KEY CONCEPTS

1. DEFINITION :

If f & g are functions of x such that $g'(x) = f(x)$ then the function g is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x) + c\} = f(x)$, where c is called the **constant of integration**.

2. STANDARD RESULTS :

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\,ln\,(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\,a}\ (a > 0) + c$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\,ln\sec(ax+b) + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\,ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \mathrm{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b)\,.\,\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \mathrm{cosec}(ax+b)\,.\,\cot(ax+b)\,dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(xiv) $\int \mathrm{cosec}\,x\,dx = \ln(\mathrm{cosec}x - \cot x) + c$ **OR** $\ln\tan\frac{x}{2} + c$ **OR** $-\ln(\mathrm{cosec}x + \cot x)$

(xv) $\int \sinh x\,dx = \cosh x + c$ **(xvi)** $\int \cosh x\,dx = \sinh x + c$ **(xvii)** $\int \mathrm{sech}^2x\,dx = \tanh x + c$

(xviii) $\int \mathrm{cosech}^2x\,dx = -\coth x + c$

(xix) $\int \mathrm{sech}\,x\,.\,\tanh x\,dx = -\mathrm{sech}\,x + c$

(xx) $\int \mathrm{cosech}\,x\,.\,\coth x\,dx = -\mathrm{cosech}\,x + c$

(xxi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xxii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xxiii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xxiv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x + \sqrt{x^2+a^2}\right]$ **OR** $\sinh^{-1}\frac{x}{a} + c$

(xxv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x + \sqrt{x^2-a^2}\right]$ **OR** $\cosh^{-1}\frac{x}{a} + c$

(xxvi) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} + c$

(xxvii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$

(xxviii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxix) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$

(xxx) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c$

(xxxi) $\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxxii) $\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

3. TECHNIQUES OF INTEGRATION :

(i) **Substitution** or change of independent variable .

Integral $I = \int f(x)\,dx$ is changed to $\int f(\phi(t))\, f'(t)\,dt$, by a suitable substitution $x = \phi(t)$ provided the later integral is easier to integrate .

(ii) **Integration by part :** $\int u.v\,dx = u\int v\,dx - \int \left[\frac{du}{dx}.\int v\,dx\right] dx$ where u & v are differentiable function . **Note :** While using integration by parts, choose u & v such that

(a) $\int v\,dx$ is simple & **(b)** $\int \left[\frac{du}{dx}.\int v\,dx\right] dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letters in **ILATE**, where ; I – Inverse function, L – Logarithmic function ,
A – Algebraic function, T – Trigonometric function & E – Exponential function

(iii) **Partial fraction** , spiliting a bigger fraction into smaller fraction by known methods .

4. INTEGRALS OF THE TYPE :

(i) $\int [f(x)]^n f'(x)\,dx$ **OR** $\int \frac{f'(x)}{[f(x)]^n}\,dx$ put $f(x) = t$ & proceed .

(ii) $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\;dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results .

(iii) $\int \frac{px+q}{ax^2+bx+c}\,dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$.

Express $px + q = A$ (differential co-efficient of denominator) + B .

(iv) $\int e^x [f(x) + f'(x)]\,dx = e^x . f(x) + c$ **(v)** $\int [f(x) + xf'(x)]\,dx = x\,f(x) + c$

(vi) $\int \frac{dx}{x(x^n+1)}$ $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(vii) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1+x^{-n} = t^n$

(viii) $\int \frac{dx}{x^n\left(1+x^n\right)^{1/n}}$ take x^n common as x and put $1 + x^{-n} = t$.

(ix) $\int \frac{dx}{a+b\sin^2 x}$ **OR** $\int \frac{dx}{a+b\cos^2 x}$ **OR** $\int \frac{dx}{a\sin^2 x+b\sin x\cos x+c\cos^2 x}$

Multiply N^{r} & D^{r} by $\sec^2 x$ & put $\tan x = t$.

(x) $\int \frac{dx}{a+b\sin x}$ **OR** $\int \frac{dx}{a+b\cos x}$ **OR** $\int \frac{dx}{a+b\sin x+c\cos x}$

Hint : Convert sines & cosines into their respective tangents of half the angles , put $\tan\frac{x}{2} = t$

(xi) $\int \frac{a.\cos x+b.\sin x+c}{\ell.\cos x+m.\sin x+n}\,dx$. Express $Nr \equiv A(Dr) + B\frac{d}{dx}(Dr) + c$ & proceed .

(xii) $\int \frac{x^2+1}{x^4+Kx^2+1}\,dx$ **OR** $\int \frac{x^2-1}{x^4+Kx^2+1}\,dx$ where K is any constant .

Hint : Divide Nr & Dr by x^2 & proceed .

(xiii) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{\left(ax^2+bx+c\right)\sqrt{px+q}}$; put $px + q = t^2$.

(xiv) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax + b = \frac{1}{t}$; $\int \frac{dx}{\left(ax^2+bx+c\right)\sqrt{px^2+qx+r}}$, put $x = \frac{1}{t}$

(xv) $\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

DEFINITE INTEGRAL

1. $\int_a^b f(x)\,dx = F(b) - F(a)$ where $\int f(x)\,dx = F(x) + c$

VERY IMPORTANT NOTE : If $\int_a^b f(x)\,dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a , b) provided f is a continuous function in (a , b) .

2. PROPERTIES OF DEFINITE INTEGRAL :

P–1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ provided f is same

P – 2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P–3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b] . This property to be used when f is piecewise continuous in (a, b) .

P–4 $\int_{-a}^a f(x)\,dx = 0$ if f(x) is an odd function i.e. $f(x) = -f(-x)$.

$= 2\int_0^a f(x)\,dx$ if f(x) is an even function i.e. $f(x) = f(-x)$.

P–5 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, In particular $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

P–6 $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

P–7 $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$; where 'a' is the period of the function i.e. $f(a+x) = f(x)$

P–8 $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$ where $f(x)$ is periodic with period T & $n \in I$.

P–9 $\int_{ma}^{na} f(x)\,dx = (n-m)\int_0^a f(x)\,dx$ if $f(x)$ is periodic with period 'a' .

P–10 If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

P–11 $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$.

P–12 If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

3. WALLI'S FORMULA :

$$\int_0^{\pi/2} \sin^n x \,.\, \cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2}K$$

Where K $= \dfrac{\pi}{2}$ if both m and n are even $(m, n \in N)$;

$= 1$ otherwise

4. DERIVATIVE OF ANTIDERIVATIVE FUNCTION :

If h(x) & g(x) are differentiable functions of x then ,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)]\,.\,h'(x) - f[g(x)]\,.\,g'(x)$$

5. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Limit}}\; h\,[f(a) + f(a+h) + f(a+2h) + ... + f(a + \overline{n-1}\,h)]$$

$$= \underset{h\to 0}{\text{Limit}}\; h\sum_{r=0}^{n-1} f(a+rh) \quad \text{where } b-a = nh$$

If $a = 0$ & $b = 1$ then , $\underset{n\to\infty}{\text{Limit}}\; h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)\,dx$; where $nh = 1$ **OR**

$$\underset{n\to\infty}{\text{Limit}}\left(\frac{1}{n}\right)\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx .$$

6. ESTIMATION OF DEFINITE INTEGRAL :

(i) For a monotonic decreasing function in (a , b) ; $f(b).(b-a) < \int_a^b f(x)\,dx < f(a).(b-a)$ &

(ii) For a monotonic increasing function in (a , b) ; $f(a).(b-a) < \int_a^b f(x)\,dx < f(b).(b-a)$

7. SOME IMPORTANT EXPANSIONS :

(i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+.....\infty=\ln 2$

(ii) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.....\infty=\frac{\pi^2}{6}$

(iii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.....\infty=\frac{\pi^2}{12}$

(iv) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.....\infty=\frac{\pi^2}{8}$

(v) $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+.....\infty=\frac{\pi^2}{24}$

EXERCISE–1

Q.1 $\int \frac{\tan 2\theta}{\sqrt{\cos^6\theta+\sin^6\theta}}\,d\theta$

Q.2 $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}\,dx$

Q.3 $\int \frac{\cos^2 x}{1+\tan x}\,dx$

Q.4 $\int \frac{dx}{(x^4-1)^2}$

Q.5 Integrate $\int \frac{dx}{x\sqrt{x^2+2x-1}}$ by the substitution $z = x+\sqrt{x^2+2x-1}$

Q.6 $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ell nx\,dx$

Q.7 $\int \cos 2\theta \,.\, \ln\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\,d\theta$

Q.8 $\int \frac{dx}{\sin^2 x+\sin 2x}$

Q.9 $\int \frac{a^2\sin^2 x+b^2\cos^2 x}{a^4\sin^2 x+b^4\cos^2 x}\,dx$

Q.10 $\int \frac{dx}{\left(x+\sqrt{x(1+x)}\right)^2}$

Q.11 $\int \sqrt{x+\sqrt{x^2+2}}\,dx$

Q.12 $\int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx$

Q.13 $\int (\sin x)^{-11/3}(\cos x)^{-1/3}dx$

Q.14 $\int \frac{\cot x\,dx}{(1-\sin x)(\sec x+1)}$

Q.15 $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Q.16 $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Q.17 $\int \left[\frac{\sqrt{x^2+1}\left[\ln(x^2+1)-2\ln x\right]}{x^4}\right]dx$

Q.18 $\int\left[\ln(\ln x)+\frac{1}{(\ln x)^2}\right]dx$

Q.19 $\int \frac{x+1}{x(1+xe^x)^2}\,dx$

Q.20 Integrate $\frac{1}{2}f'(x)$ w.r.t. x^4, where $f(x)=\tan^{-1}x+\ln\sqrt{1+x}-\ln\sqrt{1-x}$

Q.21 $\int \frac{(\sqrt{x}+1)dx}{\sqrt{x}\,(\sqrt[3]{x}+1)}$

Q.22 $\int \frac{dx}{\sin\frac{x}{2}\sqrt{\cos^3\frac{x}{2}}}$

Q.23 $\int \frac{x^2+x}{(e^x+x+1)^2}\,dx$

Q.24 $\int \sqrt{\frac{\operatorname{cosec} x-\cot x}{\operatorname{cosec} x+\cot x}}\cdot\frac{\sec x}{\sqrt{1+2\sec x}}\,dx$

Q.25 $\int \frac{\cos x-\sin x}{7-9\sin 2x}\,dx$

Q.26 $\int \frac{dx}{\sec x+\operatorname{cosec} x}\,dx$

Q.27 $\int \frac{dx}{\sin x+\sec x}$

Q.28 $\int \tan x.\tan 2x.\tan 3x\,dx$

Q.29 $\int \frac{dx}{\sin x\sqrt{\sin(2x+\alpha)}}$

Q.30 $\int \frac{x^2}{(x\cos x-\sin x)(x\sin x+\cos x)}\,dx$

Q.31 $\int \frac{3+4\sin x+2\cos x}{3+2\sin x+\cos x}\,dx$

Q.32 $\int \frac{\ell n\left(\cos x+\sqrt{\cos 2x}\right)}{\sin^2 x}\,dx$

Q.33 $\int \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Q.34 $\int \frac{dx}{\sin x+\tan x}$

Q.35 $\int \frac{3x^2+1}{(x^2-1)^3}\,dx$

Q.36 $\int \frac{e^{\cos x}(x\sin^3 x+\cos x)}{\sin^2 x}dx$ Q.37 $\int \frac{(ax^2-b)\ dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$ Q.38 $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}dx$

Q.39 $\int \frac{x}{(7x-10-x^2)^{3/2}}\ dx$ Q.40 $\int \frac{x\ln x}{\left(x^2-1\right)^{3/2}}\ dx$ Q.41 $\int \sqrt[3]{\frac{1-x}{1+x}}\ \frac{dx}{x}$

Q.42 $\int \frac{2-3x}{2+3x}\sqrt{\frac{1+x}{1-x}}\ dx$ Q.43 $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}\ dx$ Q.44 $\int \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}dx$

Q.45 $\int \frac{x+2}{\left(x^2+3x+3\right)}\frac{dx}{\sqrt{x+1}}$ Q.46 $\int \frac{\sqrt{2-x-x^2}}{x^2}\ dx$ Q.47 $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$

Q.48 $\int \frac{x\,dx}{\sqrt{x^4+4x^3-6x^2+4x+1}}$ Q.49 $\int \frac{\sqrt{\cos 2x}}{\sin x}\ dx$ Q.50 $\int \frac{(1+x^2)dx}{1-2x^2\cos\alpha+x^4}\ \alpha\in(0,\pi)$

EXERCISE–2

Q.1 $\int_0^{\pi} \frac{x\ \ dx}{9\cos^2 x+\sin^2 x}$ Q.2 $\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}\ \ dx$ Q.3 Evaluate $I_n=\int_1^e (\ln^n x)\,dx$ hence find I_3.

Q.4 $\int_0^{\pi/2} \sin 2x\cdot \text{arc}\tan(\sin x)\ dx$ Q.5 $\int_0^{\pi/2} \cos^4 3x\ .\ \sin^2 6x\ dx$ Q.6 $\int_0^{\pi/4} \frac{x\ dx}{\cos x\ (\cos x+\sin x)}$

Q.7 Let $h(x)=(f\circ g)(x)+K$ where K is any constant. If $\frac{d}{dx}\left(h(x)\right)=-\frac{\sin x}{\cos^2(\cos x)}$ then compute the value of $j(0)$ where $j(x)=\int_{g(x)}^{f(x)}\frac{f(t)}{g(t)}dt$, where f and g are trigonometric functions.

Q.8 Find the value of the definite integral $\int_0^{\pi}\left|\sqrt{2}\sin x+2\cos x\right|dx$.

Q.9 Evaluate the integral : $\int_3^5\left(\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}\right)dx$

Q.10 If $P=\int_0^{\infty}\frac{x^2}{1+x^4}dx$; $Q=\int_0^{\infty}\frac{x\,dx}{1+x^4}$ and $R=\int_0^{\infty}\frac{dx}{1+x^4}$ then prove that

(a) $Q=\frac{\pi}{4}$, (b) $P=R$, (c) $P-\sqrt{2}\ Q+R=\frac{\pi}{2\sqrt{2}}$

Q.11 Prove that $\int_a^b \frac{x^{n-1}\left\{(n-2)x^2+(n-1)(a+b)x+nab\right\}}{(x+a)^2(x+b)^2}dx=\frac{b^{n-1}-a^{n-1}}{2(a+b)}$

Q.12 $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$ Q.13 $\int_0^1 \frac{x^2\ln x}{\sqrt{1-x^2}}\ dx$ Q.14 Evaluate: $\int_{-2}^{2}\frac{x^2-x}{\sqrt{x^2+4}}dx$

Q.15 $\int_0^{\sqrt{3}}\sin^{-1}\frac{2x}{1+x^2}\ dx$ Q.16 $\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin\left(\frac{\pi}{4}+x\right)}\ dx$ Q.17 $\int_0^{2\pi}\frac{dx}{2+\sin 2x}$

Q.18 $\int_0^{2\pi}e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$ Q.19 $\int_{-\sqrt{2}}^{\sqrt{2}}\frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}dx$

Q.20 Let α, β be the distinct positive roots of the equation $\tan x = 2x$ then evaluate $\int_0^1 (\sin \alpha x \cdot \sin \beta x)\,dx$, independent of α and β.

Q.21 $\int_0^{\pi/4} \frac{\cos x - \sin x}{10 + \sin 2x}\,dx$

Q.22 $\int_0^{\pi} \frac{(ax+b)\sec x \tan x}{4 + \tan^2 x}\,dx$ (a,b>0)

Q.23 Evaluate: $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)}\,dx$

Q.24 If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$$\int_0^1 (\sin x + a\cos x)^3\,dx - \frac{4a}{\pi - 2}\int_0^1 x\cos x\,dx = 2$$

then find the value of $1000(a_1^2 + a_2^2 + a_3^2)$.

Q.25 Show that $\int_0^{p+q\pi} |\cos x|\,dx = 2q + \sin p$ where q ε N & $-\frac{\pi}{2} < p < \frac{\pi}{2}$

Q.26 Show that the sum of the two integrals $\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$ is zero.

Q.27 $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}\,dx$

Q.28 $\int_0^{\pi/2} \frac{\sin^2 x}{a^2\sin^2 x + b^2\cos^2 x}\,dx$ (a>0, b>0)

Q.29 $\int_{-1}^{1} \ell n\,\frac{1+x}{1-x}\,\frac{x^3}{\sqrt{1-x^2}}\,dx$

Q.30 $\int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]dx$

Q.31 $\displaystyle\int_{\frac{\sqrt{3a^2+b^2}}{2}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x.dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$

Q.32 Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t + k|\,dt$ depending on the value of $k \in R$.

Q.33 $\int_0^{2a} x\sin^{-1}\left[\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right]dx$

Q.34 Prove that $\int_0^{\infty} \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$ $(n > 1)$

Q.35 Show that $\int_0^x e^{zx}\cdot e^{-z^2}\,dz = e^{x^2/4}\int_0^x e^{-z^2/4}\,dz$

Q.36 $\int_0^{\pi} \frac{x^2 \sin 2x.\sin\left(\frac{\pi}{2}.\cos x\right)}{2x-\pi}\,dx$

Q.37 (a) $\int_0^1 \frac{1-x}{1+x}.\frac{dx}{\sqrt{x+x^2+x^3}}$, (b) $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1}\,ln\left(1+x-\frac{1}{x}\right)dx$

Q.38 Show that $\int_0^{\infty} \frac{dx}{x^2+2x\cos\theta+1} = 2\int_0^1 \frac{dx}{x^2+2x\cos\theta+1} = \begin{cases} \dfrac{\theta}{\sin\theta} & \text{if } \theta \in (0,\pi) \\ \dfrac{\theta - 2\pi}{\sin\theta} & \text{if } \theta \in (\pi, 2\pi) \end{cases}$

Q.39 $\int_0^{2\pi} \frac{x^2\sin x}{8+\sin^2 x}\,dx$

Q.40 $\int_0^{\pi/4} \frac{x^2(\sin 2x - \cos 2x)}{(1+\sin 2x)\cos^2 x}\,dx$

Q.41 Prove that $\int_0^x \left(\int_0^u f(t)\,dt\right)du = \int_0^x f(u).(x-u)\,du$.

Q.42 $\int_0^{\pi} \frac{dx}{(5+4\cos x)^2}$ Q.43 Evaluate $\int_0^1 \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)dx$ Q.44 $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\ dx$

Q.45 $\lim_{n\to\infty} n^2 \int_{-1/n}^{1/n} (2006\sin x + 2007\cos x)|x|dx$.   Q.46 Show that $\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ln x}{x}dx = \ln a . \int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$

Q.47 Evaluate the definite integral, $\int_{-1}^{1} \frac{(2x^{332}+x^{998}+4x^{1668}\cdot\sin x^{691})}{1+x^{666}}dx$

Q.48 Prove that

(a) $\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\ dx = \frac{(\beta-\alpha)^2\pi}{8}$ (b) $\int_{\alpha}^{\beta} \sqrt{\frac{x-\alpha}{\beta-x}}\ dx = (\beta-\alpha)\frac{\pi}{2}$

(c) $\int_{\alpha}^{\beta} \frac{dx}{x\sqrt{(x-\alpha)(\beta-x)}} = \frac{\pi}{\sqrt{\alpha\beta}}$ where $\alpha, \beta > 0$ (d) $\int_{\alpha}^{\beta} \frac{x.dx}{\sqrt{(x-\alpha)(\beta-x)}} = (\alpha+\beta)\frac{\pi}{2}$ where $\alpha < \beta$

Q.49 If $f(x) = \begin{vmatrix} 4\cos^2 x & 1 & 1 \\ (\cos x-1)^2 & (\cos x+1)^2 & (\cos x-1)^2 \\ (\cos x+1)^2 & (\cos x+1)^2 & \cos^2 x \end{vmatrix}$, find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\ dx$

Q.50 Evaluate : $\int_0^1 e^{\ln\tan^{-1}x}\cdot\sin^{-1}(\cos x)dx$.

EXERCISE–3

Q.1 If the derivative of f(x) wrt x is $\frac{\cos x}{f(x)}$ then show that f(x) is a periodic function.

Q.2 Find the range of the function, $f(x) = \int_{-1}^{1} \frac{\sin x\ dt}{1-2t\cos x+t^2}$.

Q.3 A function f is defined in [–1 , 1] as $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$; $x \neq 0$; $f(0) = 0$; $f(1/\pi) = 0$. Discuss the continuity and derivability of f at $x = 0$.

Q.4 Let $f(x) = \begin{cases} -1 & \text{if } -2 \le x \le 0 \\ |x-1| & \text{if } 0 < x \le 2 \end{cases}$ and $g(x) = \int_{-2}^{x} f(t)\ dt$. Define g (x) as a function of x and test the continuity and differentiability of g(x) in (–2, 2).

Q.5 Prove the inequalities:

(a) $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi\sqrt{2}}{8}$ **(b)** $2e^{-1/4} < \int_0^2 e^{x^2-x}\ dx < 2e^2$.

(c) $a < \int_0^{2\pi} \frac{dx}{10+3\cos x} < b$ then find a & b.   **(d)** $\frac{1}{2} \le \int_0^2 \frac{dx}{2+x^2} \le \frac{5}{6}$

Q.6 Determine a positive integer $n \le 5$, such that $\int_0^1 e^x (x-1)^n\ dx = 16 - 6e$.

Q.7 Using calculus

(a) If $|x| < 1$ then find the sum of the series $\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+......\infty$.

(b) If $|x| < 1$ prove that $\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+......\infty = \frac{1+2x}{1+x+x^2}$.

(c) Prove the identity $f(x)=\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+....+\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}}=\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}-2\cot 2x$

Q.8 If $\phi(x)=\cos x-\int_0^x (x-t)\,\phi(t)\,dt$. Then find the value of $\phi''(x)+\phi(x)$.

Q.9 If $y=\frac{1}{a}\int_0^x f(t)\cdot\sin a(x-t)\,dt$ then prove that $\frac{d^2y}{dx^2}+a^2y=f(x)$.

Q.10 If $y=x^{\int_1^x \ln t\,dt}$, find $\frac{dy}{dx}$ at $x=e$.

Q.11 If $f(x)=x+\int_0^1 [xy^2+x^2y]\,f(y)\,dy$ where x and y are independent variable. Find f(x).

Q.12 A curve C_1 is defined by: $\frac{dy}{dx}=e^x\cos x$ for $x\in[0,2\pi]$ and passes through the origin. Prove that the roots of the function (other than zero) occurs in the ranges $\frac{\pi}{2}<x<\pi$ and $\frac{3\pi}{2}<x<2\pi$.

Q.13(a) Let $g(x)=x^c\,.\,e^{2x}$ & let $f(x)=\int_0^x e^{2t}\,.\,(3t^2+1)^{1/2}\,dt$. For a certain value of 'c', the limit of $\frac{f'(x)}{g'(x)}$ as $x\to\infty$ is finite and non zero. Determine the value of 'c' and the limit.

(b) Find the constants 'a' (a > 0) and 'b' such that, $\lim_{x\to 0}\frac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx-\sin x}=1$.

Q.14 Evaluate: $\lim_{x\to+\infty}\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}}\frac{3t^4+1}{(t-3)(t^2+3)}dt$

Q.15 Given that $U_n=\{x(1-x)\}^n$ & $n\ge 2$ prove that $\frac{d^2U_n}{dx^2}=n(n-1)\,U_{n-2}-2n(2n-1)U_{n-1}$, further if $V_n=\int_0^1 e^x\,.\,U_n\,dx$, prove that when $n\ge 2$, $V_n+2n(2n-1).V_{n-1}-n(n-1)\,V_{n-2}=0$

Q.16 If $\int_0^\infty \frac{\ell n t}{x^2+t^2}dt=\frac{\pi\,\ell n\,2}{4}$ $(x>0)$ then show that there can be two integral values of 'x' satisfying this equation.

Q.17 Let $f(x)=\begin{cases}1-x & \text{if } 0\le x\le 1\\ 0 & \text{if } 1<x\le 2\\ (2-x)^2 & \text{if } 2<x\le 3\end{cases}$. Define the function $F(x)=\int_0^x f(t)\,dt$ and show that F is continuous in [0, 3] and differentiable in (0, 3).

Q.18 Let f be an injective function such that $f(x)\,f(y)+2=f(x)+f(y)+f(xy)$ for all non negative real x & y with $f'(0)=0$ & $f'(1)=2\neq f(0)$. Find f(x) & show that, $3\int f(x)\,dx-x(f(x)+2)$ is a constant.

Q.19 Evaluate: (a) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right).....\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$;

(b) $\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n+1}+\frac{2}{n+2}+.....+\frac{3n}{4n}\right]$; (c) $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$;

(d) Given $\lim_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{1/n}=\frac{a}{b}$ where a and b are relatively prime, find the value of (a + b).

Q.20 Prove that $\sin x + \sin 3x + \sin 5x + ... + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x}$, $k \in N$ and hence prove that, $\int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\,dx = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+......+\frac{1}{2k-1}$.

Q.21 If $U_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$, then show that $U_1, U_2, U_3, ..., U_n$ constitute an AP. Hence or otherwise find the value of U_n.

Q.22 Solve the equation for y as a function of x, satisfying

$$x\cdot\int_0^x y(t)\,dt = (x+1)\int_0^x t\cdot y(t)\,dt, \text{ where } x > 0, \text{ given } y(1) = 1.$$

Q.23 Prove that : (a) $I_{m,n} = \int_0^1 x^m.(1-x)^n\,dx = \frac{m!n!}{(m+n+1)!}$ $m, n \in N$.

(b) $I_{m,n} = \int_0^1 x^m.(\ln x)^n\,dx = (-1)^n \frac{n!}{(m+1)^{n+1}}$ $m, n \in N$.

Q.24 Find a positive real valued continuously differentiable functions f on the real line such that for all x

$$f^2(x) = \int_0^x \left((f(t))^2 + (f'(t))^2\right)dt + e^2$$

Q.25 Let f(x) be a continuously differentiable function then prove that, $\int_1^x [t]\,f'(t)\,dt = [x].\,f(x) - \sum_{k=1}^{[x]} f(k)$ where [·] denotes the greatest integer function and x > 1.

Q.26 Let f be a function such that $|f(u) - f(v)| \le |u - v|$ for all real u & v in an interval [a, b]. Then:

(i) Prove that f is continuous at each point of [a, b].

(ii) Assume that f is integrable on [a, b]. Prove that, $\left|\int_a^b f(x)\,dx - (b-a)\,f(c)\right| \le \frac{(b-a)^2}{2}$, where $a \le c \le b$

Q.27 Let $F(x) = \int_{-1}^x \sqrt{4+t^2}\,dt$ and $G(x) = \int_x^1 \sqrt{4+t^2}\,dt$ then compute the value of (FG)' (0) where dash denotes the derivative.

Q.28 Show that for a continuously thrice differentiable function f(x)

$$f(x) - f(0) = xf'(0) + \frac{f''(0).x^2}{2} + \frac{1}{2}\int_0^x f'''(t)(x-t)^2\,dt$$

Q.29 Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+m+1} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{k+n+1}$

Q.30 Let f and g be function that are differentiable for all real numbers x and that have the following properties:

(i) $f'(x) = f(x) - g(x)$ (ii) $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$ (iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x. (b) Find $f(x)$ and $g(x)$.

EXERCISE–4

Q.1 Find $\underset{n\to\infty}{\text{Limit}}\, S_n$, if : $S_n = \frac{1}{2n} + \frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + ... + \frac{1}{\sqrt{3n^2+2n-1}}$. [REE '97, 6]

Q.2 (a) If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals :

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$ (C) $g(x)\ g(\pi)$ (D) $[g(x)/g(\pi)]$

(b) $\underset{n\to\infty}{\text{Limit}} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ equals :

(A) $1+\sqrt{5}$ (B) $-1+\sqrt{5}$ (C) $-1+\sqrt{2}$ (D) $1+\sqrt{2}$

(c) The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x}\,dx$ is ________ .

(d) Let $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{2\, e^{\sin x^2}}{x}\,dx = F(k) - F(1)$ then one of the possible values of k is _______.

(e) Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$. [JEE '97, 2 + 2 + 2 + 2 + 5]

Q.3 (a) If $\int_0^x f(t)\,dt = x + \int_x^1 t\, f(t)\,dt$, then the value of $f(1)$ is

(A) $1/2$ (B) 0 (C) 1 (D) $-1/2$

(b) Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x\,dx$. Hence or otherwise, evaluate the integral $\int_0^1 \tan^{-1}\left(1-x+x^2\right) dx$ [JEE'98, 2 + 8]

Q.4 Evaluate $\int_0^1 \frac{1}{(5+2x-2x^2)(1+e^{(2-4x)})}\,dx$ [REE '98, 6]

Q.5 (a) If for al real number y, [y] is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\,dx$ is :

(A) $-\pi$ (B) 0 (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

(b) $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to :

(A) 2 (B) -2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

(c) Integrate : $\int \frac{x^3+3x+2}{\left(x^2+1\right)^2 (x+1)}\,dx$

(d) Integrate: $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$ [JEE '99, 2 + 2 + 7 + 3 (out of 200)]

Q.6 Evaluate the integral $\int_0^{\pi/6} \frac{\sqrt{3\cos 2x - 1}}{\cos x}\,dx.$ [REE '99, 6]

Q.7 (a) The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is :

(A) 3/2 (B) 5/2 (C) 3 (D) 5

(b) Let $g(x) = \int_0^x f(t)\,dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in (0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in (1, 2]$. Then g(2) satisfies the inequality :

(A) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (B) $0 \le g(2) < 2$ (C) $\frac{3}{2} < g(2) \le \frac{5}{2}$ (D) $2 < g(2) < 4$

(c) If $f(x) = \begin{cases} e^{\cos x} \cdot \sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$. Then $\int_{-2}^{3} f(x)\,dx$:

(A) 0 (B) 1 (C) 2 (D) 3

(d) For $x > 0$, let $f(x) = \int_1^x \frac{\ell n\, t}{1 + t}\,dt$. Find the function $f(x) + f(1/x)$ and show that, $f(e) + f(1/e) = 1/2$. [JEE 2000, 1 + 1 + 1 + 5]

Q.8 (a) $S_n = \frac{1}{1 + \sqrt{n}} + \frac{1}{2 + \sqrt{2n}} + \dots\dots + \frac{1}{n + \sqrt{n^2}}$. Find $\underset{n \to \infty}{\text{Limit}}\, S_n$.

(b) Given $\int_0^1 \frac{\sin t}{1 + t}\,dt = \alpha$, find the value of $\int_{4\pi - 2}^{4\pi} \frac{\sin \frac{t}{2}}{4\pi + 2 - t}\,dt$ in terms of α.

[REE 2000, Mains, 3 + 3 out of 100]

Q.9 Evaluate $\int \sin^{-1}\left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}}\right)dx$.

Q.10 (a) Evaluate $\int_0^{\pi/2} \frac{\cos^9 x}{\cos^3 x + \sin^3 x}\,dx$. (b) Evaluate $\int_0^{\pi} \frac{x\,dx}{1 + \cos\alpha \sin x}$

[REE 2001, 3 + 5]

Q.11 (a) Let $f(x) = \int_1^x \sqrt{2 - t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(A) ± 1 (B) $\pm \frac{1}{\sqrt{2}}$ (C) $\pm \frac{1}{2}$ (D) 0 and 1

(b) Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$ $f(x + T) = f(x)$. If $I = \int_0^T f(x)\,dx$ then the value of $\int_3^{3+3T} f(2x)\,dx$ is

(A) $\frac{3}{2}$ I (B) 2 I (C) 3 I (D) 6 I

(c) The integral $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left([x] + \ln\left(\frac{1 + x}{1 - x}\right)\right) dx$ equals

(A) $-\frac{1}{2}$ (B) 0 (C) 1 (D) $2\ln\left(\frac{1}{2}\right)$

[JEE 2002(Scr.), 3+3+3]

(d) For any natural number m, evaluate

$\int \left(x^{3m} + x^{2m} + x^m\right)\left(2x^{2m} + 3x^m + 6\right)^{\frac{1}{m}}\,dx$, where $x > 0$ [JEE 2002 (Mains),4]

Q.12 If f is an even function then prove that $\int_0^{\pi/2} f(\cos 2x) \cos x\,dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x\,dx$

[JEE 2003,(Mains) 2 out of 60]

Q.13 (a) $\int_0^1 \sqrt{\frac{1 - x}{1 + x}}\,dx =$

(A) $\frac{\pi}{2}+1$ (B) $\frac{\pi}{2}-1$ (C) π (D) 1

(b) If $\int_0^{t^2} x\,f(x)\,dx = \frac{2}{5}t^5$, $t > 0$, then $f\left(\frac{4}{25}\right) =$

(A) $\frac{2}{5}$ (B) $\frac{5}{2}$ (C) $-\frac{2}{5}$ (D) 1

[JEE 2004, (Scr.)]

(c) If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x.\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}.d\theta$ then find $\frac{dy}{dx}$ at $x = \pi$. [JEE 2004 (Mains), 2]

(d) Evaluate $\int_{-\pi/3}^{\pi/3} \frac{\pi+4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$. [JEE 2004 (Mains), 4]

Q.14 (a) If $\int_{\sin x}^{1} t^2(f(t))dt = (1-\sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is [JEE 2005 (Scr.)]

(A) 1/3 (B) $1/\sqrt{3}$ (C) 3 (D) $\sqrt{3}$

(b) $\int_{-2}^{0}\left(x^3+3x^2+3x+3+(x+1)\cos(x+1)\right)dx$ is equal to [JEE 2005 (Scr.)]

(A) – 4 (B) 0 (C) 4 (D) 6

(c) Evaluate: $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$. [JEE 2005, Mains,2]

Q.15 $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is equal to

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+C$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+C$

(C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+C$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+C$ [JEE 2006, 3]

Comprehension

Q.16 Suppose we define the definite integral using the following formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b))$, for more accurate result for $c \in (a, b)$ $F(c) = \frac{c-a}{2}(f(a)+f(c))+\frac{b-c}{2}(f(b)+f(c))$. When $c = \frac{a+b}{2}$,

$$\int_a^b f(x)dx = \frac{b-a}{4}(f(a)+f(b)+2f(c))$$

(a) $\int_0^{\pi/2}\sin x\,dx$ is equal to

(A) $\frac{\pi}{8}\left(1+\sqrt{2}\right)$ (B) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$ (C) $\frac{\pi}{8\sqrt{2}}$ (D) $\frac{\pi}{4\sqrt{2}}$

(b) If $f(x)$ is a polynomial and if $\lim\limits_{t\to a} \dfrac{\int_a^t f(x)\,dx - \frac{t-a}{2}\left(f(t)+f(a)\right)}{(t-a)^3} = 0$ for all a then the degree of $f(x)$ can atmost be

(A) 1 (B) 2 (C) 3 (D) 4

(c) If $f''(x) < 0$, $\forall\, x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which F(c) is maximum, then $f'(c)$ is equal to

(A) $\dfrac{f(b)-f(a)}{b-a}$ (B) $\dfrac{2(f(b)-f(a))}{b-a}$ (C) $\dfrac{2f(b)-f(a)}{2b-a}$ (D) 0

[JEE 2006, 5 marks each]

Q.17 Find the value of $\dfrac{5050\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ [JEE 2006, 6]

ANSWER
EXERCISE–1

Q.1 $\ln\left(\dfrac{1+\sqrt{1+3\cos^2 2\theta}}{\cos 2\theta}\right) + C$

Q.2 $-\dfrac{x+1}{x^5+x+1} + c$

Q.3 $\dfrac{1}{4}\ln(\cos x + \sin x) + \dfrac{x}{2} + \dfrac{1}{8}(\sin 2x + \cos 2x) + c$

Q.4 $\dfrac{3}{8}\tan^{-1}x - \dfrac{x}{4(x^4-1)} - \dfrac{3}{16}\ln\left(\dfrac{x-1}{x+1}\right) + c$

Q.5 $2\tan^{-1}\left(x+\sqrt{x^2+2x-1}\right) + c$

Q.6 $\left(\dfrac{x}{e}\right)^x - \left(\dfrac{e}{x}\right)^x + c$

Q.7 **(c)** $\dfrac{1}{2}(\sin 2\theta)\ \ln\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \dfrac{1}{2}\ln(\sec 2\theta) + c$

Q.8 $\dfrac{1}{2}\ln\left|\dfrac{\tan x}{\tan x+2}\right| + c$

Q.9 $\dfrac{1}{a^2+b^2}\left(x+\tan^{-1}\left(\dfrac{a^2\tan x}{b^2}\right)\right) + c$

Q.10 $2\ln\dfrac{t}{2t+1} + \dfrac{1}{2t+1} + C$ when $t = x+\sqrt{x^2+x}$

Q.11 $\dfrac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2} - \dfrac{2}{\left(x+\sqrt{x^2+2}\right)^{1/2}} + c$

Q.12 $\cos a \,.\, \text{arc}\cos\left(\dfrac{\cos x}{\cos a}\right) - \sin a \,.\, \ln\left(\sin x+\sqrt{\sin^2 x-\sin^2 a}\right) + c$

Q.13 $-\dfrac{3(1+4\tan^2 x)}{8(\tan x)^{8/3}} + c$

Q.14 $\dfrac{1}{2}\ln\left|\tan\dfrac{x}{2}\right| + \dfrac{1}{4}\sec^2\dfrac{x}{2} + \tan\dfrac{x}{2} + c$

Q.15 $\sqrt{x}\sqrt{1-x} - 2\sqrt{1-x} + \arccos\sqrt{x} + c$

Q.16 $(a+x)\,\text{arc}\tan\sqrt{\dfrac{x}{a}} - \sqrt{ax} + c$

Q.17 $\dfrac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2-3\ln\left(1+\dfrac{1}{x^2}\right)\right]$

Q.18 $x\ln(\ln x) - \dfrac{x}{\ln x} + c$

Q.19 $\ln\left(\dfrac{xe^x}{1+xe^x}\right) + \dfrac{1}{1+xe^x} + c$

Q.20 $-\ln(1-x^4)+c$

Q.21 $6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\ln(1+t^2)-\tan^{-1}t\right]+C$ where $t=x^{1/6}$

Q.22 $\frac{4}{\sqrt{\cos\frac{x}{2}}}+2\tan^{-1}\sqrt{\cos\frac{x}{2}}-\ln\frac{1+\sqrt{\cos\frac{x}{2}}}{1-\sqrt{\cos\frac{x}{2}}}+c$

Q.23 $C-\ln(1+(x+1)e^{-x})-\frac{1}{1+(x+1)e^{-x}}$

Q24. $\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)+c$

Q.25 $\frac{1}{24}\ln\frac{(4+3\sin x+3\cos x)}{(4-3\sin x-3\cos x)}+c$

Q.26 $\frac{1}{2}\left[\sin x-\cos x-\frac{1}{\sqrt{2}}\ln\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right]+c$

Q.27 $\frac{1}{2\sqrt{3}}\ln\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}+\arctan(\sin x+\cos x)+c$

Q.28 $\left[-\ell n(\sec x)-\frac{1}{2}\ell n(\sec 2x)+\frac{1}{3}\ell n(\sec 3x)\right]+c$

Q.29 $-\frac{1}{\sqrt{\sin\alpha}}\ln\left[\cot x+\cot\alpha+\sqrt{\cot^2x+2\cot\alpha\cot x-1}\right]+c$

Q.30 $\ln\left|\frac{x\sin x+\cos x}{x\cos x-\sin x}\right|$

Q.31 $2x-3\arctan\left(\tan\frac{x}{2}+1\right)+c$

Q.32 $\frac{\sqrt{\cos 2x}}{\sin x}-x-\cot x\,.\,\ln\left(e\left(\cos x+\sqrt{\cos 2x}\right)\right)+c$

Q.33 $\ln(1+t)-\frac{1}{4}\ln(1+t^4)+\frac{1}{2\sqrt{2}}\ln\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}-\frac{1}{2}\tan^{-1}t^2+c$ where $t=\sqrt{\cot x}$

Q.34 $\frac{1}{2}\ln\tan\frac{x}{2}-\frac{1}{4}\tan^2\frac{x}{2}+c$

Q.35 $c-\frac{x}{(x^2-1)^2}$

Q.36 $c-e^{\cos x}(x+\operatorname{cosec}x)$

Q.37 $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+k$

Q.38 $e^x\sqrt{\frac{1+x}{1-x}}+c$

Q.39 $\frac{2(7x-20)}{9\sqrt{7x-10-x^2}}+c$

Q.40 $\operatorname{arcsec}x-\frac{\ln x}{\sqrt{x^2-1}}+c$

Q.41 $\ell n\frac{|u^2-1|}{\sqrt{u^4+u^2+1}}+\sqrt{3}\tan^{-1}\frac{1+2u^2}{\sqrt{3}}+c$ where $u=\sqrt[3]{\frac{1-x}{1+x}}$

Q.42 $\frac{8}{3}\left[\tan^{-1}t+\frac{1}{2\sqrt{5}}\ell n\left(\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right)\right]-\left(\sin^{-1}x-\sqrt{1-x^2}\right)+c$ where $t=\sqrt{\frac{1+x}{1-x}}$

Q.43 $\tan^{-1}\left(\frac{\sqrt{2}\sin 2x}{\sin x+\cos x}\right)+c$

Q.44 $4\ln x+\frac{7}{x}+6\tan^{-1}(x)+\frac{6x}{1+x^2}+C$

Q.45 $\frac{2}{\sqrt{3}}\arctan\frac{x}{\sqrt{3(x+1)}}+c$

Q.46 $-\frac{\sqrt{2-x-x^2}}{x}+\frac{\sqrt{2}}{4}\ell n\left(\frac{4-x+2\sqrt{2}\sqrt{2-x-x^2}}{x}\right)-\sin^{-1}\left(\frac{2x+1}{3}\right)+c$

Q.47 $\frac{-2}{\alpha-\beta}.\sqrt{\frac{x-\beta}{x-\alpha}}+c$

Q.48 $\frac{1}{2}\ln\left[x+\frac{1}{x}+2+\sqrt{\left(x+\frac{1}{x}+2\right)^2-12}\right]+C$

Q.49 $\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+t}{\sqrt{2}-t}\right)-\frac{1}{2}\ln\left(\frac{1-t}{1+t}\right)$ where $t=\cos\theta$ and $\theta=\operatorname{cosec}^{-1}(\cot x)$

Q.50 $\frac{1}{2}\left(\operatorname{cosec}\frac{\alpha}{2}\right)\cdot\tan^{-1}\left(\left(\frac{x^2-1}{2x}\right)\operatorname{cosec}\frac{\alpha}{2}\right)$

EXERCISE–2

Q.1 $\frac{\pi^2}{6}$ **Q.2** $\ln 2$ **Q.3** $6 - 2e$ **Q.4** $\frac{\pi}{2} - 1$ **Q.5** $\frac{5\pi}{64}$ **Q.6** $\frac{\pi}{8}\ln 2$

Q.7 $1 - \sec(1)$ **Q.8** $2\sqrt{6}$ **Q.9** $2\sqrt{2} + \frac{4}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$ **Q.12** $\left(\frac{22}{7} - \pi\right)$ **Q.13** $\frac{\pi}{8}(1 - \ln 4)$

Q.14 $4\sqrt{2} - 4\ln(\sqrt{2}+1)$ **Q.15** $\frac{\pi\sqrt{3}}{3}$ **Q.16** $\frac{\pi(a+b)}{2\sqrt{2}}$ **Q.17** $\frac{2\pi}{\sqrt{3}}$ **Q.18** $-\frac{3\sqrt{2}}{5}(e^{2\pi}+1)$

Q.19 $\frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$ **Q.20** 0 **Q.21** $\frac{1}{3}\left(\arctan\frac{\sqrt{2}}{3} - \arctan\frac{1}{3}\right)$

Q.22 $\frac{(a\pi+2b)\pi}{3\sqrt{3}}$ **Q.23** $\frac{\pi(\pi+3)}{2}$ **Q.24** 5250 **Q.27** $\frac{\pi^2}{6\sqrt{3}}$

Q.28 $\frac{\pi}{2a(a+b)}$ **Q.29** $\frac{5\pi}{3}$ **Q.30** $\frac{3\pi^2}{16}$ **Q.31** $\frac{\pi}{12}$ **Q.32** real & distinct $\forall\ k \in R$

Q.33 $\frac{\pi a^2}{4}$ **Q.36** $\frac{8}{\pi}$ **Q.37 (a)** $\frac{\pi}{3}$; **(b)** $\frac{\pi}{8}\ln 2$ **Q.39** $-\frac{2\pi^2}{3}\ln 2$ **Q.40** $\frac{\pi^2}{16} - \frac{\pi}{4}\ln 2$

Q.42 $\frac{5\pi}{27}$ **Q.43** $\frac{1}{2}\left[\ln 2 + \frac{\pi}{2} - 1\right]$ **Q.44** $\frac{16\pi}{3} - 2\sqrt{3}$ **Q.45** 2007 **Q.47** $\frac{\pi+4}{666}$

Q.49 $-2\pi - \frac{32}{15}$ **Q.50** $\frac{\pi^2}{8} - \frac{\pi}{4}(1+\ln 2) + \frac{1}{2}$

EXERCISE–3

Q.2 $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ **Q.3** cont. & der. at $x = 0$

Q.4 g(x) is cont. in $(-2, 2)$; g(x) is der. at $x = 1$ & not der. at $x = 0$. Note that ;

$$g(x) = \begin{cases} -(x+2) & \text{for } -2 \le x \le 0 \\ -2 + x - \frac{x^2}{2} & \text{for } 0 < x < 1 \\ \frac{x^2}{2} - x - 1 & \text{for } 1 \le x \le 2 \end{cases}$$

Q.5 (c) $a = \frac{2\pi}{13}$ & $b = \frac{2\pi}{7}$ **Q.6** $n = 3$

Q.7 (a) $\frac{1}{1-x}$ **Q.8** $-\cos x$ **Q.10** $1 + e$ **Q.11** $f(x) = x + \frac{61}{119}x + \frac{80}{119}x^2$

Q.13 (a) $c = 1$ and $\lim_{x\to\infty}$ will be $\frac{\sqrt{3}}{2}$ **(b)** $a = 4$ and $b = 1$ **Q.14** 13.5

Q.16 $x = 2$ or 4 **Q.17** $F(x) = \begin{cases} x - \frac{x^2}{2} & \text{if } 0 \le x \le 1 \\ \frac{1}{2} & \text{if } 1 < x \le 2 \\ \frac{(x-2)^3}{3} + \frac{1}{2} & \text{if } 2 < x \le 3 \end{cases}$

Q.18 $f(x) = 1 + x^2$ **Q.19** **(a)** $2e^{(1/2)(\pi-4)}$; **(b)** $3 - \ln 4$; **(c)** $\frac{1}{e}$; **(d)** 43 **Q.21** $U_n = \frac{n\pi}{2}$

Q.22 $y = \frac{e}{x^3}e^{-1/x}$ **Q.24** $f(x) = e^{x+1}$ **Q.27** 0

Q.30 $f(x) = 3 + 2e^{2x}$; $g(x) = 3 - 2e^{2x}$

EXERCISE–4

Q.1 $\pi/6$ **Q.2** **(a)** A **(b)** B **(c)** 2 **(d)** 16 **(e)** π^2 **Q.3** **(a)** A **(b)** $\ln 2$

Q.4 $\frac{1}{2\sqrt{11}}\ell n\frac{\sqrt{11}+1}{\sqrt{11}-1}$

Q.5 **(a)** C, **(b)** A ; **(c)** $\frac{3}{2}\tan^{-1}x-\frac{1}{2}\ln(1+x)+\frac{1}{4}\ln(1+x^2)+\frac{x}{1+x^2}+c$, **(d)** $\frac{\pi}{2}$

Q.6 $\sqrt{\frac{2}{3}}\,\pi-2\tan^{-1}\sqrt{2}$ **Q.7** **(a)** B, **(b)** B, **(c)** C, **(d)** $\frac{1}{2}\ln^2 x$

Q.8 **(a)** $2\ln 2$, **(b)** $-\alpha$ **Q.9** $(x+1)\tan^{-1}\frac{2(x+1)}{3}-\frac{3}{4}\ell n(4x^2+8x+13)+C$

Q.10 **(a)** $\frac{1}{8}\left[\frac{5\pi}{4}-\frac{1}{3}\right]$, **(b)** $I=\begin{cases}\frac{\pi\alpha}{\sin\alpha} & \text{if } \alpha\in(0,\pi)\\ \frac{\pi}{\sin\alpha}(\alpha-2\pi) & \text{if } \alpha\in(\pi,2\pi)\end{cases}$

Q.11 **(a)** A, **(b)** C, **(c)** B, **(d)** $\frac{1}{6(m+1)}\left(2x^{3m}+3x^{2m}+6x^m\right)^{\frac{m+1}{m}}+C$

Q.13 **(a)** B, **(b)** A, **(c)** 2π, **(d)** $\frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$

Q.14 **(a)** C, **(b)** C, **(c)** $\frac{24}{5}\left(e\cos\left(\frac{1}{2}\right)+\frac{e}{2}\sin\left(\frac{1}{2}\right)-1\right)$

Q.15 D **Q.16** **(a)** A, **(b)** A, **(c)** A **Q.17** 5051

ELEMENTARY DEFINITE INTEGRAL

(SELF PRACTICE)

Evaluate the following definite integrals.

Q.1 $\displaystyle\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}\,dx$ Q.2 $\displaystyle\int_0^{\ell n 2} x\,e^{-x}\,dx$ Q.3 $\displaystyle\int_0^{3\pi/4} \frac{\sin x\,dx}{1+\cos^2 x}$

Q 4. $\displaystyle\int_0^{\pi/2} e^{2x}\,.\cos x\,dx$ Q 5. $\displaystyle\int_{-1}^{1} \frac{x\,dx}{\sqrt{5-4x}}$ Q 6. $\displaystyle\int_2^{e} \left(\frac{1}{\ell n x} - \frac{1}{\ell n^2 x}\right) dx$

Q 7. $\displaystyle\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$ Q 8. $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$ Q 9. $\displaystyle\int_0^{\pi/4} \frac{\sin^2 x\,.\cos^2 x}{\left(\sin^3 x + \cos^3 x\right)^2}\,dx$

Q 10. $\displaystyle\int_1^{2} \sqrt{(x-1)(2-x)}\,dx$ Q 11. $\displaystyle\int_2^{3} \frac{dx}{\sqrt{(x-1)(5-x)}}$ Q 12. $\displaystyle\int_0^{3/4} \frac{dx}{(x+1)\sqrt{1+x^2}}$

Q 13. $\displaystyle\int_0^{\pi/2} \sin\phi\cos\phi\sqrt{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)}\,d\phi \quad a\neq b$ Q14. $\displaystyle\int_0^{1} x^2\,.\sqrt{4-x^2}\,dx$

Q 15. $\displaystyle\int_0^{\pi/4} x\cos x\,\cos 3x\,dx$ Q16. $\displaystyle\int_0^{\pi/2} \frac{dx}{5+4\sin x}$ Q17. $\displaystyle\int_2^{3} \frac{dx}{(x-1)\sqrt{x^2-2x}}$

Q 18. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cos\theta\,.\cos x}\quad \theta\in(0,\pi)$ Q 19. $\displaystyle\int_0^{3} \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}$ Q 20. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{\left(1+x^2\right)^{3/2}}$

Q 21. $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx$ Q 22. $\displaystyle\int_0^{\pi/4} \cos 2x\sqrt{1-\sin 2x}\,dx$ Q23. $\displaystyle\int_0^{3} \sqrt{\frac{x}{3-x}}\,dx$

Q.24 $\displaystyle\int_0^{1/2} \frac{dx}{\left(1-2x^2\right)\sqrt{1-x^2}}$ Q 25. $\displaystyle\int_1^{2} \frac{dx}{x\left(x^4+1\right)}$ Q 26. $\displaystyle\int_0^{a} x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

Q27. $\displaystyle\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x + 2\cos x + 2}\,dx$ Q 28. $\displaystyle\int_0^{1} x\left(\tan^{-1}x\right)^2 dx$ Q 29. $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{\left(1-x^2\right)^{3/2}}\,dx$

Q 30. $\displaystyle\int_0^{1} \frac{dx}{x^2+2x\cos\alpha+1}$ where $-\pi<\alpha<\pi$ Q 31. $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^4}\,dx$

Q 32. $\displaystyle\int_a^{b} \frac{dx}{\sqrt{1+x^2}}$ where $a = \dfrac{e-e^{-1}}{2}$ & $b = \dfrac{e^2-e^{-2}}{2}$ Q33. $\displaystyle\int_0^{1} \frac{1-x^2}{1+x^2+x^4}\,dx$

Q 34. $\displaystyle\int_0^{1} x^5\sqrt{\frac{1+x^2}{1-x^2}}\,dx$ Q 35. $\displaystyle\int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$ Q 36. $\displaystyle\int_0^{\pi/4} \frac{\sin\theta+\cos\theta}{9+16\sin 2\theta}\,d\theta$

Q 37. $\displaystyle\int_0^{\pi} \theta\sin^2\theta\cos\theta\,d\theta$ Q 38. $\displaystyle\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\,dx$ Q 39. $\displaystyle\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\,dx$

Q 40. $\displaystyle\int_0^{\pi/2} \cos^3 x\,\sin 3x\,dx$ Q 41. $\displaystyle\int_0^{1} \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx$ Q 42. $\displaystyle\int_{-1}^{1}\left(\frac{d}{dx}\left(\frac{1}{1+e^{1/x}}\right)\right)dx$

Q 43. $\int_0^e \frac{dx}{\ln(x^x e^x)}$ Q 44. $\int_{-1}^{1} x^2\, d(\ln x)$

Q 45. If $f(\pi) = 2$ & $\int_0^{\pi} (f(x) + f''(x)) \sin x \, dx = 5$, then find $f(0)$

Q.46 $\int_a^b \frac{|x|}{x} dx$ Q.47 $\int_0^{\pi} \left[\cos^2\left(\frac{3\pi}{8} - \frac{x}{4}\right) - \cos^2\left(\frac{11\pi}{8} + \frac{x}{4}\right)\right] dx$

Q.48 $\int_0^{\pi/2} \sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}}\, \frac{\operatorname{cosec} x}{\sqrt{1 + 2\operatorname{cosec} x}} dx$ Q.49 $\int_0^1 x\, f''(x)\, dx$, where $f(x) = \cos(\tan^{-1}x)$

Q.50 $\int_{\ell n 2}^{\ell n 3} f(x)dx$, where $f(x) = e^{-x} + 2e^{-2x} + 3e^{-3x} + \ldots\ldots\infty$

ANSWER KEY

Q 1. $\frac{\pi^2}{4}$ Q 2. $\frac{1}{2}\ell n\left(\frac{e}{2}\right)$ Q3. $\frac{\pi}{4} + \tan^{-1}\frac{1}{\sqrt{2}}$ Q 4. $\frac{e^{\pi} - 2}{5}$

Q 5. $\frac{1}{6}$ Q6. $e - \frac{2}{\ell n 2}$ Q 7. $\frac{\pi}{4}$ Q 8. $\ln\frac{4}{3}$

Q 9. $\frac{1}{6}$ Q 10. $\frac{\pi}{8}$ Q 11. $\frac{\pi}{6}$ Q 12. $\frac{1}{\sqrt{2}}\ell n\left(\frac{9 + 4\sqrt{2}}{7}\right)$

Q 13. $\frac{1}{3}\,\frac{a^3 - b^3}{a^2 - b^2}$ Q 14. $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ Q 15. $\frac{\pi - 3}{16}$ Q 16. $\frac{2}{3}\tan^{-1}\frac{1}{3}$

Q 17. $\frac{\pi}{3}$ Q 18 $\frac{\theta}{\sin\theta}$ Q 19. $\frac{14}{15}$ Q 20. $\frac{\sqrt{3} - \sqrt{2}}{2}$

Q 21. $\frac{3\pi}{16}$ Q 22. $\frac{1}{3}$ Q 23. $\frac{3\pi}{2}$ Q 24. $\frac{1}{2}\ell n\left(2 + \sqrt{3}\right)$

Q 25. $\frac{1}{4}\ln\frac{32}{17}$ Q 26. $\frac{a^2}{4}(\pi - 2)$ Q 27. $\frac{\pi}{4} - \tan^{-1}2 + \frac{1}{2}\ln\frac{5}{2}$

Q 28. $\frac{\pi}{4}\left(\frac{\pi}{4} - 1\right) + \frac{1}{2}\ell n 2$ Q 29. $\frac{\pi}{4} - \frac{1}{2}\ln 2$ Q 30. $\frac{\alpha}{2\sin\alpha}$ if $\alpha \neq 0$; $\frac{1}{2}$ if $\alpha = 0$

Q 31. $\frac{\pi}{2\sqrt{2}}$ Q 32. 1 Q 33. $\frac{1}{2}\ln 3$ Q 34. $\frac{3\pi + 8}{24}$

Q 35. $\frac{\pi}{4}$ Q 36. $\frac{1}{20}\ln 3$ Q 37. $-\frac{4}{9}$ Q 38. $\frac{1}{2}$

Q 39. $\frac{\pi}{2}$ Q 40. $\frac{5}{12}$ Q 41. $\frac{\pi}{2}$ Q 42. $\frac{2}{1+e}$

Q 43. $\ln 2$ Q 44. $\frac{e^2 - e^{-2}}{2}$ Q 45. 3 Q.46 $|b| - |a|$

Q.47 $\sqrt{2}$ Q.48 $\pi/3$ Q.49 $1 - \frac{3}{2\sqrt{2}}$ Q.50 $\frac{1}{2}$