विध्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचितः मानव धर्म प्रणेता
सद्‌गुरु श्री रणछोड़दासजी महाराज

STUDY PACKAGE

Subject : Mathematics

Topic : Trigonometric Ratio & Identity

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Trigonometric Ratios & Identities

1. Basic Trigonometric Identities:

(a) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1$ $\forall$ $\theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1$ $\forall$ $\theta \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1$ $\forall$ $\theta \in R - \{n\pi,\ n \in I\}$

Solved Example # 1

Prove that

(i) $\cos^4A - \sin^4A + 1 = 2\cos^2A$

(ii) $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$

Solution

(i) $\cos^4A - \sin^4A + 1$

$= (\cos^2A - \sin^2A)(\cos^2A + \sin^2A) + 1$

$= \cos^2A - \sin^2A + 1$ $\quad[\therefore\ \cos^2A + \sin^2A = 1]$

$= 2\cos^2A$

(ii) $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$

$$= \frac{\tan A + \sec A - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1}$$

$$= \frac{(\tan A + \sec A)(1 - \sec A + \tan A)}{\tan A - \sec A + 1}$$

$$= \tan A + \sec A = \frac{1 + \sin A}{\cos A}$$

Solved Example # 2

If $\sin x + \sin^2x = 1$, then find the value of

$$\cos^{12}x + 3\cos^{10}x + 3\cos^8x + \cos^6x - 1$$

Solution

$\cos^{12}x + 3\cos^{10}x + 3\cos^8x + \cos^6x - 1$

$= (\cos^4x + \cos^2x)^3 - 1$

$= (\sin^2x + \sin x)^3 - 1$ $\quad[\because\ \cos^2x = \sin x]$

$= 1 - 1 = 0$

Solved Example # 3

If $\tan\theta = m - \dfrac{1}{4m}$, then show that $\sec\theta - \tan\theta = -2m$ or $\dfrac{1}{2m}$

Solution

Depending on quadrant in which θ falls, $\sec\theta$ can be $\pm\dfrac{4m^2+1}{4m}$

So, if $\sec\theta = \dfrac{4m^2+1}{4m} = m + \dfrac{1}{4m}$

$\Rightarrow \quad \sec\theta - \tan\theta = \dfrac{1}{2m}$ and if $\sec\theta = -\left(m + \dfrac{1}{4m}\right)$

$\Rightarrow \quad \sec\theta - \tan\theta = -2m$

Self Practice Problem

1. Prove the followings :

(i) $\cos^6 A + \sin^6 A + 3\sin^2 A\cos^2 A = 1$

(ii) $\sec^2 A + \text{cosec}^2 A = (\tan A + \cot A)^2$

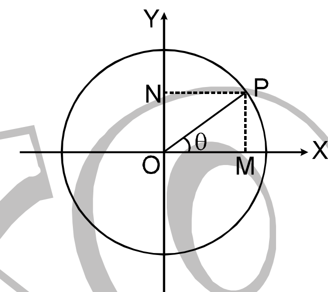
(iii) $\sec^2 A\,\text{cosec}^2 A = \tan^2 A + \cot^2 A + 2$

(iv) $(\tan\alpha + \text{cosec}\,\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha\cot\beta\,(\text{cosec}\,\alpha + \sec\beta)$

(v) $\left(\dfrac{1}{\sec^2\alpha - \cos^2\alpha} + \dfrac{1}{\text{cosec}^2\alpha - \sin^2\alpha}\right)\cos^2\alpha\sin^2\alpha = \dfrac{1 - \sin^2\alpha\cos^2\alpha}{2 + \sin^2\alpha\cos^2\alpha}$

2. If $\sin\theta = \dfrac{m^2 + 2mn}{m^2 + 2mn + 2n^2}$, then prove that $\tan\theta = \dfrac{m^2 + 2mn}{2mn + 2n^2}$

2. Circular Definition Of Trigonometric Functions:



$\sin\theta = \dfrac{PM}{OP} \qquad \cos\theta = \dfrac{OM}{OP}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$

$\sec\theta = \dfrac{1}{\cos\theta}$, $\cos\theta \neq 0 \qquad \text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, $\sin\theta \neq 0$

3. Trigonometric Functions Of Allied Angles:

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **ALLIED ANGLES**.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90° - \theta) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$

(c) $\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$

(d) $\sin(180° - \theta) = \sin\theta$; $\cos(180° - \theta) = -\cos\theta$

(e) $\sin(180° + \theta) = -\sin\theta$; $\cos(180° + \theta) = -\cos\theta$

(f) $\sin(270° - \theta) = -\cos\theta$; $\cos(270° - \theta) = -\sin\theta$

(g) $\sin(270° + \theta) = -\cos\theta$; $\cos(270° + \theta) = \sin\theta$

(h) $\tan(90° - \theta) = \cot\theta$; $\cot(90° - \theta) = \tan\theta$

Solved Example # 4

Prove that

(i) $\cot A + \tan(180° + A) + \tan(90° + A) + \tan(360° - A) = 0$

(ii) $\sec(270° - A)\sec(90° - A) - \tan(270° - A)\tan(90° + A) + 1 = 0$

Solution

(i) $\cot A + \tan(180° + A) + \tan(90° + A) + \tan(360° - A)$

$= \cot A + \tan A - \cot A - \tan A = 0$

(ii) $\sec(270° - A)\sec(90° - A) - \tan(270° - A)\tan(90° + A) + 1$

$= -\text{cosec}^2 A + \cot^2 A + 1 = 0$

Self Practice Problem

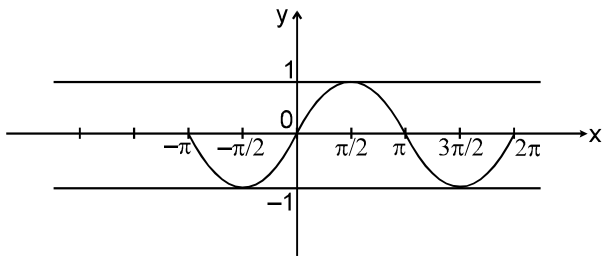
3. Prove that

(i) $\sin 420°\cos 390° + \cos(-300°)\sin(-330°) = 1$

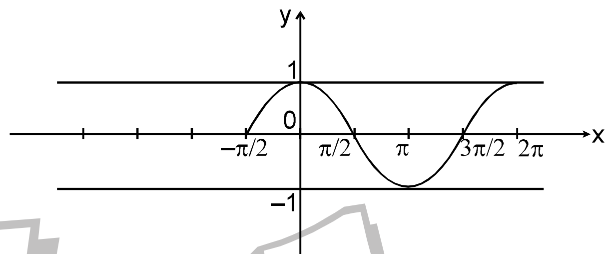
(ii) $\tan 225°\cot 405° + \tan 765°\cot 675° = 0$

4. Graphs of Trigonometric functions:

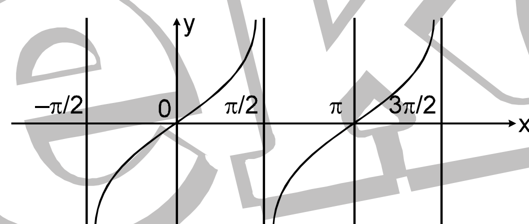
(a) $y = \sin x \quad x \in R;\ y \in [-1, 1]$



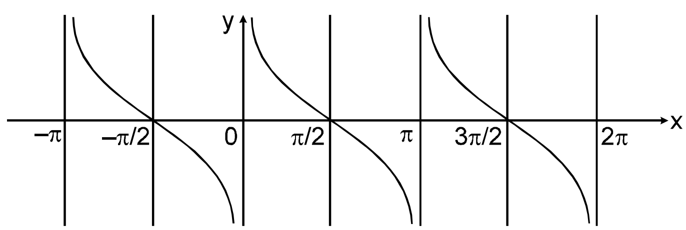
(b) $y = \cos x \quad x \in R;\ y \in [-1, 1]$



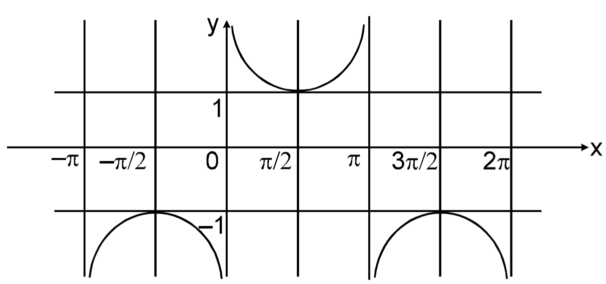
(c) $y = \tan x \quad x \in R - (2n + 1)\,\pi/2,\ n \in I;\ y \in R$



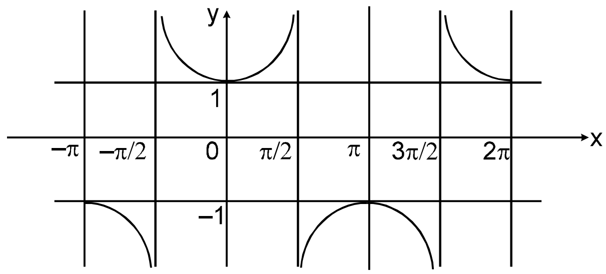
(d) $y = \cot x \quad x \in R - n\pi,\ n \in I;\ y \in R$



(e) $y = \text{cosec } x \quad x \in R - n\pi,\ n \in I;\ y \in (-\infty, -1] \cup [1, \infty)$



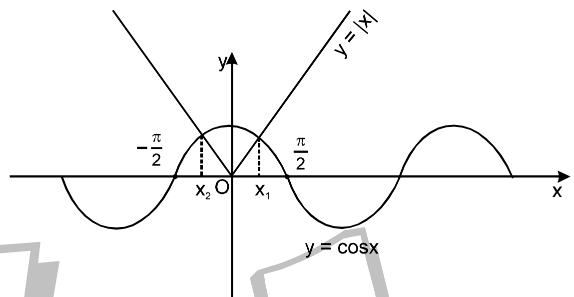
(f) $y = \sec x \quad x \in R - (2n + 1)\,\pi/2,\ n \in I;\ y \in (-\infty, -1] \cup [1, \infty)$



Solved Example # 5

Find number of solutions of the equation cos x = |x|

Solution



Clearly graph of cos x & |x| intersect at two points. Hence no. of solutions is 2

Solved Example # 6

Find range of $y = \sin^2x + 2\sin x + 3 \ \forall\ x \in R$

Solution

We know $-1 \le \sin x \le 1$

$\Rightarrow \quad 0 \le \sin x + 1 \le 2$

$\Rightarrow \quad 2 \le (\sin x + 1)^2 + 2 \le 6$

Hence range is $y \in [2, 6]$

Self Practice Problem

4. Show that the equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is only possible when $x = y \neq 0$

5. Find range of the followings.

(i) $y = 2\sin^2x + 5\sin x + 1 \ \forall\ x \in R$ **Answer** $[-2, 8]$

(ii) $y = \cos^2x - \cos x + 1 \quad \forall\ x \in R$ **Answer** $\left[\frac{3}{4}, 3\right]$

6. Find range of $y = \sin x,\ x \in \left[\frac{2\pi}{3}\ 2\pi\right]$ **Answer** $\left[-1, \frac{\sqrt{3}}{2}\right]$

5. Trigonometric Functions of Sum or Difference of Two Angles:

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2A - \sin^2B = \cos^2B - \cos^2A = \sin(A+B).\sin(A-B)$

(d) $\cos^2A - \sin^2B = \cos^2B - \sin^2A = \cos(A+B).\cos(A-B)$

(e) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(f) $\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(g) $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$.

Solved Example # 7

Prove that

(i) $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A - B)$

(ii) $\tan\left(\dfrac{\pi}{4} + \theta\right)\tan\left(\dfrac{3\pi}{4} + \theta\right) = -1$

Solution

(i) Clearly $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B)$

$= \sin(45^\circ + A + 45^\circ - B)$

$= \sin(90^\circ + A - B)$

$= \cos(A - B)$

(ii) $\tan\left(\dfrac{\pi}{4} + \theta\right) \times \tan\left(\dfrac{3\pi}{4} + \theta\right)$

$= \dfrac{1 + \tan\theta}{1 - \tan\theta} \times \dfrac{-1 + \tan\theta}{1 + \tan\theta} = -1$

Self Practice Problem

7. If $\sin\alpha = \dfrac{3}{5}$, $\cos\beta = \dfrac{5}{13}$, then find $\sin(\alpha + \beta)$ **Answer** $-\dfrac{33}{65}, \dfrac{63}{65}$

8. Find the value of $\sin 105^\circ$ **Answer** $\dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

9. Prove that $1 + \tan A \tan\dfrac{A}{2} = \tan A \cot\dfrac{A}{2} - 1 = \sec A$

6. Factorisation of the Sum or Difference of Two Sines or Cosines:

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (b) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (d) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Solved Example # 8

Prove that $\sin 5A + \sin 3A = 2\sin 4A \cos A$

Solution

L.H.S. $\sin 5A + \sin 3A = 2\sin 4A \cos A$ = R.H.S.

$[\because \sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}]$

Solved Example # 9

Find the value of $2\sin 3\theta\cos\theta - \sin 4\theta - \sin 2\theta$

Solution

$2\sin 3\theta\cos\theta - \sin 4\theta - \sin 2\theta = 2\sin 3\theta\cos\theta - [2\sin 3\theta\cos\theta] = 0$

Self Practice Problem

10. Proved that

(i) $\cos 8x - \cos 5x = -2 \sin \frac{13x}{2} \sin \frac{3x}{2}$ (ii) $\frac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot \frac{A}{2}$

(iii) $\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

(iv) $\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$

(v) $\frac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$

7. Transformation of Products into Sum or Difference of Sines & Cosines:

(a) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$ (b) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(c) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$ (d) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

Solved Example # 10

Prove that

(i) $\frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

(ii) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4 \cos 2\theta \cos 4\theta$

Solution

(i) $\frac{2\sin 8\theta \cos\theta - 2\sin 6\theta \cos 3\theta}{2\cos 2\theta \cos\theta - 2\sin 3\theta \sin 4\theta}$

$= \frac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos\theta - \cos\theta + \cos 7\theta} = \frac{2\sin 2\theta \cos 5\theta}{2\cos 5\theta \cos 2\theta} = \tan 2\theta$

(ii) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = \frac{\sin 5\theta \cos 3\theta + \sin 3\theta \cos 5\theta}{\sin 5\theta \cos 3\theta - \sin 3\theta \cos 5\theta} = \frac{\sin 8\theta}{\sin 2\theta} = 4 \cos 2\theta \cos 4\theta$

Self Practice Problem

11. Prove that $\sin \frac{\theta}{2} \sin \frac{7\theta}{2} + \sin \frac{3\theta}{2} \sin \frac{11\theta}{2} = \sin 2\theta \sin 5\theta$

12. Prove that $\cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0$

13. Prove that $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$

8. Multiple and Sub-multiple Angles :

(a) $\sin 2A = 2 \sin A \cos A$; $\sin\theta = 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2 \sin^2 A$; $2 \cos^2 \frac{\theta}{2} = 1 + \cos\theta$, $2 \sin^2 \frac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$; $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$, $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$ (f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Solved Example # 11

Prove that

(i) $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

(ii) $\tan A + \cot A = 2 \operatorname{cosec} 2A$

(iii) $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}$

Solution

(i) L.H.S. $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{2\cos^2 A} = \tan A$

(ii) L.H.S. $\tan A + \cot A = \dfrac{1+\tan^2 A}{\tan A} = 2\left(\dfrac{1+\tan^2 A}{2\tan A}\right) = \dfrac{2}{\sin 2A} = 2\operatorname{cosec} 2A$

(iii) L.H.S. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}$

$$= \frac{2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\sin\left(\frac{A}{2}+B\right)}{2\cos^2\frac{A}{2} - 2\cos\frac{A}{2}\cos\left(\frac{A}{2}+B\right)}$$

$$= \tan\frac{A}{2}\left[\frac{\sin\frac{A}{2}+\sin\left(\frac{A}{2}+B\right)}{\cos\frac{A}{2}-\cos\left(\frac{A}{2}+B\right)}\right] = \tan\frac{A}{2}\left[\frac{2\sin\frac{A+B}{2}\cos\left(\frac{B}{2}\right)}{2\sin\frac{A+B}{2}\sin\left(\frac{B}{2}\right)}\right]$$

$$= \tan\frac{A}{2}\cot\frac{B}{2}$$

Self Practice Problem

14. Prove that $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

15. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$

16. Prove that $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

17. Prove that $\tan\left(45^\circ + \dfrac{A}{2}\right) = \sec A + \tan A$

9. Important Trigonometric Ratios:

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(c) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

10. Conditional Identities:

If $A + B + C = \pi$ then :

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\tan\frac{A}{2} \tan\frac{B}{2} + \tan\frac{B}{2} \tan\frac{C}{2} + \tan\frac{C}{2} \tan\frac{A}{2} = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(ix) $A + B + C = \frac{\pi}{2}$ then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

Solved Example # 12

If $A + B + C = 180°$, Prove that, $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$.

Solution.

Let $S = \sin^2 A + \sin^2 B + \sin^2 C$

so that $2S = 2\sin^2 A + 1 - \cos 2B + 1 - \cos 2C$

$= 2 \sin^2 A + 2 - 2\cos(B + C) \cos(B - C)$

$= 2 - 2 \cos^2 A + 2 - 2\cos(B + C) \cos(B - C)$

$\therefore$ $S = 2 + \cos A [\cos(B - C) + \cos(B+ C)]$

since $\cos A = -\cos(B+C)$

$\therefore$ $S = 2 + 2 \cos A \cos B \cos C$

Solved Example # 13

If $x + y + z = xyz$, Prove that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} . \frac{2y}{1-y^2} . \frac{2z}{1-z^2}$.

Solution.

Put $x = \tan A$, $y = \tan B$ and $z = \tan C$,

so that we have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Rightarrow A + B + C = n\pi$, where $n \in I$

Hence

L.H.S.

$\therefore \quad \dfrac{2x}{1-x^2} + \dfrac{2y}{1-y^2} + \dfrac{2z}{1-z^2} = \dfrac{2\tan A}{1-\tan^2 A} + \dfrac{2\tan B}{1-\tan^2 B} + \dfrac{2\tan C}{1-\tan^2 C}$.

$= \tan 2A + \tan 2B + \tan 2C \qquad [\because A + B + C = n\pi]$

$= \tan 2A \tan 2B \tan 2C$

$= \dfrac{2x}{1-x^2} \cdot \dfrac{2y}{1-y^2} \cdot \dfrac{2z}{1-z^2}$

Self Practice Problem

18. If A + B + C = 180°, prove that

(i) $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin \dfrac{B-C}{2} \sin \dfrac{C-A}{2} \sin \dfrac{A-B}{2}$

(ii) $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$.

19. If A + B + C = 2S, prove that

(i) $\sin(S - A) \sin(S - B) + \sin S \sin(S - C) = \sin A \sin B$.

(ii) $\sin(S - A) + \sin(S - B) + \sin(S - C) - \sin S = 4\sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$.

11. Range of Trigonometric Expression:

$E = a \sin\theta + b \cos\theta$

$E = \sqrt{a^2+b^2} \sin(\theta + \alpha)$, where $\tan\alpha = \dfrac{b}{a}$

$= \sqrt{a^2+b^2} \cos(\theta - \beta)$, where $\tan\beta = \dfrac{a}{b}$

Hence for any real value of θ, $-\sqrt{a^2+b^2} \le E \le \sqrt{a^2+b^2}$

Solved Example # 14

Find maximum and minimum values of following :

(i) 3sinx + 4cosx

(ii) $1 + 2\sin x + 3\cos^2 x$

Solution.

(i) We know

$-\sqrt{3^2+4^2} \le 3\sin x + 4\cos x \le \sqrt{3^2+4^2}$

$-5 \le 3\sin x + 4\cos x \le 5$

(ii) $1 + 2\sin x + 3\cos^2 x$

$= -3\sin^2 x + 2\sin x + 4$

$= -3\left(\sin^2 x - \dfrac{2\sin x}{3}\right) + 4$

$= -3\left(\sin x - \dfrac{1}{3}\right)^2 + \dfrac{13}{3}$

Now $\quad 0 \le \left(\sin x - \dfrac{1}{3}\right)^2 \le \dfrac{16}{9}$

$\Rightarrow \quad -\dfrac{16}{3} \le -3\left(\sin x - \dfrac{1}{3}\right)^2 \le 0$

$$-1 \le -3\left(\sin x - \frac{1}{3}\right)^2 + \frac{13}{3} \le \frac{13}{3}$$

Self Practice Problem

20. Find maximum and minimum values of following

(i) $3 + (\sin x - 2)^2$ **Answer** max = 12, min = 4.

(ii) $10\cos^2 x - 6\sin x \cos x + 2\sin^2 x$ **Answer** max = 11, min = 1.

(iii) $\cos\theta + 3\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) + 6$ **Answer** max = 11, min = 1

12. Sine and Cosine Series:

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

Solved Example # 15

Find the summation of the following

(i) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(ii) $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$

(iii) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

Solution.

(i) $$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \frac{\cos\frac{\left(\frac{2\pi}{7}+\frac{6\pi}{7}\right)}{2}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$$

$$= \frac{\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$$

$$= \frac{-\cos\frac{3\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$$

$$= -\frac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}} = -\frac{1}{2}$$

(ii) $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$

$$= \frac{\cos\left(\frac{\frac{\pi}{7}+\frac{6\pi}{7}}{2}\right)\sin\frac{6\pi}{14}}{\sin\frac{\pi}{14}} = \frac{\cos\frac{\pi}{2}\sin\frac{6\pi}{14}}{\sin\frac{\pi}{14}} = 0$$

(iii) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

$$= \frac{\cos\frac{10\pi}{22}\sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}} = \frac{\sin\frac{10\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{1}{2}$$

Self Practice Problem

Find sum of the following series :

21. $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots\ldots + \text{to n terms.}$ **Answer** $\frac{1}{2}$

22. $\sin 2\alpha + \sin 3\alpha + \sin 4\alpha + \ldots.. + \sin n\alpha$, where $(n + 2)\alpha = 2\pi$ **Answer** 0.