SHORT REVISION (FUNCTIONS)

THINGS TO REMEMBER :

1. GENERAL DEFINITION :

If to every value (Considered as real unless other–wise stated) of a variable x, which belongs to some collection (Set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (Single valued) of x or a dependent variable defined on the set E ; x is the argument or independent variable .

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a multiple valued function of x defined on E.Conventionally the word **"FUNCTION"** is used only as the meaning of a single valued function, if not otherwise stated.

Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x)=y}$, y is called the image of x & x is the pre-image of y under f.

Every function from $A \to B$ satisfies the following conditions .

(i) $f \subset A \times B$ **(ii)** $\forall\ a \in A \Rightarrow (a, f(a)) \in f$ and

(iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

2. DOMAIN, CO–DOMAIN & RANGE OF A FUNCTION :

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of all f images of elements of A is known as the range of f . Thus :

Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

It should be noted that range is a subset of co–domain . If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

3. IMPORTANT TYPES OF FUNCTIONS :

(i) POLYNOMIAL FUNCTION :

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n .

NOTE : (a) A polynomial of degree one with no constant term is called an odd linear function . i.e. $f(x) = ax,\ a \neq 0$

(b) There are two polynomial functions, satisfying the relation ; $f(x).f(1/x) = f(x) + f(1/x)$. They are :

(i) $f(x) = x^n + 1$ & **(ii)** $f(x) = 1 - x^n$, where n is a positive integer .

(ii) ALGEBRAIC FUNCTION :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form

$P_0(x)\,y^n + P_1(x)\,y^{n-1} + ... + P_{n-1}(x)\,y + P_n(x) = 0$ Where n is a positive integer and $P_0(x), P_1(x)$ .... are Polynomials in x.

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called **TRANSCEDENTAL FUNCTION** .

(iii) FRACTIONAL RATIONAL FUNCTION :

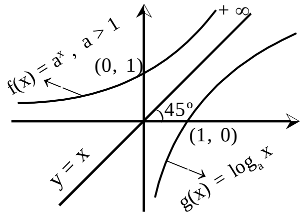
A rational function is a function of the form. $y = f(x) = \dfrac{g(x)}{h(x)}$, where

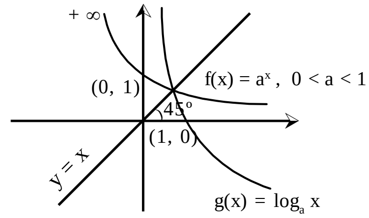
g (x) & h (x) are polynomials & $h(x) \neq 0$.

(IV) EXPONENTIAL FUNCTION :

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0,\ a \neq 1,\ x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function . i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown .





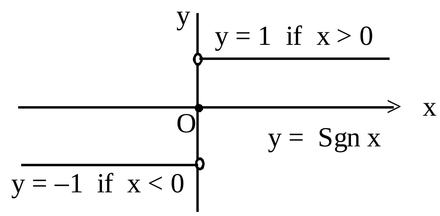
(v) ABSOLUTE VALUE FUNCTION :

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

(vi) SIGNUM FUNCTION :

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



It is also written as $\text{Sgn}\, x = |x|/x$; $x \neq 0$; $f(0) = 0$

(vii) GREATEST INTEGER OR STEP UP FUNCTION :

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x . Note that for :

$-1 \leq x < 0$; $[x] = -1$ $\quad$ $0 \leq x < 1$; $[x] = 0$

$1 \leq x < 2$; $[x] = 1$ $\quad$ $2 \leq x < 3$; $[x] = 2$

and so on .

Properties of greatest integer function :

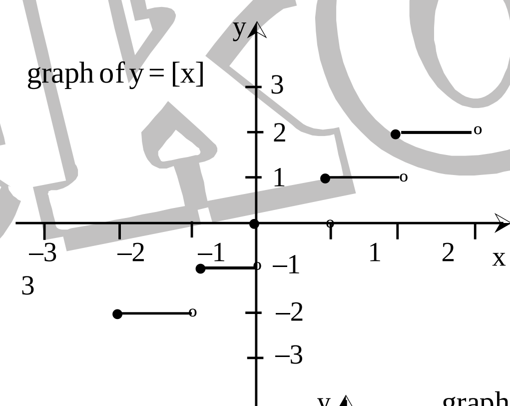
(a) $[x] \leq x < [x] + 1$ and $x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$

(b) $[x + m] = [x] + m$ if m is an integer .

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(d) $[x] + [-x] = 0$ if x is an integer
$= -1$ otherwise .

graph of $y = [x]$

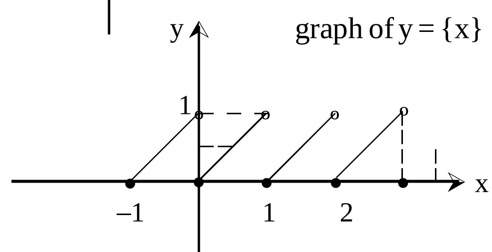


(viii) FRACTIONAL PART FUNCTION :

It is defined as :

$g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the no. 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown .



4. DOMAINS AND RANGES OF COMMON FUNCTION :

Function $(y = f(x))$	Domain (i.e. values taken by x)	Range (i.e. values taken by f (x))
A. Algebraic Functions		
(i) x^n , $(n \in N)$	R = (set of real numbers)	R , if n is odd $R^+ \cup \{0\}$, if n is even
(ii) $\dfrac{1}{x^n}$, $(n \in N)$	$R - \{0\}$	$R - \{0\}$, if n is odd

			R^+ , if n is even
(iii)	$x^{1/n}$, $(n \in N)$	R , if n is odd $R^+ \cup \{0\}$, if n is even	R , if n is odd $R^+ \cup \{0\}$, if n is even
(iv)	$\dfrac{1}{x^{1/n}}$, $(n \in N)$	$R - \{0\}$, if n is odd R^+ , if n is even	$R - \{0\}$, if n is odd R^+ , if n is even

B. Trigonometric Functions

(i)	sin x	R	$[-1, +1]$
(ii)	cos x	R	$[-1, +1]$
(iii)	tan x	$R - (2k+1)\dfrac{\pi}{2}, k \in I$	R
(iv)	sec x	$R - (2k+1)\dfrac{\pi}{2}, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v)	cosec x	$R - k\pi, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi)	cot x	$R - k\pi, k \in I$	R

C. Inverse Circular Functions (Refer after Inverse is taught)

(i)	$\sin^{-1} x$	$[-1, +1]$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
(ii)	$\cos^{-1} x$	$[-1, +1]$	$[0, \pi]$
(iii)	$\tan^{-1} x$	R	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
(iv)	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$
(v)	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$
(vi)	$\cot^{-1} x$	R	$(0, \pi)$

Function (y = f (x))	Domain (i.e. values taken by x)	Range (i.e. values taken by f (x))

D. Exponential Functions

(i)	e^x	R	R^+
(ii)	$e^{1/x}$	$R - \{0\}$	$R^+ - \{1\}$
(iii)	a^x , $a > 0$	R	R^+
(iv)	$a^{1/x}$, $a > 0$	$R - \{0\}$	$R^+ - \{1\}$

E. Logarithmic Functions

(i)	$\log_a x$, $(a > 0)$ $(a \neq 1)$	R^+	R
(ii)	$\log_x a = \dfrac{1}{\log_a x}$ $(a > 0)$ $(a \neq 1)$	$R^+ - \{1\}$	$R - \{0\}$

F. Integral Part Functions Functions

(i)	$[x]$	R	I
(ii)	$\dfrac{1}{[x]}$	R – [0, 1)	$\left\{\dfrac{1}{n}, n \in I - \{0\}\right\}$

G. Fractional Part Functions

(i)	$\{x\}$	R	[0, 1)
(ii)	$\dfrac{1}{\{x\}}$	R – I	$(1, \infty)$

H. Modulus Functions

(i)	$\lvert x \rvert$	R	$R^+ \cup \{0\}$
(ii)	$\dfrac{1}{\lvert x\rvert}$	R – { 0 }	R^+

I. Signum Function

$\text{sgn}(x) = \dfrac{\lvert x\rvert}{x}, x \neq 0$ $= 0, x = 0$	R	{–1, 0 , 1}

J. Constant Function

say $f(x) = c$	R	{ c }

5. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if :

(i) The domain of f = the domain of g.

(ii) The range of f = the range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain. eg.

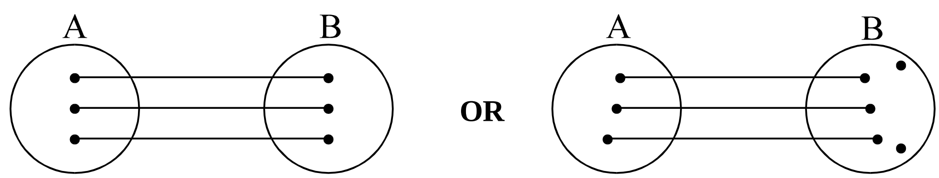
$f(x) = \dfrac{1}{x}$ & $g(x) = \dfrac{x}{x^2}$ are identical functions .

6. CLASSIFICATION OF FUNCTIONS :

One – One Function (Injective mapping) :

A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B . Thus for $x_1, x_2 \in A$ & $f(x_1)$, $f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagramatically an injective mapping can be shown as



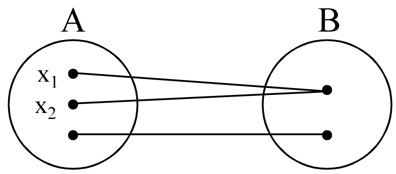
Note : (i) Any function which is entirely increasing or decreasing in whole domain, then f(x) is one–one .

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one .

Many–one function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B . Thus $f : A \to B$ is many one if for ; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagramatically a many one mapping can be shown as



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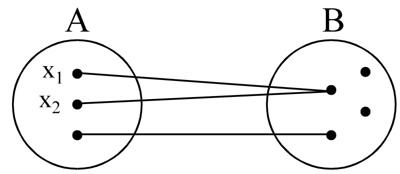


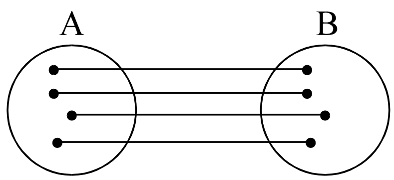
Note : (i) Any continuous function which has atleast one local maximum or local minimum, then $f(x)$ is many–one . In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one .

(ii) If a function is one–one, it cannot be many–one and vice versa .

Onto function (Surjective mapping) :

If the function $f: A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B . Thus $f: A \to B$ is surjective iff $\forall\ b \in B, \exists$ some $a \in A$ such that $f(a) = b$.

Diagramatically surjective mapping can be shown as



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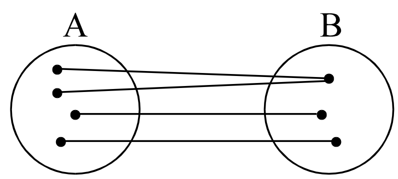
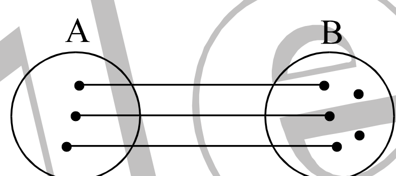


Note that : if range = co–domain, then $f(x)$ is onto.

Into function :

If $f: A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then $f(x)$ is into .

Diagramatically into function can be shown as



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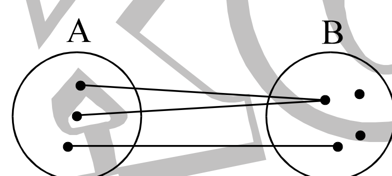
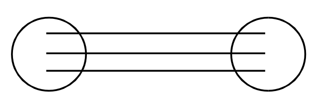


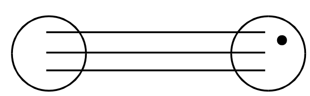
Note that : If a function is onto, it cannot be into and vice versa . A polynomial of degree even will always be into.

Thus a function can be one of these four types :

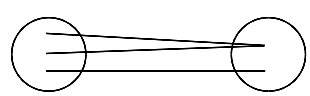
(a) one–one onto (injective & surjective)



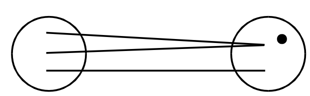
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : (i) If f is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one one.

Identity function :

The function $f: A \to A$ defined by $f(x) = x\ \forall\ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function is a bijection .

Constant function :

A function $f : A \to B$ is said to be a constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c,\ \forall\ x \in A,\ c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into.

7. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, then both f & g are defined in $A \cap B$. Now we define $f+g,\ f-g,\ (f.g)$ & (f/g) as follows :

(i) $(f \pm g)(x) = f(x) \pm g(x)$

(ii) $(f.g)(x) = f(x).g(x)$

domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B \text{ s.t } g(x) \neq 0\}$.

8. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS :

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$.

Thus the image of every $x \in A$ under the function gof is the g–image of the f–image of x.

Note that gof is defined only if $\forall\ x \in A,\ f(x)$ is an element of the domain of g so that we can take its g-image. Hence for the product gof of two functions f & g, the range of f must be a subset of the domain of g.

PROPERTIES OF COMPOSITE FUNCTIONS :

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & $(fog)oh$ are defined, then $fo(goh) = (fog)oh$.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

9. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y. Symbolically if,

$f(tx, ty) = t^n . f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

10. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

11. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **IMPLICIT FUNCTION**. For eg. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **EXPLICIT FUNCTION**.

12. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one–one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A$ & $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

PROPERTIES OF INVERSE FUNCTION :

(i) The inverse of a bijection is unique.

(ii) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.

Note that the graphs of f & g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2\ (x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

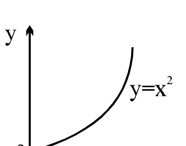


fig. 1

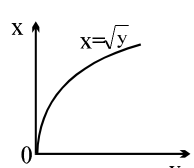


fig. 2

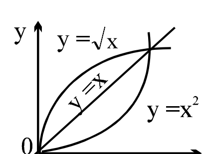


fig. 3

(iii) The inverse of a bijection is also a bijection .

(iv) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o\, g^{-1}$.

13. ODD & EVEN FUNCTIONS :

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.
If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

NOTE : (a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd .

(b) A function may neither be odd nor even .

(c) Inverse of an even function is not defined .

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin .

(e) Every function can be expressed as the sum of an even & an odd function.

e.g. $$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd .

14. PERIODIC FUNCTION :

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that $f(x+T) = f(x)$, for all values of x within the domain of x.
e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π .

NOTE : (a) $f(T) = f(0) = f(-T)$, where 'T' is the period .

(b) Inverse of a periodic function does not exist .

(c) Every constant function is always periodic, with no fundamental period .

(d) If f(x) has a period T & g (x) also has a period T then it does not mean that $f(x)+g(x)$ must have a period T . e.g. $f(x) = |\sin x| + |\cos x|$.

(e) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p .

(f) if f(x) has a period T then $f(ax + b)$ has a period T/a (a > 0) .

15. GENERAL :

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) . f(y) \Rightarrow f(x) = x^n$, $n \in R$

(iii) $f(x+y) = f(x) . f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant .

EXERCISE–1

Q.1 Find the domains of definitions of the following functions :
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x) = \sqrt{\cos 2x} + \sqrt{16 - x^2}$

(ii) $f(x) = \log_7 \log_5 \log_3 \log_2 (2x^3 + 5x^2 - 14x)$

(iii) $f(x) = \ln\left(\sqrt{x^2 - 5x - 24} - x - 2\right)$

(iv) $f(x) = \sqrt{\frac{1-5^x}{7^{-x}-7}}$

(v) $y = \log_{10} \sin(x-3) + \sqrt{16-x^2}$

(vi) $f(x) = \log_{100x}\left(\frac{2\log_{10} x+1}{-x}\right)$

(vii) $f(x) = \dfrac{1}{\sqrt{4x^2-1}} + \ln x(x^2-1)$ (viii) $f(x) = \sqrt{\log_{\frac{1}{2}} \dfrac{x}{x^2-1}}$

(ix) $f(x) = \sqrt{x^2 - |x|} + \dfrac{1}{\sqrt{9-x^2}}$ (x) $f(x) = \sqrt{(x^2-3x-10).\ln^2(x-3)}$

(xi) $f(x) = \sqrt{\log_x(\cos 2\pi x)}$ (xii) $f(x) = \dfrac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6+35x-6x^2}}$

(xiii) $f(x) = \sqrt{\log_{1/3}\left(\log_4\left([x]^2 - 5\right)\right)}$ (xiv) $f(x) = \dfrac{1}{[x]} + \log_{(2\{x\}-5)}(x^2-3x+10) + \dfrac{1}{\sqrt{1-|x|}}$,

(xv) $f(x) = \log_x \sin x$

(xvi) $f(x) = \log_2\left(-\log_{1/2}\left(1 + \dfrac{1}{\sin\left(\frac{x^\circ}{100}\right)}\right)\right) + \sqrt{\log_{10}(\log_{10} x) - \log_{10}(4 - \log_{10} x) - \log_{10} 3}$

(xvii) $f(x) = \dfrac{1}{[x]} + \log_{1-\{x\}}(x^2 - 3x + 10) + \dfrac{1}{\sqrt{2-|x|}} + \dfrac{1}{\sqrt{\sec(\sin x)}}$

(xviii) $f(x) = \sqrt{(5x-6-x^2)\left[\{\ln\{x\}\}\right]} + \sqrt{(7x-5-2x^2)} + \left(\ln\left(\dfrac{7}{2} - x\right)\right)^{-1}$

(xix) If $f(x) = \sqrt{x^2 - 5x + 4}$ & $g(x) = x + 3$, then find the domain of $\dfrac{f}{g}(x)$.

Q.2 Find the domain & range of the following functions .
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $y = \log_{\sqrt{5}}\left(\sqrt{2}(\sin x - \cos x) + 3\right)$ (ii) $y = \dfrac{2x}{1+x^2}$ (iii) $f(x) = \dfrac{x^2-3x+2}{x^2+x-6}$

(iv) $f(x) = \dfrac{x}{1+|x|}$ (v) $y = \sqrt{2-x} + \sqrt{1+x}$

(vi) $f(x) = \log_{(\operatorname{cosec} x - 1)}(2 - [\sin x] - [\sin x]^2)$ (vii) $f(x) = \dfrac{\sqrt{x+4}-3}{x-5}$

Q.3 Draw graphs of the following function , where [] denotes the greatest integer function.
(i) $f(x) = x + [x]$
(ii) $y = (x)^{[x]}$ where $x = [x] + (x)$ & $x > 0$ & $x \le 3$
(iii) $y = \text{sgn}\,[x]$ (iv) $\text{sgn}\,(x - |x|)$

Q.4 Classify the following functions f(x) definzed in $R \to R$ as injective, surjective, both or none .

(a) $f(x) = \dfrac{x^2+4x+30}{x^2-8x+18}$ (b) $f(x) = x^3 - 6x^2 + 11x - 6$ (c) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$

Q.5 Let $f(x) = \dfrac{1}{1-x}$. Let $f_2(x)$ denote $f[f(x)]$ and $f_3(x)$ denote $f[f\{f(x)\}]$. Find $f_{3n}(x)$ where n is a natural number. Also state the domain of this composite function.

Q.6 If $f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$ and $g\left(\dfrac{5}{4}\right) = 1$, then find $(gof)(x)$.

Q.7 The function f(x) is defined on the interval [0,1]. Find the domain of definition of the functions.
(a) $f(\sin x)$ (b) $f(2x+3)$

Q.8**(i)** Find whether the following functions are even or odd or none

(a) $f(x) = \log\left(x + \sqrt{1+x^2}\right)$ (b) $f(x) = \dfrac{x(a^x+1)}{a^x-1}$ (c) $f(x) = \sin x + \cos x$

(d) $f(x) = x\sin^2 x - x^3$ (e) $f(x) = \sin x - \cos x$ (f) $f(x) = \dfrac{(1+2^x)^2}{2^x}$

(g) $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$ (h) $f(x)=[(x+1)^2]^{1/3}+[(x-1)^2]^{1/3}$

(ii) If f is an even function defined on the interval (–5, 5), then find the 4 real values of x satisfying the equation $f(x) = f\left(\dfrac{x+1}{x+2}\right)$..

Q.9 Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :
(a) $10^x + 10^y = 10$ (b) $x + |y| = 2y$

Q.10 Show if $f(x) = \sqrt[n]{a-x^n}$, $x > 0$ $n \ge 2$, $n \in N$, then $(fof)(x) = x$. Find also the inverse of f(x).

Q.11 (a) Represent the function $f(x) = 3^x$ as the sum of an even & an odd function.
(b) For what values of $p \in z$, the function $f(x) = \sqrt[n]{x^p}$, $n \in N$ is even.

Q.12 A function f defined for all real numbers is defined as follows for $x \ge 0$: $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$
How is f defined for $x \le 0$ if : (a) f is even (b) f is odd?

Q.13 If $f(x) = \max\left(x, \dfrac{1}{x}\right)$ for x > 0 where max (a, b) denotes the greater of the two real numbers a and b.
Define the function $g(x) = f(x) \cdot f\left(\dfrac{1}{x}\right)$ and plot its graph.

Q.14 The function $f(x)$ has the property that for each real number x in its domain, 1/x is also in its domain and $f(x) + f\left(\dfrac{1}{x}\right) = x$. Find the largest set of real numbers that can be in the domain of $f(x)$?

Q.15 Compute the inverse of the functions:
(a) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ (b) $f(x) = 2^{\frac{x}{x-1}}$ (c) $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Q.16 A function $f : \left[\dfrac{1}{2}, \infty\right) \to \left[\dfrac{3}{4}, \infty\right)$ defined as, $f(x) = x^2 - x + 1$. Then solve the equation $f(x) = f^{-1}(x)$.

Q.17 Function f & g are defined by $f(x) = \sin x$, $x \in R$; $g(x) = \tan x$, $x \in R - \left(K + \dfrac{1}{2}\right)\pi$
where $K \in I$. Find **(i)** periods of fog & gof. **(ii)** range of the function fog & gof .

Q.18 Find the period for each of the following functions :
(a) $f(x) = \sin^4 x + \cos^4 x$ (b) $f(x) = |\cos x|$ (c) $f(x) = |\sin x| + |\cos x|$
(d) $f(x) = \cos\dfrac{3}{5}x - \sin\dfrac{2}{7}x$.

Q.19 Prove that the functions ; (a) $f(x) = \cos\sqrt{x}$ (b) $f(x) = \sin\sqrt{x}$
(c) $f(x) = x + \sin x$ (d) $f(x) = \cos x^2$ are not periodic .

Q.20 Find out for what integral values of n the number 3π is a period of the function :
$f(x) = \cos nx \cdot \sin(5/n)x$.

EXERCISE–2

Q.1 Let f be a one–one function with domain {x,y,z} and range {1,2,3}. It is given that exactly one of the following statements is true and the remaining two are false .
$f(x) = 1$; $f(y) \ne 1$; $f(z) \ne 2$. Determine $f^{-1}(1)$

Q.2 Solve the following problems from (a) to (e) on functional equation.

(a) The function $f(x)$ defined on the real numbers has the property that $f\left(f(x)\right)\cdot\left(1+f(x)\right) = -f(x)$ for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of $f(3)$.

(b) Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find the value of $f(21)$.

(c) Let f be a function defined from $R^+ \to R^+$. If $[f(xy)]^2 = x\left(f(y)\right)^2$ for all positive numbers x and y and $f(2) = 6$, find the value of $f(50)$.

(d) Let $f(x)$ be a function with two properties
(i) for any two real number x and y, $f(x + y) = x + f(y)$ and
(ii) $f(0) = 2$.
Find the value of $f(100)$.

(e) Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x - 3)$ for all x. Then find the value of $f(300)$.

Q.3(a) A function f is defined for all positive integers and satisfies $f(1) = 2005$ and $f(1)+ f(2)+ ... + f(n) = n^2 f(n)$ for all $n > 1$. Find the value of $f(2004)$.

(b) If a, b are positive real numbers such that $a - b = 2$, then find the smallest value of the constant L for which $\sqrt{x^2 + ax} - \sqrt{x^2 + bx} < L$ for all $x > 0$.

(c) Let $f(x) = x^2 + kx$; k is a real number. The set of values of k for which the equation $f(x) = 0$ and $f\left(f(x)\right) = 0$ have same real solution set.

(d) If $f(2x + 1) = 4x^2 + 14x$, then find the sum of the roots of the equation $f(x) = 0$.

Q.4 Let $f(x) = \dfrac{ax + b}{4x + c}$ for real a, b and c with $a \neq 0$. If the vertical asymptote of $y = f(x)$ is $x = -\dfrac{5}{4}$ and the vertical asymptote of $y = f^{-1}(x)$ is $x = \dfrac{3}{4}$, find the value(s) that b can take on.

Q.5 A function $f : R \to R$ satisfies the condition, $x^2 f(x) + f(1 - x) = 2x - x^4$. Find f (x) and its domain and range.

Q.6 Suppose p(x) is a polynomial with integer coefficients. The remainder when p(x) is divided by $x - 1$ is 1 and the remainder when p(x) is divided by $x - 4$ is 10. If r (x) is the remainder when p(x) is divided by $(x - 1)(x - 4)$, find the value of r (2006).

Q.7 Prove that the function defined as , $f(x) = \begin{cases} e^{-\sqrt{|\ln\{x\}|}} - \{x\}^{\sqrt{\frac{1}{|\ln\{x\}|}}} & \text{where ever it exists} \\ \{x\} & \text{otherwise} \end{cases}$, then

f (x) is odd as well as even. (where {x} denotes the fractional part function)

Q.8 In a function $2f(x) + xf\left(\frac{1}{x}\right) - 2f\left(\left|\sqrt{2}\sin\left(\pi\left(x + \frac{1}{4}\right)\right)\right|\right) = 4\cos^2\frac{\pi x}{2} + x\cos\frac{\pi}{x}$

Prove that (i) $f(2) + f(1/2) = 1$ and (ii) $f(2) + f(1) = 0$

Q.9 A function f, defined for all $x, y \in R$ is such that $f(1) = 2$; $f(2) = 8$ & $f(x+y) - kxy = f(x) + 2y^2$, where k is some constant. Find f(x) & show that :

$$f(x+y)\, f\left(\frac{1}{x+y}\right) = k \text{ for } x + y \neq 0.$$

Q.10 Let 'f' be a real valued function defined for all real numbers x such that for some positive constant 'a' the equation $f(x + a) = \frac{1}{2} + \sqrt{f(x) - \left(f(x)\right)^2}$ holds for all x. Prove that the function f is periodic.

Q.11 If $f(x) = -1 + |x - 2|$, $0 \le x \le 4$
$g(x) = 2 - |x|$, $-1 \le x \le 3$
Then find fog (x) & gof (x). Draw rough sketch of the graphs of fog (x) & gof (x).

Q.12 Find the domain of definition of the implicit function defined by the implicit equation,

$$3^y + 2^{x^4} = 2^{4x^2 - 1}.$$

Q.13 Let {x} & [x] denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$

Q.14 Let $f(x) = \frac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\frac{1}{2006}\right) + f\left(\frac{2}{2006}\right) + f\left(\frac{3}{2006}\right) + \ldots + f\left(\frac{2005}{2006}\right)$

Q.15 Let $f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 5$ where $x \in [-6, 6]$. If the range of the function is $[a, b]$ where $a, b \in N$ then find the value of $(a + b)$.

Q.16 Find a formula for a function g (x) satisfying the following conditions
(a) domain of g is $(-\infty, \infty)$
(b) range of g is $[-2, 8]$
(c) g has a period π and
(d) $g(2) = 3$

Q.17 The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and $\frac{b}{c}$ is in its lowest form. Find the value of $a + b + c + abc$.

Q.18 Find the set of real x for which the function $f(x) = \frac{1}{\left[|x-1|\right] + \left[|12-x|\right] - 11}$ is not defined, where [x] denotes the greatest integer function.

Q.19 A is a point on the circumference of a circle. Chords AB and AC divide the area of the circle into three equal parts . If the angle BAC is the root of the equation, $f(x) = 0$ then find $f(x)$.

Q.20 If for all real values of u & v, $2f(u)\cos v = f(u + v) + f(u - v)$, prove that, for all real values of x
(i) $f(x) + f(-x) = 2a\cos x$ (ii) $f(\pi - x) + f(-x) = 0$
(iii) $f(\pi - x) + f(x) = -2b\sin x$. Deduce that $f(x) = a\cos x - b\sin x$, a, b are arbitrary constants.

EXERCISE–3

Q.1 If the functions f, g, h are defined from the set of real numbers R to R such that ;
$f(x) = x^2 - 1$, $g(x) = \sqrt{x^2 + 1}$, $h(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x \ge 0 \end{cases}$; then find the composite function ho(fog) & determine whether the function (fog) is invertible & the function h is the identity function. [REE '97, 6]

Q.2(a) If $g(f(x)) = |\sin x|$ & $f(g(x)) = \left(\sin\sqrt{x}\right)^2$, then :
(A) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$ (B) $f(x) = \sin x$, $g(x) = |x|$
(C) $f(x) = x^2$, $g(x) = \sin\sqrt{x}$ (D) f & g cannot be determined

(b) If $f(x) = 3x - 5$, then $f^{-1}(x)$
(A) is given by $\frac{1}{3x-5}$ (B) is given by $\frac{x+5}{3}$
(C) does not exist because f is not one–one (D) does not exist because f is not onto
[JEE'98, 2 + 2]

Q.3 If the functions f & g are defined from the set of real numbers R to R such that $f(x) = e^x$, $g(x) = 3x - 2$, then find functions fog & gof. Also find the domains of functions $(fog)^{-1}$ & $(gof)^{-1}$.
[REE '98, 6]

Q.4 If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is : [JEE '99, 2]
(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$ (C) $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (D) not defined

Q.5 The domain of definition of the function, $y(x)$ given by the equation, $2^x + 2^y = 2$ is :
(A) $0 < x \le 1$ (B) $0 \le x \le 1$ (C) $-\infty < x \le 0$ (D) $-\infty < x < 1$

Q.6 Given $x = \{1, 2, 3, 4\}$, find all one–one, onto mappings, $f : X \to X$ such that, $f(1) = 1$, $f(2) \ne 2$ and $f(4) \ne 4$. [REE 2000, 3 out of 100]

Q.7(a) Let $g(x) = 1 + x - [x]$ & $f(x) = \begin{cases} -1 & , x < 0 \\ 0 & , x = 0 \\ 1 & , x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to
(A) x (B) 1 (C) $f(x)$ (D) $g(x)$

(b) If $f : [1, \infty) \to [2, \infty)$ is given by, $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals
(A) $\frac{x + \sqrt{x^2 - 4}}{2}$ (B) $\frac{x}{1 + x^2}$ (C) $\frac{x - \sqrt{x^2 - 4}}{2}$ (D) $1 - \sqrt{x^2 - 4}$

(c) The domain of definition of $f(x) = \frac{\log_2(x + 3)}{x^2 + 3x + 2}$ is :
(A) $R \setminus \{-1, -2\}$ (B) $(-2, \infty)$ (C) $R\setminus\{-1, -2, -3\}$ (D) $(-3, \infty) \setminus \{-1, -2\}$

(d) Let $E = \{1, 2, 3, 4\}$ & $F = \{1, 2\}$. Then the number of onto functions from E to F is
(A) 14 (B) 16 (C) 12 (D) 8

(e) Let $f(x) = \frac{\alpha x}{x + 1}$, $x \ne -1$. Then for what value of α is $f(f(x)) = x$?
(A) $\sqrt{2}$ (B) $-\sqrt{2}$ (C) 1 (D) -1.

Q.8(a) Suppose $f(x) = (x + 1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals
(A) $-\sqrt{x} - 1, x \ge 0$ (B) $\frac{1}{(x+1)^2}, x \ge -1$ (C) $\sqrt{x+1}, x \ge -1$ (D) $\sqrt{x} - 1, x \ge 0$

(b) Let function $f : R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then f is
(A) one to one and onto (B) one to one but NOT onto
(C) onto but NOT one to one (D) neither one to one nor onto

Q.9(a) Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}$ is
(A) $[1, 2]$ (B) $[1, \infty)$ (C) $\left[2, \frac{7}{3}\right]$ (D) $\left(1, \frac{7}{3}\right]$

(b) Let $f(x) = \frac{x}{1+x}$ defined from $(0, \infty) \to [0, \infty)$ then by $f(x)$ is
(A) one- one but not onto (B) one- one and onto
(C) Many one but not onto (D) Many one and onto [JEE 2003 (Scr),3+3]

Q.10 Let $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$. Thus $g(f(x))$ is invertible for $x \in$
(A) $\left[-\frac{\pi}{2}, 0\right]$ (B) $\left[-\frac{\pi}{2}, \pi\right]$ (C) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (D) $\left[0, \frac{\pi}{2}\right]$ [JEE 2004 (Screening)]

Q.11(a) If the functions $f(x)$ and $g(x)$ are defined on $R \to R$ such that
$$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}, \quad g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$$
then $(f - g)(x)$ is
(A) one-one and onto (B) neither one-one nor onto (C) one-one but not onto (D) onto but not one-one

(b) X and Y are two sets and $f : X \to Y$. If $\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x; d \subset Y, x \subset X\}$, then the true statement is
(A) $f\left(f^{-1}(b)\right) = b$ (B) $f^{-1}\left(f(a)\right) = a$
(C) $f\left(f^{-1}(b)\right) = b$, $b \subset y$ (D) $f^{-1}\left(f(a)\right) = a$, $a \subset x$ [JEE 2005 (Scr.)]

ANSWER KEY

FUNCTIONS

EXERCISE–1

Q 1. **(i)** $\left[-\frac{5\pi}{4}, \frac{-3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ **(ii)** $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$ **(iii)** $(-\infty, -3]$

(iv) $(-\infty, -1) \cup [0, \infty)$ **(v)** $(3 - 2\pi < x < 3 - \pi)$ U $(3 < x \le 4)$ **(vi)** $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{\sqrt{10}}\right)$

(vii) $(-1 < x < -1/2)$ U $(x > 1)$ **(viii)** $\left[\frac{1-\sqrt{5}}{2}, 0\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$ **(ix)** $(-3, -1]$ U $\{0\}$ U $[1, 3)$

(x) $\{4\} \cup [5, \infty)$ **(xi)** $(0, 1/4)$ U $(3/4, 1)$ U $\{x : x \in N, x \ge 2\}$ **(xii)** $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$

(xiii) $[-3, -2) \cup [3, 4)$ **(xiv)** ϕ

(xv) $2K\pi < x < (2K+1)\pi$ but $x \ne 1$ where K is non–negative integer

(xvi) $\{x \mid 1000 \le x < 10000\}$ **(xvii)** $(-2, -1)$ U $(-1, 0)$ U $(1, 2)$ **(xviii)** $(1, 2) \cup \left(2, \frac{5}{2}\right)$

(xix) $(-\infty, -3) \cup (-3, 1] \cup [4, \infty)$

Q 2.

(i) D : $x \,\varepsilon\, R$ R : $[0, 2]$ **(ii)** D = R ; range $[-1, 1]$

(iii) D : $\{x \mid x \in R;\ x \ne -3;\ x \ne 2\}$ R : $\{f(x) \mid f(x) \in R,\ f(x) \ne 1/5;\ f(x) \ne 1\}$

(iv) D : R ; R : $(-1, 1)$ **(v)** D : $-1 \le x \le 2$ R : $\left[\sqrt{3}, \sqrt{6}\right]$

(vi) D : $x \in (2n\pi, (2n+1)\pi) - \left\{2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{2}, 2n\pi + \frac{5\pi}{6}, n \in I\right\}$ and
R : $\log_a 2$; $a \in (0, \infty) - \{1\} \Rightarrow$ Range is $(-\infty, \infty) - \{0\}$

(vii) D : $[-4, \infty) - \{5\}$; R : $\left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{3}\right]$

Q.4 **(a)** neither surjective nor injective **(b)** surjective but not injective
(c) neither injective nor surjective

Q.5 $f_{3n}(x) = x$; Domain = R – $\{0, 1\}$

Q.6 1 **Q.7** **(a)** $2K\pi \le x \le 2K\pi + \pi$ where $K \in I$ **(b)** $[-3/2, -1]$

Q.8 **(i)** (a) odd, (b) even, (c) neither odd nor even, (d) odd, (e) neither odd nor even, (f) even,
(g) even, (h) even; **(ii)** $\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$

Q.9 **(a)** $y = \log(10 - 10^x)$, $-\infty < x < 1$
(b) $y = x/3$ when $-\infty < x < 0$ & $y = x$ when $0 \le x < +\infty$

Q.10 $f^{-1}(x) = (a - x^n)^{1/n}$

Q.12 **(a)** $f(x) = 1$ for $x < -1$ & $-x$ for $-1 \le x \le 0$; **(b)** $f(x) = -1$ for $x < -1$ and x for $-1 \le x \le 0$

Q.13 $g(x) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$ **Q.14** $\{-1, 1\}$

Q.15 **(a)** $\frac{e^x - e^{-x}}{2}$; **(b)** $\frac{\log_2 x}{\log_2 x - 1}$; **(c)** $\frac{1}{2}\log\frac{1+x}{1-x}$ **Q.16** $x = 1$

Q.17 **(i)** period of fog is π, period of gof is 2π ; **(ii)** range of fog is $[-1, 1]$, range of gof is $[-\tan 1, \tan 1]$

Q.18 **(a)** $\pi/2$ **(b)** π **(c)** $\pi/2$ **(d)** 70π

Q.20 $\pm1,\ \pm3,\ \pm5,\ \pm15$

EXERCISE–2

Q 1. $f^{-1}(1) = y$

Q.2 (a) – 3/4, (b) 64, (c) 30, (d) 102, (e) 5050

Q.3 (a) $\frac{1}{1002}$, (b) 1, (c) [0, 4), (d) – 5

Q 4. b can be any real number except $\frac{15}{4}$ **Q5.** $f(x) = 1 - x^2$, D = $x \in R$; range $=(-\infty, 1]$

Q.6 6016 **Q 9.** $f(x) = 2x^2$

Q 11. $fog(x) = \begin{cases} -(1+x) & , -1 \le x \le 0 \\ x-1 & , 0 < x \le 2 \end{cases}$; $gof(x) = \begin{cases} x+1 & , 0 \le x < 1 \\ 3-x & , 1 \le x \le 2 \\ x-1 & , 2 < x \le 3 \\ 5-x & , 3 < x \le 4 \end{cases}$;

$fof(x) = \begin{cases} x & , 0 \le x \le 1 \\ 4-x & , 3 \le x \le 4 \end{cases}$; $gog(x) = \begin{cases} -x & , -1 \le x \le 0 \\ x & , 0 < x \le 2 \\ 4-x & , 2 < x \le 3 \end{cases}$

Q 12. $\left(-\frac{\sqrt{3}+1}{\sqrt{2}}, \frac{1-\sqrt{3}}{\sqrt{2}}\right) \cup \left(\frac{\sqrt{3}-1}{\sqrt{2}}, \frac{\sqrt{3}+1}{\sqrt{2}}\right)$ **Q.13** x = 0 or 5/3

Q.14 1002.5 **Q.15** 5049 **Q.16** g (x) = 3 + 5 sin($n\pi$ + 2x – 4), $n \in I$

Q.17 20 **Q 18.** $(0, 1) \cup \{1, 2, ..., 12\} \cup (12, 13)$ **Q 19.** $f(x) = \sin x + x - \frac{\pi}{3}$

EXERCISE–3

Q.1 $(hofog)(x) = h(x^2) = x^2$ for $x \in R$, Hence h is not an identity function , fog is not invertible

Q.2 **(a)** A, **(b)** B

Q.3 $(fog)(x) = e^{3x-2}$; $(gof)(x) = 3e^x - 2$;
Domain of $(fog)^{-1}$ = range of fog = $(0, \infty)$; Domain of $(gof)^{-1}$ = range of gof = $(-2, \infty)$

Q.4 B **Q.5** D

Q.6 {(1, 1), (2, 3), (3, 4), (4, 2)} ; {(1, 1), (2, 4), (3, 2), (4, 3)} and {(1, 1), (2, 4), (3, 3), (4, 2)}

Q.7 **(a)** B, **(b)** A, **(c)** D, **(d)** A, **(e)** D **Q.8** **(a)** D ; **(b)** A

Q.9 **(a)** D , **(b)** A **Q.10** C **Q.11** **(a)** A ; **(b)** D

Exercise-4

Part : (A) Only one correct option

1. The domain of the function $f(x) = \frac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{x^2+2x+8}}$ is

(A) (1, 4) (B) (– 2, 4) (C) (2, 4) (D) $[2, \infty)$

2. The function $f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2+3x+1}$ is defined on the set S, where S is equal to:

(A) {0, 3} (B) (0, 3) (C) {0, – 3} (D) [– 3, 0]

3. The range of the function $f(x) = \sin^{-1}\left[x^2+\frac{1}{2}\right] + \cos^{-1}\left[x^2-\frac{1}{2}\right]$, where [ ] is the greatest integer function, is:

(A) $\left\{\frac{\pi}{2}, \pi\right\}$ (B) $\left\{0, \frac{\pi}{2}\right\}$ (C) $\{\pi\}$ (D) $\left(0, \frac{\pi}{2}\right)$

4. Range of $f(x) = \log_{\sqrt{5}}\{\sqrt{2}(\sin x - \cos x) + 3\}$ is

(A) [0, 1] (B) [0, 2] (C) $\left[0, \frac{3}{2}\right]$ (D) none of these

5. Range of $f(x) = 4^x + 2^x + 1$ is
(A) $(0, \infty)$ (B) $(1, \infty)$ (C) $(2, \infty)$ (D) $(3, \infty)$

6. If x and y satisfy the equation $y = 2[x] + 3$ and $y = 3[x - 2]$ simultaneously, the $[x + y]$ is
(A) 21 (B) 9 (C) 30 (D) 12

7. The function $f : [2, \infty) \to Y$ defined by $f(x) = x^2 - 4x + 5$ is both one–one & onto if
(A) $Y = R$ (B) $Y = [1, \infty)$ (C) $Y = [4, \infty)$ (D) $Y = [5, \infty)$

8. Let S be the set of all triangles and R^+ be the set of positive real numbers. Then the function, $f : S \to R^+$, $f(\Delta)$ = area of the Δ, where $\Delta \in S$ is :
(A) injective but not surjective (B) surjective but not injective
(C) injective as well as surjective (D) neither injective nor surjective

9. Let f(x) be a function whose domain is $[-5, 7]$. Let $g(x) = |2x + 5|$. Then domain of (fog) (x) is
(A) $[-4, 1]$ (B) $[-5, 1]$ (C) $[-6, 1]$ (D) none of these

10. The inverse of the function $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is
(A) $\frac{1}{2}\log\frac{1+x}{1-x}$ (B) $\frac{1}{2}\log\frac{2+x}{2-x}$ (C) $\frac{1}{2}\log\frac{1-x}{1+x}$ (D) $2\log(1 + x)$

11. The fundamental period of the function,
$f(x) = x + a - [x + b] + \sin \pi x + \cos 2\pi x + \sin 3\pi x + \cos 4\pi x + \ldots\ldots + \sin (2n - 1)\pi x$
$+ \cos 2n\pi x$ for every $a, b \in R$ is: (where [ ] denotes the greatest integer function)
(A) 2 (B) 4 (C) 1 (D) 0

12. The period of $e^{\cos^4 \pi x + x - [x] + \cos \pi x}$ is ______(where [ ] denotes the greatest integer function)
(A) 1 (B) 2 (C) 3 (D) 4

13. If y = f(x) satisfies the condition $f\left(x + \frac{1}{x}\right) = x^2 + \dfrac{1}{x^2}$ $(x \neq 0)$ then f(x) =
(A) $-x^2 + 2$ (B) $-x^2 - 2$ (C) $x^2 + 2$ (D) $x^2 - 2$

14. Given the function $f(x) = \dfrac{a^x + a^{-x}}{2}$ $(a > 0)$. If $f(x + y) + f(x - y) = k\, f(x).\, f(y)$ then k has the value equal to:
(A) 1 (B) 2 (C) 4 (D) 1/2

15. A function $f : R \to R$ satisfies the condition, $x^2 f(x) + f(1 - x) = 2x - x^4$. Then f(x) is:
(A) $-x^2 - 1$ (B) $-x^2 + 1$ (C) $x^2 - 1$ (D) $-x^4 + 1$

16. The domain of the function, $f(x) = \sqrt{\dfrac{1}{(|x| - 1)\cos^{-1}(2x + 1)\tan 3x}}$ is:
(A) $(-1, 0)$ (B) $(-1, 0) - \left\{-\frac{\pi}{6}\right\}$ (C) $(-1, 0] - \left\{-\frac{\pi}{6}, -\frac{\pi}{2}\right\}$ (D) $\left(-\frac{\pi}{6}, 0\right)$

17. If $f(x) = 2[x] + \cos x$, then $f: R \to R$ is: (where [ ] denotes greatest integer function)
(A) one–one and onto (B) one–one and into (C) many–one and into (D) many–one and onto

18. If $q^2 - 4pr = 0$, $p > 0$, then the domain of the function, $f(x) = \log(p x^3 + (p + q) x^2 + (q + r) x + r)$ is:
(A) $R - \left\{-\frac{q}{2p}\right\}$ (B) $R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$ (C) $R - \left[(-\infty, -1) \cap \left\{-\frac{q}{2p}\right\}\right]$ (D) none of these

19. If $[2 \cos x] + [\sin x] = -3$, then the range of the function, $f(x) = \sin x + \sqrt{3} \cos x$ in $[0, 2\pi]$ is:
(where [ ] denotes greatest integer function)
(A) $[-2, -1)$ (B) $(-2, -1]$ (C) $(-2, -1)$ (D) $[-2, -\sqrt{3})$

20. The domain of the function $f(x) = \log_{1/2}\left(-\log_2\left(1 + \dfrac{1}{\sqrt[4]{x}}\right) - 1\right)$ is:
(A) $0 < x < 1$ (B) $0 < x \le 1$ (C) $x \ge 1$ (D) null set

21. The range of the functions $f(x) = \log_{\sqrt{2}}\left(2 - \log_2\left(16\sin^2 x + 1\right)\right)$ is
(A) $(-\infty, 1)$ (B) $(-\infty, 2)$ (C) $(-\infty, 1]$ (D) $(-\infty, 2]$

22. The domain of the function, $f(x) = \sin^{-1}\left(\dfrac{1 + x^3}{2x^{3/2}}\right) + \sqrt{\sin(\sin x)} + \log_{(3\{x\} + 1)}(x^2 + 1)$,
where {x} represents fractional part function is:
(A) $x \in \{1\}$ (B) $x \in R - \{1, -1\}$ (C) $x > 3, x \neq I$ (D) none of these

23. The minimum value of $f(x) = a\tan^2 x + b\cot^2 x$ equals the maximum value of $g(x) = a\sin^2 x + b\cos^2 x$ where $a > b > 0$, when
(A) 4a = b (B) 3a = b (C) a = 3b (D) a = 4b

24. Let $f(2, 4) \to (1, 3)$ be a function defined by $f(x) = x - \left[\dfrac{x}{2}\right]$ (where [.] denotes the greatest integer function), then $f^{-1}(x)$ is equal to :
(A) 2x (B) $x + \left[\frac{x}{2}\right]$ (C) x + 1 (D) x – 1

25. The image of the interval R when the mapping $f: R \to R$ given by $f(x) = \cot^{-1}(x^2 - 4x + 3)$ is
(A) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (B) $\left[\frac{\pi}{4}, \pi\right)$ (C) $(0, \pi)$ (D) $\left(0, \frac{3\pi}{4}\right]$

26. If the graph of the function $f(x) = \frac{a^x - 1}{x^n (a^x + 1)}$ is symmetric about y-axis, then n is equal to:
(A) 2 (B) 2 / 3 (C) 1 / 4 (D) – 1 / 3

27. If $f(x) = \cot^{-1}x$ $: R^+ \to \left(0, \frac{\pi}{2}\right)$
and $g(x) = 2x - x^2$ $: R \to R$. Then the range of the function f(g(x)) wherever define is
(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{4}\right)$ (C) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left(\frac{\pi}{4}\right)$

28. Let f: $(e^2, \infty) \to R$ be defined by $f(x) = \ell n\,(\ell n(\ell n\, x))$, then
(A) f is one one but not onto (B) f is on to but not one - one (C) f is one-one and onto (D) f is neither one-one nor onto

29. Let f: $(e, \infty) \to R$ be defined by $f(x) = \ell n\,(\ell n(\ell n\, x))$, then
(A) f is one one but not onto (B) f is on to but not one - one
(C) f is one-one and onto (D) f is neither one-one nor onto

30. Let f(x) = sin x and g(x) = $|\ell n\, x|$ if composite functions fog(x) and gof (x) are defined and have ranges R_1 & R_2 respectively then.
(A) $R_1 = \{u: -1 < u < 1\}$ $R_2 = \{v: 0 < v < \infty\}$
(B) $R_1 = \{u: -\infty < u \le 0\}$ $R_2 = \{v: -1 \le v \le 1\}$
(C) $R_1 = \{u: 0 \le u < \infty\}$ $R_2 = \{v: -1 < v < 1;\ v \ne 0\}$
(D) $R_1 = \{u: -1 \le u \le 1\}$ $R_2 = \{v: 0 \le v < \infty\}$

31. Function f : $(-\infty, 1) \to (0, e^5]$ defined by $f(x) = e^{-(x^2 - 3x + 2)}$ is
(A) many one and onto (B) many one and into (C) one one and onto (D) one one and into

32. The number of solutions of the equation $[\sin^{-1} x] = x - [x]$, where [.] denotes the greatest integer function is
(A) 0 (B) 1 (C) 2 (D) infinitely many

33. The function $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ is
(A) an odd function (B) an even function
(C) neither an odd nor an even function (D) a periodic function

Part : (B) May have more than one options correct

34. For the function $f(x) = \ell n\ (\sin^{-1} \ell og_2 x)$,
(A) Domain is $\left[\frac{1}{2}, 2\right]$ (B) Range is $\left(-\infty, \ell n\frac{\pi}{2}\right]$ (C) Domain is (1, 2] (D) Range is R

35. A function ' f ' from the set of natural numbers to integers defined by,
$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases} \text{ is:}$$
(A) one-one (B) many-one (C) onto (D) into

36. Domain of $f(x) = \sin^{-1}[2 - 4x^2]$ where [x] denotes greatest integer function is:
(A) $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right) - \{0\}$ (B) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] - \{0\}$ (C) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ (D) $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$

37. If $F(x) = \frac{\sin \pi [x]}{\{x\}}$, then F (x) is:
(A) periodic with fundamental period 1 (B) even
(C) range is singleton
(D) identical to $\text{sgn}\left(\text{sgn}\,\frac{\{x\}}{\sqrt{\{x\}}}\right) - 1$, where {x} denotes fractional part function and [.] denotes greatest integer function and sgn (x) is a signum function.

38. $D \equiv [-1, 1]$ is the domain of the following functions, state which of them are injective.
(A) $f(x) = x^2$ (B) $g(x) = x^3$ (C) $h(x) = \sin 2x$ (D) $k(x) = \sin(\pi x/2)$

Exercise-5

1. Find the domain of the function $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

2. Find the domain of the function $f(x) = \sqrt{1-2x} + 3\sin^{-1}\left(\frac{3x-1}{2}\right)$

3. Find the inverse of the following functions. $f(x) = \ell n\,(x + \sqrt{1+x^2})$

4. Let f : $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \to B$ defined by $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1$. Find the B such that f^{-1} exists. Also find $f^{-1}(x)$.

5. Find for what values of x, the following functions would be identical.

$f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$.

6. If $f(x) = \frac{4^x}{4^x+2}$, then show that $f(x) + f(1-x) = 1$

7. Let f(x) be a polynomial function satisfying the relation $f(x).\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall x \in R - \{0\}$ and $f(3) = -26$. Determine $f'(1)$.

8. Find the domain of definitions of the following functions.

(i) $f(x) = \sqrt{3 - 2^x - 2^{1-x}}$ (ii) $f(x) = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1-x}{1+x}}$

(iii) $f(x) = \ell og_{10}(1 - \ell og_{10}(x^2 - 5x + 16))$

9. Find the range of the following functions.

(i) $f(x) = \frac{x^2-2x+4}{x^2+2x+4}$ (ii) $f(x) = \sin \ell og\left(\frac{\sqrt{4-x^2}}{1-x}\right)$

(iii) $f(x) = x^4 - 2x^2 + 5$ (iv) $f(x) = \sin^2 x + \cos^4 x$

10. Solve the following equation for x (where [x] & {x} denotes integral and fractional part of x)
$2x + 3[x] - 4\{-x\} = 4$

11. Draw the graph of following functions where [.] denotes greatest integer function and { .} denotes fractional part function.

(i) $y = \{\sin x\}$ (ii) $y = [x] + \sqrt{\{x\}}$

12. Draw the graph of the function $f(x) = \left|x^2 - 4|x| + 3\right|$ and also find the set of values of 'a' for which the equation f(x) = a has exactly four distinct real roots.

13. Examine whether the following functions are even or odd or none.

(i) $f(x) = \frac{(1+2^x)^7}{2^x}$ (ii) $f(x) = \begin{cases} x|x|, & x \le -1 \\ [1+x]+[1-x], & -1 < x < 1 \\ -x|x|, & x \ge 1 \end{cases}$

(iii) $f(x) = \frac{2x(\sin x + \tan x)}{2\left[\frac{x+2\pi}{\pi}\right] - 3}$, where [ ] denotes greatest integer function.

14. Find the period of the following functions.

(i) $f(x) = 1 - \frac{\sin^2 x}{1+\cot x} - \frac{\cos^2 x}{1+\tan x}$

(ii) $f(x) = \tan\frac{\pi}{2}[x]$, where [.] denotes greatest integer function.

(iii) $f(x) = \frac{1}{2}\left(\frac{|\sin x|}{\cos x} + \frac{\sin x}{|\cos x|}\right)$ (iv) $f(x) = \frac{\sin x + \sin 3x}{\cos x + \cos 3x}$

15. If $f(x) = \begin{cases} 1+x^2 & x \le 1 \\ x+1 & 1 < x \le 2 \end{cases}$ and $g(x) = 1 - x\ ;\ -2 \le x \le 1$ then define the function fog(x).

16. Find the set of real x for which the function, $f(x) = \frac{1}{[|x-1|] + [|12-x|] - 11}$ is not defined, where [x] denotes the greatest integer not greater than x.

17. Given the functions $f(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)}$, $g(x) = \operatorname{cosec}^{-1}\left(\frac{4-2\cos x}{3}\right)$ & the function h(x) = f(x) defined only for those values of x, which are common to the domains of the functions f(x) and g(x). Calculate the range of the function h(x).

18. Let 'f' be a real valued function defined for all real numbers x such that for some positive constant 'a' the equation $f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$ holds for all x. Prove that the function f is periodic.

19. If $f(x) = -1 + |x-2|,\ 0 \le x \le 4$
$g(x) = 2 - |x|,\ -1 \le x \le 3$

Then find fog (x), gof (x), fof(x) & gog(x). Draw rough sketch of the graphs of fog (x) & gof (x).

20. Find the integral solutions to the equation [x] [y] = x + y. Show that all the non-integral solutions lie on exactly two lines. Determine these lines. Here [.] denotes greatest integer function.

Exercise-4

1. D	**2.** C	**3.** C	**4.** B	**5.** B	**6.** C	**7.** B
8. B	**9.** C	**10.** A	**11.** A	**12.** B	**13.** D	**14.** B
15. B	**16.** D	**17.** C	**18.** B	**19.** D	**20.** D	**21.** D
22. D	**23.** D	**24.** C	**25.** D	**26.** D	**27.** C	**28.** A
29. C	**30.** D	**31.** D	**32.** B	**33.** B	**34.** BC	
35. AC	**36.** B	**37.** ABCD		**38.** BD		

Exercise-5

1. $[-2, 0) \cup (0, 1)$ **2.** $\left[-\frac{1}{3}, \frac{1}{2}\right]$

3. $f^{-1} = \frac{e^x - e^{-x}}{2}$

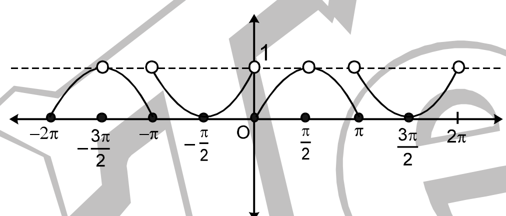
4. $B = [0, 4]$; $f^{-1}(x) = \frac{1}{2}\left(\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}\right)$

5. $(2, \infty)$ **7.** -3 **8.** (i) $[0, 1]$ (ii) ϕ (iii) $(2, 3)$

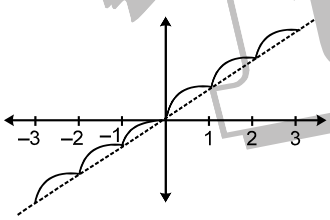
9. (i) $\left[\frac{1}{3}, 3\right]$ (ii) $[-1, 1]$ (iii) $[4, \infty)$ (iv) $\left[\frac{3}{4}, 1\right]$

10. $\left\{\frac{3}{2}\right\}$

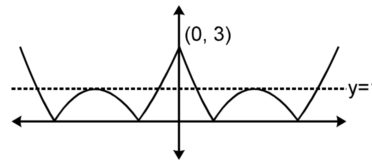
11. (i)



(ii)



12.



$a \in (1, 3) \cup \{0\}$

13. (i) neither even nor odd (ii) even (iii) odd

14. (i) π (ii) 2 (iii) 2π (iv) π

15. $f(g(x)) = \begin{cases} 2 - 2x + x^2 & 0 \le x \le 1 \\ 2 - x & -1 \le x < 0 \end{cases}$

16. (0, 1) U {1, 2,......., 12} U (12, 13) **17.** $\left[e^{\frac{\pi}{6}}, e^{\pi}\right]$

18. Period 2 a

19. $fog(x) = \begin{cases} -(1+x) & , \ -1 \le x \le 0 \\ x - 1 & , \ 0 < x \le 2 \end{cases}$;

$gof(x) = \begin{cases} x + 1 & , \ 0 \le x < 1 \\ 3 - x & , \ 1 \le x \le 2 \\ x - 1 & , \ 2 < x \le 3 \\ 5 - x & , \ 3 < x \le 4 \end{cases}$

$fof(x) = \begin{cases} x & , \ 0 \le x \le 2 \\ 4 - x & , \ 2 < x \le 2 \end{cases}$;

$gog(x) = \begin{cases} -x & , \ -1 \le x \le 0 \\ x & , \ 0 < x \le 2 \\ 4 - x & , \ 2 < x \le 3 \end{cases}$

20. Integral solution (0, 0); (2, 2). x + y = 6, x + y = 0

For 38 Years Que. from IIT-JEE(Advanced) &

14 Years Que. from AIEEE (JEE Main)

we distributed a book in class room