PERMUTATION AND COMBINATION

The notion of permutation relates to the act of arranging all the members of a set into some sequence or order, or if the set is already ordered, REARRANGING (REORDER) its elements, a process called PERMUTING

A combination is a way of selecting items from a collection, such that (unlike Permutation) the order of selection does not matter.

SO BASICALLY IT'S COUNTING!

OH! IT MEANS IT'S GOING TO BE AN EASY RIDE. I KNOW HOW TO COUNT!

REALLY??? ARE YOU SURE?

TWO PRINCIPLES APPLY TO COUNTING

The Rule of Sum

According to the rule of Sum if I own 5 long-shirts and 3 short-shirts the no. of shirts I can wear any day is $5+3$.

The Rule of PRODUCT

According to the rule of Product, If I own 8 shirts and 5 pants, the no. of outfits I can wear on any given day is 8×5.
AND if I have 10 ties, then the no. of different outfits $8 \times 5 \times 10$.

BUT THE QUESTION IS ...

WHEN TO ADD & WHEN TO MULTIPLY

Let's start with a Simple Question.

In how many ways,

letter A and B can be arranged

AB BA

A, B and C can be arranged.

ABC ACB BCA BAC CBA CAB

A, B, C and D

DADBDCD DADCDBD

⇓

OH I CAN SEE I can Fit D at 4 places by Considering ABC from last order

⇓

Oh I can see the same thing

Ahaan! WHEN

TWO THINGS → 2×1

THREE THINGS → $3 \times 2 \times 1$

FOUR THINGS → $4 \times 3 \times 2 \times 1$

WELL! THEN I CAN SAY

10 things $= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

AHUFFF! I'VE TO KEEP MULTIPLYING??

IT'S TIRING!!!

- WELL DO NOT WORRY WE CAN WORK IT OUT ON THIS ...

REALLY??? HOW???

- I AM GOING TO CALL THEM FACTORIALS!

FACTORIALS!!!!!!!! WHAT IS THAT!!!!

- !!!!!!! !!!!!! ← It's a cool symbol

!!! THE EXCLAMATION!!!

COME ON!! DON'T PLAY RIDDLE AND EDUCATE ME ABOUT FACTORIALS!!!!

- OKAY! HERE IT IS

$1! = 1$, $2! = 2\times 1$, $3! = 3\times 2\times 1$,

$4! = 4\times 3\times 2\times 1$, $5! = 5\times 4\times 3\times 2\times 1$, ...

$$n! = n\times(n-1)\times(n-2)\times\cdots\times 2\times 1$$

SO YOU WANT TO SAY

n Factorials i.e. n! is nothing but product of FIRST n NATURAL NUMBERS!!!!

- YES!!! BUT I GOTTA TELL YOU ONE MORE THING!!!!

0! oh it means the product of first zero natural number. I know the ANSWER IT'S GOING TO BE ~~0~~... RI~~GH~~T ?

WRONG !!! WHEN YOU ARE NOT MULTIPLY-ING ANY NUMBER IT DOESN'T MEAN ZERO BUT IT MEANS ONE!

I AM GETTING NO CLUE WHAT ARE YOU TALKING ABOUT?

WELL LET ME BE MORE CLEAR, IN CASE OF SUMMATION WHEN YOU'RE ADDING NOTHING, YOU'RE ADDING ZERO AND THAT IS ADDITIVE IDENTITY.

AND DO YOU REMEMBER WHAT WAS MULTIPLICATIVE IDENTITY? THAT WAS ONE! Let's say you take one number let's say, 3 and you multiply 3 by itself 5 times and divide the product by 3, 5 times! WELL THE NET NUMBER OF TIMES YOU MULTIPLY 3 BY ITSELF IS ZERO, WHICH MEANS...

$$\left[\left(\left[\left\{(3\times3\times3\times3\times3)\div3\right\}\div3\right]\div3\right)\div3\right]\div3$$

$$= \text{ONE}$$

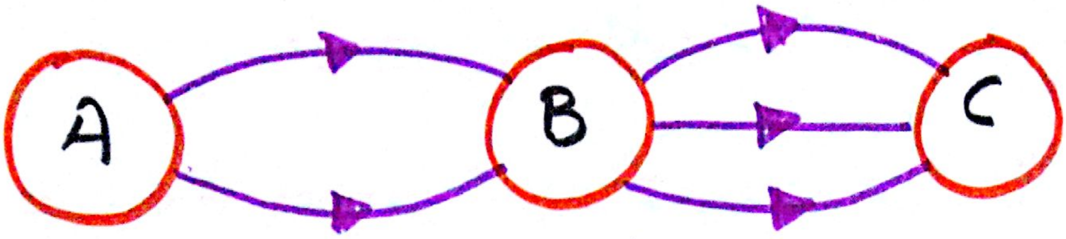
AND HENCE WE CAN SAY WHEN WE ARE MULTIPLYING NO NATURAL NUMBERS i.e. 0! WHICH MEANS ONE !!!

AND HENCE WE CAN SAY

$$0! = 1$$

WELL EVERY THING IS GOING SMOOTH BUT WE STILL DIDN'T ANSWER ~~THE~~ OUR 1st QUES. WHEN TO ADD AND WHEN TO MULTIPLY ?

Hmmm! Let's solve this question first! Hope this question will answer our Question !!!!????



Suppose I've to GO FROM CITY A TO CITY C. THER'S NO DIRECT ROAD. YOU'VE TO GO VIA B.

2 ROADS CONNECT A and B.

3 ROADS CONNECT B and C

HOW MANY WAYS YOU CAN REACH!

WELL! THIS IS EASY!

FROM A to B → 2 AND B to C → 3

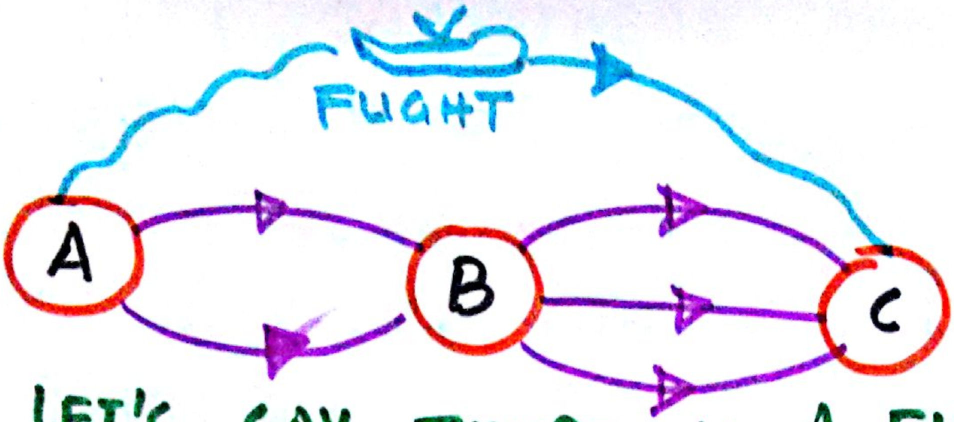
WHICH MEANS 2 CHOICES AND 3 CHOICES

WHICH MEANS $2 \times 3 = 6$

DID YOU REPLACE AND by ×

YES YOU GOT IT!

LET ME DO LITTLE CHANGES IN THE QUESTION.



NOW LET'S SAY THERE IS A FLIGHT TOO!

SO I HAVE TWO CASES

EITHER ROADWAYS OR AIRWAY

IN ROADWAYS (2 AND 3) OR 1 CHOICE IN AIRWAY

$= (2 \times 3) + 1 = 7$

DID YOU REPLACE OR BY +

YES! SO HERE WE ARE!!!

IN CASES WE ADD

WHEN WE USE EITHER/OR WE ADD

IN STAGES WE MULTIPLY

WHEN WE USE AND/THEN WE ADD

NOW LET'S SAY THERE ARE SIX STUDENTS - A, B, C, D, E & F. I'VE TO CHOOSE SOME STUDENTS OUT OF THEM! LET'S SEE HOW MANY CHOICES WE HAVE?

IF I'VE TO CHOOSE 1 student

$^6C_1 \rightarrow$ From 6 choose One

$^6C_1 = 6$, $^6C_2 = \frac{6\times5}{2}$?????

let me help you!

IF YOU'RE CHOOSING TWO

FIRST YOU CHOOSE THE 1ST MEMBER AND THE SECOND MEMBER

SO YOU'VE CHOSEN 1st member from 6 CHOICES AND 2nd member from rest of 5 CHOICES

WHICH MEANS $\rightarrow$ 6×5 BUT WHY YOU DIVIDED THE PRODUCT BY 2.

SINCE WE ARE NOT CONSIDERING THE ORDER.

ORDER???

WELL LET'S SAY I CHOSE FIRST D and 2nd F. IT'S ONE COMBINATION AND I COUNTED IT ONE WAY.

NOW LET'S SAY I CHOSE FIRST F and 2nd D. IT'S THE SAME COMBINATION BUT WE ~~ADDED~~ COUNTED IT TWICE! AND TO CORRECT THAT ~~WE~~ SINCE WE COUNTED EVERY PAIR TWICE HENCE

$$^6C_2 = \frac{6\times5}{2\times1}$$ And similarly

$$^6C_3 = \frac{6\times5\times4}{3\times2\times1}$$

Since the ABC, ACB, BCA, BAC, CBA, CAB Counted As DIFFERENT COMBINATION BUT THEY ARE NOT

$$^6C_4 = \frac{6\times5\times4\times3}{4\times3\times2\times1}$$

$$^6C_5 = \frac{6\times5\times4\times3\times2}{5\times4\times3\times2\times1}$$

$$^6C_6 = 1$$

BUT WHAT ABOUT ${}^6C_0$

WELL I DIDN'T CHOOSE ANY ONE AND THAT IS ALSO ONE WAY!

HENCE ${}^6C_0 = 1$.

OKAY! THAT'S COOL! SO WE CAN SAY:

$${}^6C_2 = \frac{6\times5}{2\times1}\times\frac{4!}{4!} = \frac{6!}{2!\,4!}$$

$${}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = \frac{6\times5\times4}{3!}\times\frac{3!}{3!} = \frac{6!}{3!\,3!}$$

$${}^6C_4 = \frac{6\times5\times4\times3}{4!} = \frac{6\times5\times4\times3\times2!}{4!\,2!} = \frac{6!}{2!\,4!}$$

$${}^6C_5 = \frac{6!}{5!\,1!} \qquad {}^6C_6 = \frac{6!}{6!\,0!} \qquad {}^6C_0 = \frac{6!}{0!\,6!}$$

$${}^6C_1 = \frac{6!}{5!\,1!}$$

HOPE YOU GOT THE EVIDENCE THAT 0! IS ONE.

SO NOW WE CAN DEFINE

${}^nC_r$ from n Choose r

So, $^nC_r = \frac{n!}{r!(n-r)!}$ And it means total no. of ways you choose r objects from n objects !!!

QUES: DID YOU NOTICE $^nC_r = {^nC_{n-r}}$??

WELL YES AND IT MAKES SENSE

IF YOU CHOOSE 2 FROM 6 YOU'RE ALSO LEAVING A GROUP OF 4.

HENCE $^6C_2 = {^6C_4}$

AND THUS $^nC_r = {^nC_{n-r}}$

And we can also show this with the calculation

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$\Rightarrow \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,r!} \Rightarrow \text{LHS} = \text{RHS}.$$

Q.E.D.

QUES. Show that ${}^{n}C_{r} \times r = n \times {}^{n-1}C_{r-1}$

AND REMEMBER IT!

Sol: ${}^{n}C_{r} \cdot r = n \times {}^{n-1}C_{r-1}$

$$\Rightarrow \frac{n!}{r!\,(n-r)!} \times r = \frac{n \times (n-1)!}{(r-1)!\,(n-1-r+1)!}$$

$$\Rightarrow \frac{n!}{(r-1)!\,(n-r)!} = \frac{n!}{(r-1)!\,(n-r)!} \Rightarrow \text{LHS} = \text{RHS}$$

BUT HOW TO KEEP THE RESULT IN MIND WELL I KNOW ANOTHER AMAZING PROOF VIA EXAMPLE AND IT IS EASY AND MORE RELATABLE TO KEEP IN MIND INSTEAD OF PUZZLING WITH VARIOUS ${}^{n}C_{r}$s.

CONSIDER A CLASS OF 50 students YOU HAVE TO CHOOSE A CRICKET TEAM AND A CAPTAIN.

Sol. SELECTION OF A CRICKET TEAM AND A CAPTAIN = SELECTION OF A CAPTAIN AND REST OF THE TEAM

$$\Rightarrow {}^{50}C_{11} \times {}^{11}C_1 = {}^{50}C_1 \times {}^{49}C_{10}$$

$$\Rightarrow {}^{50}C_{11} \times 11 = 50 \times {}^{49}C_{10}.$$

$$\Rightarrow {}^{50}C_{11} \times 11 = 50 \times {}^{50-1}C_{11-1}$$

$$\Rightarrow {}^{n}C_{r} \times r = n \times {}^{n-1}C_{r-1}$$

Q.E.D.

Ques: Show that ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

AND REMEMBER IT!!!

Sol: $^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

$$\Rightarrow \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r+1)!\,(n-r-1)!} = {}^{n+1}C_{r+1}$$

$$\Rightarrow \frac{n!}{r!}\left(\frac{1}{(n-r)!} + \frac{1}{(r+1)\,(n-r-1)!}\right) = {}^{n+1}C_{r+1}$$

$$\Rightarrow \frac{n!}{r!\,(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r+1}\right) = {}^{n+1}C_{r+1}$$

$$\Rightarrow \frac{n!}{r!\,(n-r-1)!}\cdot\frac{r+1+n-r}{(n-r)(r+1)} = {}^{n+1}C_{r+1}$$

$$\Rightarrow \frac{(n+1)!}{(r+1)!\,(n-r)!} = {}^{n+1}C_{r+1}$$

Oh! this was messy and how I am going to keep this result in my mind!!!

WELL Don't Worry I'VE EASIER WAY TO PROVE AND KEEP IN MIND!!!

SELECTION OF A CRICKET TEAM FROM A CLASS OF FIFTY STUDENTS. AND I AM GOING TO HIGHLIGHT ONE STUDENT FROM THE CLASS LET'S CALL HIM JACK! I'VE TWO Options of creating team or two Cases of creating team of 11. Either Jack in team OR Jack not in team.

JACK IN TEAM OR JACK NOT IN TEAM = TOTAL POSSIBLE WAYS TO SELECT A TEAM

$$^{49}C_{10} + {}^{49}C_{11} = {}^{50}C_{11}$$

$$^{49}C_{10} + {}^{49}C_{10+1} = {}^{49+1}C_{10+1}$$

$$^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$$

AHAAN! THIS IS SO EASY PEASY!

SUMMARY WHICH CAN BE KEPT IN MIND

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

Choose r from n.

$$^{n}C_{r} = {}^{n}C_{n-r}$$

$$^{n}C_{r} \times r = n\,{}^{n-1}C_{r-1}$$

Selection of a CRICKET TEAM & A CAPTAIN

$$^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$$

JACK IN TEAM

JACK NOT IN TEAM!

THREE TECHNIQUES OF COUNTING

SELECTION TECHNIQUE

SHUFFLING TECHNIQUE

BOX AND LETTERS TECHNIQUE

OKAY! LET'S SEE INDIVIDUALLY WHAT ARE THESE TECHNIQUES!!!

SELECTION TECHNIQUE :

Selection of objects from a collection of

DISTINCT OBJECTS
LET'S SAY
A, B, C, D, E, F, G

Let's Say
Select 3
Ans: $^7C_3$

IDENTICAL OBJECTS
LET'S SAY
A, A, A, A, A

Let's Say
Select 4
Ans: 1

MIXED COLLECTION
A, A, B, B, B,
C, C, D, E, F

PTO for mixed bag

Let's say we have to select 3 from

A, A, B, B, B, C, C, D, E, F

⇒ We'll have to categorize the selection in terms of like and distinct

WHAT DOES THAT MEAN?

I'll tell you! TAKE THE CASES!

CASE I : ALL DISTINCT

WE'VE TOTAL 6 DISTINCT i.e. A, B, C, D, E, F, HENCE $^6C_3$

CASE II : 2 IDENTICAL & 1 DISTINCT

FOR SELECTING 2 IDENTICALS I'VE CHOICES {A, A} {B, B} {C, C}

AND THE LAST DISTINCT CAN BE CHOSEN FROM THE REMAINING 5

i.e. $^3C_1$ AND $^5C_1$ = $^3C_1 \times {}^5C_1$

CASE III : ALL IDENTICAL

WE'VE ONLY ONE CHOICE {B, B, B}

HENCE 1

SHUFFLING TECHNIQUE

Shuffling of objects from a collection of

DISTICNT OBJECTS, SAY, A, B, C, D, E

Shufflin Chambers

$5 \times 4 \times 3 \times 2 \times 1$

$= 5!$

IDENTICAL OBJECTS, SAY, A, A, A, A

Shuffling Chambers

1

MIXED OBJECTS, SAY, A, A, B, C, D, D, D, E, F

Shuffling of A, A, B, C, D, D, D, E, F

Considering all 9 as distinct

$9!$

BUT A, A can shuffle 2! with each other without breaking the patter and similarly 3 Ds will shuffle 3! and hence

$$\frac{9!}{2!\,3!}$$

BOX AND LETTERS TECHNIQUE

IT IS ABOUT COUNTING THE NO. OF WAYS WHEN THE RELATION BETWEEN SHAREABLE AND UNSHARABLE IS IN THE QUESTION! WELL WHAT DOES THAT MEAN?

Let's Say when you've to Post 4 letters And there are 4 Letter Boxes

SO EACH LETTER HAS A CHOICE OF 4 BOXES HENCE $4 \times 4 \times 4 \times 4 = 4^4$

BUT WHAT ABOUT 3 LETTERS AND 4 LETTER BOXES!

3^4 OR 4^3 CONFUSED!!!

WELL HERE WHICH IS SHAREABLE? LETTER OR LETTER BOXES?

OFCOURSE THE LETTER BOXES CAN BE SHARED BY MORE THAN ONE LETTER.

A letter can't be shared! Why?

OFcourse One letter cannot go to two boxes at a time.

Hence letters are non-shareable and hence it has the power to choose. Therefore each letter has 4 choices.

Hence the ans is $4 \times 4 \times 4 = 4^3$

SO HERE I CONCLUDE WHENEVER THERE IS COUNTING OF THINGS WHICH ARE OF TYPE

SHAREABLE VS NON-SHAREABLE

THE NO. OF WAYS IS

$(\text{SHAREABLE})^{\text{UNSHAREABLE}}$

EXERCISE: #1

① Find the Number of Ways to Select 4 letters from the Word SLEEPLESSNESS.

② How many 4 letter Words can be formed from the letters of SLEEPLESSNESS.

③ 4 Boys and 4 Girls, Find the number of ways to make a team of 3 such that atleast one of them is a girl.

④ 5 choclates to be distributed among 6 boys such that no choclate can be shared by 2 or more boys. Find the number of ways the choclates can be distributed.

⑤ A train Comes at a Railway Platform with 3 compartments. 10 passengers are waiting to board.

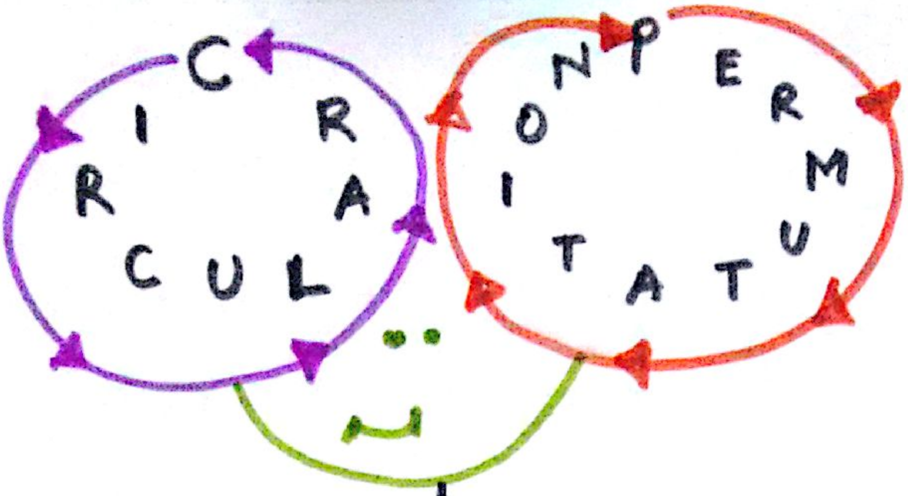
Ⓐ Find the no. of ways they can board

Ⓑ Find the no. of ways they can board if it is not necessary that all will board.

Ⓒ Find the no. of ways if it is not necessary that all compartments will be filled.

Ⓓ IF in one Compartment only one will board.

Ⓔ IF in I compartment 2 boarded, II Comp. 2 boarded and rest in III compartment.



ARRANGE
A, B, C, D, E, F, G
IN A CIRCLE

BREAK THE SYMMETRY BY ANY OF A, B, D, E, F, G, C AND THEN ARRANGE REST
HENCE 5!

ARRANGE
A, A, A, A, A, A, A
IN A CIRCLE

THE ONE

ARRANGE
G, G, G, N, N, N, P, H, H, B, B
IN A CIRCLE

ONLY P CAN BREAK THE SYMMETRY OF CIRCLE AND REST WILL ARRANGE IN $\frac{10!}{3!3!2!2!}$

WELL IN THE THIRD CASE OF CIRCULAR PERMUTATION P SAVED US BY BREAKING THE SYMMETRY!

BUT WHAT IF P WASN'T PRESENT?

HAVE LITTLE PATIENCE WE'LL DISCUSS THAT CASE IN PASCAL NUMBER WHICH WILL COME LATER!

CHALLENGE???

13 GUESTS WERE INVITED TO A DINNER PARTY IN A HALL WHICH HAS TWO ROUND TABLES. SUCH THAT IN ONE 7 GUESTS WILL BE SITTING AND REMAINING 6 ON THE OTHER. FIND THE NUMBER OF WAYS THE SITTING ARRANGEMENT CAN BE DONE.

When Number Of Objects To Be Selected Is not Specified

A, B, C, D, E — Distinct Objects

Total no. of Selection

= Zero object Selected + 1 Selected + 2 selected + ... + All Selected

$= {}^5C_0 + {}^5C_1 + {}^5C_2 + \dots + {}^5C_5 = 2^5$

A, A, A, A, A — Identical Objects

Total no. of Selection

= Zero A + 1 A + 2A + ... + 5A

= 1+1+1+1+1+1

= 6

Mixed Case — A, A, A, B, C, D

Selecting any number of A and selecting any number of B and C and D and E

$= 4 \times 2 \times 2 \times 2$

REMARK :

#1 For n distinct objects, if number of objects to be selected is not specified, then total no. of ways of selection

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

#2 For n identical objects, if no. of objects to be selected is not specified, then total no. of ways of selection

= No selection + 1 selection + 2 obj selection + . . . + n obj. selection

$$= 1 + 1 + 1 + \dots + 1 = n$$

EXERCISE :: #2

1. A child has 5 Sandwiches and 5 choclates in his lunchbox. In how many ways I can eat his lunchbox. Sandwiches are DISTINCT and choclates are IDENTICAL.

2. CONSIDER A FRUIT BASKET CONTAINING 10 DISTINCT MANGOES, 10 DISTINCT APPLES, 10 IDENTICAL BANANAS, 10 IDENTICAL GRAPES.

Find:- (a) no. of ways to pick any no. of fruit from the basket.

(b) no. of ways to pick atleast one fruit from the basket.

(c) no. of ways to pick atleast one fruit of each kind from the basket.

(d) no. of ways to pick exactly one fruit.

(e) no. of ways to pick exactly two fruits.

(f) no. of ways to pick exactly one type of flavour irrespective of number of fruits.

(g) no. of ways to pick odd no. of fruits of each type.

(h) no. of ways to pick any no. of fruit, but no. of mango should be twice of no. of grapes.

(i) no. of ways to pick maximum 2 fruits.