Matrices

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**. If a matrix has m rows and n columns then the **order** of matrix is said to be m by n (denoted as m × n).

The general m × n matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ...... & a_{1j} & ..... & a_{1n} \\ a_{21} & a_{22} & a_{23} & ...... & a_{2j} & ..... & a_{2n} \\ ..... & ..... & ..... & ..... & ..... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & ...... & a_{ij} & ...... & a_{in} \\ ..... & ..... & ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & a_{m3} & ..... & a_{mj} & ..... & a_{mn} \end{bmatrix}$$

where a_{ij} denote the element of i^{th} row & j^{th} column. The above matrix is usually denoted as $[a_{ij}]_{m \times n}$.

Note : (i) The elements $a_{11}, a_{22}, a_{33}, ...$ are called as **diagonal elements**. Their sum is called as **trace of A** denoted as $T_r(A)$

(ii) Capital letters of English alphabets are used to denote a matrix.

1. Basic Definitions

(i) **Row matrix :** A matrix having only one row is called as row matrix (or row vector). General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, ..., a_{1n}]$

(ii) **Column matrix :** A matrix having only one column is called as column matrix. (or column vector)

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{bmatrix}$

(iii) **Square matrix :** A matrix in which number of rows & columns are equal is called a square matrix. General form of a square matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$$

which we denote as $A = [a_{ij}]_n$.

(iv) **Zero matrix :** $A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \ \forall$ i & j.

(v) **Upper triangular matrix :**
$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for i > j (i.e., all the elements below the diagonal elements are zero).

(vi) **Lower triangular matrix :** $A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for i < j. (i.e., all the elements above the diagonal elements are zero.)

(vii) **Diagonal matrix :** A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)
Note : Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, ...a_{nn})$.

(viii) **Scalar matrix :** Scalar matrix is a diagonal matrix in which all the diagonal elements are same $A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for i = j.

(ix) **Unit matrix (Identity matrix) :**
Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).
i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(x) **Comparable matrices :** Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

(xi) **Equality of matrices :** Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.
Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$
A = B iff (i) m = p, n = q
(ii) $a_{ij} = b_{ij} \ \forall$ i & j.

(xii) **Multiplication of matrix by scalar :**
Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij} \ \forall$ i & j.
Note : If A is a scalar matrix, then $A = \lambda I$, where λ is the diagonal element.

(xiii) **Addition of matrices :** Let A and B be two matrices of same order (i.e. comparable matrices).

Then A + B is defined to be.

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$.

$= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}\ \forall$ i & j.

(xiv) Substraction of matrices : Let A & B be two matrices of same order. Then A – B is defined as A + – B where – B is (– 1) B.

(xv) Properties of addition & scalar multiplication :

Consider all matrices of order m × n, whose elements are from a set F (F denote Q, R or C).

Let $M_{m \times n}$ (F) denote the set of all such matrices.

Then

(a) $A \in M_{m \times n}(F)$ & $B \in M_{m \times n}(F) \Rightarrow A + B \in M_{m \times n}(F)$

(b) A + B = B + A

(c) (A + B) + C = A + (B + C)

(d) $O = [o]_{m \times n}$ is the additive identity.

(e) For every $A \in M_{m \times n}(F)$, – A is the additive inverse.

(f) $\lambda(A + B) = \lambda A + \lambda B$

(g) $\lambda A = A\lambda$

(h) $(\lambda_1 + \lambda_2) A = \lambda_1 A + \lambda_2 A$

(xvi) Multiplication of matrices : Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum\limits_{k=1}^{p} a_{ik}b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

Note - 1: The product AB is defined iff number of columns of A equals number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined $\not\Rightarrow$ BA is defined.

Note - 2 : In general AB ≠ BA, even when both the products are defined.

Note - 3 : A (BC) = (AB) C, whenever it is defined.

(xvii) Properties of matrix multiplication :

Consider all square matrices of order 'n'. Let M_n (F) denote the set of all square matrices of order n. (where F is Q, R or C). Then

(a) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(b) In general AB ≠ BA

(c) (AB) C = A(BC)

(d) I_n, the identity matrix of order n, is the multiplicative identity.

$AI_n = A = I_n A \quad \forall A \in M_n(F)$

(e) For every non singular matrix A (i.e., |A| ≠ 0) of M_n (F) there exist a unique (particular) matrix $B \in M_n(F)$ so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(f) If λ is a scalar $(\lambda A) B = \lambda(AB) = A(\lambda B)$.

(g) $A(B + C) = AB + AC \quad \forall A, B, C \in M_n(F)$

(h) $(A + B) C = AC + BC \quad \forall A, B, C \in M_n(F)$.

Note : (i) Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

(ii) For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Solved Example # 1

Let $A = \begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B = \begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$. Find θ so that A = B.

Solution.

By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ & $\tan\theta = -1 \Rightarrow \theta = (2n + 1)\pi - \frac{\pi}{4}$.

Solved Example # 2

f(x) is a quadratic expression such that

$$\begin{bmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(-1) \end{bmatrix} = \begin{bmatrix} 2a+1 \\ 2b+1 \\ 2c+1 \end{bmatrix}$$ for three unequal numbers a, b, c. Find f(x).

Solution. The given matrix equation implies

$$\begin{bmatrix} a^2 f(0) + af(1) + f(-1) \\ b^2 f(0) + bf(1) + f(-1) \\ c^2 f(0) + cf(1) + f(-1) \end{bmatrix} = \begin{bmatrix} 2a+1 \\ 2b+1 \\ 2c+1 \end{bmatrix}$$

$\Rightarrow$ $x^2 f(0) + xf(1) + f(-1) = 2x + 1$ for three unequal numbers a, b, c .....(i)

$\Rightarrow$ (i) is an identity

$\Rightarrow$ $f(0) = 0$, $f(1) = 2$ & $f(-1) = 1$ $\therefore$ $f(x) = x\,(ax + b)$

$2 = a + b$ & $-1 = -a + b$.

$\Rightarrow$ $b = \frac{1}{2}$ & $a = \frac{3}{2}$ $\Rightarrow$ $f(x) = \frac{3}{2}x^2 + \frac{1}{2}x$.

Self Practice Problems :

1. If $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, varify that $A(\alpha)\,A(\beta) = A(\alpha + \beta)$.

Hence show that in this case $A(\alpha).\,A(\beta) = A(\beta)\,.\,A(\alpha)$.

2. Let $A = \begin{bmatrix} 4 & 6 & -1 \\ 3 & 0 & 2 \\ 1 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $C = [3\ \ 1\ \ 2]$.

Then which of the products ABC, ACB, BAC, BCA, CAB, CBA are defined. Calculate the product whichever is defined. **Ans.** only CAB is defined. $CAB = [25\ \ 100]$

2. Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A'(or A^T) and is defined as

$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}$ $\forall$ i & j.

i.e. A' is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

(i) For any matrix $A = [a_{ij}]_{m \times n}$, $(A')' = A$

(ii) Let λ be a scalar & A be a matrix. Then $(\lambda A)' = \lambda A'$

(iii) $(A + B)' = A' + B'$ & $(A - B)' = A' - B'$ for two comparable matrices A and B.

(iv) $(A_1 \pm A_2 \pm \ldots\ldots \pm A_n)' = A_1' \pm A_2' \pm \ldots\ldots \pm A_n'$, where A_i are comparable.

(v) Let $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$, then $(AB)' = B'A'$

(vi) $(A_1\,A_2\,\ldots\ldots A_n)' = A_n'.\,A_{n-1}'\,\ldots\ldots\ldots\ldots A_2'.\,A_1'$, provided the product is defined.

(vii) **Symmetric & skew symmetric matrix :** A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ $\forall$ i & j.

A square matrix A is said to be skew symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$. A is skew symmetric iff $a_{ij} = -a_{ji}$ $\forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew symmetric matrix.

Note-1 In a skew symmetric matrix all the diagonal elements are zero. ($\because$ $a_{ii} = -a_{ii}$ $\Rightarrow$ $a_{ii} = 0$)

Note-2 For any square matrix A, $A + A'$ is symmetric & $A - A'$ is skew symmetric.

Note- 3 Every square matrix can be uniqualy expressed as sum of two square matrices of which one is symmetric and other is skew symmetric.

$$A = B + C, \text{ where } B = \frac{1}{2}(A + A') \ \& \ C = \frac{1}{2}(A - A').$$

Solved Example # 3 Show that BAB' is symmetric or skew symmetric according as A is symmetric or skew symmetric (where B is any square matrix whose order is same as that of A).

Solution. **Case - I** A is symmetric $\Rightarrow$ $A' = A$

$(BAB')' = (B')'A'B' = BAB'$ $\Rightarrow$ BAB' is symmetric.

Case - II A is skew symmetric $\Rightarrow$ $A' = -A$

$(BAB')' = (B')'A'B'$

$= B\,(-A)\,B'$

$= -(BAB')$ $\Rightarrow$ BAB' is skew symmetric

Self Practice Problems :

1. For any square matrix A, show that $A'A$ & AA' are symmetric matrices.

2. If A & B are symmetric matrices of same order, than show that $AB + BA$ is symmetric and $AB - BA$ is skew symmetric.

3. Submatrix, Minors, Cofactors & Determinant of a Matrix

(i) **Submatrix :** Let A be a given matrix. The matrix obtained by deleting some rows or columns of A is called as submatrix of A.

eg. $A = \begin{bmatrix} a & b & c & d \\ x & y & z & w \\ p & q & r & s \end{bmatrix}$

Then $\begin{bmatrix} a & c \\ x & z \\ p & r \end{bmatrix}$, $\begin{bmatrix} a & b & d \\ p & q & s \end{bmatrix}$, $\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ are all submatrices of A.

(ii) Determinant of a square matrix :

Let $A = [a]_{1\times1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$.

e.g. $A = [-3]_{1\times1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|$ is defined as ad – bc.

e.g. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, $|A| = 23$

(iii) Minors & Cofactors :

Let $A = [a_{ij}]_n$ be a square matrix. Then minor of element a_{ij}, denoted by M_{ij} is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of A. Cofactor of element a_{ij}, denoted by C_{ij} (or A_{ij}) is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$M_{11} = d = C_{11}$
$M_{12} = c,\ C_{12} = -c$
$M_{21} = b,\ C_{21} = -b$
$M_{22} = a = C_{22}$

e.g. 2 $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$.

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay$ etc.

(iv) Determinant of any order :

Let $A = [a_{ij}]_n$ be a square matrix (n > 1). Determinant of A is defined as the sum of products of elements of any one row (or any one column) with corresponding cofactors.

e.g.1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (using first row).

$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|A| = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ (using second column).

$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$.

(v) **Some properties of determinant**

(a) $|A| = |A'|$ for any square matrix A.

(b) If two rows are identical (or two columns are identical) then $|A| = 0$.

(c) Let λ be a scalar. Than $\lambda|A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ
Note : $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

(d) Let $A = [a_{ij}]_n$. The sum of the products of elements of any row with corresponding cofactors of any other row is zero. (Similarly the sum of the products of elements of any column with corresponding cofactors of any other column is zero).

(e) If A and B are two square matrices of same order, then $|AB| = |A|\,|B|$.
Note : As $|A| = |A'|$, we have
$|A|\,|B| = |AB'|$ (row - row method)
$|A|\,|B| = |A'B|$ (column - column method)
$|A|\,|B| = |A'B'|$ (column - row method)

(vi) **Singular & non singular matrix :** A square matrix A is said to be singular or non singular according as $|A|$ is zero or non zero respectively.

(vii) **Cofactor matrix & adjoint matrix :** Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$
then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} $\forall$ i & j.
Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ $\forall$ i & j.

(viii) **Properties of cofactor A and adj A:**
(a) $A \,.\, \text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.
(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A.
In particular, for 3 × 3 matrix, $|\text{adj } A| = |A|^2$
(c) If A is a symmetric matrix, then adj A are also symmetric matrices.
(d) If A is singular, then adj A is also singular.

(ix) **Inverse of a matrix (reciprocal matrix) :** Let A be a non singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}.
We have $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

$\Rightarrow$ $A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right) A$, for A is non singular

$\Rightarrow$ $A^{-1} = \frac{1}{|A|}$ adj A.

***Remarks*:**

1. *The necessary and sufficient condition for existence of inverse of A is that A is non singular.*

2. *A^{-1} is always non singular.*

3. *If A = dia $(a_{11}, a_{12}, \ldots\ldots, a_{nn})$ where $a_{ii} \neq 0$ $\forall$ i, then A^{-1} = diag $(a_{11}^{-1}, a_{22}^{-1}, \ldots., a_{nn}^{-1})$.*

4. *$(A^{-1})' = (A')^{-1}$ for any non singular matrix A. Also adj (A') = (adj A)$'$.*

5. *$(A^{-1})^{-1} = A$ if A is non singular.*

6. *Let k be a non zero scalar & A be a non singular matrix. Then $(kA)^{-1} = \frac{1}{k}A^{-1}$.*

7. *$|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.*

8. *Let A be a nonsingular matrix. Then AB = AC $\Rightarrow$ B = C & BA = CA $\Rightarrow$ B= C.*

9. *A is non-singular and symmetric $\Rightarrow$ A^{-1} is symmetric.*

10. *In general AB = 0 does not imply A = 0 or B = 0. But if A is non singular and AB = 0, then B = 0. Similarly B is non singular and AB = 0 $\Rightarrow$ A = 0. Therefore, AB = 0 $\Rightarrow$ either both are singular or one of them is 0.*

Solved Example # 4
For a 3×3 skew symmetric matrix A, show that adj A is a symmetric matrix.
Solution.

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \qquad \text{cof } A = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$$

$$\text{adj } A = (\text{cof } A)' = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix} \text{ which is symmetric.}$$

Solved Example # 5
For two nonsingular matrices A & B, show that adj (AB) = (adj B) (adj A)
Solution.
We have $(AB)\,(\text{adj }(AB)) = |AB|\, I_n$
$= |A|\,|B|\, I_n$
$A^{-1}\,(AB)(\text{adj }(AB)) = |A|\,|B|\, A^{-1}$

$\Rightarrow$ $B \text{ adj }(AB) = |B| \text{ adj } A$ $\qquad (\because\ A^{-1} = \frac{1}{|A|} \text{ adj } A)$

$\Rightarrow$ $B^{-1} B \text{ adj }(AB) = |B|\, B^{-1} \text{ adj } A$
$\Rightarrow$ adj (AB) = (adjB) (adj A)

Self Practice Problems :

1. If A is nonsingular, show that adj (adj A) = $|A|^{n-2}$ A.

2. Prove that adj (A^{-1}) = $(\text{adj } A)^{-1}$.

3. For any square matrix A, show that |adj (adj A) | = $|A|^{(n-1)^2}$.

4. If A and B are nonsingular matrices, show that $(AB)^{-1} = B^{-1}A^{-1}$.

4. System of Linear Equations & Matrices

Consider the system

$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2$

....

$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_n$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_n \end{bmatrix}$.

Then the above system can be expressed in the matrix form as AX = B.
The system is said to be consistent if it has atleast one solution.

(i) System of linear equations and matrix inverse:
If the above system consist of n equations in n unknowns, then we have AX = B where A is a square matrix. If A is nonsingular, solution is given by $X = A^{-1}B$.
If A is singular, (adj A) B = 0 and all the columns of A are not proportional, then the system has infinite many solution.
If A is singular and (adj A) B ≠ 0, then the system has no solution (we say it is inconsistent).

(ii) Homogeneous system and matrix inverse:
If the above system is homogeneous, n equations in n unknowns, then in the matrix form it is AX = O. (∵ in this case $b_1 = b_2 = \dots\dots b_n = 0$), where A is a square matrix.
If A is nonsingular, the system has only the trivial solution (zero solution) X = 0
If A is singular, then the system has infinitely many solutions (including the trivial solution) and hence it has non trivial solutions.

(iii) Rank of a matrix :
Let $A = [a_{ij}]_{m\times n}$. A natural number ρ is said to be the rank of A if A has a nonsingular submatrix of order ρ and it has no nonsingular submatrix of order more than ρ. Rank of zero matrix is regarded to be zero.

eg. $A = \begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$

we have $\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$ as a non singular submatrix.

The square matrices of order 3 are

$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 & 5 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 0 \\ 0 & 5 & 0 \end{bmatrix}$

and all these are singular. Hence **rank of A is 2.**

(iv) Elementary row transformation of matrix :
The following operations on a matrix are called as elementary row transformations.
(a) Interchanging two rows.
(b) Multiplications of all the elements of row by a nonzero scalar.
(c) Addition of constant multiple of a row to another row.

Note : Similar to above we have elementary column transformations also.

<u>Remark</u> :

1. *Elementary transformation on a matrix does not affect its rank.*

2. *Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write A ≈ B.*

(v) ***Echelon form of a matrix :*** *A matric is said to be in Echelon form if it satisfy the followings:*
(a) The first non-zero element in each row is 1 & all the other elements in the corresponding column (i.e. the column where 1 appears) are zeroes.
(b) The number of zeroes before the first non zero element in any non zero row is less than the number of such zeroes in succeeding non zero rows.

<u>Result</u> : *Rank of a matrix in Echelon form is the number of non zero rows (i.e. number of rows with atleast one non zero element.)*

<u>Remark</u> :

1. *To find the rank of a given matrix we may reduce it to Echelon form using elementary row transformations and then count the number of non zero rows.*

(vi) System of linear equations & rank of matrix:

Let the system be AX = B where A is an m × n matrix, X is the n-column vector & B is the m-column vector. Let [AB] denote the **augmented matrix** (i.e. matrix obtained by accepting elements of B as n + 1th column & first n columns are that of A).

$\rho(A)$ denote rank of A and $\rho([AB])$ denote rank of the augmented matrix.

Clearly $\rho(A) \le \rho([AB])$.

(a) If $\rho(A) < \rho([AB])$ then the system has no solution (i.e. system is inconsistent).

(b) If $\rho(A) = \rho([AB])$ = number of unknowns, then the system has unique solution. (and hence is consistent)

(c) If $\rho(A) = \rho([AB])$ < number of unknowns, then the systems has infinitely many solutions (and so is consistent).

(vii) Homogeneous system & rank of matrix :

Let the homogenous system be AX = 0, m equations in 'n' unknowns. In this case B = 0 and so $\rho(A) = \rho([AB])$.

Hence if $\rho(A) = n$, then the system has only the trivial solution. If $\rho(A) < n$, then the system has infinitely many solutions.

Solved Example # 6

Solve the system $\begin{matrix} x+y+z=6 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$ using matrix inverse.

Solution.

Let A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & B = $\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$.

Then the system is AX = B.

|A| = 6. Hence A is non singular.

Cofactor A = $\begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$

adj A = $\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1} B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ x = 1, y = 2, z = 3.

Solved Example # 7

Test the consistancy of the system $\begin{matrix} x-y+2z=1 \\ x+y+z=3 \\ x-3y+3z=-1 \\ 2x+4y+z=8. \end{matrix}$. Also find the solution, if any.

Solution.

A = $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -3 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 1 \\ 3 \\ -1 \\ 8 \end{bmatrix}$

[AB] = $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -3 & 3 & -1 \\ 2 & 4 & 1 & 8 \end{bmatrix}$

$$\approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 6 & -3 & 6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/2 & 1 \\ 0 & 1 & -1/2 & 1 \\ 0 & 1 & -1/2 & 1 \end{bmatrix} \begin{matrix} R_2 \to \frac{1}{2}R_2 \\ R_3 \to -\frac{1}{2}R_3 \\ R_4 \to \frac{1}{6}R_4 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & 0 & 3/2 & 2 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

This is in Echelon form.

$\rho(AB) = 2 = \rho(A) <$ number of unknowns

Hence there are infinitely many solutions $n - \rho = 1$.

Hence we can take one of the variables any value and the rest in terms of it.

Let $z = r$, where r is any number.

Then $x - y = 1 - 2r$

$x + y = 3 - r$

$\Rightarrow \quad x = \dfrac{4-3r}{2}$ & $y = \dfrac{2+r}{2}$

$\therefore \quad$ Solutions are $(x, y, z) = \left(\dfrac{4-3r}{2}, \dfrac{2+r}{2}, r\right)$.

Self Practice Problems:

1. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the inverse of A using |A| and adj A. Also find A^{-1} by solving a system of equations.

2. Find real values of λ and μ so that the following systems has

(i) unique solution (ii) infinite solution (iii) No solution.

$x + y + z = 6$
$x + 2y + 3z = 1$
$x + 2y + \lambda z = \mu$

Ans. (i) $\lambda \neq 3, \mu \in R$ (ii) $\lambda = 3, \mu = 1$ (iii) $\lambda = 3, \mu \neq 1$

3. Find λ so that the following homogeneous system have a non zero solution

$x + 2y + 3z = \lambda x$
$3x + y + 2z = \lambda y$
$2x + 3y + z = \lambda z$

Ans. $\lambda = 6$

5. More on Matrices

(i) Characteristic polynomial & Characteristic equation :

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called as characteristic equation of A.

Remark :

Every square matrix A satisfy its characteristic equation (Cayley - Hamilton Theorem).

i.e. $a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0$ is the characteristic equation of A, then

$a_0A^n + a_1A^{n-1} + \ldots\ldots + a_{n-1}A + a_n I = 0$

(ii) More Definitions on Matrices :

(a) Nilpotent matrix:

A square matrix A is said to be nilpotent (of order 2) if, $A^2 = O$.

A square matrix is said to be nilpotent of order p, if p is the least positive integer such that $A^p = O$.

(b) Idempotent matrix:

A square matrix A is said to be idempotent if, $A^2 = A$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an idempotent matrix.

(c) Involutory matrix:
A square matrix A is said to be involutory if $A^2 = I$, I being the identity matrix.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix.

(d) Orthogonal matrix:
A square matrix A is said to be an orthogonal matrix if,
$A'A = I = A'A$.

(e) Unitary matrix:
A square matrix A is said to be unitary if $A(\overline{A})' = I$, where $\overline{A}$ is the complex conjugate of A.

Solved Example # 8

If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, show that $5A^{-1} = A^2 + A - 5I$.

Solution.
We have the characteristic equation of A.

$|A - xI| = 0$

i.e. $\begin{bmatrix} 1-x & 2 & 0 \\ 2 & -1-x & 0 \\ 0 & 0 & -1-x \end{bmatrix} = 0$

i.e. $x^3 + x^2 - 5x - 5 = 0$.

Using cayley - Hamilton theorem.

$A^3 + A^2 - 5A - 5I = 0$

$\Rightarrow$ $5I = A^3 + A^2 - 5A$

Multiplying by A^{-1}, we get

$5A^{-1} = A^2 + A - 5I$

Solved Example # 9

Show that a square matrix A is involutory, iff $(I - A)(I + A) = 0$

Solution.
Let A be involutory

Then $A^2 = I$

$(I - A)(I + A) = II + IA - AI - A^2$
$= I + A - A - A^2$
$= I - A^2$
$= 0$

Conversly, let $(I - A)(I + A) = 0$

$\Rightarrow$ $II + IA - AI - A^2 = 0$

$\Rightarrow$ $I + A - A - A^2 = 0$

$\Rightarrow$ $I - A^2 = 0$

$\Rightarrow$ A is involutory

Self Practice Problems

1. If A is idempotent, show that $B = I - A$ is idempotent and that $AB = BA = 0$.

2. If A is a nilpotent matrix of index 2, show that $A(I + A)^n = A$ for all $n \in N$.

3. A is a skew symmetric matrix, such that $A^2 + I = 0$. Show that A is orthogonal and is of even order.

4. Let $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$. If $A^3 + \lambda A = 0$, find λ.

Ans. $a^2 + b^2 + c^2$.