Chapter 5
COMPLEX NUMBERS AND QUADRATIC EQUATIONS

INTRODUCTION

$\sqrt{-36}$, $\sqrt{-25}$ etc do not have values in the system of real numbers.

So we need to extend the real numbers system to a larger system.

Let us denote $\sqrt{-1}$ by the symbol i.

ie $i^2 = -1$

A number of the form a+ib where a&b are real numbers is defined to be a complex number.

Eg 2+i3, -7+$\sqrt{2}$ i, $\sqrt{3}$ i, 4+$\underline{1}$i, 5=5+0i, -7=-7+0i etc

For z = 2+i5, Re z =2 (real part)

and Im z = 5 (imaginary part)

Refer algebra of complex numbers of text book pg 98

1) **Addition of complex numbers**

$$(2+i3) + (-3+i2) = (2+-3) + i(3+2) = -1+5i$$

2) **Difference of complex numbers**

$$(2+i3)-(-3+i2) = (2+3) + i(3-2) = 5 + i$$

3) **Multiplication of two complex numbers**

$$\begin{aligned}(2+i3)(-3+i2) &= 2(-3+i2) + i3(-3+i2)\\ &= -6+4i-9i+6i^2\\ &= -6-5i-6 \qquad (i^2 = -1)\\ &= -12-5i\end{aligned}$$

4) **Division of complex numbers**

$$\begin{aligned}\frac{2+i3}{-3+i2} &= \frac{(2+i3)}{(-3+i2)} \times \frac{(-3-i2)}{(-3-i2)}\\ &= \frac{-6-4i-9i-6i^2}{(-3)^2-(i2)^2}\\ &= \frac{-6-13i+6}{9-(-1)\times 4}\\ &= \frac{-13i}{13} = \underline{-i}\end{aligned}$$

5) **Equality of 2 complex numbers**
   a+ib = c+id, iff a=c & b=d
6) a+ib =0, iff a=0 and b=0
   Refer : the square roots of a negative real no & identities (text page 100,101)

Formulas

a) IF Z=a+ib then modulus of Z ie $|Z| = (a^2+b^2)^{1/2}$
b) Conjugate of Z is a-ib
c) **Multiplicative inverse of a+ib** = $\dfrac{a}{(a^2+b^2)} - \dfrac{ib}{(a^2+b^2)}$

****d) Polar representation of a complex number**

$\boxed{a+ib = r(\cos\theta + i\sin\theta)}$

Where $r = |Z| = (a^2+b^2)^{1/2}$ and θ = arg Z(argument or amplitude of Z which has many different values but when $-\pi < \theta \leq \pi$, θ is called principal argument of Z.

Trick method to find θ

Step 1First find angle using the following

1) $\cos\theta = 1$ and $\sin\theta = 0$ then angle = 0
2) $\cos\theta = 0$ and $\sin\theta = 1$ then angle = $\pi/2$
3) $\sin\theta = \sqrt{3}/2$ and $\cos\theta = 1/2$ then angle = $\pi/3$
4) $\sin\theta = ½$ and $\cos\theta = \sqrt{3}/2$ then angle = $\pi/6$

Step 2: **To find θ**

1) If both sinθ and cosθ are positive then θ = angle (first quadrant)
2) If sinθ positive, cosθ negative then θ = π-angle (second quadrant)
3) If both sinθ and cosθ are negative the θ= π+angle (third quadrant)

4) If sinθ negative and cosθ positive then θ= 2π-angle (fourth quadrant)
   Or θ = - (angle) since $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$
5) If sinθ = 0 and cosθ = -1 then θ=π

****e) Formula needed to find square root of a complex number**

$(a+b)^2 = (a-b)^2 + 4ab$

ie $[x^2 + y^2]^2 = [x^2 - y^2]^2 + 4x^2y^2$

e) Powers of i

i) $i^{4k} = 1$

ii) $i^{4k+1} = i$

iii) $i^{4k+2} = -1$

iv) $i^{4k+3} = -i$**, for any integer k**

Examples:

$i^1 = i, i^2 = -1, i^3 = -i$ and $i^4 = 1$&

$$i^{19} = i^{16} \times i^3 = 1 \times -i = -i$$

g) Solutions of quadratic equation $ax^2+bx + c = 0$ with real coefficients a,b,c and $a \neq 0$ are given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, If $b^2 - 4ac \geq 0$

If $b^2 - 4ac < 0$ then $x = \frac{-b \pm \sqrt{4ac - b^2}\ i}{2a}$

Refer text page 102 the modulus and conjugate of a complex number properties given in the end. (i) to (v)
Ex 5.1
Q. 3*(1 mark), 8* (4 marks), 11**, 12**, 13**, 14**(4 Marks)

Polar form (very important)

Ex 5.2

Q 2**) Express $Z = -\sqrt{3}+i$ in the polar form and also write the modulus and the argument of Z

Solution Let $-\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

Here $a = -\sqrt{3}$, $b = 1$

$r = (a^2+b^2)^{1/2} = \sqrt{3+1} = \sqrt{4} = 2$

$-\sqrt{3}+i = 2\cos\theta + i \times 2\sin\theta$

Therefore $2\cos\theta = -\sqrt{3}$ and $2\sin\theta = 1$

$\cos\theta = -\sqrt{3}/2$ and $\sin\theta = ½$

Here cosθ negative and sinθ positive

Therefore $\theta = \pi - \pi/6 = 5\pi/6$ (see trick method given above)

Therefore polar form of $Z = -\sqrt{3}+i = 2(\cos 5\pi/6 + i\sin 5\pi/6)$

$|Z| = 2$ and argument of $Z = 5\pi/6$ and $-\sqrt{3} + i = 2(Cos 5\pi/6 + iSin 5\pi/6)$

Ex 5.2

Q (1 to 8)** Note: Q 1) $\theta = 4\pi/3$ or principal argument $\theta = 4\pi/3 - 2\pi = -2\pi/3$

Q 5) $\theta = 5\pi/4$ or principal argument $\theta = 5\pi/4 - 2\pi = -3\pi/4$

eg 7**, eg 8**

Ex 5.3

Q 1,8,9,10 (1 mark)

Misc examples (12 to 16)**

Misc exercise

Q 4**,5**,10**,11**,12**,13**,14**,15**,16**,17*,20**

Supplementary material

eg 12**

Ex 5.4

Q (1 to 6)**

EXTRA/HOT QUESTIONS

1** Find the square roots of the following complex numbers (4 marks)

i. $6 + 8i$
ii. $3 - 4i$
iii. $2 + 3i$ (HOT)
iv. $7 - 30\sqrt{2}i$
v. $\dfrac{3 + 4i}{3 - 4i}$ (HOT)

2** Convert the following complex numbers in the polar form

i. $3\sqrt{3} + 3i$
ii. $\dfrac{1 - i}{1 + i}$

iii. 1 + i

iv. -1 + $\sqrt{3}i$

v. -3 + 3i

vi. -2 –i

3. If a+ ib =$\frac{x+i}{x-i}$ where x is a real, prove that $a^2 + b^2 = 1$ and b/a = $2x/(x^2-1)$ 4marks

4 Find the real and imaginary part of i. (1 mark)

5 Compute : $i + i^2 + i^3 + i^4$ (1 mark)

6 Solve the following quadratic equations (I mark)

i) $x^2-(\sqrt{2}+1)x+\sqrt{2}=0$

ii) $2x^2 + 5 = 0$

7 Find the complex conjugate and multiplicative inverse of (4 mark)

i) $(2 - 5i)^2$

ii) $\frac{2 + 3i}{3 - 7i}$

8 If $|Z|$ = 2 and arg Z = π/4 then Z= _______. (1 mark)

Answers

1) i) $2\sqrt{2} + \sqrt{2}i$, $-2\sqrt{2} - \sqrt{2}i$

ii) 2 –i, -2 + i

iii) $\frac{\sqrt{\sqrt{13}+2}}{\sqrt{2}} + \frac{\sqrt{\sqrt{13}-2}\ i}{\sqrt{2}}$, $\frac{\sqrt{\sqrt{13}+2}}{\sqrt{2}} + \frac{\sqrt{\sqrt{13}-2}\ i}{\sqrt{2}}$,

iv) $5 - 3\sqrt{2}\,i$, $-5 + 3\sqrt{2}\,i$

v) 3/5 + 4/5 i, -3/5 -4/5 i

2) i) 6(cos π/6 + isinπ/6)

ii) cos (–π/2) + isin(-π/2)

iii) $\sqrt{2}$(cos π/4 + isinπ/4)

iv) 2(cos2π/3 + isin 2π/3)

iv) $3\sqrt{2}$ (cos 3π/4 + isin 3π/4)

vi) $2\sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4)$ or $2\sqrt{2}[\cos(-3\pi/4) + i\sin(-3\pi/4)]$

4) 0,1

5) 0

6) i) $\sqrt{2}$, 1

ii) $\sqrt{\frac{5}{2}}\, i$, $-\sqrt{\frac{5}{2}}\, i$

7) i) $-21 + 10i$, $\frac{-21}{541} - \frac{10}{541}i$

ii) $\frac{-15}{58} - \frac{23}{58}i$, $\frac{3 - 7i}{2 + 3i}$

8) $\sqrt{2} + i\sqrt{2}$