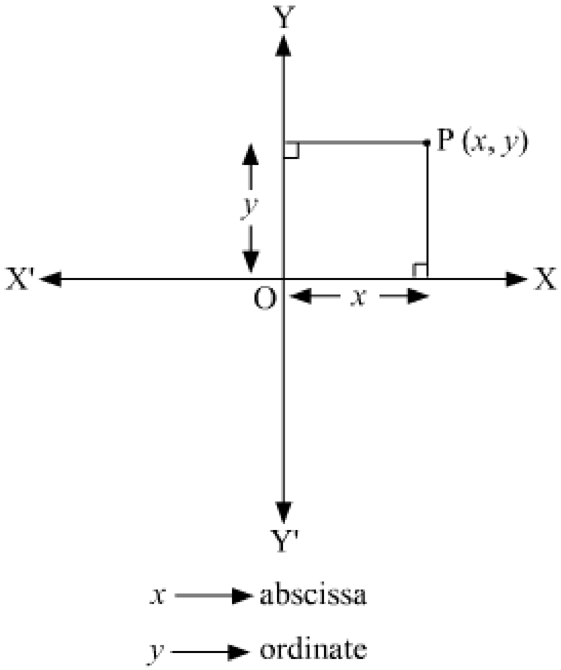
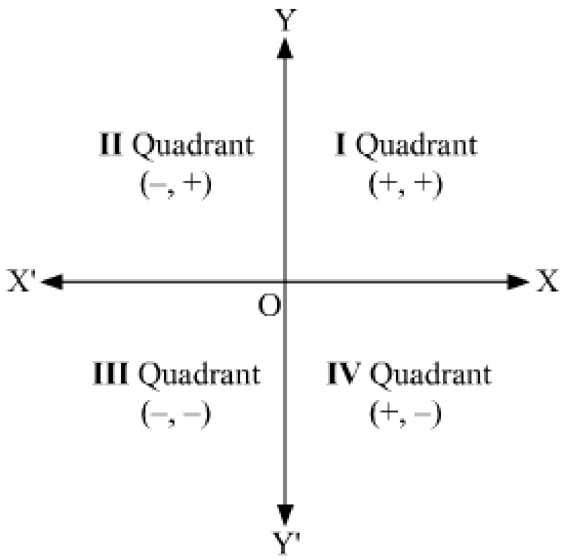
Straight Lines

Basics of Two-Dimensional Geometry

- A plane containing two mutually perpendicular lines (x-axis and y-axis) is known as the Cartesian plane. For any point on the Cartesian plane, the distance of the point from the y-axis is called the ***x*-coordinate or abscissa** and the distance from the y-axis is called the ***y*-coordinate or ordinate**.



- The coordinate axes divide the plane into four regions, i.e., XOY, X′OY, X′O Y′ and XOY′, which are known as the first, second, third and fourth quadrants, respectively.



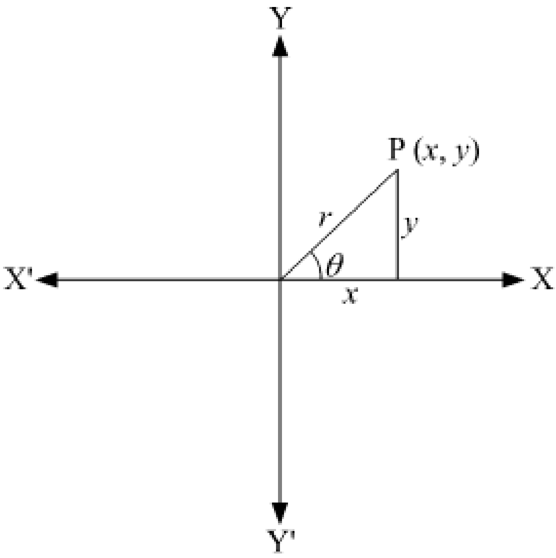
Consider the figure.

- OX is the positive x-axis and OX′ is the negative x-axis.

- OY is the positive *y*-axis and OY′ is the negative *y*-axis.

  For the point (*x*, *y*) lying in

- I quadrant: $x > 0$ and $y > 0$
- II quadrant: $x < 0$ and $y > 0$
- III quadrant: $x < 0$ and $y < 0$
- IV quadrant: $x > 0$ and $y < 0$
- The polar coordinates of a point P (*x*, *y*) on the Cartesian plane are ($r\sin\theta$, $r\cos\theta$), where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{x}{y}$.



- **Distance Formula**

The distance between any two points on the Cartesian plane can be evaluated using the distance formula. The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

- **Section Formula**

In many real-life situations, we need to divide a particular line segment in a given ratio. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the plane and P be a point on the segment joining A and B such that AP:PB = *m*:*n*. Then, the coordinates of the point P are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

If P is the midpoint of AB, then it divides AB in the ratio 1:1. Thus, its coordinates are $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$.

- **Properties of Centroid, Orthocentre, Incentre and Circumcentre of Triangle**

Consider a ΔABC whose vertices are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) and the lengths of the sides BC, CA and AB are a, b and c, respectively.

Coordinates of the centroid (G) = $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Coordinates of the orthocentre (H) = $\left(\frac{x_1 \tan \mathrm{A} + x_2 \tan B + x_3 \tan C}{\tan \mathrm{A} + \tan \mathrm{B} + \tan \mathrm{C}}, \frac{y_1 \tan \mathrm{A} + y_2 \tan \mathrm{B} + y_3 \tan \mathrm{C}}{\tan \mathrm{A} + \tan \mathrm{B} + \tan \mathrm{C}}\right)$

Coordinates of the incentre (I) = $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

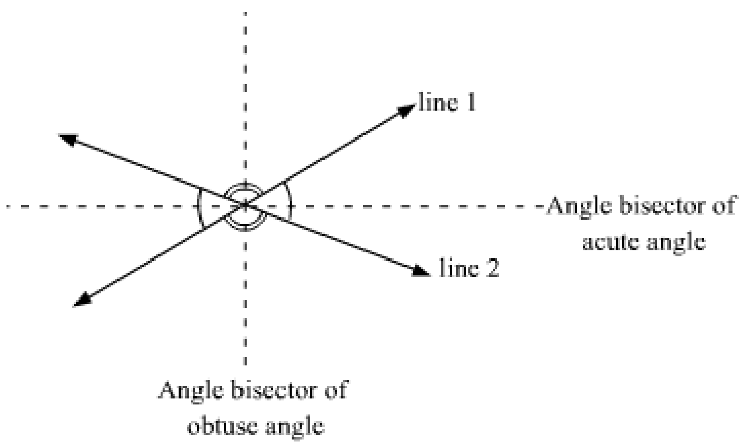
Coordinates of the circumcentre (O) =
$\left(\frac{x_1 \sin 2\mathrm{A} + x_2 \sin 2\mathrm{B} + x_3 \sin 2\mathrm{C}}{\sin 2\mathrm{A} + \sin 2\mathrm{B} + \sin 2\mathrm{C}}, \frac{y_1 \sin 2\mathrm{A} + y_2 \sin 2\mathrm{B} + y_3 \sin 2\mathrm{C}}{\sin 2\mathrm{A} + \sin 2\mathrm{B} + \sin 2\mathrm{C}}\right)$

- The circumcentre (O), centroid (G) and orthocentre (H) of a triangle are collinear.
- The centroid (G) divides the line joining the circumcentre (O) and the orthocentre (H) in the ratio 1:2.
- The area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is
  $\frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3+y_1)+x_3(y_1-y_2)\right|$.

This can also be written as $\left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$.

Angle Bisectors of Two Intersecting Lines

Two intersecting lines make four angles or two pairs of equal angles that are vertically opposite each other. These angles can be acute, obtuse or right angles. Since there are two different angles between the lines, there exist two angle bisectors.



Equation of bisectors of angle between two intersecting lines

Let us suppose the equations of the two lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.

- When $a_1a_2+b_1b_2 < 0$ then,

$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ represents the equation of the bisector of an acute angle.

$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ represents the equation of the bisector of an obtuse angle.

- When $a_1a_2+b_1b_2 > 0$ then,

$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ represents the equation of the bisector of an obtuse angle.

$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ represents the equation of the bisector of an acute angle.

- If $a_1a_2 + b_1b_2 = 0$, then lines are perpendicular to each other.

Equation of angle bisector when the pair of lines is given by $ax^2 + 2hxy + by^2 = 0$

The equation of the angle bisector is

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

Pair of straight Lines

- **Equation of a pair of straight lines**

The general form of an equation of second degree is:

$ax^2+2hxy+by^2+2gx+2fy+c=0$ … (1)

Equation (1) will represent a pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

$\Rightarrow abc+2fgh-af^2-bg^2-ch^2=0$

- **Quadratic form of an equation of a pair of straight lines**

The equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ can also be written as:

$by^2 + 2(hx + f)y + ax^2 + 2gx + c = 0$

By quadratic formula, we get:

$$y = \frac{-2(hx+f) \pm 2\sqrt{(hx+f)^2 - b(ax^2+2gx+c)}}{2b}$$

$$\Rightarrow y = \frac{-(hx+f) \pm \sqrt{(h^2-ab)x^2 + 2(hf-bg)x + f^2 - bc}}{b}$$

The nature of the lines can be determined w.r.t. the value of $h^2 - ab$ as follows:

Case	Nature of lines
$h^2 - ab > 0$	Pair of non-parallel lines
$h^2 - ab = 0$ (i) $f^2 - bc > 0$ (ii) $f^2 - bc = 0$ (iii) $f^2 - bc < 0$	Parallel non-coincident lines Coincident lines Imaginary lines
$h^2 - ab < 0$	Pair of imaginary lines

- When $a = b = 0$ and $h \neq 0$, then the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ reduces to $2hxy + 2gx + 2fy + c = 0$

$\Rightarrow \left(2x + \frac{2f}{h}\right)(hy + g) + c - \frac{2gf}{h} = 0$ will represent a pair of straight lines $iff\ c - \frac{2gf}{h} = 0$.

- **General form of a pair of straight lines passing through the origin**

The homogeneous equation of a pair of straight lines passing through the origin is

$ax^2 + 2hxy + by^2 = 0.$

- **Angle between the lines**

Let $y - m_1x = 0$ and $y - m_2x = 0$ be the lines represented by the general equation $ax^2 + 2hxy + by^2 = 0$.

Then, the angle θ between these lines is

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2} = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

i. If $a + b = 0$, the lines are perpendicular.

ii. If $h^2 = ab$, the lines are parallel.

Shifting of Origin and Rotation of Axes

When we shift the origin to a point or rotate the axes, the coordinates of the given point change with respect to that point or axes.

Shifting of Origin

- Let (x, y) be the coordinates of a point P with respect to the origin O (0, 0). Also, let (X, Y) be the coordinates of P with respect to the new origin $O'(a, b)$. The relation between both the coordinates of P is $x = X + a$ and $y = Y + b$.

- Let $f(x, y) = 0$ be the equation of a curve with respect to the origin O (0, 0). When the origin is shifted to the point (a, b), the equation of the curve with respect to the new coordinate system will be $f(x + a, y + b) = 0$.

- **Rotation of Axes**

- Let us consider a point P, whose coordinates are (x, y) in a two-dimensional system. When the coordinate axes are rotated through an angle θ about the origin, then the coordinates of point P are shifted to (X, Y) in the new system. The relation between the old and the new coordinates of P is

  $x = X \cos\theta - Y \sin\theta$

  and $y = X \sin\theta + Y \cos\theta$

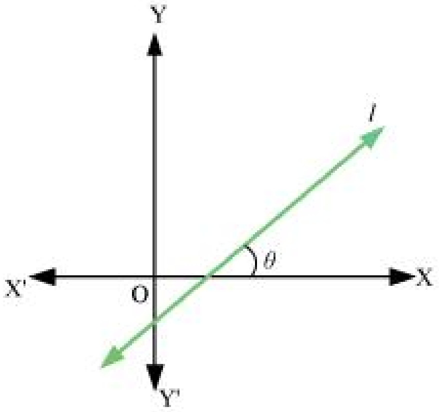
- Let us consider a curve $f(x, y) = 0$ in a two-dimensional system. When the system is rotated about the origin by an angle θ, then the equation of the curve, with respect to the new coordinate system, will be $f(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta) = 0$.

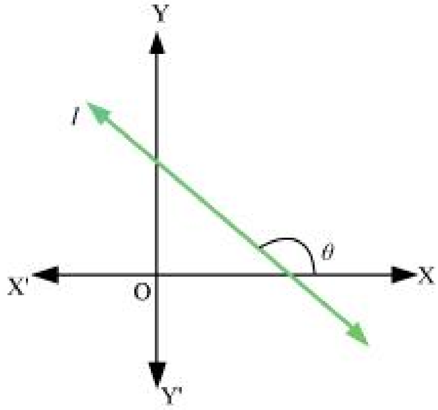
- If the axes are rotated through an angle $(-\theta)$, then the relation between the old and the new coordinates of the point P is

  $x = X \cos\theta + Y \sin\theta$

  and $y = -X \sin\theta + Y \cos\theta$

- **Slope of a line:** If θ is the inclination of a line l (the angle between positive x-axis and line l), then $m = \tan\theta$ is called the slope or gradient of line l.





- The slope of a line whose inclination is 90º is not defined. Hence, the slope of the vertical line, *y*-axis is undefined.
- The slope of the horizontal line, *x*-axis is zero.

For example, the slope of a line making an angle of 135° with the positive direction of *x*-axis is $m = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

- **Slope of line passing through two given points:**

  The slope (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$.

  For example, the slope of the line joining the points (–1, 3) and (4, –2) is given by,
  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 3}{4 - (-1)} = -\frac{5}{5} = -1$

- **Conditions for parallelism and perpendicularity of lines:**

  Suppose l_1 and l_2 are non-vertical lines having slopes m_1 and m_2 respectively.

  - l_1 is parallel to l_2 if and only if $m_1 = m_2$ i.e., their slopes are equal.
  - l_1 is perpendicular to l_2 if and only if $m_1 m_2 = -1$ i.e., the product of their slopes is –1.

Example:
Find the slope of the line which makes an angle of 45° with a line of slope 3.
Solution:
Let m be the slope of the required line.

$\therefore \tan 45^\circ = \left|\frac{m-3}{1+3m}\right|$

$\Rightarrow \left|\frac{m-3}{1+3m}\right| = 1$

$\Rightarrow \left|\frac{m-3}{1+3m}\right| = \pm 1$

$\Rightarrow \frac{m-3}{1+3m} = 1 \quad \text{or} \quad \frac{m-3}{1+3m} = -1$

$\Rightarrow m-3 = 1+3m \quad \text{or} \quad m-3 = -1-3m$

$\Rightarrow -2m = 4 \quad \text{or} \quad 4m = 2$

$\Rightarrow m = -2 \quad \text{or} \quad m = \frac{1}{2}$

- **Collinearity of three points:** Three points A, B and C are collinear if and only if slope of AB = slope of BC

- **Angle between two lines:** An acute angle, θ, between line l_1 and l_2 with slopes m_1 and m_2 respectively is given by

$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|,\ 1+m_1m_2 \neq 0$.

Example 1: Two lines AB and CB, intersect at point B. The coordinates of end points are A(–4, –3), B(0, 5), and C(10, 5). Find the measures of angles between AB and CB.

Solution: Let the angle between the lines AB and BC be θ.

Slope of line $AB = \frac{5-(-3)}{0-(-4)} = \frac{8}{4} = 2$

Slope of line $BC = \frac{5-5}{10-0} = 0$

We know that the angle between two lines with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2-m_1}{1+m_2m_1}\right|$.

Therefore, tan $\theta = \left|\frac{2-0}{1+2\times 0}\right| = 2$

$\Rightarrow \theta = \tan^{-1}(2)$.

- The equation of a horizontal line at distance a from the x-axis is either $y = a$ (above x-axis) or $y = -a$ (below x-axis).
- The equation of a vertical line at distance b from the y-axis is either $x = b$ (right of y-axis) or $x = -b$ (left of y-axis).

- **Point-slope form of the equation of a line**

The point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) if and only if its coordinates satisfy the equation. This means $y - y_0 = m(x - x_0)$.

Example: Find the equation of the line passing through (4, 5) and making an angle of 120° with the positive direction of x-axis?
Solution: Slope of the line, $m = \tan 120° = \tan\left(180° - 60°\right) = -\tan 60° = -\sqrt{3}$
Equation of the required line is,

$$y - 5 = -\sqrt{3}(x - 4)$$
$$\Rightarrow y - 5 = -\sqrt{3}x + 4\sqrt{3}$$
$$\Rightarrow \sqrt{3}x + y - \left(5 + 4\sqrt{3}\right) = 0$$

- **Two-point form of the equation of a line**

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

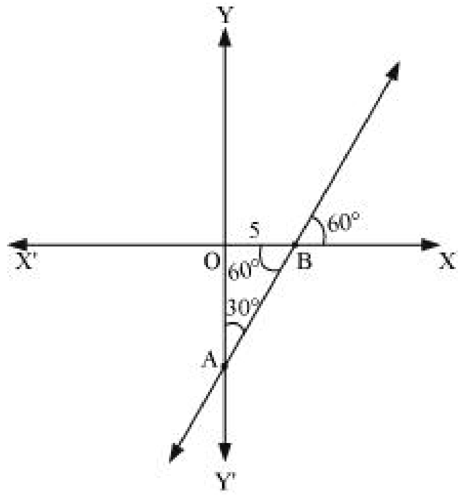
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Example: Find the equation of the line passing through the points (–5, 2) and (1, 6).
Solution: Equation of the line passing through points (–5, 2) and (1, 6) is

$$y - 2 = \frac{6-2}{1-(-5)}(x - (-5))$$
$$\Rightarrow y - 2 = \frac{4}{6}(x + 5)$$
$$\Rightarrow y - 2 = \frac{2}{3}(x + 5)$$
$$\Rightarrow 3y - 6 = 2x + 10$$
$$\Rightarrow 2x - 3y + 16 = 0$$

- **Slope-intercept form of a line**
  - The equation of the line, with slope m, which makes y-intercept c is given by $y = mx + c$.
  - The equation of the line, with slope m, which makes x-intercept d is given by $y = m(x - d)$.

Example:
Find the equation of the line which cuts off an intercept 5 on the x-axis and makes an angle of 30° with the y-axis.
Solution:



Slope of the line, $m = \tan 60° = \sqrt{3}$

OB = 5

Intercept on the x-axis, $c = –\text{OB} = –5$ and $\tan 60° = -5\sqrt{3}$

Equation of the required line is $y = \sqrt{3}x + \left(-5\sqrt{3}\right)$.

- **General equation of line**

Any equation of the form $Ax + By + C = 0$, where A and B are not zero simultaneously is called the general linear equation or general equation of line.

Slope of the line = $-\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{A}{B}$

y- intercept = $-\frac{C}{B}$

Example:

Find the slope and the y-intercept of the line $2x - 3y = –16$.

Solution:

The equation of the given line can be rewritten as $2x - 3y + 16 = 0$.

Here, $A = 2$, $B = –3$ and $C = 16$.

Slope of the line $= -\frac{A}{B} = -\frac{2}{(-3)} = \frac{2}{3}$

Intercept on the y-axis $= -\frac{C}{B} = -\frac{16}{(-3)} = \frac{16}{3}$

- **Intercept form**

The equation of the line making intercepts a and b on x-axis and y-axis respectively is

$\frac{x}{a} + \frac{y}{b} = 1$

Example:

If a line passes through (3, 2) and cuts off intercepts on the axes in such a way that the product of the intercepts is 24, then find the equation of the line.

Solution:

The equation of a line in intercept form is

$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots(1)$

Where, a and b are the intercepts on x and y axes respectively.

Since the line passes through (3, 2), we obtain

$\frac{3}{a} + \frac{2}{b} = 1$

$\Rightarrow 3b + 2a = ab$

$\Rightarrow 2a + 3b = 24 \quad \ldots(2)$ (Since product of intercepts is given as 24)
Now,
$(2a - 3b)^2 = 24ab$
$= (24)^2 - 24(24) \quad$ [From equation (2)]
$= 0$
$\therefore 2a - 3b = 0 \quad \ldots(3)$

On adding equations (2) and (3), we obtain
$4a = 24 \Rightarrow a = 6$
$\therefore 3b = 2a = 2 \times 6 = 12$
$\Rightarrow b = 4$
Hence, from (1), the required equation of line is
$\frac{x}{6} + \frac{y}{4} = 1$
$\Rightarrow 4x + 6y = 24$
$\Rightarrow 2x + 3y = 12$

- **Normal form of the equation of a line**

The equation of the line at normal distance *p* from the origin and angle *ω*, which the normal makes with the positive direction of the *x*-axis is given by $x\cos\omega + y\sin\omega = p$

Example: Reduce the equation $x - \sqrt{3y} - 6 = 0$ to normal form and hence find the length of perpendicular to the line from the origin. Also find angle between the normal and positive direction of the *x*-axis.
Solution: The given equation is $x - \sqrt{3y} - 6 = 0$.
$\Rightarrow x - \sqrt{3y} = 6 \quad \ldots(1)$
On dividing (1) by $\sqrt{\left(\sqrt{1}\right)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$, we obtain
$\frac{1}{2}x - \frac{\sqrt{2}}{2}y = 3$
$\Rightarrow x\cos 300^\circ + y\sin 300^\circ = 3 \quad \ldots (2)$

On comparing equation (2) with $x\cos\omega + y\sin\omega = p$, we obtain $\omega = 300°$ and $p = 3$

Therefore, the length of perpendicular to the line from the origin is 3 units and the angle between the normal and the positive *x*-axis is 300°.

- **Distance of a Point From a Line**

The perpendicular distance (*d*) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Example: Find the distance of point (1, –2) from the line $8x - 6y - 12 = 0$.
Solution: On comparing the equation of the given line i.e., $8x - 6y - 12 = 0$ with $Ax + By + C = 0$, we obtain
$A = 8$, $B = -6$, $C = -12$
The distance (*d*) of point (1, –2) from line $8x - 6y - 12 = 0$ is

$$d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}} = \frac{|8\times 1+(-6)(-2)+(-12)|}{\sqrt{8^2+6^2}} = \frac{|8+12-12|}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

- **Distance between parallel lines**

The distance (d) between two parallel lines i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by, $d = \frac{|C_1-C_2|}{\sqrt{A^2+B^2}}$

Example: Find the distance between the lines $4x + 3y = 11$ and $4x + 3y = 8$.
Solution: The given lines are $4x + 3y - 11 = 0$ and $4x + 3y - 8 = 0$
Slope of the line $4x + 3y - 11 = 0$ is $-\frac{4}{3}$.
Slope of the line $4x + 3y - 8 = 0$ is $-\frac{4}{3}$.
Since the slopes of the given lines are equal, the lines are parallel.
Here, $A = 4$, $B = 3$, $C_1 = -11$ and $C2 = -8$
Distance between the lines $= \left|\frac{-11-(-8)}{\sqrt{4^2+3^2}}\right| = \left|\frac{-11+8}{\sqrt{16+9}}\right| = \left|\frac{-3}{\sqrt{25}}\right| = \frac{3}{5}$