}\right)^{10}$

Solution:

In the expansion of $\left(\sqrt{x} - \dfrac{2}{x^2}\right)^{10}$

$$T_{r+1} = {}_{10}C_r \left(\sqrt{x}\right)^{10-r} \left(\frac{-2}{x^2}\right)^r$$

$$= {}_{10}C_r\, x^{\frac{10-r}{2}} \frac{(-2)^r}{x^{2r}} = {}_{10}C_r\,(-2)^r\, x^{\frac{10-r}{2}-2r}$$

$$= {}_{10}C_r\,(-2)^r\, x^{\frac{10-5r}{2}}$$

Let T_{r+1} be the constant term

Then, $\dfrac{10-5r}{2} = 0 \Rightarrow r = 2$

$$\therefore \text{ The constant term} = {}_{10}C_2\,(-2)^2\, x^{\frac{10-5(2)}{2}}$$

$$= \frac{10 \times 9}{1 \times 2} \times 4 \times x^0$$

$$= 180$$

Example 3.58: If $n \in$ N, in the expansion of $(1 + x)^n$ prove the following :

(i) Sum of the binomial coefficients $= 2^n$

(ii) Sum of the coefficients of odd terms = Sum of the coefficients of even terms = 2^{n-1}

Solution: The coefficients nC_0, nC_1, nC_2, … , nC_n in the expansion of $(1 + x)^n$ are called the binomial coefficients, we write them as C_0, C_1, C_2, … C_n,

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + \ldots + C_nx^n$$

It is an identity in x and so it is true for all values of x.

Putting $x = 1$ we get

$$2^n = C_0 + C_1 + C_2 + \dots + C_n \qquad \dots (1)$$

$$\text{put } x = -1$$

$$0 = C_0 - C_1 + C_2 - C_3 + \dots (-1)^n C_n$$

$$\Rightarrow C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$$

It is enough to prove that

$$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

Let $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = k \quad \dots (2)$

From (1), $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

$$2k = 2^n \qquad \text{From (2)}$$

$$k = 2^{n-1}$$

From (2), $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$

EXERCISE 3.7

(1) Expand the following by using binomial theorem

(i) $(3a + 5b)^5$ (ii) $(a - 2b)^5$ (iii) $(2x - 3x^2)^5$

(iv) $\left(x + \frac{1}{y}\right)^{11}$ (v) $(x^2 + 2y^3)^6$ (vi) $\left(x\sqrt{y} + y\sqrt{x}\right)^4$

(2) Evaluate the following:

(i) $\left(\sqrt{2} + 1\right)^5 + \left(\sqrt{2} - 1\right)^5$ (ii) $\left(\sqrt{3} + 1\right)^5 - \left(\sqrt{3} - 1\right)^5$

(iii) $\left(1 + \sqrt{5}\right)^5 + \left(1 - \sqrt{5}\right)^5$ (iv) $\left(2\sqrt{a} + 3\right)^6 + \left(2\sqrt{a} - 3\right)^6$

(v) $\left(2 + \sqrt{3}\right)^7 - \left(2 - \sqrt{3}\right)^7$

(3) Using Binomial theorem find the value of $(101)^3$ and $(99)^3$.

(4) Using Binomial theorem find the value of $(0.998)^3$.

(5) Find the middle term in the expansion of

(i) $\left(3x - \frac{2x^2}{3}\right)^8$ (ii) $\left(\frac{b}{x} + \frac{x}{b}\right)^{16}$

(iii) $\left(\frac{a}{x} - \sqrt{x}\right)^{16}$ (iv) $(x - 2y)^{13}$ (v) $\left(x + \frac{2}{x^2}\right)^{17}$

(6) Show that the middle term of

(i) $(1 + x)^{2n}$ is $\dfrac{1.3.5.7 \ldots (2n-1)2^n x^n}{n!}$

(ii) $\left(x + \dfrac{1}{2x}\right)^{2n}$ is $\dfrac{1.3.5. \ldots (2n-1)}{n!}$

(iii) $\left(x - \dfrac{1}{x}\right)^{2n}$ is $\dfrac{(-1)^n . 1.3.5.7. \ldots (2n-1)}{n!} \quad 2^n$

(7) Find the coefficient of x^5 in the expansion of $\left(x - \dfrac{1}{x}\right)^{11}$

(8) Find the term independent of x (constant term) in the expansion of

(i) $\left(2x^2 + \dfrac{1}{x}\right)^{12}$ (ii) $\left(\dfrac{4x^2}{3} - \dfrac{3}{2x}\right)^9$ (iii) $\left(9x - \dfrac{b}{cx^2}\right)^{17}$

(9) In the expansion of $(1 + x)^{20}$, the coefficient of r^{th} and $(r + 1)^{th}$ terms are in the ratio 1 : 6, find the value of r.

(10) If the coefficients of 5^{th}, 6^{th} and 7^{th} terms in the expansion of $(1 + x)^n$ are in A.P., find n.

4. SEQUENCE AND SERIES

4.1 Introduction

We hear statements such as “a sequence of events”, “a series of tests before the board examination”, “a cricket test match series”. In all these statements the words “sequence” and “series” are used in the same sense. They are used to suggest a succession of things or events arranged in some order. In mathematics these words have special technical meanings. The word ‘sequence’ is used as in the common use of the term to convey the idea of a set of things in order, but the word “series” is used in a different sense.

Let us consider the following example.

A rabbit and a frog are jumping on the same direction. When they started they were one metre apart. The rabbit is jumping on the frog in order to catch it. At the same time the frog is jumping forward half of the earlier distance to avoid the catch. The jumping process is going on. Can the rabbit catch the frog?

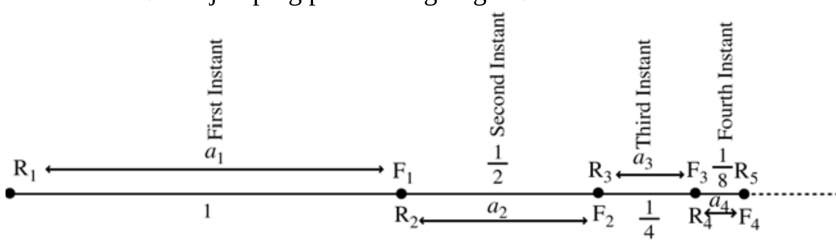


Fig. 4. 1

Let $a_1, a_2, a_3, a_4 \dots$ be the distances between the rabbit and the frog at the first, second, third, fourth instants etc,. The distance between the rabbit and the frog at the first instant is 1 metre.

$$\therefore\ a_1 = 1\ ;\ a_2 = \frac{1}{2}\ ;\ a_3 = \frac{1}{4} = \frac{1}{2^2}\ ;\ a_4 = \frac{1}{8} = \frac{1}{2^3}$$

Here $a_1, a_2, a_3 \dots$ form a sequence. There is a pattern behind the arrangement of $a_1, a_2, a_3 \dots$ Now a_n has the meaning,

(i.e.) a_n is the distance between the rabbit and the frog at the n^{th} instant

Further $a_n = \dfrac{1}{2^{n-1}}$. When a_n becomes 0 the rabbit will catch the frog.

As $n \to \infty$, $a_n \to 0$

i.e. the distance between the frog and the rabbit is zero when $n \to \infty$

At this stage the rabbit will catch the frog.

This example suggests that for each natural number there is a unique real number.

i.e.

	1	2	3	…	n
	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$
	a_1	a_2	a_3	…	a_n
	$= 1$	$= \frac{1}{2} = \frac{1}{2^1}$	$= \frac{1}{4} = \frac{1}{2^2}$	…	$= \frac{1}{2^{n-1}}$

Consider the following list of numbers

(a) 8, 15, 22, 29, …… (b) 6, 18, 54, 162, ……

In the list (a) the first number is 8, the 2nd number is 15, the 3rd number is 22, and so on. Each number in the list is obtained by adding 7 to the previous number.

In the list (*b*) the first number is 6, the 2 nd number is 18, the 3rd number is 54 etc. Each number in the list is obtained by multiplying the previous number by 3.

In these examples we observe the following:

(i) A rule by which the elements are written (pattern).

(ii) An ordering among the elements (order).

Thus a sequence means an arrangement of numbers in a definite order according to some rule.

4.2 Sequence

A sequence is a function from the set of natural numbers to the set of real numbers.

If the sequence is denoted by the letter a, then the image of $n \in \mathrm{N}$ under the sequence a is $a(n) = a_n$.

Since the domain for every sequence is the set of natural numbers, the images of 1, 2, 3, … n … under the sequence a are denoted by $a_1, a_2, a_3 \ldots a_n$, … respectively. Here $a_1, a_2, a_3 \ldots a_n, \ldots$ form the sequence.

“A sequence is represented by its range”.

Recursive formula

A sequence may be described by specifying its first few terms and a formula to determine the other terms of the sequence in terms of its preceding terms. Such a formula is called as **recursive** formula.

For example, 1, 4, 5, 9, 14, …, is a sequence because each term (except the first two) is obtained by taking the sum of preceding two terms. The corresponding recursive formula is $a_{n+2} = a_n + a_{n+1}$, $n \geq 1$ here $a_1=1$, $a_2= 4$

Terms of a sequence:

The various numbers occurring in a sequence are called its terms. We denote the terms of a sequence by $a_1, a_2, a_3, \dots , a_n, \dots$, the subscript denote the position of the term. The n^{th} term is called the general term of the sequence.

For example, in the sequence 1, 3, 5, 7, … $2n - 1$, …

the 1^{st} term is 1, 2^{nd} term is 3, … … and n^{th} term is $2n - 1$

Consider the following electrical circuit in which the resistors are indicated with saw-toothed lines.

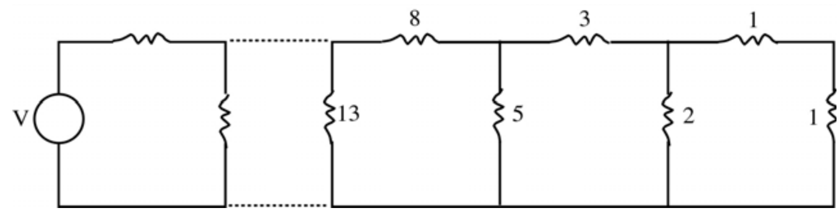


Fig. 4. 2

If all the resistors in the circuit are 1 ohm with a current of 1 ampere then the voltage across the resistors are 1, 1, 2, 3, 5, 8, 13, 21, …

In this sequence there is no fixed pattern. But we can generate the terms of the sequence recursively using a relation. Every number after the second is obtained by the sum of the previous two terms.

i.e.

$$\begin{aligned} V_1 &= 1 \\ V_2 &= 1 \\ V_3 &= V_2 + V_1 \\ V_4 &= V_3 + V_2 \\ V_5 &= V_4 + V_3 \\ &\vdots \\ V_n &= V_{n-1} + V_{n-2} \\ &\vdots \end{aligned}$$

Thus the above sequence is given by the rule:

$$V_1 = 1$$
$$V2 = 1$$
$$Vn = Vn - 1 + Vn - 2 \quad ; \quad n \geq 3$$

This sequence is called Fibonacci sequence. The numbers occurring in this sequence are called Fibonacci numbers named after the Italian Mathematician Leonardo Fibonacci.

Example 4.1:

Find the 7th term of the sequence whose n^{th} term is $(-1)^{n+1}\left(\frac{n+1}{n}\right)$

Solution:

Given $\quad a_n = (-1)^{n+1}\left(\frac{n+1}{n}\right)$

substituting $\quad n = 7$, we get

$$a_7 = (-1)^{7+1}\left(\frac{8}{7}\right) = \frac{8}{7}$$

4.3 Series

For a finite sequence 1, 3, 5, 7, 9 the familiar operation of addition gives the symbol 1 + 3 + 5 + 7 + 9 which has the value 25.

If we consider the infinite sequence 1, 3, 5, 7, … then the symbol 1 + 3 + 5 + 7 + … has no definite value, because when we add more and more terms the value steadily increases. 1 + 3 + 5 + 7 + 9 + … is called an infinite series. Thus a series is obtained by adding the terms of a sequence.

If $a_1, a_2, a_3, \ldots a_n \ldots$ is an infinite sequence then $a_1 + a_2 + \ldots + a_n + \ldots$ is called an infinite series. It is also denoted by $\sum\limits_{k=1}^{\infty} a_k$

If $S_n = a_1 + a_2 + \ldots + a_n$ then S_n is called the n^{th} partial sum of the series $\sum\limits_{k=1}^{\infty} a_k$

Example 4.2 Find the n^{th} partial sum of the series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$

Solution:

$$S_n = \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

and $S_{n+1} = \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$

$$S_{n+1} = S_n + \frac{1}{2^{n+1}} \qquad \dots (1)$$

Also we can write S_{n+1} as

$$S_{n+1} = \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right]$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right)\right]$$

$$S_{n+1} = \frac{1}{2}\,[1 + S_n] \qquad \dots (2)$$

From (1) and (2) $\quad S_n + \frac{1}{2^{n+1}} = \frac{1}{2}\,[1 + S_n]$

$$2S_n + \frac{1}{2^n} = 1 + S_n$$

$$\therefore\ S_n = 1 - \frac{1}{2^n}$$

Note: This can be obtained by using the idea of geometric series also. We know that the sum to n terms of a geometric series is $S_n = \frac{a(1 - r^n)}{(1 - r)}$

Here $a = \frac{1}{2}$, $n = n$, $r = \frac{1}{2}$ (< 1)

$$S_n = \frac{\frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

EXERCISE 4.1

(1) Write the first 5 terms of each of the following sequences:

(i) $a_n = (-1)^{n-1}\, 5^{n+1}$ (ii) $a_n = \frac{n(n^2 + 5)}{4}$ (iii) $a_n = -11n + 10$

(iv) $a_n = \frac{n+1}{n+2}$ (v) $a_n = \frac{1 - (-1)^n}{3}$ (vi) $a_n = \frac{n^2}{3^n}$

(2) Find the indicated terms of the following sequences whose n^{th} term is

(i) $a_n = 2 + \frac{1}{n}$; a_5 , a_7 (ii) $a_n = \cos\left(\frac{n\pi}{2}\right)$; a_4 , a_5

(iii) $a_n = \frac{(n+1)^2}{n}$; a_7 , a_{10} (iv) $a_n = (-1)^{n-1}\, 2^{n+1}$, a_5 , a_8

(3) Find the first 6 terms of the sequence whose general term is

$$a_n = \begin{cases} n^2 - 1 & \text{if } n \text{ is odd} \\ \dfrac{n^2+1}{2} & \text{if } n \text{ is even} \end{cases}$$

(4) Write the first five terms of the sequence given by

(i) $a_1 = a_2 = 2,\quad a_n = a_{n-1} - 1,\quad n > 2$

(ii) $a_1 = 1,\ a_2 = 2,\ a_n = a_{n-1} + a_{n-2},\ n > 2$

(iii) $a_1 = 1,\ a_n = na_{n-1}\ ,\ n \geq 2$

(iv) $a_1 = a_2 = 1,\ a_n = 2a_{n-1} + 3a_{n-2},\ n > 2$

(5) Find the n^{th} partial sum of the series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$

(6) Find the sum of first n terms of the series $\sum\limits_{n=1}^{\infty} 5^n$

(7) Find the sum of 101^{th} terms to 200^{th} term of the series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$

4.4 Some special types of sequences and their series

(1) Arithmetic progression:

An arithmetic progression (abbreviated as A.P) is a sequence of numbers in which each term, except the first, is obtained by adding a fixed number to the immediately preceding term. This fixed number is called the common difference, which is generally denoted by d.

For example, 1, 3, 5, 7, … is an A.P with common difference 2.

(2) Arithmetic series:

The series whose terms are in A.P is called an arithmetic series.

For example, 1 + 3 + 5 + 7 + … is an arithmetic series.

(3) Geometric progression

A geometric progression (abbreviated as G.P.) is a sequence of numbers in which the first term is non-zero and each term, except the first is obtained by multiplying the term immediately preceeding it by a fixed non-zero number. This fixed number is called the common ratio and it is denoted by the letter 'r'.

The general form of a G.P. is $a, ar, ar^2, \dots,$ with $a \neq 0$ and $r \neq 0$, the first term is 'a'

(4) Geometric series:

The series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ is called a geometric series because the terms of the series are in G.P. Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite (or) infinite number of terms.

(5) Harmonic progression:

A sequence of non-zero numbers is said to be in harmonic progression (abbreviated as H.P.) if their reciprocals are in A.P.

The general form of H.P is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots,$ where $a \neq 0$.

n^{th} term of H.P. is $T_n = \frac{1}{a + (n-1)d}$

For example the sequences $1, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \dots$ is a H.P., since their reciprocals 1, 5, 9, 13, … are in A.P.

Note: There is no general formula for the sum to n terms of a H.P. as we have for A.P. and G.P.

Example 4.3 If the 5^{th} and 12^{th} terms of a H.P. are 12 and 5 respectively, find the 15^{th} term.

Solution:

$$T_n = \frac{1}{a + (n-1)d}$$

Given $$T_5 = 12 \Rightarrow \frac{1}{a + (5-1)d} = 12 \Rightarrow \frac{1}{a+4d} = 12$$

$$a + 4d = \frac{1}{12} \quad \dots (1)$$

and $$T_{12} = 5 \Rightarrow \frac{1}{a + (12-1)d} = 5 \Rightarrow \frac{1}{a+11d} = 5$$

$$\Rightarrow \quad a + 11d = \frac{1}{5} \quad \dots (2)$$

$$(2) - (1) \qquad 7d = \frac{7}{60} \quad \Rightarrow \quad d = \frac{1}{60}$$

$$(1) \quad \Rightarrow \quad a + 4\left(\frac{1}{60}\right) = \frac{1}{12}$$

$$a + \frac{4}{60} = \frac{1}{12} \quad \Rightarrow \quad a = \frac{1}{12} - \frac{4}{60}$$

$$a = \frac{1}{60}$$

$$\therefore \; T_{15} = \frac{1}{a + (15-1)d} = \frac{1}{\frac{1}{60} + 14 \times \frac{1}{60}}$$

$$= \frac{1}{\frac{15}{60}} = \frac{60}{15}$$

$$T_{15} = 4$$

4.5 Means of Progressions

4.5.1 Arithmetic mean

A is called the arithmetic mean of the numbers a and b if and only if a, A, b are in A.P. If A is the A.M between a and b then a, A, b are in A.P

$$\Rightarrow \qquad A - a = b - A$$

$$\Rightarrow \qquad 2A = a + b$$

$$\Rightarrow \qquad \boxed{A = \frac{a+b}{2}}$$

$A_1, A_2, \dots, A_n$ are called n arithmetic means between two given numbers a and b if and only if $a, A_1, A_2, \dots A_n, b$ are in A.P.

Example 4.4 : Find the n arithmetic means between a and b and find their sum.

Solution:

Let $A_1, A_2, \dots, A_n$ be the n A.Ms between a and b. Then by the definition of A.Ms $\quad a, A_1, A_2, \dots, A_n, b$ are in A.P

Let the common difference be d.

$\therefore$ $A_1 = a + d$, $A_2 = a + 2d$, $A_3 = a + 3d$, $\dots$, $A_n = a + nd$ and $b = a + (n+1)d$

$$\Rightarrow \quad (n+1)d = b - a$$

$$\therefore \; d = \frac{b-a}{n+1}$$

$$\therefore \; A_1 = a + \frac{b-a}{n+1} \; ; \quad A_2 = a + \frac{2(b-a)}{n+1} \quad \dots \; A_n = a + \frac{n(b-a)}{n+1}$$

Sum of n A.Ms between a and b is

$$A_1 + A_2 + \dots + A_n = \left[a + \frac{b-a}{n+1}\right] + \left[a + \frac{2(b-a)}{n+1}\right] + \dots + \left[a + \frac{n(b-a)}{n+1}\right]$$

$$= na + \frac{(b-a)}{n+1}[1 + 2 + \dots + n]$$

$$= na + \frac{(b-a)}{(n+1)} \cdot \frac{n(n+1)}{2} = na + \frac{n(b-a)}{2}$$

$$= \frac{2na + nb - na}{2} = \frac{na + nb}{2} = n\left(\frac{a+b}{2}\right)$$

Example 4.5: Prove that the sum of n arithmetic means between two numbers is n times the single A.M between them

Solution:

Let $A_1, A_2, \dots, A_n$ be the n A.Ms between a and b.

From the example (4.4)

$$A_1 + A_2 + A_3 + \dots + A_n = n\left(\frac{a+b}{2}\right) = n \times (\text{A.M between } a \text{ and } b)$$

$$= n \text{ (single A.M between } a \text{ and } b)$$

Example 4.6: Insert four A.Ms between – 1 and 14.

Solution:

Let A_1, A_2, A_3, A_4 be the four A.Ms between – 1 and 14.

By the definition – 1, A_1, A_2, A_3, A_4, 14 are in A.P. Let d be the common difference.

$\therefore$ $A_1 = -1 + d$, ; $A_2 = -1 + 2d$; $A_3 = -1 + 3d$, ; $A_4 = -1 + 4d$; $14 = -1+5d$

$\therefore$ $d = 3$

$\therefore$ $A_1 = -1 + 3 = 2$; $A_2 = -1+2 \times 3 = 5$; $A_3 = -1+3\times3 = 8$; $A_4 = -1 + 12 = 11$

$\therefore$ The four A.Ms are 2, 5, 8 and 11.

4.5.2 Geometric Mean

G is called the geometric mean of the numbers a and b if and only if a, G, b are in G.P.

$$\Rightarrow \quad \frac{G}{a} = \frac{b}{G} = r$$

$$\Rightarrow \quad G^2 = ab$$

$$G = \pm\sqrt{ab}$$

Note:

(1) If a and b are positive then G = $+\sqrt{ab}$

(2) If a and b are negative then G = $-\sqrt{ab}$

(3) If a and b are opposite sign then their G.M is not real and it is discarded since we are dealing with real sequences.

i.e. If a and b are opposite in signs, then G.M between them does not exist.

Example 4.7: Find n geometric means between two given numbers a and b and find their product.

Solution:

Let $G_1, G_2, \ldots, G_n$ be n geometric means between a and b.

By definition $a, G_1, G_2, \ldots, G_n, b$ are in G.P. Let r be the common ratio.

Then $G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$ and $b = ar^{n+1}$

$$r^{n+1} = \frac{b}{a} \qquad \therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\Rightarrow \qquad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ \ldots\ G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

The product is

$$\begin{aligned} G_1 . G_2 . G_3 . G_n &= a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} . a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ldots a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} \\ &= a^n\left[\left(\frac{b}{a}\right)^{\frac{1+2+\ldots+n}{n+1}}\right] \\ &= a^n\left[\left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}\right] = a^n\left(\frac{b}{a}\right)^{\frac{n}{2}} \\ &= (ab)^{\frac{n}{2}} \end{aligned}$$

Example 4.8: Find 5 geometric means between 576 and 9.

Solution:

Let G_1, G_2, G_3, G_4, G_5 be 5 G.Ms between $a = 576$ and $b = 9$

Let the common ratio be r

$G_1 = 576r$, $G_2 = 576r^2$, $G_3 = 576r^3$, $G_4 = 576r^4$, $G_5 = 576r^5$, $9 = 576r^6$

$$\Rightarrow \quad r^6 = \frac{9}{576} \quad \Rightarrow \quad r = \left(\frac{9}{576}\right)^{\frac{1}{6}} = \left(\frac{1}{64}\right)^{\frac{1}{6}}$$

$$r = \frac{1}{2}$$

$$\therefore \quad G_1 = 576r = 576 \times \frac{1}{2} = 288 \qquad G_2 = 576r^2 = 576 \times \frac{1}{4} = 144$$

$$G_3 = 576r^3 = 576 \times \frac{1}{8} = 72 \qquad G_4 = 576r^4 = 576 \times \frac{1}{16} = 36$$

$$G_5 = 576r^5 = 576 \times \frac{1}{32} = 18$$

Hence 288, 144, 72, 36, 18 are the required G.Ms between 576 and 9.

Example 4.9: If b is the A.M of a and c ($a \neq c$) and ($b - a$) is the G.M of a and $c - a$, show that $a : b : c = 1 : 3 : 5$

Solution:

Given b is the A.M of a and c

$\therefore$ a, b, c are in A.P. Let the common difference be d

$$\therefore b = a + d \qquad \ldots (1)$$

$$c = a + 2d \qquad \ldots (2)$$

Given ($b - a$) is the G.M of a and ($c - a$)

$$\therefore (b - a)^2 = a(c - a)$$

$$d^2 = a(2d) \qquad \text{From (1) and (2)}$$

$$\Rightarrow \quad d = 2a \qquad [\because d \neq 0]$$

$\therefore$ $b = a + d$ | $c = a + 2d$

$b = a + 2a$ | $c = a + 2(2a)$

$\boxed{b = 3a}$ | $\boxed{c = 5a}$

$$\therefore a : b : c = a : 3a : 5a$$

$$= 1 : 3 : 5$$

4.5.3 Harmonic mean

H is called the harmonic mean between a and b if a, H, b are in H.P

If a, H, b are in H.P then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P

$$\Rightarrow \quad \frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b}}{2} \quad ; \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\boxed{H = \frac{2ab}{a+b}}$$

This H is single H.M between a and b

Definition:

$H_1, H_2, \ldots H_n$ are called n harmonic means between a and b if $a, H_1, H_2, \ldots H_n, b$ are in H.P.

Relation between A.M., G.M. and H.M.

Example 4.10: If a, b are two different positive numbers then prove that

(i) A.M., G.M., H.M. are in G.P. (ii) A.M > G.M > H.M

Proof:

$$\text{A.M.} = \frac{a+b}{2} \quad ; \quad \text{G.M.} = \sqrt{ab} \quad ; \quad \text{H.M.} = \frac{2ab}{a+b}$$

(i) $$\frac{\text{G.M}}{\text{A.M}} = \frac{\sqrt{ab}}{\frac{a+b}{2}} = \frac{2\sqrt{ab}}{a+b} \qquad \ldots (1)$$

$$\frac{\text{H.M}}{\text{G.M}} = \frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{2\sqrt{ab}}{a+b} \qquad \ldots (2)$$

From (1) and (2)

$$\frac{\text{G.M}}{\text{A.M}} = \frac{\text{H.M}}{\text{G.M}}$$

$\therefore$ A.M, G.M, H.M are in G.P

(ii) $$\text{A.M} - \text{G.M} = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} > 0 \qquad \because a > 0 \,;\, b > 0 \,;\, a \neq b$$

$$\text{A.M} > \text{G.M} \qquad \ldots (1)$$

$$\text{G.M} - \text{H.M} = \sqrt{ab} - \frac{2ab}{a+b}$$

$$= \frac{\sqrt{ab}\,(a+b) - 2ab}{a+b} = \frac{\sqrt{ab}\left[a+b-2\sqrt{ab}\right]}{a+b}$$

$$= \frac{\sqrt{ab}\left(\sqrt{a}-\sqrt{b}\right)^2}{a+b} > 0$$

$$\therefore \text{G.M} > \text{H.M} \qquad \ldots (2)$$

From (1) and (2) A.M. > G.M > H.M

EXERCISE 4.2

(1) (i) Find five arithmetic means between 1 and 19

(ii) Find six arithmetic means between 3 and 17

(2) Find the single A.M between

(i) 7 and 13 (ii) 5 and – 3 (iii) $(p + q)$ and $(p - q)$

(3) If b is the G.M of a and c and x is the A.M of a and b and y is the A.M of b and c, prove that $\frac{a}{x} + \frac{c}{y} = 2$

(4) The first and second terms of a H.P are $\frac{1}{3}$ and $\frac{1}{5}$ respectively, find the 9^{th} term.

(5) If a, b, c are in H.P., prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

(6) The difference between two positive numbers is 18, and 4 times their G.M is equal to 5 times their H.M. Find the numbers.

(7) If the A.M between two numbers is 1, prove that their H.M is the square of their G.M.

(8) If a, b, c are in A.P. and a, mb, c are in G.P then prove that a, m^2b, c are in H.P

(9) If the p^{th} and q^{th} terms of a H.P are q and p respectively, show that $(pq)^{th}$ term is 1.

(10) Three numbers form a H.P. The sum of the numbers is 11 and the sum of the reciprocals is one. Find the numbers.

4.6 Some special types of series

4.6.1 Binomial series

Binomial Theorem for a Rational Index:

In the previous chapter we have already seen the Binomial expansion for a positive integral index n. (power is a positive integer)

$$(x + a)^n = x^n + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \ldots + nC_r x^{n-r} a^r + \ldots + nC_n a^n$$

A particular form is

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots + x^n$$

When n is a positive integer the number of terms in the expansion is $(n+1)$ and so the series is a finite series. But when it is not a positive integer, the series does not terminate and it is an infinite series.

Theorem (without proof)

For any rational number n other than positive integer

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}\,x^2 + \frac{n(n-1)\,(n-2)}{1.2.3}\,x^3 + \dots\ \dots$$

provided $| x | < |$.

Here we require the condition that $| x |$ should be less than 1.

To see this, put $x = 1$ and $n = -1$ in the above formula for $(1+x)^n$

The left side of the formula $= (1+1)^{-1} = \frac{1}{2}$,

while the right side $= 1 + (-1)\,(1) + \frac{(-1)\,(-2)}{2}\,1^2 + \dots$

$= 1 - 1 + 1 - 1 + \dots$

Thus the two sides are not equal. This is because, $x = 1$ doesn't satisfy $| x | < 1$.

This extra condition $| x | < 1$ is unnecessary, if n is a positive integer.

Differences between the Binomial theorem for a positive integral index and for a rational index:

1. If $n \in N$, then $(1+x)^n$ is defined for all values of x and if n is a rational number other than the natural number, then $(1+x)^n$ is defined only when $| x | < |$.
2. If $n \in N$, then the expansion of $(1+x)^n$ contains only $n+1$ terms. If n is a rational number other than natural number, then the expansion of $(1+x)^n$ contains infinitely many terms.

Some particular expansions

We know that , when n is a rational index,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}\,x^2 + \frac{n(n-1)\,(n-2)}{3!}\,x^3 + \dots \qquad (1)$$

Replacing x by $-x$, we get

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}\,x^2 - \frac{n(n-1)\,(n-2)}{3!}\,x^3 + \dots \qquad (2)$$

Replacing n by $-n$ in (1) we get

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}\,x^2 - \frac{n(n+1)\,(n+2)}{3!}\,x^3 + \dots \qquad (3)$$

Replacing x by $-x$ in (3), we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots \qquad (4)$$

Note :

(1) If the exponent is negative then the value of the factors in the numerators are increasing uniformly by 1

(2) If the exponent is positive then the value of the factors in the numerators are decreasing uniformly by 1

(3) If the signs of x and n are same then all the terms in the expansion are positive.

(4) If the signs of x and n are different, then the terms alternate in sign

Special cases

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

General term:

For a rational number n and $|x| < 1$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots$$

In this expansion

$$\text{First term } T_1 = T_{0+1} = 1$$

$$\text{Second term } T_2 = T_{1+1} = nx = \frac{n}{1}x^1$$

$$\text{Third term } T_3 = T_{2+1} = \frac{n(n-1)}{1.2}x^2$$

$$\text{Fourth term } T_4 = T_{3+1} = \frac{n(n-1)(n-2)}{1.2.3}x^3 \text{ etc.}$$

$$(r+1)^{\text{th}} \text{ term : } T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{1.2.3\ldots r}x^r$$

The general term is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots r \text{ factors}}{r!}x^r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Example 4.11: Write the first four terms in the expansions of

(i) $(1+4x)^{-5}$ where $|x| < \frac{1}{4}$ (ii) $(1-x^2)^{-4}$ where $|x| <|$

Solution: (i) $|4x| = 4|x| < 4\left(\frac{1}{4}\right) = 1 \quad \therefore \quad |4x| < 1$

$\therefore$ $(1 + 4x)^{-5}$ can be expanded by Binomial theorem.

$$(1 + 4x)^{-5} = 1 - (5)(4x) + \frac{(5)(5+1)}{1.2}(4x)^2 - \frac{(5)(5+1)(5+2)}{1.2.3}(4x)^3 + \ldots$$

$$= 1 - 20x + 15(16x^2) - 35(64x^3) + \ldots$$

$$= 1 - 20x + 240x^2 - 2240x^3 + \ldots$$

(ii) $(1 - x^2)^{-4}$ can be expanded by Binomial theorem since $|x^2| < 1$

$$= 1 + (4)(x^2) + \frac{(4)(4+1)}{1.2}(x^2)^2 + \frac{(4)(4+1)(4+2)}{1.2.3}(x^2)^3 + \ldots$$

$$= 1 + 4x^2 + 10x^4 + 20x^6 + \ldots$$

Example 4.12: Find the expansion of $\frac{1}{(2+x)^4}$ where $|x| < 2$ upto the fourth term.

Solution:

$$\frac{1}{(2+x)^4} = (2+x)^{-4} = 2^{-4}\left(1 + \frac{x}{2}\right)^{-4} \qquad |x| < 2 \Rightarrow \left|\frac{x}{2}\right| < 1$$

$$= \frac{1}{16}\left[1 - (4)\left(\frac{x}{2}\right) + \frac{(4)(4+1)}{1.2}\left(\frac{x}{2}\right)^2 - \frac{(4)(4+1)(4+2)}{1.2.3}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{16}\left[1 - 2x + \frac{(4)(5)}{2}\left(\frac{x^2}{4}\right) - \frac{(4)(5)(6)}{1.2.3}\frac{x^3}{8} + \ldots\right]$$

$$= \frac{1}{16} - \frac{x}{8} + \frac{5}{32}x^2 - \frac{5}{32}x^3 + \ldots$$

Example 4.13: Show that $(1+x)^n = 2^n\left[1 - n\left(\frac{1-x}{1+x}\right) + n\left(\frac{n+1}{2!}\right)\left(\frac{1-x}{1+x}\right)^2 + \ldots\right]$

Solution: Let $y = \frac{1-x}{1+x}$

$$\text{R.H.S} = 2^n\left[1 - ny + \frac{n(n+1)}{2!}y^2 + \ldots\right] = 2^n[1 + y]^{-n}$$

$$= 2^n\left[1 + \frac{1-x}{1+x}\right]^{-n} \qquad = 2^n\left[\frac{1+x+1-x}{1+x}\right]^{-n}$$

$$= 2^n\left[\frac{2}{1+x}\right]^{-n} = 2^n\left[\frac{1+x}{2}\right]^{n} = (1+x)^n = \text{L.H.S.}$$

Approximation by using Binomial series

Example 4.14: Find the value of $\sqrt[3]{126}$ correct to two decimal places.
Solution:

$$\sqrt[3]{126} = (126)^{\frac{1}{3}} = (125 + 1)^{\frac{1}{3}}$$

$$= \left[125\left(1 + \frac{1}{125}\right)\right]^{\frac{1}{3}} = (125)^{\frac{1}{3}} \left(1 + \frac{1}{125}\right)^{\frac{1}{3}}$$

$$= 5\left[1 + \frac{1}{3} \cdot \frac{1}{125} + \ldots\right] \qquad \because \frac{1}{125} < 1$$

$$= 5\left[1 + \frac{1}{3}(0.008)\right] \text{ by neglecting other terms}$$

$$= 5[1 + 0.002666]$$

$$= 5.01 \text{ (correct to 2 decimal places)}$$

Example 4.15: If x is large and positive show that $\sqrt[3]{x^3 + 6} - \sqrt[3]{x^3 + 3} = \frac{1}{x^2}$ (app.)

Solution: Since x is large, $\frac{1}{x}$ is small and hence $\left|\frac{1}{x}\right| < 1$

$$\sqrt[3]{x^3 + 6} - \sqrt[3]{x^3 + 3} = (x^3 + 6)^{\frac{1}{3}} - (x^3 + 3)^{\frac{1}{3}} = x\left(1 + \frac{6}{x^3}\right)^{\frac{1}{3}} - x\left(1 + \frac{3}{x^3}\right)^{\frac{1}{3}}$$

$$= x\left[1 + \frac{1}{3} \cdot \frac{6}{x^3} + \ldots\right] - x\left[1 + \frac{1}{3} \cdot \frac{3}{x^3} + \ldots\right]$$

$$= \left[x + \frac{2}{x^2} + \ldots\right] - \left[x + \frac{1}{x^2} + \ldots\right] = \frac{2}{x^2} - \frac{1}{x^2} + \ldots$$

$$= \frac{1}{x^2} \text{ (approximately)}$$

Example 4.16: In the expansion $(1 - 2x)^{-\frac{1}{2}}$, find the coefficient of x^8.
Solution: We know that

$$(1-x)^{-n} = 1 + nx + \frac{n(n + 1)}{2!}x^2 + \frac{n(n + 1)(n + 2)}{3!}x^3 + \ldots + \frac{n(n + 1) \ldots (n + r - 1)}{r!}x^r + \ldots$$

General term $\quad T_{r + 1} = \frac{n(n + 1) \ldots (n + r - 1)}{r!} x^r$

Take $\quad n = \frac{1}{2}$ and replace x by $2x$.

$$T_{r+1} = \frac{\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2r-1}{2}\right)}{r!}\ (2x)^r = \frac{1.3.5\ \ldots\ (2r-1)}{r!\ \ 2^r}\ 2^r\ x^r$$

$\therefore$ coefficient of $x^r = \dfrac{1.3.5\ \ldots\ (2r-1)}{r!}$

put $r = 8$

$\therefore$ coefficient of $x^8 = \dfrac{1.3.5.7.9.11.13.15}{8!}$

4.6.2. Exponential series

Exponential theorem (without proof)

For all real values of x,

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots\right)^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

But $e = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots$

$\therefore$ For all real values of x, $\quad e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

Thus we have the following results:

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots$$

$$\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots$$

4.6.3 Logarithmic Series:

If $-1 < x \leq 1$ then $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

This series is called the logarithmic series.

The other forms of logarithmic series are as follows:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

$$\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

EXERCISE 4.3

(1) Write the first four terms in the expansions of the following:

(i) $\dfrac{1}{(2+x)^4}$ where $|x| > 2$ (ii) $\dfrac{1}{\sqrt[3]{6-3x}}$ where $|x| < 2$

(2) Evaluate the following:

(i) $\sqrt[3]{1003}$ correct to 2 places of decimals

(ii) $\dfrac{1}{\sqrt[3]{128}}$ correct to 2 places of decimals

(3) If x is so small show that $\sqrt{\dfrac{1-x}{1+x}} = 1 - x + \dfrac{x^2}{2}$ (app.)

(4) If x is so large prove that $\sqrt{x^2+25} - \sqrt{x^2+9} = \dfrac{8}{x}$ nearly.

(5) Find the 5^{th} term in the expansion of $(1-2x^3)^{\frac{11}{2}}$

(6) Find the $(r+1)^{th}$ term in the expansion of $(1-x)^{-4}$

(7) Show that $x^n = 1 + n\left(1-\dfrac{1}{x}\right) + \dfrac{n(n+1)}{1.2}\left(1-\dfrac{1}{x}\right)^2 + \ldots$

5. ANALYTICAL GEOMETRY

Introduction

'Geometry' is the study of points, lines, curves, surfaces etc and their properties. Geometry is based upon axioms and it was laid by the famous Greek Mathematician Euclid about 300 B.C. In the 17th century A.D., the methods of Algebra were applied in the study of Geometry and thereby 'Analytical Geometry' emerged out. The renowned French philosopher and Mathematician Rene Descartes (1596 – 1650) showed how the methods of Algebra could be applied to the study of Geometry. He thus became the founder of Analytical Geometry (also called as Cartesian Geometry, from the latinized form of his name Cartesius). To bring a relationship between Algebra and Geometry, Descartes introduces basic algebraic entity 'number' to the basic geometric concept of 'point'. This relationship is called 'system of coordinates'. Descartes relates the position of a point with its distance from fixed lines and its direction. This chapter is a continuation of the study of the concepts of Analytical Geometry to which the students had been introduced in earlier classes.

5.1 Locus

The path traced by a point when it moves according to specified geometrical conditions is called the **locus** of the point. For example, the locus of a point P(x_1, y_1) whose distance from a fixed point C (h, k) is constant 'a', is a circle (fig. 5.1). The fixed point 'C' is called the centre and the fixed distance 'a' is called the radius of the circle.

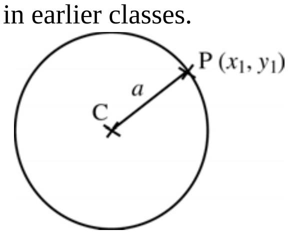


Fig. 5. 1

Example 5.1: A point in the plane moves so that its distance from (0, 1) is twice its distance from the x-axis. Find its locus.

Solution:

Let A(0, 1) be the given point. Let P(x_1, y_1) be any point on the locus. Let B be the foot of the perpendicular from P(x_1,y_1) to the x-axis. Thus PB = y_1.

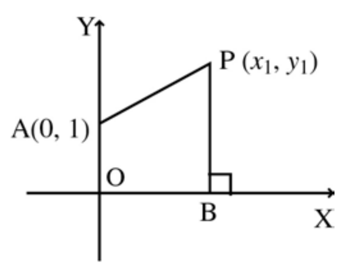


Fig. 5. 2

Given that PA = 2PB

$$\therefore \; PA^2 = 4PB^2$$

i.e. $$(x_1 - 0)^2 + (y_1 - 1)^2 = 4y_1^2$$

i.e. $\quad x_1^2 + y_1^2 - 2y_1 + 1 = 4y_1^2$

i.e. $\quad x_1^2 - 3y_1^2 - 2y_1 + 1 = 0$

$\therefore$ The locus of (x_1, y_1) is $\quad x^2 - 3y^2 - 2y + 1 = 0$

Example 5.2: Find the locus of the point which is equidistant from (– 1, 1) and (4, – 2).

Solution:

Let A(– 1, 1) and B(4, – 2) be the given points.

Let $P(x_1,y_1)$ be any point on the locus. Given that PA = PB

$$\therefore \; PA^2 = PB^2$$

i.e. $\quad (x_1 + 1)^2 + (y_1 - 1)^2 = (x_1 - 4)^2 + (y_1 + 2)^2$

i.e. $\quad x_1^2 + 2x_1 + 1 + y_1^2 - 2y_1 + 1 = x_1^2 - 8x_1 + 16 + y_1^2 + 4y_1 + 4$

i.e. $\quad 10x_1 - 6y_1 - 18 = 0 \quad$ i.e. $\; 5x_1 - 3y_1 - 9 = 0$

$\therefore$ The locus of the point (x_1, y_1) is $5x - 3y - 9 = 0$

Example 5.3: If A and B are the two points (– 2, 3) and (4, – 5), find the equation of the locus of a point such that $PA^2 - PB^2 = 20$.

Solution:

A(– 2, 3) and B(4, – 5) are the two given points. Let $P(x_1, y_1)$ be any point on the locus. Given that $PA^2 - PB^2 = 20$.

$$(x_1 + 2)^2 + (y_1 - 3)^2 - \left[(x_1 - 4)^2 + (y_1 + 5)^2\right] = 20$$

$$x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9 - [x_1^2 - 8x_1 + 16 + y_1^2 + 10y_1 + 25] = 20$$

$$12x_1 - 16y_1 - 48 = 0$$

i.e. $\quad 3x_1 - 4y_1 - 12 = 0$

The locus of (x_1,y_1) is $3x - 4y - 12 = 0$

Example 5.4: Find a point on x-axis which is equidistant from the points (7, – 6) and (3, 4) .

Solution:

Let $P(x_1, y_1)$ be the required point. Since P lies on x-axis, $y_1 = 0$. Given that A(7, – 6) and B(3, 4) are equidistant from P.

i.e. $\quad PA = PB \;\Rightarrow\; PA^2 = PB^2$

$\Rightarrow \quad (x_1 - 7)^2 + (0 + 6)^2 = (x_1 - 3)^2 + (0 - 4)^2$

$\Rightarrow \qquad x_1^2 - 14x_1 + 49 + 36 = x_1^2 - 6x_1 + 9 + 16$

$\Rightarrow \qquad 8x_1 = 60 \quad \therefore \; x_1 = 15/2$

Thus the required point is $\left(\frac{15}{2}, 0\right)$

EXERCISE 5.1

(1) A point moves so that it is always at a distance of 6 units from the point (1, – 4). Find its locus.

(2) Find the equation of the locus of the point which are equidistant from (1, 4) and (– 2, 3).

(3) If the point P($5t - 4$, $t + 1$) lies on the line $7x - 4y + 1 = 0$, find
(i) the value of t (ii) the co-ordinates of P.

(4) The distance of a point from the origin is five times its distance from the y-axis. Find the equation of the locus.

(5) Show that the equation of the locus of a point which moves such that its distance from the points (1, 2) and (0, – 1) are in the ratio 2 : 1 is $3x^2 + 3y^2 + 2x + 12y - 1 = 0$.

(6) A point P moves such that P and the points (2, 3), (1, 5) are always collinear. Show that the equation of the locus of P is $2x + y - 7 = 0$.

(7) A and B are two points (1, 0) and (– 2, 3). Find the equation of the locus of a point such that (i) $PA^2 + PB^2 = 10$ (ii) PA = 4PB.

5.2 Straight lines

5.2.1 Introduction

A straight line is the simplest geometrical curve. Every straight line is associated with an equation. To determine the equation of a straight line, two conditions are required. We have derived the equation of a straight line in different forms in the earlier classes. They are

(1) Slope-intercept form:

i.e. $y = mx + c$ where 'm' is the slope of the straight line and 'c' is the y intercept.

(2) Point-slope form:

i.e. $y - y_1 = m(x - x_1)$ where 'm' is the slope and (x_1, y_1) is the given point.

(3) Two point form:

i.e $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are the two given points.

(4) Intercept form:

i.e. $\frac{x}{a} + \frac{y}{b} = 1$ where 'a' and 'b' are x and y intercepts respectively.

In this section we shall derive and discuss other forms of equation of a straight line.

5.2.2 Normal form:

Equation of a straight line in terms of the length of the perpendicular p from the origin to the line and the angle α which the perpendicular makes with x-axis.

Let R and N be the points where the straight line cuts the x and y axes respectively.

Draw the perpendicular OL to RN.

Let OL = p and $\angle XOL = \alpha$.

Now OR and ON are the x and y intercepts respectively.

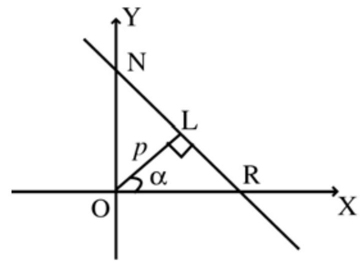


Fig. 5. 3

The equation of the straight line is $\frac{x}{OR} + \frac{y}{ON} = 1$ …(1)

From the right angled triangle OLR, $\sec\alpha = \frac{OR}{OL}$ $\therefore$ $OR = p\sec\alpha$

From the right angled triangle OLN, $\operatorname{cosec}\alpha = \sec(90 - \alpha) = \frac{ON}{OL}$

$\therefore$ $ON = p\operatorname{cosec}\alpha$

Substituting the values of OR and ON in equation (1),

we get, $\frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1$ i.e. $\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$

i.e. $x\cos\alpha + y\sin\alpha = p$ is the required equation of the straight line.

5.2.3 Parametric form

Definition: If two variables, say x and y, are functions of a third variable, say 'θ', then the functions expressing x and y in terms of θ are called the parametric representations of x and y. The variable θ is called the parameter of the function.

Equation of a straight line passing through the point (x_1, y_1) and making an angle θ with x-axis. (parametric form)

Let Q (x_1, y_1) be the given point and P(x, y) be any point on the required straight line. Assume that PQ = r.

It is given that

$\angle PTR = \theta$. But $\angle PQM = \angle PTR$

$\therefore \angle PQM = \theta$

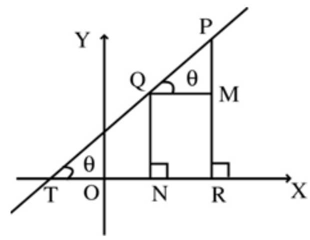


Fig. 5. 4

In the right angled triangle PQM,

… (1)

$$\cos\theta = \frac{QM}{PQ} = \frac{NR}{r} = \frac{OR - ON}{r} = \frac{x - x_1}{r}$$

$$\therefore \frac{x - x_1}{\cos\theta} = r$$

$$\text{Similarly } \sin\theta = \frac{PM}{PQ} = \frac{PR - MR}{r} = \frac{y - y_1}{r}$$

$$\therefore \frac{y - y_1}{\sin\theta} = r \qquad \dots (2)$$

From (1) and (2), $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$ which is the required equation.

Any point on this line can be taken as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ where r is the algebraic distance. Here r is the parameter.

5.2.4 General form

The equation $ax + by + c = 0$ will always represent a straight line.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any three points on the locus represented by the equation $ax + by + c = 0$. Then

$$ax_1 + by_1 + c = 0 \qquad \dots (1)$$

$$ax_2 + by_2 + c = 0 \qquad \dots (2)$$

$$ax_3 + by_3 + c = 0 \qquad \dots (3)$$

(1) × $(y_2 - y_3)$ + (2) × $(y_3 - y_1)$ + (3) × $(y_1 - y_2)$ gives

$a\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] = 0$

Since $a \neq 0$, $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

That is (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear and hence they lie on a straight line.

Thus the equation $ax + by + c = 0$ represents a straight line.

5.2.5. Perpendicular distance from a point to a straight line

The length of the perpendicular from the point (x_1, y_1) to the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Let the given line $ax + by + c = 0$ … (1) be represented by AB.

Let $P(x_1, y_1)$ be the given point.

Draw PD perpendicular to AB. Note that PD is the required distance.

Draw OM parallel to PD. Let OM = p

Assume that $\angle MOB = \alpha$.

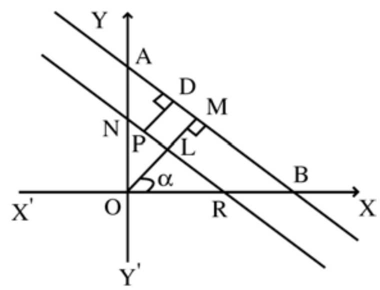


Fig. 5. 5

From 5.2.2, the equation of the straight line AB is

$x \cos\alpha + y \sin\alpha - p = 0$ … (2)

Now equations (1) and (2) are representing the same straight line. Hence their corresponding coefficients are proportional.

$$\therefore \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{-p}{c}$$

$$\cos\alpha = -\frac{ap}{c}, \quad \sin\alpha = -\frac{pb}{c}$$

We know that $\sin^2\alpha + \cos^2\alpha = 1$

$$\frac{p^2b^2}{c^2} + \frac{p^2a^2}{c^2} = 1 \quad \text{i.e.} \quad p^2a^2 + p^2b^2 = c^2$$

$$p^2(a^2 + b^2) = c^2 \quad \text{i.e} \quad p^2 = \frac{c^2}{a^2 + b^2}$$

$$p = \pm\frac{c}{\sqrt{a^2 + b^2}}$$

Hence $\cos\alpha = \mp\dfrac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \mp\dfrac{b}{\sqrt{a^2 + b^2}}$

Suppose OL = p', the equation of the straight line NR is
$x \cos\alpha + y \sin\alpha - p' = 0$
since P(x_1, y_1) is a point on NR

$x_1 \cos\alpha + y_1 \sin\alpha - p' = 0$

i.e. OL = $p' = x_1 \cos\alpha + y_1 \sin\alpha$

From the figure, the required distance

$$\begin{aligned} PD &= LM = OM - OL = p - p' \\ &= p - x_1 \cos\alpha - y_1 \sin\alpha \\ &= \pm \frac{c}{\sqrt{a^2+b^2}} \pm \frac{x_1 \,.\, a}{\sqrt{a^2+b^2}} \pm \frac{y_1 \,.\, b}{\sqrt{a^2+b^2}} = \pm \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \end{aligned}$$

The required distance $= \left| \dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$

Corollary:

The length of the perpendicular from the origin to $ax + by + c = 0$ is $\left| \dfrac{c}{\sqrt{a^2+b^2}} \right|$

Note: The general equation of the straight line is $ax+by+c = 0$ i.e. $y = -\dfrac{a}{b}x - \dfrac{c}{a}$

This is of the form $y = mx + c$.

$\therefore\ m = -\dfrac{a}{b}$ i.e. slope $= -\dfrac{\text{co-efficient of } x}{\text{co-efficient of } y}$

Example 5.5: Determine the equation of the straight line whose slope is 2 and y-intercept is 7.

Solution:

The slope – intercept form is $y = mx + c$ Here $m = 2$, $c = 7$

$\therefore$ The required equation of the straight line is $y = 2x + 7$

Example 5.6: Determine the equation of the straight line passing through (– 1, 2) and having slope $\dfrac{2}{7}$

Solution:

The point-slope form is $y - y_1 = m(x - x_1)$.

Here $(x_1, y_1) = (-1, 2)$ and $m = \dfrac{2}{7}$

$$\therefore \quad y - 2 = \frac{2}{7}(x + 1) \quad \text{i.e.} \quad 7y - 14 = 2x + 2$$

$2x - 7y + 16 = 0$ is the equation of the straight line.

Example 5.7:

Determine the equation of the straight line passing through the points (1, 2) and (3, – 4).

Solution:

The equation of a straight line passing through two points is $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, -4)$.

Substituting the above, the required line is $\dfrac{y - 2}{2 + 4} = \dfrac{x - 1}{1 - 3}$

$\Rightarrow \quad \dfrac{y - 2}{6} = \dfrac{x - 1}{-2} \quad \Rightarrow \quad \dfrac{y - 2}{3} = \dfrac{x - 1}{-1}$

$\Rightarrow \quad y - 2 = -3(x - 1) \Rightarrow y - 2 = -3x + 3$

$\Rightarrow \quad 3x + y = 5$ is the required equation of the straight line.

Example 5.8: Find the equation of the straight line passing through the point (1, 2) and making intercepts on the co-ordinate axes which are in the ratio 2 : 3.

Solution:

The intercept form is $\dfrac{x}{a} + \dfrac{y}{b} = 1$ … (1)

The intercepts are in the ratio 2 : 3 $\quad \therefore\ a = 2k, \quad b = 3k.$

(1) becomes $\dfrac{x}{2k} + \dfrac{y}{3k} = 1$ i.e. $3x + 2y = 6k$

Since (1, 2) lies on the above straight line, $3 + 4 = 6k$ i.e. $6k = 7$

Hence the required equation of the straight line is $3x + 2y = 7$

Example 5.9: Find the length of the perpendicular from (2, – 3) to the line $2x - y + 9 = 0$

Solution:

The perpendicular distance from (x_1, y_1) to the straight line $ax + by + c = 0$ is given by $\left| \dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$

$\therefore$ The length of the perpendicular from (2, –3) to the straight line $2x - y + 9 = 0$ is $\left| \dfrac{2(2) - (-3) + 9}{\sqrt{(2)^2 + (-1)^2}} \right| = \dfrac{16}{\sqrt{5}}$ units.

Example 5.10: Find the co-ordinates of the points on the straight line $y = x + 1$ which are at a distance of 5 units from the straight line $4x - 3y + 20 = 0$

Solution: Let (x_1, y_1) be a point on $y = x + 1$

$\therefore\ y_1 = x_1 + 1$ … (1)

The length of the perpendicular from (x_1, y_1) to the straight line

$4x - 3y + 20 = 0$ is $\left|\dfrac{4x_1 - 3y_1 + 20}{\sqrt{4^2 + (-3)^2}}\right| = \pm\left(\dfrac{4x_1 - 3y_1 + 20}{5}\right)$

But the length of the perpendicular is given as 5.

$$\therefore\ \pm\left(\frac{4x_1 - 3y_1 + 20}{5}\right) = 5$$

$$\therefore\ 4x_1 - 3y_1 + 20 = \pm 25$$

Considering the positive sign, $4x_1 - 3y_1 + 20 = 25$

$\Rightarrow\quad 4x_1 - 3y_1 = 5$ … (2)

Considering the negative sign, $4x_1 - 3y_1 + 20 = -25$

$\Rightarrow\quad 4x_1 - 3y_1 = -45$ … (3)

Solving (1) and (2), we get $x_1 = 8,\ y_1 = 9$

Solving (1) and (3), we get $x_1 = -42,\ y_1 = -41$.

$\therefore$ The co-ordinates of the required points are (8, 9) and (– 42, – 41).

Example 5.11: Find the equation of the straight line, if the perpendicular from the origin makes an angle of 120° with x-axis and the length of the perpendicular from the origin is 6 units.

Solution:

The normal form of a straight line is $x\cos\alpha + y\sin\alpha = p$

Here $\alpha = 120°$, $p = 6$ $\quad\therefore\ x\cos 120° + y\sin 120° = 6$

$$\Rightarrow\quad x\left(-\frac{1}{2}\right) + y\left(\frac{\sqrt{3}}{2}\right) = 6 \Rightarrow -x + \sqrt{3}\,y = 12$$

$$\Rightarrow\quad x - \sqrt{3}\,y + 12 = 0$$

$\therefore$ The required equation of the straight line is $x - \sqrt{3}\,y + 12 = 0$

Example 5.12: Find the points on y-axis whose perpendicular distance from the straight line $4x - 3y - 12 = 0$ is 3.

Solution:

Any point on y-axis will have x co-ordinate as 0.

Let the point on y-axis be $P(0, y_1)$.

The given straight line is $4x - 3y - 12 = 0$ … (1)

The perpendicular distance from the point P to the given straight line is

$$\left|\frac{-3y_1 - 12}{\sqrt{4^2 + (-3)^2}}\right| = \left|\frac{3y_1 + 12}{5}\right|$$

But the perpendicular distance is 3.

$$\text{i.e.} \quad \left|\frac{3y_1 + 12}{5}\right| = 3 \quad \Rightarrow \quad 3y_1 + 12 = \pm 15$$

$$\begin{aligned} 3y_1 + 12 &= 15 \quad \text{or} \quad & 3y_1 + 12 &= -15 \\ 3y_1 &= 3 \quad \text{or} \quad & 3y_1 &= -27 \\ y_1 &= 1 \quad \text{or} \quad & y_1 &= -9 \end{aligned}$$

Thus the required points are (0, 1) and (0, – 9).

EXERCISE 5.2

(1) Determine the equation of the straight line passing through the point (– 1, – 2) and having slope $\frac{4}{7}$

(2) Determine the equation of the line with slope 3 and *y*-intercept 4.

(3) A straight line makes an angle of 45° with *x*-axis and passes through the point (3, – 3). Find its equation.

(4) Find the equation of the straight line joining the points (3, 6) and (2, – 5).

(5) Find the equation of the straight line passing through the point (2, 2) and having intercepts whose sum is 9.

(6) Find the equation of the straight line whose intercept on the *x*-axis is 3 times its intercept on the *y*-axis and which passes through the point (– 1, 3).

(7) Find the equations of the medians of the triangle formed by the points (2, 4), (4, 6) and (– 6, – 10).

(8) Find the length of the perpendicular from (3, 2) to the straight line $3x + 2y + 1=0$.

(9) The portion of a straight line between the axes is bisected at the point (– 3, 2). Find its equation.

(10) Find the equation of the diagonals of a quadrilateral whose vertices are (1, 2), (– 2, – 1), (3, 6) and (6, 8).

(11) Find the equation of the straight line, which cut off intercepts on the axes whose sum and product are 1 and – 6 respectively.

(12) Find the intercepts made by the line $7x + 3y - 6 = 0$ on the co-ordinate axis.

(13) What are the points on x-axis whose perpendicular distance from the straight line $\frac{x}{3} + \frac{y}{4} = 1$ is 4?

(14) Find the distance of the line $4x - y = 0$ from the point (4, 1) measured along the straight line making an angle of 135° with the positive direction of the x-axis.

5.3. Family of straight lines

In the previous section, we studied about a single straight line. In this section we will discuss the profile about more than one straight line, which lie on a plane.

5.3.1 Angle between two straight lines

Let $l_1 : y = m_1x + c_1$ and

$l_2 : y = m_2x + c_2$ be the two intersecting lines and assume that P be the point of intersection of the two straight lines which makes angle θ_1 and θ_2 with the positive direction of x-axis. Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Let θ be the angle between the two straight lines.

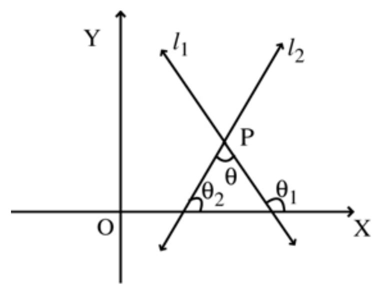


Fig. 5. 6

From the figure (5.6), $\theta_1 = \theta + \theta_2$

$$\therefore \theta = \theta_1 - \theta_2$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1.\tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Note that $\frac{m_1 - m_2}{1 + m_1 m_2}$ is either positive or negative. As convention we consider the acute angle as the angle between any two straight lines and hence we consider only the positive value (absolute value) of $\tan\theta$.

Hence $$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \therefore \theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Corollary (1) : If the two straight lines are parallel, then their slopes are equal.

Proof:

Since the two straight lines are parallel, $\theta = 0$. $\therefore$ $\tan\theta = 0$

$$\Rightarrow \quad \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \quad \Rightarrow m_1 - m_2 = 0$$

$$\text{i.e.} \quad m_1 = m_2$$

$\therefore$ If the straight lines are parallel, then the slopes are equal.

Note : If the slopes are equal, then the straight lines are parallel.

Corollary (2): If the two straight lines are perpendicular then the product of their slopes is –1.

Proof:

Since the two straight lines are perpendicular, $\theta = 90°$.

$$\therefore \tan\theta = \tan 90° = \infty \quad \Rightarrow \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

This is possible only if the denominator is zero.

$$\text{i.e. } 1 + m_1 m_2 = 0 \qquad \text{i.e. } m_1 m_2 = -1$$

$\therefore$ If the two straight lines are perpendicular then the product of their slopes is – 1.

Note (1): If the product of the slopes is – 1, then the straight lines are perpendicular.

(2): Corollary (2) is applicable only if both the slopes m_1 and m_2 are finite. It fails when the straight lines are co-ordinate axes or parallel to axes.

Corollary (3): If the straight lines are parallel, then the coefficients of x and y are proportional in their equations. In particular, the equations of two parallel straight lines differ only by the constant term.

Proof:

Let the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be parallel.

Slope of $a_1x + b_1y + c_1 = 0$ is $m_1 = -\frac{a_1}{b_1}$; Slope of $a_2x + b_2y + c_2 = 0$ is $m_2 = -\frac{a_2}{b_2}$

Since the straight lines are parallel, $m_1 = m_2$.

$$\text{i.e. } -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \quad \Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

i.e. coefficients of x and y are proportional

Let $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = \lambda$ (say)

$\therefore\ a_2 = a_1 \lambda,\ \ b_2 = b_1 \lambda$

The second equation $a_2x + b_2y + c_2 = 0$ can be written as

$$\lambda a_1x + \lambda b_1y + c_2 = 0$$

i.e. $a_1x + b_1y + \dfrac{c_2}{\lambda} = 0$ i.e. $a_1x + b_1y + k = 0$ where $k = \dfrac{c_2}{\lambda}$

i.e. If $a_1x + b_1y + c_1 = 0$ is a straight line then a line parallel to it is $a_1x + b_1y + k = 0$

$\therefore$ Equations of parallel straight lines differ by the constant term.

Note (1): In the previous section, we established a formula to find the distance between the origin and the straight line. i.e. distance = $\left|\dfrac{c}{\sqrt{a^2 + b^2}}\right|$

We can find out the distance between two parallel straight lines

$ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ by using the formula $d = \dfrac{|\,c_1 - c_2\,|}{\sqrt{a^2 + b^2}}$.

This is obtained by using the above result. Note that, we took $|\,c_1 - c_2\,|$ since $c_2 > c_1$ or $c_1 > c_2$

Note (2): To apply the above formula, write the equations of the parallel straight lines in the standard form $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$.

Corollary (4): The equation of the straight line perpendicular to the straight line $ax + by + c = 0$ is of the form $bx - ay + k = 0$ for some k.

Proof:

Let the straight lines $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ be perpendicular.

Slope of $ax + by + c = 0$ is $m_1 = -\dfrac{a}{b}$

Slope $a_1x + b_1y + c_1 = 0$ is $m_2 = -\dfrac{a_1}{b_1}$

Since the straight lines are perpendicular, $m_1 m_2 = -1$

i.e. $\left(-\dfrac{a_1}{b_1}\right)\left(-\dfrac{a}{b}\right) = -1$ i.e. $aa_1 = -bb_1$

i.e. $\dfrac{a_1}{b} = -\dfrac{b_1}{a} = \lambda$ (say) $\quad \therefore\ a_1 = b\lambda$ and $b_1 = -a\lambda$

The second equation $a_1x + b_1y + c_1 = 0$ can be written as $b\lambda x - a\lambda y + c_1 = 0$

i.e. $$bx - ay + \frac{c_1}{\lambda} = 0$$

i.e. $$bx - ay + k = 0 \quad \text{where } k = \frac{c_1}{\lambda}$$

A straight line perpendicular to $ax + by + c = 0$ is given by $bx - ay + k = 0$ for some k.

Note: To find the point of intersection of two straight lines, solve the simultaneous equations of the straight lines.

5.3.3 The condition for the three straight lines to be concurrent

Let the three straight lines be given by

$$a_1x + b_1y + c_1 = 0 \qquad \dots (1)$$

$$a_2x + b_2y + c_2 = 0 \qquad \dots (2)$$

$$a_3x + b_3y + c_3 = 0 \qquad \dots (3)$$

If the three straight lines are concurrent, then the point of intersection of any two straight lines lies on the third straight line.

Solving the equation (1) and (2), the coordinates of the point of intersection is

$$x = \frac{b_1 c_2 - b_2c_1}{a_1b_2 - a_2b_1}\ ,\ y = \frac{c_1a_2 - c_2 a_1}{a_1b_2 - a_2b_1}$$

substituting the values of x and y in the equation (3)

$$a_3\left(\frac{b_1 c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2 a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

i.e. $a_3\,(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

i.e. $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$

i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ is the condition for the three straight lines to be concurrent.

5.3.4 Equation of a straight line passing through the intersection of the two given straight lines

Let $$a_1x + b_1y + c_1 = 0 \qquad \ldots (1)$$

$$a_2x + b_2y + c_2 = 0 \qquad \ldots (2)$$

be the equations of the two given straight lines.

Consider the equation $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$ … (3)

where λ is a constant

Equation (3) is of degree one in x and y and therefore (refer 5.2.4) it represents a straight line. Let (x_1, y_1) be the point of intersection of (1) and (2)

$\therefore\ a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$

$\therefore\ a_1x_1 + b_1y_1 + c_1 + \lambda\,(a_2x_1 + b_2y_1 + c_2) = 0$

$\therefore$ Value of (x_1, y_1) satisfies equation (3) also.

Hence $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$ represents a straight line passing through the intersection of the straight lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_2y + c_2 = 0$

Example 5.13: Find the angle between the straight lines $3x - 2y + 9 = 0$ and $2x + y - 9 = 0$.

Solution:

Slope of the straight line $3x - 2y + 9 = 0$ is $m_1 = \frac{3}{2}$ $\left[\because y = \frac{3}{2}x + \frac{9}{2}\right]$

Slope of the straight line $2x + y - 9 = 0$ is $m_2 = -2$ $[\because y = -2x + 9]$

Suppose ‘θ’ is the angle between the given lines, then

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \tan^{-1}\left|\frac{\frac{3}{2} + 2}{1 + \frac{3}{2}(-2)}\right| = \tan^{-1}\left|\frac{\frac{7}{2}}{\frac{2-6}{2}}\right|$$

$$= \tan^{-1}\left|-\frac{7}{4}\right| = \tan^{-1}\left(\frac{7}{4}\right)$$

Example 5.14: Show that the straight lines $2x + y - 9 = 0$ and $2x + y - 10 = 0$ are parallel.

Solution:

Slope of the straight line $2x + y - 9 = 0$ is $m_1 = -2$

Slope of the straight line $2x + y - 10 = 0$ is $m_2 = -2$ $\quad \therefore m_1 = m_2$

$\therefore$ The given straight lines are parallel.

Example 5.15: Show that the two straight lines whose equations are

$x + 2y + 5 = 0$ and $2x + 4y - 5 = 0$ are parallel.

Solution:

The two given equations are

$$x + 2y + 5 = 0 \qquad \ldots (1)$$

$$2x + 4y - 5 = 0 \qquad \ldots (2)$$

The coefficients of x and y are proportional since $\frac{1}{2} = \frac{2}{4}$ and therefore they are parallel.

Note : This can also be done by writing the equation(2) as $x + 2y - 5/2 = 0$

Now the two equations differ by constant alone. $\therefore$ They are parallel.

Example 5.16: Find the distance between the parallel lines $2x + 3y - 6=0$ and $2x + 3y + 7 = 0$.

Solution:

The distance between the parallel lines is $\left| \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right|$.

Here $c_1 = -6$, $c_2 = 7$, $a = 2$, $b = 3$

The required distance is $\left| \frac{-6-7}{\sqrt{2^2 + 3^2}} \right| = \left| \frac{-13}{\sqrt{13}} \right| = \sqrt{13}$ units.

Example 5.17: Show that the straight lines $2x + 3y - 9 = 0$ and $3x - 2y + 10 = 0$ are at right angles.

Solution:

Slope of the straight line $2x + 3y - 9 = 0$ is $m_1 = -\frac{2}{3}$

Slope of the straight line $3x - 2y + 10 = 0$ is $m_2 = \frac{3}{2}$

$$\therefore m_1 m_2 = -\frac{2}{3} \cdot \frac{3}{2} = -1$$

$\therefore$ The two straight lines are at right angles.

Example 5.18: Find the equation of the straight line parallel to $3x + 2y = 9$ and which passes through the point $(3, -3)$.

Solution:

The straight line parallel to $3x + 2y - 9 = 0$ is of the form

$$3x + 2y + k = 0 \qquad \dots (1)$$

The point $(3, -3)$ satisfies the equation (1)

Hence $9 - 6 + k = 0$ i.e. $k = -3$

$\therefore$ $3x + 2y - 3 = 0$ is the equation of the required straight line.

Example 5. 19: Find the equation of the straight line perpendicular to the straight line $3x + 4y + 28 = 0$ and passing through the point $(-1, 4)$.

Solution:

The equation of any straight line perpendicular to $3x + 4y + 28 = 0$ is of the form

$$4x - 3y + k = 0$$

The point $(-1, 4)$ lies on the straight line $4x - 3y + k = 0$

$$\therefore\ -4 - 12 + k = 0 \quad \Rightarrow\ k = 16$$

$\therefore$ The equation of the required straight line is $4x - 3y + 16 = 0$

Example 5. 20: Show that the triangle formed by straight lines

$4x - 3y - 18 = 0$, $3x - 4y + 16 = 0$ and $x + y - 2 = 0$ is isosceles.

Solution:

Slope of the straight line $4x - 3y - 18 = 0$ is $m_1 = \frac{4}{3}$

Slope of the straight line $3x - 4y + 16 = 0$ is $m_2 = \frac{3}{4}$

Slope of the straight line $x + y - 2 = 0$ is $m_3 = -1$

Let 'α' be the angle between the straight lines $4x - 3y - 18 = 0$ and $3x - 4y + 16 = 0$

Using the formula, $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ we get

$$\alpha = \tan^{-1}\left|\frac{\frac{4}{3} - \frac{3}{4}}{1 + \frac{4}{3}\,\frac{3}{4}}\right| = \tan^{-1}\left|\frac{\frac{16-9}{12}}{2}\right|$$

$$= \tan^{-1}\left|\frac{7}{24}\right| = \tan^{-1}\left(\frac{7}{24}\right)$$

Let 'β' be the angle between the straight lines $3x - 4y + 16 = 0$ and $x + y - 2 = 0$

$$\therefore \beta = \tan^{-1}\left|\frac{\frac{3}{4}+1}{1+\frac{3}{4}(-1)}\right| = \tan^{-1}\left|\frac{7/4}{1/4}\right|$$

$$= \tan^{-1}(7)$$

Let 'γ' be the angle between the straight lines $x+y-2=0$ and $4x-3y-18=0$

$$\therefore \gamma = \tan^{-1}\left|\frac{-1-\frac{4}{3}}{1+(-1)\left(\frac{4}{3}\right)}\right| = \tan^{-1}\left|\frac{-\frac{7}{3}}{-\frac{1}{3}}\right|$$

$$= \tan^{-1}(7)$$

Therefore $\beta = \gamma$ $\therefore$ The triangle is isosceles.

Example 5.21: Find the point of intersection of the straight lines

$$5x + 4y - 13 = 0 \text{ and } 3x + y - 5 = 0$$

Solution:

To find the point of intersection, solve the given equations.

Let (x_1, y_1) be the point of intersection. Then (x_1, y_1) lies on both the straight lines.

$$\therefore \quad 5x_1 + 4y_1 = 13 \qquad \dots (1)$$

$$3x_1 + y_1 = 5 \qquad \dots (2)$$

$(2) \times 4 \Rightarrow$ $\quad 12x_1 + 4y_1 = 20 \qquad \dots (3)$

$(1) - (3) \Rightarrow$ $\quad -7x_1 = -7 \quad \therefore \quad x_1 = 1$

Substituting $x_1 = 1$ in equation (1), we get $5 + 4y_1 = 13$

$$4y_1 = 8 \qquad \therefore y_1 = 2$$

The point of intersection is (1, 2).

Example 5.22: Find the equation of the straight line passing through the intersection of the straight lines $2x + y = 8$ and $3x - y = 2$ and through the point $(2, -3)$

Solution:

The equation of the straight line passing through the intersection of the given lines is

$2x + y - 8 + \lambda(3x - y - 2) = 0 \qquad \dots (1)$

$(2, -3)$ lies on the equation (1) and hence $4 - 3 - 8 + \lambda(6 + 3 - 2) = 0$

$$\therefore \quad \lambda = 1$$

$\therefore$ (1) $\Rightarrow$ $2x + y - 8 + 3x - y - 2 = 0$ $\Rightarrow$ $5x - 10 = 0$

$x = 2$ is the equation of the required straight line.

Example 5.23: Find the equation of the straight line passing through the intersection of the straight lines $2x + y = 8$ and $3x - 2y + 7 = 0$ and parallel to $4x + y - 11 = 0$

Solution:

Let (x_1, y_1) be the point of intersection of the given straight lines

$$2x_1 + y_1 = 8 \qquad \ldots (1)$$

$$3x_1 - 2y_1 = -7 \qquad \ldots (2)$$

(1) × 2 $\Rightarrow$ $4x_1 + 2y_1 = 16$ … (3)

(2) + (3) $\Rightarrow$ $\therefore x_1 = \frac{9}{7}$ $y_1 = \frac{38}{7}$ $\therefore (x_1, y_1) = \left(\frac{9}{7}, \frac{38}{7}\right)$

The straight line parallel to $4x + y - 11 = 0$ is of the form $4x + y + k = 0$

But it passes through $\left(\frac{9}{7}, \frac{38}{7}\right)$

$$\therefore \frac{36}{7} + \frac{38}{7} + k = 0 \qquad \therefore k = -\frac{74}{7}$$

$$4x + y - \frac{74}{7} = 0$$

$28x + 7y - 74 = 0$ is the equation of the required straight line.

Example 5.24:

Find the equation of the straight line which passes through the intersection of the straight lines $5x - 6y = 1$ and $3x + 2y + 5 = 0$ and is perpendicular to the straight line $3x - 5y + 11 = 0$

Solution:

The straight line passing through the intersection of the given straight lines is

$5x - 6y - 1 + \lambda (3x + 2y + 5) = 0$ … (1)

$(5 + 3\lambda)x + (-6 + 2\lambda)y + (-1 + 5\lambda) = 0$

This straight line is perpendicular to $3x - 5y + 11 = 0$

Product of the slopes of the perpendicular straight lines is -1 i.e. $m_1 m_2 = -1$

$$\Rightarrow \quad -\left(\frac{5 + 3\lambda}{-6 + 2\lambda}\right)\left(\frac{3}{5}\right) = -1$$

$$15 + 9\lambda = -30 + 10\lambda \qquad \therefore \lambda = 45$$

(1) $\Rightarrow$ $5x - 6y - 1 + 45\,(3x + 2y + 5) = 0$ i.e. $140x + 84y + 224 = 0$

i.e. $5x + 3y + 8 = 0$ is the equation of the required straight line.

Example 5.25: Show that the straight lines $3x + 4y = 13$; $2x - 7y + 1 = 0$ and $5x - y = 14$ are concurrent.

Solution:

Let (x_1, y_1) be the point of intersection of the first two straight lines

$$3x_1 + 4y_1 = 13 \qquad \dots (1)$$

$$2x_1 - 7y_1 = -1 \qquad \dots (2)$$

(1) × 7 $\Rightarrow$ $21x_1 + 28y_1 = 91$ … (3)

(2) × 4 $\Rightarrow$ $8x_1 - 28y_1 = -4$ … (4)

(3) + (4) $\Rightarrow$ $29x_1 = 87$ $\Rightarrow$ $x_1 = 3$

(1) $\Rightarrow$ $9 + 4y_1 = 13$ $\Rightarrow$ $y_1 = 1$

The point of intersection of the first two straight lines is (3, 1).

Substitute this value in the equation $5x - y = 14$

$$\text{L.H.S.} = 5x - y$$

$$= 15 - 1 = 14 = \text{R.H.S.}$$

i.e. The point (3, 1) satisfies the third equation.

Hence the three straight lines are concurrent.

Example 5.26: Find the co-ordinates of orthocentre of the triangle formed by the straight lines

$x - y - 5 = 0$, $2x - y - 8 = 0$ and $3x - y - 9 = 0$

Solution:

Let the equations of sides AB, BC and CA of a ΔABC be represented by

$$x - y - 5 = 0 \qquad \dots (1)$$

$$2x - y - 8 = 0 \qquad \dots (2)$$

$$3x - y - 9 = 0 \qquad \dots (3)$$

Solving (1) and (3), we get A as (2, – 3)

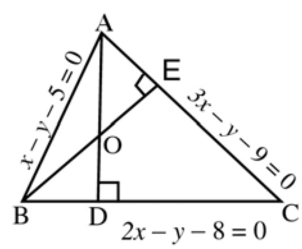


Fig. 5. 7

The equation of the straight line BC is $2x - y - 8 = 0$. The straight line perpendicular to it is of the form

$$x + 2y + k = 0 \qquad \dots (4)$$

A(2, – 3) satisfies the equation (4) $\therefore\ 2 - 6 + k = 0 \Rightarrow k = 4$

The equation of AD is $x + 2y = -4$ … (5)

Solving the equations (1) and (2), we get B as (3, – 2)

The straight line perpendicular to $3x - y - 9 = 0$ is of the form

$$x + 3y + k = 0$$

But B(3, – 2) lies on this straight line $\therefore\ 3 - 6 + k = 0 \Rightarrow k = 3$

$\therefore$ The equation of BE is $x + 3y = -3$ … (6)

Solving (5) and (6), we get the orthocentre O as (– 6, 1).

Example 5.27: For what values of 'a', the three straight lines $3x + y + 2 = 0$, $2x - y + 3 = 0$ and $x + ay - 3 = 0$ are concurrent?

Solution:

Let (x_1, y_1) be the point of concurrency. This point satisfies the first two equations.

$$\therefore\ 3x_1 + y_1 + 2 = 0 \quad \dots (1)$$

$$2x_1 - y_1 + 3 = 0 \quad \dots (2)$$

Solving (1) and (2) we get (– 1, 1) as the point of intersection. Since it is a point of concurrency, it lies on $x + ay - 3 = 0$.

$$\therefore\ -1 + a - 3 = 0$$

i.e. $a = 4$

EXERCISE 5.3

(1) Find the angle between the straight lines $2x + y = 4$ and $x + 3y = 5$

(2) Show that the straight lines $2x + y = 5$ and $x - 2y = 4$ are at right angles.

(3) Find the equation of the straight line passing through the point (1, – 2) and parallel to the straight line $3x + 2y - 7 = 0$

(4) Find the equation of the straight line passing through the point (2, 1) and perpendicular to the straight line $x + y = 9$

(5) Find the point of intersection of the straight lines $5x + 4y - 13 = 0$ and $3x + y - 5 = 0$

(6) If the two straight lines $2x - 3y + 9 = 0$, $6x + ky + 4 = 0$ are parallel, find k

(7) Find the distance between the parallel lines $2x + y - 9 = 0$ and $4x + 2y + 7 = 0$

(8) Find the values of p for which the straight lines $8px + (2 - 3p)\, y + 1 = 0$ and $px + 8y - 7 = 0$ are perpendicular to each other.

(9) Find the equation of the straight line which passes through the intersection of the straight lines $2x + y = 8$ and $3x - 2y + 7 = 0$ and is parallel to the straight line $4x + y - 11 = 0$

(10) Find the equation of the straight line passing through intersection of the straight lines $5x - 6y = 1$ and $3x + 2y + 5 = 0$ and perpendicular to the straight line $3x - 5y + 11 = 0$

(11) Find the equation of the straight line joining $(4, -3)$ and the intersection of the straight lines $2x - y + 7 = 0$ and $x + y - 1 = 0$

(12) Find the equation of the straight line joining the point of the intersection of the straight lines $3x + 2y + 1 = 0$ and $x + y = 3$ to the point of intersection of the straight lines $y - x = 1$ and $2x + y + 2 = 0$

(13) Show that the angle between $3x + 2y = 0$ and $4x - y = 0$ is equal to the angle between $2x + y = 0$ and $9x + 32y = 41$

(14) Show that the triangle whose sides are $y = 2x + 7$, $x - 3y - 6 = 0$ and $x + 2y = 8$ is right angled. Find its other angles.

(15) Show that the straight lines $3x + y + 4 = 0$, $3x + 4y - 15 = 0$ and $24x - 7y - 3 = 0$ form an isosceles triangle.

(16) Show that the straight lines $3x + 4y=13$; $2x - 7y + 1 = 0$ and $5x - y = 14$ are concurrent.

(17) Find 'a' so that the straight lines $x - 6y + a = 0$, $2x + 3y + 4 = 0$ and $x + 4y + 1 = 0$ may be concurrent.

(18) Find the value of 'a' for which the straight lines
$x + y - 4 = 0$, $3x + 2 = 0$ and $x - y + 3a = 0$ are concurrent.

(19) Find the co-ordinates of the orthocentre of the triangle whose vertices are the points $(-2, -1)$, $(6, -1)$ and $(2, 5)$

(20) If $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent, show that $a^3 + b^3 + c^3 = 3abc$

(21) Find the co-ordinates of the orthocentre of the triangle formed by the straight lines $x + y - 1 = 0$, $x + 2y - 4 = 0$ and $x + 3y - 9 = 0$

(22) The equation of the sides of a triangle are $x + 2y = 0$, $4x + 3y = 5$ and $3x + y = 0$. Find the co-ordinates of the orthocentre of the triangle.

5.4 Pair of straight lines

5.4.1 Combined equation of the pair of straight lines

We know that any equation of first degree in x and y represents a straight line. Let $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ be the individual equations of any two straight lines. Then their combined equation is

$$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0$$

$$l_1l_2x^2 + (l_1m_2 + l_2m_1)\,xy + m_1m_2y^2 + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2 = 0$$

Hence the equation of a pair of straight lines may be taken in the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where a, b, c, f, g, h are constants.

5.4.2 Pair of straight lines passing through the origin

The homogeneous equation $ax^2 + 2hxy + by^2 = 0$ of second degree in x and y represents a pair of straight lines passing through the origin.

Considering $ax^2 + 2hxy + by^2 = 0$ as a quadratic equation in x, we get

$$x = \frac{-2hy \pm \sqrt{4h^2y^2 - 4aby^2}}{2a}$$

$$= \left[\frac{-2h \pm 2\sqrt{h^2 - ab}}{2a}\right]y = \frac{-h \pm \sqrt{h^2 - ab}}{a}\,y$$

$$\therefore \quad ax = \left(-h \pm \sqrt{h^2 - ab}\right)y$$

i.e. $ax + \left(h + \sqrt{h^2 - ab}\right)y = 0$ and $ax + \left(h - \sqrt{h^2 - ab}\right)y = 0$ are the two straight lines, each passing through the origin. Hence $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines intersecting at the origin.

Note : The straight lines are (1) real and distinct if $h^2 > ab$

(2) coincident if $h^2 = ab$

(3) imaginary if $h^2 < ab$

Sum and product of the slopes of pair of straight lines

The homogeneous equation $ax^2 + 2hxy + by^2 = 0$ of second degree in x and y represents a pair of straight lines passing through the origin.

Let $y = m_1x$ and $y = m_2x$ be the two straight lines passing through the origin. Therefore the combined equation is $(y - m_1x)(y - m_2x) = 0$

$$\Rightarrow \; m_1m_2x^2 - (m_1 + m_2)\,xy + y^2 = 0$$

This equation also represents a pair of straight lines passing through the origin.

Equating the co-efficients of like terms in the above equations, we get

$$\frac{m_1m_2}{a} = -\frac{(m_1 + m_2)}{2h} = \frac{1}{b}$$

$$\therefore\; m_1m_2 = \frac{a}{b}\;; \text{ i.e. Product of the slopes} = \frac{a}{b}$$

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{i.e. Sum of the slopes} = -\frac{2h}{b}$$

5.4.3 Angle between pair of straight lines passing through the origin

The equation of the pair of straight lines passing through the origin is

$$ax^2 + 2hxy + by^2 = 0 \qquad \dots (1)$$

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b}$$

Let 'θ' be the angle between the pair of straight lines.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\left[\pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\right]\right|$$

$$= \left|\left[\pm\frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}}\right]\right| = \left|\left[\pm\frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{a+b}{b}}\right]\right|$$

$$\tan\theta = \left|\left[\frac{\pm 2\sqrt{h^2 - ab}}{a + b}\right]\right|$$

$$\theta = \tan^{-1}\left|\left[\frac{\pm 2\sqrt{h^2 - ab}}{a + b}\right]\right| \quad \text{i.e. } \theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

It is conventional to take θ to be acute.

Corollary (1):

If 'θ' is the angle between the pair of straight lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

then $\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

It is same as the angle between the pair of straight lines

$ax^2 + 2hxy + by^2 = 0$ passing through the origin.

Corollary (2): If the straight lines are parallel, then $h^2 = ab$

[since $\theta = 0°$, $\tan\theta = 0$]

Corollary (3): If the straight lines are perpendicular then

coeff. of x^2 + coeff. of $y^2 = 0$ [since $\theta = 90°$, $\tan\theta = \infty$]

The condition for a general second degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Assume that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ … (1)

represents a pair of straight lines. Treating this equation as a quadratic in x, this can be written as $ax^2 + 2(hy + g)x + (by^2 + 2fy + c) = 0$

By solving for x,we get

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2 + 2fy + c)}}{a}$$

$$\Rightarrow \qquad ax + hy + g = \pm\sqrt{(hy+g)^2 - a(by^2 + 2fy + c)}$$

$$= \pm\sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}$$

Now in order that each of these equations may be of the first degree in x and y, the expression in the R.H.S should be a perfect square. This is possible only if the discriminant of this quadratic in 'y' under the radical or within the root is zero.

$\therefore (h^2 - ab)(g^2 - ac) = (gh - af)^2$

Simplifying this we get $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ which is the required condition.

Example 5.28: Find the angle between the straight lines $x^2 + 4xy + 3y^2 = 0$

Solution:

Here $a = 1$, $2h = 4$, $b = 3$

If 'θ' is the angle between the given straight lines, then

$$\theta = \tan^{-1}\left|\left[\frac{2\sqrt{h^2 - ab}}{a+b}\right]\right| = \tan^{-1}\left|\left[\frac{2\sqrt{4-3}}{4}\right]\right| = \tan^{-1}\left(\frac{1}{2}\right)$$

Example 5.29: The slope of one of the straight lines of $ax^2 + 2hxy + by^2 = 0$ is thrice that of the other, show that $3h^2 = 4ab$

Solution:

Let 'm_1' and 'm_2' be the slopes of pair of straight lines.

Then $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$

It is given that $m_2 = 3m_1$

$$\therefore m_1 + 3m_1 = -\frac{2h}{b} \quad \Rightarrow \quad m_1 = -\frac{h}{2b}$$

But $\quad m_1 \,.\, 3m_1 = \dfrac{a}{b} \qquad \Rightarrow\; 3m_1^2 = \dfrac{a}{b} \;\Rightarrow\; 3\left(\dfrac{-h}{2b}\right)^2 = \dfrac{a}{b}$

$$\Rightarrow \qquad \frac{3h^2}{4b^2} = \frac{a}{b}$$

$$\Rightarrow \qquad 3h^2 = 4ab$$

Example 5.30: Show that $x^2 - y^2 + x - 3y - 2 = 0$ represents a pair of straight lines. Also find the angle between them.

Solution:

The given equation is

$$x^2 - y^2 + x - 3y - 2 = 0 \qquad \ldots (1)$$

Comparing this with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get $a = 1$, $h = 0$, $b = -1$, $g = \frac{1}{2}$, $f = -\frac{3}{2}$, $c = -2$. Condition for the given equation to represent a pair of straight lines is $abc+2fgh-af^2-bg^2-ch^2 = 0$

$$abc+2fgh-af^2-bg^2-ch^2 = (1)(-1)(-2)+2\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)(0)-(1)\left(\frac{9}{4}\right) - -1)\left(\frac{1}{4}\right)-(2)\,(0)$$

$$= 2 - \frac{9}{4} + \frac{1}{4} = \frac{8-9+1}{4}$$

$$= 0$$

Hence the given equation represents a pair of straight lines.

Since $a + b = 1 - 1 = 0$, the angle between the straight lines is 90°.

Example 5.31: Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines and also find the separate equation of the straight lines.

Solution:

Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 3$, $b = 2$, $h = \frac{7}{2}$, $g = \frac{5}{2}$, $f = \frac{5}{2}$, $c = 2$. The condition for the given equation to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$abc+2fgh-af^2-bg^2-ch^2 = (3)\,(2)\,(2) + 2\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right) - 3\left(\frac{25}{4}\right) - 2\left(\frac{25}{4}\right) - 2\left(\frac{49}{4}\right)$$

$$= 12 + \frac{175}{4} - \frac{75}{4} - \frac{50}{4} - \frac{98}{4} = 0$$

Hence the given equation represents a pair of straight lines.

Now, factorising the second degree terms

we get $3x^2 + 7xy + 2y^2 = (x + 2y)(3x + y)$

Let $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = (x + 2y + l)(3x + y + m)$

Comparing the coefficient of x, $3l + m = 5$;

Comparing the coefficient of y, $l + 2m = 5$

Solving these two equations, we get $l = 1$, $m = 2$

$\therefore$ The separate equations are $x + 2y + 1 = 0$ and $3x + y + 2 = 0$

Example 5.32: Show that the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines and find the distance between them.

Solution:

The given equation is $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$

Here $a = 4$, $h = 2$, $b = 1$; $ab - h^2 = 4(1) - 2^2 = 4 - 4 = 0$

$\therefore$ The given equation represents a pair of parallel straight lines.

Now $4x^2 + 4xy + y^2 = (2x + y)^2$

$\therefore$ $4x^2 + 4xy + y^2 - 6x - 3y - 4 = (2x + y + l)(2x + y + m)$

Comparing the coefficient of x, $2l + 2m = -6$ i.e. $l + m = -3$ … (1)

Comparing the constant term, $lm = -4$ … (2)

$$\therefore \quad l + \left(\frac{-4}{l}\right) = -3 \quad \Rightarrow l^2 + 3l - 4 = 0$$

$$\text{i.e.} \quad (l + 4)(l - 1) = 0 \quad \Rightarrow \quad l = -4, 1$$

$$\text{Now} \quad lm = -4 \quad \Rightarrow \quad m = 1, -4$$

$\therefore$ The separate equations are $2x + y - 4 = 0$ and $2x + y + 1 = 0$

The distance between them is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} = \left|\dfrac{-4 - 1}{\sqrt{2^2 + 1^2}}\right| = \sqrt{5}$ units

Example 5.33: Find the combined equation of the straight lines whose separate equations are $x + 2y - 3 = 0$ and $3x - y + 4 = 0$

Solution:

The combined equation of the given straight lines is

$$(x + 2y - 3)(3x - y + 4) = 0$$

i.e. $3x^2 + 6xy - 9x - xy - 2y^2 + 3y + 4x + 8y - 12 = 0$

i.e. $3x^2 + 5xy - 2y^2 - 5x + 11y - 12 = 0$ is the required combined equation.

EXERCISE 5.4

(1) If the equation $ax^2 + 3xy - 2y^2 - 5x + 5y + c = 0$ represents a pair of perpendicular straight lines, find a and c.

(2) Find the angle between the pair of straight lines given by

$(a^2 - 3b^2)\, x^2 + 8ab\, xy + (b^2 - 3a^2)y^2 = 0$

(3) Show that if one of the angles between pair of straight lines

$ax^2 + 2hxy + by^2 = 0$ is 60° then $(a + 3b)(3a + b) = 4h^2$

(4) Show that $9x^2 + 24xy + 16y^2 + 21x + 28y + 6 = 0$ represents a pair of parallel straight lines and find the distance between them.

(5) The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, show that $8h^2 = 9ab$.

(6) Find the combined equation of the straight lines through the origin, one of which is parallel to and the other is perpendicular to the straight line $2x + y + 1 = 0$

(7) Find the combined equation of the straight lines whose separate equations are $x + 2y - 3 = 0$ and $3x + y + 5 = 0$

(8) Find k such that the equation $12x^2 + 7xy - 12y^2 - x + 7y + k = 0$ represents a pair of straight lines. Find the separate equations of the straight lines and also the angle between them.

(9) If the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + c = 0$ represents a pair of straight lines, find the value of c. Find the separate equations of the straight lines and also the angle between them.

(10) For what value of k does $12x^2 + 7xy + ky^2 + 13x - y + 3 = 0$ represents a pair of straight lines? Also write the separate equations.

(11) Show that $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ represents a pair of straight lines and the angle between them is $\tan^{-1}\left(\frac{2}{11}\right)$

5.5 Circle

Definition: A circle is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

5.5.1 The equation of a circle when the centre and radius are given

Let C(h, k) be the centre and r be the radius of the circle. Let P(x, y) be any point on the circle

CP = $r \Rightarrow CP^2 = r^2 \Rightarrow (x - h)^2 + (y-k)^2 = r^2$

is the required equation of the circle.

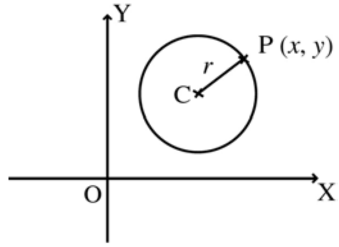


Fig. 5.8

Note :

If the centre of the circle is at the origin, i.e. (h, k) = (0, 0) then the equation of the circle is $x^2 + y^2 = r^2$.

5.5.2 The equation of a circle if the end points of a diameter are given

Let A(x_1, y_1) and B(x_2, y_2) be the end points of a diameter. Let P(x, y) be any point on the circle.

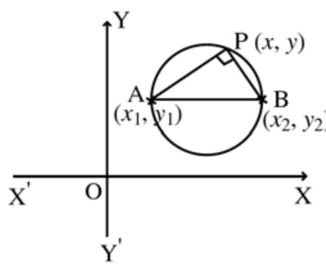


Fig. 5.9.

The angle in a semi circle is a right angle.

$\therefore$ PA is perpendicular to PB

$\therefore$ (Slope of PA) (Slope of PB) = – 1

$$\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$\therefore$ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the required equation of the circle.

5.5.3 The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Consider the equation $x^2 + y^2 + 2gx + 2fy + c = 0$

This can be written as $x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$

$$(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$[x - (-g)]^2 + [y - (-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

This is of the form $(x - h)^2 + (y - k)^2 = r^2$

$\therefore$ The considered equation represents a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$

$\therefore$ The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Note : The general second degree equation $ax^2+by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if (1) $a = b$ i.e. coefficient of x^2 = coefficient of y^2

(2) $h = 0$ i.e. no xy term

5.5.4 Parametric form

Consider a circle with radius r and centre at the origin. Let P(x, y) be any point on the circle. Assume that OP makes an angle θ with the positive direction of x-axis. Draw the perpendicular PM to the x-axis.

From the figure (5.10), $\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$.

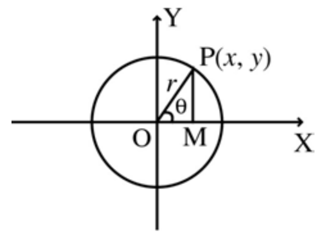


Fig. 5.10

Here x and y are the co-ordinates of any point on the circle. Note that these two co-ordinates depend on θ.

The value of r is fixed. The equations $x = r\cos\theta$, $y = r\sin\theta$ are called the parametric equations of the circle $x^2 + y^2 = r^2$. Here 'θ' is called the parameter and $0 \le \theta \le 2\pi$

Another parametric form:

We know that $\quad \sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \quad ; \quad \cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Let $t = \tan\frac{\theta}{2}$

If $0 \le \theta \le 2\pi$ then $-\infty < t < \infty$

$x = r\cos\theta \Rightarrow x = \dfrac{r(1-t^2)}{1+t^2} \quad ; \quad y = r\sin\theta \Rightarrow y = \dfrac{2rt}{1+t^2}$

Thus $x = \dfrac{r(1-t^2)}{1+t^2}$, $y = \dfrac{2rt}{1+t^2}$, $-\infty < t < \infty$ is another parametric equation of the circle $x^2 + y^2 = r^2$

Clearly $x = \dfrac{r(1-t^2)}{1+t^2}$, $y = \dfrac{2rt}{1+t^2}$ satisfy the equation $x^2 + y^2 = r^2$

Example 5.34: Find the equation of the circle if the centre and radius are (2, – 3) and 4 respectively.

Solution:

The equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

Here $(h, k) = (2, -3)$ and $r = 4$ $\therefore$ $(x-2)^2 + (y+3)^2 = 4^2$

i.e. $x^2 + y^2 - 4x + 6y - 3 = 0$ is the required equation of the circle.

Example 5.35: Find the equation of the circle if (2, – 3) and (3, 1) are the extremities of a diameter.

Solution:

The equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Here $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (3, 1)$

$\therefore$ $(x-2)(x-3) + (y+3)(y-1) = 0$

$x^2 - 5x + 6 + y^2 + 2y - 3 = 0$

$\therefore$ The required equation is $x^2 + y^2 - 5x + 2y + 3 = 0$

Example 5.36: Find the centre and radius of the circle $x^2 + y^2 + 2x - 4y + 3 = 0$

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Here $2g = 2$, $2f = -4$, $c = 3$

$\therefore$ centre is $(-g, -f) = (-1, 2)$

radius is $\sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 - 3} = \sqrt{2}$ units.

Example 5.37: Find the centre and radius of the circle $3x^2+3y^2-2x+6y-6 = 0$

Solution:

The given equation is $3x^2 + 3y^2 - 2x + 6y - 6 = 0$

Rewriting the above, $x^2 + y^2 - \frac{2}{3}x + 2y - 2 = 0$

Comparing this with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$

We get $2g = -\frac{2}{3}$, $2f = 2$, $c = -2$

$\therefore$ centre is $(-g, -f) = \left(\frac{1}{3}, -1\right)$

radius is $\sqrt{g^2 + f^2 - c} = \sqrt{\frac{1}{9} + 1 + 2} = \frac{2\sqrt{7}}{3}$ units.

Example 5.38: If (4, 1) is one extremity of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the other extremity.

Solution:

Comparing $x^2 + y^2 - 2x + 6y - 15 = 0$ with the general equation of the circle,

we get $2g = -2 \quad 2f = 6$

$\therefore$ centre is C $(-g, -f) = (1, -3)$

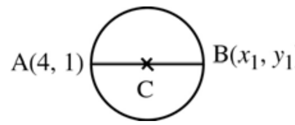


Fig. 5.11

Let B(x_1, y_1) be the other extremity and

A be (4, 1)

C is the mid point of AB

$$\therefore \quad \frac{x_1 + 4}{2} = 1, \quad \frac{y_1 + 1}{2} = -3 \Rightarrow x_1 = -2, \quad y_1 = -7$$

$\therefore$ The other extremity is $(-2, -7)$

Example 5.39: Find the equation the circle passing through the points (0,1), (2,3) and (–2, 5).

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

The points (0, 1), (2, 3) and (– 2, 5) lie on the circle

$$\therefore \quad 2f + c = -1 \quad \dots (1)$$

$$4g + 6f + c = -13 \quad \dots (2)$$

$$-4g + 10f + c = -29 \quad \dots (3)$$

$$(1) - (2) \Rightarrow -4g - 4f = 12$$

$$g + f = -3 \quad \dots (4)$$

$$(2) - (3) \Rightarrow 8g - 4f = 16$$

$$2g - f = 4 \quad \dots (5)$$

$$(4) + (5) \Rightarrow 3g = 1 \Rightarrow g = \frac{1}{3}$$

$$(4) \Rightarrow f = -3 - \frac{1}{3} = -\frac{10}{3}$$

$$(1) \Rightarrow c = \frac{17}{3}$$

$$\therefore x^2 + y^2 + 2\left(\frac{1}{3}\right)x + 2\left(-\frac{10}{3}\right)y + \frac{17}{3} = 0$$

$\therefore 3x^2 + 3y^2 + 2x - 20y + 17 = 0$ is the required equation.

Example 5.40: Find the equation of the circle passing through the points (0, 1), (2, 3) and having the centre on the line $x - 2y + 3 = 0$

Solution:

The general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

(0, 1) lies on the circle $\therefore\ 2f + c = -1 \qquad \ldots (1)$

(2, 3) lies on the circle $\therefore\ 4g + 6f + c = -13 \qquad \ldots (2)$

The centre $(-g, -f)$ lies on $x - 2y + 3 = 0$;

$$\therefore\ -g + 2f = -3 \qquad \ldots (3)$$

(1) – (2) $\Rightarrow$ $-4g - 4f = 12$

i.e. $g + f = -3 \qquad \ldots (4)$

(3) + (4) $\Rightarrow$ $3f = -6 \quad \therefore f = -2$

(3) $\Rightarrow$ $g = -1$

(1) $\Rightarrow$ $c = 3$

$\therefore$ The required equation is $x^2 + y^2 - 2x - 4y + 3 = 0$

Example 5.41:

Find the values of a and b if the equation

$(a - 4)x^2 + by^2 + (b - 3)xy + 4x + 4y - 1 = 0$ represents a circle.

Solution:

The given equation is $(a - 4)x^2 + by^2 + (b - 3)xy + 4x + 4y - 1 = 0$

(i) coefficient of $xy = 0 \Rightarrow b - 3 = 0 \quad \therefore\ b = 3$

(ii) coefficient of x^2 = co-efficient of $y^2 \quad \Rightarrow\ a - 4 = b$

$$\therefore\ a = 7$$

Thus $a = 7$, $b = 3$

Example 5.42: Find the equation of the circle with centre (2, – 3) and radius 3. Show that it passes through the point (2, 0).

Solution:

If the centre is (h, k) and radius is r, then the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$.

Here $(h, k) = (2, -3)$ and $r = 3$.

$$(x - 2)^2 + (y + 3)^2 = 3^2$$

$(x - 2)^2 + (y + 3)^2 = 9$ is the required equation of the circle.

Putting (2, 0) in the equation of the circle, we get

L.H.S. = $(2 - 2)^2 + (0 + 3)^2 = 0 + 9 = 9$ = R.H.S.

Hence the circle passes through (2, 0)

Example 5.43:

Find the equation of the circle with centre (1, – 2) and passing through the point (4, 1)

Solution:

Let C be (1, – 2) and P be (4, 1)

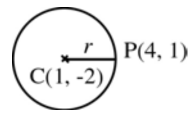


Fig. 5.12

Radius $r = \text{CP} = \sqrt{(1-4)^2 + (-2-1)^2} = \sqrt{9+9} = \sqrt{18}$

Thus the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2 = r^2$

$$\Rightarrow \quad (x-1)^2 + (y+2)^2 = \sqrt{18}^{\,2}$$

i.e. $x^2 + y^2 - 2x + 4y - 13 = 0$ is the required equation.

Example 5.44: Find the parametric equations of the circle $x^2 + y^2 = 16$

Solution:

Here $r^2 = 16 \Rightarrow r = 4$. The parametric equations of the circle $x^2 + y^2 = r^2$ in parameter θ are $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ The parametric equations of the given circle $x^2 + y^2 = 16$ are

$x = 4\cos\theta, \quad y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$

Example 5.45: Find the cartesian equation of the circle whose parametric equations are

$x = 2\cos\theta, y = 2\sin\theta, \quad 0 \le \theta \le 2\pi$

Solution:

To find the caretsian equation of the circle, eliminate the parameter 'θ' from the given equations, $\cos\theta = \frac{x}{2}$; $\sin\theta = \frac{y}{2}$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$\therefore x^2 + y^2 = 4$ is the required cartesian equation of the circle.

EXERCISE 5.5

(1) Find the centre and radius of the following circles:

(i) $x^2 + y^2 = 1$ (ii) $x^2 + y^2 - 4x - 6y - 9 = 0$

(iii) $x^2 + y^2 - 8x - 6y - 24 = 0$ (iv) $3x^2 + 3y^2 + 4x - 4y - 4 = 0$

(v) $(x-3)(x-5) + (y-7)(y-1) = 0$

(2) For what values of a and b does the equation $(a-2)x^2 + by^2 + (b-2)xy + 4x + 4y - 1 = 0$ represents a circle? Write down the resulting equation of the circle.

(3) Find the equation of the circle passing through the point (1, 2) and having its centre at (2, 3).

(4) $x + 2y = 7$, $2x + y = 8$ are two diameters of a circle with radius 5 units. Find the equation of the circle.

(5) The area of a circle is 16π square units. If the centre of the circle is $(7, -3)$, find the equation of the circle.

(6) Find the equation of the circle whose centre is $(-4, 5)$ and circumference is 8π units.

(7) Find the circumference and area of the circle $x^2 + y^2 - 6x - 8y + 15 = 0$

(8) Find the equation of the circle which passes through (2, 3) and whose centre is on x-axis and radius is 5 units.

(9) Find the equation of the circle described on the line joining the points (1, 2) and (2, 4) as its diameter.

(10) Find the equation of the circle passing through the points (1, 0), (0, – 1) and (0, 1).

(11) Find the equation of the circle passing through the points (1, 1), (2, –1) and (3, 2).

(12) Find the equation of the circle that passes through the points (4, 1) and (6, 5) and has its centre on the line $4x + y = 16$.

(13) Find the equation of the circle whose centre is on the line $x = 2y$ and which passes through the points (– 1, 2) and (3, – 2).

(14) Find the cartesian equation of the circle whose parametric equations are $x = \frac{1}{4}\cos\theta$, $y = \frac{1}{4}\sin\theta$ and $0 \le \theta \le 2\pi$

(15) Find the parametric equation of the circle $4x^2 + 4y^2 = 9$

5.6. Tangent

5.6.1 Introduction

Let us consider a circle with centre at C and a straight line AB. This straight line can be related to the circle in 3 different positions as shown in the following figures.

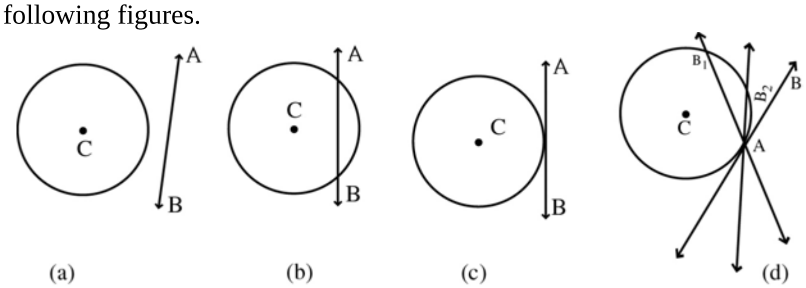


Fig. 5.13

In figure (5.13 a), the straight line AB does not touch or intersect the circle.

In figure (5.13 b), the straight line AB intersects the circle in two points and it is called a secant.

In figure (5.13 c), the straight line AB touches the circle at exactly one point, and it is called a tangent. In otherwords, the limiting form of a secant is called a tangent *(Fig. 5.13d)*

Definition : A tangent to a circle is a straight line which intersects (touches) the circle in exactly one point.

5.6.2 Equation of the tangent to a circle at a point (x_1, y_1)

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (1)$

Let $P(x_1, y_1)$ be a given point on it.

$\therefore\ x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \ldots (2)$

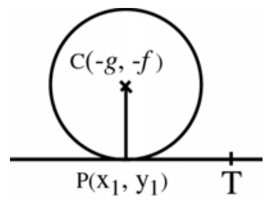


Fig. 5.14

Let PT be the tangent at P.

The centre of the circle is $C(-g, -f)$.

$$\text{Slope of the CP} = \frac{y_1 + f}{x_1 + g}$$

$$\text{Since CP is perpendicular to PT, slope of PT} = -\left(\frac{x_1 + g}{y_1 + f}\right)$$

$\therefore$ Equation of the tangent PT is

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

$$(y - y_1)(y_1 + f) = -(x - x_1)(x_1 + g)$$

$$(y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$$

$$\Rightarrow \quad yy_1 - y_1^2 + fy - fy_1 + [xx_1 - x_1^2 + gx - gx_1] = 0$$

$$\Rightarrow \quad xx_1 + yy_1 + fy + gx = x_1^2 + y_1^2 + gx_1 + fy_1$$

Add $gx_1 + fy_1 + c$ on both sides

$$xx_1 + yy_1 + gx + gx_1 + fy + fy_1 + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ is the required equation of the tangent at (x_1, y_1)

Corollary:

The equation of the tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Note: To get the equation of the tangent at (x_1, y_1), replace x^2 as xx_1, y^2 as yy_1, x as $\frac{x + x_1}{2}$ and y as $\frac{y + y_1}{2}$ in the equation of the circle.

5.6.3 Length of the tangent to the circle from a point (x_1, y_1)

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

Let PT be the tangent to the circle from $P(x_1, y_1)$ outside it. We know that the co-ordinate of the centre C is $(-g, -f)$ and

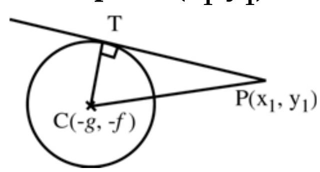


Fig. 5.15

$$\text{radius } r = \text{CT} = \sqrt{g^2 + f^2 - c}$$

From the right angled triangle PCT,

$$\begin{aligned}
\text{PT}^2 &= \text{PC}^2 - \text{CT}^2 \\
&= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\
&= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c \\
&= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c
\end{aligned}$$

$\therefore$ PT $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$, which is the length of the tangent from the point (x_1, y_1)

to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Note : (1) If the point P is on the circle then $\text{PT}^2 = 0$ (PT is zero).

(2) If the point P is outside the circle then $\text{PT}^2 > 0$ (PT is real)

(3) If the point P is inside the circle then $\text{PT}^2 < 0$ (PT is imaginary)

Corollary:

The constant c will be positive if the origin is outside the circle, zero if it is on the circle and negative if it is inside the circle.

5.6.4 The condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$

Let the line $y = mx + c$ be a tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1)

But the equation of the tangent at (x_1, y_1) to the circle

$x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

Thus the equations $y = mx + c$ and $xx_1 + yy_1 = a^2$ are representing the same straight line and hence their coefficients are proportional.

$$\therefore \frac{1}{y_1} = -\frac{m}{x_1} = \frac{c}{a^2}$$

$$\therefore x_1 = \frac{-a^2m}{c}, y_1 = \frac{a^2}{c}$$

But (x_1, y_1) is a point on the circle $x^2 + y^2 = a^2$

$$\therefore x_1^2 + y_1^2 = a^2 \Rightarrow \frac{a^4m^2}{c^2} + \frac{a^4}{c^2} = a^2$$

$$\Rightarrow \quad a^2m^2 + a^2 = c^2 \Rightarrow a^2(m^2 + 1) = c^2$$

i.e. $\quad c^2 = a^2(1 + m^2)$ is the required condition.

Note:(1)The point of contact of the tangent $y = mx + c$ to the circle $x^2+y^2= a^2$ is

$$\left[\frac{-am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right]$$

(2) The equation of any tangent to a circle is of the form

$$y = mx \pm a\sqrt{1+m^2}$$

5.6.5 Two tangents can be drawn from a point to a circle

Let (x_1, y_1) be the given point. We know that $y = mx \pm a\sqrt{1+m^2}$ is the equation of any tangent. It passes through (x_1, y_1).

$$\therefore y_1 = mx_1 \pm a\sqrt{1+m^2}$$

$$\Rightarrow \quad y_1 - mx_1 = \pm a\sqrt{1+m^2}$$

$$\Rightarrow \quad (y_1 - mx_1)^2 = a^2(1+m^2)$$

$$\Rightarrow \quad y_1^2 + m^2x_1^2 - 2mx_1y_1 - a^2 - a^2m^2 = 0$$

$$\Rightarrow \quad m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0$$

This is a quadratic equation in 'm'. Thus 'm' has two values. But 'm' is the slope of the tangent. Thus two tangents can be drawn from a point to a circle.

Note : (1) If (x_1, y_1) is an exterior point (lies outside) then both the tangents are real and visible

(2) If (x_1, y_1) is an interior point (lies inside) the circle then both the tangents are imaginary and hence not visible.

(3) If (x_1, y_1) is a boundary point (lies on) then both the tangents coincide and appears to be one.

5.6.6. Equation of the chord of contact of tangents from a point to the circle

The general equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ … (1)

Let $P(x_1, y_1)$ be a point outside the circle. Let the tangents from $P(x_1, y_1)$ touch the circle at $Q(x_2, y_2)$ and $R(x_3, y_3)$

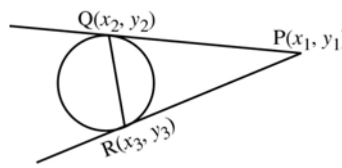


Fig. 5.16

The equation of the tangent PQ at Q (x_2, y_2) is

$$xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad \ldots (2)$$

The equation of the tangent PR at $R(x_3, y_3)$ is

$$xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0 \quad \ldots (3)$$

But (x_1, y_1) satisfy the equations (2) and (3)

$\therefore\ x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$ and … (4)

$x_1x_3 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0$ … (5)

But equations (4) and (5) show that (x_2, y_2) and (x_3, y_3) lie on the line

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Hence the straight line $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ represents the equation of QR, chord of contact of tangents from (x_1, y_1).

Example 5.46: Find the length of the tangent from (2, 3) to the circle $x^2 + y^2 – 4x – 3y + 12 = 0$.

Solution:

The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$\therefore$ Length of the tangent to the given circle is $\sqrt{x_1^2 + y_1^2 - 4x_1 - 3y_1 + 12}$

$$= \sqrt{2^2 + 3^2 - 4.2 - 3.3 + 12}$$

$$= \sqrt{4 + 9 - 8 - 9 + 12}$$

$$= \sqrt{8} = 2\sqrt{2} \text{ units}$$

Example 5.47: Show that the point (2, 3) lies inside the circle

$$x^2 + y^2 - 6x - 8y + 12 = 0.$$

Solution:

The length of the tangent PT from P(x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$PT^2 = 2^2 + 3^2 - 6.2 - 8.3 + 12 = 4 + 9 - 12 - 24 + 12$$

$$= -11 < 0$$

The point (2, 3) lies inside the circle

Example 5.48: Find the equation of the tangent to the circle $x^2+y^2=25$ at (4, 3).

Solution:

The equation of the circle is $x^2 + y^2 = 25$.

The equation of the tangent at (x_1, y_1) is $xx_1 + yy_1 = 25$. Here $(x_1, y_1) = (4, 3)$.

$\therefore$ The equation of the tangent at (4, 3) is $4x + 3y = 25$

Example 5.49: If $y=3x+c$ is a tangent to the circle $x^2+y^2=9$, find the value of c.

Solution:

The condition for the line $y = mx + c$ to be a tangent to

$$x^2 + y^2 = a^2 \text{ is } c = \pm a\sqrt{1 + m^2}$$

Here $a = 3, m = 3$

$$\therefore c = \pm 3\sqrt{10}$$

Example 5.50: Find the equation of the tangent to

$$x^2 + y^2 - 4x + 4y - 8 = 0 \text{ at } (-2, -2)$$

Solution:

The equation of the tangent at (x_1, y_1) to the given circle is

$$xx_1 + yy_1 - 4\left(\frac{x + x_1}{2}\right) + 4\left(\frac{y + y_1}{2}\right) - 8 = 0$$

$$xx_1 + yy_1 - 2(x + x_1) + 2(y + y_1) - 8 = 0$$

At $(-2, -2)$, the equation of the tangent is

$$-2x - 2y - 2(x - 2) + 2(y - 2) - 8 = 0$$

$$\Rightarrow \quad -4x - 8 = 0$$

$\Rightarrow \quad x + 2 = 0$ is the required equation of the tangent.

Example 5.51: Find the length of the chord intercepted by the circle

$$x^2 + y^2 - 2x - y + 1 = 0 \text{ and the line } x - 2y = 1.$$

Solution:

To find the end points of the chord, solve the equations of the circle and the line. Substitute $x = 2y + 1$ in the equation of the circle.

$$(2y + 1)^2 + y^2 - 2(2y + 1) - y + 1 = 0$$

$$4y^2 + 4y + 1 + y^2 - 4y - 2 - y + 1 = 0$$

$$5y^2 - y = 0 \qquad \therefore\ y(5y - 1) = 0$$

$$y = 0 \quad \Big| \quad y = \frac{1}{5}$$

$$\Rightarrow \qquad x = 1 \quad \Big| \quad x = \frac{7}{5}$$

$\therefore$ The two end points are (1, 0) and $\left(\frac{7}{5}, \frac{1}{5}\right)$

$\therefore$ Length of the chord $= \sqrt{\left(1 - \frac{7}{5}\right)^2 + \left(0 - \frac{1}{5}\right)^2} = \sqrt{\frac{4}{25} + \frac{1}{25}} = \frac{1}{\sqrt{5}}$ units

Example 5.52: Find the value of p if the line $3x + 4y - p = 0$ is a tangent to the circle $x^2 + y^2 = 16$.

Solution:

The condition for the tangency is $c^2 = a^2 (1 + m^2)$.

Here $a^2 = 16$, $m = -\frac{3}{4}$, $c = \frac{p}{4}$

$$c^2 = a^2 (1 + m^2) \quad \Rightarrow \quad \frac{p^2}{16} = 16\left(1 + \frac{9}{16}\right) = 25$$

$$p^2 = 16 \times 25$$

$$\therefore\ p = \pm 20$$

Example 5.53: Find the equation of the circle which has its centre at (2, 3) and touches the x-axis.

Solution:

Let P be a point on x-axis where it touches the circle.

Given that the centre C is (2, 3) and P is (2, 0)

$r = \text{CP} = \sqrt{(2-2)^2 + (3-0)^2} = 3$

The equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$

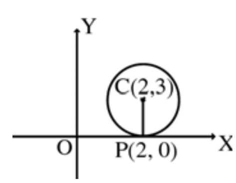


Fig. 5.17

$$(x - 2)^2 + (y - 3)^2 = 3^2$$

$$x^2 + y^2 - 4x - 6y + 4 = 0$$

EXERCISE 5.6

(1) Find the length of the tangent from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$

(2) Prove that the tangents from (0, 5) to the circles
$x^2 + y^2 + 2x - 4 = 0$ and
$x^2 + y^2 - y + 1 = 0$ are equal.

(3) Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 8y - 5 = 0$ at (2, 1).

(4) Is the point (7, – 11) lie inside or outside the circle $x^2 + y^2 - 10x = 0$?

(5) Determine whether the points (– 2, 1), (0, 0) and (4, – 3) lie outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$

(6) Find the co-ordinates of the point of intersection of the line $x + y = 2$ with the circle $x^2 + y^2 = 4$

(7) Find the equation of the tangent lines to the circle $x^2 + y^2 = 9$ which are parallel to $2x + y - 3 = 0$

(8) Find the length of the chord intercepted by the circle $x^2 + y^2 - 14x + 4y + 28 = 0$ and the line $x - 7y + 4 = 0$

(9) Find the equation of the circle which has its centre at (5, 6) and touches (i) x-axis (ii) y-axis

(10) Find the equation of the tangent to $x^2 + y^2 - 2x - 10y + 1 = 0$ at (– 3, 2)

(11) Find the equation of the tangent to the circle $x^2 + y^2 = 16$ which are (i) perpendicular and (ii) parallel to the line $x + y = 8$

(12) Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 2y - 21 = 0$ at (1, 4).

(13) Find the value of p so that the line $3x + 4y - p = 0$ is a tangent to $x^2 + y^2 - 64 = 0$

(14) Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 + 2x + y - 3 = 0$ cuts off by the line $y = x - 1$.

5.7. Family of circles

Concentric circles:

Two or more circles having the same centre are called concentric circles.

Circles touching each other:

Two circles may touch each other either internally or externally. Let C_1, C_2 be the centers of the circle and r_1, r_2 be their radii and P the point of contact.

Case (1):

The two circles touch externally.

The distance between their centres is equal to the sum of their radii.

(i.e.) $C_1C_2 = r_1 + r_2$

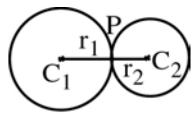


Fig. 5.18

Case (2) The two circles touch internally:

The distance between their centres is equal to the difference of their radii.

$C_1\,C_2 = C_1P - C_2P = r_1 - r_2$

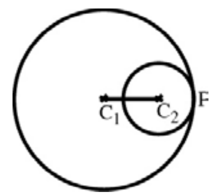


Fig. 5.19

Orthogonal circles:

Definition: Two circles are said to be orthogonal if the tangent at their point of intersection are at right angles.

Condition for two circles cut orthogonally

Let the two circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and cut each other orthogonally.

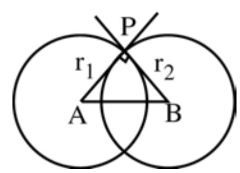


Fig. 5.20

Let A and B be the centres of the two circles

$\therefore$ A is $(-g_1, -f_1)$ and B is $(-g_2, -f_2)$ $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ and $r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$

In the right angled triangle APB, $AB^2 = AP^2 + PB^2$

i.e. $(-g_1 + g_2)^2 + (-f_1 + f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$

$\Rightarrow$ $g_1^2 + g_2^2 - 2g_1g_2 + f_1^2 + f_2^2 - 2f_1f_2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$

$\Rightarrow$ $-2g_1g_2 - 2f_1f_2 = -c_1 - c_2$

i.e. $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

is the required condition for orthogonality.

Example 5.54: Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other.

Solution:

The given circles are

$$x^2 + y^2 - 4x + 6y + 8 = 0 \quad \dots (1)$$

and $$x^2 + y^2 - 10x - 6y + 14 = 0 \quad \dots (2)$$

(1) $\Rightarrow$ $g_1 = -2 \quad f_1 = 3, \quad c_1 = 8$. Centre is A(2, – 3)

$$\text{radius } r_1 = \sqrt{g_1^2 + f_1^2 - c_1} = \sqrt{4 + 9 - 8} = \sqrt{5}$$

(2) $\Rightarrow$ $g_2 = -5, f_2 = -3, \quad c_2 = 14$. Centre is B(5, 3)

$$\text{radius } r_2 = \sqrt{25 + 9 - 14} = \sqrt{20} = 2\sqrt{5}$$

$$\begin{aligned}\text{Distance between A and B} &= \sqrt{(2-5)^2 + (-3-3)^2} \\ &= \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5} \\ &= r_1 + r_2\end{aligned}$$

$\therefore$ The circles touch each other.

Example 5.55: Find the equation of the circle, which is concentric with the circle

$x^2 + y^2 - 4x - 6y - 9 = 0$ and passing through the point (– 4, – 5).

Solution:

The given circle is $x^2 + y^2 - 4x - 6y - 9 = 0$

Centre $(-g, -f)$ is (2, 3)

The circle passes through the point (– 4, – 5).

$\therefore$ radius $= \sqrt{(2+4)^2 + (3+5)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

The equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$

Here $(h, k) = (2, 3), \quad r = 10$

$\therefore \quad (x-2)^2 + (y-3)^2 = 10^2$

$x^2 + y^2 - 4x - 6y - 87 = 0$ is the required equation of the circle.

Example 5.56: Prove that the circles $x^2 + y^2 - 8x + 6y - 23 = 0$ and

$x^2 + y^2 - 2x - 5y + 16 = 0$ are orthogonal.

Solution:

The equations of the circle are

$$x^2 + y^2 - 8x + 6y - 23 = 0 \quad \dots (1)$$

$$x^2 + y^2 - 2x - 5y + 16 = 0 \quad \dots (2)$$

(1) $\Rightarrow$ $g_1 = -4, \quad f_1 = 3, \qquad c_1 = -23$

(2) $\Rightarrow$ $g_2 = -1, \quad f_2 = -\frac{5}{2}, \quad c_2 = 16$

Condition for orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$$2g_1g_2 + 2f_1 f_2 = 2(-4)(-1) + 2(3)\left(-\frac{5}{2}\right) = 8 - 15 = -7$$

$$c_1 + c_2 = -23 + 16 = -7$$

$$\therefore \qquad 2g_1g_2 + 2f_1 f_2 = c_1 + c_2$$

$\therefore$ The two circles cut orthogonally and hence they are orthogonal circles.

Example 5.57:

Find the equation of the circle which passes through the point (1, 2) and cuts orthogonally each of the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 2x + 8y - 7 = 0$

Solution:

Let the required equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ … (1)

The point (1, 2) lies on the circle

$$\therefore \qquad 1 + 4 + 2g + 4f + c = 0$$

$$2g + 4f + c = -5 \qquad \dots (2)$$

The circle (1) cuts the circle $x^2 + y^2 = 9$ orthogonally.

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$\Rightarrow \qquad 2g(0) + 2f(0) = c - 9$$

$$\therefore \quad c = 9 \qquad \dots (3)$$

Again the circle (1) cuts $x^2 + y^2 - 2x + 8y - 7 = 0$ orthogonally.

$$\therefore\ 2g(-1) + 2f(4) = c - 7$$

$$\Rightarrow \qquad -2g + 8f = 9 - 7 = 2$$

$$\Rightarrow \qquad -g + 4f = 1 \qquad \dots (4)$$

(2) becomes $\qquad 2g + 4f = -14$

$$\therefore \quad g + 2f = -7 \qquad \dots (5)$$

(4) + (5) $\Rightarrow$ $\qquad 6f = -6 \quad \Rightarrow \quad f = -1$

(5) $\Rightarrow$ $\qquad g - 2 = -7 \quad \Rightarrow \quad g = -5$

$\therefore$ The required equation of the circle is $x^2 + y^2 - 10x - 2y + 9 = 0$

EXERCISE 5.7

(1) Show that the circles $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$ touch each other.

(2) Show that each of the circles $x^2 + y^2 + 4y - 1 = 0$, $x^2 + y^2 + 6x + y + 8 = 0$ and $x^2 + y^2 - 4x - 4y - 37 = 0$ touches the other two.

(3) Find the equation of the circle concentric with the circle $x^2 + y^2 - 2x - 6y + 4 = 0$ and having radius 7.

(4) Find the equation of the circle which is concentric with the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passes through the point (5, 4)

(5) Show that the circle $x^2 + y^2 - 8x - 6y + 21 = 0$ is orthogonal to the circle $x^2 + y^2 - 2y - 15 = 0$

(6) Find the circles which cuts orthogonally each of the following circles

(i) $x^2 + y^2 + 2x + 4y + 1 = 0$, $x^2+y^2 - 4x+3 = 0$ and $x^2 + y^2 + 6y + 5 = 0$

(ii) $x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$ and $x^2 + y^2 - x + 22y + 3 = 0$

(7) Find the equation of the circle which passes through (1, – 1) and cuts orthogonally each of the circles $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 - 2x + 3y - 7 = 0$

(8) Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$

6. TRIGONOMETRY

6.1 Introduction:

Trigonometry is one of the oldest branch of Mathematics. The word trigonometry means "triangle measurement". In olden days trigonometry was mainly used as a tool for use in astronomy. The early Babylonians divided the circle into 360 equal parts, giving us degrees, perhaps because they thought that there were 360 days in a year.

The sine function was invented in India, perhaps around 300 to 400 A.D. By the end of ninth century, all six trigometric functions and identities relating them were known to the Arabs.

In its earlier stages trigonometry was mainly concerned with establishing relations between the sides and angles of a triangle, but now it finds its application in various branches of science such as surveying, engineering, navigation etc. For every branch of higher Mathematics a knowledge of trigonometry is essential.

6.1.1 Angles:

An angle is defined as the amount of rotation of a revolving line from the initial position to the terminal position. Counter–clockwise rotations will be called positive and the clockwise will be called negative.

Consider a rotating ray OA with its end point at the origin O.

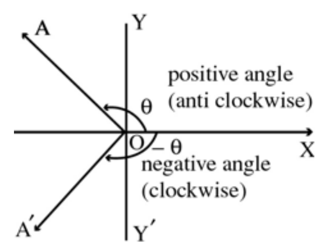


Fig. 6. 1

The rotating ray OA is often called the **terminal** side of the angle and the positive half of the *x*-axis (OX) is called the **initial side.**

The positive angle θ is $\underline{|XOA}$ (counter-clockwise rotation)

The negative angle θ is $\underline{|XOA'}$ (clockwise rotation)

Note : 1. one complete rotation (counter –clockwise) = 360° = 360 degree

2. If there is no rotation the measure of the angle is 0°.

6.1.2 Measurement of angles:

If a rotation from the initial position to the terminal position is $\left(\frac{1}{360}\right)^{th}$ of the revolution, the angle is said to have a measure of one degree and written as 1°. A degree is divided into minutes, and minute is divided into seconds.

i.e. 1 degree (1°) = 60 minutes (60′)

1 minute (1′) = 60 seconds (60″)

In theoretical work another system of measurement of angles is used which is known as circular measure. A radian is taken as the unit of measurement.

6.1.3 Radian measure:

Definition:

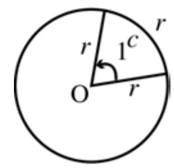


Fig. 6. 2

One radian, written as 1^c is the measure of an angle subtended at the centre O of a circle of radius r by an arc of length r.

Note :
1. To express the measure of an angle as a real number, we use radian measure.
2. The word "radians" is optional and often omitted. Thus if no unit is given for a rotation, it is understood to be in radians.
3. 'c' in 1^c indicates the circular measure.

6.1.4 Relation between Degrees and Radians

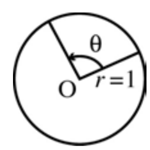


Fig. 6. 3

Since a circle of radius r has a circumference of $2\pi r$, a circle of radius 1 unit (which is referred to as an unit circle) has circumference 2π. When θ is a complete rotation, P travels the circumference of an unit circle completely.

If θ is a complete rotation (counter-clockwise) then $\theta = 2\pi$ radian. On the other hand we already know that one complete rotation (counter-clockwise) is 360°, consequently, $360° = 2\pi$ radians or $180° = \pi$ radian. It follows that $1° = \frac{\pi}{180}$ radian and $\frac{180°}{\pi} = 1$ radian. Therefore $1° = 0.01746$ radian (app.) and 1 radian $= 180° \times \frac{7}{22} = 57°\ 16'$ (app.).

Conversion for some special angles:

degrees	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

(Table 6.1)

Example 6.1: Convert

(i) 150° into radians (ii) $\frac{3\pi}{4}$ into degrees (iii) $\frac{1}{4}$ radians into degrees.

Solution:

(i) $150° = 150 \times \frac{\pi}{180}$ radians $= \frac{5}{6}\pi$

(ii) $\frac{3\pi}{4}$ radians $= \frac{3\pi}{4} \times \frac{180°}{\pi} = 135°$

(iii) $\frac{1}{4}$ radians $= \frac{1}{4} \times \frac{180}{\pi} = \frac{1}{4} \times 180 \times \frac{7}{22} = 14°\ 19'\ 5''$

6.1.5 Quadrants

Let X′OX and YOY′ be two lines at right angles to each other as in the fig. (6.4) we call X′OX and YOY′ as *x*-axis and *y*-axis respectively.

Clearly these axes divide the entire plane into four equal parts, called quadrants.

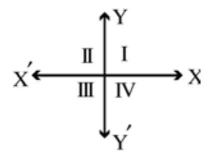


Fig. 6. 4

The parts XOY, YOX′, X′OY′ and Y′OX are known as the first, the second, the third and the fourth quadrant respectively.

Angle in standard position:

If the vertex of an angle is at O and its initial side lies along *x*-axis, then the angle is said to be in standard position.

Angle in a Quadrant:

An angle is said to be in a particular quadrant, if the terminal side of the angle in standard position lies in that quadrant.

Example 6.2: Find the quadrants in which the terminal sides of the following angles lie.

(i) – 60°

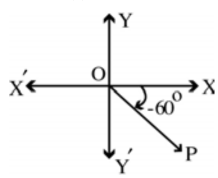


Fig. 6. 5 a

From Fig (6.5a) the terminal side of – 60° lies in IV quadrant.

(ii) – 300°

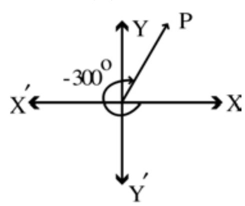


Fig. 6.5 b

From Fig (6.5b) the terminal side of – 300° lies in I quadrant.

(iii) 1295°

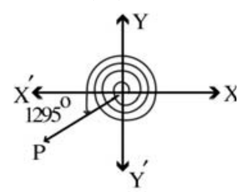


Fig. 6.5 c

From Fig (6.5c)

$1295° = 3\times360°+180° + 35°$

The terminal side lies in III quadrant.

EXERCISE 6.1

(1) Convert the following degree measure into radian measure.

(i) $30°$ (ii) $100°$ (iii) $200°$ (iv) $-320°$

(v) $-85°$ (vi) $7° 30'$

(2) Find the degree measure corresponding to the following radian measure

(i) $\left(\frac{\pi}{8}\right)$ (ii) $\left(\frac{18\pi}{5}\right)$ (iii) -3 (iv) $\left(\frac{7\pi}{12}\right)$

(3) Determine the quadrants in which the following degrees lie.

(i) $380°$ (ii) $-140°$ (iii) $1100°$

6.2. Trigonometrical ratios and Identities

6.2.1 Trigonometrical ratios:

In the co-ordinate plane, consider a point A on the positive side of x-axis. Let this point revolve about the origin in the anti clockwise direction through an angle θ and reach the point P. Now $\angle XOP = \theta$. Let the point P be (x, y). Draw PL perpendicular to OX.

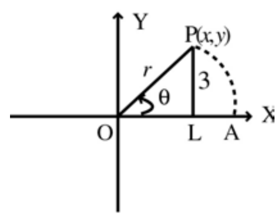


Fig. 6. 6

The triangle OLP is a right angled triangle, in which θ is in standard position. Also, from the ΔOLP, we have

OL = x = Adjacent side ; PL = y = opposite side ;

$$OP = \sqrt{x^2 + y^2} = \text{Hypotenuse } (= r > 0)$$

The trigonometrical ratios (circular functions) are defined as follows :

The sine of the angle θ is defined as the ratio $\frac{y}{r}$ it is denoted by $\sin\theta$.

i.e. $\sin\theta = \frac{y}{r}$; cosecant value at $\theta = \frac{r}{y} = \text{cosec}\,\theta$; $y \neq 0$

and $\cos\theta = \frac{x}{r}$; secant value at $\theta = \frac{r}{x} = \sec\theta$; $x \neq 0$

$\tan\theta = \frac{y}{x}$; cotangent value at $\theta = \frac{x}{y} = \cot\theta$; $y \neq 0$

Note :

1. From the definition, observe that $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$, while $\cot\theta$ and $\text{cosec}\,\theta$ are not defined if $y = 0$.
2. $\text{cosec}\,\theta$, $\sec\theta$ and $\cot\theta$ are the reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively.

Example 6.3: If (2, 3) is a point on the terminal side of θ, find all the six trigonometrical ratios.

Solution:

P(x, y) is (2, 3) and it lies in the first quadrant.

$\therefore\ x = 2, \quad y = 3\ ;\ r = \sqrt{x^2 + y^2} = \sqrt{4 + 9} = \sqrt{13}$

$\therefore \sin\theta = \dfrac{y}{r} = \dfrac{3}{\sqrt{13}}\ ;\ \cos\theta = \dfrac{x}{r} = \dfrac{2}{\sqrt{13}}\ ;\ \tan\theta = \dfrac{y}{x} = \dfrac{3}{2}$

$\text{cosec}\,\theta = \dfrac{\sqrt{13}}{3}\quad ;\ \sec\theta = \dfrac{\sqrt{13}}{2}\quad ;\quad \cot\theta = \dfrac{2}{3}$

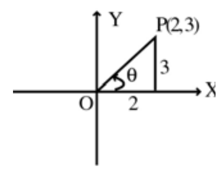


Fig. 6. 7

Note : 1. From example (6.3), we see that all the trigonometrical ratios are positive when the terminal side of angle θ lies in first quadrant.

Now, let us observe the sign of trigonometrical ratios if the point on the terminal side of θ lies in the other quadrants. (other than the first quadrant).

Example 6.4: If (– 2, – 3) is a point on the terminal side of θ. Find all the six trigonometrical ratios.

Solution:

P(x, y) is (– 2, – 3) and it lies in the third quadrant

$\therefore\ x = -2\ ,\ y = -3\ ;$

$r = \sqrt{x^2 + y^2} = \sqrt{4 + 9} = \sqrt{13}$

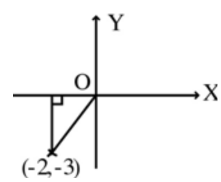


Fig. 6. 8

$\sin\theta = \dfrac{y}{r} = \dfrac{-3}{\sqrt{13}} = -\,ve\ ;\ \cos\theta = \dfrac{x}{r} = \dfrac{-2}{\sqrt{13}} = -\,ve\ ;\ \tan\theta = \dfrac{y}{x} = \dfrac{-3}{-2} = \dfrac{3}{2} = +\,ve$

$\text{cosec}\,\theta = \dfrac{r}{y} = \dfrac{\sqrt{13}}{-3} = -\,ve\ ;\ \sec\theta = \dfrac{r}{x} = \dfrac{\sqrt{13}}{-2} = -\,ve\ ;\ \cot\theta = \dfrac{-2}{-3} = \dfrac{2}{3} = +\,ve$

As example illustrates, trigonometric functions may be negative. For instance, since r is always positive, $\sin\theta = \dfrac{y}{r}$ and $\text{cosec}\,\theta = \dfrac{r}{y}$ have the same sign as y. Thus sin θ and cosec θ are positive when θ is in the first or second quadrants, and negative when θ is in the third or fourth quadrants. The signs of the other trigonometric functions can be found similarly. The following table indicates the signs depending on where θ lies.

Quadrants / Functions	I	II	III	IV
Sin	+	+	–	–
Cos	+	–	–	+
Tan	+	–	+	–
Cosec	+	+	–	–
Sec	+	–	–	+
Cot	+	–	+	–

II sin cosec	I All
III tan cot	IV cos sec

Table (6.2)

6.2.2 Trigonometrical ratios of particular angles:

Let X′OX and YOY′ be the co-ordinate axes. With O as centre and unit radius draw a circle cutting the co-ordinate axes at A, B, A′ and B′ as shown in the figure.

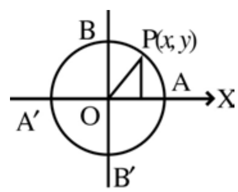


Fig. 6. 9

Suppose that a moving point starts from A and move along the circumference of the circle. Let it cover an arc length. θ and take the final position P. Let the co-ordinates of this point be P(x, y). Then by definition, $x = \cos\theta$ and $y = \sin\theta$.

We consider the arc length θ to be positive or negative according as the variable point moves in the anti clockwise or clockwise direction respectively.

Range of cos θ and sinθ :

Since for every point (x, y) on the unit circle, we have

$-1 \le x \le 1$ and $-1 \le y \le 1$, therefore $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

Values of cosθ and sinθ for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π.

We know that the circumference of a circle of unit radius is 2π. If the moving point starts from A and moves in the anti clockwise direction then at the points A, B, A′, B′ and A the arc lengths covered are $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π respectively.

Also, the co-ordinates of these points are:A(1, 0), B(0, 1), A′(–1, 0), B′(0,– 1) and A(1, 0)

At the point:

A(1, 0), $\theta = 0 \Rightarrow \cos 0 = 1$ and $\sin 0 = 0$

B (0, 1),	$\theta = \frac{\pi}{2}$	$\Rightarrow$	$\cos\frac{\pi}{2}$	$= 0$	and	$\sin\frac{\pi}{2} = 1$
A′ (–1, 0),	$\theta = \pi$	$\Rightarrow$	$\cos\pi$	$= -1$	and	$\sin\pi = 0$
B′ (0, – 1),	$\theta = 3\frac{\pi}{2}$	$\Rightarrow$	$\cos 3\frac{\pi}{2}$	$= 0$	and	$\sin 3\frac{\pi}{2} = -1$
A (1, 0),	$\theta = 2\pi$	$\Rightarrow$	$\cos 2\pi$	$= 1$	and	$\sin 2\pi = 0$

6.2.3 Trigonometrical ratios of 30°, 45° and 60° :

Consider an isosceles right-angled triangle whose equal sides are 1. Its hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Its base angle is 45°.

$\therefore \sin 45° = \frac{1}{\sqrt{2}}$; $\cos 45° = \frac{1}{\sqrt{2}}$; $\tan 45° = 1$

$\operatorname{cosec} 45° = \sqrt{2}$; $\sec 45° = \sqrt{2}$; $\cot 45° = 1$

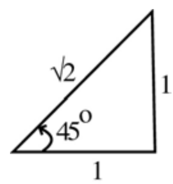


Fig. 6. 10

Opposite side = 1

adjacent side = 1

hypotenuse = $\sqrt{2}$

Consider an equilateral triangle ABC of side 2 units. Each of its angle is 60°. Let CD be the bisector of angle C. Then angle ACD is 30°. Also AD = 1 and CD = $\sqrt{2^2 - 1^2} = \sqrt{3}$. Now in the right angled triangle ACD

For 30°

opposite side = 1

adjacent side = $\sqrt{3}$

hypotenuse = 2

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$

$\therefore \operatorname{cosec} 30° = 2$

$\therefore \sec 30° = \frac{2}{\sqrt{3}}$

$\therefore \cot 30° = \sqrt{3}$

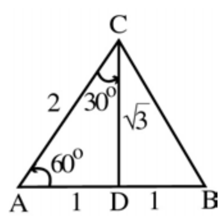


Fig. 6. 11

For 60°

opposite side = $\sqrt{3}$

adjacent side = 1

hypotenuse = 2

$\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

$\tan 60° = \sqrt{3}$

$\therefore \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}$

$\sec 60° = 2$

$\therefore \cot 60° = \frac{1}{\sqrt{3}}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

Table 6.3

Important results:

For all values of θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Proof:

Let X′OX and YOY′ be the co-ordinate axes. With O as centre and unit radius draw a circle meeting OX at A. Now let a moving point start from A and move in anti clockwise direction and take the final position P(x, y) so that arc AP = θ.

On the other hand, if the point starts from A and moves in the clockwise direction through the arc length AP′ equal to arc length AP. Then arc AP′= $-\theta$.

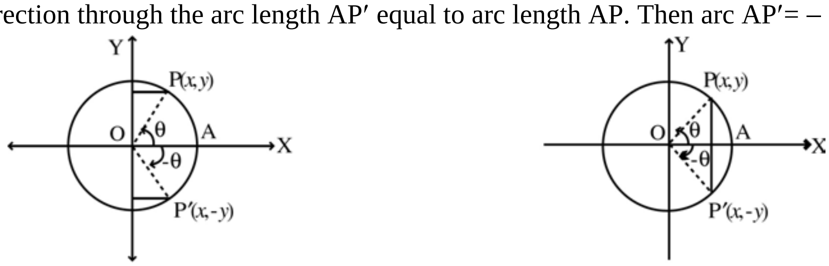


Fig. 6. 12

Thus $\angle AOP = \theta$ and $\angle AOP' = -\theta$

From the co-ordinate geometry, we know that the co-ordinates of P′ are (x, $-y$).

Clearly, $\cos\theta$ and $\cos(-\theta)$ are respectively the distances of points P and P′ from y axis and clearly each one of them is equal to x.

$\therefore \cos(-\theta) = \cos\theta$

Clearly, $\sin\theta$ and $\sin(-\theta)$ are respectively the distances of points P and P′ from x-axis. As $\sin\theta = y$ and $\sin(-\theta) = -y$, we have $\sin(-\theta) = -\sin\theta$

Deductions $\quad \text{cosec}(-\theta) = -\text{cosec}\,\theta$; $\sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = -\cot\theta$

6.2.4 T-ratios of (90° ± θ), (180° ± θ), (270° ± θ) and (360° ± θ):

It is evident that, when θ is a small angle ($0 < \theta < 90°$), then $90° - \theta$, $90° + \theta$ etc., are in the quadrants as given below:

Angle	Quadrant
$90° - \theta$	Q_1 (first quadrant)
$90° + \theta$	Q_2
$180° - \theta$	Q_2
$180° + \theta$	Q_3
$270° - \theta$	Q_3
$270° + \theta$	Q_4
$360° - \theta$; also equal to "$- \theta$"	Q_4
$360° + \theta$	Q_1

Table 6.4:

Let $P(\alpha, \beta)$ be a point in the first quadrant. Let $\angle XOP = \theta°$.

$$\therefore \sin\theta = \frac{\beta}{OP} \; ; \; \cos\theta = \frac{\alpha}{OP} \; ; \; \tan\theta = \frac{\beta}{\alpha}$$

$$\operatorname{cosec}\theta = \frac{OP}{\beta} \; ; \; \sec\theta = \frac{OP}{\alpha} \; ; \cot\theta = \frac{\alpha}{\beta}$$

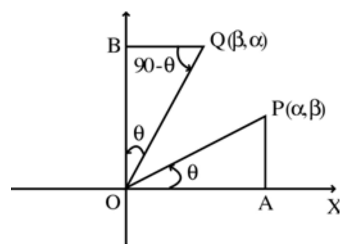


Fig. 6. 13

T-ratios of (90° – θ)

Let Q be a point in the first quadrant such that

$\angle XOQ = 90° - \theta$ and $OQ = OP$.

Let PA and QB be perpendicular to OX and OY respectively.

Then $\Delta OAP \equiv \Delta OBQ$ and Q is (β, α). Hence

$$\sin(90° - \theta) = \frac{y \text{ co-ordinate of Q}}{OQ} = \frac{\alpha}{OP} = \cos\theta$$

$$\cos(90° - \theta) = \frac{x \text{ co-ordinate of Q}}{OQ} = \frac{\beta}{OP} = \sin\theta$$

$$\tan(90° - \theta) = \frac{y \text{ co-ordinate of Q}}{x \text{ coordinate of Q}} = \frac{\alpha}{\beta} = \cot\theta$$

Similarly,

$$\operatorname{cosec}(90° - \theta) = \sec\theta$$

$$\sec(90° - \theta) = \operatorname{cosec}\theta$$

$$\cot(90° - \theta) = \tan\theta$$

T-ratios of (90° + θ)

Let R be a point in the second quadrant such that $\underline{|XOR}$ = 90° + θ and OR = OP

Let RC be perpendicular to x axis.

Then ΔOAP ≡ ΔRCO and R is (– β, α), Hence

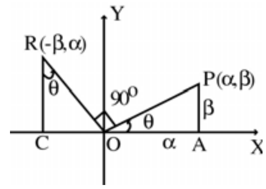


Fig. 6. 14

$$\sin(90° + \theta) = \frac{y \text{ co-ordinate of R}}{\text{OR}} = \frac{\alpha}{\text{OP}} = \cos\theta$$

$$\cos(90° + \theta) = \frac{x \text{ co-ordinate of R}}{\text{OR}} = \frac{-\beta}{\text{OP}} = -\sin\theta$$

$$\tan(90° + \theta) = \frac{y \text{ co-ordinate of R}}{x \text{ coordinate of R}} = \frac{\alpha}{-\beta} = -\cot\theta$$

Similarly, cosec (90° + θ) = sec θ

sec (90° + θ) = – cosec θ

cot (90° + θ) = – tan θ

T-ratios of (180° – θ)

Let S be a point in the second quadrant such that $\underline{|XOS}$ =180° –θ and OS =OP

Draw SD perpendicular to x-axis

Thus Δ OAP ≡ Δ ODS and S is (– α, β). Hence

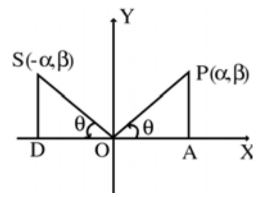


Fig. 6. 15

$$\sin(180° - \theta) = \frac{y \text{ co-ordinate of S}}{\text{OS}} = \frac{\beta}{\text{OP}} = \sin\theta$$

$$\cos(180° - \theta) = \frac{x \text{ co-ordinate of S}}{\text{OS}} = \frac{-\alpha}{\text{OP}} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{y \text{ co-ordinate of S}}{x \text{ coordinate of S}} = \frac{\beta}{-\alpha} = -\tan\theta$$

Similarly, cosec (180° – θ) = cosec θ

sec (180° – θ) = – sec θ

cot (180° – θ) = – cot θ

T-ratios of (180° + θ)

Let T be a point in the third quadrant such that $\underline{|XOT}$ = 180° + θ and OT = OP

Draw TE perpendicular to x-axis

Then ΔOAP ≡ ΔOET and T is (– α, – β). Hence

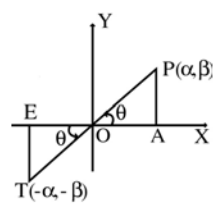


Fig. 6. 16

$$\sin(180° + \theta) = \frac{y \text{ co-ordinate of T}}{\text{OT}} = \frac{-\beta}{\text{OP}} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{x \text{ co-ordinate of T}}{\text{OT}} = \frac{-\alpha}{\text{OP}} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{y \text{ co-ordinate of T}}{x \text{ coordinate of T}} = \frac{-\beta}{-\alpha} = \tan\theta$$

Similarly, $\operatorname{cosec}(180° + \theta) = -\operatorname{cosec}\theta$

$\sec(180° + \theta) = -\sec\theta$

$\cot(180° + \theta) = \cot\theta$

Remark: To determine the trigonometric ratios of any angle, follow the procedure given below

(i) Write the angle in the form $k\frac{\pi}{2} \pm \theta$; $k \in Z$.

(ii) Determine the quadrant in which the terminal side of the angle lies.

(iii) Detrmine the sign of the given trigonometric function in that particular quadrant, using $\frac{\text{S} \quad \text{A}}{\text{T} \quad \text{C}}$ rule.

(iv) If k is even, trigonometric function of allied angle equals the same function of θ.

(v) If k is odd, then adopt the following changes:

sine $\leftrightarrow$ cos ; tan $\leftrightarrow$ cot ; sec $\leftrightarrow$ cosec

Trigonometrical ratios for related angles

Angle / function	$-\theta$	**90** $-\theta$	**90 +** θ	**180** $-\theta$	**180+**θ	**270**$-\theta$	**270+**θ	**360**$-\theta$ **or** $-\theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$
cosec	$-\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$
sec	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$
cot	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$

Table 6.5

Note : Since 360° corresponds to one full revolution, sine of the angles 360°+45°;720°+45°;1080°+45° are equal to sine of 45°. This is so for the other trigonemetrical ratios. That is, when an angle exceeds 360°, it can be reduced to an angle between 0° and 360° by wiping out integral multiples of 360°.

Example 6.5:

Simplify : (i) tan 735° (ii) cos 980° (iii) sin 2460° (iv) cos (–870°)

(v) sin (–780°) (vi) cot (–855°) (vii) cosec 2040° (viii) sec (– 1305°)

Solution:

(i) $\tan (735°) = \tan (2 \times 360° + 15°) = \tan 15°$

(ii) $\cos 980° = \cos (2 \times 360° + 260°) = \cos 260°$

$= \cos (270° - 10°) = -\sin 10°$

(iii) $\sin (2460°) = \sin (6 \times 360° + 300°) = \sin (300°)$

$= \sin (360° - 60°)$

$= -\sin 60°$

$= -\dfrac{\sqrt{3}}{2}$

(iv) $\cos (-870°) = \cos (870°) = \cos (2 \times 360° + 150°)$

$= \cos 150 = \cos (180° - 30°)$

$= -\cos 30° = -\dfrac{\sqrt{3}}{2}$

(v) $\sin (-780°) = -\sin 780°$

$= -\sin (2 \times 360° + 60°)$

$= -\sin 60° = -\dfrac{\sqrt{3}}{2}$

(vi) $\cot (-855°) = -\cot (855°) = -\cot (2 \times 360° + 135°)$

$= -\cot (135°) = -\cot (180° - 45°)$

$= \cot 45° = 1$

(vii) $\operatorname{cosec} (2040°) = \operatorname{cosec} (5 \times 360° + 240°) = \operatorname{cosec} (240°)$

$= \operatorname{cosec} (180° + 60°) = -\operatorname{cosec} (60°)$

$= -\dfrac{2}{\sqrt{3}}$

(viii) $\sec (-1305°) = \sec (1305°) = \sec (3 \times 360° + 225°)$

$= \sec (225°) = \sec (270° - 45°)$

$= -\operatorname{cosec} 45° = -\sqrt{2}$

Example 6.6: Simplify : $\dfrac{\cot (90° - \theta)\ \sin (180° + \theta)\ \sec (360° - \theta)}{\tan (180° + \theta)\ \sec (-\theta)\ \cos (90° + \theta)}$

Solution: The given expression $= \dfrac{\tan\theta\,(-\sin\theta)\,(\sec\theta)}{\tan\theta\,(\sec\theta)\,(-\sin\theta)}$

$= 1$

Example 6.7:

Without using the tables, prove that $\sin 780° \ \sin 480° + \cos 120° \cos 60° = \dfrac{1}{2}$

Solution:

$$\sin 780° = \sin(2 \times 360° + 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\sin 480° = \sin(360° + 120°)$$

$$= \sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2} \ ; \cos 60° = \frac{1}{2}$$

$$\text{L.H.S.} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \ \text{R.H.S.}$$

6.2.5 Special properties of Trigonometrical functions:

Periodic function:

A function $f(x)$ is said to be a periodic function with period α if $f(x + \alpha) = f(x)$. The least positive value of α is called the fundamental period of the function.

All the circular functions (trigonometrical functions) are periodic functions.

For example,

$$\sin(x + 2\pi) = \sin x \ ; \ \sin(x + 4\pi) = \sin x \ ; \ \sin(x + 6\pi) = \sin x$$

$$\sin(x + 2n\pi) = \sin x, \ n \in Z$$

Here $\alpha = \ldots\ldots -6\pi, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$. But the fundamental period must be the least positive quantity. Therefore $\alpha = 2\pi$ is the fundamental period.

Thus sine function is a periodic function with fundamental period 2π. Similarly one can prove that the functions $\cos x$, $\operatorname{cosec} x$ and $\sec x$ are also periodic functions with fundamental period 2π while $\tan x$ and $\cot x$ are periodic with fundamental period π.

6.2.6 Odd and even functions:

We know that, if $f(x) = f(-x)$, then the function is an even function and if $f(-x) = -f(x)$ then the function is an odd function.

Consider $f(x) = \sin x$; $f(-x) = \sin(-x) = -\sin x = -f(x)$ i.e. $f(x) = -f(-x)$ $\therefore$ $\sin x$ is an odd function. Similarly we can prove that cosec x, $\tan x$ and $\cot x$ are odd functions.

Consider $f(x) = \cos x$; $f(-x) = \cos(-x) = \cos x = f(x)$. $\therefore$ $\cos x$ is an even function. Similarly we can prove $\sec x$ is an even function.

Note : We can read more about odd and even function in Chapter 7.

EXERCISE 6.2

(1) If $\sin\theta = \frac{11}{12}$, find the value of

$\sec(360° - \theta) \cdot \tan(180° - \theta) + \cot(90° + \theta)\sin(270° + \theta)$

(2) Express the following as functions of positive acute angles:-

(i) $\sin(-840°)$ (ii) $\cos(1220°)$ (iii) $\cot(-640°)$ (iv) $\tan(300°)$

(v) $\operatorname{cosec}(420°)$ (vi) $\sin(-1110°)$ (vii) $\cos(-1050°)$

(3) Prove that $\dfrac{\sin 300° \cdot \tan 330° \cdot \sec 420°}{\cot 135° \cdot \cos 210° \cdot \operatorname{cosec} 315°} = -\sqrt{\dfrac{2}{3}}$

(4) Prove that $\left\{1+\cot\alpha - \sec\left(\alpha + \frac{\pi}{2}\right)\right\}\left\{1 + \cot\alpha + \sec\left(\alpha + \frac{\pi}{2}\right)\right\} = 2\cot\alpha$

(5) Express the following as functions of A :

(i) $\sec\left(A - \frac{3\pi}{2}\right)$ (ii) $\operatorname{cosec}\left(A - \frac{\pi}{2}\right)$ (iii) $\tan\left(A - \frac{3\pi}{2}\right)$

(iv) $\cos(720° + A)$ (v) $\tan(A + \pi)$

(6) Prove that $\dfrac{\sin(180°+A) \cdot \cos(90°-A) \cdot \tan(270°-A)}{\sec(540° - A)\cos(360°+A)\operatorname{cosec}(270°+A)} = -\sin A\cos^2 A$

(7) Prove that $\sin\theta \cdot \cos\theta\left\{\sin\left(\frac{\pi}{2} - \theta\right) \cdot \operatorname{cosec}\theta + \cos\left(\frac{\pi}{2} - \theta\right)\sec\theta\right\} = 1$

(8) Find the values of :-

(i) $\cos(135°)$ (ii) $\sin(240°)$ (iii) $\sec(225°)$ (iv) $\cos(-150°)$

(v) $\cot(315°)$ (vi) $\operatorname{cosec}(-300°)$ (vii) $\cot\frac{5\pi}{4}$ (viii) $\tan\left(-\frac{5\pi}{6}\right)$

(9) If A, B, C, D are angles of a cyclic quadrilateral prove that
$\cos A + \cos B + \cos C + \cos D = 0$.

(10) Find the values of the following expressions :

(i) $\tan^2 30° + \tan^2 45° + \tan^2 60°$ (ii) $\sin \frac{\pi}{6} \, . \cos \frac{\pi}{3} + \cos \frac{\pi}{6} \, . \sin \frac{\pi}{3}$

(iii) $\cos \frac{\pi}{6} \, . \cos \frac{\pi}{3} - \sin \frac{\pi}{6} \, . \sin \frac{\pi}{3}$ (iv) $\cos 45° . \cos 60° - \sin 45° . \sin 60°$

(v) $\tan^2 60° + 2 \tan^2 45°$ (vi) $\tan^2 45° + 4 \cos^2 60°$

(vii) $\cot 60° \, . \tan 30° + \sec^2 45° \, . \sin 90°$

(viii) $\tan^2 60° + 4 \cot^2 45° + 3 \sec^2 30° + \cos^2 90°$

(ix) $\tan^2 30° + 2 \sin 60° + \tan 45° - \tan 60° + \cos^2 30°$

(x) $\frac{1}{2} \sin^2 60° - \frac{1}{2} \sec 60° \tan^2 30° + \frac{4}{5} \sin^2 45° \, . \tan^2 60°$

(11) If $\cos \theta = -\frac{1}{2}$ and $\tan \theta > 0$ show that $\frac{5 \tan \theta + 4 \sin \theta}{\sqrt{3} \cos \theta - 3 \sin \theta} = 3$.

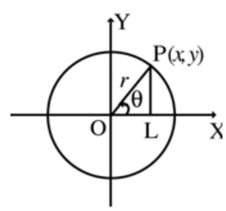
6.2.7 Trigonometrical identities:

As in variables, $\sin \theta \, . \sin \theta = (\sin \theta)^2$. This will be written as $\sin^2\theta$. Similarly

$\tan \theta \quad \tan^2\theta = \tan^3\theta$ etc. We can derive some fundamental trigonometric identities as follows:

Consider the unit circle with centre at the origin O. Let P(x, y) be any point on the circle with $\angle XOP = \theta$.

Draw PL perpendicular to OX. Now, triangle OLP is a right angled triangle in which (hypotenuse) OP = r = 1 unit, and x and y are adjacent and opposite sides respectively.



Now we have $\cos \theta = \frac{x}{1} = x$ and $\sin \theta = \frac{y}{1} = y$; $\tan \theta = \frac{y}{x}$

Fig. 6. 17

and $r^2 = x^2 + y^2 = 1$

From ΔOLP, we have $x^2 + y^2 = r^2 = 1$

i.e. $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

$$1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2} = \left(\frac{1}{x}\right)^2 = \left(\frac{1}{\cos \theta}\right)^2 = \sec^2\theta$$

$$1 + \cot^2\theta = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \left(\frac{1}{y}\right)^2 = \left(\frac{1}{\sin \theta}\right)^2 = \mathrm{cosec}^2\theta$$

Thus we have the identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \text{cosec}^2\theta$$

$\therefore$ From these we also have

$$\sec^2\theta - \tan^2\theta = 1$$
$$\text{cosec}^2\theta - \cot^2\theta = 1$$

Example 6.8: Show that $\cos^4A - \sin^4A = 1 - 2\sin^2A$

Solution:
$$\begin{aligned} \cos^4A - \sin^4A &= (\cos^2A + \sin^2A)(\cos^2A - \sin^2A) \\ &= \cos^2A - \sin^2A = 1 - \sin^2A - \sin^2A \\ &= 1 - 2\sin^2A \end{aligned}$$

Example 6.9: Prove that $\sec^2A + \text{cosec}^2A = \sec^2A \, . \, \text{cosec}^2A$

Solution:
$$\begin{aligned} \sec^2A + \text{cosec}^2A &= \frac{1}{\cos^2A} + \frac{1}{\sin^2A} \\ &= \frac{\sin^2A + \cos^2A}{\cos^2A \, . \, \sin^2A} = \frac{1}{\cos^2A \, . \, \sin^2A} \\ &= \sec^2A \, . \, \text{cosec}^2A \end{aligned}$$

Example 6.10: Show that $\cos A\sqrt{1 + \cot^2A} = \sqrt{\text{cosec}^2A - 1}$

Solution:
$$\begin{aligned} \cos A\sqrt{1 + \cot^2A} &= \cos A\sqrt{\text{cosec}^2A} = \cos A \, . \, \text{cosec}A \\ &= \frac{\cos A}{\sin A} = \cot A = \sqrt{\text{cosec}^2A - 1} \end{aligned}$$

Example 6.11: If $a\sin^2\theta + b\cos^2\theta = c$, show that $\tan^2\theta = \dfrac{c - b}{a - c}$

Solution:
$$a\sin^2\theta + b\cos^2\theta = c.$$

Dividing both sides by $\cos^2\theta$, we get $a\tan^2\theta + b = c\sec^2\theta$

$$\begin{aligned} a\tan^2\theta + b &= c(1 + \tan^2\theta) \\ \tan^2\theta(a - c) &= c - b \\ \therefore \tan^2\theta &= \frac{c - b}{a - c} \end{aligned}$$

Example 6.12: that $\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \text{cosec}\,A - \cot A$

Solution: consider, $\dfrac{1 - \cos A}{1 + \cos A} = \dfrac{1 - \cos A}{1 + \cos A} \times \dfrac{1 - \cos A}{1 - \cos A}$

$$= \frac{(1-\cos A)^2}{1-\cos^2 A} = \left(\frac{1-\cos A}{\sin A}\right)^2$$

$$\therefore \sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{1-\cos A}{\sin A} = \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$= \operatorname{cosec} A - \cot A$$

Example 6.13:

If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that $x^2 + y^2 = a^2 + b^2$

Solution:

$$\begin{aligned} x^2 + y^2 &= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 \\ &= a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta \\ &\quad + a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta \\ &= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) \\ &= a^2 + b^2 \end{aligned}$$

Example 6.14: Show that $\sin^2 A.\tan A + \cos^2 A . \cot A + 2\sin A . \cos A = \tan A + \cot A$

Solution:

$$\text{L.H.S.} = \sin^2 A . \frac{\sin A}{\cos A} + \cos^2 A . \frac{\cos A}{\sin A} + 2\sin A\cos A$$

$$= \frac{\sin^3 A}{\cos A} + \frac{\cos^3 A}{\sin A} + 2\sin A . \cos A$$

$$= \frac{\sin^4 A + \cos^4 A + 2\sin^2 A . \cos^2 A}{\sin A . \cos A}$$

$$= \frac{(\sin^2 A + \cos^2 A)^2}{\sin A . \cos A} = \frac{1}{\sin A . \cos A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A . \cos A} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

$$= \frac{\sin^2 A}{\sin A\cos A} + \frac{\cos^2 A}{\sin A\cos A}$$

Hence the result $\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A} = \tan A + \cot A =$ R.H.S.

Example 6.15: Show that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

Solution:

$$(\sin x - \cos x)^4 = \left[(\sin x - \cos x)^2\right]^2 = [\sin^2 x + \cos^2 x - 2\sin x\cos x]^2$$

$$= [1 - 2\sin x\cos x]^2$$

$$= 1 - 4\sin x\cos x + 4\sin^2 x\cos^2 x \qquad \text{...... (i)}$$

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x . \cos x$$

$$= 1 + 2\sin x\cos x \qquad \text{...... (ii)}$$

$$\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$$

$$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \,.\, \cos^2 x\,(\sin^2 x + \cos^2 x)$$

$$= 1 - 3\sin^2 x\ \cos^2 x \qquad \ldots\ldots \text{(iii)}$$

Using (i), (ii) and (iii)L.H.S.

$$= 3(1 - 4\sin x \cos x + 4\sin^2 x \,.\, \cos^2 x)$$

$$+ 6(1 + 2\sin x\ \cos x) + 4(1 - 3\sin^2 x\ \cos^2 x)$$

$$= 3 + 6 + 4$$

$$= 13 = \text{R.H.S.}$$

Example 6.16: Prove that $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$

Solution: L.H.S. $= \dfrac{\tan\theta + \sec\theta - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$

$$= \frac{\tan\theta + \sec\theta - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta)(1 - \sec\theta + \tan\theta)}{(\tan\theta - \sec\theta + 1)} = \tan\theta + \sec\theta$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = \frac{\sin\theta + 1}{\cos\theta} = \text{R.H.S.}$$

EXERCISE 6.3

(1) Prove the following:

(i) $\sin^4 A - \cos^4 A = 1 - 2\cos^2 A$

(ii) $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1 + \sin A\ \cos A)$

(iii) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

(iv) $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \mathrm{cosec}^2\theta$

(v) $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$ (vi) $\dfrac{\sec x + \tan x}{\sec x - \tan x} = (\sec x + \tan x)^2$

(vii) $\dfrac{\mathrm{cosec}\ \theta}{\cot\theta + \tan\theta} = \cos\theta$ (viii) $\dfrac{1}{\tan\theta + \sec\theta} = \sec\theta - \tan\theta$

(ix) $\dfrac{1}{\mathrm{cosec}\theta - \cot\theta} = \dfrac{1 + \cos\theta}{\sin\theta}$

(x) $(\sec\theta + \cos\theta)(\sec\theta - \cos\theta) = \tan^2\theta + \sin^2\theta$

(2) If $\tan\theta + \sec\theta = x$, show that $2\tan\theta = x - \dfrac{1}{x}$, $2\sec\theta = x + \dfrac{1}{x}$

Hence show that $\sin\theta = \dfrac{x^2 - 1}{x^2 + 1}$

(3) If $\tan\theta + \sin\theta = p$, $\tan\theta - \sin\theta = q$ and $p > q$ then show that $p^2 - q^2 = 4\sqrt{pq}$

(4) Prove that $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A}$

(5) Prove that $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A$

(6) Prove the following :

(i) $\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$ $(\sin A \neq 1)$

(ii) $\sqrt{\dfrac{1 + \cos A}{1 - \cos A}} = \operatorname{cosec} A + \cot A$ $(\cos A \neq 1)$

(iii) $\sqrt{\dfrac{1 - \sin\theta}{1 + \sin\theta}} = \sec\theta - \tan\theta$

(7) If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, show that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

(8) Prove that $(1 + \tan A + \sec A)(1 + \cot A - \operatorname{cosec} A) = 2$

6.3 Compound Angles

6.3.1 Compound Angles A + B and A – B

In the previous chapter we have found the trigonometrical ratios of angles such as $90° \pm \theta$, $180° \pm \theta$, … which involves single angle only. In this chapter we shall express the trigonometrical ratios of compound angles such as A + B, A – B, … interms of trigonometrical ratios of A, B, ….

It is important to note that the relation $f(x + y) = f(x) + f(y)$ is not true for all functions of a real variable. As an example all the six trigonometrical ratios do not satisfy the above relation.

$\cos(A + B)$ is not equal to $\cos A + \cos B$.

Let us develop the identity

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

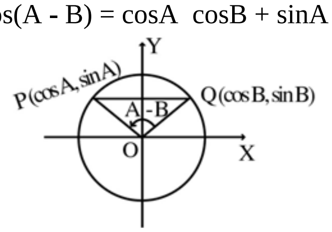


Fig. 6. 18

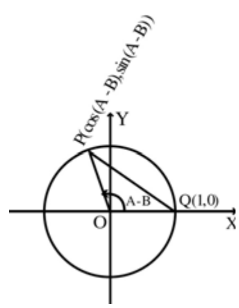


Fig. 6. 19

Let P and Q be any two points on the unit circle such that $\angle XOP = A$ and $\angle XOQ = B$. Then the coordinates of P and Q are (cos A, sinA) and (cosB, sinB) respectively.

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= (\cos^2 A - 2\cos A \cos B + \cos^2 B) + (\sin^2 A - 2\sin A \sin B + \sin^2 B)$$
$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2\cos A \cos B - 2\sin A \cos B$$
$$= 1+1 - 2\cos A \cos B - 2\sin A \sin B = 2 - 2(\cos A \cos B + \sin A \sin B) \quad \ldots (1)$$

Now imagine that the unit circle above is rotated so that the point Q is at (1, 0). The length PQ has not changed.

$$PQ^2 = [\cos(A - B) - 1]^2 + [\sin(A - B) - 0]^2$$
$$= [\cos^2(A - B) - 2\cos(A - B) + 1] + \sin^2(A - B)$$
$$= [\cos^2(A - B) + \sin^2(A - B)] + 1 - 2\cos(A - B) = 1 + 1 - 2\cos(A - B)$$
$$= 2 - 2\cos(A - B) \quad \ldots (2)$$

From (1) and (2), $2 - 2\cos(A - B) = 2 - 2(\cos A\cos B + \sin A \sin B)$

$\Rightarrow \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$

Next let us consider cos(A + B). This is equal to $\cos[A - (-B)]$ and by cosine of a difference identity, we have the following:

$$\cos(A + B) = \cos A \cos(-B) + \sin A \cdot \sin(-B)$$

But $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B.$$

To develop an identity for sin(A + B), we recall the following:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

In this identity we shall substitute A + B for θ

$$\sin(A + B) = \cos\left[\frac{\pi}{2} - (A + B)\right] = \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$$

We can now use the identity for the cosine of a difference.

$$= \cos\left(\frac{\pi}{2} - A\right) \cdot \cos B + \sin\left(\frac{\pi}{2} - A\right) \cdot \sin B$$
$$= \sin A \cdot \cos B + \cos A \cdot \sin B$$

Thus, $\sin(A + B) = \sin A \cdot \cos B + \cos A \sin B$

To find an identity for the sine of a difference, we can use the identity just derived, substituting – B for B

$$\sin (A - B) = \sin [A + (-B)]$$
$$= \sin A \cos(-B) + \cos A . \sin(-B)$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

An identity for the tangent of a sum can be derived using identities already established.

$$\tan (A + B) = \frac{\sin (A + B)}{\cos(A + B)}$$
$$= \frac{\sin A . \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Divide both Numerator and Denominator by cosA cosB

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$$

Similarly, an identity for the tangent of a difference can be established.

It is given by $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$

(1) $\sin (A + B) = \sin A \cos B + \cos A \sin B$

(2) $\sin (A - B) = \sin A \cos B - \cos A \sin B$

(3) $\cos (A + B) = \cos A \cos B - \sin A \sin B$

(4) $\cos (A - B) = \cos A \cos B + \sin A \sin B$

(5) $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(6) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$

Example 6.17: Find the values of (i) cos 15° (ii) cos 105° (iii) sin 75° (iv) tan 15°

Solution:

(i) $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) $\cos 105° = \cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$

$$= \frac{1}{2}\ \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\ \frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

(iii) $\sin 75° = \sin(45° + 30°) = \sin 45°\ \cos 30° + \cos 45°\ \sin 30°$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(iv) $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45°\ \tan 30°} = \dfrac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}}$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2 - \sqrt{3}$$

Example 6.18: If A, B are acute angles, $\sin A = \frac{3}{5}$; $\cos B = \frac{12}{13}$, find $\cos(A + B)$

Solution: $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$$

$$\therefore \cos(A + B) = \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{33}{65}$$

Example 6.19: Show that (i) $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B$

$$\begin{aligned} \sin(A+B) \sin(A-B) &= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\ &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ &= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B \\ &= \sin^2 A - \sin^2 B \end{aligned}$$

$$\begin{aligned} \cos(A + B) \cos(A - B) &= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ &= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\ &= \cos^2 A - \sin^2 B \end{aligned}$$

Example 6.20: If $A + B = 45°$, show that $(1 + \tan A)(1 + \tan B) = 2$ and hence deduce the value of $\tan 22\frac{1}{2}°$

Solution: Given $A + B = 45° \Rightarrow \tan(A + B) = \tan 45°$

$$\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = 1$$

i.e. $\tan A + \tan B = 1 - \tan A \cdot \tan B$

i.e. $1 + \tan A + \tan B = 2 - \tan A \tan B$ (add 1 on both sides)

$1 + \tan A + \tan B + \tan A \tan B = 2$

i.e. $(1 + \tan A)(1 + \tan B) = 2$

Take A = B then $2A = 45° \Rightarrow A = 22\frac{1}{2} = B$

$$\therefore \left(1 + \tan 22\frac{1}{2}\right)^2 = 2 \quad \Rightarrow \quad 1 + \tan 22\frac{1}{2} = \pm\sqrt{2}$$

$$\therefore \tan 22\frac{1}{2} = \pm\sqrt{2} - 1$$

Since $22\frac{1}{2}$ is acute, $\tan 22\frac{1}{2}$ is positive and therefore $\tan 22\frac{1}{2} = \sqrt{2} - 1$

Example 6.21:

(i) Prove that $\frac{\tan 69° + \tan 66°}{1 - \tan 69° \ \tan 66} = -1$ (ii) $\frac{\tan(A - B) + \tan B}{1 - \tan(A - B)\tan B} = \tan A$

(iii) $\frac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°} = \tan 62°$

Solution:

(i) $\frac{\tan 69° + \tan 66°}{1 - \tan 69° \ \tan 66°} = \tan(69° + 66°)$

$= \tan(135°) = \tan(90° + 45°) = -\cot 45° = -1$

(ii) $\frac{\tan(A - B) + \tan B}{1 - \tan(A - B)\tan B} = \tan[(A - B) + B] = \tan A$

(iii) L.H.S. $= \frac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°}$

Divide both Numerator and Denominator by cos17°

L.H.S. $= \frac{1 + \tan 17°}{1 - \tan 17°} = \frac{\tan 45° + \tan 17°}{1 - \tan 45° \ \tan 17°}$ $(\because \tan 45° = 1)$

$= \tan(45° + 17°) = \tan 62° =$ R.H.S.

Example 6.22: Prove that (i) $\tan\left(\frac{\pi}{4} + \theta\right) \tan\left(\frac{\pi}{4} - \theta\right) = 1$

(ii) If $\tan A = 3$ and $\tan B = \frac{1}{2}$, prove that $A - B = \frac{\pi}{4}$

Solution:

(i) L.H.S. $= \tan\left(\frac{\pi}{4} + \theta\right) \tan\left(\frac{\pi}{4} - \theta\right)$

$= \left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)\left(\frac{1 - \tan\theta}{1 + \tan\theta}\right) = 1 \qquad \left(\because \tan\frac{\pi}{4} = 1\right)$

(ii) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{3 - \frac{1}{2}}{1 + 3 \cdot \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1 = \tan\frac{\pi}{4}$

$\tan(A - B) = \tan\frac{\pi}{4} \Rightarrow A - B = \frac{\pi}{4}$

Example 6.23: If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$ where $(\alpha + \beta)$ and $(\alpha - \beta)$ are acute, find $\tan 2\alpha$.

Solution:

$\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \tan(\alpha + \beta) = \frac{3}{4}$

$\sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \tan(\alpha - \beta) = \frac{5}{12}$

$2\alpha = (\alpha + \beta) + (\alpha - \beta)$

$\therefore \tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta) \cdot \tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}} = \frac{\frac{14}{12}}{\frac{11}{16}} = \frac{56}{33}$

Example 6.24: Prove that $\tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A$

Solution:

$\tan 3A = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$

i.e. $\tan 3A(1 - \tan A \tan 2A) = \tan A + \tan 2A$

i.e. $\tan 3A - \tan A \tan 2A \tan 3A = \tan A + \tan 2A$

$\therefore \tan 3A - \tan 2A - \tan A = \tan A \tan 2A \tan 3A$

EXERCISE 6.4

(1) Find the values of (i) $\sin 15°$ (ii) $\cos 75°$ (iii) $\tan 75°$ (iv) $\sin 105°$

(2) Prove that

(i) $\sin(45°+A) = \frac{1}{\sqrt{2}}(\sin A+\cos A)$ (ii) $\cos(A+45°) = \frac{1}{\sqrt{2}}(\cos A-\sin A)$

(3) Prove that

(i) $\sin(45° + A) - \cos(45° + A) = \sqrt{2}\ \sin A$

(ii) $\sin(30° + A) + \sin(30° - A) = \cos A$

(4) Prove that (i) $\cos(A + B)\ \cos(A - B) = \cos^2 B - \sin^2 A$

(ii) $\sin(A + B)\sin(A - B) = \cos^2 B - \cos^2 A$

(5) Prove that $\cos^2 15° + \cos^2 45° + \cos^2 75° = \frac{3}{2}$

(6) Prove that (i) $\sin A + \sin(120° + A) + \sin(240° + A) = 0$

(ii) $\cos A + \cos(120° + A) + \cos(120° - A) = 0$

(7) Show that

(i) $\cos 15° - \sin 15° = \frac{1}{\sqrt{2}}$ (ii) $\tan 15° + \cot 15° = 4$ (iii) $\cot 75° + \tan 75° = 4$

(8) (i) Find $\sin 45° + \sin 30°$ and compare with $\sin 75°$

(ii) Find $\cos 45° - \cos 30°$ and compare with $\cos 15°$.

(9) Show that

(i) $\tan 70° = 2\tan 50° + \tan 20°$

(ii) $\tan 72° = \tan 18° + 2\tan 54°$ (Hint : $\tan A \tan B = 1$ if $A + B = 90°$)

(iii) $\frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \tan 56°$ (iv) $\frac{\cos 29° + \sin 29°}{\cos 29° - \sin 29°} = \tan 74°$

(10) Prove that $\frac{\sin(A - B)}{\sin A\ \sin B} + \frac{\sin(B - C)}{\sin B \sin C} + \frac{\sin(C - A)}{\sin C \sin A} = 0$

(11) (i) If $\tan A = \frac{5}{6}$, $\tan B = \frac{1}{11}$ show that $A + B = 45°$

(ii) If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, show that $\alpha + \beta = \frac{\pi}{4}$

(12) If $A + B = 45°$, show that $(\cot A - 1)(\cot B - 1) = 2$ and deduce the value of $\cot 22\frac{1}{2}^{\circ}$

(13) If $A + B + C = \pi$, prove that

(i) $\tan A + \tan B + \tan C = \tan A\ \tan B \tan C$

(ii) $\tan 2A + \tan 2B + \tan 2C = \tan 2A\ \tan 2B\ \tan 2C$

(14) If $\sin A = \frac{1}{3}$, $\sin B = \frac{1}{4}$ find $\sin(A + B)$, where A and B are acute.

(15) Prove that (i) $\sin(A + 60°) + \sin(A - 60°) = \sin A$

(ii) $\tan 4A\ \tan 3A\ \tan A + \tan 3A + \tan A - \tan 4A = 0$

6.3.2 Multiple angle identities:

Identities involving sin2A, cos2A, tan3A etc. are called multiple angle identities. To develop these identities we shall use sum identities from the preceding lesson.

We first develop an identity for sin2A.

Consider $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and put $B = A$

$$\sin 2A = \sin(A + A) = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$

Thus we have the identity $\boxed{\sin 2A = 2\sin A \,.\, \cos A}$

Identities involving cos2A and tan2A can be derived in much the same way as the identity above

$$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$

Thus we have the identity $\boxed{\cos 2A = \cos^2 A - \sin^2 A}$

Similarly we can derive $\boxed{\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}}$

The other useful identities for cos2A can easily be derived as follows:

$$\cos 2A = \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$$
$$= 1 - 2\sin^2 A$$
$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$$
$$= 2\cos^2 A - 1$$

From $\cos 2A = 1 - 2\sin^2 A$, also we have

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

Also, $\cos 2A = 2\cos^2 A - 1$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

Hence $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$

$$\sin 2A = 2\sin A \cos A$$
$$= \frac{2 \sin A}{\cos A} \cos^2 A = \frac{2 \tan A}{\sec^2 A} = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A \left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A\,(1 - \tan^2 A)$$

$$= \frac{1 - \tan^2 A}{\sec^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Thus we have $\sin 2A = 2 \sin A \,.\, \cos A$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

6.3.3: Trigonometrical ratios of A in terms of trigonometrical ratios of $\frac{A}{2}$

$$\sin A = \sin\left(2 \times \frac{A}{2}\right)$$

$$= 2 \sin \frac{A}{2} \,.\, \cos \frac{A}{2}$$

$$\cos A = \cos\left(2 \times \frac{A}{2}\right) = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 2\cos^2 \frac{A}{2} - 1$$

$$= 1 - 2\sin^2 \frac{A}{2}$$

$$\tan A = \tan\left(2 \times \frac{A}{2}\right)$$

$$= \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

Similarly, we can prove the following identities

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$$

$$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A}$$

Also note that $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$ and $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$

Example 6.25: If $\sin\theta = \frac{3}{8}$ and θ is acute, find $\sin 2\theta$?

Solution: $\sin\theta = \frac{3}{8}$; $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{64}} = \frac{\sqrt{55}}{8}$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{3}{8} \cdot \frac{\sqrt{55}}{8} = \frac{3\sqrt{55}}{32}$$

Example 6.26: Find (i) $\sin 15°$ (ii) $\tan 15°$

Solution : (i) $\sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

(ii) $\tan 15° = \tan\frac{30°}{2} = \frac{1-\cos 30°}{\sin 30°} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$

6.3.4 Trigonometrical ratios involving 3A

$$\begin{aligned}\sin 3A &= \sin(2A + A) = \sin 2A \cdot \cos A + \cos 2A \cdot \sin A\\ &= 2\sin A\cos^2 A + (1-2\sin^2 A)\sin A\\ &= 2\sin A(1-\sin^2 A) + (1-2\sin^2 A)\sin A\end{aligned}$$

$$= 3\sin A - 4\sin^3 A$$

Similarly, $\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \,.\, \tan A}$$

$$= \frac{\left(\dfrac{2\tan A}{1-\tan^2 A}\right) + \tan A}{1 - \tan A \,.\, \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Example 6.27: Prove that $\cos^4 A - \sin^4 A = \cos 2A$

Solution:

$$\text{L.H.S.} = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= 1 \,.\, \cos 2A = \cos 2A = \text{R.H.S.}$$

Example 6.28:

Show that $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

Solution:

$$\text{R.H.S.} = \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1} = \frac{\dfrac{1}{\tan^3 A} - \dfrac{3}{\tan A}}{\dfrac{3}{\tan^2 A} - 1} = \frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A}$$

$$= \frac{1}{\tan 3A} = \cot 3A = \text{L.H.S.}$$

Example 6.29:

If $\tan A = \dfrac{1-\cos B}{\sin B}$, prove that $\tan 2A = \tan B$, where A and B are acute angles.

Solution : $\text{R.H.S} = \dfrac{1-\cos B}{\sin B} = \dfrac{2\sin^2 \frac{B}{2}}{2\sin\frac{B}{2} \,.\, \cos\frac{B}{2}} = \dfrac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \tan\dfrac{B}{2}$

$$\therefore \tan\frac{B}{2} = \tan A$$

$$\Rightarrow A = \frac{B}{2} \Rightarrow B = 2A$$

Therefore $\tan 2A = \tan B$

Example 6.30: Show that $4 \sin A \sin(60° + A) . \sin(60° – A) = \sin 3A$

Solution: L.H.S. $= 4 \sin A \sin (60° + A) . \sin (60° – A)$

$= 4\sin A \{ \sin (60° + A) . \sin (60° – A)\}$

$= 4\sin A \{\sin^2 60 – \sin^2 A\}$

$= 4\sin A \left\{\frac{3}{4} – \sin^2 A\right\} = 3\sin A – 4\sin^3 A = \sin 3A$

$=$ R.H.S.

Example 6.31: Prove that $\cos 20° \ \cos 40° \ \cos 80° = \frac{1}{8}$

L.H.S. $= \cos 20° \ \cos 40° \ \cos 80°$

$= \cos 20° \ \{\cos (60° – 20°) \ \cos(60° + 20°)\}$

$= \cos 20° \ [\cos^2 60° – \sin^2 20°]$

$= \cos 20° \left[\frac{1}{4} – \sin^2 20°\right]$

$= \frac{1}{4} \cos 20° \ \{1 – 4(1 – \cos^2 20°)\}$

$= \frac{1}{4} \{4\cos^3 20° – 3 \cos 20°\} = \frac{1}{4} [\cos 3 \times 20°]$

$= \frac{1}{4} \times \cos 60° = \frac{1}{8}$ = R.H.S.

Example 6.32: Find the values of:

(i) $\sin 18°$ (ii) $\cos 18°$ (iii) $\cos 36°$ (iv) $\sin 36°$ (v) $\sin 54°$ (vi) $\cos 54°$

Solution:

(i) Let $\theta = 18°$ then $5\theta = 90° \Rightarrow 2\theta = 90° – 3\theta$

$\Rightarrow \sin 2\theta = \sin(90° – 3\theta) = \cos 3\theta$

$\Rightarrow 2\sin\theta \cos\theta = 4\cos^3\theta – 3\cos\theta$

$\Rightarrow 2\sin\theta = 4\cos^2\theta – 3 \quad (\because \cos\theta \neq 0)$

$\Rightarrow 2\sin\theta = 1 – 4\sin^2\theta$

$\Rightarrow 4\sin^2\theta + 2\sin\theta – 1 = 0$

$$\Rightarrow \quad \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

since sin 18° is positive, $\sin 18° = \frac{-1+\sqrt{5}}{4}$

(ii) $\cos 18° = \sqrt{1-\sin^2 18} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

(iii) $\cos 36° = 1 - 2\sin^2 18° = \frac{\sqrt{5}+1}{4}$

(iv) $\sin 36° = \sqrt{1-\cos^2 36°} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

(v) $\sin 54° = \sin(90° - 36°) = \cos 36° = \frac{\sqrt{5}+1}{4}$

(vi) $\cos 54° = \cos(90° - 36°) = \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}$

EXERCISE 6.5

(1) Prove the following:

(i) $2\sin 15° \cos 15° = \frac{1}{2}$ (ii) $\sin\frac{\pi}{8} \cos\frac{\pi}{8} = \frac{1}{2\sqrt{2}}$

(iii) $\sin 72° = \frac{\sqrt{10+2\sqrt{5}}}{4}$ (iv) $\cos 72° = \frac{\sqrt{5}-1}{4}$

(v) $1 - 2\sin^2 22\frac{1}{2}° = \frac{1}{\sqrt{2}}$ (vi) $\frac{2\tan 22\frac{1}{2}°}{1-\tan^2 22\frac{1}{2}°} = 1$

(2) Show that $8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} = 1$

(3) If $\tan\frac{\theta}{2} = \left(2-\sqrt{3}\right)$ find the value of $\sin\theta$

(4) Prove that $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$

(5) Prove that

(i) $\cos^2\left(\frac{\pi}{4}-\theta\right) - \sin^2\left(\frac{\pi}{4}-\theta\right) = \sin 2\theta$ (ii) $\sec 2\theta + \tan 2\theta = \tan\left(\frac{\pi}{4}+\theta\right)$

(6) (i) If $\tan\theta = 3$ find $\tan 3\theta$ (ii) If $\sin A = \frac{3}{5}$ find $\sin 3A$

(7) If $\tan\alpha = \frac{1}{3}$ and $\tan\beta = \frac{1}{7}$ show that $2\alpha + \beta = \frac{\pi}{4}$

(8) If $2\cos\theta = x + \frac{1}{x}$ then prove that $\cos 2\theta = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$

6.3.5 Transformation of a product into a sum or difference

We know that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad \dots (1)$$

and

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad \dots (2)$$

Adding (1) and (2), we get

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \qquad \dots (I)$$

Subtracting (2) from (1)

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B \qquad \dots (II)$$

Again

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad \dots (3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \qquad \dots (4)$$

$(3) + (4) \Rightarrow \quad \cos(A + B) + \cos(A - B) = 2 \cos A \cos B \qquad \dots (III)$

$(4) - (3) \quad \cos(A + B) - \cos(A - B) = -2\sin A \sin B \qquad \dots (IV)$

Now, let $A + B = C$ and $A - B = D$ then

$2A = C + D$ (OR) $A = \frac{C + D}{2}$ and $2B = C - D$ (OR) $B = \frac{C - D}{2}$

Putting these values of A and B in the above four formulae I, II, III and IV, we get

1) $\sin C + \sin D = 2 \sin \frac{C + D}{2} . \cos \frac{C - D}{2}$

2) $\sin C - \sin D = 2 \cos \frac{C + D}{2} . \sin \frac{C - D}{2}$

3) $\cos C + \cos D = 2 \cos \frac{C + D}{2} . \cos \frac{C - D}{2}$

4) $\cos D - \cos C = 2\sin \frac{C + D}{2} . \sin \frac{C - D}{2}$

Example 6.33: Express as sum or difference of following expressions.

(i) $2\sin 2\theta . \cos\theta$ (ii) $2 \cos 2\theta \cos\theta$ (iii) $2 \sin 3A . \sin A$

(iv) $\cos 7\theta.\cos 5\theta$ (v) $\cos \frac{3A}{2} .\cos \frac{5A}{2}$ (vi) $\cos 3\theta.\sin 2\theta$ (vii) $2\cos 3A . \sin 5A$

Solution:

(i) $2\sin 2\theta \,.\, \cos\theta = \sin(2\theta + \theta) + \sin(2\theta - \theta) = \sin 3\theta + \sin\theta$

(ii) $2\cos 2\theta \,.\, \cos\theta = \cos(2\theta + \theta) + \cos(2\theta - \theta) = \cos 3\theta + \cos\theta$

(iii) $2\sin 3A \,.\, \sin A = \cos(3A - A) - \cos(3A + A) = \cos 2A - \cos 4A$

(iv) $\cos 7\theta \,.\, \cos 5\theta = \frac{1}{2}\,[\cos(7\theta + 5\theta) + \cos(7\theta - 5\theta)] = \frac{1}{2}\,[\cos 12\theta + \cos 2\theta]$

(v) $$\cos\frac{3A}{2}\,.\,\cos\frac{5A}{2} = \frac{1}{2}\left[\cos\left(\frac{3A}{2} + \frac{5A}{2}\right) + \cos\left(\frac{3A}{2} - \frac{5A}{2}\right)\right]$$
$$= \frac{1}{2}\,[\cos 4A + \cos(-A)] = \frac{1}{2}\,[\cos 4A + \cos A]$$

(vi) $\cos 3\theta \,.\, \sin 2\theta = \frac{1}{2}\,[\sin(3\theta + 2\theta) - \sin(3\theta - 2\theta)] = \frac{1}{2}\,[\sin 5\theta - \sin\theta]$

(vii) $$2\cos 3A \,.\, \sin 5A = \sin(3A + 5A) - \sin(3A - 5A) = \sin 8A - \sin(-2A)$$
$$= \sin 8A + \sin 2A$$

Example 6.34: Express the following in the form of a product:

(i) sin4A + sin2A (ii) sin5A – sin3A (iii) cos3A + cos7A

(iv) cos2A – cos4A (v) cos60° – cos20° (vi) cos55° + sin55°

Solution:

(i) $$\sin 4A + \sin 2A = 2\sin\left(\frac{4A + 2A}{2}\right)\cos\left(\frac{4A - 2A}{2}\right) = 2\sin 3A\ \cos A$$

(ii) $$\sin 5A - \sin 3A = 2\cos\left(\frac{5A + 3A}{2}\right)\sin\left(\frac{5A - 3A}{2}\right) = 2\cos 4A\ \sin A$$

(iii) $$\cos 3A + \cos 7A = 2\cos\left(\frac{3A + 7A}{2}\right)\cos\left(\frac{3A - 7A}{2}\right)$$
$$= 2\cos 5A\ \cos(-2A) = 2\cos 5A\cos 2A$$

(iv) $$\cos 2A - \cos 4A = -2\sin\left(\frac{2A + 4A}{2}\right)\sin\left(\frac{2A - 4A}{2}\right)$$
$$= -2\sin 3A\ \sin(-A) = 2\sin 3A\ \sin A$$

(v) $$\cos 60^\circ - \cos 20^\circ = -2\sin\frac{(60^\circ + 20^\circ)}{2}\sin\left(\frac{60^\circ - 20^\circ}{2}\right) = -2\sin 40^\circ\sin 20^\circ$$

(vi) $$\cos 55^\circ + \sin 55^\circ = \cos 55^\circ + \cos(90^\circ - 55^\circ) = \cos 55^\circ + \cos 35^\circ$$
$$= 2\cos\frac{55^\circ + 35^\circ}{2}\cos\frac{55^\circ - 35^\circ}{2} = 2\cos 45^\circ\ \cos 10^\circ$$
$$= 2\frac{1}{\sqrt{2}}\cos 10^\circ = \sqrt{2}\ \cos 10^\circ$$

Example 6.35: Show that $\sin 20° \sin 40° \ \sin 80° = \dfrac{\sqrt{3}}{8}$

Solution:

$$\text{L.H.S.} = \sin 20 \ \sin 40° \ \sin 80° = \sin 20° \ \frac{1}{2} \ \{\cos 40° - \cos 120°\}$$

$$= \frac{1}{2} \ \sin 20° \left\{\cos 40° + \frac{1}{2}\right\}$$

$$= \frac{1}{2} \ \sin 20° \cos 40° + \frac{1}{4} \ \sin 20°$$

$$= \frac{1}{4} \ (\sin 60° - \sin 20°) + \frac{1}{4} \ \sin 20° = \frac{1}{4} \ \sin 60°$$

$$= \frac{\sqrt{3}}{8} \ = \text{R.H.S.}$$

Example 6.36: Prove that $4(\cos 6° + \sin 24°) = \sqrt{3} \ + \sqrt{15}$

Solution:

$$4(\cos 6° + \sin 24°) = 4\ (\sin 84° + \sin 24°) \quad [\because \cos 6° = \cos(90°–84) = \sin 84°]$$

$$= 4 \ . \ 2\sin\left(\frac{84° + 24°}{2}\right) \ \cos\left(\frac{84° - 24°}{2}\right)$$

$$= 8 \ \sin 54° \ . \ \cos 30° = 8\left(\frac{\sqrt{5} + 1}{4}\right) . \left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{15} \ + \sqrt{3}$$

Example 6.37:

Prove that (i) $\cos 20° + \cos 100° + \cos 140° = 0$ (ii) $\sin 50° - \sin 70° + \sin 10° = 0$

Solution:

(i) $\quad \text{L.H.S.} = \cos 20° + (\cos 100° + \cos 140°)$

$$= \cos 20° + 2\cos\left(\frac{100 + 140°}{2}\right) \ . \cos\left(\frac{100 - 140°}{2}\right)$$

$$= \cos 20° + 2\cos 120° \cos(-20°) = \cos 20° + 2\left(-\frac{1}{2}\right) \ \cos 20°$$

$$= \cos 20° - \cos 20° = 0 = \text{R.H.S.}$$

(ii) $\quad \text{L.H.S.} = \sin 50° - \sin 70° + \sin 10°$

$$= 2 \ \cos\left(\frac{50 + 70°}{2}\right) \ . \sin\left(\frac{50 - 70°}{2}\right) \ + \sin 10°$$

$$= 2\cos 60° \,.\, \sin(-10°) + \sin 10° = 2 \times \frac{1}{2}\,(-\sin 10°) + \sin 10°$$

$$= -\sin 10° + \sin 10° = 0 = \text{R.H.S.}$$

6.3.6 Conditional Identities

Example 6.38:

If $A + B + C = \pi$, prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A\,\sin B\,\sin C$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \sin 2A + \sin 2B + \sin 2C = (\sin 2A + \sin 2B) + \sin 2C \\
&= 2\sin(A + B)\cos(A - B) + \sin 2C \\
&= 2\sin(\pi - C)\,\cos(A - B) + \sin 2C \\
&= 2\sin C\,\cos(A - B) + 2\sin C\,\cos C \\
&= 2\sin C\,\{\cos(A - B) + \cos C\} \\
&= 2\sin C\,\left\{\cos(A - B) + \cos(180 - \overline{A + B})\right\} \\
&= 2\sin C\,\{\cos(A - B) - \cos(A + B)\} = 2\sin C\,\{2\sin A\,\sin B\} \\
&= 4\sin A\sin B\sin C = \text{R.H.S.}
\end{aligned}$$

Example 6.39:

If $A + B + C = 180°$ Prove that $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \cos 2A + (\cos 2B - \cos 2C) \\
&= 1 - 2\sin^2 A + \{-2\sin(B + C)\sin(B - C)\} \\
&= 1 - 2\sin^2 A - 2\sin(180° - A)\,\sin(B - C) \\
&= 1 - 2\sin^2 A - 2\sin A\sin(B - C) \\
&= 1 - 2\sin A\,[\sin A + \sin(B - C)] \\
&= 1 - 2\sin A\,[\sin(B + C) + \sin(B - C)],\ [\because A = 180° - (B + C)] \\
&= 1 - 2\sin A\,[2\sin B\,\cos C] \\
&= 1 - 4\sin A\sin B\cos C = \text{R.H.S.}
\end{aligned}$$

Example 6.40:

If $A+B+C = \pi$, prove that $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A\sin B\cos C$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \cos^2 A + \cos^2 B - \cos^2 C = (1 - \sin^2 A) + \cos^2 B - \cos^2 C \\
&= 1 + (\cos^2 B - \sin^2 A) - \cos^2 C \\
&= 1 + \cos(A + B)\,.\,\cos(A - B) - \cos^2 C \\
&= 1 + \cos(\pi - C)\cos(A - B) - \cos^2 C \\
&= 1 - \cos C.\,\cos(A - B) - \cos^2 C
\end{aligned}$$

$= 1 - \cos C\ [\cos(A - B) + \cos C]$

$= 1 - \cos C\ [\cos(A - B) - \cos(A + B)] = 1 - \cos C\ [2\sin A \sin B]$

$= 1 - 2\sin A \sin B \cos C = \text{R.H.S.}$

EXERCISE 6.6

(1) Express in the form of a sum or difference:

(i) $2\sin 4\theta \cos 2\theta$ (ii) $2\cos 8\theta \cos 6\theta$ (iii) $2\cos 7\theta \sin 3\theta$

(iv) $2\sin 3A \sin A$ (v) $2\cos 6A \sin 3A$ (vi) $\cos 4\theta \sin 9\theta$

(vii) $\cos \frac{3A}{2} \sin \frac{A}{2}$ (viii) $\sin \frac{7A}{2} \cos \frac{5A}{2}$ (ix) $\cos \frac{5\theta}{3} \cos \frac{4\theta}{3}$

(2) Express in the form of a product:

(i) $\sin 13A + \sin 5A$ (ii) $\sin 13A - \sin 5A$ (iii) $\cos 13A + \cos 5A$

(iv) $\cos 13A - \cos 5A$ (v) $\sin 52° - \sin 32°$ (vi) $\cos 51° + \cos 23°$

(vii) $\sin 80° - \cos 70°$ (viii) $\sin 50° + \cos 80°$ (ix) $\sin 20° + \cos 50°$

(x) $\cos 35° + \sin 72°$

(3) Prove that $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

(4) Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

(5) Prove that $\sin 50° - \sin 70° + \cos 80° = 0$

(6) Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\left(\frac{\alpha + \beta}{2}\right)$

(7) Prove that (i) $\frac{\sin 3A - \sin A}{\cos A - \cos 3A} = \cot 2A$ (ii) $\frac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} = \tan \frac{A}{2}$

(8) $A + B + C = \pi$, prove that $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$

(9) If $A + B + C = 180°$,

prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

(10) If $A+B+C = \pi$, prove that $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$

(11) If $A + B + C = 90°$, show that $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \cot A \cot B$

(12) Prove that $A + B + C = \pi$, prove that $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}$

$= 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

6.4 Trigonometrical Equations

An equation involving trigonometrical function is called a trigonometrical equation.

$\cos\theta = \frac{1}{2}$, $\tan\theta = 0$, $\cos^2\theta - 2\sin\theta = \frac{1}{2}$ are some examples for trigonometrical equations. To solve these equations we find all replacements for the variable θ that make the equations true.

A solution of a trigonometrical equation is the value of the unknown angle that satisfies the equation. A trigonometrical equation may have infinite number of solutions. The solution in which the absolute value of the angle is the least is called **principal solution.** Note that trigonometrical equations are different from trigonometrical identities. It is possible that some equations may not have solution. For example $\cos\theta = 4$ has no solution. The expression involving integer '*n*' which gives all solutions of a trigonometrical equation is called the general solution.

6.4.1 General solutions of sin θ = 0 ; cosθ = 0 ; tan θ = 0

Consider the unit circle with centre at O(0, 0)

Let a revolving line OP, starting from OX, trace $\angle XOP = \theta$ Draw PM perpendicular to OX

(1) $\sin\theta = 0$

In the right angled triangle OMP we have OP = 1 unit,

$\sin\theta = \frac{MP}{OP} \Rightarrow \sin\theta = MP$

If $\sin\theta = 0$, then MP = 0, i.e. OP coincides with OX or OX′

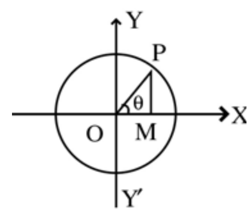


Fig. 6.20

$\therefore \angle XOP = \theta = 0, \pi, 2\pi, 3\pi, \ldots$[in the anti clockwise direction]

or $\theta = -\pi, -2\pi, -3\pi, \ldots\ldots$ [in the clockwise direction]

i.e. θ = 0 or any + *ve* or – *ve* integral multiple of π.

Hence the general solution of $\sin\theta = 0$ is given by $\theta = n\pi$, $n \in Z$,

where Z is the set of all integers.

(2) $\cos\theta = 0$

In the right angled triangle OMP we have $\cos\theta = \frac{OM}{OP} = OM$ ($\because OP = 1$ unit)

If $\cos\theta = 0$, then OM = 0

i.e. OP coincides with OY or OY′

i.e. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, 5\frac{\pi}{2}$, …. …. [in anticlockwise direction)

or $\theta = -\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}$ ……[in clockwise direction]

i.e. $\theta = \pm\left(\text{odd multiple of } \frac{\pi}{2}\right)$

Hence the general value of θ is given by $\theta = (2n + 1)\frac{\pi}{2}$, $n \in Z$

(3) $\tan\theta = 0$

In the right angled triangle OMP, if $\tan\theta = 0$ then $\frac{MP}{OM} = 0$ or MP = 0

Proceeding as in (1), we get $\theta = n\pi$, $n \in Z$

Thus, (1) If $\sin\theta = 0$, $\theta = n\pi$, $n \in Z$

(2) If $\cos\theta = 0$, $\theta = (2n + 1)\frac{\pi}{2}$, $n \in Z$

(3) If $\tan\theta = 0$, $\theta = n\pi$, $n \in Z$

When a trigonometrical equation is solved, among all solutions the solution which is in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ for sine, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for tangent and in $[0, \pi]$ for cosine, are the principal values of those functions.

Example 6.41:

Find the principal value of the following :

(i) $\cos x = \frac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\frac{\sqrt{3}}{2}$ (iii) $\operatorname{cosec}\theta = -\frac{2}{\sqrt{3}}$

(iv) $\cot\theta = -1$ (v) $\tan\theta = \sqrt{3}$

Solution: (i) $\cos x = \frac{\sqrt{3}}{2} > 0$

$\therefore$ x lies in the first or fourth quadrant. Principal value of x must be in $[0, \pi]$. Since $\cos x$ is positive the principal value is in the first quadrant $\cos x = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ and $\frac{\pi}{6} \in [0, \pi]$

$\therefore$ The principal value of x is $\frac{\pi}{6}$.

(ii) $\cos\theta = -\frac{\sqrt{3}}{2} < 0$

Since cos θ is negative, θ lies in the second or third quadrant. But the principal value must be in [0, π] i.e. in first or second quadrant. The principal value is in the second quadrant.

$\therefore \cos\theta = -\frac{\sqrt{3}}{2} = \cos(180° - 30°) = \cos 150°.$

The principal value is $\theta = 150° = \frac{5\pi}{6}$.

(iii) $\text{cosec}\theta = -\frac{2}{\sqrt{3}} \Rightarrow \sin\theta = -\frac{\sqrt{3}}{2} < 0$

∴ θ lies in the third or fourth quadrant. But principal value must be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

i.e. in first or fourth quadrant. $\therefore \ \theta = -\frac{\pi}{3}$

(iv) $\cot\theta = -1 \quad \therefore \tan\theta = -1 < 0$

∴ θ is in the second or fourth quadrant. Principal value of θ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

∴ the solution is in the fourth quadrant.

$\cot\left(-\frac{\pi}{4}\right) = -1 \Rightarrow \theta = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

6.4.2 General solutions of sin θ = sin α ; cos θ = cos α ; tan θ = tan α

(1) $\sin\theta = \sin\alpha \qquad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ i.e. $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin\theta - \sin\alpha = 0$

$\Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right) . \sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \frac{\theta+\alpha}{2} = (2n+1)\frac{\pi}{2}$, or $\frac{\theta-\alpha}{2} = n\pi, \quad n \in Z$

$\Rightarrow$ θ + α = odd multiple of π or θ − α = even multiple of π

$\Rightarrow$ θ = (odd multiple of π) − α … (1)

or θ = (even multiple of π) + α …(2)

Combining (1) and (2), we have

$$\theta = n\pi + (-1)^n.\alpha, \text{ where } n \in Z$$

(2) $\cos\theta = \cos\alpha$ $\quad 0 \le \alpha \le \pi$ i.e. $\alpha \in [0, \pi]$

$$\Rightarrow \cos\theta - \cos\alpha = 0$$

$$\Rightarrow -2\sin\left(\frac{\theta+\alpha}{2}\right) . \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = n\pi ; n\in Z \text{ or } \frac{\theta-\alpha}{2} = n\pi ; n \in Z$$

$$\Rightarrow \theta = 2n\pi - \alpha \text{ or } \theta = 2n\pi + \alpha$$

Hence $\theta = 2n\,\pi \pm \alpha$ ‘ $n \in Z$.

(3) $\tan\theta = \tan\alpha$ $\quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ i.e. $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\Rightarrow \sin(\theta - \alpha) = 0$$

$$\Rightarrow \theta - \alpha = n\pi, n \in Z$$

$$\Rightarrow \theta = n\pi + \alpha, n \in Z$$

Thus, we have $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha ; n \in Z$

$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha ; n \in Z$

$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha ; n \in Z$

Example 6.42: Find the general solution of the following :

(i) $\sin\theta = \frac{1}{2}$ (ii) $\sec\theta = -\sqrt{2}$ (iii) $\cos^2\theta = \frac{1}{4}$ (iv) $\cot^2\theta = 3$ (v) $\sec^2\theta = \frac{4}{3}$

Solution: (i) $\sin\theta = \frac{1}{2}$

$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$ which is of the form $\sin\theta = \sin\alpha$ where $\alpha = \frac{\pi}{6}$

$\therefore$ The general solution is $\theta = n\pi + (-1)^n . \frac{\pi}{6} ; n \in Z$

(ii) $\sec\theta = -\sqrt{2} \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} < 0$

Principal value of θ lies in $[0, \pi]$

As $\cos\theta$ is negative, the principal value of θ lies in second quadrant.

$$\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\therefore\ \theta = 2n\pi \pm \frac{3\pi}{4}\ ;\ n \in Z$$

(iii) We know that $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\left(\frac{1}{4}\right) - 1 = \frac{1}{2} - 1 = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\frac{2\pi}{3}$$

$$\therefore\ 2\theta = 2n\pi \pm \frac{2\pi}{3}\ ;\ n \in Z$$

$$\theta = n\pi \pm \frac{\pi}{3}\ ;\ n \in Z$$

(iv) We know that $1 + \cot^2\theta = \operatorname{cosec}^2\theta \Rightarrow 1 + 3 = \operatorname{cosec}^2\theta$

$$\therefore\ \operatorname{cosec}^2\theta = 4 \quad \text{or } \sin^2\theta = \frac{1}{4}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{1}{4}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\therefore\ 2\theta = 2n\pi \pm \frac{\pi}{3}\ ;\ n \in Z$$

$$\theta = n\pi \pm \frac{\pi}{6}\ ;\ n \in Z$$

(v) We know that $\tan^2\theta = \sec^2\theta - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2}$$

$$\cos 2\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$2\theta = 2n\pi \pm \frac{\pi}{3}\ ;\quad n \in Z$$

$$\theta = n\pi \pm \frac{\pi}{6}\ ;\quad n \in Z$$

Note : Solve : $\sin\theta = \frac{-\sqrt{3}}{2}$

There are two solutions in $0 \le \theta < 2\pi$ i.e. $\theta = -\frac{\pi}{3}$ and $\frac{4\pi}{3}$

The general solution is

$$\theta = n\pi + (-1)^n \left(-\frac{\pi}{3}\right) \; ; \quad n \in Z \qquad \ldots (1)$$

Even if we take $$\theta = n\,\pi + (-1)^n \left(\frac{4\pi}{3}\right) \; ; n \in Z \qquad \ldots (2)$$

The solution will be the same although these two structures are different.

Here the solution sets of (1) and (2) are same. But the order in which they occur are different.

For example Put $n = 1$ in (1), we get, $\theta = \frac{4\pi}{3}$

Put $n = 0$ in (2), we get, $\theta = \frac{4\pi}{3}$

It is a convention to take that value of θ whose absolute value is least as α (principal value) to define the general solution.

Example 6.43: Solve : $2\cos^2\theta + 3\sin\theta = 0$

Solution:

$$\begin{aligned}
2\cos^2\theta + 3\sin\theta = 0 \quad & \Rightarrow \quad 2\,(1 - \sin^2\theta) + 3\,\sin\theta = 0 \\
& \Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0 \\
& \Rightarrow \quad (2\,\sin\theta + 1)\,(\sin\theta - 2) = 0 \\
& \Rightarrow \quad \sin\theta = \frac{-1}{2} \qquad (\because \sin\theta = 2 \text{ is not possible}) \\
& \Rightarrow \quad \sin\theta = -\sin\frac{\pi}{6} \\
& \Rightarrow \quad \sin\theta = \sin\left(-\frac{\pi}{6}\right) \\
& \Rightarrow \quad \theta = -\frac{\pi}{6} \\
& \Rightarrow \quad \theta = n\pi + (-1)^n \cdot \left(-\frac{\pi}{6}\right) \; ; \; n \in Z
\end{aligned}$$

Example 6.44: Solve : $2\tan\theta - \cot\theta = -1$

Solution:

$$2\,\tan\theta - \cot\theta = -1$$

$$2\,\tan\theta - \frac{1}{\tan\theta} = -1$$

$$\Rightarrow 2\,\tan^2\theta + \tan\theta - 1 = 0$$

$$(2\,\tan\theta - 1)(\tan\theta + 1) = 0$$

$$2\tan\theta - 1 = 0 \quad \text{or } \tan\theta + 1 = 0$$

$$\tan\theta = \frac{1}{2} \quad \text{or } \tan\theta = -1$$

When $\tan\theta = -1 = -\tan\frac{\pi}{4}$

$$\tan\theta = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \theta = n\pi + \left(-\frac{\pi}{4}\right)$$

$$= n\pi - \frac{\pi}{4} \;;\; n \in Z$$

When $\tan\theta = \frac{1}{2} = \tan\beta$ (say)

$$\therefore \theta = n\pi + \beta$$

$$= n\pi + \tan^{-1}\left(\frac{1}{2}\right)$$

Hence $\theta = n\pi - \frac{\pi}{4}$ or $\theta = n\pi + \tan^{-1}\left(\frac{1}{2}\right)$; $n \in Z$

Example 6.45: Solve : $\sin 2x + \sin 6x + \sin 4x = 0$

Solution:

$\sin 2x + \sin 6x + \sin 4x = 0$ or $(\sin 6x + \sin 2x) + \sin 4x = 0$

or $2\sin 4x \,.\, \cos 2x + \sin 4x = 0$

$$\sin 4x\,(2\cos 2x + 1) = 0$$

when $\sin 4x = 0 \Rightarrow 4x = n\pi$ or $x = \frac{n\pi}{4}$; $n \in Z$

When $2\cos 2x + 1 = 0 \Rightarrow \cos 2x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$$\therefore 2x = 2n\pi \pm \frac{2\pi}{3} \quad \text{or} \quad x = n\pi \pm \frac{\pi}{3}$$

Hence $x = \frac{n\pi}{4}$ or $x = n\pi \pm \frac{\pi}{3}$; $n \in Z$

Example 6.46: Solve : $2\sin^2 x + \sin^2 2x = 2$

Solution:

$$2\sin^2 x + \sin^2 2x = 2$$

$$\therefore \sin^2 2x = 2 - 2\sin^2 x$$

$$= 2(1 - \sin^2 x)$$

$$\sin^2 2x = 2\cos^2 x$$

$\Rightarrow \quad 4\sin^2 x \cos^2 x - 2\cos^2 x = 0$

$\Rightarrow \quad 2(1 - \cos^2 x)\cos^2 x - \cos^2 x = 0$

$\Rightarrow \quad 2\cos^4 x - \cos^2 x = 0$

$\Rightarrow \quad \cos^2 x\,(2\cos^2 x - 1) = 0$

$\Rightarrow \quad \cos^2 x = 0$	$\cos^2 x = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$
$\Rightarrow \quad \cos^2 x = \cos^2 \frac{\pi}{2}$	$\cos^2 x = \cos^2 \frac{\pi}{4}$
$\Rightarrow \quad x = n\pi \pm \frac{\pi}{2},\ n \in Z$	$x = m\pi \pm \frac{\pi}{4},\ m \in Z$

Example 6.47: Solve : $\tan^2\theta + \left(1 - \sqrt{3}\right)\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan^2\theta + \tan\theta - \sqrt{3}\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan\theta\,(\tan\theta + 1) - \sqrt{3}\,(\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

$\Rightarrow \quad \tan\theta = -1$	$\tan\theta = \sqrt{3}$
$\Rightarrow \quad \tan\theta = \tan\left(-\frac{\pi}{4}\right)$	$\tan\theta = \tan\frac{\pi}{3}$
$\Rightarrow \theta = n\pi - \frac{\pi}{4},\quad n \in Z$	$m\pi + \frac{\pi}{3},\quad m \in Z$

6.4.3 Solving equation of the form $a\cos\theta + b\sin\theta = c$. where $c^2 \le a^2 + b^2$

$$a\cos\theta + b\sin\theta = c \qquad \ldots (1)$$

Divide each term by $\sqrt{a^2 + b^2}$,

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Choose $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$; $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$

$\therefore$ (1) becomes $\cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos\beta$

$\Rightarrow \quad \cos(\theta - \alpha) = \cos\beta$

$\Rightarrow \quad \theta - \alpha = 2n\pi \pm \beta$

$\Rightarrow \quad \theta = 2n\pi + \alpha \pm \beta,\ n \in Z$

Example 6.48: Solve : $\sqrt{3}\sin x + \cos x = 2$

Solution: This is of the form a cos$x$ + b sin$x$ = c, where $c^2 \le a^2 + b^2$

So dividing the equation by $\sqrt{(\sqrt{3})^2 + 1^2}$ or 2

We get $\frac{\sqrt{3}}{2}$ sin x + $\frac{1}{2}$ cos$x$ = 1 $\Rightarrow$ sin $\frac{\pi}{3}$. sin$x$ + cos $\frac{\pi}{3}$. cos x = 1

$$\text{i.e.} \quad \cos\left(x - \frac{\pi}{3}\right) = 1$$

$$\cos\left(x - \frac{\pi}{3}\right) = \cos 0$$

$$x - \frac{\pi}{3} = 2n\pi \pm 0$$

$$\text{i.e.} \quad x = 2n\pi + \frac{\pi}{3}, \quad n \in Z$$

EXERCISE 6.7

(1) Find the principal value of the following equations:

(i) $\sin\theta = \frac{1}{\sqrt{2}}$ (ii) $2\cos\theta - 1 = 0$ (iii) $\sqrt{3}\cot\theta = 1$

(iv) $\sqrt{3}\sec\theta = 2$ (v) $\sin x = -\frac{\sqrt{3}}{2}$ (vi) $\tan\theta = -\frac{1}{\sqrt{3}}$

(vii) $\sec x = 2$

(2) Find the general solution of the following equation:

(i) $\sin 2\theta = \frac{1}{2}$ (ii) $\tan\theta = -\sqrt{3}$ (iii) $\cos 3\theta = \frac{-1}{\sqrt{2}}$

(3) Solve the following :

(i) $\sin 3x = \sin x$ (ii) $\sin 4x + \sin 2x = 0$ (iii) $\tan 2x = \tan x$

(4) Solve the following:

(i) $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$ (ii) $\cos^2 x + \sin^2 x + \cos x = 0$

(iii) $\cos x + \cos 2x + \cos 3x = 0$ (iv) $\sin 2x + \sin 4x = 2\sin 3x$

(5) Solve the following:

(i) $\sin\theta + \cos\theta = \sqrt{2}$ (ii) $\sin\theta - \cos\theta = -\sqrt{2}$

(iii) $\sqrt{2}\sec\theta + \tan\theta = 1$ (iv) $\mathrm{cosec}\theta - \cot\theta = \sqrt{3}$

6.5 Properties of Triangles

Consider a triangle ABC.

It has three angles A, B and C.

The sides opposite to the angles A, B, C are denoted by the corresponding small letters a, b, c respectively.

Thus a = BC, b = CA, c = AB.

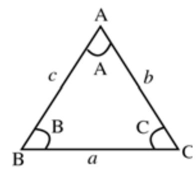


Fig. 6.21

We can establish number of formulae connecting these three angles and sides.

I. Sine formula:

In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$. Where R is the radius of the circum circle of the triangle ABC.

In fig(6.22) O is the circumcentre of the triangle ABC. R is the radius of the circumcircle. Draw OD perpendicular to BC. Now BC = a, BD = $\dfrac{a}{2}$

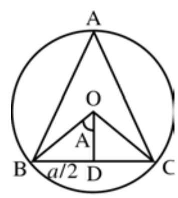


Fig. 6.22

Clearly ΔBOC is an isosceles triangle.

We know that $\angle BOC = 2\angle BAC = 2A$

$\therefore \angle BOD = A$

From the right angled triangle BOD,

$$\sin A = \frac{BD}{R} = \frac{a/2}{R} = \frac{a}{2R}$$

$$\therefore 2R\sin A = a \quad \text{or} \quad \frac{a}{\sin A} = 2R$$

Similarly, we can prove $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

II. Napier's formulae

In any triangle ABC

(1) $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

(2) $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(3) $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$ are true

These are called Napier's formulae

Result (1): $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

Proof: From sine formulae

$$\frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}$$

$$= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}$$

$$= \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}\cot\frac{C}{2}$$

$$= \cot\left(\frac{A+B}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \cot\left(90-\frac{C}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \tan\frac{C}{2}\tan\frac{A-B}{2}\cot\frac{C}{2} = \tan\frac{A-B}{2}$$

$$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

Similarly, we can prove other two results (2) and (3)

III. Cosine formulae

In any triangle ABC, the following results are true with usual notation.

Results:

(1) $a^2 = b^2 + c^2 - 2bc\cos A$ (2) $b^2 = c^2 + a^2 - 2ca\cos B$

(3) $c^2 = a^2 + b^2 - 2ab\cos C$

These are called cosine formulae

Result (1): $a^2 = b^2 + c^2 - 2bc\cos A$

Proof:

Draw CD perpendicular to AB.

Now $a^2 = BC^2 = CD^2 + BD^2$

$= (AC^2 - AD^2) + (AB - AD)^2$

$= AC^2 - AD^2 + AB^2 + AD^2 - 2AB \times AD$

$= AC^2 + AB^2 - 2AB \times (AC\cos A)$

$a^2 = b^2 + c^2 - 2bc\cos A$

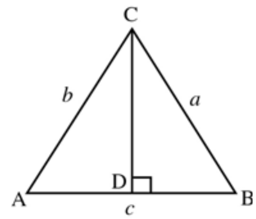


Fig. 6.23

Similarly we can prove the other results (2) and (3)

We can rewrite the formulae in different formats.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \; ; \cos B = \frac{c^2 + a^2 - b^2}{2ca} \; ; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

IV. Projection formulae

In any triangle ABC

(1) $a = b \cos C + c \cos B$ (2) $b = c \cos A + a \cos C$ (3) $c = a \cos B + b \cos A$

are true with usual notations and these are called projection formulae.

Result (1): $a = b \cos C + c \cos B$

Proof:

In triangle ABC, draw AD perpendicular to BC.

From the right angled triangles ABD and ADC,

$$\cos B = \frac{BD}{AB} \Rightarrow BD = AB \times \cos B$$

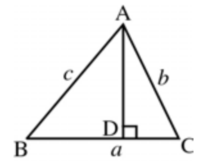


Fig. 6.24

$$\cos C = \frac{DC}{AC} \Rightarrow DC = AC \times \cos C$$

But $\quad BC = BD + DC = AB \cos B + AC \cos C$

$\quad a = c \cos B + b \cos C$

or $\quad a = b \cos C + c \cos B$

Similarly, we can prove the other formulae (2) and (3)

V. Sub-multiple (half) angle formulae

In any triangle ABC, the following results are true.

(1) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (2) $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(3) $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ (4) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(5) $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (6) $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(7) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (8) $\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(9) $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

where $s = \frac{a+b+c}{2}$

The above results are called sub-multiple angles (or half angle) formulae.

Result (1): $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Proof : We know that $\cos 2A = 1 - 2\sin^2 A$

$$2\sin^2 A = 1 - \cos 2A$$

Replacing A by $\frac{A}{2}$, $2\sin^2\frac{A}{2} = 1 - \cos A$

$$= 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

$$= \frac{(a+b+c-2c)(a+b+c-2b)}{2bc}$$

$$= \frac{(2s-2c)(2s-2b)}{2bc} \qquad \because a+b+c = 2s$$

$$2\sin^2\frac{A}{2} = \frac{2(s-c)\,2(s-b)}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{bc}}$$

Since $\frac{A}{2}$ is acute, $\sin\frac{A}{2}$ is always positive.

Thus $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Similarly we can prove the other two sine related formulae (2) and (3)

Result (4): $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Proof : We know that $\cos 2A = 2\cos^2 A - 1$

$$2\cos^2 A = 1 + \cos 2A$$

Replacing A by $\frac{A}{2}$, $2\cos^2\frac{A}{2} = 1 + \cos A$

$$= 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{(b + c + a)\ (b + c + a - 2a)}{2bc} = \frac{2s(2s - 2a)}{2bc}$$

$$2\cos^2\frac{A}{2} = \frac{2s \times 2(s - a)}{2bc}$$

$$\cos^2\frac{A}{2} = \frac{s(s - a)}{bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

Similarly we can prove other two cosine related formulae (5) and (6)

Result (7): $$\tan\frac{A}{2} = \sqrt{\frac{(s - b)\ (s - c)}{s(s - a)}}$$

Proof: $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{\frac{(s - b)\ (s - c)}{bc}}}{\sqrt{\frac{s(s - a)}{bc}}}$$

$$= \sqrt{\frac{(s - b)\ (s - c)}{s(s - a)}}$$

Similarly we can prove other two tangent related formulae (8) and (9)

VI. Area formulae (Δ denotes area of a triangle)

In any triangle ABC

(1) $\Delta = \frac{1}{2} ab \sin C$ (2) $\Delta = \frac{1}{2} bc \sin A$ (3) $\Delta = \frac{1}{2} ca \sin B$

(4) $\Delta = \frac{abc}{4R}$ (5) $\Delta = 2R^2 \sin A \sin B \sin C$ (6) $\Delta = \sqrt{s(s - a)\ (s - b)\ (s - c)}$

are true with the usual notations and these are called Area formulae.

Result (1): $\Delta = \frac{1}{2} ab \sin C$

Proof :

Draw AD perpendicular to BC

Δ = Area of triangle ABC

$= \frac{1}{2} \times BC \times AD = \frac{1}{2} \times BC \times AC \times \sin C$

$= \frac{1}{2} ab \sin C \quad [\because \sin C = \frac{AD}{AC} \Rightarrow AD = AC \times \sin C]$

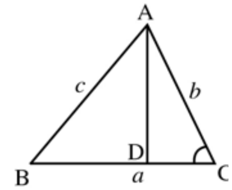


Fig. 6.25

Similarly we can prove the results (2) and (3)

Result (4): $\Delta = \dfrac{abc}{4R}$

Proof:

We know that $\Delta = \dfrac{1}{2}\ ab \sin C$

$= \dfrac{1}{2}\ ab\, \dfrac{c}{2R} \qquad \because \dfrac{c}{\sin C} = 2R$

$= \dfrac{abc}{4R}$

Result (5): $\Delta = 2R^2 \sin A \sin B \sin C$

Proof:

We know that $\Delta = \dfrac{1}{2}\ ab \sin C$

$= \dfrac{1}{2}\ 2R \sin A\ 2R \sin B \sin C \qquad \because a = 2R \sin A$

$= 2R^2 \sin A \sin B \sin C \qquad b = 2R \sin B$

Result (6) Prove that $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Proof:

We know that $\Delta = \dfrac{1}{2}\ ab \sin C$

$= \dfrac{1}{2}\ ab\ 2\sin\dfrac{C}{2}\ \cos\dfrac{C}{2}$

$= ab\sqrt{\dfrac{(s-a)(s-b)}{ab}}\ \sqrt{\dfrac{s(s-c)}{ab}}$

$= \sqrt{s(s-a)(s-b)(s-c)}$

Example 6.49: In a triangle ABC prove that $a \sin A - b \sin B = c \sin(A - B)$

Solution:

By sine formulae we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$\therefore\ a = 2R \sin A,\quad b = 2R \sin B,\quad c = 2R \sin C$

$$\begin{aligned} a \sin A - b \sin B &= 2R\sin A \sin A - 2R \sin B \sin B \\ &= 2R(\sin^2 A - \sin^2 B) \\ &= 2R \sin(A + B) \sin(A - B) \\ &= 2R \sin(180 - C) \sin(A - B) \end{aligned}$$

$$= 2R \sin C \sin(A - B)$$
$$= c \sin(A - B)$$

Example 6.50: Prove that $\dfrac{\sin(A - B)}{\sin(A + B)} = \dfrac{a^2 - b^2}{c^2}$

Solution:

By sine formula $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$$\frac{a^2 - b^2}{c^2} = \frac{(2R \sin A)^2 - (2R \sin B)^2}{(2R \sin C)^2}$$
$$= \frac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 C} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$
$$= \frac{\sin(A + B) \sin(A - B)}{\sin^2 C} \qquad [\sin C = \sin(A + B)]$$
$$= \frac{\sin(A + B) \sin(A - B)}{\sin^2 (A + B)} = \frac{\sin(A - B)}{\sin(A + B)}$$

Example 6.51: Prove that $\sum a \sin(B - C) = 0$

Solution:

$$\begin{aligned}
\sum a \sin(B - C) &= a \sin(B - C) + b \sin(C - A) + c \sin(A - B) \\
&= 2R \sin A \sin(B - C) + 2R \sin B \sin(C - A) + 2R \sin C \sin(A - B) \\
\sin A &= \sin(B + C),\ \sin B = \sin(C + A)\ ;\ \sin C = \sin(A + B) \\
&= 2R \sin(B + C) \sin(B - C) + 2R \sin(C + A) \sin(C - A) \\
&\quad + 2R \sin(A + B) \sin(A - B) \\
&= 2R [\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B] \\
&= 0
\end{aligned}$$

Example 6.52: Prove that $\cos \dfrac{B - C}{2} = \dfrac{b + c}{a} \sin \dfrac{A}{2}$

Solution: $\dfrac{b + c}{a} \sin \dfrac{A}{2} = \dfrac{2R \sin B + 2R \sin C}{2R \sin A} \sin \dfrac{A}{2}$

$$= \frac{\sin B + \sin C}{\sin A} \sin \frac{A}{2}$$
$$= \frac{2 \sin \frac{B + C}{2} \cos \frac{B - C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \sin \frac{A}{2}$$

$$= \frac{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\sin\left(\frac{180-A}{2}\right)\cos\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\sin\left(90-\frac{A}{2}\right)\cos\frac{B-C}{2}}{\cos\frac{A}{2}}$$

$$= \cos\frac{B-C}{2} \qquad \because \sin\left(90-\frac{A}{2}\right) = \cos\frac{A}{2}$$

Example 6.53: In any triangle ABC prove that

$$\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$$

Solution:

$$\frac{a^2 \sin(B-C)}{\sin A} = \frac{(2R \sin A)^2 \sin(B-C)}{\sin A} = \frac{4R^2 \sin^2 A \sin(B-C)}{\sin A}$$

$$= 4R^2 \sin A \sin(B-C) = 4R^2 \sin(B+C)\sin(B-C)$$

$$= 4R^2 (\sin^2 B - \sin^2 C) = 4R^2 \sin^2 B - 4R^2 \sin^2 C$$

$$= b^2 - c^2$$

Similarly $\frac{b^2 \sin(C-A)}{\sin B} = c^2 - a^2$

$$\frac{c^2 \sin(A-B)}{\sin C} = a^2 - b^2$$

$$\therefore \frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C}$$

$$= b^2 - c^2 + c^2 - a^2 + a^2 - b^2$$

$$= 0$$

EXERCISE 6.8

In any triangle ABC prove that

(1) $a^2 = (b+c)^2 \sin^2\frac{A}{2} + (b-c)^2 \cos^2\frac{A}{2}$

(2) $\sum a(b^2 + c^2) \cos A = 3abc$

(3) $\sum a(\sin B - \sin C) = 0$

(4) $\sum (b + c) \cos A = a + b + c$

(5) $a^3 \sin(B - C) + b^3 \sin(C - A) + c^3 \sin(A - B) = 0$

(6) $a(b \cos C - c \cos B) = b^2 - c^2$

(7) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$

(8) $\dfrac{\tan A}{\tan B} = \dfrac{c^2 + a^2 - b^2}{b^2 + c^2 - a^2}$

(9) If $a \cos A = b \cos B$ then show that the triangle is either an isosceles triangle or right angled triangle?

6.6. Solution of triangles

We know that a triangle has six parts (or elements). Consider a triangle ABC. With usual symbols, the sides a, b, c and the angles A, B, C are parts of the triangle ABC.

The process of finding the unknown parts of a triangle is called the solution of triangle. If three parts of a triangle (atleast one of which is a side) are given then the other parts can be found. Here, we shall discuss the following three types.

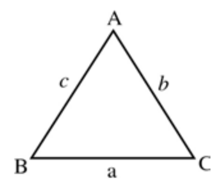


Fig. 6.26

1) Any three sides (SSS) are given.

2) Any one side and two angles (SAA) are given.

3) Any two sides and the included angle (SAS) are given.

Type I: Given three sides (SSS)

To solve this type, we can use any one of the following formulae.

(a) Cosine formula (b) Sine formula (c) Half angle formula .

It is better to use cosine formula if the sides are small, while use half angle formula if the sides are large.

Example 6.54: Given $a = 8$, $b = 9$, $c = 10$, find all the angles.

Solution: To find A, use the formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{81 + 100 - 64}{180} = \frac{117}{180}$$

$$A = 49° 28'$$

Similarly

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{100 + 64 - 81}{160} = \frac{83}{160}$$

$$B = 58° \, 51'$$

But $\quad A + B + C = 180°$

$$\therefore C = 180° - (49° \, 28' + 58° \, 51')$$

$$= 71° \, 41'$$

Thus $\quad A = 49° \, 28', B = 58° \, 51', C = 71° \, 41'$

Note: In the above example the numbers are smaller and hence we used cosine formula.

Example 6.55: Given $a = 31$, $b = 42$, $c = 57$, find all the angles.

Solution: Since the sides are larger quantities, use half angle formulae

$$s = \frac{a + b + c}{2} = 65$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}} = \left(\frac{23 \times 8}{65 \times 34}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \log\left[\tan\frac{A}{2}\right] = \frac{1}{2}\,[\log 23 + \log 8 - \log 65 - \log 34]$$

$$= \frac{1}{2}\,[1.3617 + 0.9031 - 1.8129 - 1.5315]$$

$$= \frac{1}{2}\,[-1.0796] = \frac{1}{2}\,[-2 + 0.9204]$$

$$= \frac{1}{2}\left[\bar{2} + 0.9204\right] = \bar{1}.4602$$

$$\Rightarrow \quad \frac{A}{2} = 16° \, 6' \Rightarrow A = 32° \, 12'$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{s(s-b)}} = \left(\frac{8 \times 34}{65 \times 23}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \log\left[\tan\frac{B}{2}\right] = \frac{1}{2}\,[\log 8 + \log 34 - \log 65 - \log 23]$$

$$= \frac{1}{2}\,[-0.7400] = \frac{1}{2}\,[-2 + 1.2600]$$

$$= \frac{1}{2}\left[\bar{2} + 1.2600\right] = \bar{1}.6300$$

$$\Rightarrow \qquad \frac{B}{2} = 23° 6' \Rightarrow B = 46° 12'$$

$$C = 180 - (A + B) = 101° 36'$$

Thus $\quad A = 32° 12' \quad B = 46° 12' \quad C = 101° 36'$

Type II: Given one side and any two angles (SAA)

To solve this type, draw a sketch of the triangle roughly, for better understanding and use sine formula.

Example 6.56: In a triangle ABC, A = 35° 17′, C = 45° 13′, b = 42.1. Solve the triangle.

Solution:

The unknown parts are B, a, c

B = 180 – (A + C) = 180 – (35° 17′ + 45° 13′)

= 99° 30′

To find the sides, use sine formula

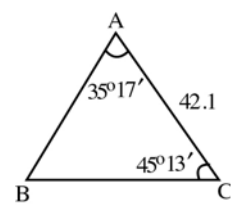


Fig. 5.27

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \qquad a = \frac{b \sin A}{\sin B} = \frac{42.1 \times \sin 35° 17'}{\sin 99° 30'}$$

$$\log a = \log 42.1 + \log \sin 35° 17' - \log \sin 99° 30'$$

$$= 1.6243 + \bar{1}.7616 - \bar{1}.9940$$

$$= 1.3859 - \bar{1}.9940$$

$$= 1.3859 - [-1 + 0.9940] = 1.3919$$

$$\Rightarrow \qquad a = 24.65$$

$$\text{Again} \qquad c = \frac{b \sin C}{\sin B} = \frac{42.1 \times \sin 45° 13'}{\sin 99' 30'}$$

$$\log c = \log 42.1 + \log \sin 45° 13' - \log \sin 99° 30'$$

$$= 1.6243 + \bar{1}.8511 - \bar{1}.9940$$

$$= 1.4754 - \bar{1}.9940$$

$$= 1.4754 - [-1 + 0.9940] = 1.4814$$

$$\Rightarrow \qquad c = 30.3$$

Thus $\quad B = 99° 30', \quad a = 24.65, \quad c = 30.3$

Type III: Given two sides and the included angle (SAS)

Since two sides and the included angle are given, the third side can be found by using the proper cosine formula. Then one can apply the sine formula to calculate the other elements.

Example 6.57: Solve the triangle ABC if $a = 5$, $b = 4$ and $C = 68°$.

Solution: To find c, use $c^2 = a^2 + b^2 - 2ab \cos C$

$$\begin{aligned} c^2 &= 25 + 16 - 2 \times 5 \times 4 \cos 68° \\ &= 41 - 40 \times 0.3746 = 26.016 \\ c &= 5.1 \end{aligned}$$

To find the other two angles, use sine formula.

$$\begin{aligned} \Rightarrow \quad \sin B &= \frac{b \sin C}{c} = \frac{4 \times \sin 68°}{5.1} \\ \log \sin B &= \log 4 + \log \sin 68° - \log 5.1 \\ &= 0.6021 + \overline{1}.9672 - .7075 \\ &= 0.5693 - 0.7075 = -0.1382 \\ &= \overline{1}.8618 \\ B &= 46°\ 40' \\ \Rightarrow \quad A &= 180 - (B + C) = 180 - (114°\ 40') \\ &= 65°\ 20' \end{aligned}$$

Thus $\quad B = 46°\ 40'$, $A = 65°\ 20'$, $c = 5.1$

Note: To find the angles A and B one can also use the tangent formula

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$$

6.7 Inverse Trigonometrical functions (Inverse circular functions)

The quantities $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, … are called inverse circular functions. **$\sin^{-1}x$ is an angle θ, whose sine is x.** Similarly $\cos^{-1}x$ denotes an angle whose cosine is x and so on. The principal value of an inverse function is that value of the general value which is numerically the least. It may be positive or negative. When there are two values, one is positive and the other is negative such that they are numerically equal, then the principal value is the positive one.

For example the principal values of $\cos^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{3}$ and not $-\frac{\pi}{3}$ though $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

Note : $\sin^{-1}x$ is different from $(\sin x)^{-1}$. $\sin^{-1}$ in $\sin^{-1}x$ denotes the inverse of the circular function. But $(\sin x)^{-1}$ is the reciprocal of $\sin x$ i.e. $\frac{1}{\sin x}$.

The Domain and Range of Inverse Trigonometrical functions are given below:

	Function	Domain	Range (Principal Value)
1.	$y = \sin^{-1}x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
2.	$y = \cos^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
3.	$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4.	$y = \text{cosec}^{-1}x$	$x \geq 1$ or $x \leq -1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$
5.	$y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$0 < y \leq \pi$; $y \neq \frac{\pi}{2}$
6.	$y = \cot^{-1} x$	R	$0 < y < \pi$

Table 6.6

Example 6.58: Find the principal values of:

(i) $\text{Sin}^{-1}\left(\frac{1}{2}\right)$ (ii) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (iv) $\sin^{-1}(-1)$

(v) $\cos^{-1}\left(-\frac{1}{2}\right)$ (vi) $\text{cosec}^{-1}(-2)$

Solution:

(i) Let $\sin^{-1}\left(\frac{1}{2}\right) = y$, where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$.

Then $\sin^{-1}\left(\frac{1}{2}\right) = y \Rightarrow \sin y = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$

$\therefore$ The principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$

(ii) Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$, where $0 < y < \frac{\pi}{2}$, then,

$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y \Rightarrow \sec y = \frac{2}{\sqrt{3}} = \sec\frac{\pi}{6} \Rightarrow y = \frac{\pi}{6}$

$\therefore$ The principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Then $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = y \Rightarrow \tan y = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right) \Rightarrow y = \frac{-\pi}{6}$

$\therefore$ The principal values of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $\frac{-\pi}{6}$

(iv) Let $\sin^{-1}(-1) = y$, where $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

Then, $\sin^{-1}(-1) = y \Rightarrow \sin y = -1$

$-1 = \sin\left(-\frac{\pi}{2}\right) \Rightarrow y = -\frac{\pi}{2}$

$\therefore$ The principal value of $\sin^{-1}(-1)$ is $-\frac{\pi}{2}$

(v) Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$, where $0 \le y \le \pi$, then

$\cos^{-1}\left(-\frac{1}{2}\right) = y \Rightarrow \cos y = -\frac{1}{2}$

$\cos y = -\cos\frac{\pi}{3} \Rightarrow \cos y = \cos\left(\pi - \frac{\pi}{3}\right) \Rightarrow y = \left(\frac{2\pi}{3}\right)$

$\therefore$ The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$

(vi) Let $\text{cosec}^{-1}(-2) = y$, where $-\frac{\pi}{2} \le y < 0$

$\text{cosec}^{-1}(-2) = y \Rightarrow \text{cosec}\, y = -2 = \text{cosec}\left(\frac{-\pi}{6}\right) \Rightarrow y = \frac{-\pi}{6}$

$\therefore$ The principal value of $\text{cosec}^{-1}(-2)$ is $\frac{-\pi}{6}$

Example 6.59:

(i) If $\cot^{-1}\left(\frac{1}{7}\right) = \theta$, find the value of $\cos\theta$ (ii) If $\sin^{-1}\left(\frac{1}{2}\right) = \tan^{-1}x$, find the value of x

Solution:

(i) $\cot^{-1}\left(\frac{1}{7}\right) = \theta \Rightarrow \cot\theta = \frac{1}{7} \therefore \tan\theta = 7$

$\Rightarrow \sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + 49}$

$$\sec\theta = 5\sqrt{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{5\sqrt{2}}$$

(ii) $\tan^{-1}x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \quad \therefore \tan^{-1}x = \frac{\pi}{6}$

$$\Rightarrow \quad x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \Rightarrow x = \frac{1}{\sqrt{3}}$$

Properties of principal inverse Trigonometric functions:

Property (1):

(i) $\sin^{-1}(\sin x) = x$ (ii) $\cos^{-1}(\cos x) = x$ (iii) $\tan^{-1}(\tan x) = x$

(iv) $\cot^{-1}(\cot x) = x$ (v) $\sec^{-1}(\sec x) = x$ (vi) $\mathrm{cosec}^{-1}(\mathrm{cosec}\, x) = x$

Proof:

(i) Let $\sin x = y$, then $x = \sin^{-1}(y)$ … (1)

$\therefore x = \sin^{-1}(\sin x)$ by (1)

Similarly, the other results may be proved.

Property (2):

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x$ (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x$ (iv) $\mathrm{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x$

(v) $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x$ (vi) $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}x$

Proof:

(i) Let $\sin^{-1}\left(\frac{1}{x}\right) = \theta \Rightarrow \sin\theta = \frac{1}{x}$

$$\Rightarrow \mathrm{cosec}\theta = x$$

$$\Rightarrow \theta = \mathrm{cosec}^{-1}(x)$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x$$

Similarly the other results can be proved.

Property (3):

(i) $\sin^{-1}(-x) = -\sin^{-1}x$ (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x$ (iv) $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$ (vi) $\cot^{-1}(-x) = -\cot^{-1}x$

Proof:

(i) Let $\sin^{-1}(-x) = \theta \quad \therefore -x = \sin\theta$

$\Rightarrow \quad x = -\sin\theta$

$x = \sin(-\theta)$

$\Rightarrow \quad -\theta = \sin^{-1} x$

$\Rightarrow \quad \theta = -\sin^{-1}x$

$\Rightarrow \quad \sin^{-1}(-x) = -\sin^{-1}x$

(ii) Let $\cos^{-1}(-x) = \theta \quad \Rightarrow \quad -x = \cos\theta$

$\Rightarrow \quad x = -\cos\theta = \cos(\pi - \theta)$

$\Rightarrow \quad \pi - \theta = \cos^{-1}x$

$\Rightarrow \quad \theta = \pi - \cos^{-1}x$

$\Rightarrow \quad \cos^{-1}(-x) = \pi - \cos^{-1}x$

Similarly the other results may be proved.

Property (4):

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ (iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$

Proof:

(i) Let $\sin^{-1}x = \theta \quad \Rightarrow \quad x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \theta$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

$\Rightarrow \quad \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Similarly (ii) and (iii) can be proved.

Property (5):

If $xy < 1$, then $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

Proof: Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$ then $\tan\theta_1 = x$ and $\tan\theta_2 = y$

$$\Rightarrow \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \,.\, \tan\theta_2} = \frac{x + y}{1 - xy}$$

$$\Rightarrow \theta_1 + \theta_2 = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Note: Similarly, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Property (6): $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

Proof: Let $\theta_1 = \sin^{-1}x$ and $\theta_2 = \sin^{-1}y$ then $\sin\theta_1 = x$ and $\sin\theta_2 = y$

$$\Rightarrow \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

$$= \left(\sin\theta_1\sqrt{1-\sin^2\theta_2} + \sqrt{1-\sin^2\theta_1}\sin\theta_2\right)$$

$$= \left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\Rightarrow \theta_1 + \theta_2 = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

Example 6.60:

Prove that (i) $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\frac{2}{9}$ (ii) $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$

Solution:

(i) $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\cdot\frac{1}{13}}\right) = \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

(ii) Let $\cos^{-1}\frac{4}{5} = \theta$ then $\cos\theta = \frac{4}{5}$ $\therefore$ $\tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4}$

$\cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$

$$\therefore \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{3}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)}\right\} = \tan^{-1}\frac{27}{11}$$

Example 6.61: Show that $\tan^{-1} + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-yz-zx-xy}\right)$

Solution: $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y}{1-xy}\right] + \tan^{-1}z$

$$= \tan^{-1}\frac{\frac{x+y}{1-xy}+z}{1-\frac{(x+y)z}{1-xy}} = \tan^{-1}\left[\frac{\frac{x+y+z-xyz}{1-xy}}{\frac{1-xy-xz-yz}{1-xy}}\right]$$

$$= \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$$

Example 6.62: Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$$\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4} \Rightarrow \tan^{-1}\left[\frac{2x+3x}{1-6x^2}\right] = \tan^{-1}(1)$$

$$\therefore \frac{5x}{1-6x^2} = 1 \Rightarrow 1-6x^2 = 5x \;\therefore\; 6x^2+5x-1 = 0$$

$$\text{i.e.} \quad (x+1)(6x-1) = 0 \Rightarrow x = -1 \text{ or } \frac{1}{6}$$

The negative value of x is rejected since it makes R.H.S. negative. $\therefore x = \frac{1}{6}$

Example 6.63:

Evaluate : (i) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$ (ii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iii) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

Solution: (i) Let $\cos^{-1}\frac{3}{5} = \theta$. Then, $\cos\theta = \frac{3}{5}$

$$\therefore \sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

(ii) Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta$ then, $\tan\theta = \frac{3}{4}$

$$\therefore \cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{4}{5}$$

(iii) Let $\cos^{-1}\frac{4}{5} = \theta$; then $\cos\theta = \frac{4}{5}$

$$\sin\left[\frac{1}{2}\cos^{-1}\frac{4}{5}\right] = \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{\sqrt{10}}$$

Example 6.64: Evaluate : $\cos\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right]$

Solution: Let $\sin^{-1}\frac{3}{5} = A \;\therefore\; \sin A = \frac{3}{5} \Rightarrow \cos A = \frac{4}{5}$

Let $\sin^{-1}\frac{5}{13} = B \;\therefore\; \sin B = \frac{5}{13} \Rightarrow \cos B = \frac{12}{13}$

$$\therefore \cos\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right] = \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \left(\frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13}\right) = \frac{33}{65}$$

EXERCISE 6.9

(1) Find the principal value of

(i) $\sin^{-1}\frac{\sqrt{3}}{2}$ (ii) $\cos^{-1}\left(\frac{1}{2}\right)$ (iii) $\text{cosec}^{-1}(-1)$

(iv) $\sec^{-1}(-\sqrt{2})$ (v) $\tan^{-1}(\sqrt{3})$ (vi) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(2) Prove that (i) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\frac{3}{4}$ (ii) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1 + x^2}$

(iii) $\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

(3) Evaluate:

(i) $\cos\left(\sin^{-1}\frac{5}{13}\right)$ (ii) $\cos\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$ (iii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$ (iv) $\sin\left[\cos^{-1}\frac{1}{2}\right]$

(4) Prove the following:

(i) $\tan^{-1}\left[\sqrt{\frac{1 - \cos x}{1 + \cos x}}\right] = \frac{x}{2}$ (ii) $\cos^{-1}(2x^2 - 1) = 2\cos^{-1}x$

(iii) $\tan^{-1}\left[\frac{3x - x^3}{1 - 3x^2}\right] = 3\tan^{-1}x$ (iv) $\sin^{-1}\left(2x\sqrt{1 - x^2}\right) = 2\sin^{-1}x$

(5) Prove that $2\tan^{-1}\frac{2}{3} = \tan^{-1}\left(\frac{12}{5}\right)$

(6) Prove that $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m - n}{m + n}\right) = \frac{\pi}{4}$

(7) Solve : $\tan^{-1}\left(\frac{x - 1}{x - 2}\right) + \tan^{-1}\left(\frac{x + 1}{x + 2}\right) = \frac{\pi}{4}$

(8) Solve $\tan^{-1}\left(\frac{2x}{1 - x^2}\right) + \cot^{-1}\left(\frac{1 - x^2}{2x}\right) = \frac{\pi}{3}$, where $x > 0$

(9) Solve : $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\frac{4}{7}$

(10) Prove the following:

(i) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1 - x^2}\sqrt{1 - y^2}\right]$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1 - y^2} - y\sqrt{1 - x^2}\right]$

(iii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1 - x^2}\,.\,\sqrt{1 - y^2}\right]$

7. FUNCTIONS AND GRAPHS

7.1 Introduction:

The most prolific mathematician whoever lived, Leonhard Euler (1707–1783) was the first scientist to give the function concept the prominence in his work that it has in Mathematics today. The concept of functions is one of the most important tool in Calculus.

To define the concept of functions, we need certain pre-requisites.

Constant and variable:

A quantity, which retains the same value throughout a mathematical process, is called a constant. A variable is a quantity which can have different values in a particular mathematical process.

It is customary to represent constants by the letters a, b, c, … and variables by x, y, z.

Intervals:

The real numbers can be represented geometrically as points on a number line called the real line (fig. 7.1)

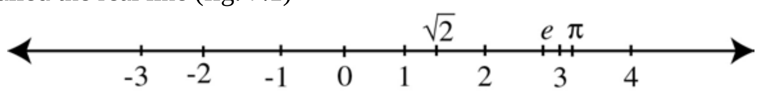


Fig 7. 1

The symbol R denotes either the real number system or the real line. A subset of the real line is called an interval if it contains atleast two numbers and contains all the real numbers lying between any two of its elements.

For example,

(a) the set of all real numbers x such that $x > 6$

(b) the set of all real numbers x such that $-2 \leq x \leq 5$

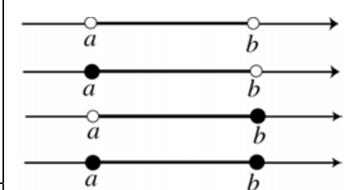
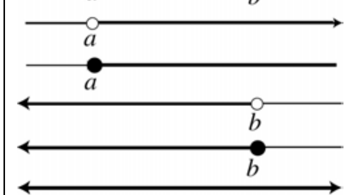
(c) the set of all real numbers x such that $x < 5$ are some intervals.

But the set of all natural numbers is not an interval. Between any two rational numbers there are infinitely many real numbers which are not included in the given set. Hence the set of natural numbers is not an interval. Similarly the set of all non zero real numbers is also not an interval. Here the real number 0 is absent. It fails to contain every real number between any two real numbers say -1 and 1.

Geometrically, intervals correspond to rays and line segments on the real line. The intervals corresponding to line segments are finite intervals and intervals corresponding to rays and the real line are infinite intervals. Here finite interval does not mean that the interval contains only a finite number of real numbers.

A finite interval is said to be closed if it contains both of its end points and open if it contains neither of its end points. To denote the closed set, the square bracket [ ] is used and the paranthesis ( ) is used to indicate open set. For example $3 \notin (3, 4)$, $3 \in [3, 4]$

Type of intervals

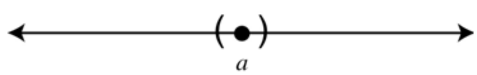
	Notation	Set	Graph
Finite	(a, b)	$\{x / a < x < b\}$	
	$[a, b)$	$\{x / a \le x < b\}$	
	$(a, b]$	$\{x / a < x \le b\}$	
	$[a, b]$	$\{x / a \le x \le b\}$	
Infinite	(a, ∞)	$\{x / x > a\}$	
	$[a, \infty)$	$\{x / x \ge a\}$	
	$(-\infty, b)$	$\{x / x < b\}$	
	$(-\infty, b]$	$\{x / x \le b\}$	
	$(-\infty, \infty)$	$\{x / -\infty < x < \infty\}$ or the set of real numbers	

Note :

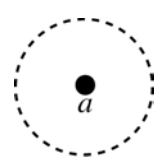
We can't write a closed interval by using ∞ or $-\infty$. These two are not representatives of real numbers.

Neighbourhood

In a number line the neighbourhood of a point (real number) is defined as an open interval of very small length.



In the plane the neighbourhood of a point is defined as an open disc with very small radius.



In the space the neighbourhood of a point is defined as an open sphere with very small radius.

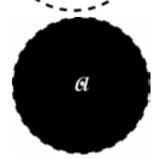


Fig 7. 2

Independent / dependent variables:

In the lower classes we have come across so many formuale. Among those, let us consider the following formulae:

(a) $V = \frac{4}{3}\pi r^3$ (volume of the sphere) (b) $A = \pi r^2$ (area of a circle)

(c) $S = 4\pi r^2$ (surface area of a sphere) (d) $V = \frac{1}{3}\pi r^2 h$ (volume of a cone)

Note that in (a), (b) and (c) for different values of r, we get different values of V, A and S. Thus the quantities V, A and S depend on the quantity r. Hence we say that V, A and S are dependent variables and r is an independent variable. In (d) the quantities r and h are independent variables while V is a dependent variable.

A variable is an independent variable when it has any arbitrary (independent) value.

A variable is said to be dependent when its value depends on other variables (independent).

"Parents pleasure depends on how their children score marks in Examination"

Cartesian product:

Let $A=\{a_1, a_2, a_3\}$, $B=\{b_1, b_2\}$. The Cartesian product of the two sets A and B is denoted by $A \times B$ and is defined as

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$

Thus the set of all ordered pairs (a, b) where $a \in A$, $b \in B$ is called the Cartesian product of the sets A and B.

It is noted that $A \times B \neq B \times A$ (in general), since the ordered pair (a, b) is different from the ordered pair (b, a). These two ordered pairs are same only if a = b.

Example 7.1: Find $A \times B$ and $B \times A$ if $A = \{1, 2\}$, $B = \{a, b\}$

Solution: $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

Relation:

In our everyday life we use the word 'relation' to connect two persons like 'is son of', 'is father of', 'is brother of', 'is sister of', etc. or to connect two objects by means of 'is shorter than', 'is bigger than', etc. When comparing (relate) the objects (human beings) the concept of relation becomes very important. In a similar fashion we connect two sets (set of objects) by means of relation.

Let A and B be any two sets. A relation from $A \to B$ (read as A to B) is a subset of the Cartesian product $A \times B$.

Example 7.2: Let $A = \{1, 2\}$, $B = \{a, b\}$. Find some relations from $A \to B$ and $B \to A$.

Solution:

Since relation from A to B is a subset of the Cartesian product

$A \times B = \{(1 , a) , (1, b) , (2 , a) , (2 , b)\}$ any subset of $A \times B$ is a relation from $A \to B$.

$\therefore \{(1 , a), (1 , b), (2 , a), (2 , b)\}$, $\{(1, a), (1, b)\}$, $\{(1, b, (2, b)\}$, $\{(1 , a)\}$ are some relations from A to B.

Similarly any subset of $B \times A = \{(a , 1), (a , 2), (b , 1), (b , 2)\}$ is a relation from B to A.

$\{(a , 1), (a , 2), (b , 1), (b , 2)\}$, $\{(a, 1), (b, 1)\}$, $\{(a, 2), (b, 1)\}$ are some relations from B to A.

7.2 Function:

A function is a special type of relation. In a function, no two ordered pairs can have the same first element and a different second element. That is, for a function, corresponding to each first element of the ordered pairs, there must be a different second element. i.e. In a function we cannot have ordered pairs of the form (a_1, b_1) and (a_2, b_2) with $a_1 = a_2$ and $b_1 \neq b_2$.

Consider the set of ordered pairs (relation) $\{(3 , 2), (5 , 7), (1 , 0), (10 , 3)\}$. Here no two ordered pairs have the same first element and different second element. It is very easy to check this concept by drawing a proper diagram *(fig. 7.3)*.

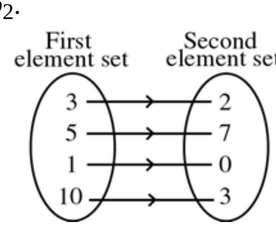


Fig 7. 3

$\therefore$ This relation is a function.

Consider another set of ordered pairs (relation) $\{(3, 5), (3, -1), (2, 9)\}$. Here the ordered pairs (3, 5) and (3, − 1) have the same first element but different second element *(fig. 7.4)*.

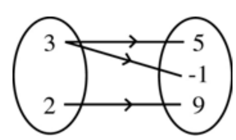


Fig 7. 4

This relation is not a function.

Thus, a function f from a set A to B is a rule (relation) that assigns a unique element $f(x)$ in B to each element x in A.

Symbolically, $f : A \to B$

i.e. $x \to f(x)$

To denote functions, we use the letters *f*, *g*, *h* etc. Thus for a function, each element of A is associated with exactly one element in B. The set A is called the **domain** of the function *f* and B is called **co-domain** of *f*. If *x* is in A, the element of B associated with *x* is

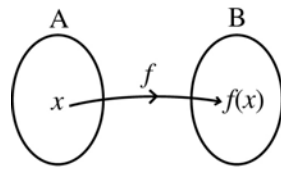


Fig 7. 5

called the **image** of *x* under *f*. i.e. $f(x)$. The set of all images of the elements of A is called the **range** of the function *f*. Note that range is a subset of the co-domain. The range of the function *f* need not be equal to the co-domain B. **Functions** are also known as **mappings**.

Example 7.3 : Let A = {1, 2, 3}, B ={3, 5, 7, 8} and *f* from A to B is defined by $f : x \to 2x + 1$ i.e. $f(x) = 2x + 1$.

(a) Find $f(1)$, $f(2)$, $f(3)$

(b) Show that *f* is a function from A to B

(c) Identify domain, co-domain, images of each element in A and range of *f*

(d) Verify that whether the range is equal to codomain

Solution:

(a) $f(x) = 2x + 1$

$f(1) = 2 + 1 = 3$, $f(2) = 4 + 1 = 5$, $f(3) = 6 + 1 = 7$

(b) The relation is {(1,3), (2, 5), (3, 7)}

Clearly each element of A has a unique image in B. Thus *f* is a function.

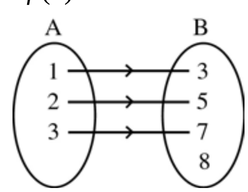


Fig 7. 6

(c) The domain set is A = {1, 2, 3}

The co-domain set is B = {3, 5, 7, 8}

Image of 1 is 3 ; 2 is 5 ; 3 is 7

The range of *f* is {3, 5, 7}

(d) $\{3, 5, 7\} \neq \{3, 5, 7, 8\}$

$\therefore$ The range is not equal to the co-domain

Example 7.4:

A father '*d*' has three sons *a*, *b*, *c*. By assuming sons as a set A and father as a singleton set B, show that

(i) the relation 'is a son of' is a function from $A \to B$ and

(ii) the relation 'is a father of' from $B \to A$ is not a function.

Solution:

(i) A = {a, b, c}, B = {d}

a is son of d

b is son of d

c is son of d

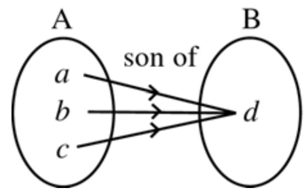


Fig 7. 7

The ordered pairs are (a, d), (b, d), (c, d). For each element in A there is a unique element in B. Clearly the relation 'is son of' from A to B is a function.

(ii) d is father of a

d is father of b

d is father of c

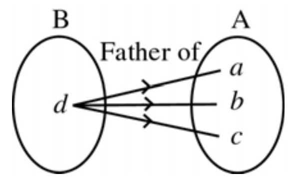


Fig 7. 8

The ordered pairs are (d, a), (d, b), (d, c). The first element d is associated with three different elements (not unique)

Clearly the relation'is father of' from B to A is not a function.

Example 7.5: A classroom consists of 7 benches. The strength of the class is 35. Capacity of each bench is 6. Show that the relation 'sitting' between the set of students and the set of benches is a function. If we interchange the sets, what will be happened?

Solution:

The domain set is the set of students and the co-domain set is the set of benches. Each student will occupy only one bench. Each student has seat also. By principle of function, ''each student occupies a single bench'. Therefore the relation 'sitting' is a function from set of Students to set of Benches.

If we interchange the sets, the set of benches becomes the domain set and the set of students becomes co-domain set. Here atleast one bench consists of more than one student. This is against the principle of function i.e. each element in the domain should have associated with only one element in the co-domain. Thus if we interchange the sets, it is not possible to define a function.

Note :

Consider the function $f : A \rightarrow B$

i.e. $x \rightarrow f(x)$ where $x \in A$, $f(x) \in B$.

Read '$f(x)$' as 'f of x'. The meaning of $f(x)$ is the value of the function f at x (which is the image of x under the function f). If we write $y = f(x)$, the symbol f represents the function name, x denotes the independent variable (argument) and y denotes the dependent variable.

Clearly, in $f(x)$, f is the name of the function and not $f(x)$. However we will often refer to the function as $f(x)$ in order to know the variable on which f depends.

Example 7.6: Identify the name of the function, the domain, co-domain, independent variable, dependent variable and range if $f : R \to R$ defined by $y = f(x) = x^2$

Solution:

Name of the function is a square function.

Domain set is R.

Co-domain set is R.

Independent variable is x.

Dependent variable is y.

x can take any real number as its value. But y can take only positive real number or zero as its value, since it is a square function.

$\therefore$ Range of f is set of non negative real numbers.

Example 7.7: Name the function and independent variable of the following function:

(i) $f(\theta) = \sin\theta$ (ii) $f(x) = \sqrt{x}$ (iii) $f(y) = e^y$ (iv) $f(t) = \log_e t$

Solution:

	Name of the function	independent variable
(i)	sine	θ
(ii)	square root	x
(iii)	exponential	y
(iv)	logarithmic	t

The domain conversion

If the domain is not stated explicitly for the function $y = f(x)$, the domain is assumed to be the largest set of x values for which the formula gives real y values. If we want to restrict the domain, we must specify the condition.

The following table illustrates the domain and range of certain functions.

Function	Domain (x)	Range (y or $f(x)$)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	R – {0} Non zero Real numbers	R – {0}
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = \sin x$	$(-\infty, \infty)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ principal domain	$[-1, 1]$
$y = \cos x$	$(-\infty, \infty)$ $[0, \pi]$ principal domain	$[-1. 1]$
$y = \tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ principal domain	$(-\infty, \infty)$
$y = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = \log_e^x$	$(0, \infty)$	$(-\infty, \infty)$

7.2.1 Graph of a function:

The graph of a function f is a graph of the equation $y = f(x)$

Example 7.8: Draw the graph of the function $f(x) = x^2$

Solution:

Draw a table of some pairs (x, y) which satisfy $y = x^2$

x	0	1	2	3	– 1	– 2	– 3
y	0	1	4	9	1	4	9

Plot the points and draw a smooth curve passing through the plotted points.

Note:

Note that if we draw a vertical line to the above graph, it meets the curve at only one point i.e. for every x there is a unique y

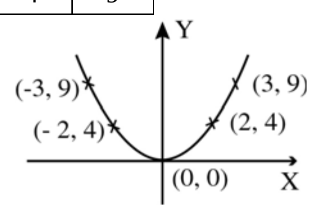


Fig 7. 9

Functions and their Graphs (Vertical line test)

Not every curve we draw is the graph of a function. A function f can have only one value $f(x)$ i.e. y for each x in its domain. Thus no vertical line can intersect the graph of a function more than once. Thus if 'a' is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$ only.

Consider the following graphs:

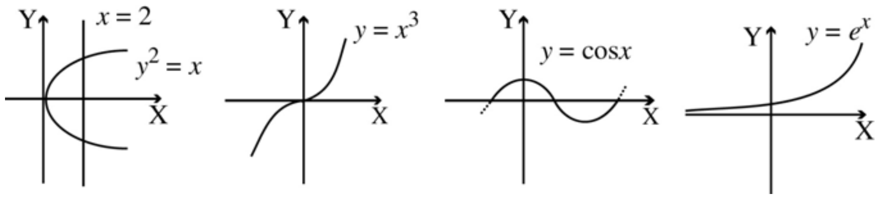


Fig 7. 10

Except the graph of $y^2 = x$, (or $y = \pm\sqrt{x}$) all other graphs are graphs of functions. But for $y^2 = x$, if we draw a vertical line $x = 2$, it meets the curve at two points $(2, \sqrt{2})$ and $(2, -\sqrt{2})$Therefore the graph of $y^2 = x$ is not a graph of a function.

Example 7.9: Show that the graph of $x^2 + y^2 = 4$ is not the graph of a function.

Solution:

Clearly the equation $x^2 + y^2 = 4$ represents a circle with radius 2 and centre at the origin.

Take
$$x = 1$$
$$y^2 = 4 - 1 = 3$$
$$y = \pm\sqrt{3}$$

For the same value $x = 1$, we have two y-values $\sqrt{3}$ and $-\sqrt{3}$. It violates the definition of a function. In the *fig* 7.11 the line $x = 1$ meets the curve at two places

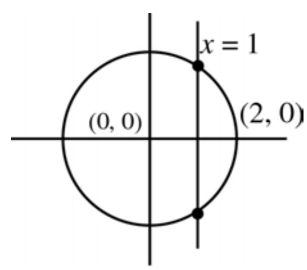


Fig 7. 11

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$. Hence, the graph of $x^2 + y^2 = 4$ is not a graph of a function.

7.2.2 Types of functions:

1. Onto function

If the range of a function is equal to the co-domain then the function is called an onto function. Otherwise it is called an into function.

In f:A$\to$B, the range of f or the image set f(A) is equal to the co-domain B i.e. f(A) = B then the function is onto.

Example 7.10

Let A = {1, 2, 3, 4}, B = {5, 6}. The function f is defined as follows:f(1) = 5, f(2) = 5, f(3) = 6, f(4) = 6. Show that f is an onto function.

Solution:

f = {(1, 5), (2, 5), (3, 6), (4, 6)}

The range of f, f(A) = {5, 6}

co-domain B = {5, 6}

i.e. f(A) = B

$\Rightarrow$ the given function is onto

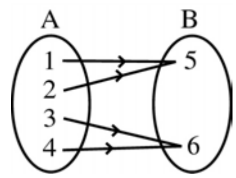


Fig 7. 12

Example 7.11: Let X = {a, b}, Y = {c, d, e} and f = {(a, c), (b, d)}. Show that f is not an onto function.

Solution:

Draw the diagram

The range of f is {c, d}

The co-domain is {c, d, e}

The range and the co-domain are not equal, and hence the given function is not onto

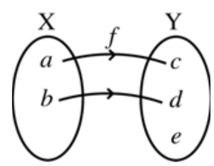


Fig 7. 13

Note :

(1) For an onto function for each element (image) in the co-domain, there must be a corresponding element or elements (pre-image) in the domain.

(2) Another name for onto function is surjective function.

Definition: A function f is onto if to each element b in the co-domain, there is atleast one element a in the domain such that $b = f(a)$

2. One-to-one function:

A function is said to be one-to-one if each element of the range is associated with exactly one element of the domain.

i.e. two different elements in the domain (A) have different images in the co-domain (B).

i.e. $a_1 \neq a_2 \quad \Rightarrow \quad f(a_1) \neq f(a_2) \quad a_1, a_2 \in$ A,

Equivalently $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

The function defined in 7.11 is one-to-one but the function defined in 7.10 is not one-to-one.

Example 7.12: Let A = {1, 2, 3}, B = {a, b, c}. Prove that the function f defined by f = {(1, a), (2, b), (3,c)} is a one-to-one function.

Solution:

Here 1, 2 and 3 are associated with a, b and c respectively.

The different elements in A have different images in B under the function f. Therefore f is one-to-one.

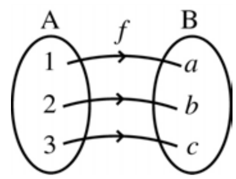


Fig 7. 14

Example 7.13: Show that the function $y = x^2$ is not one-to-one.

Solution:

For the different values of x (say 1, − 1) we have the same value of y. i.e. different elements in the domain have the same element in the co-domain. By definition of one-to-one, it is not one-to-one (OR)

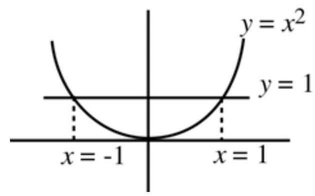


Fig 7. 15

$$y = f(x) = x^2$$
$$f(1) = 1^2 = 1$$
$$f(-1) = (-1)^2 = 1$$
$$\Rightarrow \quad f(1) = f(-1)$$

But $1 \neq -1$. Thus different objects in the domain have the same image.

$\therefore$ The function is not one-to-one.

Note: (1) A function is said to be injective if it is one-to-one.

(2) It is said to be bijective if it is both one-to-one and onto.

(3) The function given in example 7.12 is bijective while the functions given in 7.10, 7.11, 7.13 are not bijective.

Example 7.14. Show that the function f : R $\rightarrow$ R defined by $f(x) = x + 1$ is bijective.

Solution:

To prove that f is bijective, it is enough to prove that the function f is

(i) onto (ii) one-to-one

(i) Clearly the image set is R, which is same as the co-domain R. Therefore, it is onto. i.e. take $b \in$ R. Then we can find $b - 1 \in$ R such that $f(b - 1) = (b - 1) + 1 = b$. So f is onto.

(ii) Further two different elements in the domain R have different images in the co-domain R. Therefore, it is one-to-one.

i.e. $f(a_1) = f(a_2) \Rightarrow a_1 + 1 = a_2 + 1 \Rightarrow a_1 = a_2$. So f is one-to-one.

Hence the function is bijective.

3. Identity function:

A function f from a set A to the same set A is said to be an identity function if $f(x) = x$ for all $x \in$ A i.e. f : A $\rightarrow$ A is defined by $f(x) = x$ for all $x \in$ A. Identity function is denoted by I_A or simply I. Therefore $I(x) = x$ always.

Graph of identity function:

The graph of the identity function $f(x) = x$ is the graph of the function $y = x$. It is nothing but the straight line $y = x$ as shown in the fig. (7.16)

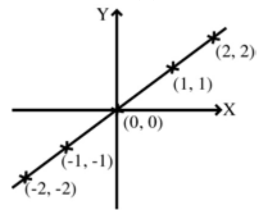


Fig 7. 16

4. Inverse of a function:

To define the inverse of a function f i.e. f^{-1} (read as 'f inverse'), the function f must be one-to-one and onto.

Let A = {1, 2, 3}, B = {a, b, c, d}. Consider a function f = {(1, a), (2, b), (3, c)}. Here the image set or the range is {a, b, c} which is not equal to the co-domain {a, b, c, d}. Therefore, it is not onto.

For the inverse function f^{-1} the co-domain of f becomes domain of f^{-1}.

i.e. If f : A $\rightarrow$ B then f^{-1} : B $\rightarrow$ A . According to the definition of domain, each element of the domain must have image in the co-domain. In f^{-1}, the element 'd' has no image in A. Therefore f^{-1} is not a function. This is because the function f is not onto.

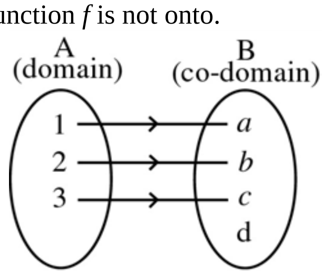


Fig 7. 17 a

$f(1) = a$
$f(2) = b$
$f(3) = c$

All the elements in A have images

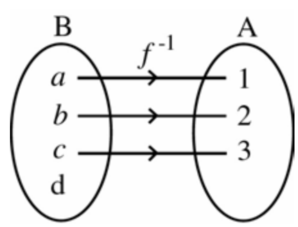


Fig 7.17 b

$f^{-1}(a) = 1$
$f^{-1}(b) = 2$
$f^{-1}(c) = 3$
$f^{-1}(d) = ?$

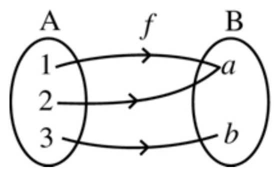
The element d has no image.

Again consider a function which is not one-to-one. i.e. consider

f = {(1, a), (2, a), (3, b)} where A = {1, 2, 3}, B = {a, b}

Here the two different elements '1' and '2' have the same image '*a*'. Therefore the function is not one-to-one.

The range = $\{a, b\}$ = B. $\therefore$ The function is onto.



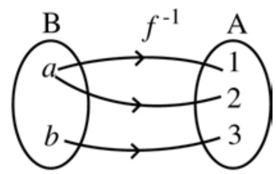


Fig 7. 18

$f(1) = a$

$f(2) = a$

$f(3) = b$

Here all the elements in A has unique image

$f^{-1}(a) = 1$

$f^{-1}(a) = 2$

$f^{-1}(b) = 3$

The element '*a*' has two images 1 and 2. It violates the principle of the function that each element has a unique image.

This is because the function is not one-to-one.

Thus, 'f^{-1} exists if and only if f is one-to-one and onto'.

Note:

(1) Since all the function are relations and inverse of a function is also a relation. We conclude that for a function which is not one-to-one and onto, the inverse f^{-1} does not exist

(2) To get the graph of the inverse function, interchange the co-ordinates and plot the points.

To define the mathematical definition of inverse of a function, we need the concept of composition of functions.

5. Composition of functions:

Let A, B and C be any three sets and let $f : A \to B$ and $g : B \to C$ be any two functions. Note that the domain of g is the co-domain of f. Define a new function $(gof) : A \to C$ such that $(gof)(a) = g(f(a))$ for all $a \in A$. Here $f(a)$ is an element of B. $\therefore$ $g(f(a))$ is meaningful. The function gof is called the composition of two functions f and g.

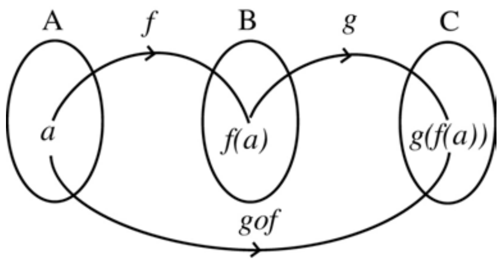


Fig 7. 19

Note:

The small circle o in *gof* denotes the composition of g and *f*

Example 7.15: Let A = {1, 2}, B = {3, 4} and C = {5, 6} and $f : A \to B$ and $g : B \to C$ such that $f(1) = 3$, $f(2) = 4$, $g(3) = 5$, $g(4) = 6$. Find *gof*.

Solution:

gof is a function from $A \to C$.

Identify the images of elements of A under the function *gof*.

$(gof)\ (1) = g(f(1)) = g(3) = 5$

$(gof)\ (2) = g(f(2)) = g(4) = 6$

i.e. image of 1 is 5 and

image of 2 is 6 under *gof*

$\therefore\ gof = \{(1, 5), (2, 6)\}$

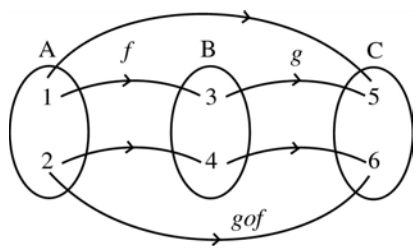


Fig 7. 20

Note:

For the above definition of *f* and *g*, we can't find *fog*. For some functions *f* and *g*, we can find both *fog* and *gof*. In certain cases *fog* and *gof* are equal. In general $fog \neq gof$ i.e. the composition of functions need not be commutative always.

Example 7.16: The two functions $f : R \to R$, $g : R \to R$ are defined by

$f(x) = x^2 + 1$, $g(x) = x - 1$. Find *fog* and *gof* and show that $fog \neq gof$.

Solution:

$$(fog)\ (x) = f(g(x)) = f(x - 1) = (x - 1)^2 + 1 = x^2 - 2x + 2$$

$$(gof)\ (x) = g(f(x)) = g(x^2 + 1) = (x^2 + 1) - 1 = x^2$$

Thus $(fog)\ (x) = x^2 - 2x + 2$

$(gof)\ (x) = x^2$

$\Rightarrow \quad fog \neq gof$

Example 7.17: Let f, g : R $\to$ R be defined by $f(x) = 2x + 1$, and $g(x) = \dfrac{x-1}{2}$.

Show that $(fog) = (gof)$.

Solution:

$$(fog)(x) = f(g(x)) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = x$$

$$(gof)(x) = g(f(x)) = g(2x+1) = \frac{(2x+1)-1}{2} = x$$

Thus $(fog)(x) = (gof)(x)$

$\Rightarrow$ $fog = gof$

In this example f and g satisfy $(fog)(x) = x$ and $(gof)(x) = x$

Consider the example 7.17. For these f and g, $(fog)(x) = x$ and $(gof)(x) = x$. Thus by the definition of identity function $fog = \mathrm{I}$ and $gof = \mathrm{I}$ i.e. $fog = gof = \mathrm{I}$

Now we can define the inverse of a function f.

Definition:

Let f : A $\to$ B be a function. If there exists a function g : B $\to$ A such that $(fog) = \mathrm{I}_B$ and $(gof) = \mathrm{I}_A$, then g is called the inverse of f. The inverse of f is denoted by f^{-1}

Note:

(1) The domain and the co-domain of both f and g are same then the above condition can be written as $fog = gof = \mathrm{I}$.

(2) If f^{-1} exists then f is said to be invertible.

(3) $f \, o \, f^{-1} = f^{-1} o \, f = \mathrm{I}$

Example 7.18: Let f : R $\to$ R be a function defined by $f(x) = 2x + 1$. Find f^{-1}

Solution:

$$\text{Let } g = f^{-1}$$

$$(gof)(x) = x \qquad \because gof = \mathrm{I}$$

$$g(f(x)) = x \quad \Rightarrow \quad g(2x+1) = x$$

$$\text{Let } 2x + 1 = y \quad \Rightarrow \quad x = \frac{y-1}{2}$$

$$\therefore \quad g(y) = \frac{y-1}{2} \quad \text{or} \quad f^{-1}(y) = \frac{y-1}{2}$$

Replace y by x

$$f^{-1}(x) = \frac{x-1}{2}$$

6. Sum, difference, product and quotient of two functions:

Just like numbers, we can add, subtract, multiply and divide the functions if both are having same domain and co-domain.

If f, g : A $\rightarrow$ B are any two functions then the following operations are true.

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\, g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0$$

$$(cf)(x) = c.f(x) \text{ where } c \text{ is a constant}$$

Note: Product of two functions is different from composition of two functions.

Example 7.19: The two functions f, g : R$\rightarrow$R are defined by $f(x)=x + 1$, $g(x)=x^2$.

Find $f + g$, $f - g$, fg, $\frac{f}{g}$, $2f$, $3g$.

Solution:

Function	Definition
f	$f(x) = x + 1$
g	$g(x) = x^2$
$f + g$	$(f + g)(x) = f(x) + g(x) = x + 1 + x^2$
$f - g$	$(f - g)(x) = f(x) - g(x) = x + 1 - x^2$
fg	$(fg)(x) = f(x)\, g(x) = (x + 1)x^2$
$\frac{f}{g}$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{x^2}$, (it is defined for $x \neq 0$)
$2f$	$(2f)(x) = 2f(x) = 2(x + 1)$
$3g$	$(3g)(x) = 3g(x) = 3x^2$

7. Constant function:

If the range of a function is a singleton set then the function is called a constant function.

i.e. f : A $\rightarrow$ B is such that $f(a) = b$ for all $a \in$ A, then f is called a constant function.

Let A = {1, 2, 3}, B = {a, b}. If the function f is defined by $f(1) = a$, $f(2) = a$, $f(3) = a$ then f is a constant function.

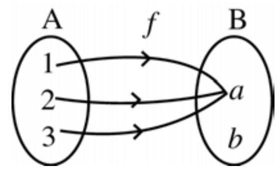


Fig 7. 21

Simply, $f : R \to R$, defined by $f(x) = k$ is a constant function and the graph of this constant function is given in fig. (7.22)

Note that 'is a son of' is a constant function between set of sons and the singleton set consisting of their father.

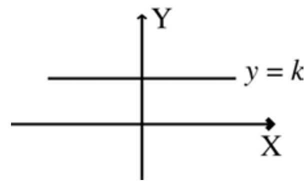


Fig 7. 22

8. Linear function:

If a function $f : R \to R$ is defined in the form $f(x) = ax + b$ then the function is called a linear function. Here a and b are constants.

Example 7.20: Draw the graph of the linear function $f : R \to R$ defined by $f(x) = 2x + 1$.

Solution:

Draw the table of some pairs $(x, f(x))$ which satisfy $f(x) = 2x + 1$.

x	0	1	-1	2
$f(x)$	1	3	-1	5

Plot the points and draw a curve passing through these points. Note that, the curve is a straight line.

Note:

(1) The graph of a linear function is a straight line.

(2) Inverse of a linear function always exists and also linear.

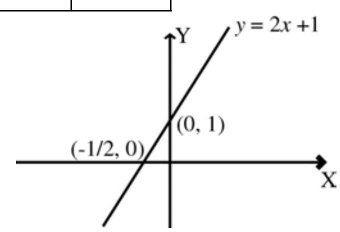


Fig 7. 23

9. Polynomial function:

If $f : R \to R$ is defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers, $a_n \neq 0$ then f is a polynomial function of degree n.

The function $f : R \to R$ defined by $f(x) = x^3 + 5x^2 + 3$ is a cubic polynomial function or a polynomial function of degree 3.

10. Rational function:

Let $p(x)$ and $q(x)$ be any two polynomial functions. Let S be a subset of R obtained after removing all values of x for which $q(x) = 0$ from R.

The function $f : \text{S} \to \text{R}$, defined by $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ is called a rational function.

Example 7.21: Find the domain of the rational function $f(x) = \dfrac{x^2 + x + 2}{x^2 - x}$.

Solution:

The domain S is obtained by removing all the points from R for which $g(x) = 0 \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, 1$

$\therefore$ S = R − {0, 1}

Thus this rational function is defined for all real numbers except 0 and 1.

11. Exponential functions:

For any number $a > 0$, $a \neq 1$, the function $f : \text{R} \to \text{R}$ defined by $f(x) = a^x$ is called an exponential function.

Note: For exponential function the range is always R^+ (the set of all positive real numbers)

Example 7.22: Draw the graphs of the exponential functions $f : \text{R} \to \text{R}^+$ defined by (1) $f(x) = 2^x$ (2) $f(x) = 3^x$ (3) $f(x) = 10^x$.

Solution:

For all these function $f(x) = 1$ when $x = 0$. Thus they cut the y axis at $y = 1$. For any real value of x, they never become zero. Hence the corresponding curves to the above functions do not meet the x-axis for real x. (or meet the x-axis at $-\infty$)

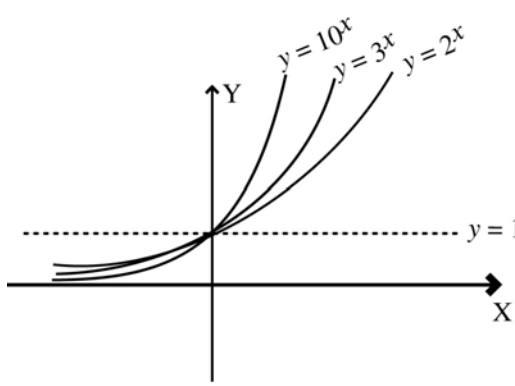


Fig 7. 24

In particular the curve corresponding to $f(x) = e^x$ lies between the curves corresponding to 2^x and 3^x, as $2 < e < 3$.

Example 7.23:

Draw the graph of the exponential function $f(x) = e^x$.

Solution:

For $x = 0$, $f(x)$ becomes 1 i.e. the curve cuts the y axis at $y = 1$. For no real value of x, $f(x)$ equals to 0. Thus it does not meet x-axis for real values of x.

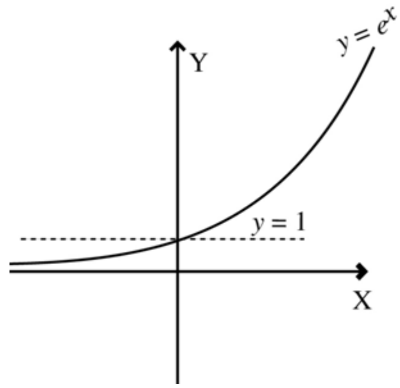


Fig 7. 25

Example 7.24:

Draw the graphs of the logarithmic functions

(1) $f(x) = \log_2 x$ (2) $f(x) = \log_e x$ (3) $f(x) = \log_3 x$

Solution:

The logarithmic function is defined only for positive real numbers. i.e. $(0, \infty)$

Domain : $(0, \infty)$

Range : $(-\infty, \infty)$

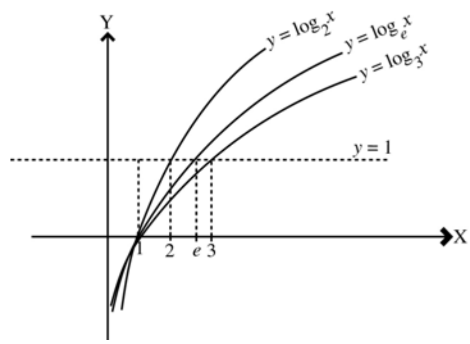


Fig 7. 26

Note:

The inverse of exponential function is a logarithmic function. The general form is $f(x) = \log_a x$, $a \neq 1$, a is any positive number. The domain $(0, \infty)$ of logarithmic function becomes the co-domain of exponential function and the co-domain $(-\infty, \infty)$ of logarithmic function becomes the domain of exponential function. This is due to inverse property.

11. Reciprocal of a function:

The function $g : S \rightarrow R$, defined by $g(x) = \dfrac{1}{f(x)}$ is called reciprocal function of $f(x)$. Since this function is defined only for those x for which $f(x) \neq 0$, we see that the domain of the reciprocal function of $f(x)$ is $R - \{x \ : \ f(x) = 0\}$.

Example 7.25: Draw the graph of the reciprocal function of the function $f(x) = x$.

Solution:

The reciprocal function of $f(x)$ is $\dfrac{1}{f(x)}$

Thus $g(x) = \dfrac{1}{f(x)} = \dfrac{1}{x}$

Here the domain of

$g(x) = R - \{$set of points x for which $f(x) = 0\}$

$= R - \{0\}$

The graph of $g(x) = \dfrac{1}{x}$ is as shown in fig 7.27.

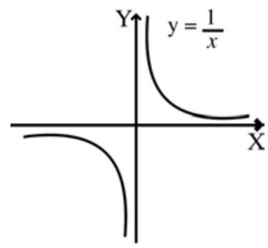


Fig 7. 27

Note:

(1) The graph of $g(x) = \dfrac{1}{x}$ does not meet either axes for finite real number. Note that the axes x and y meet the curve at infinity only. Thus x and y axes are the asymptotes of the curve $y = \dfrac{1}{x}$ or $g(x) = \dfrac{1}{x}$ [Asymptote is a tangent to a curve at infinity. Detailed study of asymptotes is included in XII Standard].

(2) Reciprocal functions are associated with product of two functions.
i.e. if f and g are reciprocals of each other then $f(x)\, g(x) = 1$.
Inverse functions are associated with composition of functions.
i.e.if f and g are inverses of each other then $fog = gof = I$

12. Absolute value function (or modulus function)

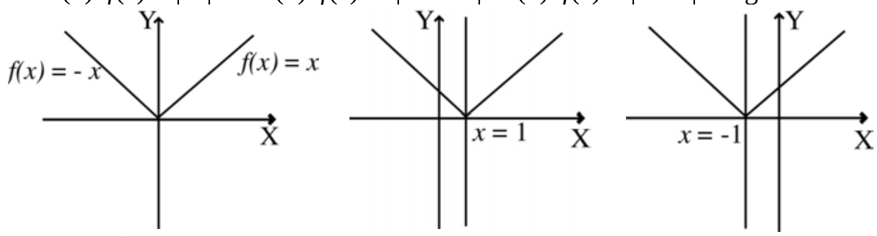
If $f : R \rightarrow R$ defined by $f(x) = |x|$ then the function is called absolute value function of x.

where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

The domain is R and co-domain is set of all non-negative real numbers.

The graphs of the absolute functions

(1) $f(x) = |x|$ (2) $f(x) = |x - 1|$ (3) $f(x) = |x + 1|$ are given below.



$f(x) = |x|$ $f(x) = |x - 1|$ $f(x) = |x + 1|$

Fig 7. 28

13. Step functions:

(a) **Greatest integer function**

The function whose value at any real number x is the greatest integer less than or equal to x is called the greatest integer function. It is denoted by $\lfloor x \rfloor$

i.e. $f : R \to R$ defined by $f(x) = \lfloor x \rfloor$

Note that $\lfloor 2.5 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$, $\lfloor -2.1 \rfloor = -3$, $\lfloor .5 \rfloor = 0$, $\lfloor -.2 \rfloor = -1$, $\lfloor 4 \rfloor = 4$

The domain of the function is R and the range of the function is Z (the set of all integers).

(b) Least integer function

The function whose value at any real number x is the smallest integer greater than or equal to x is called the least integer function and is denoted by $\lceil x \rceil$

i.e. $f : R \to R$ defined by $f(x) = \lceil x \rceil$.

Note that $\lceil 2.5 \rceil = 3$, $\lceil 1.09 \rceil = 2$, $\lceil -2.9 \rceil = -2$, $\lceil 3 \rceil = 3$

The domain of the function is R and the range of the function is Z.

Graph of $f(x) = \lfloor x \rfloor$

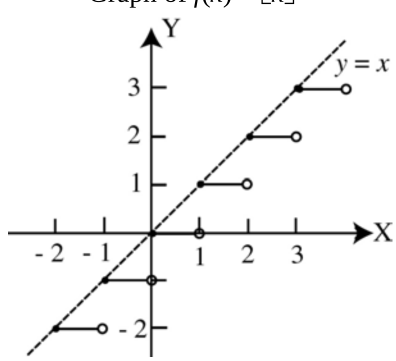


Fig 7. 29

Graph of $f(x) = \lceil x \rceil$

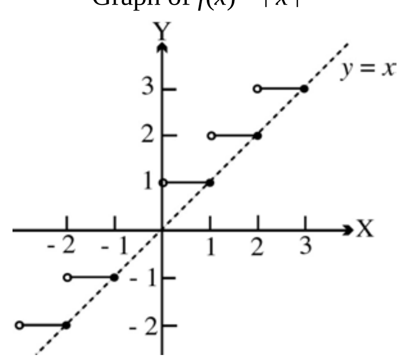


Fig 7. 30

14. Signum function:

If f:R→R is defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then f is called signum function.

The domain of the function is R and the range is {– 1, 0, 1}.

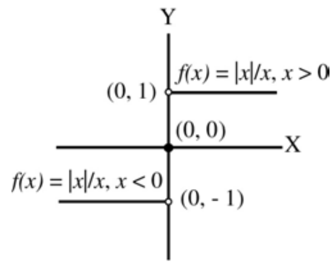


Fig 7. 31

15. Odd and even functions

If $f(x) = f(-x)$ for all x in the domain then the function is called an even function.

If $f(x) = -f(-x)$ for all x in the domain then the function is called an odd function.

For example, $f(x) = x^2$, $f(x) = x^2 + 2x^4$, $f(x) = \frac{1}{x^2}$, $f(x) = \cos x$ are some even functions.

and $f(x) = x^3$, $f(x) = x - 2x^3$, $f(x) = \frac{1}{x}$, $f(x) = \sin x$ are some odd functions.

Note that there are so many functions which are neither even nor odd. For even function, y axis divides the graph of the function into two exact pieces (symmetric). The graph of an even function is symmetric about y-axis. The graph of an odd function is symmetrical about origin.

Properties:

(1) Sum of two odd functions is again an odd function.
(2) Sum of two even functions is an even function.
(3) Sum of an odd and an even function is neither even nor odd.
(4) Product of two odd functions is an even function.
(5) Product of two even functions is an even function.
(6) Product of an odd and an even function is an odd function.
(7) Quotient of two even functions is an even function. (Denominator function ≠ O)
(8) Quotient of two odd functions is an even function. (Denominator function ≠ O)

(9) Quotient of a even and an odd function is an odd function. (Denominator function $\neq$ O)

16. Trigonometrical functions:

In Trigonometry, we have two types of functions.

(1) Circular functions (2)Hyperbolic functions.

We will discuss circular functions only. The circular functions are

(a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = \tan x$

(d) $f(x) = \sec x$ (e) $f(x) = \operatorname{cosec} x$ (f) $f(x) = \cot x$

The following graphs illustrate the graphs of circular functions.

(a) $y = \sin x$ or $f(x) = \sin x$

Domain$(-\infty, \infty)$

Range $[-1, 1]$

Principal domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

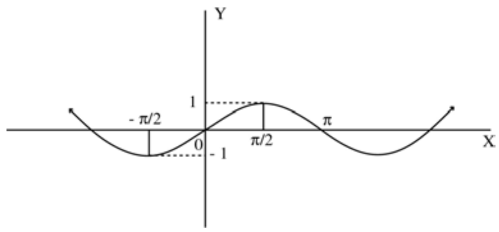


Fig 7. 32

(b) $y = \cos x$

Domain $(-\infty, \infty)$

Range $[-1, 1]$

Principal domain $[0 \ \ \pi]$

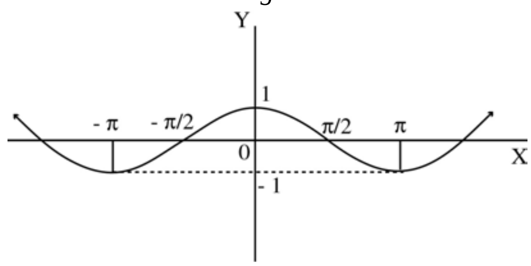


Fig 7. 33

(*c*) $y = \tan x$

Since $\tan x = \frac{\sin x}{\cos x}$, $\tan x$ is defined only for all the values of x for which $\cos x \neq 0$.

i.e. all real numbers except odd integer multiples of $\frac{\pi}{2}$ (tan$x$ is not obtained for $\cos x = 0$ and hence not defined for x, an odd multiple of $\frac{\pi}{2}$)

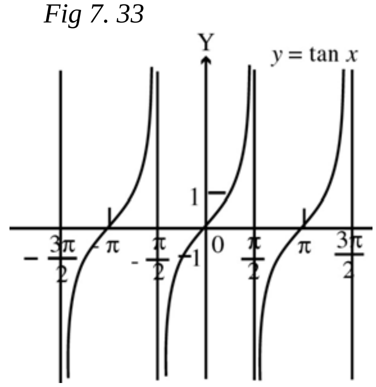


Fig 7. 34

Domain = R – $\left\{(2\,k + 1)\,\frac{\pi}{2}\right\}$, $k \in Z$

Range = $(-\infty, \infty)$

(d) $y = \text{cosec}\ x$

Since cosec x is the reciprocal of sin x, the function cosec x is not defined for values of x for which sin $x = 0$.

$\therefore$ Domain is the set of all real numbers except multiples of π

Domain = R – $\{k\pi\}$, $k \in Z$

Range = $(-\infty, -1] \cup [1, \infty)$

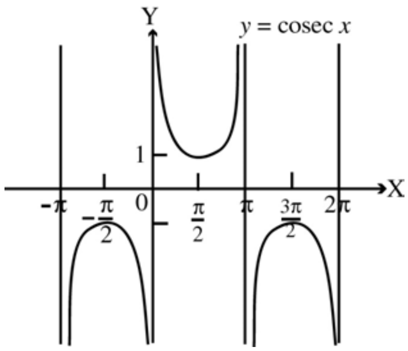


Fig 7. 35

(e) $y = \sec x$

Since sec x is reciprocal of cos$x$, the function sec$x$ is not defined for all values of x for which cos $x = 0$.

$\therefore$ Domain = R – $\left\{(2k + 1)\,\frac{\pi}{2}\right\}$, $k \in Z$

Range = $(-\infty, -1] \cup [1, \infty)$

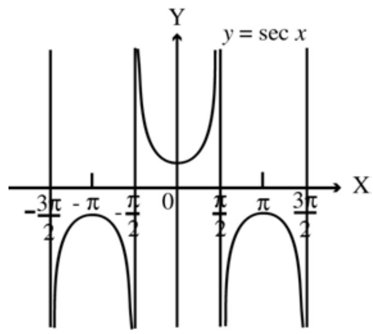


Fig 7. 36

(f) $y = \cot x$

since $\cot x = \frac{\cos x}{\sin x}$, it is not defined for the values of x for which sin $x = 0$

$\therefore$ Domain = R – $\{k\,\pi\}$, $k \in Z$

Range = $(-\infty, \infty)$

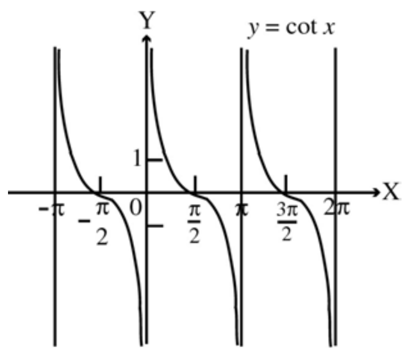


Fig 7. 37

17.Quadratic functions

It is a polynomial function of degree two.

A function $f : R \to R$ defined by $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$ is called a quadratic function. The graph of a quadratic function is always a parabola.

7.3 Quadratic Inequations:

Let $f(x) = ax^2 + bx + c$, be a quadratic function or expression. $a, b, c \in R$, $a \neq 0$

Then $f(x) \geq 0$, $f(x) > 0$, $f(x) \leq 0$ and $f(x) < 0$ are known as quadratic inequations.

The following general rules will be helpful to solve quadratic inequations.

General Rules:

1. If $a > b$, then we have the following rules:
   (i) $(a + c) > (b + c)$ for all $c \in R$
   (ii) $(a - c) > (b - c)$ for all $c \in R$
   (iii) $-a < -b$
   (iv) $ac > bc$, $\frac{a}{c} > \frac{b}{c}$ for any positive real number c
   (v) $ac < bc$, $\frac{a}{c} < \frac{b}{c}$ for any negative real number c.

The above properties, also holds good when the inequality < and > are replaced by ≤ and ≥ respectively.

2. (i) If $ab > 0$ then either $a > 0, b > 0$ (or) $a < 0, b < 0$
   (ii) If $ab \geq 0$ then either $a \geq 0, b \geq 0$ (or) $a \leq 0, b \leq 0$
   (iii) If $ab < 0$ then either $a > 0, b < 0$ (or) $a < 0, b > 0$
   (iv) If $ab \leq 0$ then either $a \geq 0, b \leq 0$ (or) $a \leq 0, b \geq 0$. $a, b, c \in R$

Domain and range of quadratic functions

Solving a quadratic inequation is same as finding the domain of the function $f(x)$ under the given inequality condition.

Different methods are available to solve a quadratic inequation. We can choose any one method which is suitable for the inequation.

Note : Eventhough the syllabus does not require the derivation, it has been derived for better understanding.

Method I: Factorisation method:

Let $ax^2 + bx + c \geq 0$ … (1)

be a quadratic inequation in x where $a, b, c \in R$ and $a \neq 0$.

The quadratic equation corresponding to this inequation is $ax^2 + bx + c = 0$. The discriminant of this equation is $b^2 - 4ac$.

Now three cases arises:

Case (i): $b^2 - 4ac > 0$

In this case, the roots of $ax^2 + bx + c = 0$ are real and distinct. Let the roots be α and β .

$\therefore\ ax^2 + bx + c = a(x - \alpha)(x - \beta)$

But $\quad ax^2 + bx + c \geq 0 \qquad$ from (1)

$\Rightarrow \qquad a(x - \alpha)(x - \beta) \geq 0$

$\Rightarrow \qquad (x - \alpha)(x - \beta) \geq 0$ if $a > 0$ (or)

$\qquad (x - \alpha)(x - \beta) \leq 0$ if $a < 0$

This inequality is solved by using the general rule (2).

Case (ii): $b^2 - 4ac = 0$

In this case, the roots of $ax^2 + bx + c = 0$ are real and equal. Let the roots be α and α

$\therefore\ ax^2 + bx + c = a(x - \alpha)^2$.

$\Rightarrow\ a(x - \alpha)^2 \geq 0$

$\Rightarrow\ (x - \alpha)^2 \geq 0$ if $a > 0$ (or) $(x - \alpha)^2 \leq 0$ if $a < 0$

This inequality is solved by using General rule-2

Case (iii): $b^2 - 4ac < 0$

In this case the roots of $ax^2 + bx + c = 0$ are non-real and distinct.

Here

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \end{aligned}$$

$\therefore$ The sign of $ax^2 + bx + c$ is same as that of a for all values of x because $\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$ is a positive real number for all values of x.

In the above discussion, we found the method of solving quadratic inequation of the type $ax^2 + bx + c \geq 0$.

Method: II

A quadratic inequality can be solved by factorising the corresponding polynomials.

1. Consider $ax^2 + bx + c > 0$

   Let $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

   Let $\alpha < \beta$

Case (i) : If $x < \alpha$ then $x - \alpha < 0$ & $x - \beta < 0$

$\therefore (x - \alpha)(x - \beta) > 0$

Case (ii): If $x > \beta$ then $x - \alpha > 0$ & $x - \beta > 0$

$\therefore (x - \alpha)(x - \beta) > 0$

Hence If $(x - \alpha)(x - \beta) > 0$ then the values of x lies outside α and β.

2. Consider $ax^2 + bx + c < 0$

   Let $ax^2 + bx + c = a(x - \alpha)(x - \beta)$; $\alpha, \beta \in R$

   Let $\alpha < \beta$ and also $\alpha < x < \beta$

   Then $x - \alpha > 0$ and $x - \beta < 0$

   $\therefore (x - \alpha)(x - \beta) < 0$

   Thus if $(x - \alpha)(x - \beta) < 0$, then the values of x lies between α and β

Method: III

Working Rules for solving quadratic inequation:

Step:1 If the coefficient of x^2 is not positive multiply the inequality by -1. Note that the sign of the inequality is reversed when it is multiplied by a negative quantity.

Step: 2 Factorise the quadratic expression and obtain its solution by equating the linear factors to zero.

Step: 3 Plot the roots obtained in step 2 on real line. The roots will divide the real line in three parts.

Step: 4 In the right most part, the quadratic expression will have positive sign and in the left most part, the expression will have positive sign and in the middle part, the expression will have negative sign.

Step: 5 Obtain the solution set of the given inequation by selecting the appropriate part in 4

Step: 6 If the inequation contains equality operator (i.e. $\leq$, $\geq$), include the roots in the solution set.

Example 7.26: Solve the inequality $x^2 - 7x + 6 > 0$

Method I:

Solution: $x^2 - 7x + 6 > 0$

$\Rightarrow$ $(x - 1)(x - 6) > 0$ [Here $b^2 - 4ac = 25 > 0$]

Now use General rule-2 :

Either $x - 1 > 0, x - 6 > 0$ (or) $(x - 1) < 0, (x - 6) < 0$

$\Rightarrow x > 1, \; x > 6$ | $\Rightarrow x < 1, x < 6$

we can omit $x > 1$ | we can omit $x < 6$

$\Rightarrow x > 6$ | $\Rightarrow x < 1$

$\therefore x \in (-\infty, 1) \cup (6, \infty)$

Method II:

$x^2 - 7x + 6 > 0$

$\Rightarrow$ $(x - 1)(x - 6) > 0$

(We know that if $(x - \alpha)(x - \beta) > 0$ then the values of x lies outside of (α, β)

(i.e.) x lies outside of $(1, 6)$

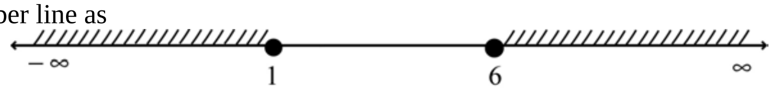
$\Rightarrow$ $x \in (-\infty, 1) \cup (6, \infty)$

Method III:

$x^2 - 7x + 6 > 0$

$\Rightarrow$ $(x - 1)(x - 6) > 0$

On equating the factors to zero, we see that $x = 1$, $x = 6$ are the roots of the quadratic equation. Plotting these roots on real line and marking positive and negative alternatively from the right most part we obtain the corresponding number line as



We have three intervals $(-\infty, 1)$, $(1, 6)$ and $(6, \infty)$. Since the sign of $(x - 1)(x - 6)$ is positive, select the intervals in which $(x - 1)(x - 6)$ is positive.

$\Rightarrow$ $x < 1$ (or) $x > 6$

$\Rightarrow$ $x \in (-\infty, 1) \cup (6, \infty)$

Note : Among the three methods, the third method, is highly useful.

Example 7.27: Solve the inequation $-x^2 + 3x - 2 > 0$

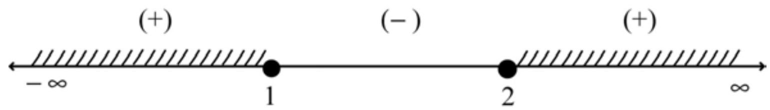
Solution :

$-x^2 + 3x - 2 > 0 \quad \Rightarrow \quad -(x^2 - 3x + 2) > 0$

$\Rightarrow \quad x^2 - 3x + 2 < 0$

$\Rightarrow \quad (x - 1)(x - 2) < 0$

On equating the factors to zero, we obtain $x = 1$, $x = 2$ are the roots of the quadratic equation. Plotting these roots on number line and making positive and negative alternatively from the right most part we obtain the corresponding numberline as given below.



The three intervals are $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$. Since the sign of $(x - 1)(x - 2)$ is negative, select the interval in which $(x - 1)(x - 2)$ is negative.

$\therefore\ x \in (1, 2)$

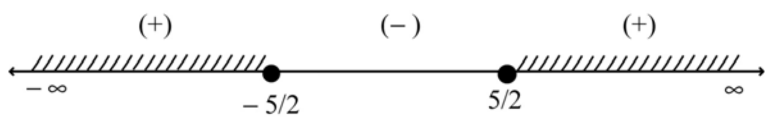
Note : We can solve this problem by the first two methods also.

Example 7.28: Solve : $4x^2 - 25 \geq 0$

Solution : $4x^2 - 25 \geq 0$

$\Rightarrow \quad (2x - 5)(2x + 5) \geq 0$

On equating the factors to zero, we obtain $x = \frac{5}{2}$, $x = -\frac{5}{2}$ are the roots of the quadratic equation. Plotting these roots on number line and making positive and negative alternatively from the right most part we obtain the corresponding number line as given below.



The three intervals are $\left(-\infty, -\frac{5}{2}\right), \left(-\frac{5}{2}, \frac{5}{2}\right) \left(\frac{5}{2}, \infty\right)$

Since the value of $(2x - 5)(2x + 5)$ is positive or zero. Select the intervals in which $f(x)$ is positive and include the roots also. The intervals are $\left(-\infty, -\frac{5}{2}\right)$ and $\left(\frac{5}{2}, \infty\right)$. But the inequality operator contains equality ($\geq$) also. $\therefore$ The solution set or the domain set should contain the roots $-\frac{5}{2}$, $\frac{5}{2}$.

Thus the solution set is $(-\infty, \frac{-5}{2}\ ] \cup [\ \frac{5}{2}, \infty)$

Example 7.29: Solve the quadratic inequation $64x^2 + 48x + 9 < 0$

Solution:

$64x^2 + 48x + 9 = (8x + 3)^2$

$(8x + 3)^2$ is a perfect square. A perfect square cannot be negative for real x.

$\therefore$ The given quadratic inequation has no solution.

Example 7.30: Solve $f(x)=x^2+2x+6 > 0$ or find the domain of the function $f(x)$

$x^2 + 2x + 6 > 0$

$(x + 1)^2 + 5 > 0$

This is true for all values of x. $\therefore$ The solution set is R

Example 7.31: Solve $f(x) = 2x^2 - 12x + 50 \le 0$ or find the domain of the function $f(x)$.

Solution:

$2x^2 - 12x + 50 \le 0$

$2(x^2 - 6x + 25) \le 0$

$x^2 - 6x + 25 \le 0$

$(x^2 - 6x + 9) + 25 - 9 \le 0$

$(x - 3)^2 + 16 \le 0$

This is not true for any real value of x.

$\therefore$ Given inequation has no solution.

Some special problems (reduces to quadratic inequations)

Example 7.32: Solve: $\dfrac{x+1}{x-1} > 0,\ x \ne 1$

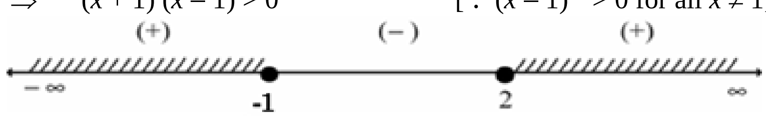
Solution:

$$\frac{x+1}{x-1} > 0$$

Multiply the numerator and denominator by $(x - 1)$

$\Rightarrow \quad \dfrac{(x+1)(x-1)}{(x-1)^2}$

$\Rightarrow \quad (x + 1)(x - 1) > 0 \qquad [\because (x - 1)^2 > 0 \text{ for all } x \ne 1]$



Since the value of $(x + 1)(x - 1)$ is positive or zero select the intervals in which $(x + 1)(x - 1)$ is positive.

$\therefore \quad x \in (-\infty, -1) \cup (1, \infty)$

Example 7.33: Solve : $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$

Solution: $\dfrac{x-1}{4x+5} < \dfrac{x-3}{4x-3}$

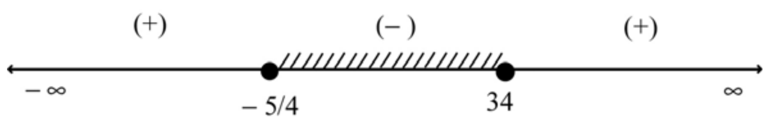
$\Rightarrow \quad \dfrac{x-1}{4x+5} - \dfrac{x-3}{4x-3} < 0$ (Here we cannot cross multiply)

$\Rightarrow \quad \dfrac{(x-1)(4x-3)-(x-3)(4x+5)}{(4x+5)(4x-3)} < 0$

$\Rightarrow \quad \dfrac{18}{(4x+5)(4x-3)} < 0$

$\Rightarrow \quad (4x+5)(4x-3) < 0$ since $18 > 0$

On equating the factors to zero, we obtain $x = \dfrac{-5}{4}$, $x = \dfrac{3}{4}$ are the roots of the quadratic equation. Plotting these roots on number line and making positive and negative alternatively from the right most part we obtain as shown in figure.



Since the value of $(4x + 5)(4x - 3)$ is negative, select the intervals in which $(4x + 5)(4x - 3)$ is negative. $\therefore\ x \in \left(\dfrac{-5}{4}, \dfrac{3}{4}\right)$

Example 7.34 : If $x \in$ R, prove that the range of the function $f(x) = \dfrac{x^2-3x+4}{x^2+3x+4}$ is $\left[\dfrac{1}{7}, \ 7\right]$

Solution:

$$\text{Let } y = \frac{x^2-3x+4}{x^2+3x+4}$$

$$(x^2+3x+4)y = x^2-3x+4$$

$$\Rightarrow \quad x^2(y-1) + 3x(y+1) + 4(y-1) = 0$$

Clearly, this is a quadratic equation in x. It is given that x is real.

$\Rightarrow \quad$ Discriminant ≥ 0

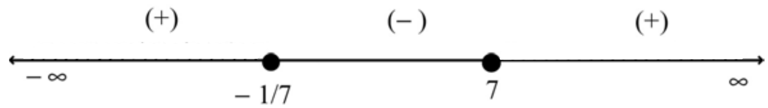
$\Rightarrow \quad 9(y+1)^2 - 16(y-1)^2 \geq 0$

$\Rightarrow \quad [3(y+1)]^2 - [4(y-1)]^2 \geq 0$

$\Rightarrow \quad [3(y+1)+4(y-1)]\ [3(y+1)-4(y-1)] \geq 0$

$\Rightarrow \quad (7y-1)(-y+7) \geq 0$

$$\Rightarrow \quad -(7y-1)(y-7) \geq 0$$
$$\Rightarrow \quad (7y-1)(y-7) \leq 0$$



The intervals are $\left(-\infty, \frac{1}{7}\right)$, $\left(\frac{1}{7}, 7\right)$ and $(7, \infty)$. Since the value of $(7y-1)(y-7)$ is negative or zero, select the intervals in which $(7y-1)(y-1)$ is negative and include the roots $\frac{1}{7}$ and 7.

$\therefore y \in \left[\frac{1}{7}, 7\right]$ i.e. the value of $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ lies between $\frac{1}{7}$ and 7

i.e. the range of $f(x)$ is $\left[\frac{1}{7}, 7\right]$

EXERCISE 7.1

(1) If $f, g : R \to R$, defined by $f(x) = x + 1$ and $g(x) = x^2$, find (i) $(fog)(x)$ (ii) $(gof)(x)$ (iii) $(fof)(x)$ (iv) $(gog)(x)$ (v) $(fog)(3)$ (vi) $(gof)(3)$

(2) For the functions f, g as defined in (1) define

(i) $(f+g)(x)$ (ii) $\left(\frac{f}{g}\right)(x)$ (iii) $(fg)(x)$ (iv) $(f-g)(x)$ (v) $(gf)(x)$

(3) Let $f : R \to R$ be defined by $f(x) = 3x + 2$. Find f^{-1} and show that $fof^{-1} = f^{-1}of = I$

(4) Solve each of the following inequations:

(i) $x^2 \leq 9$ (ii) $x^2 - 3x - 18 > 0$ (iii) $4 - x^2 < 0$

(iv) $x^2 + x - 12 < 0$ (v) $7x^2 - 7x - 84 \geq 0$ (vi) $2x^2 - 3x + 5 < 0$

(vii) $\frac{3x-2}{x-1} < 2,\ x \neq 1$ (viii) $\frac{2x-1}{x} > -1,\ x \neq 0$ (ix) $\frac{x-2}{3x+1} > \frac{x-3}{3x-2}$

(5) If x is real, prove that $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ cannot have any value between 5 and 9.

(6) If x is real, prove that the range of $f(x) = \frac{x^2 - 2x + 4}{x^2 + 2x + 4}$ is between $\left[\frac{1}{3}, 3\right]$

(7) If x is real, prove that $\frac{x}{x^2 - 5x + 9}$ lies between $-\frac{1}{11}$ and 1.

8. DIFFERENTIAL CALCULUS

Calculus is the mathematics of motion and change. When increasing or decreasing quantities are made the subject of mathematical investigation, it frequently becomes necessary to estimate their rates of growth or decay. Calculus was invented for the purpose of solving problems that deal with continuously changing quantities. Hence, the primary objective of the Differential Calculus is to describe an instrument for the measurement of such rates and to frame rules for its formation and use.

Calculus is used in calculating the rate of change of velocity of a vehicle with respect to time, the rate of change of growth of population with respect to time, etc. Calculus also helps us to maximise profits or minimise losses.

Isacc Newton of England and Gottfried Wilhelm Leibnitz of Germany invented calculus in the 17th century, independently. Leibnitz, a great mathematician of all times, approached the problem of settling tangents geometrically; but Newton approached calculus using physical concepts. Newton, one of the greatest mathematicians and physicists of all time, applied the calculus to formulate his laws of motion and gravitation.

8.1 Limit of a Function

The notion of limit is very intimately related to the intuitive idea of nearness or closeness. Degree of such closeness cannot be described in terms of basic algebraic operations of addition and multiplication and their inverse operations subtraction and division respectively. It comes into play in situations where one quantity depends on another varying quantity and we have to know the behaviour of the first when the second is very close to a fixed given value.

Let us look at some examples, which will help in clarifying the concept of a limit. Consider the function $f : \mathrm{R} \rightarrow \mathrm{R}$ given by

$$f(x) = x + 4.$$

Look at tables 8.1 and 8.2 These give values of $f(x)$ as x gets closer and closer to 2 through values less than 2 and through values greater than 2 respectively.

x	1	1.5	1.9	1.99	1.999
$f(x)$	5	5.5	5.9	5.99	5.999

Table 8.1

x	3	2.5	2.1	2.01	2.001
$f(x)$	7	6.5	6.1	6.01	6.001

Table 8.2

From the above tables we can see that as x approaches 2, $f(x)$ approaches 6. In fact, the nearer x is chosen to 2, the nearer $f(x)$ will be to 6. Thus 6 is the value of $(x + 4)$ as x approaches 2. We call such a value the limit of $f(x)$ as x tends to 2 and denote it by $\lim_{x \to 2} f(x)=6$. In this example the value $\lim_{x \to 2} f(x)$ coincides with the value $(x + 4)$ when $x = 2$, that is, $\lim_{x \to 2} f(x) = f(2)$.

Note that there is a difference between '$x \to 0$' and '$x = 0$'. $x \to 0$ means that x gets nearer and nearer to 0, but never becomes equal to 0. $x = 0$ means that x takes the value 0.

Now consider another function f given by $f(x) = \dfrac{x^2 - 4}{(x - 2)}$. This function is not defined at the point $x = 2$, since division by zero is undefined. But $f(x)$ is defined for values of x which approach 2. So it makes sense to evaluate $\lim_{x \to 2} \dfrac{x^2 - 4}{(x - 2)}$. Again we consider the following tables 8.3 and 8. 4 which give the values of $f(x)$ as x approaches 2 through values less than 2 and through values greater than 2, respectively.

x	1	1.5	1.9	1.99	1.999
$f(x)$	3	3.5	3.9	3.99	3.999

Table 8.3

x	3	2.5	2.1	2.01	2.001
$f(x)$	5	4.5	4.1	4.01	4.001

Table 8.4

We see that $f(x)$ approaches 4 as x approaches 2. Hence $\lim_{x \to 2} f(x) = 4$.

You may have noticed that $f(x) = \dfrac{x^2 - 4}{(x - 2)} = \dfrac{(x + 2)\,(x - 2)}{(x - 2)} = x + 2$, if $x \neq 2$.

In this case a simple way to calculate the limit above is to substitute the value $x = 2$ in the expression for $f(x)$, when $x \neq 2$, that is, put $x = 2$ in the expression x + 2.

Now take another example. Consider the function given by $f(x) = \dfrac{1}{x}$. We see that $f(0)$ is not defined. We try to calculate $\lim_{x \to 0} f(x)$. Look at tables 8.5 and 8.6

x	1/2	1/10	1/100	1/1000
$f(x)$	2	10	100	1000

Table 8.5

x	– 1/2	– 1/10	– 1/100	– 1/1000
$f(x)$	– 2	– 10	– 100	– 1000

Table 8.6

We see that $f(x)$ does not approach any fixed number as x approaches 0. In this case we say that $\lim_{x \to 0} f(x)$ does not exist. This example shows that there are cases when the limit may not exist. Note that the first two examples show that such a limit exists while the last example tells us that such a limit may not exist. These examples lead us to the following.

Definition

Let f be a function of a real variable x. Let c, l be two fixed numbers. If $f(x)$ approaches the value l as x approaches c, we say l is the limit of the function $f(x)$ as x tends to c. This is written as $\lim_{x \to c} f(x) = l$.

Left Hand and Right Hand Limits

While defining the limit of a function as x tends to c, we consider values of $f(x)$ when x is very close to c. The values of x may be greater or less than c. If we restrict x to values less than c, then we say that x tends to c from below or from the left and write it symbolically as $x \to c - 0$ or simply $x \to c\_$. The limit of f with this restriction on x, is called the left hand limit. This is written as

$$\mathrm{L}f(c) = \lim_{x \to c\_} f(x), \text{ provided the limit exists.}$$

Similarly if x takes only values greater than c, then x is said to tend to c from above or from right, and is denoted symbolically as $x \to c + 0$ or $x \to c_+$. The limit of f is then called the right hand limit. This is written as $\mathrm{R}f(c) = \lim_{x \to c_+} f(x)$.

It is important to note that for the existence of $\lim_{x \to c} f(x)$ it is necessary that both $\mathrm{L}f(c)$ and $\mathrm{R}f(c)$ exists and $\mathrm{L}f(c) = \mathrm{R}f(c) = \lim_{x \to c} f(x)$. These left and right hand limits are also known as one sided limits.

8.1.1 Fundamental results on limits

(1) If $f(x) = k$ for all x, then $\lim\limits_{x \to c} f(x) = k$.

(2) If $f(x) = x$ for all x, then $\lim\limits_{x \to c} f(x) = c$.

(3) If f and g are two functions possessing limits and k is a constant then

(i) $\lim\limits_{x \to c} k\,f(x) = k \lim\limits_{x \to c} f(x)$

(ii) $\lim\limits_{x \to c} [f(x) + g(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$

(iii) $\lim\limits_{x \to c} [f(x) - g(x)] = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$

(iv) $\lim\limits_{x \to c} [f(x) \,.\, g(x)] = \lim\limits_{x \to c} f(x) \,.\, \lim\limits_{x \to c} g(x)$

(v) $\lim\limits_{x \to c} \left[\dfrac{f(x)}{g(x)}\right] = \lim\limits_{x \to c} f(x) \Big/ \lim\limits_{x \to c} g(x)$, $\quad g(x) \neq 0$

(vi) If $f(x) \leq g(x)$ then $\lim\limits_{x \to c} f(x) \leq \lim\limits_{x \to c} g(x)$.

Example 8.1 :

Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ if it exists.

Solution:

Let us evaluate the left hand and right hand limits.

When $x \to 1_{-}$, put $x = 1 - h$, $h > 0$.

$$\text{Then } \lim_{x \to 1-} \frac{x^2-1}{x-1} = \lim_{h \to 0} \frac{(1-h)^2 - 1}{1 - h - 1} = \lim_{h \to 0} \frac{1 - 2h + h^2 - 1}{-h}$$

$$= \lim_{h \to 0} (2 - h) = \lim_{h \to 0} (2) - \lim_{h \to 0} (h) = 2 - 0 = 2$$

When $x \to 1_{+}$ put $x = 1 + h$, $h > 0$

$$\text{Then } \lim_{x \to 1+} \frac{x^2-1}{x-1} = \lim_{h \to 0} \frac{(1+h)^2 - 1}{1 + h - 1} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h}$$

$$= \lim_{h \to 0} (2 + h) = \lim_{h \to 0} (2) + \lim_{h \to 0} (h)$$

$= 2 + 0 = 2$, using (1) and (2) of 8.1.1

So that both, the left hand and the right hand, limits exist and are equal. Hence the limit of the function exists and equals 2.

(i.e.) $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2.$

Note: Since $x \neq 1$, division by $(x - 1)$ is permissible.

$$\therefore \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2 .$$

Example 8.2: Find the right hand and the left hand limits of the function at $x = 4$

$$\boldsymbol{f(x)} = \begin{cases} \dfrac{|x - 4|}{x - 4} & \text{for } x \neq 4 \\ 0, & \text{for } x = 4 \end{cases}$$

Solution:

Now, when $x > 4$, $|x - 4| = x - 4$

Therefore $\lim_{x \to 4+} f(x) = \lim_{x \to 4+} \dfrac{|x - 4|}{x - 4} = \lim_{x \to 4+} \dfrac{x - 4}{x - 4} = \lim_{x \to 4} (1) = 1$

Again when $x < 4$, $|x - 4| = -(x - 4)$

Therefore $\lim_{x \to 4-} f(x) = \lim_{x \to 4-} \dfrac{-(x - 4)}{(x - 4)} = \lim_{x \to 4-} (-1) = -1$

Note that both the left and right hand limits exist but they are not equal.

i.e. $\mathrm{R}f(4) = \lim_{x \to 4+} f(x) \neq \lim_{x \to 4-} f(x) = \mathrm{L}f(4)$.

Example 8.3

Find $\lim_{x \to 0} \dfrac{3x + |x|}{7x - 5|x|}$, if it exists.

Solution:

$$\mathrm{R}f(0) = \lim_{x \to 0+} \frac{3x + |x|}{7x - 5|x|} = \lim_{x \to 0+} \frac{3x + x}{7x - 5x} \quad (\text{since } x > 0, |x| = x)$$

$$= \lim_{x \to 0+} \frac{4x}{2x} = \lim_{x \to 0+} 2 = 2 .$$

$$\mathrm{L}\,f(0) = \lim_{x \to 0-} \frac{3x + |x|}{7x - 5|x|} = \lim_{x \to 0-} \frac{3x - x}{7x - 5(-x)} \quad (\text{since } x < 0, |x| = -x)$$

$$= \lim_{x \to 0-} \frac{2x}{12x} = \lim_{x \to 0-} \left(\frac{1}{6}\right) = \frac{1}{6} .$$

Since $\mathrm{R}f(0) \neq \mathrm{L}f(0)$, the limit does not exist.

Note: Let $f(x) = g(x) / h(x)$.

Suppose at $x = c$, $g(c) \neq 0$ and $h(c) = 0$, then $f(c) = \dfrac{g(c)}{0}$.

In this case $\lim\limits_{x \to c} f(x)$ does not exist.

Example 8.4 : Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 + 7x + 11}{x^2 - 9}$.

Solution:

Let $f(x) = \dfrac{x^2 + 7x + 11}{x^2 - 9}$. This is of the form $f(x) = \dfrac{g(x)}{h(x)}$,

where $g(x) = x^2 + 7x + 11$ and $h(x) = x^2 - 9$. Clearly $g(3) = 41 \neq 0$ and $h(3) = 0$.

Therefore $f(3) = \dfrac{g(3)}{h(3)} = \dfrac{41}{0}$. Hence $\lim\limits_{x \to 3} \dfrac{x^2 + 7x + 11}{x^2 - 9}$ does not exist.

Example 8.5: Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - 1}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \lim_{x \to 0} \frac{\left(\sqrt{1 + x} - 1\right)\left(\sqrt{1 + x} + 1\right)}{x\left(\sqrt{1 + x} + 1\right)}$$

$$= \lim_{x \to 0} \frac{(1 + x) - 1}{x\left(\sqrt{1 + x} + 1\right)} = \lim_{x \to 0} \frac{1}{\left(\sqrt{1 + x} + 1\right)}$$

$$= \frac{\lim\limits_{x \to 0} (1)}{\lim\limits_{x \to 0} \left(\sqrt{1 + x} + 1\right)} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} .$$

8.1.2 Some important Limits

Example 8.6 :

For $\left|\dfrac{\Delta x}{a}\right| < 1$ and for any rational index n,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \qquad (a \neq 0)$$

Solution:

Put $\Delta x = x - a$ so that $\Delta x \to 0$ as $x \to a$ and $\left|\dfrac{\Delta x}{a}\right| < 1$.

Therefore $\dfrac{x^n - a^n}{x - a} = \dfrac{(a + \Delta x)^n - a^n}{\Delta x} = \dfrac{a^n\left(1 + \dfrac{\Delta x}{a}\right)^n - a^n}{\Delta x}$

Applying Newton's Binomial Theorem for rational index we have

$$\left(1 + \frac{\Delta x}{a}\right)^n = 1 + \binom{n}{1}\left(\frac{\Delta x}{a}\right) + \binom{n}{2}\left(\frac{\Delta x}{a}\right)^2 + \binom{n}{3}\left(\frac{\Delta x}{a}\right)^3 + \ldots + \binom{n}{r}\left(\frac{\Delta x}{a}\right)^r + \ldots$$

$$\therefore \frac{x^n - a^n}{x - a} = \frac{a^n\left[1 + \binom{n}{1}\left(\frac{\Delta x}{a}\right) + \binom{n}{2}\left(\frac{\Delta x}{a}\right)^2 + \ldots + \binom{n}{r}\left(\frac{\Delta x}{a}\right)^r + \ldots\right] - a^n}{\Delta x}$$

$$= \frac{\left[\binom{n}{1} a^{n-1}\,\Delta x + \binom{n}{2} a^{n-2} (\Delta x)^2 + \ldots + \binom{n}{r} a^{n-r} (\Delta x)^r + \ldots\right]}{\Delta x}$$

$$= \binom{n}{1} a^{n-1} + \binom{n}{2} a^{n-2} (\Delta x) + \ldots + \binom{n}{r} a^{n-r} (\Delta x)^{r-1} + \ldots$$

$$= \binom{n}{1} a^{n-1} + \text{terms containing } \Delta x \text{ and higher powers of } \Delta x .$$

Since $\Delta x = x - a$, $x \to a$ means $\Delta x \to 0$ and therefore

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{\Delta x \to 0} \binom{n}{1} a^{n-1} + \lim_{\Delta x \to 0}$$

(terms containing Δx and higher powers of Δx)

$$= \binom{n}{1} a^{n-1} + 0 + 0 + \ldots = na^{n-1} \qquad \text{since } \binom{n}{1} = n .$$

As an illustration of this result, we have the following examples.

Example 8.7: Evaluate $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Solution: $\displaystyle\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3(1)^{3-1} = 3(1)^2 = 3$

Example 8.8: Find $\displaystyle\lim_{x \to 0} \frac{(1 + x)^4 - 1}{x}$

Solution: Put $1 + x = t$ so that $t \to 1$ as $x \to 0$

$$\therefore \lim_{x \to 0} \frac{(1 + x)^4 - 1}{x} = \lim_{t \to 1} \frac{t^4 - 1^4}{t - 1} = 4(1)^3 = 4$$

Example 8.9: Find the positive integer n so that $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 32$

Solution: We have $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = n2^{n-1}$

$$\therefore\ n2^{n-1} = 32 = 4 \times 8 = 4 \times 2^3 = 4 \times 2^{4-1}$$

Comparing on both sides we get $\quad n = 4$

Example 8.10: $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Solution:

We take $y = \dfrac{\sin\theta}{\theta}$. This function is defined for all θ, other than $\theta = 0$, for which both numerator and denominator become zero. When θ is replaced by $-\theta$, the magnitude of the fraction $\dfrac{\sin\theta}{\theta}$ does not change since $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$. Therefore it is enough to find the limit of the fraction as θ tends to 0 through positive values. i.e. in the first quadrant. We consider a circle with centre at O radius unity. A, B are two points on this circle so OA = OB = 1. Let θ be the angle subtended at the centre by the arc AE. Measuring angle in radians, we have $\sin\theta$ = AC, C being a point on AB such that OD passes through C.

$\cos\theta = OC$, $\theta = \dfrac{1}{2}$ arc AB, $\angle OAD = 90°$

In triangle OAD, AD = $\tan\theta$.

Now length of arc AB = 2θ and length of the chord AB = $2\sin\theta$

sum of the tangents = AD + BD = $2\tan\theta$

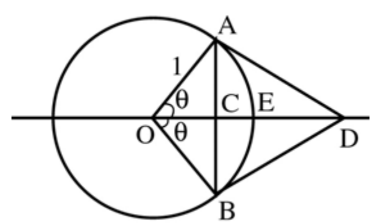


Fig. 8.1

Since the length of the arc is intermediate between the length of chord and the sum of the tangents we can write $2\sin\theta < 2\theta < 2\tan\theta$.

Dividing by $2\sin\theta$, we have $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ or $1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

But as $\theta \to 0$, $\cos\theta$, given by the distance OC, tends to 1

That is, $\lim\limits_{\theta \to 0} \cos\theta = 1$.

Therefore $1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > 1$, by 3(vi) of 8.1.1

That is, the variable $y = \frac{\sin\theta}{\theta}$ always lies between unity and a magnitude tending to unity, and hence $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

The graph of the function $y = \frac{\sin\theta}{\theta}$ is shown in fig. 8.2

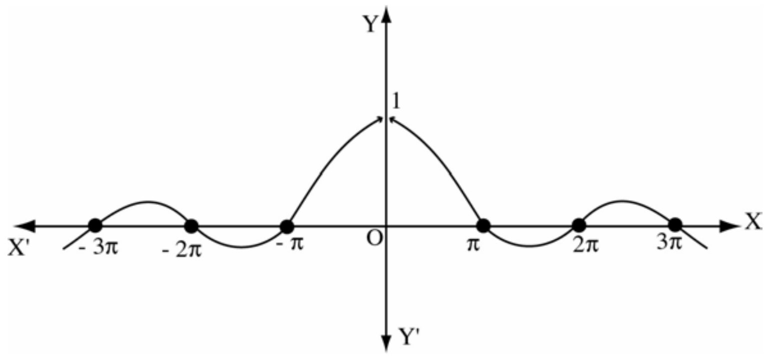


Fig. 8.2

Example 8.11: Evaluate $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2}$.

Solution:

$$\frac{1-\cos\theta}{\theta^2} = \frac{2\sin^2\frac{\theta}{2}}{\theta^2} = \frac{1}{2}\,\frac{\sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)^2} = \frac{1}{2}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2$$

If $\theta \to 0$, $\alpha = \frac{\theta}{2}$ also tends to 0 and $\lim_{\theta \to 0} \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} = \lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$ and

hence $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta^2} = \lim_{\theta \to 0} \frac{1}{2}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 = \frac{1}{2}\left(\lim_{\theta \to 0} \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 = \frac{1}{2} \times 1^2 = \frac{1}{2}$

Example 8.12: Evaluate $\lim\limits_{x \to 0^+} \dfrac{\sin x}{\sqrt{x}}$

Solution:

$$\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)\sqrt{x}$$

$$= \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right) \cdot \lim_{x \to 0^+} (\sqrt{x}) = 1 \times 0 = 0 .$$

Note: For the above problem left hand limit does not exist since $\sqrt{x}$ is not real for $x < 0$.

Example 8.13: Compute $\lim\limits_{x \to 0} \dfrac{\sin \beta x}{\sin \alpha x}$, $\alpha \neq 0$.

Solution:

$$\lim_{x \to 0} \frac{\sin \beta x}{\sin \alpha x} = \lim_{x \to 0} \frac{\beta \cdot \dfrac{\sin \beta x}{\beta x}}{\alpha \cdot \dfrac{\sin \alpha x}{\alpha x}} = \frac{\beta \lim\limits_{x \to 0}\left(\dfrac{\sin \beta x}{\beta x}\right)}{\alpha \lim\limits_{x \to 0}\left(\dfrac{\sin \alpha x}{\alpha x}\right)}$$

$$= \frac{\beta \lim\limits_{\theta \to 0}\left(\dfrac{\sin\theta}{\theta}\right)}{\alpha \lim\limits_{y \to 0}\left(\dfrac{\sin y}{y}\right)} = \frac{\beta \times 1}{\alpha \times 1} = \frac{\beta}{\alpha} . \quad \begin{array}{l}\text{since } \theta = \beta x \to 0 \text{ as } x \to 0 \\ \text{and } y = \alpha x \to 0 \text{ as } x \to 0\end{array}$$

Example 8.14: Compute $\lim\limits_{x \to \pi/6} \dfrac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

Solution:

We have

$$2 \sin^2 x + \sin x - 1 = (2 \sin x - 1) (\sin x + 1)$$

$$2 \sin^2 x - 3 \sin x + 1 = (2 \sin x - 1) (\sin x - 1)$$

Now

$$\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1} = \lim_{x \to \pi/6} \frac{(2 \sin x - 1) (\sin x + 1)}{(2 \sin x - 1) (\sin x - 1)}$$

$$= \lim_{x \to \pi/6} \frac{\sin x + 1}{\sin x - 1} \quad \left(2 \sin x - 1 \neq 0 \text{ for } x \to \frac{\pi}{6}\right)$$

$$= \frac{\sin \pi/6 + 1}{\sin \pi/6 - 1} = \frac{1/2 + 1}{1/2 - 1} = -3 .$$

Example 8.15: $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$.

Solution: We know that $e^x = 1 + \dfrac{x}{\lfloor 1} + \dfrac{x^2}{\lfloor 2} + \ldots + \dfrac{x^n}{\lfloor n} + \ldots$

and so $e^x - 1 = \dfrac{x}{\lfloor 1} + \dfrac{x^2}{\lfloor 2} + \ldots + \dfrac{x^n}{\lfloor n} + \ldots$

i.e. $\dfrac{e^x - 1}{x} = \dfrac{1}{\lfloor 1} + \dfrac{x}{\lfloor 2} + \ldots + \dfrac{x^{n-1}}{\lfloor n} + \ldots$

($\because x \neq 0$, division by x is permissible)

$\therefore \lim_{x \to 0} \dfrac{e^x - 1}{x} = \dfrac{1}{\lfloor 1} = 1$.

Example 8.16: Evaluate $\lim_{x \to 3} \dfrac{e^x - e^3}{x - 3}$.

Solution: Consider $\dfrac{e^x - e^3}{x - 3}$. Put $y = x - 3$. Then $y \to 0$ as $x \to 3$.

Therefore $\lim_{x \to 3} \dfrac{e^x - e^3}{x - 3} = \lim_{y \to 0} \dfrac{e^{y+3} - e^3}{y} = \lim_{y \to 0} \dfrac{e^3 . e^y - e^3}{y}$

$= e^3 \lim_{y \to 0} \dfrac{e^y - 1}{y} = e^3 \times 1 = e^3$.

Example 8.17: Evaluate $\lim_{x \to 0} \dfrac{e^x - \sin x - 1}{x}$.

Solution:

Now $\dfrac{e^x - \sin x - 1}{x} = \left(\dfrac{e^x - 1}{x}\right) - \left(\dfrac{\sin x}{x}\right)$

and so $\lim_{x \to 0} \dfrac{e^x - \sin x - 1}{x} = \lim_{x \to 0} \left(\dfrac{e^x - 1}{x}\right) - \lim_{x \to 0} \left(\dfrac{\sin x}{x}\right) = 1 - 1 = 0$

Example 8.18: Evaluate $\lim_{x \to 0} \dfrac{e^{\tan x} - 1}{\tan x}$

Solution: Put $\tan x = y$. Then $y \to 0$ as $x \to 0$

Therefore $\lim_{x \to 0} \dfrac{e^{\tan x} - 1}{\tan x} = \lim_{y \to 0} \dfrac{e^y - 1}{y} = 1$

Example 8.19: $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$

Solution: We know that $\log_e(1+x) = \dfrac{x}{1} - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots$

$$\frac{\log_e(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

Therefore $\lim\limits_{x \to 0} \dfrac{\log_e(1+x)}{x} = 1.$

Note: log*x* means the natural logarithm log*ₑx*.

Example 8.20: Evaluate $\lim\limits_{x \to 1} \dfrac{\log x}{x-1}$.

Solution: Put $x - 1 = y$. Then $y \to 0$ as $x \to 1$.

Therefore $\lim\limits_{x \to 1} \dfrac{\log x}{x-1} = \lim\limits_{y \to 0} \dfrac{\log(1+y)}{y}$

$= 1$ (by example 8.19)

Example 8.21: $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log a, \quad a > 0$

Solution: We know that $f(x) = e^{\log f(x)}$ and so $a^x = e^{\log a^x} = e^{x \log a}$.

Therefore $\dfrac{a^x - 1}{x} = \dfrac{e^{x \log a} - 1}{x \log a} \times \log a$

Now as $x \to 0$, $y = x \log a \to 0$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{e^y - 1}{y} \times \log a = \log a \lim_{y \to 0} \left(\frac{e^y - 1}{y}\right)$$

$= \log a.$ (since $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$)

Example 8.22: Evaluate $\lim\limits_{x \to 0} \dfrac{5^x - 6^x}{x}$

Solution:

$$\lim_{x \to 0} \frac{5^x - 6^x}{x} = \lim_{x \to 0} \frac{(5^x - 1) - (6^x - 1)}{x}$$

$$= \lim_{x \to 0} \left(\frac{5^x - 1}{\mathrm{x}}\right) - \lim_{x \to 0} \left(\frac{6^x - 1}{x}\right)$$

$$= \log 5 - \log 6 = \log\left(\frac{5}{6}\right) .$$

Example 8.23: Evaluate $\lim\limits_{x \to 0} \dfrac{3^x + 1 - \cos x - e^x}{x}$.

Solution:

$$\lim_{x \to 0} \frac{3^x + 1 - \cos x - e^x}{x} = \lim_{x \to 0} \frac{(3^x - 1) + (1 - \cos x) - (e^x - 1)}{x}$$

$$= \lim_{x \to 0} \left(\frac{3^x - 1}{x}\right) + \lim_{x \to 0} \left(\frac{1 - \cos x}{x}\right) - \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)$$

$$= \log 3 + \lim_{x \to 0} \frac{2 \sin^2 x/2}{x} - 1$$

$$= \log 3 + \lim_{x \to 0} \frac{x}{2} \left(\frac{\sin x/2}{x/2}\right)^2 - 1$$

$$= \log 3 + \frac{1}{2} \lim_{x \to 0} (x) \lim_{x \to 0} \left(\frac{\sin x/2}{x/2}\right)^2 - 1$$

$$= \log 3 + \frac{1}{2} \times 0 \times 1 - 1 = \log 3 - 1.$$

Some important limits :

(1) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$ exists and we denote this limit by e

(2) $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$ [by taking $x = \dfrac{1}{y}$ in (1)]

(3) $\lim\limits_{x \to \infty} \left(1 + \dfrac{k}{x}\right)^x = e^k$

Note : (1) The value of e lies between 2 & 3 i.e., $2 < e < 3$

(2) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$ is true for all real x

Thus $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$ for all real values of x.

Note that $e = e^1 = 1 + \dfrac{1}{1!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots + \dfrac{1}{r!} + \ldots$ This number e is also known as transcendental number in the sense that e never satisfies a polynomial (algebraic) equation of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$.

Example 8.24: Compute $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{3x}$.

Solution: Now $\left(1+\frac{1}{x}\right)^{3x} = \left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}$ and so

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\left(1+\frac{1}{x}\right)^{x}.\left(1+\frac{1}{x}\right)^{x}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e.\ e.\ e. = e^3 .$$

Example 8.25: Evaluate $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3}$.

Solution:

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3} &= \lim_{x\to\infty}\left(\frac{x-1+4}{x-1}\right)^{(x-1)+4}\\ &= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{(x-1)+4}\\ &= \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y+4} \quad (\because y = x-1 \to \infty \text{ as } x\to\infty)\\ &= \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}\left(1+\frac{4}{y}\right)^{4}\\ &= \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}.\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{4} = e^4.\ 1 = e^4\end{aligned}$$

Example 8.26: Evaluate $\lim\limits_{x\to\pi/2}(1+\cos x)^{3\sec x}$.

Solution: Put $\cos x = \frac{1}{y}$. Now $y\to\infty$ as $x\to\frac{\pi}{2}$.

$$\lim_{x\to\pi/2}(1+\cos x)^{3\sec x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{3y} = \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{3}$$

$$= \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{3} = e^3.$$

Example 8.27. Evaluate $\lim\limits_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$

Solution :

$$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{2^x-1}{(1+x-1)}\left(\sqrt{1+x}+1\right)$$

$$= \lim_{x \to 0} \frac{2^x - 1}{x} \cdot \lim_{x \to 0} \left(\sqrt{1 + x} + 1\right)$$

$$= \log 2 \cdot \left(\sqrt{1} + 1\right) \qquad \left(\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right)$$

$$= 2 \log 2 = \log 4 .$$

Example 8.28: Evaluate $\lim_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{\sin^{-1} x}$

Solution:

Put $\sin^{-1} x = y$. Then $x = \sin y$ and $y \to 0$ as $x \to 0$.

Now $\lim_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{\sin^{-1} x} = \lim_{x \to 0} \dfrac{(1 + x) - (1 - x)}{\sin^{-1} x} \left(\dfrac{1}{\sqrt{1 + x} + \sqrt{1 - x}}\right)$

$$= \lim_{y \to 0} \frac{2 \sin y}{y} \cdot \lim_{y \to 0} \frac{1}{\sqrt{1 + \sin y} + \sqrt{1 - \sin y}}$$

$$= 2 \lim_{y \to 0} \left(\frac{\sin y}{y}\right) \left(\frac{1}{\sqrt{1 + 0} + \sqrt{1 - 0}}\right)$$

$$= 2 \times 1 \times \frac{1}{2} = 1$$

EXERCISE 8.1

Find the indicated limits.

(1) $\lim_{x \to 1} \dfrac{x^2 + 2x + 5}{x^2 + 1}$

(2) $\lim_{x \to 2^-} \dfrac{x - 2}{\sqrt{2 - x}}$

(3) $\lim_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

(4) $\lim_{x \to 1} \dfrac{x^m - 1}{x - 1}$

(5) $\lim_{x \to 4} \dfrac{\sqrt{2x + 1} - 3}{\sqrt{x - 2} - \sqrt{2}}$

(6) $\lim_{x \to 0} \dfrac{\sqrt{x^2 + p^2} - p}{\sqrt{x^2 + q^2} - q}$

(7) $\lim_{x \to a} \dfrac{\sqrt[m]{x} - \sqrt[m]{a}}{x - a}$

(8) $\lim_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$

(9) $\lim_{x \to 0} \dfrac{\sqrt{1 + x + x^2} - 1}{x}$

(10) $\lim_{x \to 0} \dfrac{\sin^2 (x/3)}{x^2}$

(11) $\lim_{x \to 0} \dfrac{\sin (a + x) - \sin (a - x)}{x}$

(12) $\lim_{x \to 0} \dfrac{\log (1 + \alpha x)}{x}$

(13) $\lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^{n + 5}$

(14) Evaluate the left and right limits of $f(x) = \dfrac{x^3 - 27}{x - 3}$ at $x = 3$. Does the limit of $f(x)$ as $x \to 3$ exist? Justify your answer.

(15) Find the positive integer n such that $\lim\limits_{x \to 3} \dfrac{x^n - 3^n}{x - 3} = 108$.

(16) Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$. Hint : Take e^x or $e^{\sin x}$ as common factor in numerator.

(17) If $f(x) = \dfrac{ax^2 + b}{x^2 - 1}$, $\lim\limits_{x \to 0} f(x) = 1$ and $\lim\limits_{x \to \infty} f(x) = 1$,

then prove that $f(-2) = f(2) = 1$.

(18) Evaluate $\lim\limits_{x \to 0^-} \dfrac{|x|}{x}$ and $\lim\limits_{x \to 0^+} \dfrac{|x|}{x}$.

What can you say about $\lim\limits_{x \to 0} \dfrac{|x|}{x}$?

(19) Compute $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x}$, $a, b > 0$. Hence evaluate $\lim\limits_{x \to 0} \dfrac{5^x - 6^x}{x}$

(20) Without using the series expansion of $\log(1 + x)$,

prove that $\lim\limits_{x \to 0} \dfrac{\log(1 + x)}{x} = 1$

8.2 Continuity of a function

Let f be a function defined on an interval $I = [a, b]$. A continuous function on I is a function whose graph $y = f(x)$ can be described by the motion of a particle travelling along it from the point $(a, f(a))$ to the point $(b, f(b))$ without moving off the curve.

Continuity at a point

Definition: A function f is said to be continuous at a point c, $a < c < b$, if

$$\lim_{x \to c} f(x) = f(c)$$

A function f is said to be continuous from the left at c if $\lim\limits_{x \to c^-} f(x) = f(c)$.

Also f is continuous from the right at c if $\lim\limits_{x \to c^+} f(x) = f(c)$. Clearly a function is continuous at c if and only if it is continuous from the left as well as from the right.

Continuity at an end point

A function f defined on a closed interval $[a, b]$ is said to be continuous at the end point a if it is continuous from the right at a, that is,

$$\lim_{x \to a+} f(x) = f(a).$$

Also the function is continuous at the end point b of $[a, b]$ if

$$\lim_{x \to b-} f(x) = f(b).$$

It is important to note that a function is continuous at a point c if

(i) f is well defined at $x = c$ i.e. $f(c)$ exists. (ii) $\lim\limits_{x \to c} f(x)$ exists, and (iii) $\lim\limits_{x \to c} f(x) = f(c)$.

Continuity in an interval

A function f is said to be continuous in an interval $[a, b]$ if it is continuous at each and every point of the interval.

Discontinuous functions

A function f is said to be discontinuous at a point c of its domain if it is not continuous at c. The point c is then called a point of discontinuity of the function.

Theorem 8.1: If f, g be continuous functions at a point c then the functions
$f + g$, $f - g$, fg are also continuous at c and if $g(c) \neq 0$ then f / g is also continuous at c.

Example 8.29: Every constant function is continuous.

Solution: Let $f(x) = k$ be the constant function.

Let c be a point in the domain of f.

Then $f(c) = k$.

Also $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (k) = k$,

Thus $\lim\limits_{x \to c} f(x) = f(c)$.

Hence $f(x) = k$ is continuous at c.

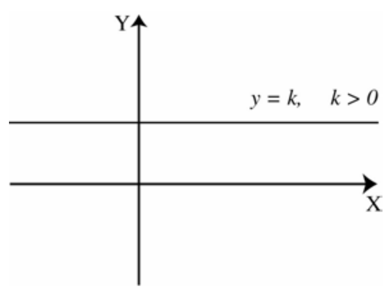


Fig. 8.3

Note : The graph of $y = f(x) = k$ is a straight line parallel to x-axis and which does not have any break. That is, continuous functions are functions, which do not admit any break in its graph.

Example 8.30: The function $f(x) = x^n$, $x \in \mathrm{R}$ is continuous.

Solution. Let x_0 be a point of R.

$$\text{Then} \quad \lim_{x \to x_0} f(x) = \lim_{x \to x_0} (x^n) = \lim_{x \to x_0} (x.\, x \ldots n \text{ factors})$$

$$= \lim_{x \to x_0} (x). \lim_{x \to x_0} (x) \cdots \lim_{x \to x_0} (x) \ldots (n \text{ factors})$$

$$= x_0.x_0 \ldots x_0 \qquad (n \text{ factors}) = x_0^{\,n}$$

$$\text{Also} \quad f(x_0) = x_0^{\,n}. \ \text{Thus} \ \lim_{x \to x_0} f(x) = f(x_0) = x_0^{\,n}$$

$$\Rightarrow \quad f(x) = x^n \text{ is continuous at } x_0$$

Example 8.31: The function $f(x) = kx^n$ is continuous where $k \in \mathrm{R}$ and $k \neq 0$.

Solution. Let $g(x) = k$ and $h(x) = x^n$.

By the example 8.29, g is continuous and by example 8.30, h is continuous and hence by Theorem 8.1, $f(x) = g(x) \,.\, h(x) = kx^n$ is continuous.

Example 8.32: Every polynomial function of degree n is continuous.

Solution. Let $f(x) = a_0x^n + a_1 x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1} x + a_n$, $a_0 \neq 0$ be a polynomial function of degree n.

Now by example 8.31 a_ix^i, $i = 0, 1, 2, \ldots n$ are continuous. By theorem 8.1 sum of continuous functions is continuous and hence the function $f(x)$ is continuous.

Example 8.33: Every rational function of the form $p(x) / q(x)$ where $p(x)$ and $q(x)$ are polynomials, is continuous ($q(x) \neq 0$).

Solution. Let $r(x) = p(x) / q(x)$, $q(x) \neq 0$ be a rational function of x. Then we know that $p(x)$ and $q(x) \neq 0$ are polynomials. Also, $p(x)$ and $q(x)$ are continuous, being polynomials. Hence by theorem 8.1 the quotient $p(x) / q(x)$ is continuous. i.e. the rational function $r(x)$ is continuous.

Results without proof :

(1) The exponential function is continuous at all points of R.
In particular the exponential function $f(x) = e^x$ is continuous.

(2) The function $f(x) = \log x$, $x > 0$ is continuous at all points of R^+, where R^+ is the set of positive real numbers.

(3) The sine function $f(x) = \sin x$ is continuous at all points of R.

(4) The cosine function $f(x) = \cos x$ is continuous at all points of R.

Note : One may refer the SOLUTION BOOK for proof.

Example 8.34: Is the function $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 1 . & \text{when } x = 0 \end{cases}$ continuous at $x = 0$?

Justify your answer.

Solution. Note that $f(0) = 1$.

$$\text{Now} \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} \quad \left(\because \text{for } x \neq 0,\ f(x) = \frac{\sin 2x}{x}\right)$$

$$= \lim_{x \to 0} 2\left(\frac{\sin 2x}{2x}\right) = 2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)$$

$$= 2 \lim_{2x \to 0} \left(\frac{\sin 2x}{2x}\right) = 2.1 = 2 .$$

Since $\lim_{x \to 0} f(x) = 2 \neq 1 = f(0)$, the function is not continuous at $x = 0$. That is, the function is discontinuous at $x = 0$.

Note that the discontinuity of the above function can be removed if we define

$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ so that for this function $\lim_{x \to 0} f(x) = f(0)$.

Such points of discontinuity are called removable discontinuities.

Example 8.35:. Investigate the continuity at the indicated point:

$$f(x) = \begin{cases} \dfrac{\sin (x - c)}{x - c} & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases} \quad \text{at } x = c$$

Solution. We have $f(c) = 0$.

$$\text{Now} \lim_{x \to c} f(x) = \lim_{x \to c} \frac{\sin (x - c)}{x - c} = \lim_{h \to 0} \frac{\sin h}{h} \quad (\because h = x - c \to 0 \text{ as } x \to c)$$

$$= 1 .$$

Since $f(c) = 0 \neq 1 = \lim_{x \to c} f(x)$, the function $f(x)$ is discontinuous at $x = c$.

Note: This discontinuity can be removed by re-defining the function as

$$f(x) = \begin{cases} \dfrac{\sin(x - c)}{x - c} & \text{if } x \neq c \\ 1 & \text{if } x = c \end{cases}$$

Thus the point $x = c$ is a removable discontinuity.

Example 8.36: A function f is defined on by $f(x) = \begin{cases} -x^2 & \text{if } x \le 0 \\ 5x-4 & \text{if } 0 < x \le 1 \\ 4x^2-3x & \text{if } 1 < x < 2 \\ 3x+4 & \text{if } x \ge 2 \end{cases}$

Examine f for continuity at $x = 0, 1, 2$.

Solution.

(i) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x^2) = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (5x-4) = (5.0-4) = -4$

Since $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$, $f(x)$ is discontinuous at $x = 0$

(ii) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (5x-4) = 5 \times 1 - 4 = 1.$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (4x^2-3x) = 4 \times 1^2 - 3 \times 1 = 1$

Also $f(1) = 5 \times 1 - 4 = 5 - 4 = 1$

Since $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = f(1)$, $f(x)$ is continuous at $x = 1$.

(iii) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (4x^2-3x)$

$= 4 \times 2^2 - 3 \times 2 = 16 - 6 = 10$.

and $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3x+4) = 3 \times 2 + 4 = 6 + 4 = 10$.

Also $f(2) = 3 \times 2 + 4 = 10$.

Since $f(2) = \lim\limits_{x \to 2} f(x)$, the function $f(x)$ is continuous at $x = 2$.

Example 8.37: Let $\lfloor x \rfloor$ denote the greatest integer function. Discuss the continuity at $x = 3$ for the function $f(x) = x - \lfloor x \rfloor$, $x \ge 0$.

Solution. Now $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} x - \lfloor x \rfloor = 3 - 2 = 1,$

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} x - \lfloor x \rfloor = 3 - 3 = 0,$

and $f(3) = 0$.

Note that $f(3) = \lim\limits_{x \to 3^+} f(x) \neq \lim\limits_{x \to 3^-} f(x)$.

Hence $f(x) = x - \lfloor x \rfloor$ is discontinuous at $x = 3$.

EXERCISE 8.2

Examine the continuity at the indicated points

(1) $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$ at $x = 2$

(2) $f(x) = x - |x|$ at $x = 0$

(3) $f(x) = \begin{cases} 2x \text{ when } 0 \leq x < 1 \\ 3 \text{ when } x = 1 \\ 4x \text{ when } 1 < x \leq 2 \end{cases}$ at $x = 1$

(4) $f(x) = \begin{cases} 2x - 1, & \text{if } x < 0 \\ 2x + 6, & \text{if } x \geq 0 \end{cases}$ at $x = 0$

(5) Find the values of a and b so that the function f given by

$f(x) = \begin{cases} 1, \text{ if } x \leq 3 \\ ax + b, \text{ if } 3 < x < 5 \\ 7, \text{ if } x \geq 5 \end{cases}$ is continuous at $x = 3$ and $x = 5$

(6) Let f be defined by $f(x) = \begin{cases} \dfrac{x^2}{2}, \text{ if } 0 \leq x \leq 1 \\ 2x^2 - 3x + \dfrac{3}{2}, \text{ if } 1 < x \leq 2 \end{cases}$

Show that f is continuous at $x = 1$.

(7) Discuss continuity of the function f, given by $f(x) = |x - 1| + |x - 2|$, at $x = 1$ and $x = 2$.

8.3 Concept of Differentiation

Having defined and studied limits, let us now try and find the exact rates of change at a point. Let us first define and understand what are increments?

Consider a function $y = f(x)$ of a variable x. Suppose x changes from an initial value x_0 to a final value x_1 . Then the increment in x is defined to be the amount of change in x. It is denoted by Δx (read as delta x).That is $\Delta x = x_1 - x_0$.

Thus $x_1 = x_0 + \Delta x$

If x increases then $\Delta x > 0$, since $x_1 > x_0$.

If x decreases then $\Delta x < 0$, since $x_1 < x_0$.

As x changes from x_0 to $x_1 = x_0 + \Delta x$, y changes from $f(x_0)$ to $f(x_0 + \Delta x)$. We put $f(x_0) = y_0$ and $f(x_0 + \Delta x) = y_0 + \Delta y$. The increment in y namely Δy depends on the values of x_0 and Δx. Note that Δy may be either positive, negative or zero (depending on whether y has increased, decreased or remained constant when x changes from x_0 to x_1).

If the increment Δy is divided by Δx, the quotient $\frac{\Delta y}{\Delta x}$ is called the average rate of change of y with respect to x, as x changes from x_0 to $x_0 + \Delta x$. The quotient is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

This fraction is also called a difference quotient.

Example 8.38: A worker is getting a salary of Rs. 1000/- p.m. She gets an increment of Rs. 100/- per year. If her house rent is half her salary, what is the annual increment in her house rent? What is the average rate of change of the house rent with respect to the salary?

Solution:

Let the salary be given by x and the house rent by y. Then $y = \frac{1}{2}x$. Also $\Delta x = 100$. Therefore, $\Delta y = \frac{1}{2}(x + \Delta x) - \frac{1}{2}x = \frac{\Delta x}{2} = \frac{100}{2} = 50$.

Thus, the annual increment in the house rent is Rs. 50/-.

Then the required average rate of change is $\frac{\Delta y}{\Delta x} = \frac{50}{100} = \frac{1}{2}$.

Example 8.39: If $y = f(x) = \frac{1}{x}$, find the average rate of change of y with respect to x as x changes from x_1 to $x_1 + \Delta x$.

Solution: $\Delta y = f(x_1 + \Delta x) - f(x_1) = \frac{1}{x_1 + \Delta x} - \frac{1}{x_1}$

$$= \frac{-\Delta x}{x_1(x_1 + \Delta x)}$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{-1}{x_1(x_1 + \Delta x)}.$$

8.3.1 The concept of derivative

We consider a point moving in a straight line. The path s traversed by the point, measured from some definite point of the line, is evidently a function of time,

$$s = f(t) .$$

A corresponding value of s is defined for every definite value of t. If t receives an increment Δt, the path $s + \Delta s$ will then correspond to the new instant $t + \Delta t$, where Δs is the path traversed in the interval Δt.

In the case of uniform motion, the increment of the path is proportional to the increment of time, and the ratio $\frac{\Delta s}{\Delta t}$ represents the constant velocity of the motion. This ratio is in general dependent both on the choice of the instant t and on the increment Δt, and represents the average velocity of the motion during the interval from t to $t + \Delta t$.

The limit of the ratio $\frac{\Delta s}{\Delta t}$, if it exists with Δt tending to zero, defines the velocity v at the given instant : $v = \lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$. That is $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ is the instantaneous velocity v.

The velocity v, like the path s, is a function of time t; this function is called the derivative of function $f(t)$ with respect to t, thus, the velocity is the derivative of the path with respect to time.

Suppose that a substance takes part in certain chemical reaction. The quantity x of this substance, taking part in the reaction at the instant t, is a function of t. There is a corresponding increment Δx of magnitude x for an increment of time Δt, and the ratio $\frac{\Delta x}{\Delta t}$ gives the average speed of the reaction in the interval Δt while the limit of this ratio as Δt tends to zero gives the speed of the chemical reaction of the given instant t.

The above examples lead us to the following concept of the derivative of a function.

Definition

The derivative of a given function $y = f(x)$ is defined as the limit of the ratio of the increment Δy of the function to the corresponding increment Δx of the independent variable, when the latter tends to zero.

The symbols y' or $f'(x)$ or $\frac{dy}{dx}$ are used to denote derivative:

$$\frac{dy}{dx} = y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It is possible for the above limit, not to exist in which case the derivative does not exist. We say that the function $y = f(x)$ is differentiable if it has a derivative.

Note.

(1) The operation of finding the derivative is called differentiation.

Further it should be noted, the notation $\frac{dy}{dx}$ does not mean $dy \div dx$. It simply means $\frac{d(y)}{dx}$ or $\frac{d}{dx}$ $f(x)$, the symbol $\frac{d}{dx}$ is an operator meaning that differentiation with respect to x whereas the fraction $\frac{\Delta y}{\Delta x}$ stands for $\Delta y \div \Delta x$. Although the notation $\frac{dy}{dx}$ suggests the ratio of two numbers dy and dx (denoting infinitesimal changes in y and x), it is really a single number, the limit of a ratio $\frac{\Delta y}{\Delta x}$ as both the terms approach 0.

(2) The differential coefficient of a given function $f(x)$ for any particular value of x say x_0 is denoted by $f'(x_0)$ or $\left(\frac{dy}{dx}\right)_{x = x_0}$ and stands for

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$ provided this limit exists.

(3) If the limit of $\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists when $\Delta x \to 0$ from the right hand side i.e. $\Delta x \to 0$ through positive values alone, it is known as right or progressive differential coefficient and is denoted by $f'(x_0+) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \mathrm{R}f'(x_0)$. Similarly the limit of $\frac{f(x_0 - \Delta x) - f(x_0)}{- \Delta x}$ as $\Delta x \to 0$ from the left hand side i.e. from negative values alone is known as the left or regressive differential coefficient and is denoted by

$$f'(x_0-) = \lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} = Lf'(x_0) .$$

If $Rf'(x_0) = Lf'(x_0)$, then f is said to be differentiable at $x = x_0$ and the common value is denoted by $f'(x_0)$. If $R f'(x_0)$ and $Lf'(x_0)$ exist but are unequal, then $f(x)$ is not differentiable at x_0. If none of them exists then also $f(x)$ is not differentiable at x_0.

Geometrically this means that the graph of the function has a corner and hence no tangent at the point $(x_0, f(x_0))$.

8.3.2 Slope or gradient of a curve $\left(\text{Geometrical meaning of } \frac{dy}{dx}\right)$

In this section we shall define what we mean by the slope of a curve at a point P on the curve.

Let P be any fixed point on a curve $y = f(x)$, and let Q be any other point on the same curve. Let PQ be the corresponding secant. If we let Q move along the curve and approach P, the secant PQ will in general rotate about the point P and may approach a limiting position PT. (Fig 8.4).

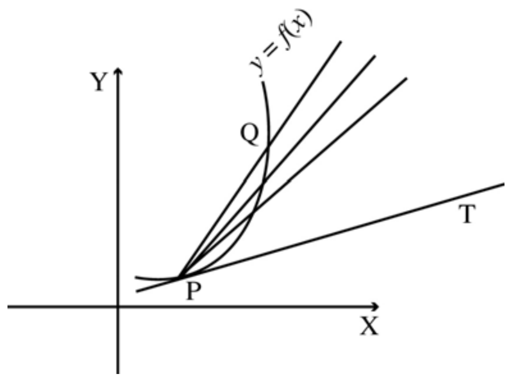


Fig. 8.4

Definition

The tangent to a curve at a point P on the curve is the limiting position PT of a secant PQ as the point Q approaches P by moving along the curve, if this limiting position exists and is unique.

If P_0 is (x_0, y_0) and P is $(x_0 + \Delta x, y_0 + \Delta y)$ are two points on a curve defined by $y = f(x)$, as in Fig. 8.5, then the slope of the secant through these two points is given by

$m' = \tan \alpha_0' = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{-\Delta x}$, where α_0' is the inclination of the secant.

As Δx approaches 0, P moves along the curve towards P_0 ; and if $f'(x_0)$ exists, the slope of the tangent at P_0 is the limit of the slope of the secant P_0 P, or

$m_0 = \tan\alpha_0 = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0) = \left(\frac{dy}{dx}\right)_{x = x_0}$ where α_0 is the inclination of the tangent P_0T and m_0 is its slope. The slope of the tangent to a curve at a point P_0 is often called the slope of the curve at that point.

Thus, geometrically we conclude that the difference ratio (or the difference coefficient) $\frac{\Delta y}{\Delta x}$ is the slope of the secant through the point $P_0(x_0,y_0)$ whereas the differential coefficient or the derivative of $y = f(x)$ at $x = x_0$ is the slope or gradient of the tangent to the curve at $P_0(x_0,y_0)$.

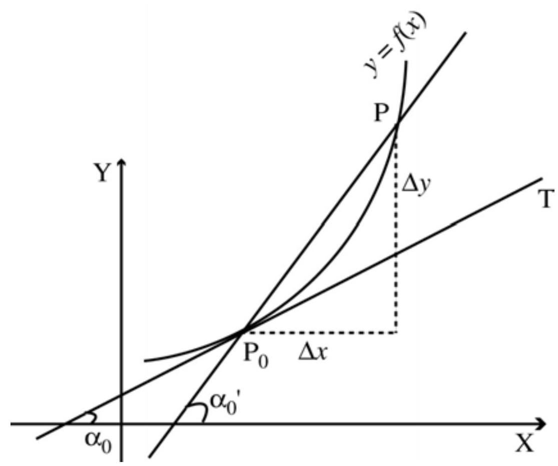


Fig. 8.5

Definition

If $f(x)$ is defined in the interval $x_0 \le x < b$, its right hand derivative at x_0 is defined as $f'(x_0+) = \lim_{x \to x_0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ provided this limit exists; if $f(x)$ is defined in the interval $a < x \le x_0$ its left hand derivative at x_0 is defined as

$$f'(x_{0-}) = \lim_{x \to x_0 -} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x}$$ provided this limit exists.

If $f(x)$ is defined in the interval $a \le x \le b$, then we can write $f'(a)$ for $f'(a+)$, and we write $f'(b)$ for $f'(b\_)$

Relationship between differentiability and continuity.

Theorem 8.2 Every differentiable function is continuous.

Proof. Let a function f be differentiable at $x = c$. Then $f'(c)$ exists and $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

Now $f(x) - f(c) = (x - c)\ \dfrac{[f(x) - f(c)]}{(x - c)}$, $x \neq c$

Taking limit as $x \to c$, we have

$$\lim_{x \to c} \{f(x) - f(c)\} = \lim_{x \to c} (x - c) . \frac{[f(x) - f(c)]}{(x - c)}$$

$$= \lim_{x \to c} (x - c) . \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c} (x - c) . f'(c) = 0. f'(c) = 0.$$

Now $f(x) = f(c) + [f(x) - f(c)]$ and $\lim\limits_{x \to c} f(x) = f(c) + 0 = f(c)$

and therefore f is continuous at $x = c$.

The converse need not be true. i.e. a function which is continuous at a point need not be differentiable at that point. We illustrate this by the following example.

Example 8.40: A function $f(x)$ is defined in an interval [0, 2] as follows :

$$f(x) = x \text{ when } 0 \leq x \leq 1$$
$$= 2x - 1 \text{ when } 1 < x \leq 2$$

Show that $f(x)$ is continuous at 1 but not differentiable at that point.

The graph of this function is as shown in fig. 8.6

This function is continuous at $x = 1$.

For, $\lim\limits_{x \to 1-} f(x) = \lim\limits_{h \to 0} f(1 - h)$

$$= \lim_{h \to 0} (1 - h)$$
$$= 1 - 0 = 1$$

$$\lim_{x \to 1_+} f(x) = \lim_{h \to 0} f(1 + h)$$
$$= \lim_{h \to 0} \big(2(1 + h) - 1\big)$$
$$= \lim_{h \to 0} (2h + 1)$$
$$= 1 .$$

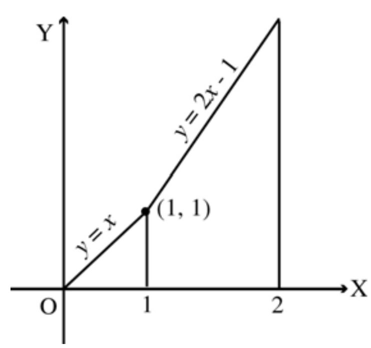


Fig. 8.6

Thus $f(x)$ is continuous at $x = 1$

$$\text{Now} \quad Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[2(1+h) - 1] - [2(1) - 1]}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \text{ and}$$

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{(1-h) - 1} = \lim_{h \to 0} \frac{(1-h) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h} = 1.$$

Since $Rf'(1) \neq Lf'(1)$, the given function is not differentiable at $x = 1$. Geometrically this means that the curve does not have a tangent line at the point (1, 1).

Example 8.41:

Show that the function $y = x^{1/3} = f(x)$ is not differentiable at $x = 0$.

[This function is defined and continuous for all values of the independent variable x. The graph of this function is shown in fig. 8.7]

Solution:

This function does not have derivative at $x = 0$

For, we have $y + \Delta y = \sqrt[3]{x + \Delta x}$

$\Delta y = \sqrt[3]{x + \Delta x} - \sqrt[3]{x}$

At $x = 0$, $y = 0$ and $\Delta y = \sqrt[3]{\Delta x}$.

$$\text{Now} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x}$$

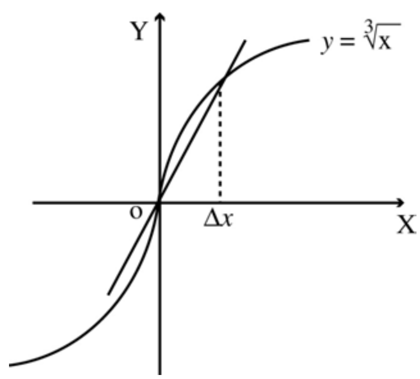


Fig. 8.7

$$= \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt[3]{(\Delta x)^2}} = +\infty.$$

Consequently this function is not differentiable at the point $x = 0$. The tangent to the curve at this point forms with the x-axis, an angle $\frac{\pi}{2}$, which means that it coincides with the y-axis.

Example 8.42: Show that the function $f(x) = x^2$ is differentiable on [0, 1].

Solution. Let c be any point such that $0 < c < 1$.

Then $$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c .$$

At the end points we have

$$f'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x^2}{x} = \lim_{x \to 0} (x) = 0$$

and $$f'(1) = \lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1-} (x + 1) = 2 .$$

Since the function is differentiable at each and every point of [0, 1], $f(x) = x^2$ is differentiable on [0, 1].

EXERCISE 8.3

(1) A function f is defined on R^+ by $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$.

Show that $f'(1)$ does not exist.

(2) Is the function $f(x) = | x |$ differentiable at the origin. Justify your answer.

(3) Check the continuity of the function $f(x) = |x| + |x - 1|$ for all $x \in R$. What can you say its differentiability at $x = 0$, and $x = 1$?

(4) Discuss the differentiability of the functions

(i) $f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ x, & x > 1 \end{cases}$ at $x = 1$ (ii) $f(x) = \begin{cases} 2x - 3, & 0 \leq x \leq 2 \\ x^2 - 3, & 2 < x \leq 4 \end{cases}$ at $x = 2, x = 4$

(5) Compute $Lf'(0)$ and $Rf'(0)$ for the function $f(x) = \begin{cases} \dfrac{x(e^{1/x} - 1)}{(e^{1/x} + 1)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

8.4. Differentiation Techniques

In this section we discuss different techniques to obtain the derivatives of given functions. In order to find the derivative of a function $y = f(x)$ from first principles (on the basis of the general definition of a derivative) it is necessary to carry out the following operations :

1) increase the argument x by Δx, calculate the increased value of the function

   $y + \Delta y = f(x + \Delta x)$.
2) find the corresponding increment of the function $\Delta y = f(x + \Delta x) - f(x)$;
3) form the ratio of the increment of the function to the increment of the argument $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$;
4) find the limit of this ratio as $\Delta x \to 0$;

   $$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We shall apply this general method for evaluating the derivatives of certain elementary (standard) functions. As a matter of convenience we denote $\dfrac{dy}{dx} = f'(x)$ by y'.

8.4.1 Derivatives of elementary functions from first principles

I. The derivative of a constant function is zero.

That is, $\dfrac{d}{dx}(c) = 0$, where c is a constant … (1)

Proof. Let $f(x) = c$ Then $f(x + \Delta x) = c$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\therefore \frac{d}{dx}(c) = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0 .$$

II. The derivative of x^n is nx^{n-1}, where n is a rational number

i.e. $\dfrac{d}{dx}(x^n) = nx^{n-1}$. … (2)

Proof: Let $f(x) = x^n$. Then $f(x + \Delta x) = (x + \Delta x)^n$

Now $$\frac{d\,f(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\therefore \frac{d(x^n)}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} x^n \left[\frac{\left(1+\frac{\Delta x}{x}\right)^n - 1}{\Delta x}\right]$$

$$= x^{n-1} \lim_{\Delta x \to 0} \left[\frac{\left(1+\frac{\Delta x}{x}\right)^n - 1}{\frac{\Delta x}{x}}\right].$$

Put $y = 1 + \frac{\Delta x}{x}$ As $\Delta x \to 0$, $y \to 1$.

$$\therefore \frac{d(x^n)}{dx} = x^{n-1} \lim_{y \to 1} \left(\frac{y^n - 1}{y - 1}\right)$$

$$= n\, x^{n-1}$$

$$= nx^{n-1}. \left[\because \lim_{y \to a} \frac{y^n - a^n}{y - a} = na^{n-1}\right]$$

Note. This result is also true for any real number n.

Example 8.43: If y = x^5 , find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = 5x^{5-1} = 5x^4.$

Exampl 8.44: If $y = x$ find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = 1.x^{1-1} = 1x^\circ = 1$.

Example 8.45: If $y = \sqrt{x}$ find $\frac{dy}{dx}$.

Solution:

Let us represent this function in the form of a power: $y = x^{\frac{1}{2}}$;
Then by formula (II) we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}\ .$$

Example 8.46: If $y = \frac{1}{x\sqrt{x}}$, find $\frac{dy}{dx}$.

Solution:

Represent y in the form of a power. i.e. $y = x^{-\frac{3}{2}}$.

$$\text{Then} \quad \frac{dy}{dx} = -\frac{3}{2}\, x^{-\frac{3}{2} - 1} = -\frac{3}{2}\, x^{-\frac{5}{2}}$$

III. The derivative of sin$x$ is cos$x$

i.e. if $y = \sin x$ then $\frac{dy}{dx} = \cos x$ … (3)

Proof:

Let $y = \sin x$. Increase the argument x by Δx, then

$y + \Delta y = \sin(x + \Delta x)$

$$\Delta y = \sin(x + \Delta x) - \sin x = 2 \sin \frac{(x + \Delta x - x)}{2} \cos \frac{(x + \Delta x + x)}{2}$$

$$= 2 \sin \frac{\Delta x}{2} \,.\, \cos\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = \frac{2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} = \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cos\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \,.\, \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right)$$

$$= 1. \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) .$$

Since $f(x) = \cos x$ is continuous

$$= 1. \cos x \qquad \lim_{\Delta x \to 0} f(x + \Delta x) = \lim_{\Delta x \to 0} \cos(x + \Delta x)$$

$$= \cos x . \qquad = \cos x$$

IV. The derivative of cos$x$ is $-$ sin$x$

ie. if $y = \cos x$, then $\frac{dy}{dx} = -\sin x$. … (4)

Proof: Let $y = \cos x$ Increase the argument x by the increment Δx.

Then $y + \Delta y = \cos(x + \Delta x)$;

$$\Delta y = \cos(x + \Delta x) - \cos x$$

$$= -2\sin\frac{x + \Delta x - x}{2}\ \sin\frac{x + \Delta x + x}{2}$$

$$= -2\sin\frac{\Delta x}{2}\ \sin\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = -\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \sin\left(x + \frac{\Delta x}{2}\right) ;$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \sin\left(x + \frac{\Delta x}{2}\right)$$

$$= -\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right)$$

Since $\sin x$ is continuous, $\lim\limits_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) = \sin x$ and $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$$\therefore \quad \frac{dy}{dx} = -\sin x .$$

Theorem 8.3

If f and g are differentiable functions of x and c is any constant, then the following are true.

(i) $$\frac{d(cf(x))}{dx} = c\,\frac{d\,(f(x))}{dx} \qquad \dots (5)$$

(ii) $$\frac{d\big(f(x) \pm g(x)\big)}{dx} = \frac{d(f(x))}{dx} \pm \frac{d(g(x))}{dx} . \qquad \dots (6)$$

Example 8.47: If $y = \frac{3}{\sqrt{x}}$, find $\frac{dy}{dx}$

Solution: $$y = 3\,x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 3\left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} = -\frac{3}{2}\,x^{-\frac{3}{2}}$$

Example 8.48: If $y = 3x^4 - 1/\sqrt[3]{x}$, find $\frac{dy}{dx}$

Solution:

$$y = 3x^4 - x^{-1/3}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^4 - x^{-1/3}\right) = 3\,\frac{d(x^4)}{dx} - \frac{d}{dx}\,(x^{-1/3})$$

$$= 3 \times 4x^{4-1} - \left(-\frac{1}{3}\right)\, x^{-\frac{1}{3}-1}$$

$$= 12x^3 + \frac{1}{3}\, x^{-\frac{4}{3}}$$

V. If $y = \log_a x$ then $\frac{dy}{dx} = \frac{1}{x}\ \log_a e$ … **(7)**

Corollary : If $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x}$ … **(8)**

Proof: In the previous result take $a = e$. Then $\frac{d}{dx}\,(\log_e x) = \frac{1}{x}\ \log_e e = \frac{1}{x}\ .\ 1 = \frac{1}{x}$.

Example 8.49: Find y' if $y = x^2 + \cos x$.

Solution: We have $y = x^2 + \cos x$.

Therefore $$y' = \frac{dy}{dx} = \frac{d}{dx}\,(x^2 + \cos x)$$

$$= \frac{d(x^2)}{dx} + \frac{d(\cos x)}{dx}$$

$$= 2x^{2-1} + (-\sin x)$$

$$= 2x - \sin x$$

Example 8.50:

Differentiate $1/\sqrt[3]{x} + \log_5 x + 8$ with respect to x.

Solution: Let $$y = x^{-1/3} + \log_5 x + 8$$

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left(x^{-\frac{1}{3}} + \log_5 x + 8\right)$$

$$= \frac{d\left(x^{-\frac{1}{3}}\right)}{dx} + \frac{d(\log_5 x)}{dx} + \frac{d(8)}{dx}$$

$$= -\frac{1}{3}\, x^{-\frac{1}{3}-1} + \frac{1}{x}\log_5{}^{e} + 0,$$

$$= -\frac{1}{3}\, x^{-\frac{4}{3}} + \frac{1}{x}\log_5 e$$

Example 8.51 : Find the derivative of $x^5 + 4x^4 + 7x^3 + 6x^2 + 2$ w.r. to x .

Solution: Let $\quad y = x^5 + 4x^4 + 7x^3 + 6x^2 + 8x + 2$

$$y' = \frac{d}{dx}\,(x^5 + 4x^4 + 7x^3 + 6x^2 + 8x + 2)$$

$$= \frac{d(x^5)}{dx} + \frac{d(4x^4)}{dx} + \frac{d(7x^3)}{dx} + \frac{d(6x^2)}{dx} + \frac{d\,(8x)}{dx} + \frac{d(2)}{dx}$$

$$= 5x^4 + 4 \times 4x^3 + 7 \times 3x^2 + 6 \times 2x + 8 \times 1 + 0$$

$$= 5x^4 + 16x^3 + 21x^2 + 12x + 8\ .$$

Example 8.52: Find the derivative of $y = e^{7x}$ from first principle.

Solution: We have $\quad y = \mathrm{e}^{7x}$

$$y + \Delta y = e^{7\,(x + \Delta x)}$$

$$\frac{\Delta y}{\Delta x} = \frac{e^{7x}\,.\,\mathrm{e}^{7\Delta x} - \mathrm{e}^{7x}}{\Delta x}$$

$$= \mathrm{e}^{7x}\left(\frac{e^{7\Delta x} - 1}{\Delta x}\right)$$

$$y' = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} e^{7x}\left(\frac{e^{7\Delta x} - 1}{\Delta x}\right) = e^{7x}\lim_{\Delta x \to 0} 7\left(\frac{e^{7\Delta x} - 1}{7\Delta x}\right)$$

$$= 7\,e^{7x}\lim_{t \to 0}\left(\frac{e^{t} - 1}{t}\right) \qquad (\because\ t = 7\Delta x \to 0 \text{ as } \Delta x \to 0)$$

$$= 7\,e^{7x} \times 1 = 7e^{7x}. \qquad \left(\because\ \lim_{t \to 0}\frac{e^{t} - 1}{t} = 1\right)$$

In particular, if $y = e^x$, then $\frac{d}{dx}\,(e^x) = e^x$ … (9)

Similarly we can prove

VI. The derivative of $y = \tan x$ w.r. to x is $y' = \sec^2 x$. … (10)

VII. The derivative of $y = \sec x$ w.r. to x is $y' = \sec x \tan x$ … (11)

VIII. The derivative of $y = \operatorname{cosec} x$ as $y' = -\operatorname{cosec} x \cot x$ … (12)

IX. The derivative of $y = \cot x$ as $y' = -\operatorname{cosec}^2 x$ … (13)

Note : One may refer the SOLUTION BOOK for the proof.

EXERCISE 8.4

1. Find $\frac{dy}{dx}$ if $y = x^3 - 6x^2 + 7x + 6$.
2. If $f(x) = x^3 - 8x + 10$, find $f'(x)$ and hence find $f'(2)$ and $f'(10)$.
3. If for $f(x) = ax^2 + bx + 12$, $f'(2) = 11$, $f'(4) = 15$ find a and b.
4. Differentiate the following with respect to x:

   (i) $x^7 + e^x$ (ii) $\log_7 x + 200$

   (iii) $3 \sin x + 4 \cos x - e^x$ (iv) $e^x + 3\tan x + \log x^6$

   (v) $\sin 5 + \log_{10} x + 2\sec x$ (vi) $x^{-3/2} + 8e + 7 \tan x$

   (vii) $\left(x + \frac{1}{x}\right)^3$ (viii) $\frac{(x-3)(2x^2-4)}{x}$

Theorem 8.4: (Product rule for differentiation)

Let u and v be differentiable functions of x. Then the product function

$$y = u(x)\ v(x) \text{ is differentiable and}$$

$$y' = u(x)\ v'(x) + v(x)\ u'(x) \quad \ldots (14)$$

Proof: We have

$$y = u(x)\ v(x)$$

$$y + \Delta y = u(x + \Delta x)\ v(x + \Delta x)$$

$$\Delta y = u(x + \Delta x)\ v(x + \Delta x) - u(x)\ v(x)$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)\ v(x + \Delta x) - u(x)\ v(x)}{\Delta x}.$$

Adding and subtracting $u(x + \Delta x)\ v(x)$ in the numerator and then re-arranging we get:

$$y' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x)\, v(x + \Delta x) - u(x + \Delta x)\ v(x) + u(x + \Delta x)\ v(x) - u(x)\, v(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)\left[v(x + \Delta x) - v(x)\right] + v(x)\left[u(x + \Delta x) - u(x)\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} u(x+\Delta x) . \lim_{\Delta x \to 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} + v(x) \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

Now, since u is differentiable, it is continuous and hence

$$\lim_{\Delta x \to 0} u(x + \Delta x) = u(x)$$

Since u and v are differentiable we have

$$u'(x) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

and $$v'(x) = \lim_{\Delta x \to 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} .$$

Therefore $y' = u(x)\, v'(x) + v(x)\, u'(x)$.

Similarly, if u, v and w are differentiable and if $y = u(x)\, v(x)\, w(x)$ then

$$y' = u(x)\, v(x)\, w'(x) + u(x)\, v'(x)\, w(x) + u'(x)\, v(x)\, w(x)$$

Note (1). The above product rule for differentiation can be remembered as :

Derivative of the product of two functions

= (1st funct.) (derivative of 2nd funct.)+(2nd funct.) (derivative of 1st funct.).

Note (2). The product rule can be rewritten as follows :

$$\big(u(x) \,.\, v(x)\big)' = u(x) \,.\, v'(x) + v(x) \,.\, u'(x)$$

$$\frac{(u(x) \,.\, v(x))'}{u(x) \,.\, v(x)} = \frac{u'(x)}{u(x)} + \frac{v'(x)}{v(x)} . \qquad \ldots (15)$$

It can be generalised as follows:

If $u_1, u_2, \ldots, u_n$ are differentiable functions with derivatives $u_1', u_2', \ldots, u_n'$ then

$$\frac{(u_1 \,.\, u_2 \ldots u_n)'}{u_1 \,.\, u_2 \ldots u_n} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \frac{u_3'}{u_3} + \ldots + \frac{u_n'}{u_n} . \qquad \ldots (16)$$

Example 8.53: Differentiate $e^x \tan x$ w.r. to x.

Solution: Let $y = e^x \,.\, \tan x$.

Then $$y' = \frac{d}{dx}(e^x \,.\, \tan x) = e^x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(e^x)$$

$$= e^x . \sec^2 x + \tan x \,.\, e^x$$

$$= e^x (\sec^2 x + \tan x) .$$

Example 8.54: If $y = 3x^4 e^x + 2\sin x + 7$ find y'.

Solution: $$y' = \frac{dy}{dx} = \frac{d(3x^4 e^x + 2\sin x + 7)}{dx}$$

$$= \frac{d(3x^4 e^x)}{dx} + \frac{d(2\sin x)}{dx} + \frac{d(7)}{dx}$$

$$= 3\frac{d(x^4 e^x)}{dx} + 2\frac{d(\sin x)}{dx} + 0$$

$$= 3\left[x^4 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^4)\right] + 2\cos x$$

$$= 3\,[x^4 . e^x + e^x . 4x^3] + 2\cos x$$

$$= 3x^3 e^x (x + 4) + 2\cos x .$$

Example 8.55: Differentiate $(x^2 + 7x + 2)\,(e^x - \log x)$ with respect to x.

Solution: Let $y = (x^2 + 7x + 2)\,(e^x - \log x)$

$$y' = \frac{d}{dx}\left[(x^2 + 7x + 2)\,(e^x - \log x)\right]$$

$$= (x^2 + 7x + 2)\frac{d}{dx}(e^x - \log x) + (e^x - \log x)\frac{d}{dx}(x^2 + 7x + 2)$$

$$= (x^2 + 7x + 2)\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(\log x)\right]$$

$$+ (e^x - \log x)\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(7x) + \frac{d}{dx}(2)\right]$$

$$= (x^2 + 7x + 2)\left(e^x - \frac{1}{x}\right) + (e^x - \log x)\,(2x + 7 + 0)$$

$$= (x^2 + 7x + 2)\left(e^x - \frac{1}{x}\right) + (e^x - \log x)\,(2x + 7) .$$

Example 8.56: Differentiate $(x^2 - 1)\,(x^2 + 2)$ w.r. to x using product rule. Differentiate the same after expanding as a polynomial. Verify that the two answers are the same.

Solution: Let $y = (x^2 - 1)\,(x^2 + 2)$

Now $y' = \frac{d}{dx}\,[(x^2 - 1)\,(x^2 + 2)]$

$$= (x^2 - 1)\frac{d}{dx}(x^2 + 2) + (x^2 + 2)\frac{d}{dx}(x^2 - 1)$$

$$= (x^2 - 1)\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2)\right] + (x^2 + 2)\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(-1)\right]$$

$$= (x^2 - 1)\,(2x + 0) + (x^2 + 2)\,(2x + 0)$$

$$= 2x\,(x^2 - 1) + 2x\,(x^2 + 2)$$

$$= 2x\,(x^2 - 1 + x^2 + 2) = 2x\,(2x^2 + 1)\,.$$

Another method

$$y = (x^2 - 1)\,(x^2 + 2) = x^4 + x^2 - 2$$

$$y' = \frac{d}{dx}\,(x^4 + x^2 - 2) = 4x^3 + 2x = 2x\,(2x^2 + 1)$$

We observe that both the methods give the same answer.

Example 8.57: Differentiate $e^x \log x \; \cot x$

Solution: Let $y = e^x \log x \cot x$

$$= u_1 \,.\, u_2 \,.\, u_3 \;\text{(say)}$$

where $u_1 = e^x\,;\, u_2 = \log x,\, u_3 = \cot x.$

$$y' = u_1\, u_2\, u_3' + u_1\, u_3\, u_2' + u_2\, u_3\, u_1'$$

$$= e^x \log x\,(-\operatorname{cosec}^2 x) + e^x \cot x \,.\, \frac{1}{x} + \log x \,.\, \cot x \,.\, e^x$$

$$= e^x \left[\cot x \,.\, \log x + \frac{1}{x}\cot x - \log x \,.\, \operatorname{cosec}^2 x\right]$$

Note: Solve this problem by using Note 2.

EXERCISE 8.5

Differentiate the following functions with respect to x.

(1) $e^x \cos x$ (2) $\sqrt[n]{x} \;\; \log \sqrt{x}\;,\; x > 0$

(3) $6 \sin x \log_{10} x + e$ (4) $(x^4 - 6x^3 + 7x^2 + 4x + 2)\,(x^3 - 1)$

(5) $(a - b \sin x)\,(1 - 2 \cos x)$ (6) $\operatorname{cosec} x \,.\, \cot x$

(7) $\sin^2 x$ (8) $\cos^2 x$

(9) $(3x^2 + 1)^2$ (10) $(4x^2 - 1)\,(2x + 3)$

(11) $(3 \sec x - 4 \operatorname{cosec} x)\,(2 \sin x + 5 \cos x)$

(12) $x^2\, e^x \sin x$ (13) $\sqrt{x}\;\; e^x \log x.$

Theorem: 8.5 (Quotient rule for differentiation)

If u and v are differentiable function and if $v(x) \neq 0$, then

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \ldots (17)$$

i.e. $$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}\ .$$

Exampl 8.58:

Differentiate $\frac{x^2 - 1}{x^2 + 1}$ with respect to x .

Solution:

Let $y = \frac{x^2-1}{x^2+1} = \frac{u}{v}$, $u = x^2 - 1$; $v = x^2 + 1$

$$y' = \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{(x^2+1)\,(x^2-1)' - (x^2-1)\,(x^2+1)'}{(x^2+1)^2} \quad \text{Using (17)}$$

$$= \frac{(x^2+1)\,(2x) - (x^2-1)\,(2x)}{(x^2+1)^2} \quad \frac{\left[(x^2+1) - (x^2-1)\right]2x}{(x^2+1)^2}$$

$$= 2x\frac{2}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}\ .$$

Example 8.59: Find the derivative of $\frac{x^2 + e^x \sin x}{\cos x + \log x}$ with respect to x

Solution:

Let $y = \frac{x^2 + e^x \sin x}{\cos x + \log x} = \frac{u}{v}$, $u = x^2 + e^x \sin x$, $v = \cos x + \log x$

Now $y' = \frac{vu' - uv'}{v^2}$

$$= \frac{(\cos x + \log x)\,(x^2 + e^x \sin x)' - (x^2 + e^x \sin x)\,(\cos x + \log x)'}{(\cos x + \log x)^2}$$

$$= \frac{(\cos x + \log x)\left[(x^2)' + (e^x \sin x)'\right] - (x^2 + e^x \sin x)\left[(\cos x)' + (\log x)'\right]}{(\cos x + \log x)^2}$$

$$= \frac{(\cos x + \log x)\left[2x + e^x \cos x + \sin x\, e^x\right] - (x^2 + e^x \sin x)\left(-\sin x + \frac{1}{x}\right)}{(\cos x + \log x)^2}$$

$$= \frac{(\cos x + \log x)\left[2x + e^x(\cos x + \sin x)\right] - (x^2 + e^x \sin x)\left(\frac{1}{x} - \sin x\right)}{(\cos x + \log x)^2}.$$

Example 8.60: Differentiate $\frac{\sin x + \cos x}{\sin x - \cos x}$ with respect to x.

Solution:

Let $y = \frac{\sin x + \cos x}{\sin x - \cos x} = \frac{u}{v}$, $u = \sin x + \cos x$, $v = \sin x - \cos x$

$$y' = \frac{vu' - uv'}{v^2} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-\left[(\sin x - \cos x)^2 + (\sin x + \cos x)\right]^2}{(\sin x - \cos x)^2}$$

$$= \frac{-\left(\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x\right)}{(\sin x - \cos x)^2}$$

$$= -\frac{2}{(\sin x - \cos x)^2}$$

EXERCISE 8.6

Differentiate the following functions using quotient rule.

(1) $\frac{5}{x^2}$ (2) $\frac{2x - 3}{4x + 5}$ (3) $\frac{x^7 - 4^7}{x - 4}$

(4) $\frac{\cos x + \log x}{x^2 + e^x}$ (5) $\frac{\log x - 2x^2}{\log x + 2x^2}$ (6) $\frac{\log x}{\sin x}$

(7) $\frac{1}{ax^2 + bx + c}$ (8) $\frac{\tan x + 1}{\tan x - 1}$ (9) $\frac{\sin x + x \cos x}{x \sin x - \cos x}$ (10) $\frac{\log x^2}{e^x}$

The derivative of a composite function (Chain rule)

If $u = f(x)$ and $y = F(u)$, then $y = F(f(x))$ is the composition of f and F.

In the expression $y = F(u)$, u is called the intermediate argument.

Theorem 8.6: If $u = f(x)$ has the derivative $f'(x)$ and $y = F(u)$ has the derivative $F'(u)$, then the function of a function $F(f(x))$ has the derivative equal to $F'(u)\, f'(x)$, where in place of u we must substitute $u = f(x)$.

Proof: We have $u = f(x)$, $y = F(u)$.

Now $u + \Delta u = f(x + \Delta x)$, $y + \Delta y = F(u + \Delta u)$

Therefore $\dfrac{\Delta u}{\Delta x} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ and $\dfrac{\Delta y}{\Delta u} = \dfrac{F(u + \Delta u) - F(u)}{\Delta u}$

If $f'(x) = \dfrac{du}{dx} \neq 0$, then $\Delta u, \Delta x \neq 0$.

Since f is differentiable, it is continuous and hence when $\Delta x \to 0$, $x + \Delta x \to x$ and $f(x + \Delta x) \to f(x)$. That is, $\lim\limits_{\Delta x \to 0} (x+\Delta x) = x$ and $\lim\limits_{\Delta x \to 0} f(x+\Delta x) = f(x)$.

Therefore $\lim\limits_{\Delta x \to 0} (u + \Delta u) = \Delta u$

Since $\Delta u \neq 0$ as $\Delta x \to 0$, we may write $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

Since both f and F are continuous functions

we have $\Delta u \to 0$ when $\Delta x \to 0$ and $\Delta y \to 0$ when $\Delta u \to 0$.

Therefore $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta u \to 0} \dfrac{\Delta y}{\Delta u} \cdot \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x}$

$$= y'(u)\ u'(x) = F'(u)\ f'(x) = F'(f(x))\ f'(x) \qquad \ldots (18)$$

This chain rule can further be extended to

i.e. if $y = F(u)$, $u = f(t)$, $t = g(x)$ then

$$\frac{dy}{dx} = F'(u) \,.\, u'(t) \,.\, t'(x)$$

i.e. $$\frac{dy}{dx} = \frac{dF}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx} . \qquad \ldots (19)$$

Example 8.61: Differentiate $\log\sqrt{x}$ with respect to x.

Solution: Let $y = \log\sqrt{x}$

Take $u = \sqrt{x}$, and so $y = \log u$, Then by chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{\mathrm{du}}{\mathrm{dx}}$

Now $\dfrac{dy}{du} = \dfrac{1}{u}$; $\dfrac{du}{dx}$; $\dfrac{1}{2\sqrt{x}}$

Therefore by chain rule $\frac{dy}{dx} = \frac{1}{u} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x} \,.\, 2\sqrt{x}} = \frac{1}{2x}$.

Example 8.62: Differentiate sin (log x)

Solution: Let $y = \sin u$, where $u = \log x$

Then by chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$,

$$\text{Now } \frac{dy}{du} = \cos u \; ; \frac{du}{dx} = \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \cos u \,.\, \frac{1}{x} = \frac{\cos(\log x)}{x} .$$

Example 8.63:

Differentiate $e^{\sin x^2}$

Solution: Let $y = e^{\sin x^2}$; $u = \sin x^2$; $t = x^2$

Then $y = e^u$, $u = \sin t$, $t = x^2$

$\therefore$ By chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx} = e^u \,.\, \cos t .\, 2x$$

$$= e^{\sin x^2} \,.\, \cos(x^2) \,.\, 2x = 2x\, e^{\sin(x^2)} \cos(x^2)$$

$$= 2x\, e^{\sin(x^2)} \cos(x^2) .$$

Example 8.64: Differentiate sin ($ax + b$) with respect to x

Solution: Let $y = \sin(ax + b) = \sin u$, $u = ax + b$

$$\frac{dy}{du} = \cos u \; ; \; \frac{du}{dx} = a$$

$$\therefore \frac{dy}{dx} = \cos u \,.\, a = a \cos(ax + b).$$

EXERCISE 8.7

Differentiate the following functions with respect to *x*

(1) $\log(\sin x)$ (2) $e^{\sin x}$ (3) $\sqrt{1 + \cot x}$

(4) $\tan(\log x)$ (5) $\frac{e^{bx}}{\cos(ax + b)}$ (6) $\log \sec\left(\frac{\pi}{4} + \frac{x}{2}\right)$

(7) $\log \sin(e^x + 4x + 5)$ (8) $\sin\left(x^{\frac{3}{2}}\right)$ (9) $\cos\left(\sqrt{x}\right)$ (10) $e^{\sin(\log x)}$.

8.4.2 Derivatives of inverse functions

If for the function $y = f(x)$ there exists an inverse function $x = \phi(y)$ and if $\frac{dx}{dy} = \phi'(y) \neq 0$, then $y = f(x)$ has derivative $f'(x)$ equal to $\frac{1}{\phi'(y)}$; that is

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \ldots (20)$$

Proof. We have $x = \phi(y)$ Then $\frac{dx}{dx} = \frac{d(\phi(y))}{dx}$

That is, $$1 = \phi'(y)\frac{dy}{dx} \quad \text{(by chain rule)}$$

$$1 = \frac{dx}{dy}\cdot\frac{dy}{dx}. \quad \text{Hence, } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Derivatives of inverse trigonometrical functions.

I. The derivative of $y = \sin^{-1}x$ is $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \ldots (21)$$

Proof: We have $y = \sin^{-1}x$ and $x = \sin y$

$$\text{Then } \frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\frac{d(\sin^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\sqrt{1-x^2}}.$$

II. The derivative of $y = \cos^{-1}x$ is $$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \qquad \ldots (22)$$

Proof: We have $y = \cos^{-1}x$ and $x = \cos y$

$$\therefore \quad \frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$

$$\frac{d(\cos^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{-1}{\sqrt{1-x^2}}.$$

Aliter : We know that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

This implies $\frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\cos^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$

$$\frac{1}{\sqrt{1-x^2}} + \frac{d(\cos^{-1}x)}{dx} = 0 \quad \therefore \frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}} .$$

III. The derivative of the function $\boldsymbol{y = \tan^{-1}x}$ is $\boldsymbol{\frac{dy}{dx} = \frac{1}{1+x^2}}$ … (23)

Proof: We have $y = \tan^{-1}x$ and $x = \tan y$

This implies $x' = \frac{d}{dy}(\tan y) = \sec^2 y = 1 + \tan^2 y = 1 + x^2$

$$y' = \frac{1}{x'} = \frac{1}{1+x^2}$$

IV. The derivative of $\boldsymbol{y = \cot^{-1}x}$ is $\boldsymbol{y' = -\frac{1}{1+x^2}}$. … (24)

Proof: We have $y = \cot^{-1}x$ and $x = \cot y$.

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1+\cot^2 y) = -(1+x^2)$$

$\therefore$ by (20), $\quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{1+x^2}$.

Aliter : We know that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

Differentiating with respect to x on both sides,

$$\frac{d(\tan^{-1}x)}{dx} + \frac{d(\cot^{-1}x)}{dx} = \frac{d\left(\frac{\pi}{2}\right)}{dx}$$

$$\frac{1}{1+x^2} + \frac{d(\cot^{-1}x)}{dx} = 0$$

$$\therefore \frac{d(\cot^{-1}x)}{dx} = -\frac{1}{1+x^2} .$$

V. The derivative of $\boldsymbol{y = \sec^{-1}x}$ is $\boldsymbol{\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}}$ … (25)

Proof: We have $y = \sec^{-1}x$ and $x = \sec y$

$$\frac{dx}{dy} = \sec y \tan y = \sec y \sqrt{\sec^2 y - 1}$$

$\therefore$ by (20), $$\frac{d\,(\sec^{-1}x)}{dx} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x\sqrt{x^2-1}}\ .$$

VI. The derivative of $y = \operatorname{cosec}^{-1}x$ is $\dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$ … (26)

Proof: We have $y = \operatorname{cosec}^{-1}x$ and $x = \operatorname{cosec} y$

$$\frac{dx}{dy} = \frac{d\,(\operatorname{cosec} y)}{dy} = -\operatorname{cosec} y \cot y$$

$$= -\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1} = -x\sqrt{x^2-1}$$

Therefore by (20) $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{x\sqrt{x^2-1}}\ .$$

Example 8.65: Differentiate $y = \sin^{-1}(x^2 + 2x)$ with respect to x.

Solution: We have $y = \sin^{-1}(x^2 + 2x)$

Take $u = x^2 + 2x$ Then $y = \sin^{-1}(u)$, a function of function.

Therefore by chain rule,

$$y' = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\ \frac{d\,(x^2+2x)}{dx},\ \text{by (21)}$$

$$= \frac{1}{\sqrt{1-(x^2+2x)^2}}\,(2x+2) = \frac{2(x+1)}{\sqrt{1-x^2(x+2)^2}}\ .$$

Example 8.66: Find $\dfrac{dy}{dx}$ if $y = \cos^{-1}\left(\dfrac{1-x}{1+x}\right)$.

Solution: We have $y = \cos^{-1}\left(\dfrac{1-x}{1+x}\right)$.

Take $u = \dfrac{1-x}{1+x}$. Therefore $y = \cos^{-1}(u)$, a function of function.

By chain rule $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}\ .$$

$$\therefore\ \frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}}\cdot\frac{d\left(\frac{1-x}{1+x}\right)}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}}\left[\frac{(1+x)\,(-1)-(1-x)\,(1)}{(1+x)^2}\right] = -\frac{1}{\sqrt{1-\left(\frac{1-x}{1+x}\right)^2}}\cdot\frac{-2}{(1+x)^2}$$

$$= -\frac{1}{\frac{\sqrt{(1+x)^2-(1-x)^2}}{1+x}} \quad \frac{-2}{(1+x)^2} = \frac{(1+x)}{\sqrt{4x}} \quad \frac{2}{(1+x)^2} = \frac{1}{\sqrt{x}\,(1+x)} \; .$$

Example 8.67: Find y' if $y = \tan^{-1}(e^x)$

Solution: We have $y = \tan^{-1}(e^x)$. Take $u = e^x$ then $y = \tan^{-1}(u)$.

$$\text{By chain rule,} \quad y' = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \; \frac{d\,(e^x)}{dx} \; = \frac{e^x}{1+e^{2x}} \; .$$

EXERCISE 8.8

Find the derivatives of the following functions:

(1) $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ (2) $\cot^{-1}(e^{x^2})$

(3) $\tan^{-1}(\log x)$ (4) $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

8.4.3 Logarithmic Differentiation

We also consider the differentiation of a function of the form:

$y = u^v$ where u and v are functions of x.

We can write $y = e^{\log u^v} = e^{v \log u}$

Now y falls under the category of function of a function.

$$y' = e^{v\log u} \quad \frac{d\,(v \log u)}{dx}$$

$$= e^{v \log u} \left[v \cdot \frac{1}{u} u' + \log u . v'\right] = u^v \left[\frac{v}{u} u' + v' \log u\right]$$

$$= vu^{v-1} u' + u^v (\log u)\, v'. \qquad \ldots (27)$$

Another method:

$y = u^v$ Taking logarithm on both sides

$\log y = \log u^v \;\Rightarrow\; \log y = v \log u$

Diff. both sides with respect to x

$$\frac{1}{y} \; \frac{dy}{dx} = v \frac{1}{u} \; u' + v' \log u$$

$$\frac{dy}{dx} = y \left(\frac{v}{u} u' + v' \log u\right) = u^v \left(\frac{v}{u} u' + v' \log u\right)$$

Example 8.68: Find the derivative of $y = x^{\alpha}$, α is real .

Solution . We have $y = x^{\alpha}$

Then by (27) $y' = \alpha\, x^{\alpha - 1} \,.\, 1 + x^{\alpha} \,.\, (\log x) \,.\, 0$

$= \alpha x^{\alpha - 1} \quad (\because u = x,\ v = \alpha\ ,\ v' = 0)$

Note: From example (8.74),we observe that the derivative of $x^n = nx^{n-1}$ is true for any real n.

Example 8.69: Find the derivative of $x^{\sin x}$ w.r. to x.

Solution: Let $y = x^{\sin x}$. Here $u = x$; $v = \sin x$; $u' = 1$; $v' = \cos x$.

Therefore by (27), $y' = \dfrac{dy}{dx} = \sin x \,.\, x^{\sin x - 1} \,.\, 1 + x^{\sin x} (\log x) \cos x$

$$= x^{\sin x}\left(\frac{\sin x}{x} + \cos x\,(\log x)\right).$$

Example 8.70: Differentiate : $\dfrac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}$

Solution: Let $y = \dfrac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}$

In such cases we take logarithm on both sides and differentiate.

$$\log y = \log (1-x)\sqrt{x^2+2} - \log (x+3)\sqrt{x-1}$$

$$= \log (1-x) + \frac{1}{2}\log (x^2+2) - \log (x+3) - \frac{1}{2}\log (x-1).$$

Differentiating w.r. to x we get:

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{-1}{1-x} + \frac{2x}{2(x^2+2)} - \frac{1}{x+3} - \frac{1}{2}\cdot\frac{1}{x-1}$$

$$= \frac{x}{x^2+2} + \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{x+3}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{x}{x^2+2} + \frac{1}{2(x-1)} - \frac{1}{x+3}\right].$$

$$= \frac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}\left[\frac{x}{x^2+2} + \frac{1}{2(x-1)} - \frac{1}{x+3}\right]$$

EXERCISE 8.9

Differentiate the following functions w.r. to *x*.

(1) $x^{\sqrt{2}}$ (2) x^{x^2} (3) $x^{\tan x}$ (4) $\sin x^{\ \sin x}$

(5) $(\tan^{-1}x)^{\log x}$ (6) $(\log x)^{\sin^{-1}x}$ (7) $\dfrac{(x^2+2)\,(x+\sqrt{2})}{(\sqrt{x+4})\,(x-7)}$

(8) $(x^2+2x+1)^{\sqrt{x-1}}$ (9) $\dfrac{\sin x \cos (e^x)}{e^x + \log x}$ (10) $x^{\sin x} + (\sin x)^x$

8.4.4 The method of substitution

Sometimes, a substitution facilitates differentiation. Following example will demonstrate this method.

Example 8.71: Differentiate the following w.r. to x

(i) $(ax+b)^n$ (ii) $\log (ax+b)^n$

(iii) $\sin^{-1} \dfrac{2x}{1+x^2}$ (iv) $\cos^{-1} \dfrac{1-x^2}{1+x^2}$ (v) $\sin^2 (ax+b)$

Solution: (i) We have $y = (ax+b)^n$. Put $u = ax+b$. Then $y = u^n$.

Now y is a function of u and u is a function of x. By chain rule,

$$y' = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} . \frac{d\,(ax+b)}{dx}$$

$$= n\,(ax+b)^{n-1} . a = na\,(ax+b)^{n-1}.$$

(ii) Let $y = \log (ax+b)^n$. Put $ax+b = u$. Then as in (i) $y' = \dfrac{na}{ax+b}$.

(iii) Let $y = \sin^{-1} \dfrac{2x}{1+x^2}$. Put $x = \tan\theta$ so that $\theta = \tan^{-1}x$.

$$\therefore\ y = \sin^{-1} \frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta) \qquad \left(\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= 2\theta \qquad (\because \sin^{-1}(\sin\theta) = \theta)$$

$$= 2\tan^{-1} x .$$

$$\therefore \frac{dy}{dx} = 2 . \frac{d}{dx} (\tan^{-1}x) = \frac{2}{1+x^2} .$$

(iv) Let $y = \cos^{-1} \dfrac{1-x^2}{1+x^2}$. Put $x = \tan\theta$.

Then $\theta = \tan^{-1}x$ and $\dfrac{1-x^2}{1+x^2} = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$\therefore y = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2} .$$

(v) Let $y = \sin^2 (ax + b)$. Put $ax + b = u$ and $v = \sin u$

Then $y = v^2$, $v = \sin u$ and $u = ax + b$.

Therefore by chain rule,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = 2\, v \,.\, \cos u \,.\, a$$

$$= 2\, a \sin u \,.\, \cos u = a \sin 2u = a \sin 2\,(ax + b).$$

Example 8.72:

Differentiate (i) $\sin^{-1} (3x - 4x^3)$ (ii) $\cos^{-1} (4x^3 - 3x)$ (iii) $\tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2}\right)$.

Solution:

(i) Let $y = \sin^{-1} (3x - 4x^3)$

put $x = \sin \theta$, so that $\theta = \sin^{-1} x$.

Now $y = \sin^{-1} (3\sin\theta - 4 \sin^3\theta)$

$= \sin^{-1} (\sin 3\theta) = 3\theta = 3 \sin^{-1} x$. ($\because \sin 3\theta = 3 \sin \theta - 4 \sin^3\theta$)

$$\frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-x^2}} = \frac{3}{\sqrt{1-x^2}}$$

(ii) Let $y = \cos^{-1} (4x^3 - 3x)$

Put $x = \cos \theta$, so that $\theta = \cos^{-1} x$.

Now $y = \cos^{-1} (4 \cos^3\theta - 3 \cos \theta)$

$= \cos^{-1} (\cos 3\theta)$ ($\therefore \cos 3\theta = 4\cos^3\theta - 3 \cos \theta$)

$= 3\theta = 3 \cos^{-1} x$.

$$\therefore \frac{dy}{dx} = -\frac{3}{\sqrt{1-x^2}} .$$

(iii) Let $y = tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2}\right)$

Put $x = \tan\theta$, so that $\theta = \tan^{-1} x$.

$y = \tan^{-1} \left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1} (\tan 3\theta) = 3\theta = 3 \tan^{-1} x.$

$$\therefore \frac{dy}{dx} = \frac{3}{1+x^2} .$$

EXERCISE 8.10

Differentiate

(1) $\cos^{-1}\sqrt{\dfrac{1+\cos x}{2}}$ (2) $\sin^{-1}\sqrt{\dfrac{1-\cos 2x}{2}}$ (3) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

(4) $\tan^{-1}\left(\dfrac{\cos x+\sin x}{\cos x-\sin x}\right)$ (5) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ (6) $\tan^{-1}\dfrac{1+x^2}{1-x^2}$

(7) $\tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}$ (8) $\tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

(9) $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ Hint:$\sin^2 x/2+\cos^2 x/2=1$; $\sin x=2\sin x/2\cos x/2$)

8.4.5 Differentiation of parametric functions

Definition

If two variables, say, x and y are functions of a third variable, say, t, then the functions expressing x and y in terms of t are called a parametric functions. The variable 't' is called the parameter of the function.

Let $x = f(t)$, $y = g(t)$ be the parametric equations.

Let Δx, Δy be the increments in x and y respectively corresponding to an increment Δt in t.

Therefore $x + \Delta x = f(t + \Delta t)$ and $y + \Delta y = g(t + \Delta t)$

and $\Delta x = f(t + \Delta t) - f(t)$ $\Delta y = g(t + \Delta t) - g(t)$.

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}\right] = \frac{\lim\limits_{\Delta t \to 0}\frac{\Delta y}{\Delta x}}{\lim\limits_{\Delta t \to 0}\frac{\Delta x}{\Delta t}} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \ldots (28)$$

where $\dfrac{dx}{dt} \neq 0$. Note that $\Delta x \to 0 \Rightarrow f(t + \Delta t) \to f(t) \Rightarrow \Delta t \to 0$.

Example 8.73: Find $\dfrac{dy}{dx}$ when $x = a\cos^3 t$, $y = a\sin^3 t$.

Solution: We have $x = a\cos^3 t$, $y = a\sin^3 t$.

Now $\therefore \dfrac{dx}{dt} = -3a\cos^2 t\sin t$ and $\dfrac{dy}{dt} = 3a\sin^2 t\cos t$.

Therefore by (28) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t} = -\tan t.$$

Example 8.74: Find $\frac{dy}{dx}$, if $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

Solution: We have $\frac{dx}{d\theta} = a(1 + \cos\theta)$ $\quad \frac{dy}{d\theta} = a(0 + \sin\theta)$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}.$$

EXERCISE 8.11

Find $\frac{dy}{dx}$ if x and y are connected parametrically by the equations (without eliminating the parameter) .

(1) $x = a\cos\theta$, $y = b\sin\theta$

(2) $x = at^2$, $y = 2at$

(3) $x = a\sec^3\theta$, $y = b\tan^3\theta$

(4) $x = 4t$, $y = \frac{4}{t}$

(5) $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$

(6) $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$

(7) $x = \frac{3at}{1+t^3}$, $y = \frac{3at^2}{1+t^3}$

8.4.6 Differentiation of implicit functions

If the relation between x and y is given by an equation of the form $f(x, y) = 0$ and this equation is not easily solvable for y, then y is said to be an implicit function of x. In case y is given in terms of x, then y is said to be an explicit function of x. In case of implicit function also, it is possible to get $\frac{dy}{dx}$ by mere differentiation of the given relation, without solving it for y first. The following examples illustrate this method.

Example 8.75: Obtain $\frac{dy}{dx}$ when $x^3 + 8xy + y^3 = 64$.

Solution . We have $x^3 + 8xy + y^3 = 64$.

Differentiating with respect to x on both sides,

$$3x^2 + 8\left[x\frac{dy}{dx} + y \cdot 1\right] + 3y^2\frac{dy}{dx} = 0$$

$$3x^2 + 8y + 8x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3x^2 + 8y) + (8x + 3y^2)\frac{dy}{dx} = 0$$

$$(8x + 3y^2)\frac{dy}{dx} = -(3x^2 + 8y) \qquad \therefore \frac{dy}{dx} = -\frac{(3x^2 + 8y)}{(8x + 3y^2)}$$

Example 8.76: Find $\frac{dy}{dx}$ when $\tan(x + y) + \tan(x - y) = 1$

Solution: We have $\tan(x + y) + \tan(x - y) = 1$.

Differentiating both sides w.r.t. to x,

$$\sec^2(x + y)\left(1 + \frac{dy}{dx}\right) + \sec^2(x - y)\left(1 - \frac{dy}{dx}\right) = 0$$

$$[\sec^2(x + y) + \sec^2(x - y)] + [\sec^2(x + y) - \sec^2(x - y)]\frac{dy}{dx} = 0$$

$$[\sec^2(x + y) - \sec^2(x - y)]\frac{dy}{dx} = -[\sec^2(x + y) + \sec^2(x - y)]$$

$$\therefore \frac{dy}{dx} = -\frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x + y) - \sec^2(x - y)} = \frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x - y) - \sec^2(x + y)}.$$

Example 8.77: Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if $\mathrm{xy} + \mathrm{xe}^{-y} + ye^x = x^2$.

Solution: We have $xy + xe^{-y} + ye^x = x^2$

Differentiating both sides w.r.t. to x,

$$x\frac{dy}{dx} + y.1 + xe^{-y}\left(-\frac{dy}{dx}\right) + e^{-y}.1 + \mathrm{y}.\mathrm{e}^x + e^x\frac{dy}{dx} = 2x$$

$$(y + e^{-y} + ye^x) + (x - xe^{-y} + e^x)\frac{dy}{dx} = 2x$$

$$(ye^x + y + e^{-y} - 2x) + (e^x - xe^{-y} + x)\frac{dy}{dx} = 0$$

$$(e^x - xe^{-y} + x)\frac{dy}{dx} = -(ye^x + y + e^{-y} - 2x)$$

$$\therefore \frac{dy}{dx} = -\frac{(ye^x + y + e^{-y} - 2x)}{(e^x - xe^{-y} + x)} = \frac{(ye^x + y + e^{-y} - 2x)}{(xe^{-y} - e^x - x)}.$$

EXERCISE 8.12

Find $\frac{dy}{dx}$ for the following implicit functions.

(1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (2) $y = \mathrm{x}\, sin\, \mathrm{y}$ (3) $x^4 + y^4 = 4a^2x^3y^3$

(4) $y \tan x - y^2 \cos x + 2x = 0$ (5) $(1 + y^2) \sec x - y \cot x + 1 = x^2$

(6) $2y^2 + \frac{y}{1 + x^2} + \tan^2 x + \sin y = 0$ (7) $xy = \tan(xy)$ (8) $x^m y^n = (x + y)^{m+n}$

(9) $e^x + e^y = e^{x+y}$ (10) $xy = 100(x + y)$ (11) $x^y = y^x$

(12) If $ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$, show that $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0$

8.4.7 Higher order Derivatives.

Let $y = f(x)$ be a differentiable function of x.

Then we know its derivative $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ is called first order derivative of $y = f(x)$ with respect to x. This first order derivative $f'(x)$, a function of x may or may not be differentiable. If $f'(x)$ is differentiable then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$ is called second order derivative of $y = f(x)$ with respect to x. It is denoted by $\frac{d^2y}{dx^2}$.

Other symbols like y_2, y'', $\ddot{y}$ or D^2y where $D^2 = \frac{d^2}{dx^2}$ also used to denote the second order derivative. Similarly, we can define third order derivative of $y = f(x)$ as

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \lim_{\Delta x \to 0} \frac{f''(x + \Delta x) - f''(x)}{\Delta x}$ provided $f''(x)$ is differentiable.

As before, y_3, y''', $\dddot{y}$ or D^3y is used to denote third order derivative.

Example 8.78: Find y_3, if $y = x^2$

Solution:

$$y_1 = \frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2x) = 2$$

$$y_3 = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(2) = 0.$$

Example 8.79:

Let $y = A\cos 4x + B\sin 4x$, A and B are constants. Show that $y_2 + 16y = 0$

Solution:

$$y_1 = \frac{dy}{dx} = (A\cos 4x + B\sin 4x)' = -4A\sin 4x + 4B\cos 4x$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\,(-\,4\text{ A}\sin 4x + 4\text{B}\cos 4\,x)$$

$$= -\,16\text{ A}\cos 4x - 16\text{ B}\sin 4x$$

$$= -\,16\,(\text{A}\cos 4x + \text{B}\sin 4x) = -\,16y$$

$$\therefore\ y_2 + 16y = 0$$

Example 8.80: Find the second derivative of the function log (log x)

Solution: Let $y = \log\ (\log x)$

By chain rule, $\frac{dy}{dx} = \frac{1}{\log x}\cdot\frac{d\,(\log x)}{dx} = \frac{1}{\log x}\cdot\frac{1}{x}$

$$= \frac{1}{x\log x} = (x\log x)^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(x\log x)^{-1}}{dx} = -\,(x\log x)^{-2}\,\frac{d\,(x\log x)}{dx}$$

$$= -\,\frac{1}{(x\log x)^2}\left[x\,.\,\frac{1}{x} + \log x\,.\,1\right] = -\,\frac{1+\log x}{(x\log x)^2}\,.$$

Example 8.81: If $y = \log\,(\cos x)$, find y_3

Solution: We have $y = \log\,(\cos x)$

$$y_1 = \frac{d\,[\log\,(\cos x)]}{dx} = \frac{1}{\cos x}\ \frac{d\,(\cos x)}{dx},\ \text{by chain rule}$$

$$= \frac{1}{\cos x}\,.\,(-\sin x) = -\tan x$$

$$y_2 = \frac{d\,y_1}{dx} = \frac{d\,(-\tan x)}{dx} = -\sec^2 x$$

$$y_3 = \frac{d\,(y_2)}{dx} = \frac{d\,(-\sec^2 x)}{dx} = -\,2\sec x\,.\,\frac{d\,(\sec x)}{dx}$$

$$= -\,2\sec x\,.\,\sec x\,.\,\tan x = -\,2\sec^2 x\tan x.$$

Example 8.82: If $y = e^{ax}\sin bx$, prove that $\frac{d^2y}{dx^2} - 2a\,.\,\frac{dy}{dx} + (a^2 + b^2)\,y = 0$

Solution: We have $y = e^{ax}\sin bx$

$$\frac{dy}{dx} = e^{ax}\,.\,b\cos bx + a\,e^{ax}\sin bx$$

$$= e^{ax}\,(b\cos bx + a\sin bx)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{e^{ax}(b\cos bx + a\sin bx)\right\}$$

$$= e^{ax}\left\{-b^2\sin bx + ab\cos bx\right\} + (b\cos bx + a\sin bx)a\,e^{ax}$$

$$= -b^2(e^{ax}\sin bx) + a\;be^{ax}\cos bx + a.e^{ax}(b\cos bx + a\sin bx)$$

$$= -b^2\;y + a\;\left(\frac{dy}{dx} - ae^{ax}\sin bx\right) + a\,\frac{dy}{dx}$$

$$= -b^2\;y + a\left(\frac{dy}{dx} - a.y\right) + a\,\frac{dy}{dx}$$

$$= 2a\;\frac{dy}{dx} - (a^2 + b^2)y$$

Therefore, $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2 + b^2)y = 0$.

Example 8.83: If $y = \sin(ax + b)$, prove that $y_3 = a^3\sin\left(ax + b + \frac{3\pi}{2}\right)$.

Solution: We have $y = \sin(ax + b)$

$$y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + 2.\frac{\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax + b + 2.\frac{\pi}{2}\right) = a^3\sin\left(ax + b + 2.\frac{\pi}{2} + \frac{\pi}{2}\right) = a^3\sin\left(ax + b + 3\,\frac{\pi}{2}\right)$$

Example 8.84: If $y = \cos(m\sin^{-1}x)$, prove that $(1 - x^2)y_3 - 3xy_2 + (m^2 - 1)y_1 = 0$

Solution: We have $y = \cos(m\sin^{-1}x)$

$$y_1 = -\sin(m\sin^{-1}x)\,.\,\frac{m}{\sqrt{1 - x^2}}$$

$$y_1^{\,2} = \sin^2(m\sin^{-1}x)\,\frac{m^2}{(1 - x^2)}$$

This implies $\quad (1 - x^2)y_1^{\,2} = m^2\sin^2(m\sin^{-1}x) = m^2\left[1 - \cos^2(m\sin^{-1}x)\right]$

That is, $\quad (1 - x^2)\,y_1^{\,2} = m^2(1 - y^2)$.

Again differentiating,

$$(1 - x^2)2y_1\frac{d\,y_1}{dx} + y_1^{\,2}(-2x) = m^2\left(-2y\,\frac{dy}{dx}\right)$$

$$(1 - x^2)\,2y_1y_2 - 2xy_1^{\,2} = -2m^2yy_1$$

$$(1 - x^2)\, y_2 - xy_1 = -\, m^2 y$$

Once again differentiating,

$$(1 - x^2)\frac{d\,y_2}{dx} + y_2\,(-\,2x) - \left[x\,.\frac{d\,y_1}{dx} + y_1\,.\,1\right] = -\,m^2\,\frac{dy}{dx}$$

$$(1 - x^2)\, y_3 - 2xy_2 - xy_2 - y_1 = -\,m^2 y_1$$

$$(1 - x^2)\, y_3 - 3xy_2 + (m^2 - 1)\, y_1 = 0.$$

EXERCISE 8.13

(1) Find $\dfrac{d^2y}{dx^2}$ if $y = x^3 + \tan x$.

(2) Find $\dfrac{d^3y}{dx^3}$ if $y = x^2 + \cot x$.

(3) Find the second order derivative of:

(i) $x^2 + 6x + 5$ (ii) x sinx (iii) $\cot^{-1} x$.

(4) Find the third order derivatives of:

(i) $e^{mx} + x^3$ (ii) $x \cos x$.

(5) If $y = 500\, e^{7x} + 600 e^{-\,7x}$, show that $\dfrac{d^2y}{dx^2} = 49y$.

(6) If $y = e^{\tan^{-1} x}$ prove that $(1 + x^2)\, y_2 + (2x - 1) y_1 = 0$.

(7) If $y = \log\,(x^2 - a^2)$, prove that $y_3 = 2\left[\dfrac{1}{(x + a)^3} + \dfrac{1}{(x - a)^3}\right]$.

(8) If $x = \sin t$; $y = \sin pt$ show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2 y = 0$.

(9) If $x = a\,(\cos\theta + \theta \sin\theta)$, $y = a\,(\sin\theta - \theta\cos\theta)$,

show that $a\,\theta\,\dfrac{d^2y}{dx^2} = \sec^3\theta$.

(10) If $y = (x^3 - 1)$, prove that $x^2\, y_3 - 2xy_2 + 2y_1 = 0$.

TABLE OF DERIVATIVES

	Function	Derivative
1.	k ; (k is a cosntant)	$(k)' = 0$
2.	k$f(x)$	$(kf(x))' = kf'(x)$
3.	$u \pm v$	$(u \pm v)' = u' \pm v'$
4.	$u_1 + u_2 + \ldots + u_n$	$(u_1 + u_2 + \ldots u_n)' = u_1' + u_2' + \ldots + u_n'$
5.	$u \cdot v$	$(uv)' = uv' + vu'$ $\frac{(uv)'}{uv} = \frac{u'}{u} + \frac{v'}{v}$
6.	$u_1.u_2 \ldots u_n$	$(u_1.u_2 .. u_n)' = u_1' u_2 u_3 .. u_n + u_1 u_2' .. u_n$ $+ \ldots + u_1 u_2 \ldots u_{n-1} u_n'$ $\frac{(u_1.u_2 .. u_n)'}{u_1.u_2 .. u_n} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \ldots + \frac{u_n'}{u_n}$
7.	x^n ($n \in$ R)	$(x^n)' = \text{n}x^{n-1}$
8.	$\log_a^x$	$(\log_a^x)' = \frac{\log a^e}{x}$
9.	$\log_e^x$	$(\log x)' = \frac{1}{x}$
10.	$\sin x$	$(\sin x)' = \cos x$
11.	$\cos x$	$(\cos x)' = -\sin x$
12.	$\tan x$	$(\tan x)' = \sec^2 x$
13.	$\cot x$	$(\cot x)' = -\text{cosec}^2 x$
14.	$\sec x$	$(\sec x)' = \sec x \cdot \tan x$
15.	$\text{cosec}\, x$	$(\text{cosec}\, x)' = -\text{cosec}\, x \cdot \cot x$
	Function	**Derivative**
16.	$\sin^{-1} x$	$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}$

17. $\cos^{-1}x$ $\quad (\cos^{-1}x)' = \dfrac{-1}{\sqrt{1-x^2}}$

18. $\tan^{-1}x$ $\quad (\tan^{-1}x)' = \dfrac{1}{1+x^2}$

19. $\cot^{-1}x$ $\quad (\cot^{-1}x)' = -\dfrac{1}{1+x^2}$

20. $\sec^{-1}x$ $\quad (\sec^{-1}x)' = \dfrac{1}{x\sqrt{x^2-1}}$

21. $\mathrm{cosec}^{-1}x$ $\quad (\mathrm{cosec}^{-1}x)' = -\dfrac{1}{x\sqrt{x^2-1}}$

22. $\dfrac{u}{v}$ $\quad \left(\dfrac{u}{v}\right)' = \dfrac{v.u' - u.v'}{v^2}$

23. e^x $\quad (e^x)' = e^x$

24. u^v $\quad (u^v)' = vu^{v-1}.\, u' + u^v\,(\log u)v'$

25. a^x $\quad (a^x)' = a^x(\log a)$

26. $\left.\begin{array}{r} y = f(x) \\ x = \varphi\,(y)\ \text{(inverse of } f) \end{array}\right\}$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$.

27. $y = f(u),\ u = \varphi\,(x)$ $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

28. $\left.\begin{array}{r} y = f(u) \\ u = g(t) \\ t = h(x) \end{array}\right\}$ $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}$.

29. $\left.\begin{array}{r} y = g\,(t) \\ x = f(t) \end{array}\right\}$ $\quad \dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{y'(t)}{x'(t)}$

30. $f(x, y) = k$ $\quad \dfrac{dy}{dx} = \dfrac{f_1\,(x,y)}{f_2\,(x, y)}\ ,\ f_2\,(x, y) \neq 0$

Note : In the above formulae from 1 to 25 $(\,.\,)' = \dfrac{d\,(\,.\,)}{dx}$.

9. INTEGRAL CALCULUS

9.1 Introduction:

Calculus deals principally with two geometric problems.

(i) The problem of finding SLOPE of the tangent line to the curve, is studied by the limiting process known as differentiation and

(ii) Problem of finding the AREA of a region under a curve is studied by another limiting process called Integration.

Actually integral calculus was developed into two different directions over a long period independently.

(i) Leibnitz and his school of thought approached it as the anti derivative of a differentiable function.

(ii) Archimedes, Eudoxus and others developed it as a numerical value equal to the area under the curve of a function for some interval. However as far back as the end of the 17^{th} century it became clear that a general method for solution of finding the area under the given curve could be developed in connection with definite problems of integral calculus.

In the first section of this chapter, we study integration, the process of obtaining a function from its derivative, and in the second we examine certain limit of sums that occur frequently in applications.

We are already familiar with inverse operations. (+, −) ; (×, ÷), $\left((\ )^n,\ \sqrt[n]{\ } \right)$ are some pairs of inverse operations. Similarly differentiation and integrations are also inverse operations. In this section we develop the inverse operation of differentiation called anti differentiation.

Definition

A function F(x) is called an anti derivative or integral of a function $f(x)$ on an interval **I** if

$$F'(x) = f(x) \text{ for every value of x in } \mathbf{I}$$

i.e. If the derivative of a function F(x) w.r.t. to x is $f(x)$, then we say that the integral of $f(x)$ w.r.t. to x is F(x).

i.e. $$\int f(x)\,dx = F(x)$$

For example we know that

$$\frac{d}{dx}(\sin x) = \cos x, \quad \text{then} \int \cos x \, dx = \sin x.$$

Also $$\frac{d}{dx}(x^5) = 5x^4, \quad \text{gives} \int 5x^4 \, dx = x^5$$

The symbol '$\int$' is the sign of integration. '$\int$' is elongated S, which is the first letter of the word sum.

The function ***f*(*x*)** is called **Integrand.**

The variable ***x*** in ***dx*** is called **variable of integration** or **integrator.**

The process of finding the integral is called **integration.**

Constant of integration:

Consider the following two examples.

Example 9.1:

$$\left.\begin{aligned} \frac{d}{dx}(2x+5) &= 2 \\ \frac{d}{dx}(2x) &= 2 \\ \frac{d}{dx}(2x-4) &= 2 \\ \frac{d}{dx}\left(2x-\sqrt{7}\right) &= 2 \end{aligned}\right\} \Rightarrow \int 2dx = 2x + ? = 2x + C$$

Where this 'C' may be 5, 0, – 4 or – $\sqrt{7}$ as shown in the above example. (See fig. 1(*a*)).

Example 9.2:

$$\left.\begin{aligned} \frac{d}{dx}(x^2+1) &= 2x \\ \frac{d}{dx}(x^2) &= 2x \\ \frac{d}{dx}(x^2-4) &= 2x \end{aligned}\right\} \Rightarrow \int 2xdx = x^2 + ? = x^2 + C$$

'C' is any constant (See fig 1(b))

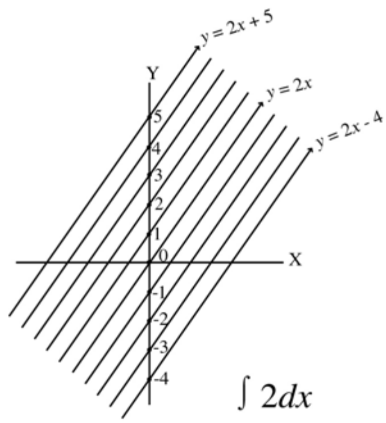


fig. 9.1.a

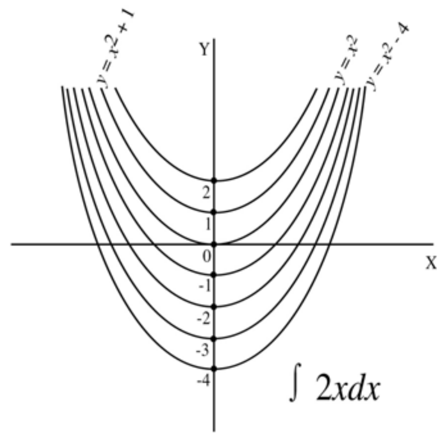


fig 9.1.b

By the way it is accepted to understand that the expression $\int f(x)\,dx$ is not a particular integral, but family of integrals of that function.

If F(x) is one such integral, it is customary to write $\int f(x)\,dx = F(x) + C$

Where 'C' is an arbitrary constant. 'C' is called **'the constant of integration'.** Since C is arbitrary, $\int f(x)\,dx$ is called the **"indefinite integral".**

Formulae

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)\,x^{n-1}} + c \quad (n \neq 1]$$

$$\int \frac{1}{x}\,dx = \log x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{a^x}{\log a} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

Example 9.3 – 9.7: Integrate the following with respect to x.

(3) x^6 (4) x^{-2} (5) $\frac{1}{x^{10}}$ (6) $\sqrt{x}$ (7) $\frac{1}{\sqrt{x}}$

Solution:

(3) $\int x^6\,dx = \frac{x^{6+1}}{6+1} = \frac{x^7}{7} + c$

(4) $\int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x} + c$

(5) $\int \frac{1}{x^{10}}\,dx = \int x^{-10}\,dx$

$= \frac{x^{-10+1}}{-10+1} + c$

$= \frac{x^{-9}}{-9} + c$

$\int \frac{1}{x^{10}}\,dx = -\frac{1}{9x^9} + c$

[Here we can also use the formula

$\int \frac{1}{x^n}\,dx = -\frac{1}{(n-1)\,x^{n-1}}$ where $n \neq 1$]

(6) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$\int \sqrt{x}\,dx = \frac{2}{3}\,x^{\frac{3}{2}} + c$

(7) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx$

$= \frac{x^{-1/2+1}}{-1/2+1} + c$

$= \frac{x^{1/2}}{+1/2} + c$

$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$

Example 9.8 – 9.10: Integrate:

(8) $\frac{\sin x}{\cos^2 x}$ (9) $\frac{\cot x}{\sin x}$ (10) $\frac{1}{\sin^2 x}$

Solution:

(8) $\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx = \int \tan x \sec x\,dx = \sec x + c$

(9) $\int \frac{\cot x}{\sin x}\,dx = \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(10) $\int \frac{1}{\sin^2 x}\,dx = \int \operatorname{cosec}^2 x\,dx = -\cot x + c$

EXERCISE 9.1

Integrate the following with respect to x

(1) (i) x^{16} (ii) $x^{\frac{5}{2}}$ (iii) $\sqrt{x^7}$ (iv) $\sqrt[3]{x^4}$ (v) $(x^{10})^{\frac{1}{7}}$

(2) (i) $\frac{1}{x^5}$ (ii) x^{-1} (iii) $\frac{1}{x^{\frac{5}{2}}}$ (iv) $\frac{1}{\sqrt[3]{x^5}}$ (v) $\left(\frac{1}{x^3}\right)^{\frac{1}{4}}$

(3) (i) $\frac{1}{\operatorname{cosec} x}$ (ii) $\frac{\tan x}{\cos x}$ (iii) $\frac{\cos x}{\sin^2 x}$ (iv) $\frac{1}{\cos^2 x}$ (v) $\frac{1}{e^{-x}}$

9.2 Integrals of function containing linear functions of x i.e. $\int f(ax + b)\, dx$

We know that

$$\frac{d}{dx}\left[\frac{(x-a)^{10}}{10}\right] = (x-a)^9 \quad \Rightarrow \int (x-a)^9 dx = \frac{(x-a)^{10}}{10}$$

$$\frac{d}{dx}[\sin(x+k)] = \cos(x+k) \quad \Rightarrow \int \cos(x+k)\, dx = \sin(x+k)$$

It is clear that whenever a constant is added to the independent variable x or subtracted from x the fundamental formulae remain the same.

But

$$\frac{d}{dx}\left[\frac{1}{\boldsymbol{l}}(e^{\boldsymbol{l}x+m})\right] = e^{\boldsymbol{l}x+m} \quad \Rightarrow \int e^{\boldsymbol{l}x+m}\, dx = \frac{1}{\boldsymbol{l}} e^{(lx+m)}$$

$$\frac{d}{dx}\left[\frac{1}{\boldsymbol{a}}\sin(\boldsymbol{a}x+b)\right] = \cos(\boldsymbol{a}x+b) \Rightarrow \int \cos(\boldsymbol{a}x+b)\, dx = \frac{1}{\boldsymbol{a}}\sin(ax+b)$$

Here, if any constant is multiplied with the independent variable x, then the same fundamental formula can be used after dividing it by the coefficient of x

i.e.

$$\boxed{\text{if } \int f(x)\, dx = g(x) + c, \text{ then } \int f(\boldsymbol{a}x+b)\, dx = \frac{1}{\boldsymbol{a}}\, g(\boldsymbol{a}x+b) + c}$$

The extended forms of fundamental formulae

$$\int (\boldsymbol{ax+b})^{\boldsymbol{n}} \boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\left[\frac{(\boldsymbol{ax+b})^{\boldsymbol{n}+1}}{\boldsymbol{n}+1}\right] + \boldsymbol{c} \quad (\boldsymbol{n} \neq -1)$$

$$\int \frac{\mathbf{1}}{\boldsymbol{ax+b}}\, \boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{a}} \log(\boldsymbol{ax+b}) + \boldsymbol{c}$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$$

$$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$$

$$\int \frac{1}{1+(ax)^2}\,dx = \frac{1}{a}\tan^{-1}(ax) + c$$

$$\int \frac{1}{\sqrt{1-(ax)^2}}\,dx = \frac{1}{a}\sin^{-1}(ax) + c$$

The above formulae can also be derived by using substitution method, which will be studied later.

Example 9.11 – 9.17: Integrate the following with respect to x.

(11) $(3-4x)^7$ (12) $\frac{1}{3+5x}$ (13) $\frac{1}{(lx+m)^n}$ (14) e^{8-4x}

(15) $\sin(lx+m)$ (16) $\sec^2(p-qx)$ (17) $\operatorname{cosec}(4x+3)\ \cot(4x+3)$

Solution:

(11) $$\int (3-4x)^7dx = \left(-\frac{1}{4}\right)\frac{(3-4x)^8}{8} + c$$

$$\int (3-4x)^7dx = -\frac{1}{32}(3-4x)^8 + c$$

(12) $$\int \frac{1}{3+5x}\,dx = \frac{1}{5}\log(3+5x) + c$$

(13) $$\int \frac{1}{(lx+m)^n}\,dx = \left(\frac{1}{l}\right)\left[\frac{(-1)}{(n-1)(lx+m)^{n-1}}\right] + c$$

$$\therefore \int \frac{1}{(lx+m)^n}\,dx = -\left(\frac{1}{l(n-1)}\right)\frac{1}{(lx+m)^{n-1}} + c$$

(14) $\int e^{8-4x} dx = \left(\frac{1}{-4}\right) e^{8-4x} + c$

$\int e^{8-4x} dx = -\frac{1}{4} e^{8-4x} + c$

(15) $\int \sin (lx + m)\, dx = \left(\frac{1}{l}\right) [-\cos (lx + m)] + c$

$= -\frac{1}{l} \cos (lx + m) + c$

(16) $\int \sec^2 (p - qx)\, dx = \left(-\frac{1}{q}\right) [\tan(p - qx)] + c$

(17) $\int \operatorname{cosec} (4x + 3) \cot (4x + 3)\, dx = -\frac{1}{4} \operatorname{cosec} (4x + 3) + c$

EXERCISE 9.2

Integrate the following with respect to x

(1) (i) x^4 (ii) $(x + 3)^5$ (iii) $(3x + 4)^6$ (iv) $(4 - 3x)^7$ (v) $(lx + m)^8$

(2) (i) $\frac{1}{x^6}$ (ii) $\frac{1}{(x + 5)^4}$ (iii) $\frac{1}{(2x + 3)^5}$ (iv) $\frac{1}{(4 - 5x)^7}$ (v) $\frac{1}{(ax + b)^8}$

(3) (i) $\frac{1}{x + 2}$ (ii) $\frac{1}{3x + 2}$ (iii) $\frac{1}{3 - 4x}$ (iv) $\frac{1}{p + qx}$ (v) $\frac{1}{(s - tx)}$

(4) (i) $\sin (x + 3)$ (ii) $\sin (2x + 4)$ (iii) $\sin (3 - 4x)$
(iv) $\cos (4x + 5)$ (v) $\cos (5 - 2x)$

(5) (i) $\sec^2(2 - x)$ (ii) $\operatorname{cosec}^2(5 + 2x)$ (iii) $\sec^2 (3 + 4x)$
(iv) $\operatorname{cosec}^2(7-11x)$ (v) $\sec^2(p - qx)$

(6) (i) $\sec (3 + x) \tan (3 + x)$ (ii) $\sec (3x + 4) \tan (3x + 4)$
(iii) $\sec (4-x) \tan (4 - x)$ (iv) $\sec (4 - 3x) \tan (4 - 3x)$
(v) $\sec (ax + b) \tan (ax + b)$

(7) (i) $\operatorname{cosec} (2 - x) \cot (2 - x)$ (ii) $\operatorname{cosec} (4x + 2) \cot (4x + 2)$
(iii) $\operatorname{cosec} (3-2x) \cot (3 - 2x)$ (iv) $\operatorname{cosec} (lx + m) \cot (lx + m)$
(v) $\cot (s-tx) \operatorname{cosec} (s-tx)$

(8) (i) e^{3x} (ii) $e^{x + 3}$ (iii) $e^{3x + 2}$ (iv) $e^{5 - 4x}$ (v) $e^{ax + b}$

(9) (i) $\frac{1}{\cos^2 (px + a)}$ (ii) $\frac{1}{\sin^2 (l - mx)}$ (iii) $(ax + b)^{-8}$ (iv) $(3 - 2x)^{-1}$ (v) e^{-x}

(10) **(i)** $\dfrac{\tan(3-4x)}{\cos(3-4x)}$ **(ii)** $\dfrac{1}{e^{p+qx}}$ **(iii)** $\dfrac{1}{\tan(2x+3)\sin(2x+3)}$

(iv) $(lx+m)^{\frac{1}{2}}$ **(v)** $\sqrt{(4-5x)}$

Properties of integrals

> **(1)** **If k is any constant then $\int kf(x)\,dx = k\int f(x)\,dx$**
>
> **(2)** **If $f(x)$ and $g(x)$ are any two functions in x then**
> $\int [f(x)+g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$

Example 9.18 – 9.21: Integrate the following with respect to x

(18) $10x^3 - \dfrac{4}{x^5} + \dfrac{2}{\sqrt{3x+5}}$ (19) $k\sec^2(ax+a) - \sqrt[3]{(4x+5)^2} + 2\sin(7x-2)$

(20) $a^x + x^a + 10 - \text{cosec}\,2x\,\cot 2x$ (21) $\dfrac{1}{5}\cos\left(\dfrac{x}{5}+7\right) + \dfrac{3}{(lx+m)} + e^{\frac{x}{2}+3}$

Solution:

(18) $$\int\left(10x^3 - \frac{4}{x^5} + \frac{2}{\sqrt{3x+5}}\right)dx = 10\int x^3\,dx - 4\int\frac{dx}{x^5} + 2\int\frac{1}{\sqrt{3x+5}}\,dx$$

$$= 10\left(\frac{x^4}{4}\right) - 4\left(-\frac{1}{4x^4}\right) + 2\,\frac{\left[2\sqrt{3x+5}\right]}{3}$$

$$= \frac{5}{2}x^4 + \frac{1}{x^4} + \frac{4}{3}\sqrt{3x+5} + c$$

(19) $$\int [k\sec^2(ax+b) - \sqrt[3]{(4x+5)^2} + 2\sin(7x-2)]\,dx$$

$$= k\int \sec^2(ax+b)\,dx - \int (4x+5)^{\frac{2}{3}}\,dx + 2\int \sin(7x-2)\,dx$$

$$= k\,\frac{\tan(ax+b)}{a} - \frac{1}{4}\,\frac{(4x+5)^{\frac{2}{3}+1}}{\left(\frac{2}{3}+1\right)} + (2)\left(\frac{1}{7}\right)(-\cos(7x-2)) + c$$

$$= \frac{k}{a}\tan(ax+b) - \frac{3}{20}(4x+5)^{\frac{5}{3}} - \frac{2}{7}\cos(7x-2) + c$$

(20) $\int \left(a^x + x^a + 10 - \text{cosec } 2x \ \cot 2x\right)dx$

$$= \int a^x dx + \int x^a dx + 10\int dx - \int \text{cosec } 2x \ \cot 2x \, dx$$

$$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + 10x + \frac{\text{cosec } 2x}{2} + c$$

(21) $\int \left(\frac{1}{5}\cos\left(\frac{x}{5} + 7\right) + \frac{3}{lx + m} + e^{\frac{x}{2} + 3}\right) dx$

$$= \frac{1}{5}\int \cos\left(\frac{x}{5} + 7\right) dx + 3\int \frac{1}{lx + m} dx + \int e^{\frac{x}{2} + 3} dx$$

$$= \frac{1}{5} \cdot \frac{1}{(1/5)} \sin\left(\frac{x}{5} + 7\right) + 3\left(\frac{1}{l}\right) \log (lx + m) + \frac{1}{(1/2)} e^{\frac{x}{2} + 3} + c$$

$$= \sin\left(\frac{x}{5} + 7\right) + \frac{3}{l} \log (lx + m) + 2e^{\frac{x}{2} + 3} + c$$

EXERCISE 9.3

Integrate the following with respect to *x*

(1) $5x^4 + 3(2x + 3)^4 - 6(4 - 3x)^5$ (2) $\frac{3}{x} + \frac{m}{4x + 1} - 2(5 - 2x)^5$

(3) $4 - \frac{5}{x + 2} + 3\cos 2x$ (4) $3e^{7x} - 4\sec(4x + 3)\tan(4x + 3) + \frac{11}{x^5}$

(5) $p\,\text{cosec}^2 (px - q) - 6(1 - x)^4 + 4e^{3 - 4x}$

(6) $\frac{4}{(3 + 4x)} + (10x + 3)^9 - 3\text{cosec}(2x + 3)\cot(2x + 3)$

(7) $6\sin 5x - \frac{l}{(px + q)^m}$ (8) $a\sec^2 (bx + c) + \frac{q}{e^{l - mx}}$

(9) $\frac{1}{\left(3 + \frac{2}{3}x\right)} - \frac{2}{3}\cos\left(x - \frac{2}{3}\right) + 3\left(\frac{x}{3} + 4\right)^6$

(10) $7\sin\frac{x}{7} - 8\sec^2\left(4 - \frac{x}{4}\right) + 10\left(\frac{2x}{5} - 4\right)^{\frac{3}{2}}$ (11) $2x^e + 3e^x + e^e$

(12) $(ae)^x - a^{-x} + b^x$

9.3 Methods of Integration

Integration is not as easy as differentiation. This is first due to its nature. Finding a derivative of a given function is facilitated by the fact that the differentiation itself has a constructive character. A derivative is simply defined as

$$\underset{\Delta x \to 0}{\mathrm{L}t} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Suppose we are asked to find the derivative of **log*x*,** we know in all details how to proceed in order to obtain the result.

When we are asked to find the integral of **log*x*,** we have no constructive method to find integral or even how to start.

In the case of differentiation we use the laws of differentiation of several functions in order to find derivatives of their various combinations, e.g. their sum, product, quotient, composition of functions etc.

There are very few such rules available in the theory of integration and their application is rather restricted. But the significance of these methods of integration is very great.

In every case one must learn to select the most appropriate method and use it in the most convenient form. This skill can only be acquired after long practice.

Already we have seen two important properties of integration. The following are the four important methods of integrations.

(1) Integration by decomposition into sum or difference.

(2) Integration by substitution.

(3) Integration by parts

(4) Integration by successive reduction.

Here we discuss only the first three methods of integration and the other will be studied in higher classes.

9.3.1 Decomposition method

Sometimes it is impossible to integrate directly the given function. But it can be integrated after decomposing it into a sum or difference of number of functions whose integrals are already known.

For example $(1 + x^2)^3$, $\sin 5x \cos 2x$, $\dfrac{x^2 - 5x + 1}{x}$, $\sin^5 x$, $\dfrac{e^x + 1}{e^x}$, $(\tan x + \cot x)^2$

do not have direct formulae to integrate. But these functions can be decomposed into a sum or difference of functions whose individual integrals are known. In most of the cases the given integrand will be any one of the algebraic, trigonometric or exponential forms, and sometimes combinations of these functions.

Example 9.22 - Integrate

(22) $\int (1 + x^2)^3 dx = \int \left(1 + 3x^2 + 3x^4 + x^6\right) dx$

$= x + \frac{3x^3}{3} + \frac{3x^5}{5} + \frac{x^7}{7} + c$

$= x + x^3 + \frac{3}{5} x^5 + \frac{x^7}{7} + c$

(23) $\int \sin 5x \cos 2x \; dx = \int \frac{1}{2} [\sin (5x + 2x) + \sin (5x - 2x)] \; dx$

$[\because 2\sin A \cos B = \sin (A+B) + \sin(A - B)]$

$= \frac{1}{2} \int [\sin 7x + \sin 3x] \, dx$

$= \frac{1}{2} \left[\frac{-\cos 7x}{7} - \frac{\cos 3x}{3}\right] + c$

$\therefore \int \sin 5x \cos 2x \; dx = -\frac{1}{2} \left[\frac{\cos 7x}{7} + \frac{\cos 3x}{3}\right] + c$

(24) $\int \frac{x^2 - 5x + 1}{x} dx = \int \left(\frac{x^2}{x} - \frac{5x}{x} + \frac{1}{x}\right) dx = \int \left(x - 5 + \frac{1}{x}\right) dx$

$= \int x dx - 5 \int dx + \int \frac{1}{x} dx$

$= \frac{x^2}{2} - 5x + \log x + c$

(25) $\int \cos^3 x \, dx = \int \frac{1}{4} [3\cos x + \cos 3x] \, dx$

$= \frac{1}{4} \left[3 \int \cos x \, dx + \int \cos 3x \, dx\right]$

$= \frac{1}{4} \left[3 \sin x + \frac{\sin 3x}{3}\right] + c$

(26) $\int \frac{e^x + 1}{e^x} dx \quad = \quad \int \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx \ = \ \int 1dx + \int e^{-x} dx$

$= \quad x - e^{-x} + c$

(27) $\int (\tan x + \cot x)^2 dx \quad = \quad \int \left(\tan^2 x + 2\tan x \cot x + \cot^2 x\right) dx$

$= \quad \int [\left(\sec^2 x - 1\right) + 2 + (\mathrm{cosec}^2 x - 1)] \, dx$

$= \quad \int \left(\sec^2 x + \mathrm{cosec}^2 x\right) dx$

$= \quad \tan x + (- \cot x) + c$

$= \quad \tan x - \cot x + c$

(28) $\int \frac{1}{1 + \cos x} dx = \int \frac{(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} \ dx$

$= \int \frac{1 - \cos x}{1 - \cos^2 x} dx \ = \int \frac{1 - \cos x}{\sin^2 x} dx$

$= \int \left[\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right] dx \ = \int \left[\mathrm{cosec}^2 x - \mathrm{cosec} x \cot x\right] dx$

$= \int \mathrm{cosec}^2 x \, dx \ - \int \mathrm{cosec} x \cot x \, dx$

$= - \cot x - (- \mathrm{cosec}\ x) + c$

$= \mathrm{cosec}\ x - \cot x + c$

Note: Another method

$$\left(\int \frac{1}{1 + \cos x} dx = \int \frac{1}{2\cos^2 \frac{x}{2}} dx = \frac{1}{2} \int \sec^2 \frac{x}{2} dx = \frac{1}{2} \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c = \tan \frac{x}{2} + c\right)$$

(29) $\int \frac{1 - \cos x}{1 + \cos x} dx \quad = \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx \ = \int \tan^2 \frac{x}{2} dx$

$$= \int \left(\sec^2 \frac{x}{2} - 1\right) dx = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + c$$

$$= 2 \tan \frac{x}{2} - x + c \quad \ldots \text{(i)}$$

Another method:

$$\int \frac{1 - \cos x}{1 + \cos x} dx = \int \frac{(1 - \cos x)}{(1 + \cos x)} \frac{(1 - \cos x)}{(1 - \cos x)} dx$$

$$= \int \frac{(1 - \cos x)^2}{1 - \cos^2 x} dx = \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x} dx$$

$$= \int \left[\frac{1}{\sin^2 x} - \frac{2 \cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right] dx$$

$$= \int \left(\operatorname{cosec}^2 x - 2\operatorname{cosec} x \cot x + \cot^2 x\right) dx$$

$$= \int \left[\operatorname{cosec}^2 x - 2\operatorname{cosec} x \cot x + (\operatorname{cosec}^2 x - 1)\right] dx$$

$$= \int \left[2 \operatorname{cosec}^2 x - 2 \operatorname{cosec} x \cot x - 1\right] dx$$

$$= 2 \int \operatorname{cosec}^2 x \, dx - 2 \int \operatorname{cosec} x \cot x \, dx - \int dx$$

$$= -2 \cot x - 2 (-\operatorname{cosec} x) - x + c$$

$$\int \frac{1 - \cos x}{1 + \cos x} dx = 2 \operatorname{cosec} x - 2 \cot x - x + c \quad \ldots \text{(ii)}$$

Note: From (i) and (ii) both $2 \tan \frac{x}{2} - x + c$ and $2 \operatorname{cosec} x - 2 \cot x - x + c$ are trigonometrically equal.

(30) $$\int \sqrt{1 + \sin 2x} \, dx = \int \sqrt{(\cos^2 x + \sin^2 x) + (2 \sin x \cos x)} \, dx$$

$$= \int \sqrt{(\cos x + \sin x)^2} \, dx = \int (\cos x + \sin x) \, dx$$

$$= [\sin x - \cos x] + c$$

(31) $\int \frac{x^3+2}{x-1}\,dx \quad = \int \frac{x^3-1+3}{x-1}\,dx = \int \left(\frac{x^3-1}{x-1} + \frac{3}{x-1}\right) dx$

$$= \int \left[\frac{(x-1)\,(x^2+x+1)}{x-1} + \frac{3}{x-1}\right] dx$$

$$= \int \left(x^2 + x + 1 + \frac{3}{x-1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + 3\log(x-1) + c$$

(32) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\,dx \quad = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x}\,dx$

$$= \int \left(\frac{\cos^2 x}{\sin^2 x \cos^2 x} - \frac{\sin^2 x}{\sin^2 x \cos^2 x}\right) dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int (\operatorname{cosec}^2 x - \sec^2 x)\,dx$$

$$= -\cot x - \tan x + c$$

(33) $\int \frac{3^x - 2^{x+1}}{6^x}\,dx \quad = \int \left(\frac{3^x}{6^x} - \frac{2^{x+1}}{6^x}\right) dx = \int \left[\left(\frac{3}{6}\right)^x - 2.\left(\frac{2}{6}\right)^x\right] dx$

$$= \int \left[\left(\frac{1}{2}\right)^x - 2\left(\frac{1}{3}\right)^x\right] dx = \int (2^{-x} - 2.3^{-x})\,dx$$

$$= \frac{-2^{-x}}{\log 2} - 2\,.\,\frac{(-3^{-x})}{\log 3} + c$$

$$= \frac{2}{\log 3}\,3^{-x} - \frac{1}{\log 2}\,2^{-x} + c$$

(34) $\int e^{x\log 2}\,.\,e^x\,dx \quad = \int e^{\log 2^x}\,e^x\,dx = \int 2^x e^x\,dx$

$$= \int (2e)^x\,dx = \frac{(2e)^x}{\log 2e} + c$$

(35) $\displaystyle\int \frac{dx}{\sqrt{x+3}-\sqrt{x-4}} = \int \frac{\sqrt{x+3}+\sqrt{x-4}}{\left\{\sqrt{x+3}-\sqrt{x-4}\right\}\left\{\sqrt{x+3}+\sqrt{x-4}\right\}}\,dx$

$$= \int \frac{\sqrt{x+3}+\sqrt{x-4}}{(x+3)-(x-4)}\,dx = \int \frac{\sqrt{x+3}+\sqrt{x-4}}{7}\,dx$$

$$= \frac{1}{7} \int \left[(x+3)^{1/2} + (x-4)^{1/2}\right] dx$$

$$\int \frac{dx}{\sqrt{x+3}-\sqrt{x-4}} = \frac{1}{7} \left[\frac{2}{3}(x+3)^{3/2} + \frac{2}{3}(x-4)^{3/2}\right] + c$$

(36) $\displaystyle\int (x-1)\sqrt{x+1}\,dx = \int \{(x+1)-2\}\left(\sqrt{x+1}\right) dx$

$$= \int \left[(x+1)^{3/2} - 2(x+1)^{1/2}\right] dx$$

$$= \frac{2}{5}(x+1)^{5/2} - 2.\frac{2}{3}(x+1)^{3/2} + c$$

$$\int (x-1)\sqrt{x+1}\,dx = \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + c$$

(37) $\displaystyle\int (3x+4)\sqrt{3x+7}\,dx = \int \{(3x+7)-3\}\sqrt{3x+7}\,dx$

$$= \int \left((3x+7)\sqrt{3x+7} - 3\sqrt{3x+7}\right) dx$$

$$= \int \left((3x+7)^{3/2} - 3(3x+7)^{1/2}\right) dx$$

$$= \frac{1}{3}\,\frac{(3x+7)^{5/2}}{5/2} - 3\,.\,\frac{1}{3}\,\frac{(3x+7)^{3/2}}{3/2} + c$$

$$= \frac{2}{15}(3x+7)^{5/2} - \frac{2}{3}(3x+7)^{3/2} + c$$

(37a) $\displaystyle\int \frac{9}{(x-1)(x+2)^2}\,dx = \int\left[\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\right] dx$ resolve into partial fraction

$$= \int \left[\frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}\right] dx$$

$$= \log(x-1) - \log(x+2) + \frac{3}{(x+2)} + c$$

EXERCISE 9.4

Integrate the following

(1) $(2x - 5)(36 + 4x)$ (2) $(1 + x^3)^2$ (3) $\dfrac{x^3 + 4x^2 - 3x + 2}{x^2}$

(4) $\dfrac{x^4 - x^2 + 2}{x + 1}$ (5) $\dfrac{(1 + x)^2}{\sqrt{x}}$ (6) $\dfrac{e^{2x} + e^{-2x} + 2}{e^x}$

(7) $\sin^2 3x + 4\cos 4x$ (8) $\cos^3 2x - \sin 6x$ (9) $\dfrac{1}{1 + \sin x}$

(10) $\dfrac{1}{1 - \cos x}$ (11) $\sqrt{1 - \sin 2x}$ (12) $\sqrt{1 + \cos 2x}$

(13) $\dfrac{1}{\sin^2 x \cos^2 x}$ (14) $\dfrac{\sin^2 x}{1 + \cos x}$ (15) $\sin 7x \cos 5x$

(16) $\cos 3x \cos x$ (17) $\cos 2x \sin 4x$ (18) $\sin 10x \sin 2x$

(19) $\dfrac{1 + \cos 2x}{\sin^2 2x}$ (20) $(e^x - 1)^2 e^{-4x}$ (21) $\dfrac{1 - \sin x}{1 + \sin x}$

(22) $\dfrac{2^{x+1} - 3^{x-1}}{6^x}$ (23) $e^{x\log a} e^x$ (24) $\dfrac{a^{x+1} - b^{x-1}}{c^x}$

(25) $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$ (26) $\sin mx \cos nx$ $(m > n)$ (27) $\cos px \cos qx$ $(p > q)$

(28) $\cos^2 5x \sin 10x$ (29) $\dfrac{1}{\sqrt{x+1} - \sqrt{x-2}}$ (30) $\dfrac{1}{\sqrt{ax+b} - \sqrt{ax+c}}$

(31) $(x + 1)\sqrt{x + 3})$ (32) $(x - 4)\sqrt{x + 7}$ (33) $(2x + 1)\sqrt{2x + 3}$

(34) $\dfrac{x + 1}{(x + 2)(x + 3)}$ (35) $\dfrac{x^2 + 1}{(x - 2)(x + 2)(x^2 + 9)}$

9.3.2 Method of substitution or change of variable

Sometimes the given functions may not be in an integrable form and the variable of integration (x in dx) can be suitably changed into a new variable by substitution so that the new function will be found integrable.

Suppose $\quad F(u) = \int f(u)\, du,$

then $\quad \dfrac{dF(u)}{du} = f(u)$

Put $u = \phi(x)$, then $\frac{du}{dx} = \phi'(x)$

Also we know that

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du} \cdot \frac{du}{dx}$$

$$= f(u)\, \phi'(x)$$

i.e. $$\frac{dF(u)}{dx} = f[\phi(x)]\, \phi'(x)$$

$$\Rightarrow \quad F(u) = \int f[\phi(x)]\, \phi'(x)\, dx$$

$$\therefore \quad \int f(u) du = \int f[\phi(x)]\, \phi'(x)\, dx$$

$$\boxed{\int f[\phi(x)]\phi'(x)\, dx = \int f(u)\, du}$$

The success of the above method depends on the selection of suitable substitution either $x = \phi(u)$ or $u = g(x)$.

Example 9.38 – 9.41: Integrate

(38) $\int 5x^4 e^{x^5} dx$ (39) $\int \frac{\cos x}{1 + \sin x}\, dx$ (40) $\int \frac{1}{\sqrt{1 - x^2}}\, dx$ (41) $\int \frac{1}{1 + x^2}\, dx$

For the first two problems (38) and (39) the substitution in the form $u = \phi(x)$ and for (40) and (41) the substitution in the form $x = \phi(u)$.

(38) Let $I = \int 5x^4 e^{x^5} dx$

Put $x^5 = u$ … (i)

$\therefore$ $5x^4\, dx = du$ … (ii)

Since the variable of integration is changed from x to u, we have to convert entire integral in terms of the new variable u.

$\therefore$ We get $I = \int \left(e^{x^5}\right)\left(5x^4\, dx\right)$

$= \int e^u\, du$ (by (i) and (ii))

$= e^u + c$

$= e^{x^5} + c$ (replacing u by x^5, as the function of given variable)

(39) Let $\quad I = \int \frac{\cos x}{1+\sin x}\, dx$

Put $(1+\sin x) = u \quad \ldots$ (i)

$\cos x\, dx = du \quad \ldots$ (ii)

$\therefore\ I = \int \frac{1}{(1+\sin x)} (\cos x\, dx)$

$= \int \frac{1}{u}\, du \quad$ (by (i) and (ii))

$= \log u + c$

$$\int \frac{\cos x}{1+\sin x}\, dx = \log(1+\sin x) + c$$

(40) Let $\quad I = \int \frac{1}{\sqrt{1-x^2}}\, dx$

Put $x = \sin u \quad \ldots$.(i) $\Rightarrow u = \sin^{-1} x$

$dx = \cos u\, du \quad \ldots$ (ii)

$\therefore \quad I = \int \frac{1}{\sqrt{1-x^2}}\, dx$

$= \int \frac{1}{\sqrt{1-\sin^2 u}} (\cos u\, du)$ by (i) and (ii)

$= \int \frac{1}{\sqrt{\cos^2 u}} (\cos u\, du)$

$= \int du = u + c$

$$\therefore \int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c \quad (\because u = \sin^{-1} x)$$

(41) Let $\quad I = \int \frac{1}{1+x^2}\, dx$

put $x = \tan u \quad \Rightarrow u = \tan^{-1} x$

$dx = \sec^2 u\, du$

$$\therefore \quad I = \int \frac{1}{1+\tan^2 u} \sec^2 u \, du$$

$$= \int \frac{1}{\sec^2 u} \sec^2 u \, du = \int du$$

$$I = u + c$$

$$\therefore \quad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c$$

Some standard results of integrals

(i) $\displaystyle \int \frac{f'(x)}{f(x)} \, dx = \log [f(x)] + c$

(ii) $\displaystyle \int \frac{f'(x)}{\sqrt{f(x)}} \, dx = 2\sqrt{f(x)} + c$

(iii) $\displaystyle \int f'(x) \, [f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ **where** $n \neq -1$

Proof :

(i) Let $\displaystyle I = \int \frac{f'(x)}{f(x)} \, dx$

Put $f(x) = u$

$\therefore \quad f'(x)dx = du$

$\displaystyle \therefore \quad I = \int \frac{1}{u} \, du = \log u + c = \log [f(x)] + c$

i.e. $\displaystyle \int \frac{f'(x)}{f(x)} \, dx = \log [f(x)] + c$

(ii) Let $\displaystyle I = \int \frac{f'(x)}{\sqrt{f(x)}} \, dx$

$\displaystyle = \int \frac{1}{\sqrt{u}} \, du$ where $u = f(x)$ and $du = f'(x) \, dx$

$= 2\sqrt{u} + c = 2\sqrt{f(x)} + c$

$$\therefore \quad \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$$

(iii) Let $I = \int f'(x)\,[f(x)]^n\, dx$ where $n \neq -1$

Put $f(x) = u$

$\therefore \quad f'(x)\, dx = du$

$$\therefore \quad I = \int \{f(x)\}^n \left(f'(x)\, dx \right)$$

$$= \int u^n\, du = \frac{u^{n+1}}{n+1} + c \qquad (\because n \neq -1)$$

$$\therefore \quad \int f'(x)\,[f(x)]^n\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Examples 9.42 – 9.47: Integrate the following

(42) $\dfrac{2x+1}{x^2+x+5}$ (43) $\dfrac{e^x}{5+e^x}$ (44) $\dfrac{6x+5}{\sqrt{3x^2+5x+6}}$ (45) $\dfrac{\cos x}{\sqrt{\sin x}}$

(46) $(4x-1)(2x^2-x+5)^4$ (47) $(3x^2+6x+7)(x^3+3x^2+7x-4)^{11}$

Solution

(42) Let $I = \int \frac{2x+1}{x^2+x+5}\, dx = \int \frac{1}{(x^2+x+5)}\{(2x+1)\, dx\}$

Put $x^2 + x + 5 = u$

$(2x+1)\, dx = du$

$$\therefore \quad I = \int \frac{1}{u}\, du = \log u + c = \log (x^2+x+5) + c$$

$$\therefore \int \frac{2x+1}{x^2+x+5}\, dx = \log (x^2+x+5) + c$$

(43) Let $I = \int \frac{e^x}{5+e^x}\, dx$

put $5 + e^x = u$

$e^x\, dx = du$

$$\therefore \quad I = \int \frac{1}{5+e^x}(e^x\,dx)$$

$$\therefore \quad = \int \frac{1}{u}\,du$$

$$I = \log u + c = \log(5+e^x) + c$$

$$\text{i.e.} \int \frac{e^x}{5+e^x}\,dx = \log(5+e^x) + c$$

(44) Let $$I = \int \frac{6x+5}{\sqrt{3x^2+5x+6}}\,dx$$

put $3x^2 + 5x + 6 = t$

$(6x + 5)\,dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t} + c = 2\sqrt{3x^2+5x+6} + c$$

$$\therefore \quad \int \frac{6x+5}{\sqrt{3x^2+5x+6}}\,dx = 2\sqrt{3x^2+5x+6} + c$$

(45) Let $$I = \int \frac{\cos x}{\sqrt{\sin x}}\,dx$$

put $\sin x = t$

$\cos x\,dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t}}\,dt$$

$$\text{i.e.} \quad I = 2\sqrt{t} + c = 2\sqrt{\sin x} + c$$

$$\text{i.e.} \quad \int \frac{\cos x}{\sqrt{\sin x}}\,dx = 2\sqrt{\sin x} + c$$

(46) Let $$I = \int (4x-1)(2x^2-x+5)^4\,dx$$

put $2x^2 - x + 5 = u$

$(4x - 1)\,dx = du$

$$\therefore \quad I = \int (2x^2-x+5)^4\big((4x-1)\,dx\big)$$

$$= \int u^4\,du = \frac{u^5}{5} + c = \frac{(2x^2-x+5)^5}{5} + c$$

i.e. $\int (4x-1)(2x^2-x+5)^4\,dx = \dfrac{(2x^2-x+5)^5}{5} + c$

(47) Let $\quad I = \int (3x^2+6x+7)(x^3+3x^2+7x-4)^{11}\,dx$

put $x^3+3x^2+7x-4 = u$

$\therefore \quad (3x^2+6x+7)\,dx = du$

$$\therefore \quad I = \int (x^3+3x^2+7x-4)^{11}\left\{(3x^2+6x+7)dx\right\}$$

$$= \int u^{11} du$$

$$I = \frac{u^{12}}{12} + c = \frac{(x^3+3x^2+7x-4)^{12}}{12} + c$$

$$\therefore \int (x^3+3x^2+7x-4)^{11}\,(3x^2+6x+7)\,dx = \frac{(x^3+3x^2+7x-4)^{12}}{12} + c$$

Example 9.48 – 9.67: Integrate the following

(48) $x^{16}(1+x^{17})^4$ (49) $\dfrac{x^{24}}{(1+x^{25})^{10}}$ (50) $\dfrac{x^{15}}{1+x^{32}}$ (51) $x(a-x)^{17}$

(52) $\cot x$ (53) $\operatorname{cosec} x$ (54) $\dfrac{\log \tan x}{\sin 2x}$ (55) $\sin^{15}x \cos x$

(56) $\sin^7 x$ (57) $\tan x\sqrt{\sec x}$ (58) $\dfrac{e^{\tan x}}{\cos^2 x}$ (59) $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$

(60) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ (61) $e^{2\log x}\, e^{x^3}$ (62) $\dfrac{\log x}{x}$ (63) $\dfrac{1}{x \log x}$

(64) $\dfrac{1}{x+\sqrt{x}}$ (65) $\dfrac{e^{x/2}-e^{-x/2}}{e^x-e^{-x}}$ (66) $\dfrac{x^{e-1}+e^{x-1}}{x^e+e^x}$

(67) $\alpha\beta\, x^{\alpha-1}\, e^{-\beta x^{\alpha}}$ (68) $(2x-3)\sqrt{4x+1})$

Solution:

(48) $\int x^{16}(1+x^{17})^4\,dx$

Let $\quad I = \int x^{16}(1+x^{17})^4\,(dx)$

put $\quad 1+x^{17} = u \quad \ldots$ (i)

$$17x^{16}dx = du$$

$$dx = \frac{1}{17x^{16}}\,du \qquad \ldots \text{(ii)}$$

$$\therefore\ \mathrm{I} = \int x^{16}(u)^4\left(\frac{1}{17x^{16}}\,dx\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{17}\int u^4 du \quad = \quad \frac{1}{17}\,\frac{u^5}{5} + c$$

$$\int x^{16}\,(1 + x^{17})^4\,dx = \frac{1}{85}\,(1 + x^{17})^5 + c$$

(49) $\displaystyle\int \frac{x^{24}}{(1 + x^{25})^{10}}\,dx$

Let $\quad \mathrm{I} = \displaystyle\int \frac{x^{24}}{(1 + x^{25})^{10}}\,dx$

put $\quad 1 + x^{25} = u \qquad \ldots$ (i)

$$25x^4\,dx = du$$

$$dx = \frac{1}{25x^{24}}\,du \qquad \ldots \text{(ii)}$$

$$\therefore \qquad \mathrm{I} = \int \frac{x^{24}}{u^{10}}\left(\frac{1}{25x^{24}}\,du\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{25}\int \frac{1}{u^{10}}\,du \quad = \frac{1}{25}\left[-\frac{1}{9u^9}\right] + c$$

$$\therefore \quad \int \frac{x^{24}}{(1 + x^{25})^{10}}\,dx = -\frac{1}{225\,(1 + x^{25})^9} + c$$

(50) $\displaystyle\int \frac{x^{15}}{1 + x^{32}}\,dx$

Let $\quad \mathrm{I} = \displaystyle\int \frac{x^{15}}{1 + x^{32}}\,dx$

$$\begin{aligned}
\text{put} \quad x^{16} &= u && \dots \text{(i)} \\
16x^{15}dx &= du \\
dx &= \frac{1}{16x^{15}}\, du && \dots \text{(ii)} \\
\therefore \quad &= \int \frac{x^{15}}{1+u^2}\left(\frac{1}{16x^{15}}\, du\right) && \text{by (i) and (ii)} \\
&= \frac{1}{16} \int \frac{1}{1+u^2}\, du \\
\mathrm{I} &= \frac{1}{16} \tan^{-1} u + c \\
\int \frac{x^{15}}{1+x^{32}}\, dx &= \frac{1}{16} \tan^{-1}(x^{16}) + c
\end{aligned}$$

(51) $\int x(a-x)^{17}\, dx$

$$\begin{aligned}
\text{Let} \quad \mathrm{I} &= \int x(a-x)^{17}\, dx \\
\text{put} \quad (a-x) &= u \quad \Rightarrow \; x = a-u \\
dx &= -\, du \\
\therefore \quad \mathrm{I} &= \int (a-u)u^{17}\, (-\, du) \\
&= \int (u^{18} - au^{17})\, du \\
\mathrm{I} &= \frac{u^{19}}{19} - a\frac{u^{18}}{18} + c \\
\therefore \quad \int x(a-x)^{17}\, dx &= \frac{(a-x)^{19}}{19} - \frac{a(a-x)^{18}}{18} + c
\end{aligned}$$

(52) $\int \cot x\, dx$

$$\begin{aligned}
\text{Let} \quad \mathrm{I} &= \int \cot x\, dx \\
\text{put} \quad \sin x &= u \\
\cos x\, dx &= du
\end{aligned}$$

$$\therefore \quad I = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{u} du = \log u + c$$

$$\therefore \quad \int \boldsymbol{\cot x \, dx} = \boldsymbol{\log \sin x + c}$$

(53) $\int \text{cosec } x \, dx$

$$\text{Let } I = \int \text{cosec } x \, dx = \int \frac{\text{cosec}x\,[\text{cosec}x - \cot x]}{[\text{cosec}x - \cot x]} dx$$

$$\text{Put } \text{cosec}x - \cot x = u \qquad \ldots (1)$$

$$(-\text{cosec } x \cot x + \text{cosec}^2 x)dx = du$$

$$\text{cosec}x\,(\text{cosec}x - \cot x)\,dx = du \qquad \ldots(2)$$

$$\therefore \quad I = \int \frac{\text{cosec}x\,[\text{cosec}x - \cot x]}{[\text{cosec}x - \cot x]} dx$$

$$= \int \frac{du}{u} = \log u + c$$

$$\therefore \int \text{cosec } x \, dx = \log(\text{cosec}x - \cot x) + c$$

$$\int \boldsymbol{\text{cosec } x \, dx = \log \tan \frac{x}{2} + c}$$

(54) $\int \frac{\log \tan x}{\sin 2x} dx$

$$\text{Let} \quad I = \int \frac{\log \tan x}{\sin 2x} dx$$

$$\text{Put} \quad \log \tan x = u \qquad \ldots (i)$$

$$\therefore \quad \frac{1}{\tan x} \sec^2 x dx = du \Rightarrow \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} dx = du$$

$$\text{i.e.} \quad \frac{2}{2\sin x \cos x} dx = du \Rightarrow \frac{2}{\sin 2x} dx = du$$

$$dx = \frac{\sin 2x}{2} du \qquad \ldots (ii)$$

$$\therefore \quad I = \int \frac{u}{\sin 2x} \cdot \left(\frac{\sin 2x}{2} du\right) \qquad \text{by (i) and (ii)}$$

$$= \frac{1}{2}\int u\,du = \frac{1}{2}\left[\frac{u^2}{2}\right] + c$$

$$\int \frac{\log \tan x}{\sin 2x}\,dx = \frac{1}{4}\,[\log \tan x]^2 + c$$

(55) $\int \sin^{15}x \cos x\, dx$

Let $\quad I = \int \sin^{15}x \cos x\, dx$

Put $\quad \sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore \quad I = \int t^{15}\, dt = \frac{t^{16}}{16} + c$

$\therefore \quad \int \sin^{15}x \cos x\, dx = \frac{\sin^{16}x}{16} + c$

(56) $\int \sin^7 x\, dx$

Let $\quad I = \int \sin^7 x\, dx$

$\therefore \quad = \int \sin^6 x \sin x\, dx = \int (1 - \cos^2 x)^3 (\sin x\, dx)$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\sin x\, dx = (-dt)$

$\therefore \quad I = \int (1 - t^2)^3 (-dt)$

$= \int (1 - 3t^2 + 3t^4 - t^6)(-dt)$

$= \int (t^6 - 3t^4 + 3t^2 - 1)\, dt$

$= \frac{t^7}{7} - 3\frac{t^5}{5} + 3\frac{t^3}{3} - t + c$

$\therefore \quad \int \sin^7 x\, dx = \frac{\cos^7 x}{7} - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + c$

(Note : This method is applicable only when the power is odd).

(57) $\int \tan x \sqrt{\sec x}\, dx$

Let $I = \int \tan x \sqrt{\sec x}\, dx$

Put $\sec x = t$

$$\sec x \tan x \, dx = dt \qquad \therefore\ dx = \frac{dt}{\sec x \tan x}$$

Converting everything in terms of t.

$$\therefore\ \mathrm{I} = \int \tan x \ (\sqrt{t}) \left(\frac{1}{\sec x \tan x} dt\right)$$

$$= \int \frac{\sqrt{t}}{\sec x} dt = \int \frac{\sqrt{t}}{t} dt = \int \frac{1}{\sqrt{t}} dt = 2\sqrt{t} + c$$

$$\therefore\ \int \tan x \sqrt{\sec x}\, dx = 2\sqrt{\sec x} + c$$

(When the integrand is with $e^{f(x)}$ and $f(x)$ is not a linear function in x, substitute $f(x) = u$.)

(58) $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x} dx$

$$\text{Let I} = \int \frac{e^{\tan x}}{\cos^2 x} dx$$

$$\text{Put } \tan x = t$$

$$\sec^2 x \, dx = dt \qquad \therefore\ dx = \cos^2 x \, dt$$

$$\therefore\ \mathrm{I} = \int \frac{e^t}{\cos^2 x} . \cos^2 x \, dt = \int e^t \, dt = e^t + c$$

$$\therefore\ \int \frac{e^{\tan x}}{\cos^2 x} dx = e^{\tan x} + c$$

(59) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\text{Let I} = \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\text{Put } \sqrt{x} = t \qquad \therefore\ x = t^2 \ \Rightarrow\ dx = 2t dt$$

$$\therefore\ \mathrm{I} = \int \frac{e^t}{t} . 2t \ dt = 2\int e^t \, dt = 2e^t + c$$

$$\therefore\ \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^{\sqrt{x}} + c$$

(60) $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$

$$\text{Let } I = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$$

$$\text{put } \sin^{-1}x = t$$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt \qquad \Rightarrow \quad dx = \sqrt{1-x^2}\ dt$$

$$\therefore\ I = \int \frac{e^t}{\sqrt{1-x^2}}\sqrt{1-x^2}\ dt$$

$$= \int e^t\,dt = e^t + c$$

$$\therefore\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = e^{\sin^{-1}x} + c$$

(61) $\int e^{2\log x}\, e^{x^3}\, dx$

$$\text{Let } I = \int e^{2\log x}\, e^{x^3}\, dx$$

$$\text{put } x^3 = t \qquad \Rightarrow \quad 3x^2\,dx = dt \qquad \therefore\ dx = \frac{1}{3x^2}\ dt$$

$$\therefore\ I = \int e^{\log x^2}\, e^{x^3}\,dx = \int x^2\, e^{x^3}\,dx$$

$$= \int x^2\, e^t \left(\frac{1}{3x^2}\,dt\right)$$

$$= \frac{1}{3}\int e^t\ dt = \frac{1}{3}\ e^t + c$$

$$\therefore\ \int e^{2\log x}\, e^{x^3}\,dx = \frac{1}{3}\ e^{x^3} + c$$

(62) $\int \frac{\log x}{x}\, dx$

$$\text{Let I} = \int \frac{\log x}{x}\, dx$$

$$\text{put } \log x = u \Rightarrow \frac{1}{x}\, dx = du \qquad \therefore\ dx = x\, du$$

$$\therefore\ \text{I} = \int \frac{u}{x}(x\, du) = \int u\, du = \frac{u^2}{2} + c$$

$$\int \frac{\log x}{x}\, dx = \frac{1}{2}[\log x]^2 + c$$

(63) $\int \frac{1}{x \log x}\, dx$

$$\text{Let I} = \int \frac{1}{x \log x}\, dx$$

$$\text{put } \log x = u$$

$$\frac{1}{x}\, dx = du \qquad \therefore\ dx = x\, du$$

$$\therefore\ \text{I} = \int \frac{1}{xu}(x\, du) = \int \frac{1}{u}\, du = \log u + c$$

$$\int \frac{1}{x \log x}\, dx = \log(\log x) + c$$

(64) $\int \frac{1}{x + \sqrt{x}}\, dx$

$$\text{Let I} = \int \frac{1}{x + \sqrt{x}}\, dx$$

$$\text{put } \sqrt{x} = t \Rightarrow x = t^2$$

$$dx = 2t\, dt$$

$$\therefore\ \text{I} = \int \frac{1}{t^2 + t}\, 2t\, dt = 2\int \frac{t}{t(t+1)}\, dt$$

$$= 2\int \left(\frac{1}{1+t}\right) dt = 2\log(1+t) + c$$

$$\therefore\ \int \frac{1}{x + \sqrt{x}}\, dx = 2\log\left(1 + \sqrt{x}\right) + c$$

(65) $\int \frac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}} dx$

$$\text{Let I} = \int \frac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}} dx$$

$$\text{put } e^{x\backslash 2} = t \quad \Rightarrow \quad \frac{1}{2} e^{x\backslash 2} dx = dt$$

$$dx = \frac{2}{e^{x/2}} dt = \frac{2}{t} dt$$

$$\therefore \quad \text{I} = \int \frac{t - 1/t}{t^2 - 1/t^2} \left(\frac{2dt}{t}\right)$$

$$= 2 \int \frac{\frac{(t^2 - 1)}{t}}{\frac{(t^4 - 1)}{t^2}} \frac{dt}{t} = 2 \int \frac{(t^2 - 1)}{t^4 - 1} dt$$

$$= 2 \int \frac{t^2 - 1}{(t^2 - 1)(t^2 + 1)} dt = 2 \int \frac{1}{1 + t^2} dt = 2 \tan^{-1} t + c$$

$$\therefore \int \frac{e^{x/2} - e^{-x/2}}{e^x - e^{-x}} dx = 2 \tan^{-1}\left(e^{x/2}\right) + c$$

(66) $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx$

$$\text{Let I} = \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx$$

$$\text{Put } x^e + e^x = t \quad \ldots \text{(i)}$$

$$(ex^{e-1} + e^x)\, dx = dt, \qquad e(x^{e-1} + e^{x-1})\, dx = dt$$

$$\therefore \quad dx = \frac{1}{e(x^{e-1} + e^{x-1})} dt \quad \ldots \text{(ii)}$$

$$\therefore\ I = \int \frac{(x^{e-1} + e^{x-1})}{t}\left(\frac{1}{e\left(x^{e-1} + e^{x-1}\right)}\right) dt \quad \text{by (i) and (ii)}$$

$$= \frac{1}{e}\int \frac{1}{t}\, dt = \frac{1}{e}\log t + c$$

$$\therefore \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x}\, dx = \frac{1}{e}\log(x^e + e^x) + c$$

(67) $\int \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^{\alpha}}\, dx$

$$\text{Let } I = \int \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^{\alpha}}\, dx$$

$$\text{Put } -\beta x^{\alpha} = u \Rightarrow -\alpha\beta x^{\alpha-1} dx = du \quad \therefore dx = -\frac{1}{\alpha\beta\, x^{\alpha-1}}\, du$$

$$\therefore\ I = \int \alpha\beta x^{\alpha-1} e^{u}\left(\frac{-1}{\alpha\beta\, x^{\alpha-1}}\right) du = -\int e^{u}\, du = -e^{u} + c$$

$$\therefore \int \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^{\alpha}}\, dx = -e^{-\beta x^{\alpha}} + c$$

(68) $\int (2x-3)\sqrt{4x+1}\, dx$

$$\text{Let } I = \int (2x-3)\sqrt{4x+1}\, dx$$

$$\text{Put } (4x+1) = t^2 \Rightarrow x = \frac{1}{4}(t^2 - 1) \qquad \therefore dx = \frac{t}{2}\, dt$$

$$\therefore\ I = \int \left\{2 \cdot \frac{1}{4}(t^2-1) - 3\right\}(t)\left(\frac{t}{2}\right) dt = \int \frac{1}{2}(t^2 - 1 - 6) \cdot \frac{t^2}{2}\, dt$$

$$= \frac{1}{4}\int (t^4 - 7t^2)\, dt = \frac{1}{4}\left(\frac{t^5}{5} - \frac{7}{3}t^3\right) + c$$

$$\int (2x-3)\sqrt{4x+1}\, dx = \frac{1}{20}(4x+1)^{5/2} - \frac{7}{12}(4x+1)^{3/2} + c$$

EXERCISE 9.5

Integrate the following

(1) $x^5(1 + x^6)^7$ (2) $\dfrac{(2lx + m)}{lx^2 + mx + n}$ (3) $\dfrac{4ax + 2b}{(ax^2 + bx + c)^{10}}$

(4) $\dfrac{x}{\sqrt{x^2 + 3}}$ (5) $(2x + 3)\sqrt{x^2 + 3x - 5}$ (6) $\tan x$

(7) $\sec x$ (8) $\cos^{14}x \sin x$ (9) $\sin^5 x$

(10) $\cos^7 x$ (11) $\dfrac{1 + \tan x}{x + \log \sec x}$ (12) $\dfrac{e^{m\tan^{-1}x}}{1 + x^2}$

(13) $\dfrac{x\sin^{-1}(x^2)}{\sqrt{1 - x^4}}$ (14) $\dfrac{5(x + 1)(x + \log x)^4}{x}$ (15) $\dfrac{\sin(\log x)}{x}$

(16) $\dfrac{\cot x}{\log \sin x}$ (17) $\sec^4 x \tan x$ (18) $\tan^3 x \sec x$

(19) $\dfrac{\sin x}{\sin(x + a)}$ (20) $\dfrac{\cos x}{\cos(x - a)}$ (21) $\dfrac{\sin 2x}{a\cos^2 x + b\sin^2 x}$

(22) $\dfrac{1 - \tan x}{1 + \tan x}$ (23) $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$ (24) $\dfrac{(\log x)^2}{x}$

(25) $e^{3\log x}\, e^{x^4}$ (26) $\dfrac{x^{e-1} + e^{x-1}}{x^e + e^x + e^e}$ (27) $x(l - x)^{16}$

(28) $x(x - a)^m$ (29) $x^2(2 - x)^{15}$ (30) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$

(31) $(x + 1)\sqrt{2x + 3}$ (32) $(3x + 5)\sqrt{2x + 1}$ (33) $(x^2 + 1)\sqrt{x + 1}$

9.3.3 Integration by parts

Integration by parts method is generally used to find the integral when the integrand is a product of two different types of functions or a single logarithmic function or a single inverse trigonometric function or a function which is not integrable directly.

From the formula for derivative of product of two functions we obtain this useful method of integration.

If $f(x)$ and $g(x)$ are two differentiable functions then we have

$$\frac{d}{dx}\,[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$$

By definition of anitiderivative

$$f(x)\,g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

rearranging we get

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g\,(x)\,dx \qquad \dots (1)$$

For computational purpose a more convenient form of writing this formula is obtained by

letting $\qquad u = f(x) \quad \text{and } v = g(x)$

$$\therefore \quad du = f'(x)\,dx \quad \text{and} \quad dv = g'(x)\,dx$$

So that (1) becomes $\qquad \boxed{\int u\,dv = uv - \int v\,du}$

The above formula expresses the integral.

$\int u\,dv$ interms of another integral $\int v\,du$ and does not give a final expression for the integral $\int u\,dv$. It only partially solves the problem of integrating the product uv'. Hence the term **'Partial Integration'** has been used in many European countries. The term **"Integration by Parts"** is established in many other languages as well as in our own.

The success of this method depends on the proper choice of u

(i) If integrand contains any non integrable functions directly from the formula, like $\log x$, $\tan^{-1}x$ etc., we have to take these unintegrable functions as u and other as dv.

(ii) If the integrand contains both the integrable function, and one of these is x^n (where n is a positive integer). Then take $u = x^n$.

(iii) For other cases choice of u is ours.

Examples **Suitable substitution for *u***

No.	Given Integrals	u	dv	Reason for u
1.	$\int \log x\, dx$ $\int \tan^{-1} x\, dx$	$\log x$ $\tan^{-1}x$	dx dx	$\log x$ and $\tan^{-1}x$ are not integrable directly from the formula.
2.	$\int x^n \log x\, dx$	$\log x$	$x^n\, dx$	
3.	$\int x^n \tan^{-1} x\, dx$	$\tan^{-1}x$	$x^n\, dx$	
4.	$\int x^n e^{ax}\, dx$ (n is a positive integer)	x^n	$e^{ax}\, dx$	both are integrable and power of x will be reduced by successive differentiation
5.	$\int x^n(\sin x \text{ or } \cos x)dx$	x^n	$\sin x\, dx$ or $\cos x dx$	both are integrable and power of x will be reduced by successive differentiation
6.	$\int e^{ax} \cos bx\, dx$ or $\int e^{ax} \sin bx\, dx$	e^{ax} or $\cos bx$ / $\sin bx$	Remains	–

Example 9.69 – 9.84: Integrate

(69) xe^x (70) $x \sin x$ (71) $x \log x$ (72) $x \sec^2 x$

(73) $x \tan^{-1}x$ (74) $\log x$ (75) $\sin^{-1}x$ (76) $x \sin^2 x$

(77) $x \sin 3x \cos 2x$ (78) $x\, 5^x$ (79) $x^3 e^{x^2}$ (80) $e^{\sqrt{x}}$

(81) $\displaystyle\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\, dx$ (82) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (83) $x^2 e^{3x}$ (84) $x^2 \cos 2x$

Solution:

(69) $\int xe^x\, dx = \int (x)\,(e^x dx)$

We apply integration by parts by taking

$$u = x \quad \text{and} \quad dv = e^x\, dx$$

Then $\qquad du = dx$ and $\quad v = \int e^x\, dx = e^x$

$$\therefore \quad \int xe^x\, dx = xe^x - \int e^x dx = xe^x - e^x + c$$

(70) $\int x \sin x\, dx = \int (x)\, (\sin x\, dx)$

We use integration by parts with

$$u = x \quad \text{and} \quad dv = \sin dx$$

$$du = dx \text{ and } v = -\cos x$$

$$\therefore \int x \sin x\, dx = (x)\,(-\cos x) - \int (-\cos x)\,(dx)$$

$$= -x \cos x + \int \cos x\, dx$$

$$\therefore \qquad \int x \sin x\, dx = -x \cos x + \sin x + c$$

(71) $\int x \log x = \int (\log x)\, (x\, dx)$

Since $\log x$ is not integrable take

$$u = \log x \quad \text{and} \quad dv = x\, dx$$

$$\therefore\ du = \frac{1}{x}\, dx \qquad v = \frac{x^2}{2}$$

$$\therefore \qquad \int x \log x = (\log x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right) \left(\frac{1}{x}\, dx\right)$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} \int x\, dx$$

$$\therefore \qquad \int x \log x = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + c$$

(72) $\int x \sec^2 x\, dx = \int (x)\, (\sec^2 x\, dx)$

Applying integration by parts, we get $\qquad dv = \sec^2 x\, dx$

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx \qquad v = \tan x$$

$$= x \tan x - \log \sec x + c \qquad u = x$$

$$\therefore \quad \int x \sec^2 x\, dx = x \tan x + \log \cos x + c \qquad du = dx$$

(73) $\int x \tan^{-1} x\, dx = \int (\tan^{-1} x)\, (x\, dx)$

Applying integration by parts, we get

$u = \tan^{-1}x \quad dv = x\,dx \quad du = \dfrac{1}{1+x^2}\,dx \quad v = \dfrac{x^2}{2}$

$$\int x \tan^{-1}x\, dx = (\tan^{-1}x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left[\frac{(x^2+1)-1}{1+x^2}\right] dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$\mathrm{I} = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - (\tan^{-1}x)\right] + c$$

$$\therefore \int x \tan^{-1}x\, dx = \frac{1}{2}\,[x^2 \tan^{-1}x + \tan^{-1}x - x] + c$$

(74) $\int \log x\, dx = \int (\log x)\,(dx)$

Applying integration by parts, we get

$u = \log x \quad dv = dx \quad du = \dfrac{1}{x}\,dx \quad v = x$

$$= (\log x)\,(x) - \int x \cdot \frac{1}{x}\, dx$$

$$= x \log x - \int dx$$

$$\therefore \quad \int \log x\, dx = x \log x - x + c$$

(75) $\int \sin^{-1}x\, dx = \int (\sin^{-1}x)\,(dx)$

Applying integration by parts, we get

$u = \sin^{-1}x \quad dv = dx \quad du = \dfrac{1}{\sqrt{1-x^2}}\,dx \quad v = x$

$$\int \sin^{-1}x\, dx = (\sin^{-1}x)\,(x) - \int x \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= x \sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

Applying substitution method by substituting

$$\sqrt{1-x^2} = t$$
$$1-x^2 = t^2$$
$$-2x\,dx = 2t\,d\,t$$
$$dx = \frac{2tdt}{-2x} = \frac{-t}{x}\,dt$$
$$\therefore \quad \int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{t}\left(\frac{-t}{x}\,dt\right)$$
$$= x\sin^{-1}x + \int dt \quad = x\ \sin^{-1}x + t + c$$
$$\therefore \quad \int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + c$$

(76) $\int x\sin^2 x\,dx$

Let $\quad I = \int x\sin^2 x\,dx$

[To eliminate power of $\sin x$, $\sin^2 x = \frac{1}{2}\,(1-\cos 2x)$]

$$= \int x\left\{\frac{1}{2}(1-\cos 2x)\right\}dx$$
$$= \frac{1}{2}\int (x - x\cos 2x)\,dx$$
$$= \frac{1}{2}\left[\int x dx - \int x\cos 2x\,dx\right]$$
$$I = \frac{1}{2}\left[\frac{x^2}{2} - I_1\right] \qquad \dots (1)$$

where $\quad I_1 = \int x\cos 2x\,dx$

Applying integration by parts for I_1

$$dv = \cos 2x\,dx$$
$$u = x \qquad v = \frac{\sin 2x}{2}$$
$$du = dx$$

$$I_1 = \int (x)\ (\cos 2x\,dx)$$
$$= \left[\frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}\,dx\right]$$
$$= \frac{x}{2}\ \sin 2x - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$
$$I_1 = \frac{x}{2}\ \sin 2x + \frac{1}{4}\ \cos 2x$$

substituting I_1 in (1) we get

$$I = \frac{1}{2}\left[\frac{x^2}{2} - I_1\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left(\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)\right] + c$$

$$\therefore \int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2x}{8} + c$$

(77) $\int x \sin 3x \cos 2x \, dx = \int x \frac{1}{2}[\sin(3x + 2x) + \sin(3x - 2x)] \, dx$

$$\left(\because \sin A \cos B = \frac{1}{2}\{\sin(A + B) + \sin(A - B)\}\right)$$

$$= \int x \frac{1}{2}[\sin(3x + 2x) + \sin(3x - 2x)] \, dx$$

Applying integration by parts, we get $\quad u = x \quad dv = (\sin 5x + \sin x)dx$

$$= \frac{1}{2}\int x(\sin 5x + \sin x) \, dx \qquad du = dx \quad v = \left(-\frac{\cos 5x}{5} - \cos x\right)$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 5x}{5} - \cos x\right) - \int\left(-\frac{\cos 5x}{5} - \cos x\right)dx\right]$$

$$= \frac{1}{2}\left[-x\left(\frac{\cos 5x}{5} + \cos x\right) + \int\left(\frac{\cos 5x}{5} + \cos x\right)dx\right]$$

$$= \frac{1}{2}\left[-x\left(\frac{\cos 5x}{5} + \cos x\right) + \left(\frac{\sin 5x}{5 \times 5} + \sin x\right)\right] + c$$

$$\therefore \int x \sin 3x \cos 2x \, dx = \frac{1}{2}\left[-x\left(\frac{\cos 5x}{5} + \cos x\right) + \frac{\sin 5x}{25} + \sin x\right] + c$$

(78) $\int x\, 5^x \, dx = \int (x)(5^x \, dx)$

Applying integration by parts, we get $\qquad dv = 5^x \, dx$

$$\int x\, 5^x \, dx = x\frac{5^x}{\log 5} - \int \frac{5^x}{\log 5} dx \qquad u = x \qquad v = \frac{5^x}{\log 5}$$

$$= \frac{x 5^x}{\log 5} - \frac{1}{\log 5} \cdot \frac{5^x}{\log 5} + c \qquad du = dx$$

$$\therefore \int x\, 5^x \, dx = \frac{x 5^x}{\log 5} - \frac{5^x}{(\log 5)^2} + c$$

For the following problems (79) to (82), first we have to apply substitution method to convert the given problem into a convenient form to apply integration by parts.

(79) $\int x^3 e^{x^2} \, dx$

Let $\qquad I = \int x^3 e^{x^2}\, dx$

$$\text{put } x^2 = t$$

$$\therefore\ 2x\, dx = dt$$

$$\therefore\ dx = \frac{dt}{2\text{x}}$$

$$\therefore\ I = \int x^3 \,.\, e^t \,.\, \frac{dt}{2x}$$

$$= \frac{1}{2} \int x^2\, e^t\, dt \quad = \quad \frac{1}{2} \int (t)\, (e^t\, dt)$$

$dv = e^t\, dt$

$u = t \qquad v = e^t$

Now let us use integration by parts method

$du = dt$

$$\therefore\ I = \frac{1}{2} \left(te^t - \int e^t\, dt \right)$$

$$= \frac{1}{2} \left(te^t - e^t + c \right) \ = \ \frac{1}{2} (x^2 e^{x^2} - e^{x^2} + c)$$

$$\therefore \int x^3 e^{x^2}\, dx \ = \frac{1}{2}\ (x^2 e^{x^2} - e^{x^2}) + c$$

(80) $\int e^{\sqrt{x}}\, dx$

$$\text{Let } I = \int e^{\sqrt{x}}\, dx$$

$$\text{put } \sqrt{x} = t$$

$$\therefore\ x = t^2 \quad \Rightarrow \quad dx = 2t\, dt$$

$$I = \int e^t\, 2tdt$$

$$= 2 \int (t)\, (\text{e}^\text{t}\, dt)$$

$dv = e^t\, dt$

Now applying integration by parts, we get

$u = t \qquad v = e^t$

$du = dt$

$$I = 2 \left(te^t - \int e^t\, dt \right)$$

$$= 2\ (te^t - e^t) + c$$

$(\because\ t = \sqrt{x}\ )$

$$\therefore \qquad \int e^{\sqrt{x}}\, dx = 2\left(\sqrt{x}\, e^{\sqrt{x}} - e^{\sqrt{x}}\right) + c$$

(81) $\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\, dx$

Let $\quad I = \int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\, dx$

put $\sin^{-1}x = t \quad \Rightarrow \quad x = \sin t$

$$\frac{1}{\sqrt{1-x^2}}\, dx = dt$$

$$dx = \sqrt{1-x^2}\; dt$$

$$\therefore \quad I = \int x \frac{t}{\sqrt{1-x^2}} \cdot \left(\sqrt{1-x^2}\, dt\right)$$

$$= \int xt\, dt$$

$$= \int (\sin t)\,(t)\, dt$$

$$I = \int (t)\;(\sin t\, dt)$$

$u = t \qquad dv = \sin t\, dt$

$du = dt \qquad v = -\cos t$

Applying integration by parts, we get

$$= t\,(-\cos t) - \int (-\cos t)\, dt$$

$$= -t\cos t + \int \cos t\, dt$$

$$= -t\cos t + \sin t + c$$

$$I = -(\sin^{-1}x)\left(\sqrt{1-x^2}\right) + x + c$$

$$\therefore \quad \int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\, dx = x - \sqrt{1-x^2}\;\sin^{-1}x + c$$

$\because t = \sin^{-1}x \Rightarrow \sin t = x$

$\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$

(82) $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Let $\quad I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\; d\theta$

$$\therefore \quad I = \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \sec^2\theta \, d\theta$$

$$= \int \tan^{-1}(\tan 2\theta) \sec^2\theta \, d\theta$$

$$= \int 2\theta \sec^2\theta \, d\theta$$

$$= 2\int (\theta)(\sec^2\theta \, d\theta)$$

$dv = \sec^2\theta \, d\theta$

$u = \theta \qquad v = \tan\theta$

$du = d\theta$

Applying integration by parts

$$\therefore \; I = 2\left[\theta \tan\theta - \int \tan\theta \, d\theta\right]$$

$$= 2\theta \tan\theta - 2 \log \sec\theta + c$$

$$I = 2(\tan^{-1}x)(x) - 2\log\sqrt{1+\tan^2\theta} + c$$

$$\therefore \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx = 2x \tan^{-1}x - 2\log\sqrt{1+x^2} + c$$

For the following problems (83) and (84) we have to apply the integration by parts twice to find the solution.

(83) $\int x^2 e^{3x} \, dx = \int (x^2)(e^{3x} \, dx)$

Applying integration by parts, we get

$dv = e^{3x} \, dx$

$u = x^2 \qquad v = \dfrac{e^{3x}}{3}$

$du = 2x \, dx$

$$\int x^2 e^{3x} \, dx = \frac{x^2 e^{3x}}{3} - \int \frac{e^{3x}}{3} \, 2x \, dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int (x)(e^{3x} \, dx)$$

again applying integration by parts, we get

$dv = e^{3x} \, dx$

$u = x \qquad v = \dfrac{e^{3x}}{3}$

$du = dx$

$$\int x^2 e^{3x} \, dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left\{x.\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \, dx\right\}$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9}\int e^{3x} \, dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{27} e^{3x} + c$$

$$\therefore \int x^2 e^{3x} \, dx = \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + c$$

(84) $\int x^2 \cos 2x\, dx = \int (x^2)\ (\cos 2x\, dx)$

Applying integration by part, we get

$$u = x^2 \qquad dv = \cos 2x\, dx$$
$$du = 2x\, dx \qquad v = \frac{\sin 2x}{2}$$

$$\int x^2 \cos 2x\, dx = x^2 \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} . 2x\, dx$$
$$= x^2 \frac{\sin 2x}{2} - \int (x)\ (\sin 2x\, dx)$$

again applying integration by parts we get

$$u = x \qquad dv = \sin 2x\, dx$$
$$du = dx \qquad v = \frac{-\cos 2x}{2}$$

$$\int x^2 \cos 2x\, dx = x^2 \frac{\sin 2x}{2} - \left\{\frac{x(-\cos 2x)}{2} - \int\left(\frac{-\cos 2x}{2}\, dx\right)\right\}$$
$$= \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{1}{2}\int \cos 2x\, dx$$
$$\mathrm{I} = \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{1}{4}\sin 2x + c$$
$$\therefore \int x^2 \cos 2x\, dx = \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x \cos 2x - \frac{1}{4}\sin 2x + c$$

The following examples illustrate that there are some integrals whose integration continues forever.

Example 9.85 – 9.87: Evaluate the following

(85) $\int e^x \cos x\, dx$ (86) $\int e^{ax} \sin bx\, dx$ (87) $\int \sec^3 x\, dx$

Solution:

(85) $\int e^x \cos x\, dx = \int (e^x)\ (\cos x\, dx)$

Here both the functions in the integrand are integrable directly from the formula. Hence the choice of u is ours.

Applying the integration by parts

$$u = e^x \qquad dv = \cos x\, dx$$
$$du = e^x\, dx \qquad v = \sin x$$

$$\int e^x \cos x\, dx = e^x \sin x - \int \sin x\, e^x\, dx$$
$$= e^x \sin x - \int (e^x)\ (\sin x\, dx) \quad \ldots (1)$$

Again applying integration by parts we get

$$u = e^x \qquad dv = \sin x\, dx$$
$$du = e^x\, dx \qquad v = -\cos x$$

$$\int e^x \cos x\, dx = e^x \sin x - \left[e^x(-\cos x) - \int(-\cos x)(e^x\, dx)\right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

i.e. $\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx \quad \ldots (2)$

Note that $\int e^x \cos x\, dx$ appears on both the sides.

$\therefore$ rearranging, we get

$$2\int e^x \cos x\, dx = (e^x \sin x + e^x \cos x)$$

$$\therefore \int e^x \cos x\, dx = \frac{1}{2}[e^x \sin x + e^x \cos x] + c$$

$$\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x) + c$$

(86) $\int \boldsymbol{e^{ax} \sin bx\, dx} = \int (\sin bx)(e^{ax} dx)$

since both functions are integrable,
we can take any one of them as u

$u = \sin bx$

$du = b \cos bx\, dx$

$dv = e^{ax}\, dx$

$v = \dfrac{e^{ax}}{a}$

$$\int e^{ax} \sin bx\, dx = (\sin bx)\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}(b \cos bx)\, dx$$

$$= \frac{1}{a} \quad e^{ax} \quad \sin \quad bx \quad - \quad \frac{b}{a}$$

$\int \cos bx \,.\, e^{ax}\, dx$

$dv = e^{ax}\, dx$

$u = \cos bx$

$du = -b \sin bx\, dx$

$v = \dfrac{e^{ax}}{a}$

Again applying integration by parts we get

$$\int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a}\left[(\cos bx)\left(\frac{e^{ax}}{a}\right) - \int \frac{e^{ax}}{a}(-b \sin bx\, dx)\right]$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2}\int e^{ax} \sin bx\, dx$$

$$\int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2}\int e^{ax} \sin bx\, dx$$

The integral on the right hand side is same as the integral on the left hand side.

$\therefore$ Rearranging we get

$$\int e^{ax} \sin bx\, dx + \frac{b^2}{a^2}\int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx$$

i.e. $$\left[1+\frac{b^2}{a^2}\right]\int e^{ax}\sin bx\,dx = \left[\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right]$$

$$\left(\frac{a^2+b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = e^{ax}\left(\frac{a\sin bx - b\cos bx}{a^2}\right)$$

$$\therefore \quad \int e^{ax}\sin bx\,dx = \left(\frac{a^2}{a^2+b^2}\right)\times\frac{e^{ax}}{a^2}\,(a\sin bx - b\cos bx)$$

$$\boxed{\therefore \int \boldsymbol{e^{ax}}\,\mathbf{sin}\,\boldsymbol{bx\,dx} = \left(\frac{\boldsymbol{e^{ax}}}{\boldsymbol{a^2+b^2}}\right)(\boldsymbol{a}\,\mathbf{sin}\,\boldsymbol{bx} - \boldsymbol{b}\,\mathbf{cos}\,\boldsymbol{bx}) + \boldsymbol{c}}$$

Whenever we integrate function of the form $e^{ax}\cos bx$ or $e^{ax}\sin bx$, we have to apply the Integration by Parts rule twice to get the similar integral on both sides to solve.

Caution:

In applying integration by parts to specific integrals, once pair of choice for u and dv initially assumed should be maintained for the successive integrals on the right hand side. (See the above two examples). The pair of choice should not be interchanged.

Consider the example: $\int e^x \sin x\,dx$ **Initial assumption**

$$\int e^x\sin x\,dx = -e^x\cos x + \int \cos x\,e^x\,dx$$

$dv = \sin x\,dx$

$u = e^x \qquad v = -\cos x$

$du = e^x\,dx$

Again applying integration by parts for R.H.S by interchanging the initial assumption we get

$$\int e^x\sin x\,dx = -e^x\cos x + \int \cos x\,e^x - \int e^x(-\sin x)\,dx$$

$dv = e^x\,dx$

$u = \cos x \qquad v = e^x$

$du = -\sin x\,dx$

$$\int e^x\sin x\,dx = -e^x\cos x + \cos x\,e^x + \int e^x\sin x\,dx$$

$$\int e^x\sin x\,dx = \int e^x\sin x\,dx \quad ?$$

Finally we have arrived at the same given problem on R.H.S!

87) $\int \sec^3 x\,dx = \int(\sec x)\,(\sec^2 x\,dx)$

Applying integration by parts, we get

$$\int \sec^3 x\,dx = \sec x\tan x - \int(\tan x)(\sec x\tan x\,dx)$$

$$= \sec x\tan x - \int \tan^2 x\sec x\,dx$$

$dv = \sec^2 x\,dx$

$u = \sec x \qquad v = \tan x$

$du = \sec x\tan x\,dx$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \log (\sec x + \tan x)$$

Rearranging we get,

$$2 \int \sec^3 x \, dx = \sec x \tan x + \log (\sec x + \tan x)$$

$$\int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \log (\sec x + \tan x)] + c$$

EXERCISE 9.6

Integrate the followings with respect to x

(1) xe^{-x} (2) $x \cos x$ (3) $x \operatorname{cosec}^2 x$ (4) $x \sec x \tan x$

(5) $\tan^{-1} x$ (6) $x \tan^2 x$ (7) $x \cos^2 x$ (8) $x \cos 5x \cos 2x$

(9) $2x\, e^{3x}$ (10) $x^2 e^{2x}$ (11) $x^2 \cos 3x$ (12) $(\sin^{-1} x) \dfrac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$

(13) $x^5 e^{x^2}$ (14) $\tan^{-1} \left(\dfrac{3x - x^3}{1 - 3x^2} \right)$ (15) $x \sin^{-1}(x^2)$ (16) $\operatorname{cosec}^3 x$

(17) $e^{ax} \cos bx$ (18) $e^{2x} \sin 3x$ (19) $e^x \cos 2x$ (20) $e^{3x} \sin 2x$

(21) $\sec^3 2x$ (22) $e^{4x} \cos 5x \sin 2x$ (23) $e^{-3x} \cos^3 x$

Type I: 9.88 – 9.93: Standard integrals

(88) $\displaystyle\int \frac{dx}{a^2 - x^2}$ **(89)** $\displaystyle\int \frac{dx}{x^2 - a^2}$ **(90)** $\displaystyle\int \frac{dx}{a^2 + x^2}$

(91) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$ **(92)** $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$ **(93** $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}}$

Solution:

(88)
$$\int \frac{dx}{a^2 - x^2} = \int \frac{1}{(a-x)(a+x)}\,dx$$

$$= \frac{1}{2a}\int \frac{2a}{(a-x)(a+x)}\,dx$$

$$= \frac{1}{2a}\int \frac{(a-x)+(a+x)}{(a-x)(a+x)}\,dx$$ or use Partial fraction method

$$= \frac{1}{2a}\int \left[\frac{1}{a+x} + \frac{1}{a-x}\right] dx$$

$$= \frac{1}{2a}\left[\log(a+x) - \log(a-x)\right]$$

$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$$

(89)
$$\int \frac{dx}{x^2 - a^2}\,dx = \int \frac{dx}{(x-a)(x+a)}$$

$$= \frac{1}{2a}\int \frac{2a}{(x-a)(x+a)}\,dx = \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}\,dx$$

$$= \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

$$= \frac{1}{2a}\left[\log(x-a) - \log(x+a)\right]$$

$$\therefore \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$$

(90) Let $I = \int \frac{dx}{a^2 + x^2}$

put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}(x/a)$

$dx = a\sec^2\theta\, d\theta$

$$\therefore\ I = \int \frac{a \sec^2\theta d\theta}{a^2 + a^2 \tan^2\theta} = \int \frac{a \sec^2\theta d\theta}{a^2 \sec^2\theta} = \frac{1}{a} \int d\theta$$

$$I = \frac{1}{a}\ \theta + c$$

$$\therefore\ \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\ \tan^{-1} \frac{x}{a} + c$$

(91) Let $I = \int \frac{dx}{\sqrt{a^2 - x^2}}$

$$\text{put } x = a \sin\theta \Rightarrow \theta = \sin^{-1} (x / a)$$

$$dx = a \cos\theta\ d\theta$$

$$\therefore\ I = \int \frac{a \cos\theta\ d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} = \int \frac{a \cos\theta\ d\theta}{a \sqrt{1 - \sin^2\theta}}$$

$$= \int \frac{1}{\cos\theta} \cos\theta\ d\ \theta = \int d\theta$$

$$I = \theta + c$$

$$\therefore\ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$$

(92) Let $I = \int \frac{1}{\sqrt{x^2 - a^2}}\ dx$

$$\text{put } u = x + \sqrt{x^2 - a^2}$$

$$du = \left(1 + \frac{(2x)}{2\sqrt{x^2 - a^2}}\right) dx = \left(\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}}\right) dx$$

$$\therefore\ dx = \frac{\sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}}\ du = \frac{\sqrt{x^2 - a^2}}{u}\ du$$

$$\therefore\ I = \int \frac{1}{\sqrt{x^2 - a^2}} \cdot \left(\frac{\sqrt{x^2 - a^2}}{u} du\right)$$

$$= \int \frac{1}{u}\ du$$

$$\mathrm{I} = \log u + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left(x + \sqrt{x^2 - a^2}\right) + c$$

(Try the above problem by substituting $x = a \sec\theta$)

(93) $$\text{Let } \mathrm{I} = \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\text{put } u = x + \sqrt{x^2 + a^2}$$

$$du = \left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right) dx = \left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right) dx$$

$$\therefore\ dx = \frac{\sqrt{x^2 + a^2}}{x + \sqrt{x^2 + a^2}}\, du = \frac{\sqrt{x^2 + a^2}}{u}\, du$$

$$\therefore\ \mathrm{I} = \int \frac{1}{\sqrt{x^2 + a^2}} \cdot \left(\frac{\sqrt{x^2 + a^2}}{u}\, du\right)$$

$$= \int \frac{1}{u}\, du$$

$$\mathrm{I} = \log u + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right) + c$$

(Try the above problem by substituting $x = a \tan\theta$)

Remark: Remember the following useful substitution of the given integral as a functions of

Given	Substitution
$a^2 - x^2$	$x = a \sin\theta$
$a^2 + x^2$	$x = a \tan\theta$
$x^2 - a^2$	$x = a \sec\theta$

Example 9.94 – 9.105 :

Integrate :

(94) $\dfrac{1}{1 + 9x^2}$ (95) $\dfrac{1}{1 - 9x^2}$ (96) $\dfrac{1}{1 + \dfrac{x^2}{16}}$ (97) $\dfrac{1}{1 - 4x^2}$

(98) $\dfrac{1}{(x+2)^2-4}$ (99) $\dfrac{1}{(2x+1)^2-9}$ (100) $\dfrac{1}{\sqrt{25-x^2}}$ (101) $\dfrac{1}{\sqrt{1-\dfrac{x^2}{16}}}$

(102) $\dfrac{1}{\sqrt{1-16x^2}}$ (103) $\dfrac{1}{\sqrt{x^2-9}}$ (104) $\dfrac{1}{\sqrt{4x^2-25}}$ (105) $\dfrac{1}{\sqrt{9x^2+16}}$

Solution:

(94)
$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$$
$$= \left[\tan^{-1}\left(\frac{3x}{1}\right)\right] \times \frac{1}{3} + c$$
$$= \frac{1}{3}\tan^{-1} 3x + c$$

(95)
$$\int \frac{1}{1-9x^2}\,dx = \int \frac{1}{1-(3x)^2}\,dx$$
$$= \frac{1}{2.1}\log\left(\frac{1+3x}{1-3x}\right) \times \frac{1}{3}$$
$$= \frac{1}{6}\log\left(\frac{1+3x}{1-3x}\right) + c$$

(96)
$$\int \frac{1}{1+\dfrac{x^2}{16}}\,dx = \int \frac{1}{1+\left(\dfrac{x}{4}\right)^2}\,dx$$
$$= \left[\frac{1}{1}\tan^{-1}\left(\frac{x}{4}\right)\right]\frac{1}{(1/4)}$$
$$= 4\tan^{-1}\left(\frac{x}{4}\right) + c$$

(97)
$$\int \frac{1}{1-4x^2}\,dx = \int \frac{1}{1-(2x)^2}\,dx$$
$$= \left[\frac{1}{2.1}\log\left(\frac{1+2x}{1-2x}\right)\right] \times \frac{1}{2}$$

$$= \frac{1}{4} \log\left(\frac{1 + 2x}{1 - 2x}\right) + c$$

(98) $$\int \frac{dx}{(x+2)^2 - 4} = \int \frac{dx}{(x+2)^2 - 2^2}$$

$$= \frac{1}{2.(2)} \log\left(\frac{(x+2) - 2}{(x+2) + 2}\right)$$

$$= \frac{1}{4} \log\left(\frac{x}{x+4}\right) + c$$

(99) $$\int \frac{1}{(2x+1)^2 - 9} \, dx = \int \frac{1}{(2x+1)^2 - 3^2} \, dx$$

$$= \left[\frac{1}{2.(3)} \log\left(\frac{(2x+1) - 3}{(2x+1) + 3}\right)\right] \times \frac{1}{2}$$

$$= \frac{1}{12} \log\left(\frac{2x - 2}{2x + 4}\right)$$

$$= \frac{1}{12} \log\left(\frac{x+1}{x+2}\right) + c$$

(100) $$\int \frac{1}{\sqrt{25 - x^2}} \, dx = \int \frac{1}{\sqrt{5^2 - x^2}} \, dx$$

$$= \sin^{-1} \frac{x}{5} + c$$

(101) $$\int \frac{1}{\sqrt{1 - \frac{x^2}{16}}} \, dx = \int \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \, dx$$

$$= \left[\sin^{-1}\left(\frac{x}{4}\right)\right] \cdot \frac{1}{1/4}$$

$$= 4 \sin^{-1}\left(\frac{x}{4}\right) + c$$

(102) $$\int \frac{1}{\sqrt{1 - 16x^2}} \, dx = \int \frac{1}{\sqrt{1 - (4x)^2}} \, dx$$

$$= \left[\sin^{-1}(4x)\right] \frac{1}{4}$$

$$= \frac{1}{4} \sin^{-1}(4x) + c$$

(103) $$\int \frac{1}{\sqrt{x^2 - 9}}\, dx = \int \frac{1}{\sqrt{x^2 - 3^2}}\, dx$$

$$= \log\left(x + \sqrt{x^2 - 9}\right) + c$$

(104) $$\int \frac{1}{\sqrt{4x^2 - 25}}\, dx = \int \frac{1}{\sqrt{(2x)^2 - 5^2}}\, dx$$

$$= \log\left[2x + \sqrt{(2x)^2 - 5^2}\right] \times \frac{1}{2} + c$$

$$= \frac{1}{2} \log\left[2x + \sqrt{4x^2 - 25}\right] + c$$

(105) $$\int \frac{1}{\sqrt{9x^2 + 16}}\, dx = \int \frac{1}{\sqrt{(3x)^2 + 4^2}}\, dx$$

$$= \log\left[3x + \sqrt{(3x)^2 + 4^2}\right] \times \frac{1}{3} + c$$

$$= \frac{1}{3} \log\left[3x + \sqrt{9x^2 + 16}\right] + c$$

Type II: integral of the form $\int \frac{dx}{ax^2 + bx + c}$ **and** $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

In this case, we have to express $ax^2 + bx + c$ as sum or difference of two square terms to get the integrand in one of the standard forms of Type 1 mentioned earlier.

We first make the co-efficient of x^2 numerically one. Complete the square interms containing x^2 and x by adding and subtracting the square of half the coefficient of x.

i.e. $$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$$

OR We can directly use a formula for

$$\boxed{ax^2 + bx + c = \frac{1}{4a}\left[(2ax + b)^2 + (4ac - b^2)\right]}$$

Example 9.106 – 9.113: Integrate the following:

(106) $\dfrac{1}{x^2 + 5x + 7}$ (107) $\dfrac{1}{x^2 - 7x + 5}$ (108) $\dfrac{1}{\sqrt{x^2 + 16x + 100}}$

(109) $\dfrac{1}{\sqrt{9 + 8x - x^2}}$ (110) $\dfrac{1}{\sqrt{6 - x - x^2}}$ (111) $\dfrac{1}{3x^2 + 13x - 10}$

(112) $\dfrac{1}{2x^2 + 7x + 13}$ (113) $\dfrac{1}{\sqrt{18 - 5x - 2x^2}}$

Solution:

(106) $$\int \frac{1}{x^2 + 5x + 7}\,dx = \int \frac{1}{\left(x + \frac{5}{2}\right)^2 + 7 - \left(\frac{5}{2}\right)^2}\,dx = \int \frac{1}{\left(x + \frac{5}{2}\right)^2 + \frac{3}{4}}\,dx$$

$$= \int \frac{1}{\left(x + \frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x + \frac{5}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$\int \frac{1}{x^2 + 5x + 7}\,dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 5}{\sqrt{3}}\right) + c$$

(107) $$\int \frac{1}{x^2 - 7x + 5}\,dx = \int \frac{1}{\left(x - \frac{7}{2}\right)^2 + 5 - \left(\frac{7}{2}\right)^2}\,dx = \int \frac{1}{\left(x - \frac{7}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2}\,dx$$

$$= \frac{1}{2 . \frac{\sqrt{29}}{2}}\log\left(\frac{\left(x - \frac{7}{2}\right) - \frac{\sqrt{29}}{2}}{\left(x - \frac{7}{2}\right) + \frac{\sqrt{29}}{2}}\right) + c$$

$$\int \frac{1}{x^2 - 7x + 5}\,dx = \frac{1}{\sqrt{29}}\log\left(\frac{2x - 7 - \sqrt{29}}{2x - 7 + \sqrt{29}}\right) + c$$

(108) $$\int \frac{1}{\sqrt{x^2 + 16x + 100}}\,dx = \int \frac{1}{\sqrt{(x + 8)^2 + 100 - (8)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{(x + 8)^2 + 6^2}}\,dx$$

$$= \log\left[(x+8) + \sqrt{(x+8)^2 + 6^2}\right] + c$$

$$= \log\left((x+8) + \sqrt{x^2 + 16x + 100}\right) + c$$

(109) $$\int \frac{1}{\sqrt{9+8x-x^2}}\,dx = \int \frac{1}{\sqrt{9-(x^2-8x)}}\,dx = \int \frac{1}{\sqrt{9-\left\{(x-4)^2-4^2\right\}}}\,dx$$

$$= \int \frac{1}{\sqrt{9+16-(x-4)^2}}\,dx = \int \frac{1}{\sqrt{5^2-(x-4)^2}}\,dx$$

$$\int \frac{1}{\sqrt{9+8x-x^2}}\,dx = \sin^{-1}\frac{x-4}{5} + c$$

(110) $$\int \frac{1}{\sqrt{6-x-x^2}}\,dx = \int \frac{1}{\sqrt{6-(x^2+x)}}\,dx = \int \frac{1}{\sqrt{6-\left\{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right\}}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(6+\frac{1}{4}\right)-\left(x+\frac{1}{2}\right)^2}}\,dx = \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{5}{2}}\right) + c = \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

$$\int \frac{1}{\sqrt{6-x-x^2}}\,dx = \sin^{-1}\left(\frac{2x+1}{5}\right) + c$$

For the following problems 111 to 113 the direct formula

$ax^2 + bx + c = \frac{1}{4a}\left[(2ax+b)^2 + (4ac-b^2)\right]$ is used.

(111) $$\int \frac{1}{3x^2+13x-10}\,dx = \int \frac{4\times 3}{(2\times 3x+13)^2 - 4\times 3\times 10 - 13^2}\,dx$$

$$= \int \frac{12}{(6x+13)^2-289}\,dx = 12\int \frac{1}{(6x+13)^2-17^2}\,dx$$

$$= 12\times\frac{1}{2\times 17}\left[\log\left(\frac{6x+13-17}{6x+13+17}\right)\right]\times\left(\frac{1}{6}\right)+c \qquad \begin{pmatrix}\text{6 is the coefficient}\\ \text{of } x\end{pmatrix}$$

$$= \frac{1}{17}\log\left(\frac{6x-4}{6x+30}\right)+c = \frac{1}{17}\log\left(\frac{3x-2}{3x+15}\right)+c$$

$$\int\frac{1}{3x^2+13x-10}\,dx = \frac{1}{17}\log\left(\frac{3x-2}{3x+15}\right)+c$$

(112) $$\int\frac{1}{2x^2+7x+13}\,dx = \int\frac{4\times 2}{(4x+7)^2+104-49}\,dx = 8\int\frac{1}{(4x+7)^2+\sqrt{55}^2}\,dx$$

$$= 8.\frac{1}{\sqrt{55}}\times\tan^{-1}\left(\frac{4x+7}{\sqrt{55}}\right)\times\left(\frac{1}{4}\right) \qquad \begin{pmatrix}\text{4 is the coefficient}\\ \text{of } x\end{pmatrix}$$

$$\int\frac{1}{2x^2+7x+13}\,dx = \frac{2}{\sqrt{55}}\tan^{-1}\left(\frac{4x+7}{\sqrt{55}}\right)+c$$

(113) $$\int\frac{1}{\sqrt{18-5x-2x^2}}\,dx = \int\frac{1}{\sqrt{-\left\{2x^2+5x-18\right\}}}dx \qquad \begin{pmatrix}\text{negative sign}\\ \text{should not be taken}\\ \text{outside from the}\\ \text{square root}\end{pmatrix}$$

$$= \int\frac{\sqrt{4\times 2}}{\sqrt{-\left\{(4x+5)^2-18\times 8-5^2\right\}}}\,dx$$

$$= \int\frac{2\sqrt{2}}{\sqrt{13^2-(4x+5)^2}}\,dx$$

$$= 2\sqrt{2}\left\{\sin^{-1}\left(\frac{4x+5}{13}\right)\right\}\times\left(\frac{1}{4}\right)+c$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+5}{13}\right)+c$$

$$\therefore \quad \int\frac{1}{\sqrt{18-5x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+5}{13}\right)+c$$

Type III :Integrals of the form $\displaystyle\int\frac{px+q}{ax^2+bx+c}\,dx$ **and** $\displaystyle\int\frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

To evaluate the above integrals, we have to express the numerator $px + q$ into two parts with suitable constants. One in terms of differential coefficient of denominator and the other without 'x' term.

Then the integrals will be separated into two standard form of known integrals and can easily be evaluated.

Let $\quad (px + q) = A \frac{d}{dx}(ax^2 + bx + c) + B$

i.e. $\quad (px + q) = A(2ax + b) + B$

(A & B can be found by equating coefficients of x and constant terms separately.

(i) $$\int \frac{px + q}{ax^2 + bx + c} = \int \frac{A(2ax + b) + B}{ax^2 + bx + c}\, dx$$

$$= A \int \left(\frac{2ax + b}{ax^2 + bx + c}\right) dx + B \int \frac{1}{ax^2 + bx + c}\, dx$$

$$\left(\because \int \frac{f'(x)}{f(x)}\, dx = \log f(x) \Rightarrow \int \left(\frac{2ax + b}{ax^2 + bx + c}\right) dx = [\log(ax^2 + bx + c)]\right)$$

$$\boxed{\therefore \int \frac{px + q}{ax^2 + bx + c}\, dx = A\,[\log(ax^2 + bx + c)] + B \int \frac{1}{ax^2 + bx + c}\, dx}$$

(ii) $$\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx = A \int \frac{(2ax + b)}{\sqrt{ax^2 + bx + c}}\, dx + B \int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

$$\left(\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} \Rightarrow \int \frac{(2ax + b)}{\sqrt{ax^2 + bx + c}}\, dx = 2\sqrt{ax^2 + bx + c}\right)$$

$$\boxed{\int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx = A\left(2\sqrt{ax^2 + bx + c}\right) + B \int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx}$$

Example 114:

Integrate the followings:

(114) $\frac{4x - 3}{x^2 + 3x + 8}$ $\qquad$ (115) $\frac{3x + 2}{x^2 + x + 1}$ $\qquad$ (116) $\frac{5x - 2}{x^2 - x - 2}$

(117) $\frac{3x + 1}{\sqrt{2x^2 + x + 3}}$ $\qquad$ (118) $\frac{x + 1}{\sqrt{8 + x - x^2}}$ $\qquad$ (119) $\frac{4x - 3}{\sqrt{x^2 + 2x - 1}}$

Solution:

(114) $\displaystyle\int \frac{4x-3}{x^2+3x+8}\,dx$

let $\quad 4x-3 = A\frac{d}{dx}(x^2+3x+8)+B$

$4x-3 = A(2x+3)+B \quad \ldots \text{(i)}$

rearranging $\quad 4x-3 = (2A)\,x+(3A+B)$

Equating like terms $\quad 2A = 4 \quad \Rightarrow \quad A = 2$

$3A+B = -3 \quad \Rightarrow \quad B = -3-3A = -9$

$\therefore$ (i) $\Rightarrow$

$$(4x-3) = 2(2x+3)+(-9)$$

$$\therefore \int \frac{4x-3}{x^2+3x+8}\,dx = \int \frac{2(2x+3)+(-9)}{x^2+3x+8}\,dx$$

$$= 2\int \frac{(2x+3)}{x^2+3x+8}\,dx - 9\int \frac{dx}{x^2+3x+8}$$

$$\int \frac{4x-3}{x^2+3x+8}\,dx = 2I_1 - 9I_2 \qquad \ldots (1)$$

Where $\quad I_1 = \displaystyle\int \frac{(2x+3)}{x^2+3x+8}\,dx$ and $I_2 = \displaystyle\int \frac{dx}{x^2+3x+8}\,dx$

$$I_1 = \int \frac{(2x+3)}{x^2+3x+8}\,dx$$

put $\quad x^2+3x-18 = u \quad \therefore \quad (2x+3)dx = du$

$$\therefore \quad I_1 = \int \frac{du}{u} = \log(x^2+3x+8) \qquad \ldots (2)$$

$$I_2 = \int \frac{dx}{x^2+3x+8} = \int \frac{4(1)}{(2x+3)^2+4\times 8-3^2}\,dx$$

$$= \int \frac{4}{(2x+3)^2+\left(\sqrt{23}\right)^2}\,dx = 4\times\frac{1}{\sqrt{23}}\times\frac{1}{2}\tan^{-1}\frac{2x+3}{\sqrt{23}}$$

$$I_2 = \frac{2}{\sqrt{23}}\tan^{-1}\frac{2x+3}{\sqrt{23}} \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$\therefore \quad \int \frac{4x-3}{x^2+3x+8}\,dx = 2\log(x^2+3x+8) - \frac{18}{\sqrt{23}}\tan^{-1}\frac{2x+3}{\sqrt{23}}$

(115) $\int \frac{3x+2}{x^2+x+1}\,dx$

Let $\quad 3x+2 = A\frac{d}{dx}(x^2+x+1) + B$

$(3x + 12) = \quad A(2x+1) + B \qquad \dots (1)$

i.e. $\quad 3x+2 = (2A)x + (A+B)$

Equating like terms

$2A = 3 \quad ; \quad A + B = 2$

$\therefore A = \frac{3}{2} \quad ; \quad \frac{3}{2} + B = 2 \qquad \Rightarrow B = 2 - \frac{3}{2} = \frac{1}{2}$

Substituting $A = \frac{3}{2}$ and $B = \frac{1}{2}$ in (1) we get

$\therefore (3x+2) = \frac{3}{2}(2x+1) + \left(\frac{1}{2}\right)$

$$\therefore \int \frac{3x+2}{x^2+x+1}\,dx = \int \frac{\frac{3}{2}(2x+1) + \left(\frac{1}{2}\right)}{x^2+x+1}\,dx$$

$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$\therefore \int \frac{3x+2}{x^2+x+1}\,dx = \frac{3}{2}\left\{\log(x^2+x+1)\right\} + I \quad \dots (2)$$

Where $\quad I = \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{4\times 1}{(2x+1)^2 + 4\times 1\times 1 - 1^2}\,dx$

$$= 2\int \frac{1}{(2x+1)^2 + (\sqrt{3})^2} = 2\times\frac{1}{\sqrt{3}}\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Substituting above I in (2), we get

$$\therefore \int \frac{3x+2}{x^2+x+1}\,dx = \frac{3}{2}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

(116) $\displaystyle\int \frac{5x-2}{x^2-x-2}\, dx$

$$\text{Let} \quad 5x-2 = A\frac{d}{dx}(x^2-x-2) + B$$

$$5x-2 = A(2x-1) + B \qquad \dots (1)$$

$$5x-2 = (2A)x - A + B$$

equating like terms $\quad 2A = 5 \;;\; -A + B = -2$

$$\therefore\ A = \frac{5}{2} \quad ; \; -\frac{5}{2} + B = -2 \Rightarrow B = -2 + \frac{5}{2} = \frac{1}{2}$$

Substituting $A = \frac{5}{2}$ and $B = \frac{1}{2}$ in (1), we get

$$(5x-2) = \frac{5}{2}(2x-1) + \frac{1}{2}$$

$$\therefore \int \frac{5x-2}{x^2-x-2}\, dx = \int \frac{\frac{5}{2}(2x-1) + \left(\frac{1}{2}\right)}{x^2-x-2}\, dx$$

$$= \frac{5}{2}\int \frac{2x-1}{x^2-x-2}\, dx + \frac{1}{2}\int \frac{1}{x^2-x-2}\, dx$$

$$\therefore \int \frac{5x-2}{x^2-x-2}\, dx = \frac{5}{2}\left\{\log(x^2-x-2)\right\} + I \dots (2)$$

Where $\quad I = \dfrac{1}{2}\displaystyle\int \frac{1}{x^2-x-2}\, dx = \frac{1}{2}\int \frac{4\times 1}{(2x-1)^2-8-1}\, dx$

$$= \frac{1}{2}\int \frac{4}{(2x-1)^2-3^2} = \frac{4}{2}\times\frac{1}{2\times 3}\,\frac{1}{2}\log\left[\frac{2x-1-3}{2x-1+3}\right]$$

$$I = \frac{1}{3\times 2}\log\left[\frac{2x-4}{2x+2}\right] = \frac{1}{6}\log\left(\frac{x-2}{x+1}\right)$$

Substituting I in (2), we get

$$\int \frac{5x-2}{x^2-x-2}\, dx = \frac{5}{2}\log(x^2-x-2) + \frac{1}{6}\log\left(\frac{x-2}{x+1}\right) + c$$

Note : Resolve into partial fractions and then integrate.

(117) $\displaystyle\int \frac{3x+1}{\sqrt{2x^2+x+3}}\, dx$

$$\text{Let } 3x + 1 = A \frac{d}{dx} (2x^2 + x + 3) + B$$

$$3x + 1 = A(4x + 1) + B \qquad \ldots (1)$$

$$3x + 1 = 4Ax + A + B$$

equating like terms $\quad 4A = 3 \; ; \; A + B = 1$

$$\therefore A = \frac{3}{4} \qquad B = 1 - A = 1 - \frac{3}{4} = \frac{1}{4}$$

by (i) $\quad \Rightarrow \; \therefore 3x + 1 = \frac{3}{4}(4x + 1) + \frac{1}{4}$

$$\therefore \int \frac{3x + 1}{\sqrt{2x^2 + x + 3}} \, dx = \int \frac{\frac{3}{4}(4x + 1) + \frac{1}{4}}{\sqrt{2x^2 + x + 3}} \, dx$$

$$= \frac{3}{4} \int \frac{4x + 1}{\sqrt{2x^2 + x + 3}} \, dx + \frac{1}{4} \int \frac{1}{\sqrt{2x^2 + x + 3}} \, dx$$

$$\therefore \int \frac{3x + 1}{\sqrt{2x^2 + x + 3}} \, dx = \frac{3}{4} \left\{ 2\sqrt{2x^2 + x + 3} \right\} + \mathbf{I} \ldots (2) \quad \left(\because \int \frac{f'(x)}{\sqrt{f(x)}} \, dx = 2\sqrt{f(x)} \right)$$

Where
$$\mathbf{I} = \frac{1}{4} \int \frac{1}{\sqrt{2x^2 + x + 3}} \, dx$$

$$= \frac{1}{4} \int \frac{\sqrt{4.2}}{\sqrt{(4x + 1)^2 + 24 - 1}} \, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(4x + 1)^2 + (\sqrt{23})^2}} \, dx$$

$$\mathbf{I} = \frac{1}{\sqrt{2}} \left[\log (4x + 1) + \sqrt{(4x + 1)^2 + 23} \right] \times \frac{1}{4}$$

substituting in (2) we get

$$\int \frac{3x + 1}{\sqrt{2x^2 + x + 3}} \, dx = \frac{3}{2} \sqrt{2x^2 + x + 3} + \frac{1}{4\sqrt{2}} \left\{ \log (4x+1) + \sqrt{(4x+1)^2 + 23} \right\} + c$$

(118) $\displaystyle \int \frac{x + 1}{\sqrt{8 + x - x^2}} \, dx$

$$\text{Let } x + 1 = A \frac{d}{dx} (8 + x - x^2) + B$$

$$x + 1 = A(1 - 2x) + B \qquad \ldots (1)$$

$$= (-2A)x + A + B$$

equating like terms $\quad -2A = 1 \;;\quad A + B = 1$

$$\therefore\ A = -\frac{1}{2} \qquad B = 1 - A = 1 - \frac{1}{2} = \frac{3}{2}$$

Substituting $\quad A = -\frac{1}{2}$ and $B = \frac{3}{2}$

by (1) $\quad x + 1 = -\frac{1}{2}(1 - 2x) + \frac{3}{2}$

$$\therefore \int \frac{x+1}{\sqrt{8 + x - x^2}}\,dx = \int \frac{-\frac{1}{2}(1-2x) + \frac{3}{2}}{\sqrt{8 + x - x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{(1-2x)}{\sqrt{8 + x - x^2}}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{8 + x - x^2}}\,dx$$

$$\therefore \int \frac{x+1}{\sqrt{8 + x - x^2}}\,dx = -\frac{1}{2}\left\{2\sqrt{8 + x - x^2}\right\} + \quad \mathbf{I} \ldots (2)$$

Where $\quad \mathbf{I} = \frac{3}{2}\int \frac{1}{\sqrt{8 + x - x^2}}\,dx$

$$= \frac{3}{2}\int \frac{1}{\sqrt{-\{x^2 - x - 8\}}}\,dx$$

$$= \frac{3}{2}\int \frac{\sqrt{4 \times 1}}{\sqrt{-\{(2x-1)^2 - 32 - 1\}}}\,dx$$

$$= \frac{3}{2}\int \frac{2}{\sqrt{\left(\sqrt{33}\right)^2 - (2x-1)^2}}\,dx$$

$$= 3\left[\left(\frac{1}{2}\right)\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)\right]$$

$$\mathbf{I} = \frac{3}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$$

substituting in (2) we get

$$\int \frac{x+1}{\sqrt{8 + x - x^2}}\,dx = -\sqrt{8 + x - x^2} + \frac{3}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + c$$

(119) $\int \frac{4x-3}{\sqrt{x^2+2x-1}}\, dx$

$$\text{Let } 4x-1 = A(2x+2)+B \qquad \ldots (1)$$

$$4x-3 = (2A)x + 2A + B$$

equating like terms

$$4 = 2A \;; \qquad 2A + B$$

$$\therefore A = 2, \qquad B = -3 - 2A = -3 - 4 = -7$$

Substituting $A = 2$ and $B = -7$ in (1), we get

$$4x-3 = 2(2x+2) - 7$$

$$\therefore \int \frac{4x-3}{\sqrt{x^2+2x-1}}\, dx = \int \frac{2(2x+2)-7}{\sqrt{x^2+2x-1}}\, dx$$

$$= 2\int \frac{2x+2}{\sqrt{x^2+2x-1}}\, dx + (-7)\int \frac{1}{\sqrt{x^2+2x-1}}\, dx$$

$$\therefore \int \frac{4x-3}{\sqrt{x^2+2x-1}}\, dx = 2\left\{2\sqrt{x^2+2x-1}\right\} + \mathbf{I} \qquad \ldots (2)$$

Where

$$\mathbf{I} = -7\int \frac{1}{\sqrt{x^2+2x-1}}\, dx = -7\frac{dx}{\sqrt{(x+1)^2-1-1}}$$

$$= -7 \int \frac{dx}{\sqrt{(x+1)^2 - \left(\sqrt{2}\right)^2}}$$

$$= -7 \log \left\{(x+1) + \sqrt{(x+1) - \left(\sqrt{2}^2\right)}\right\}$$

$$\mathbf{I} = -7 \log \left\{(x+1) + \sqrt{x^2+2x-1}\right\}$$

substituting in (2) we get

$$\int \frac{4x-3}{\sqrt{x^2+2x-1}}\, dx = 4\sqrt{x^2+2x-1} - 7 \log \left\{(x+1) + \sqrt{x^2+2x-1}\right\} + c$$

We have already seen that

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left[x + \sqrt{x^2 - a^2}\right] + c$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log\left[x + \sqrt{x^2 + a^2}\right] + c$$

The three more standard forms similar to the above are

Type IV:

(120) $\displaystyle \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(121) $\displaystyle \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c$

(122) $\displaystyle \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left[x + \sqrt{x^2 + a^2}\right] + c$

(120) Let **I** $= \int \sqrt{a^2 - x^2}\, dx$

Applying integration by parts rule

$$\mathbf{I} = x\sqrt{a^2 - x^2} - \int x\left(-\frac{x}{\sqrt{a^2 - x^2}}\right) dx$$

let $u = \sqrt{a^2 - x^2}$ $\qquad dv = dx$

$du = \dfrac{-2x}{2\sqrt{a^2 - x^2}}\, dx$ $\qquad v = x$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} + \frac{(-a^2)}{\sqrt{a^2 - x^2}}\right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$\mathbf{I} = x\sqrt{a^2 - x^2} - \mathbf{I} \quad + a^2 \int \frac{1}{\sqrt{a^2 - x^2}}\, dx$$

$$\mathbf{I} + \mathbf{I} = x\sqrt{a^2 - x^2} + a^2 \,.\, \sin^{-1}\frac{x}{a}$$

$$\therefore\ 2\mathbf{I} = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}$$

$$\mathbf{I} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\boxed{\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c}$$

(121) Let $\mathbf{I} = \int \sqrt{x^2 - a^2}\, dx$

Applying integration by parts rule

$$\mathbf{I} = x\sqrt{x^2 - a^2} - \int x\left(\frac{x}{\sqrt{x^2 - a^2}}\right) dx$$

let $u = \sqrt{x^2 - a^2}$ $\qquad dv = dx$

$du = \dfrac{2x}{2\sqrt{x^2 - a^2}}\, dx$ $\qquad v = x$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}}\, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int\sqrt{x^2 - a^2}\, dx - a^2 \int \frac{1}{\sqrt{x^2 - a^2}}\, dx$$

$$\mathrm{I} = x\sqrt{x^2 - a^2} - \mathrm{I} - a^2 \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\therefore\ 2\mathrm{I} = x\sqrt{x^2 - a^2} - a^2 \log\left[x + \sqrt{x^2 - a^2}\right]$$

$$\therefore\ \mathrm{I} = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c$$

$$\boxed{\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c}$$

(122) Let $\mathbf{I} = \int \sqrt{x^2 + a^2}\, dx$

Applying integration by parts rule $\qquad dv = dx$

$$\mathbf{I} = x\sqrt{x^2+a^2} - \int\left(\frac{x^2}{\sqrt{x^2+a^2}}\right) dx \qquad \text{let } u = \sqrt{x^2+a^2} \qquad v = x$$

$$du = \frac{2x}{2\sqrt{x^2+a^2}}\, dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\, dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2}{\sqrt{x^2+a^2}}\, dx + \int \frac{a^2}{\sqrt{x^2+a^2}}\, dx$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\, dx + a^2 \int \frac{1}{\sqrt{x^2+a^2}}\, dx$$

$$\mathbf{I} = x\sqrt{x^2+a^2} - \mathbf{I} + a^2 \log\left[x+\sqrt{x^2+a^2}\right] + c$$

$$\therefore\ 2\mathbf{I} = x\sqrt{x^2+a^2} + a^2 \log\left[x+\sqrt{x^2+a^2}\right] + c$$

$$\therefore\ \mathbf{I} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right] + c$$

$$\boxed{\therefore \int\sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right] + c}$$

Example: 9.123 – 9.131:

Integrate the following :

(123) $\sqrt{4-9x^2}$ (124) $\sqrt{16x^2-25}$ (125) $\sqrt{9x^2+16}$ (126) $\sqrt{2x-x^2}$

(127) $\sqrt{x^2-4x+6}$ (128) $\sqrt{x^2+4x+1}$ (129) $\sqrt{4+8x-5x^2}$

(130) $\sqrt{(2-x)(1+x)}$ (131) $\sqrt{(x+1)(x-2)}$

Solution:

(123) $\sqrt{4-9x^2}\ dx = \int\sqrt{2^2-(3x)^2}dx = \frac{1}{3}\left[\frac{(3x)}{2}\sqrt{2^2-(3x)^2} + \frac{2^2}{2}\sin^{-1}\frac{3x}{2}\right]+c$

$$= \frac{1}{3}\left[\frac{3x}{2}\sqrt{4-9x^2} + 2\sin^{-1}\frac{3x}{2}\right] + c$$

(124) $\int\sqrt{16x^2-25}\ dx = \int\sqrt{(4x)^2-5^2}\,dx$

$$= \frac{1}{4}\left[\frac{(4x)}{2}\sqrt{(4x)^2-5^2} - \frac{25}{2}\log\left[4x+\sqrt{(4x)^2-5^2}\right]\right]$$

$$= \frac{1}{8}\ \left[4x\sqrt{16x^2-25} - 25\log\left(4x+\sqrt{16x^2-25}\right)\right] + c$$

(125) $\int\sqrt{9x^2+16}\ \ dx = \int\sqrt{(3x)^2+4^2}\,dx$

$$= \frac{1}{3}\left[\frac{(3x)}{2}\sqrt{(3x)^2+4^2} + \frac{4^2}{2}\log\left[3x+\sqrt{(3x)^2+4^2}\right]\right]$$

$$= \frac{1}{6}\ \left[3x\sqrt{9x^2+16} + 16\log\left(3x+\sqrt{9x^2+16}\right)\right] + c$$

(126) $\int\sqrt{2x-x^2}\ \ dx = \int\sqrt{1-\{x^2-2x+1\}}\,dx = \int\sqrt{1^2-(x-1)^2}\ \ dx$

$$= \frac{(x-1)}{2}\ \sqrt{1-(x-1)^2}\ + \frac{1^2}{2}\ \sin^{-1}\left(\frac{x-1}{1}\right) + c$$

$$= \frac{x-1}{2}\ \sqrt{2x-x^2}\ + \frac{1}{2}\ \sin^{-1}(x-1) + c$$

(127) $\int\sqrt{x^2-4x+6}\,dx = \int\sqrt{x^2-4x+4+2}\ \ dx = \int\sqrt{(x-2)^2+\left(\sqrt{2}^2\right)}\ \ dx$

$$= \frac{(x-2)}{2}\ \sqrt{(x-2)^2+\left(\sqrt{2}\right)^2} + \frac{\left(\sqrt{2}\right)^2}{2}\log\left[(x-2)+\sqrt{(x-2)^2+\left(\sqrt{2}\right)^2}\right] + c$$

$$= \frac{(x-2)}{2}\sqrt{x^2-4x+6}\ + \log\left[(x-2)+\sqrt{x^2-4x+6}\right]\ + c$$

(128) $\int\sqrt{x^2+4x+1}\,dx = \int\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$

$$= \frac{(x+2)}{2}\ \sqrt{(x+2)^2-\left(\sqrt{3}\right)^2} - \frac{\left(\sqrt{3}\right)^2}{2}\ \log\left[(x+2)+\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\right] + c$$

$$= \frac{(x+2)}{2}\ \sqrt{x^2+4x+1}\ - \frac{3}{2}\ \log\left[(x+2)+\sqrt{x^2+4x+1}\right] + c$$

(129) $\int\sqrt{4+8x-5x^2}\,dx = \int\sqrt{-\{5x^2-8x-4\}}\;dx$

$$\left(\because ax^2+bx+c = \frac{1}{4a}\,[(2ax+b)^2+(4ac-b^2])\right)$$

$$= \int \frac{1}{\sqrt{4\times 5}}\sqrt{-\{(10x-8)^2-80-64\}}\;dx$$

$$= \frac{1}{\sqrt{20}}\int\sqrt{12^2-(10x-8)^2}\;dx$$

$$= \frac{1}{\sqrt{20}}\left[\left(\frac{1}{10}\right)\left(\frac{10x-8}{2}\sqrt{12^2-(10x-8)^2}+\left(\frac{12^2}{2}\right)\sin^{-1}\frac{10x-8}{12}\right)\right]$$

$$= \frac{1}{\sqrt{20}}\left[\frac{1}{10}(5x-4)\sqrt{80+16x-100x^2}+\frac{36}{5}\sin^{-1}\left(\frac{5x-4}{6}\right)\right]$$

$$= \frac{1}{\sqrt{20}}\left[\left(\frac{5x-4}{10}\right)\sqrt{20}\sqrt{(4+8x-5x^2)}+\frac{36}{5}\sin^{-1}\frac{5x-4}{6}\right]$$

$$= \frac{5x-4}{10}\sqrt{4+8x-5x^2}+\frac{36}{\sqrt{20}\times 5}\sin^{-1}\frac{5x-4}{6}$$

$$\therefore \int\sqrt{4+8x-5x^2}\,dx = \frac{5x-4}{10}\sqrt{4+8x-5x^2}+\frac{18}{5\sqrt{5}}\sin^{-1}\frac{5x-4}{6}+c$$

(130) $\int\sqrt{(2-x)(1+x)}\,dx = \int\sqrt{2+x-x^2}\;dx = \int\sqrt{-(x^2-x-2)}\;dx$

$$= \int\frac{\sqrt{-\{(2x-1)^2-8-1\}}}{\sqrt{4.1}}\;dx = \frac{1}{2}\int\sqrt{3^2-(2x-1)^2}\;dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{(2x-1)}{2}\sqrt{3^2-(2x-1)^2}+\left(\frac{1}{2}\right)\frac{3^2}{2}\sin^{-1}\left(\frac{2x-1}{3}\right)\right]$$

$$= \frac{1}{8}\left[(2x-1)\sqrt{8+4x-4x^2}+9\sin^{-1}\left(\frac{2x-1}{3}\right)\right]$$

$$= \frac{1}{8}\left[2(2x-1)\sqrt{2+x-x^2}+9\sin^{-1}\left(\frac{2x-1}{3}\right)\right]$$

(131) $\int\sqrt{(x+1)(x-2)}\,dx = \int\sqrt{x^2-x-2}\;dx = \int\frac{\sqrt{(2x-1)^2-8-1}}{\sqrt{4}}\;dx$

$$= \frac{1}{2}\int\sqrt{(2x-1)^2-3^2}\;dx$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\right)\left(\frac{2x-1}{2}\right)\sqrt{(2x-1)^2-3^2}-\left(\frac{1}{2}\right)\left(\frac{3^2}{2}\right)\log\left\{(2x-1)+\sqrt{(2x-1)^2-3^2}\right\}\right]$$

$$\int\sqrt{(x+1)(x-2)}dx = \frac{1}{2}\left[\frac{(2x-1)}{4}\sqrt{(2x-1)^2-9}-\frac{9}{4}\log\left\{(2x-1)+\sqrt{(2x-1)^2-9}\right\}\right]$$

EXERCISE 9.7

Integrate the followings

(1) $\frac{1}{x^2+25}$, $\frac{1}{(x+2)^2+16}$, $\frac{1}{(3x+5)^2+4}$, $\frac{1}{2x^2+7x+13}$, $\frac{1}{9x^2+6x+10}$

(2) $\frac{1}{16-x^2}$, $\frac{1}{9-(3-x)^2}$, $\frac{1}{7-(4x+1)^2}$, $\frac{1}{1+x-x^2}$, $\frac{1}{5-6x-9x^2}$

(3) $\frac{1}{x^2-25}$, $\frac{1}{(2x+1)^2-16}$, $\frac{1}{(3x+5)^2-7}$, $\frac{1}{x^2+3x-3}$, $\frac{1}{3x^2-13x-10}$

(4) $\frac{1}{\sqrt{x^2+1}}$, $\frac{1}{\sqrt{(2x+5)^2+4}}$, $\frac{1}{\sqrt{(3x-5)^2+6}}$, $\frac{1}{\sqrt{x^2+3x+10}}$, $\frac{1}{\sqrt{x^2+5x+26}}$

(5) $\frac{1}{\sqrt{x^2-91}}$, $\frac{1}{\sqrt{(x+1)^2-15}}$, $\frac{1}{\sqrt{(2x+3)^2-16}}$, $\frac{1}{\sqrt{x^2+4x-12}}$, $\frac{1}{\sqrt{x^2+8x-20}}$

(6) $\frac{1}{\sqrt{4-x^2}}$, $\frac{1}{\sqrt{25-(x-1)^2}}$, $\frac{1}{\sqrt{11-(2x+3)^2}}$, $\frac{1}{\sqrt{1+x-x^2}}$, $\frac{1}{\sqrt{8-x-x^2}}$

(7) $\frac{3-2x}{x^2+x+1}$, $\frac{x-3}{x^2+21x+3}$, $\frac{2x-1}{2x^2+x+3}$, $\frac{1-x}{1-x-x^2}$, $\frac{4x+1}{x^2+3x+1}$

(8) $\frac{x+2}{\sqrt{6+x-2x^2}}$, $\frac{2x-3}{\sqrt{10-7x-x^2}}$, $\frac{3x+2}{\sqrt{3x^2+4x+7}}$, $\sqrt{\frac{1+x}{1-x}}$, $\frac{6x+7}{\sqrt{(x-4)(x-5)}}$

(9) $\sqrt{1+x^2}$, $\sqrt{(x+1)^2+4)}$, $\sqrt{(2x+1)^2+9}$, $\sqrt{(x^2-3x+10)}$

(10) $\sqrt{4-x^2}$, $\sqrt{25-(x+2)^2}$, $\sqrt{169-(3x+1)^2}$, $\sqrt{1-3x-x^2}$, $\sqrt{(2-x)(3+x)}$

9.4 Definite integrals

A basic concept of integral calculus is limit, an idea applied by the Greeks in geometry.

To find the area of a circle, Archimedes inscribed an equilateral polygon in a circle. Upon increasing the numbers of sides, the area of the polygon approaches the area of the circle as a limit. The area of an irregular shaped plate also can be found by subdividing it into rectangles of equal width. If the number of rectangles is made larger and larger by reducing the width, the sum of the area of rectangles approaches the required area as a limit. The beauty and importance of the integral calculus is that it provides a systematic way for the exact calculations of many areas, volumes and other quantities.

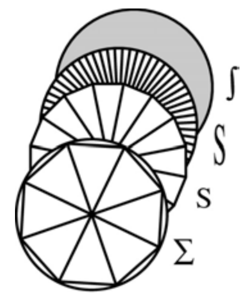


Fig. 9.2

Integration as summation

To understand the concept of definite integral, let us take a simple case.

Consider the region R in the plane showing figure 9.3. The region R is bounded by the curve $y = f(x)$, the x-axis, and two vertical lines $x = a$ and $x = b$, where $b > a$

For simplicity, we assume $y = f(x)$ to be a continuous and increasing function on the closed interval $[a, b]$.

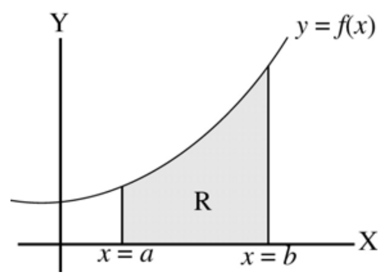


Fig. 9.3

We first define a polygon contained in R. Divided the closed interval $[a, b]$ into n sub intervals of equal length say Δx.

$$\therefore \Delta x = \frac{b-a}{n}$$

Denote the end points of these sub intervals by $x_0, x_1, x_2 \ldots\ldots\ldots x_r \ldots\ldots x_n$.

Where $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, …… $x_r = a + r\Delta x$, ……, $x_n = b$

The area of the polygon shown in figure 9.4 is the sum of the area of the rectangles (by taking the left hand ***x*** values of the such intervals).

$S_n = A_1 + A_2 + \ldots\ldots + A_n$

$= f(x_0)\,\Delta x + f(x_1)\Delta x \ldots + f(x_{n-1})\,\Delta x$

$= [f(a) + f(a + \Delta x) + \ldots f(a + r\ \Delta x) \ldots$

$+ f(a + (n-1)\Delta x)]\,\Delta x$

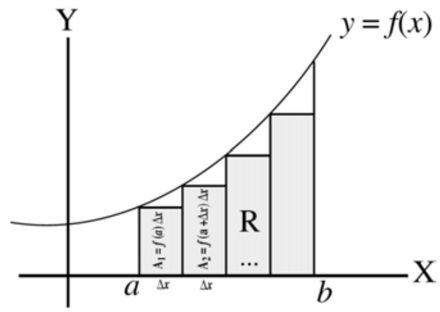


Fig. 9.4

$$= \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\}.(\Delta x) \quad = \Delta x \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\}$$

$$S_n = \frac{b-a}{n} \sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\} \qquad \left(\because \Delta x = \frac{b-a}{n}\right)$$

Now increase the number of sub intervals multiply n by 2, then the number of rectangles is doubled, and width of each rectangle is halved as shown in figure. 9.5. By comparing the two figures, notice that the shaded region in fig.9.5 appears more approximate to the region R than in figure 9.4.

So sum of the areas of the rectangles S_n, approaches to the required region R as n increases.

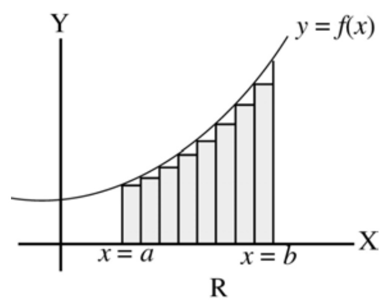


Fig. 9.5

Finally we get, $\underset{n \to \infty}{\mathrm{L}t}\ S_n = \underset{n \to \infty}{\mathrm{L}t}\ \dfrac{b-a}{n} \displaystyle\sum_{r=1}^{n} f\{a + (r-1)\,\Delta x\} \to \mathrm{R}$

Similarly, by taking the right hand values of x of the sub intervals, we can have,

$$\underset{n \to \infty}{\mathrm{L}t}\ S_n = \underset{n \to \infty}{\mathrm{L}t}\ \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x) \ \to\ \mathrm{R}$$

$$\text{i.e. } \boxed{\mathrm{R} = \underset{n \to \infty}{\mathrm{L}t}\ \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x)} \quad - \ \mathbf{I}$$

Definition: If a function $f(x)$ is defined on a closed interval $[a, b]$, then the definite integral of $f(x)$ from a to b is given by

$$\boxed{\underset{n \to \infty}{\mathrm{L}t} \ \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x)}, \quad \text{where } \Delta x = \frac{b-a}{n} \quad \text{(provided the limit exists)}$$

On the other hand the problem of finding the area of the region R is the problem of arguing from the derivative of a function back to the function itself.

Anti-derivative approach to find the area of the region R.

Let us consider the same region R (considering 9.6) bounded by the curve $y = f(x)$ the x-axis and the two vertical lines, $x = a$ and $x = b$, where $b > a$.

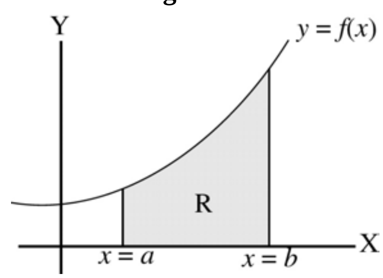


Fig. 9.6

To evaluate the area of R, we need to consider the total area between the curve $y = f(x)$ and the x-axis from the left to the arbitrary point $P(x, y)$ on the curve.

Let us denote this area by $\mathbf{A}_x$.

Let Q $(x + \Delta x, y + \Delta y)$ be another point very close to $P(x, y)$.

Let $\Delta\mathbf{A}_x$ is the area enclosed by the strip under the arc PQ and x-axis.

If the strip is approximated by a rectangle of length y and width Δx, then the area of the strip is $y \cdot \Delta x$.

Since P and Q are very close

$\Delta\mathbf{A}_x \approx y \cdot \Delta x \qquad \therefore \quad \dfrac{\Delta\mathbf{A}_x}{\Delta x} \approx y$

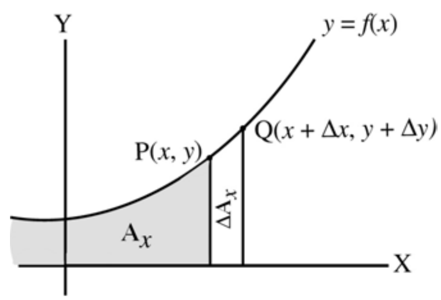


Fig. 9.7

If the width Δx is reduced, then the error is accordingly reduced.

If $\Delta x \to 0$ then $\Delta\mathbf{A}_x \to 0$

$$\therefore \underset{\Delta x \to 0}{\mathrm{L}t} \ \frac{\Delta\mathbf{A}_x}{\Delta x} = y \quad \Rightarrow \quad \frac{d\mathbf{A}_x}{dx} = y$$

$\therefore$ By definition of anti derivative $\dfrac{d\mathbf{A}_x}{dx} = y \Rightarrow \mathbf{A}_x = \int y\,dx$

is the total area $\mathbf{A}_x$, between the curve and x-axis upto the point P is given by the indefinite integral $\int y\,dx$

Let $\int y\,dx = F(x) + c$

If $x = a$, then the area upto $x = a$, $\mathbf{A}_a$ is

$\int y\,dx = F(a) + c$

If $x = b$, then the area upto $x = b$, $\mathbf{A}_b$ is

$\int y\,dx = F(b) + c$

$\therefore$ The required area of the region R is $\mathbf{A}_b - \mathbf{A}_a$

given by

$$\underset{\text{upto } x = b}{\int y\,dx} - \underset{\text{upto } x = a}{\int y\,dx}$$

$$= (F(b) + c) - (F(a) + c)$$

by notation $\int_a^b y\,dx = F(b) - F(a)$

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} \qquad - \qquad \text{II}$$

gives the area of the region R bounded by the curve $y = f(x)$, x axis and between the lines $x = a$ and $x = b$.

***a* & *b* are called the lower and upper limits of the integral.**

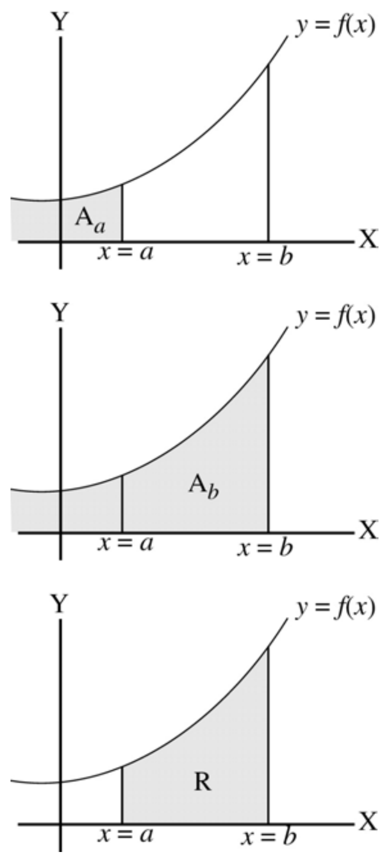


Fig. 9.8

From I & II, it is clear that

$$\boxed{R = \underset{n \to \infty}{\mathrm{L}t}\ \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\Delta x) = \int_a^b f(x)\,dx = F(b) - F(a)} \text{ , if the limit exists}$$

To evaluate the definite integrals under this method, the following four formulae will be very much useful.

(i) $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(ii) $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) $$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(iv) $$\sum_{r=1}^{n} a^r = a\left(\frac{a^n-1}{a-1}\right) \; ; (a \neq 1)$$

Illustration:

Consider the area A below the straightline $y = 3x$ above the x-axis and between the lines $x = 2$ and $x = 6$. as shown in the figure.

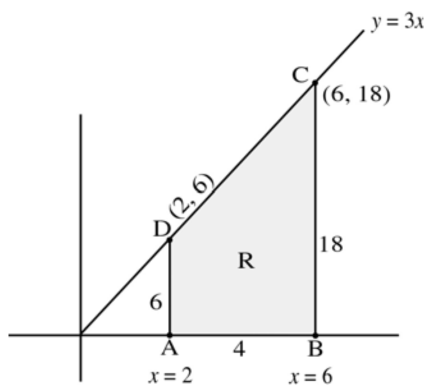


Fig. 9.9

(1) Using the formula for the area of the trapezium ABCD

$$R = \frac{h}{2}[a+b]$$

$$= \frac{4}{2}[6+18] = 2 \times 24$$

$$R = 48 \text{ sq. units} \quad \ldots \text{(i)}$$

(2) Integration as summation

Let us divide the area ABCD into n strips with equal widths. Here $a = 2$, $b = 6$

$\therefore$ width of each strip

$$\Delta x = \frac{b-a}{n}$$

i.e. $$\Delta x = \frac{6-2}{n}$$

$$\Delta x = \frac{4}{n}$$

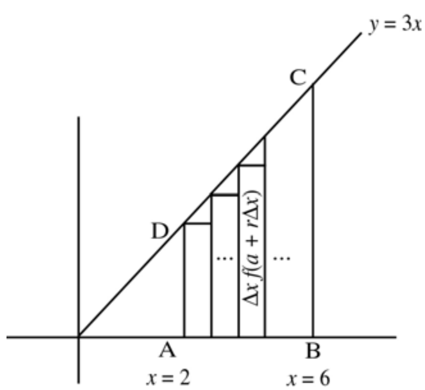


Fig. 9.10

By definite integral formula

$$R = \underset{n \to \infty}{\text{Lt}} \frac{b-a}{n} \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{4}{n} \sum_{r=1}^{n} f\left(2 + r\left(\frac{4}{n}\right)\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \sum_{r=1}^{n} \left(2+\frac{4r}{n}\right) \quad ; \quad \left[\because f(x) = 3x, f\left(2+r\,\frac{4}{n}\right)=3\left[2+r\left(\frac{4}{n}\right)\right]\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \left[\sum_{r=1}^{n} 2 + \frac{4}{n} \sum_{r=1}^{n} r\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \left[2n + \frac{4}{n}\frac{(n)\,(n+1)}{2}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{12}{n} \left[2n + 2(n+1)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} 12\left[2 + 2\,\frac{(n+1)}{n}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} 12\left[2 + 2\left(1 + \frac{1}{n}\right)\right]$$

$$= 12\,[2 + 2\,(1 + 0)] \qquad \text{as } n \to \infty, \quad \frac{1}{n} \to 0$$

$$= 12 \times 4$$

$$R = 48 \text{ square units} \qquad \ldots \text{(ii)}$$

(3) By anti derivative method

$$R = \int_a^b f(x)\,dx = \int_2^6 3x\,dx = 3\int_2^6 x\,dx = 3\left[\frac{x^2}{2}\right]_2^6$$

$$= 3\left[\frac{6^2 - 2^2}{2}\right] = 3\left[\frac{36-4}{2}\right] = 3 \times \frac{32}{2}$$

$$R = 48 \text{ square units} \qquad \ldots \text{(iii)}$$

From (i), (ii) and (iii) it is clear that the area of the region is

$$\boxed{R = \underset{n \to \infty}{\text{Lt}} \frac{b-a}{n} \sum f(a + r\,\Delta x) = \int_a^b f(x)\,dx}, \quad \text{if the limit exists}$$

Examples 9.132 – 9.134:

Evaluate the following definite integrals as limit of sums

(132) $\int_1^2 (2x + 5)\, dx$ (133) $\int_1^3 x^2\, dx$ (134) $\int_2^5 (3x^2 + 4)\, dx$

(132) $\int_1^2 (2x + 5)\, dx$

$$\text{Let } f(x) = 2x + 5 \text{ and } [a, b] = [1, 2]$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\therefore \Delta x = \frac{1}{n}$$

$$f(x) = 2x + 5$$

$$\therefore f(a + r\,\Delta x) = f\left(1 + r\ \frac{1}{n}\right) = 2\left(1 + \frac{r}{n}\right) + 5$$

Let us divide the closed interval [1, 2] into n equal sub intervals of each length Δx.

By the formula

$$\int_a^b f(x)\, dx = \underset{\Delta x \to 0}{\text{Lt}}\ \Delta x \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$\int_1^2 (2x + 5) = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{n}\right) \sum_{r=1}^{n} \left(2\left(1 + \frac{r}{n}\right) + 5\right)$$

$$= \underset{n \to \infty}{\text{Lt}}\ \frac{1}{n} \sum_{r=1}^{n} \left(7 + \frac{2}{n} r\right)$$

$$= \underset{n \to \infty}{\text{Lt}}\ \frac{1}{n} \cdot \left[\sum_{r=1}^{n} 7 + \frac{2}{n} \sum_{r=1}^{n} r\right]$$

$$= \underset{n \to \infty}{\text{Lt}}\ \frac{1}{n} \left[7n + \frac{2}{n} \cdot \frac{n(n+1)}{2}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \left[7 + \frac{n+1}{n}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \left[7 + \left(1 + \frac{1}{n}\right)\right]$$

$$= (7 + 1) \qquad 1/n \to 0 \quad \text{as } n \to \infty$$

$$\therefore \int_1^2 (2x + 5) = 8 \text{ square units}$$

Verification : $\displaystyle\int_1^2 (2x + 5)\, dx = \left[2\left(\frac{x^2}{2}\right) + 5x\right]_1^2$

$$= (2^2 - 1^2) + 5(2 - 1) = (4 - 1) + (5 \times 1)$$

$$\int_1^2 (2x + 5)\, dx = 8 \text{ square units}$$

(133) $\displaystyle\int_1^3 x^2\, dx$

Let $f(x) = x^2$ and $[a, b] = [1, 3]$

Let us divide the closed interval [1, 3] into n equal sub intervals of each length Δx.

$$\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$$

$$\therefore \Delta x = \frac{2}{n}$$

$$f(x) = x^2$$

$$\therefore f(a + r\,\Delta x) = f\left(1 + r\,\frac{2}{n}\right)$$

$$= \left(1 + r\,\frac{2}{n}\right)^2$$

$$f(a + r\,\Delta x) = \left(1 + \frac{4}{n}r + \frac{4}{n^2}r^2\right)$$

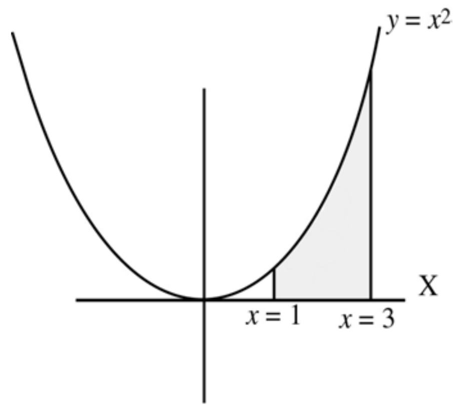


Fig. 9.11

By the formula

$$\int_a^b f(x)\, dx = \underset{\Delta x \to 0}{\text{Lt}} \Delta x \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$\int_1^3 x^2\,dx = \underset{n\to\infty}{\text{Lt}}\ \frac{2}{n}\sum_{r=1}^{n}\left(1+\frac{4}{n}r+\frac{4}{n^2}r^2\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\ \frac{2}{n}\left[\Sigma 1+\frac{4}{n}\Sigma r+\frac{4}{n^2}\Sigma r^2\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\ \frac{2}{n}\left[n+\frac{4}{n}\cdot\frac{n(n+1)}{2}+\frac{4}{n^2}\frac{(n)\,(n+1)\,(2n+1)}{6}\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\ 2\left[1+\frac{2(n+1)}{n}+\frac{2}{3}\left(\frac{n+1}{n}\right).\left(\frac{2n+1}{n}\right)\right]$$

$$= \underset{n\to\infty}{\text{Lt}}\ 2\left[1+2\left(1+\frac{1}{n}\right)+\frac{2}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 2\left[1+2+\frac{2}{3}(1)\,(2)\right] \text{ as } \underset{n\to\infty}{\text{Lt}} \to \frac{1}{n}\to 0$$

$$= 2\left[3+\frac{4}{3}\right]$$

$$\therefore \quad \int_1^3 x^2\,dx = \frac{26}{3} \text{ square units.}$$

(134) $\int_2^5 (3x^2+4)\,dx$

Let $f(x) = 3x^2 + 4$ and $[a, b] = [2, 5]$

Let us divide the closed interval [2, 5] into n equal sub intervals of each length Δx.

$$\Delta x = \frac{5-2}{n}$$

$$\therefore\ \Delta x = \frac{3}{n}$$

$$f(x) = 3x^2 + 4$$

$$\therefore\ f(a + r\,\Delta x) = f\left(2 + r\,.\,\frac{3}{n}\right)$$

$$= 3\left(2+\frac{3r}{n}\right)^2 + 4$$

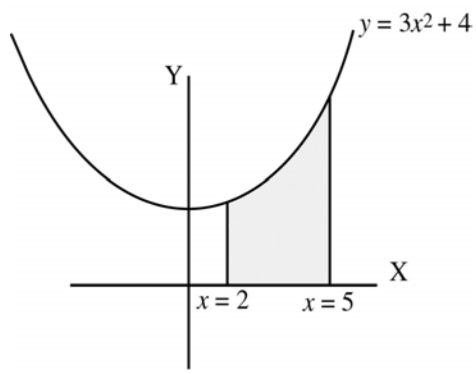


Fig. 9.12

By the formula

$$\int_a^b f(x)\, dx = \underset{\Delta x \to 0}{\text{Lt}} \; \Delta x \sum_{r=1}^{n} f(a + r\,\Delta x)$$

$$\int_2^5 (3x^2 + 4)\; dx = \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left(3\left(2 + \frac{3r}{n}\right)^2 + 4\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left(3\left(4 + \frac{12}{n}\, r + \frac{9}{n^2}\, r^2\right) + 4\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left(12 + \frac{36}{n}\, r + \frac{27}{n^2}\, r^2 + 4\right)$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \sum_{r=1}^{n} \left[16 + \frac{36}{n}\,(r) + \frac{27}{n^2}\,(r^2)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \left[\Sigma 16 + \frac{36}{n}\,\Sigma r + \frac{27}{n^2}\,\Sigma r^2\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; \frac{3}{n} \left[16n + \frac{36}{n}\,\frac{(n)\,(n+1)}{2} + \frac{27}{n^2}\,\frac{n(n+1)\,(2n+1)}{6}\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; 3\left[16 + 18\,\frac{(n+1)}{n} + \frac{9}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$

$$= \underset{n \to \infty}{\text{Lt}} \; 3\left[16 + 18\left(1 + \frac{1}{n}\right) + \frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right]$$

$$= 3\left[16 + (18 \times 1) + \frac{9}{2}\,(1)\,(2)\right] = 3\,[43]$$

$$\int_2^5 (3x^2 + 4)\; dx = 129 \text{ square units.}$$

10. PROBABILITY

"The theory of probability is nothing more than good sense confirmed by calculation"

– Pierre Laplace

10.1 Introduction:

The word probability and chance are quite familiar to everyone. Many a time we come across statements like "There is a bright **chance** for Indian cricket team to win the World Cup this time".

"It is **possible** that our school students may get state ranks in forthcoming public examination".

"**Probably** it may rain today".

The word chance, possible, probably, likely etc. convey some sense of uncertainty about the occurrence of some events. Our entire world is filled with uncertainty. We make decisions affected by uncertainty virtually every day.

In order to think about and measure uncertainty, we turn to a branch of mathematics called probability.

Before we study the theory of probability let us learn the definition of certain terms, which will be frequently used.

Experiment: An experiment is defined as a process for which its result is well defined.

Deterministic experiment: An experiment whose outcomes can be predicted with certain, under identical conditions.

Random experiment: An experiment whose all possible outcomes are known, but it is not possible to predict the outcome.

Example: (i) A fair coin is "tossed" (ii) A die is "rolled" are random experiments, since we cannot predict the outcome of the experiment in any trial.

A simple event (or elementary event): The most basic possible outcome of a random experiment and it cannot be decomposed further.

Sample space: The set of all possible outcomes of a random experiment is called a sample space.

Event: Every non-empty subset of the sample space is an event.

The sample space S is called **Sure event** or **Certain event**. The null set in S is called **Impossible event**.

Example: When a single, regular die is rolled once, the associated sample space is

S = {1, 2, 3, 4, 5, 6}

{1}, {2}, {3}, {4}, {5}, {6} are the simple events or elementary events.

{1}, {2, 3}, {1, 3, 5}, {2, 4, 5, 6} are some of the events.

Mutually exclusive events (or disjoint events)

Two or more events are said to be mutually exclusive if they have no simple events (or outcomes) in common. (i.e. They cannot occur simultaneously).

Example: When we roll a die the events {1, 2, 3} and {4, 5, 6} are mutually exclusive event

Exhaustive events:

A set of events is said to be exhaustive if no event outside this set occurs and atleast one of these events must happen as a result of an experiment.

Example:

When a die is rolled, the set of events {1, 2, 3}, {2, 3, 5}, {5, 6} and {4, 5} are exhaustive events.

Equally likely events:

A set of events is said to be equally likely if none of them is expected to occur in preference to the other.

Example:When a coin is tossed, the events {head} and {tail} are equally likely.

Example:

Trial	Random Experiment	Total Number of Outcomes	Sample space
(1)	Tossing of a fair coin	$2^1 = 2$	{H, T}
(2)	Tossing of two coins	$2^2 = 4$	{HH, HT, TH, TT}
(3)	Tossing of three coins	$2^3 = 8$	{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
(4)	Rolling of fair die	$6^1 = 6$	{1, 2, 3, 4, 5, 6}
(5)	Rolling of two dice	$6^2 = 36$	{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}
(6)	Drawing a card from a pack of 52 playing cards	$52^1 = 52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

Notations:

Let A and B be two events.

(i) $A \cup B$ stands for the occurrence of A or B or both.

(ii) $A \cap B$ stands for the simultaneous occurrence of A and B.

(iii) $\bar{A}$ or A' or A^c stands for non-occurrence of A

(iv) $(A \cap \overline{B})$ stands for the occurrence of only A.

Example: Suppose a fair die is rolled, the sample space is S = {1, 2, 3, 4, 5, 6}

Let A = {1, 2}, B = {2, 3}, C = {3, 4}, D = {5, 6}, E = {2, 4, 6} be some events.

(1) The events A, B, C and D are equally likely events, because they have equal chances to occur (but not E).

(2) The events A, C, D are mutually exclusive because

$A \cap C = C \cap D = A \cap D = \phi$.

(3) The events B and C are not mutually exclusive since $B \cap C = \{3\} \neq \phi$.

(4) The events A, C and D are exhaustive events, since $A \cup C \cup D = S$

(5) The events A, B and C are not exhaustive events since the event {5, 6} occurs outside the totality of the events A, B and C.

(i.e. $A \cup B \cup C \neq S$).

10.2 Classical definition of probability:

If there are ***n*** exhaustive, mutually exclusive and equally likely outcomes of an experiment and ***m*** of them are favourable to an event A, then the mathematical probability of A is defined as the ratio $\frac{m}{n}$ i.e. $P(A) = \frac{m}{n}$

In other words,

let S be the sample space and A be an event associated with a random experiment.

Let $n(S)$ and $n(A)$ be the number of elements of S and A respectively. Then the probability of event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to A}}{\text{Exhaustive Number of cases in S}}$$

Axioms of probability

Given a finite sample space S and an event A in S, we define P(A), the probability of A, satisfies the following three conditions.

(1) O ≤ P(A) ≤ 1
(2) P(S) = 1
(3) If A and B are mutually exclusive events, P(A ∪ B) = P(A) + P(B)

Note:

If A_1, A_2 …… A_n are mutually exclusively events in a sample space S, then

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$

Example 10.1:

If an experiment has exactly the three possible mutually exclusive outcomes A, B and C, check in each case whether the assignment of probability is permissible.

(i) $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{3}$

(ii) $P(A) = \frac{1}{4}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{1}{4}$

(iii) P(A) = 0.5 , P(B) = 0.6 , P(C) = − 0.1

(iv) P(A) = 0.23 , P(B) = 0.67 , P(C) = 0.1

(v) P(A) = 0.51 , P(B) = 0.29 , P(C) = 0.1

Solution:

(i) The values of P(A), P(B) and P(C) are all lying in the interval from [0, 1]

Also their sum $P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

∴ The assignment of probability is permissible.

(ii) Given that O ≤ P(A), P(B), P(C) ≤ 1

But the sum $P(A) + P(B) + P(C) = \frac{1}{4} + \frac{3}{4} + \frac{1}{4} = \frac{5}{4} > 1$

∴ The assignment is not permissible.

(iii) Since P(C) = − 0.1, is negative, the assignment is not permissible.

(iv) The assignment is permissible because 0 ≤ P(A), P(B), P(C) ≤ 1 and their sum P(A) + P(B) + P(C) = 0.23 + 0.67 + 0.1 = 1

(v) Eventhough 0 ≤ P(A), P(B), P(C) ≤ 1, their sum P(A) + P(B) + P(C) = 0.51 + 0.29 + 0.1 = 0.9 ≠ 1. Therefore, the assignment is not permissible

Note:

In the above examples each experiment has exactly three possible outcomes. Therefore they must be exhaustive events (i.e. totality must be sample space) and the sum of probabilities is equal to 1.

Examples 10.2: Two coins are tossed simultaneously, what is the probability of getting

(i) exactly one head (ii) atleast one head (iii) atmost one head.

Solution:

The sample space is S = {HH, HT, TH, TT}, $n(\text{S}) = 4$

Let A be the event of getting one head, B be the event of getting atleast one head and C be the event of getting atmost one head.

$$\therefore \text{A} = \{\text{HT, TH}\}, \quad n(\text{A}) = 2$$
$$\text{B} = \{\text{HT, TH, HH}\}, \quad n(\text{B}) = 3$$
$$\text{C} = \{\text{HT, TH, TT}\}, \quad n(\text{C}) = 3$$

(i) $\text{P(A)} = \dfrac{n(\text{A})}{n(\text{S})} = \dfrac{2}{4} = \dfrac{1}{2}$ (ii) $\text{P(B)} = \dfrac{n(\text{B})}{n(\text{S})} = \dfrac{3}{4}$ (iii) $\text{P(C)} = \dfrac{n(\text{C})}{n(\text{S})} = \dfrac{3}{4}$

Example 10.3: When a pair of balanced dice is rolled, what are the probabilities of getting the sum (i) 7 (ii) 7 or 11 (iii) 11 or 12

Solution:

The sample space $\text{S} = \{(1,1), (1,2) \ldots (6,6)\}$

Number of possible outcomes $= 6^2 = 36 = n(\text{S})$

Let A be the event of getting sum 7, B be the event of getting the sum 11 and C be the event of getting sum 12

$$\therefore \text{A} = \{(1,6), (2,5), (3,4), (4,3), (5,2),(6,1)\}, \; n(\text{A}) = 6.$$
$$\text{B} = \{(5,6), (6,5)\}, \; \text{n(B)} = 2$$
$$\text{C} = \{(6, 6)\}, \; n(\text{C}) = 1$$

(i) P(getting sum 7) $= \text{P(A)} = \dfrac{n(\text{A})}{n(\text{S})} = \dfrac{6}{36} = \dfrac{1}{6}$

(ii) P (7 or 11) $= \text{P(A or B)} = \text{P}(\text{A} \cup \text{B})$

$= \text{P(A)} + \text{P(B}$ ($\because$ A and B are mutually exclusive i.e. $\text{A} \cap \text{B} = \phi$)

$= \dfrac{6}{36} + \dfrac{2}{36} = \dfrac{8}{36} = \dfrac{2}{9}$

P(7 or 11) $= \dfrac{2}{9}$

(iii) P (11 or 12) $= \text{P(B or C)} = \text{P}(\text{B} \cup \text{C})$

$= \text{P(B)} + \text{P(C)}$ ($\because$ B and C are mutually exclusive)

$= \dfrac{2}{36} + \dfrac{1}{36} = \dfrac{3}{36} = \dfrac{1}{12}$

P(11 or 12) $= \dfrac{1}{12}$

Example 10.4: Three letters are written to three different persons and addresses on three envelopes are also written. Without looking at the addresses, what is the probability that (i) all the letters go into right envelopes, (ii) none of the letters goes into right envelopes?

Solution:

Let A, B and C denote the envelopes and 1, 2 and 3 denote the corresponding letters.

The different combination of letters put into the envelopes are shown below:

Outcomes

		c_1	c_2	c_3	c_4	c_5	c_6
Envelopes	A	1	1	2	2	3	3
	B	2	3	1	3	1	2
	C	3	2	3	1	2	1

Let X be the event of putting the letters go into right envelopes.

Y be the event of putting none of the letters go into right envelope.

$S = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $n(S) = 6$

$X = \{c_1\}$, $n(X) = 1$ $\quad$ $Y = \{c_4, c_5\}$, $n(Y) = 2$

$$\therefore P(X) = \frac{1}{6}$$

$$P(Y) = \frac{2}{6} = \frac{1}{3}$$

Example 10.5: A cricket club has 15 members, of whom only 5 can bowl. What is the probability that in a team of 11 members atleast 3 bowlers are selected?

Let A, B and C be the three possible events of selection. The number of combinations are shown below.

Event	Combination of 11 players		Number of ways the combination formed		Total number of ways the selection can be done
	5 Bowlers	10 Others	5 Bowlers	10 Others	
A	3	8	$5c_3$	$10c_8$	$5c_3 \times 10c_8$
B	4	7	$5c_4$	$10c_7$	$5c_4 \times 10c_7$
C	5	6	$5c_5$	$10c_6$	$5c_5 \times 10c_6$

Solution:

Total number of exhaustive cases = Combination of 11 players from 15 members

$n(S) = 15c_{11}$

$$
\begin{aligned}
\text{P (atleast 3 bowlers)} &= \text{P[A or B or C]}\\
&= \text{P}[A \cup B \cup C]\\
&= \text{P(A)} + \text{P(B)} + \text{P(C)} \quad \left(\because \begin{array}{l}\text{A, B and C are}\\ \text{mutually exclusive events}\end{array}\right)\\
&= \frac{5C_3 \times 10C_8}{15C_{11}} + \frac{5C_4 \times 10C_7}{15C_{11}} + \frac{5C_5 \times 10C_6}{15C_{11}}\\
&= \frac{5C_2 \times 10C_2}{15C_4} + \frac{5C_1 \times 10C_3}{15C_4} + \frac{5C_0 \times 10C_4}{15C_4} \quad (\because nCr = nCn{-}r)\\
&= \frac{450}{1365} + \frac{600}{1365} + \frac{210}{1365} = \frac{1260}{1365}
\end{aligned}
$$

$$\text{P (atleast 3 bowlers)} = \frac{12}{13}$$

EXERCISE 10.1

(1) An experiment has the four possible mutually exclusive outcomes A, B, C and D. Check whether the following assignments of probability are permissible.

(i) P(A) = 0.37, P(B) = 0.17, P(C) = 0.14, P(D) = 0.32

(ii) P(A) = 0.30, P(B) = 0.28, P(C) = 0.26, P(D) = 0.18

(iii) P(A) = 0.32, P(B) = 0.28, P(C) = – 0.06, P(D) = 0.46

(iv) P(A) = 1/2, P(B) = 1/4, P(C) = 1/8, P(D) = 1/16

(v) P(A) = 1/3, P(B) = 1/6, P(C) = 2/9, P(D) = 5/18

(2) In a single throw of two dice, find the probability of obtaining (i) sum of less than 5 (ii) a sum of greater than 10, (iii) a sum of 9 or 11.

(3) Three coins are tossed once. Find the probability of getting (i) exactly two heads (ii) atleast two heads (iii) atmost two heads.

(4) A single card is drawn from a pack of 52 cards. What is the probability that

(i) the card is a jack or king (ii) the card will be 5 or smaller

(iii) the card is either queen or 7.

(5) A bag contains 5 white and 7 black balls. 3 balls are drawn at random. Find the probability that (i) all are white (ii) one white and 2 black.

(6) In a box containing 10 bulbs, 2 are defective. What is the probability that among 5 bulbs chosen at random, none is defective.

(7) 4 mangoes and 3 apples are in a box. If two fruits are chosen at random, the probability that (i) one is a mango and the other is an apple (ii) both are of the same variety.

(8) Out of 10 outstanding students in a school there are 6 girls and 4 boys. A team of 4 students is selected at random for a quiz programme. Find the probability that there are atleast 2 girls.

(9) What is the chance that (i) non-leap year (ii) leap year should have fifty three Sundays?

(10) An integer is chosen at random from the first fifty positive integers. What is the probability that the integer chosen is a prime or multiple of 4.

10.3 Some basic theorems on probability

In the development of probability theory, all the results are derived directly or indirectly using only the axioms of probability. Here we study some of the important theorems on probability.

Theorem 10.1: The probability of the impossible event is zero i.e. $\boxed{\mathbf{P(\phi) = 0}}$

Proof:

Impossible event contains no sample point.

$$\therefore S \cup \phi = S$$

$$P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S)$$ ($\because$ S and ϕ are mutually exclusive)

$$\therefore P(\phi) = 0$$

Theorem 10.2:

If $\bar{A}$ is the complementary event of A, $\boxed{\mathbf{P(\bar{A}) = 1 - P(A)}}$

Proof:

Let S be a sample space, we have

$$A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = 1$$

($\because$ A and $\bar{A}$ are mutually exclusive and P(S) = 1)

$$\therefore P(A) = 1 - P(\bar{A})$$

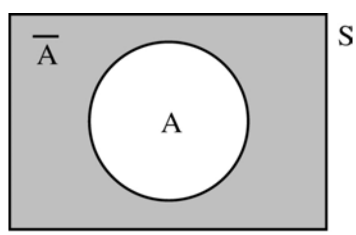


Fig. 10. 1

Theorem 10.3: If A and B are any two events and $\overline{B}$ is the complimentary event of B

$$\boxed{\mathbf{P(A \cap \overline{B}) = P(A) - P(A \cap B)}}$$

Proof: A is the union of two mutually exclusive events $(A \cap \overline{B})$ and $(A \cap B)$ (see fig 10.2)

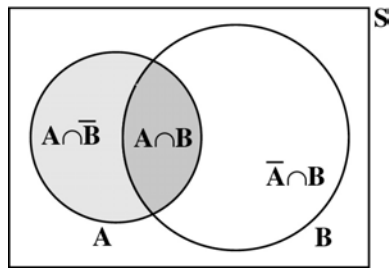


Fig. 10. 2

i.e. $A = (A \cap \overline{B}) \cup (A \cap B)$

$\therefore$ P(A) = $P[(A \cap \overline{B}) \cup (A \cap B)]$

($\because (A \cap \overline{B})$ and $(A \cap B)$ are mutually exclusive)

$$P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

rearranging, we get $$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

Similarly $$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

Theorem 10.4: (Additive theorem on probability) If A and B are any two events

$$\boxed{\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}}$$

Proof: We have

$A \cup B = (A \cap \overline{B}) \cup B$ (See fig. 10.3)

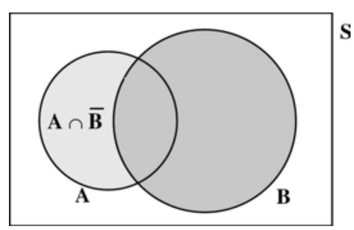


Fig. 10. 3

$P(A \cup B) = P[(A \cap \overline{B}) \cup B]$

($\because A \cap \overline{B}$ and B are mutually exclusive event)

$= P(A \cap \overline{B}) + P(B)$

$= [P(A) - P(A \cap B)] + P(B)$ (by theorem 3)

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note: The above theorem can be extended to any 3 events.

$$P(A\cup B\cup C) = \{P(A)+P(B)+P(C)\} - \{P(A \cap B)+P(B \cap C)+P(C \cap A)\} + P(A\cap B\cap C)$$

Example 10.6:

Given that P(A) = 0.35, P(B) = 0.73 and $P(A \cap B) = 0.14$, find

(i)$P(A\cup B)$ (ii) $P(\overline{A} \cap B)$ (iii) $P(A\cap\overline{B})$ (iv) $P(\overline{A} \cup \overline{B})$ (v) $P(\overline{A\cup B})$

Solution:

(i)
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.35 + 0.73 - 0.14 = 0.94$$
$$P(A \cup B) = 0.94$$

(ii)
$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$
$$= 0.73 - 0.14 = 0.59$$
$$P(\overline{A} \cap B) = 0.59$$

(iii)
$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$
$$= 0.35 - 0.14 = 0.21$$
$$P(A \cap \overline{B}) = 0.21$$

(iv)
$$P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.14$$
$$P(\overline{A} \cup \overline{B}) = 0.86$$

(v)
$$P(\overline{A\cup B}) = 1 - P(A \cup B) = 1 - 0.94 = 0.06 \qquad \text{(by (1))}$$
$$P(\overline{A\cup B}) = 0.06$$

Example 10.7: A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability of drawing (i) a king or a queen (ii) a king or a spade (iii) a king or a black card

Solution:

$$\text{Total number of cases} = 52$$
$$\text{i.e. } n(S) = 52$$

Let A be the event of drawing a king ; B be the event of drawing a queen

C be the event of drawing a spade; D be the event of drawing a black card

$\therefore$ $n(A) = 4$, $n(B) = 4$, $n(C) = 13$, $n(D) = 26$

also we have $n(A \cap C) = 1$, $n(A \cap D) = 2$

(i) P [king or queen] = [A or B] = $P(A \cup B)$

$$= P(A) + P(B) \qquad \left(\begin{array}{l}\because \text{A and B are mutually} \\ \text{exclusive i.e. } A \cap B = \phi\end{array}\right)$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$= \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$$

(ii) P[king or spade] $= P(A \text{ or } C) = P(A \cup C)$

$$= P(A)+P(C)-P(A\cap C)$$

$\left(\begin{array}{l}\because \text{A and C are not mutually} \\ \text{exclusive}\end{array}\right)$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$$

$$= \frac{4}{13}$$

(iii)P[king or black card]$= P(A \text{ or } D) = P(A \cup D)$

$$= P(A) + P(D) - P(A \cap D) \qquad \left(\begin{array}{l}\because \text{ A and D are not} \\ \text{mutually exclusive}\end{array}\right)$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

$$= \frac{7}{13}$$

Example 10.8: The probability that a girl will get an admission in IIT is 0.16, the probability that she will get an admission in Government Medical College is 0.24, and the probability that she will get both is 0.11. Find the probability that (i) She will get atleast one of the two seats (ii) She will get only one of the two seats

Solution:

Let I be the event of getting admission in IIT and M be the event of getting admission in Government Medical College.

$\therefore$ P(I) = 0.16, P(M) = 0.24 and $P(I \cap M) = 0.11$

(i) P(atleast one of the two seats)

$$= P(I \text{ or } M) = P(I \cup M)$$
$$= P(I) + P(M) - P(I \cap M)$$
$$= 0.16 + 0.24 - 0.11$$
$$= 0.29$$

(ii) P(only one of two seats) = P[only I or only M].

$= P[(I \cap \overline{M}) \cup (\overline{I} \cap M)]$

$= P(I \cap \overline{M}) + P(\overline{I} \cap M)$

$= \{P(I) - P(I \cap M)\} + \{P(M) - P(I \cap M)\}$

$= \{0.16 - 0.11\} + \{0.24 - 0.11\}$

$= 0.05 + 0.13$

$= 0.18$

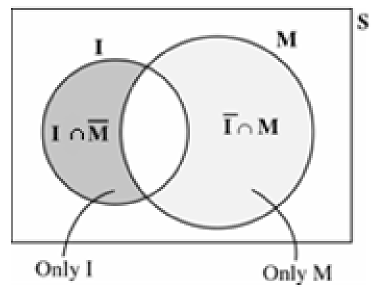


Fig. 10. 4

EXERCISE 10.2

(1) A and B are two events associated with random experiment for which P(A) = 0.36, P(A or B) = 0.90 and P(A and B) = 0.25. Find (i) P(B), (ii) $P(\overline{A} \cap \overline{B})$

(2) If A and B are mutually exclusive events P(A) = 0.28, P(B) = 0.44, find

(i) $P(\overline{A})$ (ii) $P(A \cup B)$ (iii) $(A \cap \overline{B})$ (iv) $P(\overline{A} \cap \overline{B})$

(3) Given P(A) = 0.5, P(B) = 0.6 and $P(A \cap B) = 0.24$.

Find (i) $P(A \cup B)$ (ii) $P(\overline{A} \cap B)$ (iii) $P(A \cap \overline{B})$

(iv) $P(\overline{A} \cup \overline{B})$ (v) $P(\overline{A} \cap \overline{B})$

(4) A die is thrown twice. Let A be the event. "First die shows 4' and B be the event, 'second die shows 4'. Find $P(A \cup B)$.

(5) The probability of an event A occurring is 0.5 and B occurring is 0.3. If A and B are mutually exclusive events, then find the probability of neither A nor B occurring

(6) A card is drawn at random from a deck of 52 cards. What is the probability that the drawn card is (i) a queen or club card (ii) a queen or a black card

(7) The probability that a new ship will get an award for its design is 0.25, the probability that it will get an award for the efficient use of materials is 0.35, and that it will get both awards is 0.15. What is the probability, that

(i) it will get atleast one of the two awards (ii) it will get only one of the awards

10.4 Conditional probability:

Consider the following example to understand the concept of conditional probability.

Suppose a fair die is rolled once. The sample space is S = {1, 2, 3, 4, 5, 6}.

Now we ask two questions:

Q 1: What is the probability that getting an even number which is less than 4?

Q2 : If the die shows an even number, then what is the probability that it is less than 4?

Case 1:

The event of getting an even number which is less than 4 is {2}

$$\therefore\ P_1 = \frac{n(\{2\})}{n(\{1, 2, 3, 4, 5, 6\})} = \frac{1}{6}$$

Case 2:

Here first we restrict our sample space S to a subset containing only even number i.e. to {2, 4, 6}. Then our interest is to find the probability of the event getting a number less than 4 i.e. to {2}.

$$\therefore\ P_2 = \frac{n(\{2\})}{n(\{2, 4, 6\})} = \frac{1}{3}$$

In the above two cases the favourable events are the same, but the number of exhaustive outcomes are different. In case 2, we observe that we have first imposed a condition on sample space, then asked to find the probability. This type of probability is called conditional probability.

Definition: (**Conditional probability**) : The conditional probability of an event B, assuming that the event A has already happened; is denoted by P (B/A) and defined as

$$\boxed{\mathbf{P(B/A) = \frac{P(A \cap B)}{P(A)}}} \quad \text{provided } P(A) \neq 0$$

Similarly

$$\boxed{\mathbf{P(A/B) = \frac{P(A \cap B)}{P(B)}}} \quad \text{provided } P(B) \neq 0$$

Example 10.9: If $P(A) = 0.4 \quad P(B) = 0.5 \quad P(A \cap B) = 0.25$

Find (i) $P(A/B)$ (ii) $P(B/A)$ (iii) $P(\overline{A}/B)$

(iv) $P(B/\overline{A})$ (v) $P(A/\overline{B})$ (vi) $P(\overline{B}/A)$

Solution:

(i) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.25}{0.50} = 0.5$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.25}{0.40} = 0.625$

(iii) $P(\bar{A}/B) = \dfrac{P(\bar{A} \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{0.5 - 0.25}{0.5} = 0.5$

(iv) $P(B/\bar{A}) = \dfrac{P(B \cap \bar{A})}{P(\bar{A})} = \dfrac{P(B) - P(A \cap B)}{1 - P(A)} = \dfrac{0.5 - 0.25}{1 - 0.4} = 0.4167$

(v) $P(A/\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})} = \dfrac{P(A) - P(A \cap B)}{1 - P(B)} = \dfrac{0.4 - 0.25}{1 - 0.5} = 0.3$

(vi) $P(\bar{B}/A) = \dfrac{P(A \cap \bar{B})}{P(A)} = \dfrac{P(A) - P(A \cap B)}{P(A)} = \dfrac{0.4 - 0.25}{0.4} = 0.375$

Theorem 10.6 : (Multiplication theorem on probability)

The probability of the simultaneous happening of two events A and B is given by

$$\boxed{\begin{array}{c} \mathbf{P(A \cap B) = P(A) \,.\, P(B/A)} \\ \textbf{or}\quad \mathbf{P(A \cap B) = P(B) \,.\, P(A/B)} \end{array}}$$

Note: Rewriting the definition of conditional probability, we get the above 'multiplication theorem on probability'.

Independent Events:

Events are said to be independent if the occurrence or non occurrence of any one of the event does not affect the **probability** of occurrence or non-occurrence of the other events.

Definition: Two events A and B are **independent** if $\boxed{\mathbf{P(A \cap B) = P(A) \,.\, P(B)}}$

This definition is exactly equivalent to

$$P(A/B) = P(A), \quad P(B/A) = P(B)$$

Note: The events $A_1, A_2 \ldots\ldots A_n$ are mutually independent if

$$P(A_1 \cap A_2 \cap A_3 \ldots\ldots A_n) = P(A_1).\, P(A_2) \ldots (A_n)$$

Corollary 1: If A and B are independent then A and $\bar{B}$ are also independent.

Proof:

Since A and B are independent

$$P(A \cap B) = P(A) . (PB) \qquad \ldots (1)$$

To prove A and $\overline{B}$ are independent, we have to prove

$$P(A \cap \overline{B}) = P(A) . P(\overline{B})$$

We know

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$
$$= P(A) - P(A) . P(B) \quad \text{(by (1))}$$
$$= P(A) [1 - P(B)]$$

$$P(A \cap \overline{B}) = P(A) . P(\overline{B})$$

$\therefore$ A and $\overline{B}$ are independent.

Similarly, the following corollary can easily be proved.

Corollary 2: If A and B are independent, then $\overline{A}$ and $\overline{B}$ are also independent.

Note: If A_1, A_2… A_n are mutually independent then $\overline{A}_1$, $\overline{A}_2$, … $\overline{A}_n$ are mutually independent.

Example 10.10: Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are kings when

(i) The first drawn card is replaced (ii) The card is not replaced

Solution:

Let A be the event of drawing a king in the first draw.

B be the event of drawing a king in the second draw.

Case i: Card is replaced:

$n(A) = 4$ (king)

$n(B) = 4$ (king)

and $n(S) = 52$ (Total)

Clearly the event A will not affect the probability of the occurrence of event B and therefore A and B are independent.

$$P(A \cap B) = P(A) . P(B)$$
$$= \frac{4}{52} \times \frac{4}{52}$$
$$P(A \cap B) = \frac{1}{169}$$

Case ii: (Card is not replaced)

In the first draw, there are 4 kings and 52 cards in total. Since the king, drawn at the first draw is not replaced, in the second draw there are only 3 kings and 51 cards in total. Therefore the first event A affects the probability of the occurrence of the second event B.

$\therefore$ A and B are not independent they are dependent events.

$$\therefore P(A \cap B) = P(A) . P(B/A)$$

$$P(A) = \frac{4}{52} \quad ; \quad P(B/A) = \frac{3}{51}$$

$$P(A \cap B) = P(A) . P(B/A) = \frac{4}{52} \cdot \frac{3}{51}$$

$$P(A \cap B) = \frac{1}{221}$$

Example 10.11: A coin is tossed twice. Event E and F are defined as follows : E = Head on first toss, F = head on second toss.

Find (i) $P(E \cap F)$ (ii) $P(E \cup F)$ (iii) $P(E/F)$

(iv) $P(\bar{E}/F)$ (v) Are the events E and F independent ?

Solution: The sample space is

$$S = \{(H,H), (H, T), (T, H), (T, T)\}$$

$$\text{and} \quad E = \{(H, H), (H\ T)\}$$

$$F = \{(H, H), (T, H)\}$$

$$\therefore E \cap F = \{(H, H)\}$$

(i) $$P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{4}$$

(ii) $$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4}$$

$$P(E \cup F) = \frac{3}{4}$$

(iii) $$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(iv) $$P(\bar{E}/F) = \frac{P(\bar{E} \cap F)}{P(F)} = \frac{P(F) - P(E \cap F)}{P(F)}$$

$$= \frac{2/4 - 1/4}{2/4} = \frac{1}{2}$$

$$P(\bar{E}/F) = \frac{1}{2}$$

(v) $$P(E) = \frac{2}{4} = \frac{1}{2}, \quad P(F) = \frac{2}{4} = \frac{1}{2}$$

$$P(E \cap F) = \frac{1}{4}$$

$$\therefore \; P(E)\,P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Since $P(E \cap F) = P(E) \,.\, P(F)$, E and F are independent.

In the above example the events E and F are not mutually exclusive but they are independent.

Important Note:

Independence is a property of probability but **mutually exclusion is a set-theoretic property**. Therefore independent events can be identified by their probabilities and mutually exclusive events can be identified by their events.

Theorem 10.7: Suppose A and B are two events, such that $P(A) \neq 0$, $P(B) \neq 0$

(i) If A and B are mutually exclusive, they cannot be independent.

(ii) If A and B are independent they cannot be mutually exclusive.

(Proof not required)

Example 10.12: If A and B are two independent events such that $P(A) = 0.5$ and $P(A \cup B) = 0.8$. Find P(B).

Solution:

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = p(A)+P(B)-P(A).P(B)$ ($\because$ A and B are independent)

i.e. $0.8 = 0.5 + P(B) - (0.5)\,P(B)$

$0.8 - 0.5 = (1 - 0.5)\,P(B)$

$$\therefore P(B) = \frac{0.3}{0.5} = 0.6$$

$P(B) = 0.6$

Example 10.13: A problem is given to 3 students X, Y and Z whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{5}$ respectively. What is the probability that the problem is solved?

Solution:

Let A, B and C be the events of solving the problem by X, Y and Z respectively.

$$\therefore\ P(A) = \frac{1}{2}\ ;\ P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(B) = \frac{1}{3}\ ;\ P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{2}{5}\ ;\ P(\overline{C}) = 1 - P(C) = 1 - \frac{2}{5} = \frac{3}{5}$$

P[problem is solved] = P[the problem is solved by atleast one of them]

$$= P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C})$$

$$= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})\ \text{(By De Morgan's Law)}$$

$$= 1 - P(\overline{A}) . P(\overline{B}) . P(\overline{C})$$

($\because$ A, B, C are independent $\overline{A}$, $\overline{B}$, $\overline{C}$ are also independent)

$$= 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5} = 1 - \frac{1}{5}$$

$$\text{P[problem is solved]} = \frac{4}{5}$$

Examples 10.14 : X speaks truth in 95 percent of cases, and Y in 90 percent of cases. In what percentage of cases are they likely to contradict each other in stating the same fact.

Solution: Let A be the event of X speaks the truth, B be the event of Y speaks the truth.

$\therefore$ $\overline{A}$ and $\overline{B}$ are the events of not speaking the truth by X and Y respectively.

Let C be the event that they will contradict each other.

Given that

$$P(A) = 0.95 \quad \therefore P(\overline{A}) = 1 - P(A) = 0.05$$

$$P(B) = 0.90 \quad \therefore P(\overline{B}) = 1 - P(B) = 0.10$$

C = (A speaks truth and B does not speak truth

or

B Speaks truth and A does not speak truth)

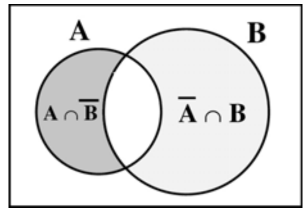


Fig. 10. 5

$$C = [(A \cap \overline{B}) \cup (\overline{A} \cap B)]$$

$$\therefore P(C) = P[(A \cap \overline{B}) \cup (\overline{A} \cap B)]$$

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

($\because \overline{A} \cap B$ and $A \cap \overline{B}$ are mutually exclusive)

$$= P(A) \,.\, P(\overline{B}) + P(\overline{A}) \,.\, P(B)$$ ($\because$ A, $\overline{B}$ are independent event also $\overline{A}$, B are independent events)

$$= (0.95) \times (0.10) + (0.05)(0.90)$$

$$= 0.095 + 0.045$$

$$= 0.1400$$

P(C) = 14%

EXERCISE 10.3

(1) Define independent and mutually exclusive events. Can two events be mutually exclusive and independent simultaneously.

(2) If A and B are independent, prove that $\overline{A}$ and $\overline{B}$ are independent.

(3) If P(A) = 0.4, P(B) = 0.7 and P(B / A) = 0.5 find P(A / B) and $P(A \cup B)$.

(4) If for two events A and B, P(A) = 2/5, P(B) = 3/4 and $A \cup B$ = (sample space), find the conditional probability P(A / B).

(5) If A and B are two independent events such that $P(A \cup B) = 0.6$, P(A) = 0.2 find P(B)

(6) If A and B are two events such that $P(A \cup B) = 5/6$, $P(A \cap B) = 1/3$, $P(\overline{B}) = 1/2$ show that A and B are independent.

(7) if the events A and B are independent and P(A) = 0.25, P(B) = 0.48,

find (i) $P(A \cap B)$ (ii) P(B / A) (iii) $P(\overline{A} \cap \overline{B})$

(8) Given P(A) = 0.50, P(B) = 0.40 and $P(A \cap B) = 0.20$.

Verify that (i) P(A / B) = P(A), (ii) $P(A / \overline{B}) = P(A)$

(iii) P(B / A) = P(B) (iv) $P(B / \overline{A}) = P(B)$

(9) $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.25$

Find (i) $P(A \cup B)$ (ii) $P(A/B)$ (iii) $P(B/\bar{A})$ (iv) $P(\bar{A}/B)$ (v) $P(\bar{A}/\bar{B})$

(10) Given $P(A) = 0.45$ and $P(A \cup B) = 0.75$.

Find P(B) if (i) A and B are mutually exclusive (ii) A and B are independent events (iii) $P(A / B) = 0.5$ (iv) $P(B / A) = 0.5$

(11) Two cards are drawn one by one at random from a deck of 52 playing cards. What is the probability of getting two jacks if (i) the first card is replaced before the second is drawn (ii) the first cards is not replaced before the second card is drawn.

(12) If a card is drawn from a deck of 52 playing cards, what is the probability of drawing (i) a red king (ii) a red ace or a black queen.

(13) One bag contains 5 white and 3 black balls. Another bag contains 4 white and 6 black balls. If one ball is drawn from each bag, find the probability that (i) both are white (ii) both are black (iii) one white and one black.

(14) A husband and wife appear in an interview for two vacancies in the same post. The probability of husbands' selection is 1/6 and that of wife's selection is 1/5. What is the probability that

(i) both of them will be selected (ii) only one of them will be selected

(iii) none of them will be selected

(15) A problem in Mathematics is given to three students whose chances of solving it are 1/2, 1/3 and 1/4 (i) What is the probability that the problem is solved (ii) what is the probability that exactly one of them will solve it.

(16) A year is selected at random. What is the probability that (i) it contains 53 Sundays (ii) it is a leap year contains 53 Sundays

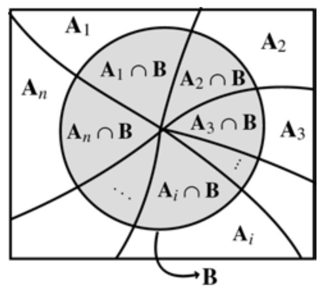
(17) For a student the probability of getting admission in IIT is 60% and probability of getting admission in Anna University is 75%. Find the probability that (i) getting admission in only one of these (ii) getting admission in atleast one of these.

(18) A can hit a target 4 times in 5 shots, B 3 times in 4 shots, C 2 times in 3 shots, they fire a volley. What is the chance that the target is damaged by exactly 2 hits?

(19) Two thirds of students in a class are boys and rest girls. It is known that the probability of a girl getting a first class is 0.75 and that of a boys is 0.70. Find the probability that a student chosen at random will get first class marks.

(20) A speaks truth in 80% cases and B in 75% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

10.5 Total probability of an event



If A_1, A_2 … A_n are mutually exclusive and exhaustive events and B is any event in S then

$P(B) = P(A_1) . P(B/A_1)$

$+P(A_2) . P(B/A_2) \ \dots + P(A_n) P(B/A_n)$

P(B) is called the total probability of event B

Fig. 10.7

Example 10.15: An urn contains 10 white and 5 black balls. While another urn contains 3 white and 7 black balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are white.

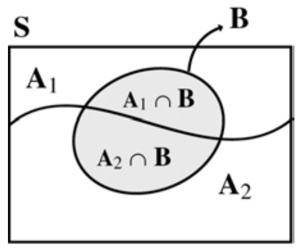


Fig. 10.8

Solution:

Let A_1 be the event of selecting urn-I and A_2 be the event of selecting urn-II. Let B be the event of selecting 2 white balls.

We have to find the total probability of event B i.e. P(B). Clearly A_1 and A_2 are mutually exclusive and exhaustive events.

$$P(B) = P(A_1) . P(B/A_1) + P(A_2) . P(B/A_2) \quad \dots (1)$$

$$P(A_1) = \frac{1}{2} \ ; P(B/A_1) = \frac{10C_2}{15C_2}$$

$$P(A_2) = \frac{1}{2} \ ; \ P(B/A_2) = \frac{3C_2}{10C_2}$$

	White	Black	Total
Urn I	10	5	(15)
Urn II	3	7	(10)

Fig. 10.9

Substituting in (1), $P(B) = P(A_1) . P(B/A_1) + P(A_2) . P(B/A_2)$

$$= \left(\frac{1}{2}\right)\left(\frac{10C_2}{15C_2}\right) + \left(\frac{1}{2}\right)\left(\frac{3C_2}{10C_2}\right) = \frac{1}{2}\left[\frac{3}{7} + \frac{1}{15}\right]$$

$$P(B) = \frac{26}{105}$$

Example 10.16: A factory has two machines I and II. Machine I produces 30% of items of the output and Machine II produces 70% of the items. Further 3% of items produced by Machine I are defective and 4% produced by Machine II are defective. If an item is drawn at random, find the probability that it is a defective item.

Solution:

Let A_1 be the event that the items are produced by Machine I, A_2 be the event that items are produced by Machine II. Let B be the event of drawing a defective item.

$$\therefore P(A_1) = \frac{30}{100} \ ; \ P(B/A_1) = \frac{3}{100}$$

$$P(A_2) = \frac{70}{100} \ ; \ P(B/A_2) = \frac{4}{100}$$

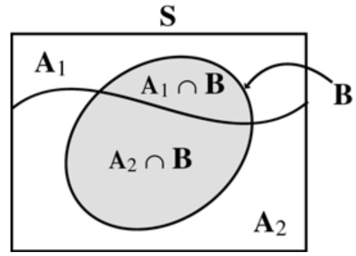


Fig. 10.10

We are asked to find the total probability of event B.

Since A_1, A_2 are mutually exclusive and exhaustive.

We have
$$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)$$

$$= \left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{70}{100}\right)\cdot\left(\frac{4}{100}\right)$$

$$= \frac{90 + 280}{10000}$$

$$P(B) = 0.0370$$

Theorem 10.8: (Bayes' Theorem):

Suppose $A_1, A_2, \ldots A_n$ are n mutually exclusive and exhaustive events such that $P(A_i) > 0$ for $i = 1, 2 \ldots n$. Let B be any event with $P(B) > 0$ then

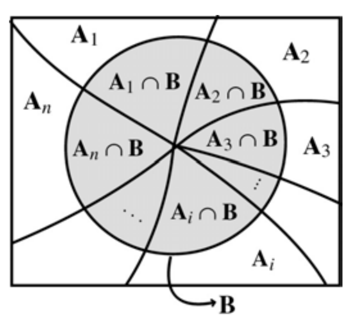


Fig. 10.11

$$P(A_i/B) = \frac{P(A_i)\, P(B/A_i)}{P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + \ldots + P(A_n)\, P(B/A_n)}$$

(Proof not required)

The above formula gives the relationship between $P(A_i/B)$ and $P(B/ A_i)$

Example 10.17: A factory has two machines I and II. Machine I and II produce 30% and 70% of items respectively. Further 3% of items produced by Machine I are defective and 4% of items produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II. (See the previous example, compare the questions).

Solution:

Let A_1 and A_2 be the events that the items produced by Machine I & II respectively.

Let B be the event of drawing a defective item.

$$\therefore P(A_1) = \frac{30}{100} \quad ; P(B/ A_1) = \frac{3}{100}$$

$$P(A_2) = \frac{70}{100} \quad ; P(B/ A_2) = \frac{4}{100}$$

Now we are asked to find the conditional probability $P(A_2/ B)$

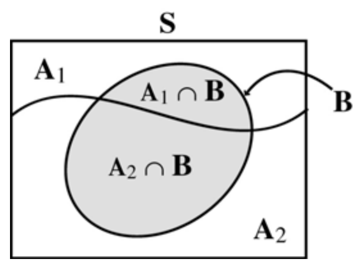


Fig. 10.12

Since A_1, A_2 are mutually exclusive and exhaustive events by Bayes' theorem

$$P(A_2/ B) = \frac{P(A_2)\,.\,P(B/ A_2)}{P(A_1)\,.\,P(B/ A_1) + P(A_2)\,.\,P(B/ A_2)}$$

$$= \frac{\left(\frac{70}{100}\right) \times \left(\frac{4}{100}\right)}{\left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{70}{100}\right)\left(\frac{4}{100}\right)} = \frac{0.0280}{0.0370} = \frac{28}{37}$$

$$P(A_2/ B) = \frac{28}{37}$$

Example 10.18: The chances of X, Y and Z becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus scheme will be introduced if X, Y and Z become managers are 0.3, 0.5 and 0.4 respectively. If the bonus scheme has been introduced, what is the probability that Z is appointed as the manager.

Solution:

Let A_1, A_2 and A_3 be the events of X, Y and Z becoming managers of the company respectively. Let B be the event that the bonus scheme will be introduced.

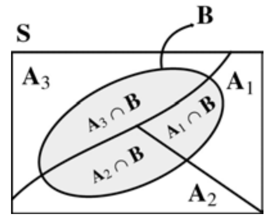


Fig. 10.13

$$\therefore\ P(A_1) = \frac{4}{9} \quad ; P(B/ A_1) = 0.3$$

$$P(A_2) = \frac{2}{9} \quad ; P(B/ A_2) = 0.5$$

$$P(A_3) = \frac{3}{9} \quad ; P(B/ A_3) = 0.4$$

We have to find the conditional probability $P(A_3/ B)$

A_1, A_2 and A_3 are mutually exclusive and exhaustive events. Applying Bayes' formula

$$P(A_3/ B) = \frac{P(A_3) \,.\, P(B/ A_3)}{P(A_1) \,.\, P(B/ A_1) + P(A_2) \,.\, P(B/ A_2) + P(A_3) \,.\, P(B/ A_3)}$$

$$= \frac{\left(\frac{3}{9}\right)(0.4)}{\left(\frac{4}{9}\right)(0.3) + \left(\frac{2}{9}\right)(0.5) + \left(\frac{3}{9}\right)(0.4)} = \frac{12}{34}$$

$$P(A_3/ B) = \frac{6}{17}$$

Example 10.19: A consulting firm rents car from three agencies such that 20% from agency X, 30% from agency Y and 50% from agency Z. If 90% of the cars from X, 80% of cars from Y and 95% of the cars from Z are in good conditions (1) what is the probability that the firm will get a car in good condition? Also (ii) If a car is in good condition, what is probability that it has came from agency Y?

Solution:

Let A_1, A_2, A_3 be the events that the cars are rented from the agencies X, Y and Z respectively.

Let G be the event of getting a car in good condition.

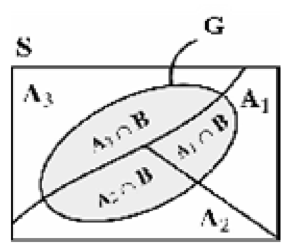


Fig. 10.14

$$\therefore\ P(A_1) = 0.20 \ ; P(G/A_1) = 0.90$$
$$P(A_2) = 0.30 \ ; P(G/A_2) = 0.80$$
$$P(A_3) = 0.50 \ ; P(G/A_3) = 0.95$$

(i) We have to find the total probability of event G i.e. P(G)

Since A_1, A_2, A_3 are mutually exclusive and exhaustive events and G is an event in S.

We have
$$\begin{aligned} P(G) &= P(A_1) \,.\, P(G/A_1) + P(A_2) \,.\, P(G/A_2) + P(A_3) \,.\, P(G/A_3) \\ &= (0.2)\,(0.90) + (0.3)\,(0.80) + (0.5)\,(0.95) \\ &= 0.180 + 0.240 + 0.475 \\ P(G) &= 0.895 \end{aligned}$$

(ii) We have to find the conditional probability A_2 given G i.e. $P(A_2/G)$

By Bayes' formula

$$P(A_2/G) = \frac{P(A_2) \,.\, P(G/A_2)}{P(A_1) \,.\, P(G/A_1) + P(A_2) \,.\, P(G/A_2) + P(A_3) \,.\, P(G/A_3)}$$

$$= \frac{(0.3)\,(0.80)}{(0.895)} \qquad \text{(by (1) Dr } = P(G) = 0.895)$$

$$= \frac{0.240}{0.895}$$

$$P(A_2/G) = 0.268 \text{ (Approximately)}$$

EXERCISE 10.4

(1) Bag A contains 5 white, 6 black balls and bag B contains 4 white, 5 black balls. One bag is selected at random and one ball is drawn from it. Find the probability that it is white.

(2) A factory has two Machines-I and II. Machine-I produces 25% of items and Machine-II produces 75% of the items of the total output. Further 3% of the items produced by Machine-I are defective whereas 4% produced by Machine-II are defective. If an item is drawn at random what is the probability that it is defective?

(3) There are two identical boxes containing respectively 5 white and 3 red balls, 4 white and 6 red balls. A box is chosen at random and a ball is drawn from it (i) find the probability that the ball is white (ii) if the ball is white, what is the probability that it is from the first box?

(4) In a factory, Machine-I produces 45% of the output and Machine-II produces 55% of the output. On the average 10% items produced by I and 5% of the items produced by II are defective. An item is drawn at random from a day's output. (i) Find the probability that it is a defective item (ii) If it is defective, what is the probability that it was produced by Machine-II.

(5) Three urns are given each containing red and white chips as given below.
Urn I : 6 red 4 white ; Urn II : 3 red 5 white ; Urn III : 4 red 6 white
An urn is chosen at random and a chip is drawn from the urn.
(i) Find the probability that it is white
(ii) If the chip is white find the probability that it is from urn II

OBJECTIVE TYPE QUESTIONS

(1) Identify the correct statement

(1) The set of real numbers is a closed set

(2) The set of all non-negative real numbers is represented by $(0, \infty)$

(3) The set $[3, 7]$ indicates the set of all natural numbers between 3 and 7

(4) $(2, 3)$ is a subset of $[2, 3]$.

(2) Identify the correct statements of the following

(i) a relation is a function

(ii) a function is a relation

(iii) 'a function which is not a relation' is not possible

(iv) 'a relation which is not a function' is possible

(1) (ii), (iii) and (iv) (2) (ii) and (iii) (3) (iii) and (iv) (4) all

(3) Which one of the following is a function which is 'onto'?

(1) $f : R \to R$; $f(x) = x^2$ (2) $f : R \to [1,\infty)$; $f(x) = x^2 + 1$

(3) $f : R \to \{1, -1\}$; $f(x) = \frac{|x|}{x}$ (4) $f : R \to R$; $f(x) = -x^2$

(4) Which of the following is a function which is not one-to-one?

(1) $f : R \to R$; $f(x) = x + 1$ (2) $f : R \to R$; $f(x) = x^2 + 1$

(3) $f : R \to \{1, -1\}$; $f(x) = x - 1$ (4) $f : R \to R$; $f(x) = -x$

(5) The inverse of $f : R \to R^+$; $f(x) = x^2$ is

(1) not onto (2) not one-to-one

(3) not onto and not one-to-one (4) not at all a function

(6) Identify the correct statements

(i) a constant function is a polynomial function.

(ii) a polynomial function is a quadratic function.

(iii) for linear function, inverse always exists.

(iv) A constant function is one-to-one only if the domain is a singleton set.

(1) (i) and (iii) (2) (i), (iii) and (iv) (3) (ii) and (iii) (4) (i) and (iii)

(7) Identify the correct statements

(i) the domain of circular functions are always R.

(ii) The range of tangent function is R.

(iii) The range of cosine function is same as the range of sine function.

(iv) The domain of cotangent function is $R - \{k\pi\}$

(1) all (2) (i) and (iii) (3) (ii), (iii) and (iv) (4) (iii) and (iv)

(8) The true statements of the following are

(i) The composition of function *fog* and the product of functions *fg* are same.

(ii) For the composition of functions *fog*, the co-domain of *g* must be the domain of *f*.

(iii) If *fog*, *gof* exist then *fog* = *gof*.

(iv) If the function *f* and *g* are having same domain and co-domain then *fg* = *gf*

(1) all (2) (ii), (iii) and (iv) (3) (iii) and (iv) (4) (ii) and (iv)

(9) $\lim\limits_{x \to -6} (-6)$ is

(1) 6 (2) – 6 (3) 36 (4) – 36

(10) $\lim\limits_{x \to -1} (x)$ is

(1) – 1 (2) 1 (3) 0 (4) 0.1

(11) The left limit as $x \to 1$ of $f(x) = -x + 3$ is

(1) 2 (2) 3 (3) 4 (4) – 4

(12) Rf(0) for $f(x) = |x|$ is

(1) x (2) 0 (3) $-x$ (4) 1

(13) $\lim\limits_{x \to 1} \dfrac{x^{1/3} - 1}{x - 1}$ is

(1) $\frac{2}{3}$ (2) $-\frac{2}{3}$ (3) $\frac{1}{3}$ (4) $-\frac{1}{3}$

(14) $\lim\limits_{x \to 0} \dfrac{\sin 5x}{x}$ is

(1) 5 (2) $\frac{1}{5}$ (3) 0 (4) 1

(15) $\lim\limits_{x \to 0} x \cot x$ is

(1) 0 (2) – 1 (3) ∞ (4) 1

(16) $\lim\limits_{x \to 0} \dfrac{2^x - 3^x}{x}$ is

(1) $\log\left(\frac{3}{2}\right)$ (2) $\log\left(\frac{2}{3}\right)$ (3) log 2 (4) log 3

(17) $\lim\limits_{x \to 1} \dfrac{e^x - e}{x - 1}$ is

(1) 1 (2) 0 (3) ∞ (4) e

(18) $\lim\limits_{x \to \infty} \left(1 - \dfrac{1}{x}\right)^x$ is

(1) e (2) $-e$ (3) $\dfrac{1}{e}$ (4) 0

(19) The function $f(x) = |x|$ is

(1) continuous at $x = 0$

(2) discontinuous at $x = 0$

(3) not continuous from the right at $x = 0$

(4) not continuous from the left at $x = 0$

(20) The function $f(x) = \begin{cases} \dfrac{\sin(x-2)}{x-2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$ is discontinuous at

(1) $x = 0$ (2) $x = -1$ (3) $x = -2$ (4) $x = 2$

(21) The function $f(x) = \dfrac{x^2 + 1}{x^2 - 3x + 2}$ is continuous at all points of R except at

(1) $x = 1$ (2) $x = 2$ (3) $x = 1, 2$ (4) $x = -1, -2$

(22) Let $f(x) = \lfloor x \rfloor$ be the greatest integer function. Then

(1) $f(x)$ is continuous at all integral values

(2) $f(x)$ is discontinuous at all integral values

(3) $x = 0$ is the only discontinuous point

(4) $x = 1$ is the only continuous point

(23) The function $y = \tan x$ is continuous at

(1) $x = 0$ (2) $x = \dfrac{\pi}{2}$ (3) $x = \dfrac{3\pi}{2}$ (4) $x = -\dfrac{\pi}{2}$

(24) $f(x) = |x| + |x - 1|$ is

(1) continuous at $x = 0$ only (2) continuous at $x = 1$ only

(3) continuous at both $x = 0$ and $x = 1$ (4) discontinuous at $x = 0, 1$

(25) If $f(x) = \begin{cases} kx^2 \text{ for } x \leq 2 \\ 3 \text{ for } x > 2 \end{cases}$ is continuous at $x = 2$, the value of k is

(1) $\dfrac{3}{4}$ (2) $\dfrac{4}{3}$ (3) 1 (4) 0

(26) $Rf'(0)$ for the function $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$ is

(1) 1 (2) 0 (3) – 1 (4) 2

(27) $Lf'(\alpha)$ for the function $f(x) = |x - \alpha|$ is

(1) α (2) $-\alpha$ (3) – 1 (4) 1

(28) The function $f(x) = \begin{cases} 2, & x \le 1 \\ x, & x > 1 \end{cases}$ is not differentiable at

(1) $x = 0$ (2) $x = -1$ (3) $x = 1$ (4) $x = -2$

(29) The derivative of $f(x) = x^2 |x|$ at $x = 0$ is

(1) 0 (2) – 1 (3) – 2 (4) 1

(30) $\int \sin^2 x \, dx =$

(1) $\frac{\sin^3 x}{3} + c$ (2) $-\frac{\cos^2 x}{2} + c$

(3) $\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$ (4) $\frac{1}{2}[1 + \sin 2x] + c$

(31) $\int \sin 7x \cos 5x \, dx =$

(1) $\frac{1}{35} \cos 7x \sin 5x + c$ (2) $-\frac{1}{2}\left[\frac{\cos 12x}{12} + \frac{\cos 2x}{2}\right] + c$

(3) $-\frac{1}{2}\left[\frac{\cos 6x}{6} + \cos x\right] + c$ (4) $\frac{1}{2}\left[\frac{\cos 12x}{12} + \frac{\cos 2x}{2}\right] + c$

(32) $\int \frac{e^x}{e^x + 1} \, dx =$

(1) $\frac{1}{2} x + c$ (2) $\frac{1}{2}\left(\frac{e^x}{1 + e^x}\right)^2 + c$ (3) $\log(e^x + 1) + c$ (4) $x + e^x + c$

(33) $\int \frac{1}{e^x} \, dx =$

(1) $\log e^x + c$ (2) $-\frac{1}{e^x} + c$ (3) $\frac{1}{e^x} + c$ (4) $x + c$

(34) $\int \log x \, dx =$

(1) $\frac{1}{x} + c$ (2) $\frac{(\log x)^2}{2} + c$ (3) $x \log x + x + c$ (4) $x \log x - x + c$

(35) $\int \frac{x}{1+x^2}\,dx =$

(1) $\tan^{-1}x + c$ (2) $\frac{1}{2}\log(1+x^2) + c$ (3) $\log(1+x^2) + c$ (4) $\log x + c$

(36) $\int \tan x\,dx =$

(1) $\log \cos x + c$ (2) $\log \sec x + c$ (3) $\sec^2 x + c$ (4) $\frac{\tan^2 x}{2} + c$

(37) $\int \frac{1}{\sqrt{3+4x}}\,dx =$

(1) $\frac{1}{2}\sqrt{3+4x} + c$ (2) $\frac{1}{4}\log\sqrt{3+4x} + c$

(3) $2\sqrt{3+4x} + c$ (4) $-\frac{1}{2}\sqrt{3+4x} + c$

(38) $\int \left(\frac{x-1}{x+1}\right) dx =$

(1) $\frac{1}{2}\left(\frac{x-1}{x+1}\right)^2 + c$ (2) $x - 2\log(x+1) + c$

(3) $\frac{(x-1)^2}{2}\log(x+1) + c$ (4) $x + 2\log(x+1) + c$

(39) $\int \operatorname{cosec} x\,dx =$

(1) $\log \tan\frac{x}{2} + c$ (2) $-\log(\operatorname{cosec} x + \cot x) + c$

(3) $\log(\operatorname{cosec} x - \cot x) + c$ (4) all of them

(40) When three dice are rolled , number of elementary events are

(1) 2^3 (2) 3^6 (3) 6^3 (4) 3^2

(41) Three coins are tossed. The probability of getting atleast two heads is

(1) $\frac{3}{8}$ (2) $\frac{7}{8}$ (3) $\frac{1}{8}$ (4) $\frac{1}{2}$

(42) If $P(A) = 0.35$, $P(B) = 0.73$ and $P(A \cap B) = 0.14$. Then $P(\bar{A} \cup \bar{B}) =$

(1) 0.94 (2) 0.06 (3) 0.86 (4) 0.14

(43) If A and B are two events such that $P(A) = 0.16$, $P(B) = 0.24$ and $P(A \cap B) = 0.11$, then the probability of obtaining only one of the two events is

(1) 0.29 (2) 0.71 (3) 0.82 (4) 0.18

(44) Two events A and B are independent, then $P(A/B) =$

(1) $P(A)$ (2) $P(A \cap B)$ (3) $P(A) = P(B)$ (4) $\frac{P(A)}{P(B)}$

(45) A and B are two events such that $P(A) \neq 0$, $P(B) \neq 0$. If A and B are mutually exclusive, then

(1) $P(A \cap B) = P(A)\, P(B)$ (2) $P(A \cap B) \neq P(A) \,.\, P(B)$

(3) $P(A/B) = P(A)$ (4) $P(B/A) = P(A)$

(46) X speaks truth in 95 percent of cases and Y in 80 percent of cases. The percentage of cases they likely to contradict each other in stating same fact is

(1) 14% (2) 86% (3) 23% (4) 85.5%

(47) A problem is given to 3 students A, B and C whose chances of solving it are $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{1}{4}$. The probability to solve is

(1) $\frac{4}{5}$ (2) $\frac{3}{10}$ (3) $\frac{7}{10}$ (4) $\frac{1}{30}$

(48) Given $P(A) = 0.50$, $P(B) = 0.40$ and $P(A \cap B) = 0.20$ then $P(A/\overline{B}) =$

(1) 0.50 (2) 0.40

(3) 0.70 (4) 0.10

(49) An urn contains 10 white and 10 black balls. While another urn contains 5 white and 10 black balls. One urn is chosen at random and a ball is drawn from it. The probability that it is white, is

(1) $\frac{5}{11}$ (2) $\frac{5}{12}$ (3) $\frac{3}{7}$ (4) $\frac{4}{7}$