fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
i#"k flg ldYi dj] lgr foifr vud] ^cuk^ u NkM /;; dk] j?kcj jk[k VdAA
jfpr% ekuo /ke i.krk
lnx# Jh j.kNkMnkl th egkjkt

STUDY PACKAGE

Subject : Mathematics

Topic : COMPLEX NUMBER

Available Online : www.MathsBySuhag.com

Teko CLASSES ®
.....the support

Index

Student's Name :______________________

Class :______________________

Roll No. :______________________

Address : Plot No. 27, III- Floor, Near Patidar Studio,
Above Bond Classes, Zone-2, M.P. NAGAR, Bhopal
☎ : (0755) 32 00 000, 98930 58881, WhatsApp 9009 260 559
www.TekoClasses.com www.MathsBySuhag.com

Circle Theory

A circle is a locus of a point whose distance from a fixed point (called centre) is always constant (called radius).

1. Equation of a Circle in Various Form:

(a) The circle with centre as origin & radius 'r' has the equation; $x^2 + y^2 = r^2$.

(b) The circle with centre (h, k) & radius 'r' has the equation; $(x - h)^2 + (y - k)^2 = r^2$.

(c) The general equation of a circle is
$x^2 + y^2 + 2gx + 2fy + c = 0$
with centre as $(-g, -f)$ & radius $= \sqrt{g^2+f^2-c}$. If:
$g^2 + f^2 - c > 0 \Rightarrow$ real circle.
$g^2 + f^2 - c = 0 \Rightarrow$ point circle.
$g^2 + f^2 - c < 0 \Rightarrow$ imaginary circle, with real centre, that is $(-g, -f)$

Note : that every second degree equation in x & y, in which coefficient of x^2 is equal to coefficient of y^2 & the coefficient of xy is zero, always represents a circle.

(d) The equation of circle with (x_1, y_1) & (x_2, y_2) as extremeties of its diameter is:
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
Note that this will be the circle of least radius passing through (x_1, y_1) & (x_2, y_2).

Example : Find the equation of the circle whose centre is (1, –2) and radius is 4.

Solution : The equation of the circle is $(x - 1)^2 + (y - (-2))^2 = 4^2$
$\Rightarrow \quad (x - 1)^2 + (y + 2)^2 = 16$
$\Rightarrow \quad x^2 + y^2 - 2x + 4y - 11 = 0$ **Ans.**

Example : Find the equation of the circle which passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is (2, – 3).

Solution : Let P be the point of intersection of the lines AB and LM whose equations are respectively
$3x - 2y - 1 = 0$ ....(i)
and $4x + y - 27 = 0$ ....(ii)
Solving (i) and (ii), we get x = 5, y = 7. So, coordinates of P are (5, 7). Let C(2, –3) be the centre of the circle. Since the circle passes through P, therefore
CP = radius $\Rightarrow \sqrt{(5-2)^2+(7+3)^2}$ = radius $\Rightarrow$ radius $= \sqrt{109}$.
Hence the equation of the required circle is
$(x - 2)^2 + (y + 3)^2 = \left(\sqrt{109}\right)^2$

Example : Find the centre & radius of the circle whose equation is $x^2 + y^2 - 4x + 6y + 12 = 0$

Solution : Comparing it with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have
$2g = -4 \quad \Rightarrow \quad g = -2$
$2f = 6 \quad \Rightarrow \quad f = 3$
& $\quad c = 12$
$\therefore$ centre is $(-g, -f)$ i.e. (2, –3)
and radius $= \sqrt{g^2+f^2-c} = \sqrt{(-2)^2+(3)^2-12} = 1$

Example : Find the equation of the circle, the coordinates of the end points of whose diameter are (–1, 2) and (4, –3)

Solution : We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
Here, $x_1 = -1$, $x_2 = 4$, $y_1 = 2$ and $y_2 = -3$.
So, the equation of the required circle is
$(x + 1)(x - 4) + (y - 2)(y + 3) = 0 \quad \Rightarrow \quad x^2 + y^2 - 3x + y - 10 = 0$.

Self Practice Problems :

1. Find the equation of the circle passing through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ and whose centre is the point of intersection of the lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.
Ans. $x^2 + y^2 + 4x - 2y = 0$

2. Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6)
Ans. $x^2 + y^2 - 2x - 4y - 20 = 0$

3. Find the equation of a circle whose radius is 6 and the centre is at the origin.
Ans. $x^2 + y^2 = 36$.

2. Intercepts made by a Circle on the Axes:

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co–ordinate axes are $2\sqrt{g^2-c}$ & $2\sqrt{f^2-c}$ respectively. If
$g^2 - c > 0 \quad \Rightarrow$ circle cuts the x axis at two distinct points.
$g^2 = c \quad \Rightarrow$ circle touches the x–axis.
$g^2 < c \quad \Rightarrow$ circle lies completely above or below the x–axis.

Example : Find the equation to the circle touching the y-axis at a distance – 3 from the origin and intercepting a length 8 on the x-axis.

Solution : Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it touches y-axis at (0, –3) and (0, – 3) lies on the circle.

$\therefore$ $c = f^2$ ...(i) $9 - 6f + c = 0$ ....(ii)

From (i) and (ii), we get $9 - 6f + f^2 = 0 \Rightarrow (f - 3)^2 = 0 \Rightarrow f = 3$.

Putting f = 3 in (i) we obtain c = 9.

It is given that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepts length 8 on x-axis

$\therefore$ $2\sqrt{g^2 - c} = 8 \Rightarrow 2\sqrt{g^2 - 9} = 8 \Rightarrow g^2 - 9 = 16 \Rightarrow g = \pm 5$

Hence, the required circle is $x^2 + y^2 \pm 10x + 6y + 9 = 0$.

Self Practice Problems :

1. Find the equation of a circle which touches the axis of y at a distance 3 from the origin and intercepts a distance 6 on the axis of x.

Ans. $x^2 + y^2 \pm 6\sqrt{2}\, x - 6y + 9 = 0$

2. Find the equation of a circle which touches y-axis at a distance of 2 units from the origin and cuts an intercept of 3 units with the positive direction of x-axis.

Ans. $x^2 + y^2 \pm 5x - 4y + 4 = 0$

3. Parametric Equations of a Circle:

The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are: $x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$ where (h, k) is the centre, r is the radius & θ is a parameter.

Example : Find the parametric equations of the circle $x^2 + y^2 - 4x - 2y + 1 = 0$

Solution : We have : $x^2 + y^2 - 4x - 2y + 1 = 0 \Rightarrow (x^2 - 4x) + (y^2 - 2y) = -1$

$\Rightarrow (x - 2)^2 + (y - 1)^2 = 2^2$

So, the parametric equations of this circle are

$x = 2 + 2\cos\theta$, $y = 1 + 2\sin\theta$.

Example : Find the equations of the following curves in cartesian form. Also, find the centre and radius of the circle $x = a + c\cos\theta$, $y = b + c\sin\theta$

Solution : We have : $x = a + c\cos\theta$, $y = b + c\sin\theta \Rightarrow \cos\theta = \frac{x-a}{c}$, $\sin\theta = \frac{y-b}{c}$

$\Rightarrow \left(\frac{x-a}{c}\right)^2 + \left(\frac{y-b}{c}\right)^2 = \cos^2\theta + \sin^2\theta \Rightarrow (x - a)^2 + (y - b)^2 = c^2$

Clearly, it is a circle with centre at (a, b) and radius c.

Self Practice Problems :

1. Find the parametric equations of circle $x^2 + y^2 - 6x + 4y - 12 = 0$

Ans. $x = 3 + 5\cos\theta$, $y = -2 + 5\sin\theta$

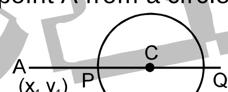
2. Find the cartesian equations of the curve $x = -2 + 3\cos\theta$, $y = 3 + 3\sin\theta$

Ans. $(x + 2)^2 + (y - 3)^2 = 9$

4. Position of a point with respect to a circle:

The point (x_1, y_1) is inside, on or outside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c <\ =$ or $>$ 0.

NOTE : The greatest & the least distance of a point A from a circle with centre C & radius r is AC+r & AC – r respectively.



Example : Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

Solution : We have $x^2 + y^2 - 4x + 2y - 11 = 0$ or S = 0, where $S = x^2 + y^2 - 4x + 2y - 11$.

For the point (1, 2), we have $S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0$

For the point (6, 0), we have $S_2 = 6^2 + 0^2 - 4 \times 6 + 2 \times 0 - 11 > 0$

Hence, the point (1, 2) lies inside the circle and the point (6, 0) lies outside the circle.

Self Practice Problem :

1. How are the points (0, 1) (3, 1) and (1, 3) situated with respect to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$?

Ans. (0, 1) lies on the circle ; (3, 1) lies outside the circle ; (1, 3) lies inside the circle.

5. Line and a Circle:

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then:

(i) $p > r \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.
(ii) $p = r \Leftrightarrow$ the line touches the circle. (It is tangent to the circle)
(iii) $p < r \Leftrightarrow$ the line is a secant of the circle.
(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.

Also, if $y = mx + c$ is line and $x^2 + y^2 = a^2$ is circle then

(i) $c^2 > a^2(1 + m^2) \Leftrightarrow$ the line is a secant of the circle.
(ii) $c^2 = a^2(1 + m^2) \Leftrightarrow$ the line touches the circle. (It is tangent to the circle)
(iii) $c^2 < a^2(1 + m^2) \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

Example : For what value of c will the line $y = 2x + c$ be a tangent to the circle $x^2 + y^2 = 5$?

Solution : We have : $y = 2x + c$ or $2x - y + c = 0$ ......(i) and $x^2 + y^2 = 5$ ....(ii)

If the line (i) touches the circle (ii), then

length of the $\perp$ from the centre (0, 0) = radius of circle (ii)

$\Rightarrow \left|\frac{2\times 0 - 0 + c}{\sqrt{2^2 + (-1)^2}}\right| = \sqrt{5} \Rightarrow \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5}$

$\Rightarrow \quad \frac{c}{\sqrt{5}} = \pm\sqrt{5} \quad \Rightarrow \quad c = \pm 5$

Hence, the line (i) touches the circle (ii) for $c = \pm 5$

Self Practice Problem :

1. For what value of λ, does the line $3x + 4y = \lambda$ touch the circle $x^2 + y^2 = 10x$. **Ans.** 40, –10

6. Tangent :

(a) Slope form :

$y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Hence, equation of tangent is $y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

(b) Point form :

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $x x_1 + y y_1 = a^2$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is: $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$.

NOTE : In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x x_1$, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{x_1y+xy_1}{2}$ and c remains as c.

(c) Parametric form :

The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = a$.

NOTE : The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is $\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Example : Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at $(4, -1)$.

Solution : Equation of tangent is

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y+(-1)}{2}\right) + 109 = 0$$

or $4x - y - 15x - 60 + 3y - 3 + 109 = 0$ or $-11x + 2y + 46 = 0$

or $11x - 2y - 46 = 0$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$

Example : Find the equation of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x + 3y + 5 = 0$

Solution : Given circle is $x^2 + y^2 - 6x + 4y - 12 = 0$ ....(i)

and given line is $4x + 3y + 5 = 0$ ....(ii)

Centre of circle (i) is (3, –2) and its radius is 5. Equation of any line $4x + 3y + k = 0$ parallel to the line (ii) ....(iii)

If line (iii) is tangent to circle, (i) then

$\frac{|4.3+3(-2)+k|}{\sqrt{4^2+3^2}} = 5$ or $|6 + k| = 25$

or $6 + k = \pm 25 \quad \therefore \quad k = 19, -31$

Hence equation of required tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$

Self Practice Problem :

1. Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are (i) parallel, (ii) perpendicular to the line $3x - 4y - 1 = 0$

Ans. (i) $3x - 4y + 20 = 0$ and $3x - 4y - 10 = 0$ (ii) $4x + 3y + 5 = 0$ and $4x + 3y - 25 = 0$

7. Normal :

If a line is normal / orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is; $y - y_1 = \frac{y_1+f}{x_1+g}(x - x_1)$.

Exercise : Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Solution : The equation of the tangent to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at (5, 6) is

$$5x + 6y - 5\left(\frac{x+5}{2}\right) + 2\left(\frac{x+6}{2}\right) - 48 = 0 \Rightarrow 10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$$

$$\Rightarrow 5x + 14y - 109 = 0$$

$\therefore$ Slope of the tangent $= -\frac{5}{14} \Rightarrow$ Slope of the normal $= \frac{14}{5}$

Hence, the equation of the normal at (5, 6) is

$y - 6 = (14/5)(x - 5) \Rightarrow 14x - 5y - 40 = 0$

Self Practice Problem :

1. Find the equation of the normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point (2, 3).

Ans. $x - y + 1 = 0$

8. Pair of Tangents from a Point:

The equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

Ex. : Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

Solution : Given circle is $S = x^2 + y^2 - 2x + 4y = 0$ ....(i)

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5$

Clearly P lies outside the circle

and $T \equiv x . 0 + y . 1 - (x + 0) + 2 (y + 1)$

i.e. $T \equiv -x + 3y + 2$.

Now equation of pair of tangents from P(0, 1) to circle (1) is $SS_1 = T^2$

or $5 (x^2 + y^2 - 2x + 4y) = (-x + 3y + 2)^2$

or $5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

or $4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

or $2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ....(ii)

Note : Separate equation of pair of tangents : From (ii), $2x^2 + 3(y - 1) x - 2(2y^2 - 4y + 2) = 0$

$\therefore \quad x = \dfrac{3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$

or $4x - 3y + 3 = \pm \sqrt{25y^2 - 50y + 25} = \pm 5(y - 1)$

$\therefore$ Separate equations of tangents are $x - 2y + 2 = 0$ and $2x + y - 1 = 0$

Self Practice Problems :

1. Find the equation of the tangents through (7, 1) to the circle $x^2 + y^2 = 25$.
Ans. $12x^2 - 12y^2 + 7xy - 175x - 25y + 625 = 0$

9. Length of a Tangent and Power of a Point:

The length of a tangent from an external point (x_1, y_1) to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$.

Square of length of the tangent from the point P is also called the power of point w.r.t. a circle.
Power of a point w.r.t. a circle remains constant.
Power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

Exercise : Find the length of the tangent drawn from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$

Solution : Given circle is $x^2 + y^2 + 6x - 4y - 3 = 0$ ....(i)

Given point is (5, 1). Let P = (5, 1)

Now length of the tangent from P(5, 1) to circle (i) = $\sqrt{5^2 + 1^2 + 6.5 - 4.1 - 3} = 7$

Self Practice Problems :

1. Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii. **Ans.** 8 sq. units

2. If the length of the tangent from a point (f, g) to the circle $x^2 + y^2 = 4$ be four times the length of the tangent from it to the circle $x^2 + y^2 = 4x$, show that $15f^2 + 15g^2 - 64f + 4 = 0$

10. Director Circle:

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

Example : Find the equation of director circle of the circle $(x - 2)^2 + (y + 1)^2 = 2$.

Solution : Centre & radius of given circle are (2, –1) & $\sqrt{2}$ respectively.

Centre and radius of the director circle will be (2, –1) & $\sqrt{2} \times \sqrt{2} = 2$ respectively.

$\therefore$ equation of director circle is $(x - 2)^2 + (y + 1)^2 = 4$

$\Rightarrow$ $x^2 + y^2 - 4x + 2y + 1 = 0$ **Ans.**

Self Practice Problems :

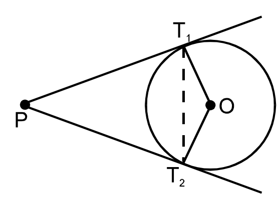
1. Find the equation of director circle of the circle whose diameters are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and area is 154 square units. **Ans.** $(x + 3)^2 + (y + 2)^2 = 98$

11. Chord of Contact:

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is: $xx_1 + yy_1 + g (x + x_1) + f (y + y_1) + c = 0$.

NOTE : Here R = radius; L = length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.



(b) Length of chord of contact $T_1 T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(c) Area of the triangle formed by the pair of the tangents & its chord of contact = $\dfrac{RL^3}{R^2 + L^2}$

(d) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\dfrac{2RL}{L^2 - R^2}\right)$

(e) Equation of the circle circumscribing the triangle PT_1T_2 is:
$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

Example : Find the equation of the chord of contact of the tangents drawn from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$

Solution : Given circle is $x^2 + y^2 - 2x + 4y + 7 = 0$ ....(i)

Let P = (1, 2)

For point P (1, 2), $x^2 + y^2 - 2x + 4y + 7 = 1 + 4 - 2 + 8 + 7 = 18 > 0$

Hence point P lies outside the circle

For point P (1, 2), $T = x \cdot 1 + y \cdot 2 - (x + 1) + 2(y + 2) + 7$

i.e. $T = 4y + 10$

Now equation of the chord of contact of point P(1, 2) w.r.t. circle (i) will be

$4y + 10 = 0$ or $2y + 5 = 0$

Example : Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the point of intersection of these tangents.

Solution : Given circles are $S_1 \equiv x^2 + y^2 - 12 = 0$ .... (i)

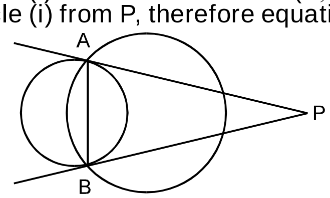
and $S_2 = x^2 + y^2 - 5x + 3y - 2 = 0$ .... (ii)

Now equation of common chord of circle (i) and (ii) is

$S_1 - S_2 = 0$ i.e. $5x - 3y - 10 = 0$ .... (iii)

Let this line meet circle (i) [or (ii)] at A and B

Let the tangents to circle (i) at A and B meet at P(α, β), then AB will be the chord of contact of the tangents to the circle (i) from P, therefore equation of AB will be



$x\alpha + y\beta - 12 = 0$ .... (iv)

Now lines (iii) and (iv) are same, therefore, equations (iii) and (iv) are identical

$$\therefore \quad \frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10} \qquad \therefore \quad \alpha = 6,\ \beta = -\frac{18}{5}$$

Hence $P = \left(6, -\frac{18}{5}\right)$

Self Practice Problems :

1. Find the co-ordinates of the point of intersection of tangents at the points where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$ **Ans.** (1, – 2)

2. Find the area of the triangle formed by the tangents drawn from the point (4, 6) to the circle $x^2 + y^2 = 25$ and their chord of contact. **Ans.** $\frac{405\sqrt{3}}{52}$; $4x + 6y - 25 = 0$

12. Pole and Polar:

(i) If through a point P in the plane of the circle there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q & R is called the Polar of the point P; also P is called the Pole of the Polar.

(ii) The equation to the polar of a point P (x_1, y_1) w.r.t. the circle $x^2 + y^2 = a^2$ is given by $xx_1 + yy_1 = a^2$, & if the circle is general then the equation of the polar becomes $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ i.e. T = 0. Note that if the point (x_1, y_1) be on the circle then the tangent & polar will be represented by the same equation. Similarly if the point (x_1, y_1) be outside the circle then the chord of contact & polar will be represented by the same equation.

(iii) Pole of a given line Ax + By + C = 0 w.r.t. circle $x^2 + y^2 = a^2$ is $\left(-\frac{Aa^2}{C}, -\frac{Ba^2}{C}\right)$.

(iv) If the polar of a point P pass through a point Q then the polar of Q passes through P.

(v) Two lines L_1 & L_2 are conjugate of each other if Pole of L_1 lies on L_2 & vice versa. Similarly two points P & Q are said to be conjugate of each other if the polar of P passes through Q & vice-versa.

Example : Find the equation of the polar of the point (2, –1) with respect to the circle $x^2 + y^2 - 3x + 4y - 8 = 0$

Solution : Given circle is $x^2 + y^2 - 3x + 4y - 8 = 0$ ....(i)

Given point is (2, –1) let P = (2, –1). Now equation of the polar of point P with respect to circle (i)

$$x.2 + y(-1) - 3\left(\frac{x+2}{2}\right) + 4\left(\frac{y-1}{2}\right) - 8 = 0$$

or $4x - 2y - 3x - 6 + 4y - 4 - 16 = 0$ or $x + 2y - 26 = 0$

Example : Find the pole of the line $3x + 5y + 17 = 0$ with respect to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$

Solution : Given circle is $x^2 + y^2 + 4x + 6y + 9 = 0$ ....(i)

and given line is $3x + 5y + 17 = 0$ ....(ii)

Let P(α, β) be the pole of line (ii) with respect to circle (i)

Now equation of polar of point P(α, β) with respect to circle (i) is

$x\alpha + y\beta + 2(x + \alpha) + 3(y + \beta) + 9 = 0$

or $(\alpha + 2)x + (\beta + 3)y + 2\alpha + 3\beta + 9 = 0$ ....(iii)

Now lines (ii) and (iii) are same, therefore,

$$\frac{\alpha+2}{3} = \frac{\beta+3}{5} = \frac{2\alpha+3\beta+9}{17}$$
(i) (ii) (iii)

From (i) and (ii), we get
$5\alpha + 10 = 3\beta + 9$ or $5\alpha - 3\beta = -1$ ....(iv)
From (i) and (iii), we get
$17\alpha + 34 = 6\alpha + 9\beta + 27$ or $11\alpha - 9\beta = -7$ ....(v)
Solving (iv) & (v), we get $\alpha = 1$, $\beta = 2$ Hence required pole is (1, 2).

Self Practice Problems :

1. Find the co-ordinates of the point of intersection of tangents at the points where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$. **Ans.** (1, – 2)

2. Find the pole of the straight line $2x - y + 10 = 0$ with respect to the circle $x^2 + y^2 - 7x + 5y - 1 = 0$
Ans. $\left(\frac{3}{2}, \frac{3}{2}\right)$

13. Equation of the Chord with a given Middle Point:

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point M (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

NOTE : (i) The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.
(ii) The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

Ex. : Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is (1, –1)

Solution : Equation of given circle is $S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$
Let $L \equiv (1, -1)$
For point L(1, –1), $S_1 = 1^2 + (-1)^2 + 6.1 + 8(-1) - 11 = -11$ and
$T \equiv x.1 + y(-1) + 3(x + 1) + 4(y - 1) - 11$ i.e. $T \equiv 4x + 3y - 12$
Now equation of the chord of circle (i) whose middle point is L(1, –1) is
$T = S_1$ or $4x + 3y - 12 = -11$ or $4x + 3y - 1 = 0$

Second Method : Let C be the centre of the given circle, then $C \equiv (-3, -4)$. $L \equiv (1, -1)$ slope of CL $= \frac{-4+1}{-3-1} = \frac{3}{4}$

$\therefore$ Equation of chord of circle whose middle point is L, is

$\therefore$ $y + 1 = -\frac{4}{3}(x - 1)$ [$\because$ chord is perpendicular to CL)

or $4x + 3y - 1 = 0$

Self Practice Problems :

1. Find the equation of that chord of the circle $x^2 + y^2 = 15$, which is bisected at (3, 2) **Ans.** $3x + 2y - 13 = 0$

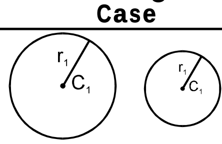
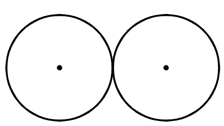
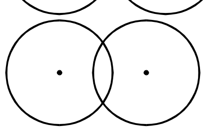
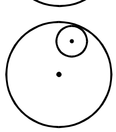
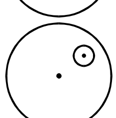
2. Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts off on the line $y = x + 2$. **Ans.** $\left(-\frac{3}{2}, \frac{1}{2}\right)$

14. Equation of the chord joining two points of circle :

The equation of chord PQ to the circle $x^2 + y^2 = a^2$ joining two points P(α) and (β) on it is given by. The equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

15. Common Tangents to two Circles:

	Case	Number of Tangents	Condition
(i)		4 common tangents (2 direct and 2 transverse)	$r_1 + r_2 < c_1c_2$.
(ii)		3 common tangents.	$r_1 + r_2 = c_1c_2$.
(iii)		2 common tangents.	$\lvert r_1 - r_2\rvert < c_1c_2 < r_1 + r_2$
(iv)		1 common tangent.	$\lvert r_1 - r_2\rvert = c_1c_2$.
(v)		No common tangent.	$c_1c_2 < \lvert r_1 - r_2\rvert$.

(Here C_1C_2 is distance between centres of two circles.)

IMPORTANT NOTE :

(i) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.
Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

(ii) Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles are given by: $L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$,
where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.

Example: Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.

Solution : Given circles are $x^2 + y^2 - 2x - 4y = 0$ ....(i)
and $x^2 + y^2 - 8y - 4 = 0$ ....(ii)
Let A and B be the centres and r_1 and r_2 the radii of circles (i) and (ii) respectively, then
$A \equiv (1, 2)$, $B \equiv (0, 4)$, $r_1 = \sqrt{5}$, $r_2 = 2\sqrt{5}$
Now $AB = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$ and $r_1 + r_2 = 3\sqrt{5}$, $|r_1 - r_2| = \sqrt{5}$
Thus $AB = |r_1 - r_2|$, hence the two circles touch each other internally.

Self Practice Problems :

1. Find the position of the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ with respect to each other.
Ans. One circle lies completely outside the other circle.

16. Orthogonality Of Two Circles:

Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

NOTE : (a) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.

(c) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

Example : Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 6 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4y + 1 = 0$.

Solution. Given circles are $x^2 + y^2 + 3x - 5y + 6 = 0$ ....(i)
and $4x^2 + 4y^2 - 28x + 29 = 0$
or $x^2 + y^2 - 7x + \frac{29}{4} = 0$. ....(ii)
Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(iii)
Since circle (iii) cuts circles (i) and (ii) orthogonally
$\therefore$ $2g\left(\frac{3}{2}\right) + 2f\left(-\frac{5}{2}\right) = c + 6$ or $3g - 5f = c + 6$ ....(iv)
and $2g\left(-\frac{7}{2}\right) + 2f.0 = c + \frac{29}{4}$ or $-7g = c + \frac{29}{4}$ ....(v)
From (iv) & (v), we get $10g - 5f = -\frac{5}{4}$
or $40g - 20f = -5$. ....(vi)
Given line is $3x + 4y = -1$ ....(vii)
Since centre $(-g, -f)$ of circle (iii) lies on line (vii),
$\therefore$ $-3g - 4g = -1$ ....(viii)
Solving (vi) & (viii), we get $g = 0$, $f = \frac{1}{4}$
$\therefore$ from (5), $c = -\frac{29}{4}$
$\therefore$ from (iii), required circle is
$x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0$ or $4(x^2 + y^2) + 2y - 29 = 0$

Self Practice Problems :

1. For what value of k the circles $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + k = 0$ cut orthogonally.
Ans. -18

2. Find the equation to the circle which passes through the origin and has its centre on the line $x + y + 4 = 0$ and cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.
Ans. $3x^2 + 3y^2 + 4x + 20y = 0$

17. Radical Axis and Radical Centre:

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given by

$S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.
The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles. Note that the length of tangents from radical centre to the three circles are equal.

NOTE:

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the two circles.

(d) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(e) Radical axis bisects a common tangent between the two circles.

(f) A system of circles, every two which have the same radical axis, is called a coaxal system.

(g) Pairs of circles which do not have radical axis are concentric.

Example : Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal.

$$3x^2 + 3y^2 + 4x - 6y - 1 = 0$$
$$2x^2 + 2y^2 - 3x - 2y - 4 = 0$$
$$2x^2 + 2y^2 - x + y - 1 = 0$$

Solution : Here we have to find the radical centre of the three circles. First reduce them to standard form in which coefficients of x^2 and y^2 be each unity. Subtracting in pairs the three radical axes are

$$\frac{17}{6}x - y + \frac{5}{3} = 0 \quad ; \quad -x - \frac{3}{2}y - \frac{3}{2} = 0$$

$$-\frac{11}{6}x + \frac{5}{2}y - \frac{1}{6} = 0.$$

solving any two, we get the point $\left(-\frac{16}{21}, \frac{31}{63}\right)$ which satisfies the third also. This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

Self Practice Problem :

1. Find the point from which the tangents to the three circles $x^2 + y^2 - 4x + 7 = 0$, $2x^2 + 2y^2 - 3x + 5y + 9 = 0$ and $x^2 + y^2 + y = 0$ are equal in length. Find also this length.
Ans. (2, – 1) ; 2.

18. Family of Circles:

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0$
($K \neq -1$ provided the co–efficient of x^2 & y^2 in S_1 & S_2 are same)

(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 & a line L = 0 is given by S + KL = 0.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \text{ where K is a parameter.}$$

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided co–efficient of xy = 0 and co–efficient of x^2 = co–efficient of y^2.

(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $u L_1L_3 + \lambda L_2L_4 = 0$ where values of u & λ can be found out by using condition that co–efficient of x^2 = co–efficient of y^2 and co–efficient of xy = 0.

Example : Find the equations of the circles passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

Solution : Any circle through the intersection of given circles is $S_1 + \lambda S_2 = 0$
or $(x^2 + y^2 - 2x - 4y - 4) + l(x^2 + y^2 - 10x - 12y + 40) = 0$

or $$(x^2 + y^2) - 2\frac{(1+5\lambda)}{1+\lambda}x - 2\frac{(2+6\lambda)}{1+\lambda}y + \frac{40\lambda - 4}{1+\lambda} = 0 \quad ....(i)$$

$r = \sqrt{g^2 + f^2 - c} = 4$, given

$$\therefore \quad 16 = \frac{(1+5\lambda)^2}{(1+\lambda)^2} + \frac{(2+6\lambda)^2}{(1+\lambda)^2} - \frac{40\lambda - 4}{1+\lambda}$$

$16(1 + 2\lambda + \lambda^2) = 1 + 10\lambda + 25\lambda^2 + 4 + 24\lambda + 36\lambda^2 - 40\lambda^2 - 40\lambda + 4 + 4\lambda$
or $16 + 32\lambda + 16\lambda^2 = 21\lambda^2 - 2\lambda + 9$ or $5\lambda^2 - 34\lambda - 7 = 0$
$\therefore$ $(\lambda - 7)(5\lambda + 1) = 0$ $\therefore$ $\lambda = 7, -1/5$
Putting the values of λ in (i) the required circles are
$2x^2 + 2y^2 - 18x - 22y + 69 = 0$ and $x^2 + y^2 - 2y - 15 = 0$

Example : Find the equations of circles which touche $2x - y + 3 = 0$ and pass through the points of intersection of the line $x + 2y - 1 = 0$ and the circle $x^2 + y^2 - 2x + 1 = 0$.

Solution : The required circle by $S + \lambda P = 0$ is

$x^2 + y^2 - 2x + 1 + \lambda (x + 2y - 1) = 0$

or $x^2 + y^2 - x (2 - \lambda) + 2\lambda y + (1 - \lambda) = 0$

centre $(-g, -f)$ is $[\{2 - \lambda)/2, -\lambda]$

$$r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(2-\lambda)^2/4 + \lambda^2 - (1-\lambda)} = \frac{1}{2}\sqrt{5\lambda^2} = (\lambda/2)\sqrt{5}.$$

Since the circle touches the line $2x - y + 3 = 0$ therefore perpendicular from centre is equal to

radius $\dfrac{2.[(2-\lambda)/2] - (-\lambda) + 3}{\pm\sqrt{5}} = \dfrac{\lambda}{2}\sqrt{5}$. or $5 = \pm \dfrac{\lambda}{2}.5$ $\therefore$ $\lambda = \pm 2$

Puttin the values of λ in (i) the required circles are

$x^2 + y^2 + 4y - 1 = 0$ and $x^2 + y^2 - 4x - 4y + 3 = 0$.

Example : Find the equation of circle pasing through the points A(1, 1) & B(2, 2) and whose radiu is 1.

Solution : Equation of AB is $x - y = 0$ $\therefore$ equation of circle is

$(x - 1)(x - 2) + (y - 1)(y - 2) + \lambda(x - y) = 0$ or $x^2 + y^2 + (\lambda - 3)x - (\lambda + 3)y + 4 = 0$

$$\text{radius} = \sqrt{\frac{(\lambda-3)^2}{4} + \frac{(\lambda+3)^2}{4} - 4}$$

But radius = 1 (given) $\therefore$ $\sqrt{\dfrac{(\lambda-3)^2}{4} + \dfrac{(\lambda+3)^2}{4} - 4} = 1$

or $(\lambda - 3)^2 + (\lambda + 3)^2 - 16 = 4$. or $2\lambda^2 = 2$ or $\lambda = \pm 1$

$\therefore$ equation of circle is

$x^2 + y^2 - 2x - 4y + 4 = 0$ & $x^2 + y^2 - 4x - 2y + 4 = 0$ **Ans.**

Example : Find the equation of the circle passing through the point (2, 1) and touching the line $x + 2y - 1 = 0$ at the point (3, – 1).

Solution : Equation of circle is

$(x - 3)^2 + (y + 1)^2 + \lambda(x + 2y - 1) = 0$

Since it passes through the point (2, 1)

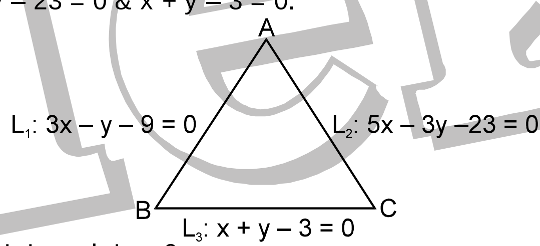
$1 + 4 + \lambda (2 + 2 - 1) = 0$ $\Rightarrow$ $\lambda = -5/3$

$\therefore$ circle is

$(x - 3)^2 + (y + 1)^2 - \dfrac{5}{3}(x + 2y - 1) = 0 \Rightarrow$ $3x^2 + 3y^2 - 23x - 4y + 35 = 0$ **Ans.**

Example : Find the equation of circle circumcscribing the triangle whose sides are $3x - y - 9 = 0$, $5x - 3y - 23 = 0$ & $x + y - 3 = 0$.

Solution :



$L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$

$(3x - y - 9)(5x - 3y - 23) + \lambda(5x - 3y - 23)(x + y - 3) + \mu(3x - y - 9)(x + y - 3) = 0$

$(15x^2 + 3y^2 - 14xy - 114x + 50y + 207) + \lambda(5x^2 - 3y^2 + 2xy - 38x - 14y + 69)$
$+ \mu(3x^2 - y^2 + 2xy - 18x - 6y + 27) = 0$

$(5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2 + xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu) + y(50 - 14\lambda - 6\mu)$
$+ (207 + 69\lambda + 27\mu) = 0$ ....(i)

coefficient of x^2 = coefficient of y^2

$\Rightarrow$ $5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu$

$8\lambda + 4\mu + 12 = 0$

$2\lambda + \mu + 3 = 0$ ....(ii)

coefficient of $xy = 0$ $\Rightarrow$ $2\lambda + 2\mu - 14 = 0$

$\Rightarrow$ $\lambda + \mu - 7 = 0$ ....(iii)

Solving (ii) and (iii), we have

$\lambda = -10, \mu = 17$

Puting these values of λ & μ in equation (i), we get

$2x^2 + 2y^2 - 5x + 11y - 3 = 0$

Self Practice Problems :

1. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

Ans. $7x^2 + 7y^2 - 10x - 10y - 12 = 0$

2. Find the equation of circle circumcribing the quadrilateral whose sides are $5x + 3y = 9$, $x = 3y$, $2x = y$ and $x + 4y + 2 = 0$.

Ans. $9x^2 + 9y^2 - 20x + 15y = 0$.

SHORT REVISION

STANDARD RESULTS :

1. EQUATION OF A CIRCLE IN VARIOUS FORM :

(a) The circle with centre (h, k) & radius 'r' has the equation ;
$(x - h)^2 + (y - k)^2 = r^2$.

(b) The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre as :
$(-g, -f)$ & radius $= \sqrt{g^2 + f^2 - c}$.

Remember that every second degree equation in x & y in which coefficient of x^2 = coefficient of y^2 & there is no xy term always represents a circle.

If $g^2 + f^2 - c > 0 \Rightarrow$ real circle.
$g^2 + f^2 - c = 0 \Rightarrow$ point circle.
$g^2 + f^2 - c < 0 \Rightarrow$ imaginary circle.

Note that the general equation of a circle contains three arbitrary constants, g, f & c which corresponds to the fact that a unique circle passes through three non collinear points.

(c) The equation of circle with (x_1 , y_1) & (x_2 , y_2) as its diameter is :
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Note that this will be the circle of least radius passing through (x_1 , y_1) & (x_2 , y_2).

2. INTERCEPTS MADE BY A CIRCLE ON THE AXES :

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co-ordinate axes are $2\sqrt{g^2 - c}$ & $2\sqrt{f^2 - c}$ respectively.

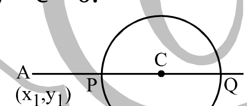
NOTE :

If $g^2 - c > 0 \Rightarrow$ circle cuts the x axis at two distinct points.
If $g^2 = c \Rightarrow$ circle touches the x-axis.
If $g^2 < c \Rightarrow$ circle lies completely above or below the x-axis.

3. POSITION OF A POINT w.r.t. A CIRCLE :

The point (x_1 , y_1) is inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.
according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \Leftrightarrow 0$.

Note : The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC – r respectively.



4. LINE & A CIRCLE :

Let L = 0 be a line & S = 0 be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then :

(i) $p > r \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

(ii) $p = r \Leftrightarrow$ the line touches the circle.

(iii) $p < r \Leftrightarrow$ the line is a secant of the circle.

(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.

5. PARAMETRIC EQUATIONS OF A CIRCLE :

The parametric equations of $(x - h)^2 + (y - k)^2 = r^2$ are :
$x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$ where (h, k) is the centre, r is the radius & θ is a parameter.

Note that equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is
$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}$.

6. TANGENT & NORMAL :

(a) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1 , y_1) is,
$x x_1 + y y_1 = a^2$. Hence equation of a tangent at $(a\cos\alpha, a\sin\alpha)$ is ;
$x\cos\alpha + y\sin\alpha = a$. The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is
$\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$.

(b) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1 , y_1) is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

(c) $y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact is $\left(-\frac{a^2 m}{c}, \frac{a^2}{c}\right)$.

(d) If a line is normal/orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1).$$

7. A FAMILY OF CIRCLES :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K\,S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.
In case the line through (x_1, y_1) is parallel to y - axis the equation of the family of circles touching it at (x_1, y_1) becomes $(x - x_1)^2 + (y - y_1)^2 + K(x - x_1) = 0$.
Also if line is parallel to x - axis the equation of the family of circles touching it at (x_1, y_1) becomes $(x - x_1)^2 + (y - y_1)^2 + K(y - y_1) = 0$.

(e) Equation of circle circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda\, L_2L_3 + \mu\, L_3L_1 = 0$ provided co-efficient of xy = 0 & co-efficient of x^2 = co-efficient of y^2.

(f) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda\, L_2L_4 = 0$ provided co-efficient of x^2 = co-efficient of y^2 and co-efficient of xy = 0.

8. LENGTH OF A TANGENT AND POWER OF A POINT :

The length of a tangent from an external point (x_1, y_1) to the circle
$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^{\,2} + y_1^{\,2} + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$.
Square of length of the tangent from the point P is also called **THE POWER OF POINT** w.r.t. a circle. Power of a point remains constant w.r.t. a circle.
Note that : power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

9. DIRECTOR CIRCLE :

The locus of the point of intersection of two perpendicular tangents is called the **DIRECTOR CIRCLE** of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

10. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$. This on simplication can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^{\,2} + y_1^{\,2} + 2gx_1 + 2fy_1 + c$
which is designated by $T = S_1$.
Note that : the shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

11. CHORD OF CONTACT :

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle
$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

REMEMBER :

(a) Chord of contact exists only if the point 'P' is not inside .

(b) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(c) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$
Where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(d) Angle between the pair of tangents from $(x_1, y_1) = \tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)$

where R = radius ; L = length of tangent.

(e) Equation of the circle circumscribing the triangle PT_1T_2 is :
$(x - x_1)\ (x + g) + (y - y_1)\ (y + f) = 0$.

(f) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.
Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

12. POLE & POLAR :

(i) If through a point P in the plane of the circle, there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q & R is called the **POLAR OF THE POINT P** ; also P is called the **POLE OF THE POLAR**.

(ii) The equation to the polar of a point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 = a^2$ is given by $\mathbf{xx_1 + yy_1 = a^2}$, & if the circle is general then the equation of the polar becomes $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$. Note that if the point (x_1, y_1) be on the circle then the chord of contact, tangent & polar will be represented by the same equation.

(iii) Pole of a given line $Ax + By + C = 0$ w.r.t. any circle $x^2 + y^2 = a^2$ is $\left(-\frac{Aa^2}{C}, -\frac{Ba^2}{C}\right)$.

(iv) If the polar of a point P pass through a point Q, then the polar of Q passes through P.

(v) Two lines L_1 & L_2 are conjugate of each other if Pole of L_1 lies on L_2 & vice versa Similarly two points P & Q are said to be conjugate of each other if the polar of P passes through Q & vice-versa.

13. COMMON TANGENTS TO TWO CIRCLES :

(i) Where the two circles neither intersect nor touch each other, there are FOUR common tangents, two of them are transverse & the others are direct common tangents.

(ii) When they intersect there are two common tangents, both of them being direct.

(iii) When they touch each other :

(a) **EXTERNALLY :** there are three common tangents, two direct and one is the tangent at the point of contact .

(b) **INTERNALLY :** only one common tangent possible at their point of contact.

(iv) Length of an external common tangent & internal common tangent to the two circles is given by:

$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$.

Where d = distance between the centres of the two circles . r_1 & r_2 are the radii of the two circles.

(v) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

14. RADICAL AXIS & RADICAL CENTRE :

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. The equation of radical axis of the two circles $S_1 = 0$ & $S_2 = 0$ is given ;
$S_1 - S_2 = 0$ i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

NOTE THAT :

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the two circles.

(d) Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

(e) Radical axis bisects a common tangent between the two circles.

(f) The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.

(g) A system of circles, every two which have the same radical axis, is called a coaxal system.

(h) Pairs of circles which do not have radical axis are concentric.

15. ORTHOGONALITY OF TWO CIRCLES :

Two circles $S_1 = 0$ & $S_2 = 0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is : $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Note : (a) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles .

(b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P . Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.

EXERCISE–I

Q.1 Determine the nature of the quadrilateral formed by four lines $3x + 4y - 5 = 0$; $4x - 3y - 5 = 0$; $3x + 4y + 5 = 0$ and $4x - 3y + 5 = 0$. Find the equation of the circle inscribed and circumscribing this quadrilateral.

Q.2 Suppose the equation of the circle which touches both the coordinate axes and passes through the point with abscissa -2 and ordinate 1 has the equation $x^2 + y^2 + Ax + By + C = 0$, find all the possible ordered triplet (A, B, C).

Q.3 A circle S = 0 is drawn with its centre at (–1, 1) so as to touch the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ externally. Find the intercept made by the circle S = 0 on the coordinate axes.

Q.4 The line $lx + my + n = 0$ intersects the curve $ax^2 + 2hxy + by^2 = 1$ at the point P and Q. The circle on PQ as diameter passes through the origin. Prove that $n^2(a^2 + b^2) = l^2 + m^2$.

Q.5 One of the diameters of the circle circumscribing the rectangle ABCD is $4y = x + 7$. If A & B are the points (–3, 4) & (5,4) respectively, then find the area of the rectangle.

Q.6 Find the equation to the circle which is such that the length of the tangents to it from the points (1, 0), (2, 0) and (3, 2) are 1, $\sqrt{7}$, $\sqrt{2}$ respectively.

Q.7 A circle passes through the points (–1, 1), (0, 6) and (5, 5). Find the points on the circle the tangents at which are parallel to the straight line joining origin to the centre.

Q.8 Find the equations of straight lines which pass through the intersection of the lines $x - 2y - 5 = 0$, $7x + y = 50$ & divide the circumference of the circle $x^2 + y^2 = 100$ into two arcs whose lengths are in the ratio 2 : 1.

Q.9 A(–a, 0) ; B (a, 0) are fixed points. C is a point which divides AB in a constant ratio $\tan\alpha$. If AC & CB subtend equal angles at P, prove that the equation of the locus of P is $x^2 + y^2 + 2ax \sec 2\alpha + a^2 = 0$.

Q.10 A circle is drawn with its centre on the line $x + y = 2$ to touch the line $4x - 3y + 4 = 0$ and pass through the point (0, 1). Find its equation.

Q.11(a) Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

(b) If the line $x \sin\alpha - y + a \sec\alpha = 0$ touches the circle with radius 'a' and centre at the origin then find the most general values of 'α' and sum of the values of 'α' lying in $[0, 100\pi]$.

Q.12 A point moving around circle $(x + 4)^2 + (y + 2)^2 = 25$ with centre C broke away from it either at the point A or point B on the circle and moved along a tangent to the circle passing through the point D (3, – 3). Find the following.

(i) Equation of the tangents at A and B. (ii) Coordinates of the points A and B.

(iii) Angle ADB and the maximum and minimum distances of the point D from the circle.

(iv) Area of quadrilateral ADBC and the ΔDAB.

(v) Equation of the circle circumscribing the ΔDAB and also the intercepts made by this circle on the coordinate axes.

Q.13 Find the locus of the mid point of the chord of a circle $x^2 + y^2 = 4$ such that the segment intercepted by the chord on the curve $x^2 - 2x - 2y = 0$ subtends a right angle at the origin.

Q.14 Find the equation of a line with gradient 1 such that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal length on it.

Q.15 Find the locus of the middle points of portions of the tangents to the circle $x^2 + y^2 = a^2$ terminated by the coordinate axes.

Q.16 Tangents are drawn to the concentric circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ at right angle to one another. Show that the locus of their point of intersection is a 3rd concentric circle. Find its radius.

Q.17 Find the equation of the circle passing through the three points (4, 7), (5, 6) and (1, 8). Also find the coordinates of the point of intersection of the tangents to the circle at the points where it is cut by the straight line $5x + y + 17 = 0$.

Q.18 Consider a circle S with centre at the origin and radius 4. Four circles A, B, C and D each with radius unity and centres (–3, 0), (–1, 0), (1, 0) and (3, 0) respectively are drawn. A chord PQ of the circle S touches the circle B and passes through the centre of the circle C. If the length of this chord can be expressed as $\sqrt{x}$, find x.

Q.19 Obtain the equations of the straight lines passing through the point A(2, 0) & making 45° angle with the tangent at A to the circle $(x + 2)^2 + (y - 3)^2 = 25$. Find the equations of the circles each of radius 3 whose centres are on these straight lines at a distance of $5\sqrt{2}$ from A.

Q.20 Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line is drawn from the point P intersects the curve at points Q & R. If the product PQ. PR is independent of the slope of the line, then show that the curve is a circle.

Q.21 The line $2x - 3y + 1 = 0$ is tangent to a circle $S = 0$ at (1, 1). If the radius of the circle is $\sqrt{13}$. Find the equation of the circle S.

Q.22 Find the equation of the circle which passes through the point (1, 1) & which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point (2, 3) on it.

Q.23 Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a \neq 0,\ b \neq 0)$. Find the condition on a & b if two chords, each bisected by the x-axis, can be drawn to the circle from the point $\left(a, \frac{b}{2}\right)$.

Q.24 Show that the equation of a straight line meeting the circle $x^2 + y^2 = a^2$ in two points at equal distances 'd' from a point (x_1, y_1) on its circumference is $xx_1 + yy_1 - a^2 + \frac{d^2}{2} = 0$.

Q.25 The radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x - 2y + 1 = 0$. Show that either $g = 3/4$ or $f = 2$.

Q.26 Find the equation of the circle through the points of intersection of circles $x^2 + y^2 - 4x - 6y - 12=0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ & cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Q.27 The centre of the circle $S = 0$ lie on the line $2x - 2y + 9 = 0$ & $S = 0$ cuts orthogonally the circle $x^2 + y^2 = 4$. Show that circle $S = 0$ passes through two fixed points & find their coordinates.

Q.28(a) Find the equation of a circle passing through the origin if the line pair, $xy - 3x + 2y - 6 = 0$ is orthogonal to it. If this circle is orthogonal to the circle $x^2 + y^2 - kx + 2ky - 8=0$ then find the value of k.

(b) Find the equation of the circle which cuts the circle $x^2 + y^2 - 14x - 8y + 64 = 0$ and the coordinate axes orthogonally.

Q.29 Find the equation of the circle whose radius is 3 and which touches the circle $x^2 + y^2 - 4x - 6y - 12=0$ internally at the point $(-1, -1)$.

Q.30 Show that the locus of the centres of a circle which cuts two given circles orthogonally is a straight line & hence deduce the locus of the centers of the circles which cut the circles $x^2 + y^2 + 4x - 6y + 9=0$ & $x^2 + y^2 - 5x + 4y + 2 = 0$ orthogonally. Interpret the locus.

EXERCISE–II

Q.1 A variable circle passes through the point A (a, b) & touches the x-axis; show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.

Q.2 Find the equation of the circle passing through the point $(-6, 0)$ if the power of the point (1, 1) w.r.t. the circle is 5 and it cuts the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ orthogonally.

Q.3 Consider a family of circles passing through two fixed points A (3, 7) & B(6, 5). Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

Q.4 Find the equation of circle passing through (1, 1) belonging to the system of co–axal circles that are tangent at (2, 2) to the locus of the point of intersection of mutually perpendicular tangent to the circle $x^2 + y^2 = 4$.

Q.5 Find the locus of the mid point of all chords of the circle $x^2 + y^2 - 2x - 2y = 0$ such that the pair of lines joining (0, 0) & the point of intersection of the chords with the circles make equal angle with axis of x.

Q.6 The circle $C : x^2 + y^2 + kx + (1 + k)y - (k + 1) = 0$ passes through the same two points for every real number k. Find(i) the coordinates of these two points.(ii) the minimum value of the radius of a circle C.

Q.7 Find the equation of a circle which is co-axial with circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ & $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of these two circles.

Q.8 Show that the locus of the point the tangents from which to the circle $x^2 + y^2 - a^2 = 0$ include a constant angle α is $(x^2 + y^2 - 2a^2)^2 \tan^2\alpha = 4a^2(x^2 + y^2 - a^2)$.

Q.9 A circle with center in the first quadrant is tangent to $y = x + 10$, $y = x - 6$, and the y-axis. Let (h, k) be the center of the circle. If the value of $(h + k) = a + b\sqrt{a}$ where $\sqrt{a}$ is a surd, find the value of a + b.

Q.10 A circle is described to pass through the origin and to touch the lines $x = 1$, $x + y = 2$. Prove that the radius of the circle is a root of the equation $\left(3 - 2\sqrt{2}\right)t^2 - 2\sqrt{2}\,t + 2 = 0$.

Q.11 Find the condition such that the four points in which the circle $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + a'x + b'y + c' = 0$ are intercepted by the straight lines $Ax + By + C = 0$ & $A'x + B'y + C' = 0$ respectively, lie on another circle.

Q.12 A circle C is tangent to the x and y axis in the first quadrant at the points P and Q respectively. BC and AD are parallel tangents to the circle with slope – 1. If the points A and B are on the y-axis while C and D are on the x-axis and the area of the figure ABCD is $900\sqrt{2}$ sq. units then find the radius of the circle.

Q.13 The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + K\sqrt{x^2 + y^2} = 0$. Find K.

Q.14 Let A, B, C be real numbers such that
(i) (sin A, cos B) lies on a unit circle centred at origin. (ii) tan C and cot C are defined.
If the minimum value of $(\tan C - \sin A)^2 + (\cot C - \cos B)^2$ is $a + b\sqrt{2}$ where $a, b \in I$, find the value of $a^3 + b^3$.

Q.15 An isosceles right angled triangle whose sides are 1, 1, $\sqrt{2}$ lies entirely in the first quadrant with the ends of the hypotenuse on the coordinate axes. If it slides prove that the locus of its centroid is $(3x - y)^2 + (x - 3y)^2 = \frac{32}{9}$.

Q.16 Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the axis of x, equidistant from the point (k, 0). Show that the locus of their intersection is $ky^2 = a^2(k - x)$.

Q.17 Find the equation of a circle which touches the lines $7x^2 - 18xy + 7y^2 = 0$ and the circle $x^2 + y^2 - 8x - 8y = 0$ and is contained in the given circle.

Q.18 Let W_1 and W_2 denote the circles $x^2 + y^2 + 10x - 24y - 87 = 0$ and $x^2 + y^2 - 10x - 24y + 153 = 0$ respectively. Let *m* be the smallest possible value of 'a' for which the line y = ax contains the centre of a circle that is externally tangent to W_2 and internally tangent to W_1. Given that $m^2 = \frac{p}{q}$ where *p* and *q* are relatively prime integers, find $(p + q)$.

Q.19 Find the equation of the circle which passes through the origin, meets the x-axis orthogonally & cuts the circle $x^2 + y^2 = a^2$ at an angle of 45°.

Q.20 The ends A, B of a fixed straight line of length 'a' & ends A′ & B′ of another fixed straight line of length 'b' slide upon the axis of x & the axis of y (one end on axis of x & the other on axis of y). Find the locus of the centre of the circle passing through A, B, A′ & B′.

EXERCISE–III

Q.1 (a) The intercept on the line y = x by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is ______.

(b) The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of the locus of the point P is
(A) $x^2 + y^2 + 4x - 6y + 4 = 0$ (B) $x^2 + y^2 + 4x - 6y - 9 = 0$
(C) $x^2 + y^2 + 4x - 6y - 4 = 0$ (D) $x^2 + y^2 + 4x - 6y + 9 = 0$

(c) Find the intervals of values of a for which the line y + x = 0 bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle; $2x^2 + 2y^2 - (1+\sqrt{2}a)\,x - (1-\sqrt{2}a)\,y = 0$. [JEE '96, 1+1+5]

Q.2 A tangent drawn from the point (4, 0) to the circle $x^2 + y^2 = 8$ touches it at a point A in the first quadrant. Find the coordinates of the another point B on the circle such that AB = 4. [REE '96, 6]

Q.3 (a) The chords of contact of the pair of tangents drawn from each point on the line 2x + y = 4 to the circle $x^2 + y^2 = 1$ pass through the point ______.

(b) Let C be any circle with centre $\left(0, \sqrt{2}\right)$. Prove that at the most two rational point can be there on C. (A rational point is a point both of whose co-ordinate are rational numbers).[JEE'97, 2+5]

Q.4 (a) The number of common tangents to the circle $x^2 + y^2 = 4$ & $x^2 + y^2 - 6x - 8y = 24$ is :
(A) 0 (B) 1 (C) 3 (D) 4

(b) C_1 & C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA & PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1.[JEE '98, 2 + 8

Q.5 Find the equation of a circle which touches the line x + y = 5 at the point (–2, 7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally. [REE '98, 6]

Q.6 (a) If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq q$) are bisected by the x – axis, then :
(A) $p^2 = q^2$ (B) $p^2 = 8q^2$ (C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

(b) Let L_1 be a straight line through the origin and L_2 be the straight line x + y = 1 . If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal, then which of the following equations can represent L_1 ?
(A) x + y = 0 (B) x – y = 0 (C) x + 7y = 0 (D) x – 7y = 0

(c) Let T_1, T_2 be two tangents drawn from (– 2, 0) onto the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time.[JEE '99, 2 + 3 + 10]

Q.7 (a) The triangle PQR is inscribed in the circle, $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) &

(– 4, 3) respectively, then $\angle$ QPR is equal to :

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

(b) If the circles, $x^2 + y^2 + 2x + 2ky + 6 = 0$ & $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then 'k' is : [JEE '2000 (Screening) 1 + 1]

(A) 2 or $-3/2$ (B) -2 or $-3/2$ (C) 2 or $3/2$ (D) -2 or $3/2$

Q.8 (a) Extremities of a diagonal of a rectangle are (0, 0) & (4, 3). Find the equation of the tangents to the circumcircle of a rectangle which are parallel to this diagonal.

(b) Find the point on the straight line, $y = 2x + 11$ which is nearest to the circle, $16(x^2 + y^2) + 32x - 8y - 50 = 0$.

(c) A circle of radius 2 units rolls on the outerside of the circle, $x^2 + y^2 + 4x = 0$, touching it externally. Find the locus of the centre of this outer circle. Also find the equations of the common tangents of the two circles when the line joining the centres of the two circles makes on angle of 60^o with x-axis. [REE '2000 (Mains) 3 + 3 + 5]

Q.9 (a) Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle then 2r equals

(A) $\sqrt{PQ \cdot RS}$ (B) $\frac{PQ + RS}{2}$ (C) $\frac{2PQ \cdot RS}{PQ + RS}$ (D) $\sqrt{\frac{(PQ)^2 + (RS)^2}{2}}$

(b) Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin 'O' to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA.

Q.10 (a) Find the equation of the circle which passes through the points of intersection of circles $x^2 + y^2 - 2x - 6y + 6 = 0$ and $x^2 + y^2 + 2x - 6y + 6 = 0$ and intersects the circle $x^2 + y^2 + 4x + 6y + 4 = 0$ orthogonally. [REE '2001 (Mains) 3 out of 100]

(b) Tangents TP and TQ are drawn from a point T to the circle $x^2 + y^2 = a^2$. If the point T lies on the line $px + qy = r$, find the locus of centre of the circumcircle of triangle TPQ.

Q.11 (a) If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is

(A) 4 (B) $2\sqrt{5}$ (C) 5 (D) $3\sqrt{5}$

(b) If $a > 2b > 0$ then the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is [JEE '2002 (Scr)3 + 3 out of 270]

(A) $\frac{2b}{\sqrt{a^2 - 4b^2}}$ (B) $\frac{\sqrt{a^2 - 4b^2}}{2b}$ (C) $\frac{2b}{a - 2b}$ (D) $\frac{b}{a - 2b}$

Q.12 The radius of the circle, having centre at (2, 1), whose one of the chord is a diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$

(A) 1 (B) 2 (C) 3 (D) $\sqrt{3}$ [JEE '2004 (Scr)]

Q.13 Line $2x + 3y + 1 = 0$ is a tangent to a circle at (1, -1). This circle is orthogonal to a circle which is drawn having diameter as a line segment with end points (0, –1) and (– 2, 3). Find equation of circle.

Q.14 A circle is given by $x^2 + (y - 1)^2 = 1$, another circle C touches it externally and also the x-axis, then the locus of its centre is [JEE '2005 (Scr)]

(A) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \le 0\}$ (B) $\{(x, y) : x^2 + (y - 1)^2 = 4\} \cup \{x, y) : y \le 0\}$

(C) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \le 0\}$ (D) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$

ANSWER KEY EXERCISE–I

Q.1 square of side 2; $x^2 + y^2 = 1$; $x^2 + y^2 = 2$

Q.2 $x^2 + y^2 + 10x - 10y + 25 = 0$ **OR** $x^2 + y^2 + 2x - 2y + 1 = 0$, (10, – 10, 25) (2, – 2, 1)

Q.3 zero, zero Q.5 32 sq. unit Q.6 $2(x^2 + y^2) + 6x - 17y - 6 = 0$]

Q.7 (5, 1) & (–1, 5) Q.8 $4x - 3y - 25 = 0$ **OR** $3x + 4y - 25 = 0$

Q.10 $x^2 + y^2 - 2x - 2y + 1 = 0$ **OR** $x^2 + y^2 - 42x + 38y - 39 = 0$

Q.11 (a) $\frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$; (b) $\alpha = n\pi$; 5050π

Q.12 (i) $3x - 4y = 45$; $4x + 3y = 3$; (ii) A(0, 1) and B (–1, – 6); (iii) 90°, $5(\sqrt{2} \pm 1)$ units
(iv) 12.5 sq. units; (v) $x^2 + y^2 + x + 5y - 6$, x intercept 5; y intercept 7]

Q.13 $x^2 + y^2 - 2x - 2y = 0$ Q.14 $2x - 2y - 3 = 0$ Q.15 $a^2(x^2 + y^2) = 4x^2y^2$

Q.16 $x^2 + y^2 = a^2 + b^2$; $r = \sqrt{a^2 + b^2}$ Q.17 (– 4, 2), $x^2 + y^2 - 2x - 6y - 15 = 0$ Q.18 63

Q.19 $x - 7y = 2$, $7x + y = 14$; $(x - 1)^2 + (y - 7)^2 = 3^2$; $(x - 3)^2 + (y + 7)^2 = 3^2$;
$(x - 9)^2 + (y - 1)^2 = 3^2$; $(x + 5)^2 + (y + 1)^2 = 3^2$

Q.21 $x^2 + y^2 - 6x + 4y=0$ **OR** $x^2 + y^2 + 2x - 8y + 4=0$ Q.22 $x^2 + y^2 + x - 6y + 3 = 0$

Q.23 $a^2 > 2b^2$ Q.26 $x^2 + y^2 + 16x + 14y - 12 = 0$

Q.27 (– 4, 4) ; (– 1/2, 1/2) Q.28 (a) $x^2 + y^2 + 4x - 6y = 0$; k = 1; (b) $x^2 + y^2 = 64$

Q.29 $5x^2 + 5y^2 - 8x - 14y - 32 = 0$ Q.30 $9x - 10y + 7 = 0$; radical axis

EXERCISE–II

Q.2 $x^2 + y^2 + 6x - 3y = 0$ Q.3 $\left(2, \frac{23}{3}\right)$ Q.4 $x^2 + y^2 - 3x - 3y + 4 = 0$

Q.5 $x + y = 2$ Q.6 $(1, 0)$ & $(1/2, 1/2)$; $r = \frac{1}{2\sqrt{2}}$ Q.7 $4x^2 + 4y^2 + 6x + 10y - 1 = 0$

Q.9 10 Q.11 $\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}$ Q.12 $r = 15$ Q.13 $K = 1$

Q.14 19 Q.17 $x^2 + y^2 - 12x - 12y + 64 = 0$ Q.18 169

Q.19 $x^2 + y^2 \pm a\sqrt{2}\,x = 0$ Q.20 $(2ax - 2by)^2 + (2bx - 2ay)^2 = (a^2 - b^2)^2$

EXERCISE–III

Q.1 (a) $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$, **(b)** D, **(c)** $(-\infty, -2) \cup (2, \infty)$ **Q.2** (2, -2) or (-2, 2) **Q.3 (a)** (1/2, 1/4)

Q.4 (a) B **Q.5** $x^2 + y^2 + 7x - 11y + 38 = 0$

Q.6 **(a)** D **(b)** B, C **(c)** $c_1 : (x - 4)^2 + y^2 = 9$; $c_2 : \left(x + \frac{4}{3}\right)^2 + y^2 = \frac{1}{9}$

common tangent between c & c_1 : $T_1 = 0$; $T_2 = 0$ and $x - 1 = 0$;
common tangent between c & c_2 : $T_1 = 0$; $T_2 = 0$ and $x + 1 = 0$;

common tangent between c_1 & c_2 : $T_1 = 0$; $T_2 = 0$ and $y = \pm \frac{5}{\sqrt{39}}\left(x + \frac{4}{5}\right)$

where $T_1 : x - \sqrt{3}\,y + 2 = 0$ and $T_2 : x + \sqrt{3}\,y + 2 = 0$

Q.7 **(a)** C **(b)** A

Q.8 **(a)** $6x - 8y + 25 = 0$ & $6x - 8y - 25 = 0$; **(b)** $(-9/2, 2)$

(c) $x^2 + y^2 + 4x - 12 = 0$, T_1: $\sqrt{3}x - y + 2\sqrt{3} + 4 = 0$, T_2: $\sqrt{3}x - y + 2\sqrt{3} - 4 = 0$ (D.C.T.)

T_3: $x + \sqrt{3}\,y - 2 = 0$, T_4: $x + \sqrt{3}\,y + 6 = 0$ (T.C.T.)

Q.9 **(a)** A; **(b)** $OA = 3(3 + \sqrt{10})$ **Q.10** **(a)** $x^2 + y^2 + 14x - 6y + 6 = 0$; **(b)** $2px + 2qy = r$

Q.11 **(a)** C; **(b)** A **Q.12** C **Q.13** $2x^2 + 2y^2 - 10x - 5y + 1 = 0$ **Q.14** D

EXERCISE–IV

Part : (A) Only one correct option

1. If (–3, 2) lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, which is concentric with the circle $x^2 + y^2 + 6x + 8y - 5 = 0$, then c is
(A) 11 (B) –11 (C) 24 (D) none of these

2. The circle $x^2 + y^2 - 6x - 10y + c = 0$ does not intersect or touch either axis & the point (1, 4) is inside the circle. Then the range of possible values of c is given by:
(A) $c > 9$ (B) $c > 25$ (C) $c > 29$ (D) $25 < c < 29$

3. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + p = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + q = 0$ is:
(A) $\sqrt{q-p}$ (B) $\sqrt{p-q}$ (C) $\sqrt{q+p}$ (D) none

4. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ equals
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) none

5. The circumference of the circle $x^2 + y^2 - 2x + 8y - q = 0$ is bisected by the circle $x^2 + y^2 + 4x + 12y + p = 0$, then p + q is equal to:
(A) 25 (B) 100 (C) 10 (D) 48

6. If $\left(a, \frac{1}{a}\right), \left(b, \frac{1}{b}\right), \left(c, \frac{1}{c}\right)$ & $\left(d, \frac{1}{d}\right)$ are four distinct points on a circle of radius 4 units then, abcd is equal to:
(A) 4 (B) 16 (C) 1 (D) none

7. The centre of a circle passing through the points (0, 0), (1, 0) & touching the circle $x^2 + y^2 = 9$ is :
(A) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \frac{3}{2}\right)$ (C) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, -\sqrt{2}\right)$

8. Two thin rods AB & CD of lengths 2a & 2b move along OX & OY respectively, when 'O' is the origin. The equation of the locus of the centre of the circle passing through the extremities of the two rods is:
(A) $x^2 + y^2 = a^2 + b^2$ (B) $x^2 - y^2 = a^2 - b^2$ (C) $x^2 + y^2 = a^2 - b^2$ (D) $x^2 - y^2 = a^2 + b^2$

9. The value of 'c' for which the set,
$\{(x, y) \mid x^2 + y^2 + 2x \le 1\} \cap \{(x, y) \mid x - y + c \ge 0\}$ contains only one point in common is:

(A) $(-\infty, -1] \cup [3, \infty)$ (B) $\{-1, 3\}$ (C) $\{-3\}$ (D) $\{-1\}$

10. Let x & y be the real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M & m respectively, then the numerical value of M – m is:
(A) 2 (B) 8 (C) 15 (D) none of these

11. A line meets the co–ordinate axes in A & B. A circle is circumscribed about the triangle OAB. If d_1 & d_2 are the distances of the tangent to the circle at the origin O from the points A and B respectively, the diameter of the circle is:
(A) $\frac{2d_1 + d_2}{2}$ (B) $\frac{d_1 + 2d_2}{2}$ (C) $d_1 + d_2$ (D) $\frac{d_1 d_2}{d_1 + d_2}$

12. The distance between the chords of contact of tangents to the circle; $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin & the point (g, f) is:
(A) $\sqrt{g^2 + f^2}$ (B) $\frac{\sqrt{g^2 + f^2 - c}}{2}$ (C) $\frac{g^2 + f^2 - c}{2\sqrt{g^2 + f^2}}$ (D) $\frac{\sqrt{g^2 + f^2 + c}}{2\sqrt{g^2 + f^2}}$

13. If tangent at (1, 2) to the circle c_1: $x^2 + y^2 = 5$ intersects the circle c_2: $x^2 + y^2 = 9$ at A & B and tangents at A & B to the second circle meet at point C, then the co–ordinates of C are:
(A) (4, 5) (B) $\left(\frac{9}{15}, \frac{18}{5}\right)$ (C) (4, –5) (D) $\left(\frac{9}{5}, \frac{18}{5}\right)$

14. The locus of the mid points of the chords of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ which subtend an angle of $\frac{\pi}{3}$ radians at its circumference is:
(A) $(x - 2)^2 + (y + 3)^2 = 6.25$ (B) $(x + 2)^2 + (y - 3)^2 = 6.25$
(C) $(x + 2)^2 + (y - 3)^2 = 18.75$ (D) $(x + 2)^2 + (y + 3)^2 = 18.75$

15. If the length of a common internal tangent to two circles is 7, and that of a common external tangent is 11, then the product of the radii of the two circles is:
(A) 36 (B) 9 (C) 18 (D) 4

16. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is:
(A) $\frac{16}{\sqrt{5}}$ (B) 8 (C) $4\sqrt{6}$ (D) $\frac{8\sqrt{5}}{5}$

17. A circle touches a straight line $lx + my + n = 0$ & cuts the circle $x^2 + y^2 = 9$ orthogonally. The locus of centres of such circles is:
(A) $(lx + my + n)^2 = (l^2 + m^2)(x^2 + y^2 - 9)$ (B) $(lx + my - n)^2 = (l^2 + m^2)(x^2 + y^2 - 9)$
(C) $(lx + my + n)^2 = (l^2 + m^2)(x^2 + y^2 + 9)$ (D) none of these

18. If a circle passes through the point (a, b) & cuts the circle $x^2 + y^2 = K^2$ orthogonally, then the equation of the locus of its centre is:
(A) $2ax + 2by - (a^2 + b^2 + K^2) = 0$ (B) $2ax + 2by - (a^2 - b^2 + K^2) = 0$
(C) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - K^2) = 0$ (D) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - K^2) = 0$

19. The circle $x^2 + y^2 = 4$ cuts the circle $x^2 + y^2 + 2x + 3y - 5 = 0$ in A & B. Then the equation of the circle on AB as a diameter is:
(A) $13(x^2 + y^2) - 4x - 6y - 50 = 0$ (B) $9(x^2 + y^2) + 8x - 4y + 25 = 0$
(C) $x^2 + y^2 - 5x + 2y + 72 = 0$ (D) none of these

20. The length of the tangents from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$ and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$ are in the ratio
(A) 1 : 2 (B) 2 : 3 (C) 3 : 4 (D) none of these

21. The normal at the point (3, 4) on a circle cuts the circle at the point (–1, –2). Then the equation of the circle is
(A) $x^2 + y^2 + 2x - 2y - 13 = 0$ (B) $x^2 + y^2 - 2x - 2y - 11 = 0$
(C) $x^2 + y^2 - 2x + 2y + 12 = 0$ (D) $x^2 + y^2 - 2x - 2y + 14 = 0$

22. The locus of poles whose polar with respect to $x^2 + y^2 = a^2$ always passes through (K, 0) is:
(A) $Kx - a^2 = 0$ (B) $Kx + a^2 = 0$ (C) $Ky + a^2 = 0$ (D) $Ky - a^2 = 0$

23. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + q)$ (where $pq \neq 0$) are bisected by the x-axis, then **[IIT - 1999]**
(A) $p^2 = q^2$ (B) $p^2 = 8q^2$ (C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

24. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (–4, 3) respectively, the $\angle$ QPR is equal to **[IIT - 2000]**
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

25. Let PQ and RS be tangents at the extremities of diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then 2r equals **[IIT- 2001]**
(A) $\sqrt{PQ \cdot RS}$ (B) $\frac{PQ + RS}{2}$ (C) $\frac{2PQ + RS}{PQ + RS}$ (D) $\frac{\sqrt{PQ^2 + RS^2}}{2}$

26. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, locus of the centroid of the triangle PAB as P moves on the circles is **[IIT- 2001]**
(A) a parabola (B) a circle (C) an ellipse (D) a pair of straight line

27. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is **[IIT- 2002]**
(A) 4 (B) $2\sqrt{5}$ (C) 5 (D) $3\sqrt{5}$

28. Tangent to the curve $y = x^2 + 6$ at a point P(1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point

Q. Then, the coordinates of Q are **[IIT- 2005]**
(A) (– 6, –11) (B) (– 9, – 13) (C) (– 10, – 15) (D) (– 6, – 7)

Part : (B) May have more than one options correct

29. A circle passes through the point $\left(3, \sqrt{\frac{7}{2}}\right)$ and touches the line pair $x^2 - y^2 - 2x + 1 = 0$. The co–ordinates of the centre of the circle are:
(A) (4, 0) (B) (5, 0) (C) (6, 0) (D) (0, 4)

30. The equation of the circle which touches both the axes and the line $\frac{x}{3} + \frac{y}{4} = 1$ and lies in the first quadrant is $(x - c)^2 + (y - c)^2 = c^2$ where c is
(A) 1 (B) 2 (C) 4 (D) 6

EXERCISE–V

1. If y = 2x is a chord of the circle $x^2 + y^2 - 10x = 0$, find the equation of a circle with this chord as diameter.

2. Find the points of intersection of the line x – y + 2 = 0 and the circle $3x^2 + 3y^2 - 29x - 19y + 56 = 0$. Also determine the length of the chord intercepted.

3. Show that two tangents can be drawn from the point (9, 0) to the circle $x^2 + y^2 = 16$; also find the equation of the pair of tangents and the angle between them.

4. Given the three circles $x^2 + y^2 - 16x + 60 = 0$, $3x^2 + 3y^2 - 36x + 81 = 0$ and $x^2 + y^2 - 16x - 12y + 84 = 0$, find (1) the point from which the tangents to them are equal in length, and (2) this length.

5. On the line joining (1, 0) and (3, 0) an equilateral triangle is drawn having its vertex in the first quadrant. Find the equation to the circles described on its sides as diameter.

6. One of the diameters of the circle circumscribing the rectangle ABCD is 4 y = x + 7. If A & B are the points (–3, 4) & (5 4) respectively. Then find the area of the rectangle.

7. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points B (1, 7) & D (4, – 2) on the circle meet at the point C. Find the area of the quadrilateral ABCD.

8. Let a circle be given by 2x (x – a) + y (2y – b) = 0, (a ≠ 0, b ≠ 0). Find the condition on a & b if two chords, each bisected by the x–axis, can be drawn to the circle from (a, b/2).

9. Find the equation of the circle which cuts each of the circles, $x^2 + y^2 = 4$, $x^2 + y^2 - 6x - 8y + 10 = 0$ & $x^2 + y^2 + 2x - 4y - 2 = 0$ at the extremities of a diameter.

10. Find the equation and the length of the common chord of the two circles given by the equations, $x^2 + y^2 + 2x + 2y + 1 = 0$ & $x^2 + y^2 + 4x + 3y + 2 = 0$.

11. Find the values of a for which the point (2a, a + 1) is an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$ made by the chord whose equation is x – y + 1 = 0.

12. If $4l^2 - 5m^2 + 6l + 1 = 0$. Prove that $lx + my + 1 = 0$ touches a definite circle. Find the centre & radius of the circle.

13. A circle touches the line y = x at a point P such that OP = $4\sqrt{2}$ where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord on the line x + y = 0 is $6\sqrt{2}$. Find the equation of the circle.

14. Show that the equation of a straight line meeting the circle $x^2 + y^2 = a^2$ in two points at equal distances 'd' from a point (x_1, y_1) on its circumference is $xx_1 + yy_1 - a^2 + \frac{d^2}{2} = 0$.

15. For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, α-particle moves k centrimetres in the counter - clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then n = __________. **[IIT 1997]**

16. Let C be any circle with centre $\left(0, \sqrt{2}\right)$. Prove that at the most two rational point can be there on C. (A rational point is a point both of whose co–ordinate are rational numbers). **[IIT - 1997]**

17. Let T_1, T_2 be two tangents drawn from (– 2, 0) onto the circle C: $x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time. **[IIT - 1999]**

18. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally C_1 internally and C_2 externally. Identify the locus of the centre of C. **[IIT 2001]**

19. Circles with raddi 3, 4 and 5 touch each other externally. If P is the point of intersection of tangents to these circles at their points of contact, find the distance of P from the points of contact. **[IIT - 2005]**

EXERCISE–IV

1. B **2.** D **3.** A **4.** C **5.** C **6.** C **7.** D **8.** B **9.** D **10.** B **11.** C **12.** C **13.** D **14.** B
15. C **16.** A **17.** A **18.** A **19.** A **20.** A **21.** B **22.** A **23.** D **24.** C **25.** A **26.** B **27.** C **28.** D
29. AC **30.** AD

PTO

EXERCISE–V

1. $x^2 + y^2 - 2x - 4y = 0$ **2.** $(1, 3), (5, 7), 4\sqrt{2}$

3. $16x^2 - 65y^2 - 288x + 1296 = 0, \tan^{-1}\left(\frac{8\sqrt{65}}{49}\right)$

4. $\left(\frac{33}{4}, 2\right)$; $\frac{1}{4}$

5. $x^2 + y^2 - 3x - \sqrt{3}\,y + 2 = 0;$

$x^2 + y^2 - 5x - \sqrt{3}\,y + 6 = 0;$

$x^2 + y^2 - 4x + 3 = 0$

6. 32 sq. unit **7.** 75 sq. units **8.** $(a^2 > 2b^2)$

9. $x^2 + y^2 - 4x - 6y - 4 = 0$

10. $2x + y + 1 = 0, \frac{2}{\sqrt{5}}$ **11.** $a \in (0, 9/5)$

12. Centre $\equiv (3, 0)$, (radius) $= \sqrt{5}$

13. $x^2 + y^2 + 18x - 2y + 32 = 0$

15. 7

17. c_1: $(x - 4)^2 + y^2 = 9$; c_2: $\left(x + \frac{4}{3}\right)^2 + y^2 = \frac{1}{9}$

common tangent between c & c_1: $T_1 = 0$; $T_2 = 0$ and $x - 1 = 0$; common tangent between c & c_2: $T_1 = 0$; $T_2 = 0$ and $x + 1 = 0$; common tangent between c_1 & c_2: $T_1 = 0$; $T_2 = 0$ and

$y = \pm \frac{5}{\sqrt{39}}\left(x + \frac{4}{5}\right)$ where T_1: $x - \sqrt{3}\,y + 2 = 0$

and T_2: $x + \sqrt{3}\,y + 2 = 0$

18. ellipse **19.** $\sqrt{5}$