विघ्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचितः मानव धर्म प्रणेता
सद्गुरु श्री रणछोड़दासजी महाराज

STUDY PACKAGE

Subject : Mathematics

Topic : VECTORS & 3 DIMENSIONAL GEOMETRY

Index

Student's Name :______________________

Class :______________________

Roll No. :______________________

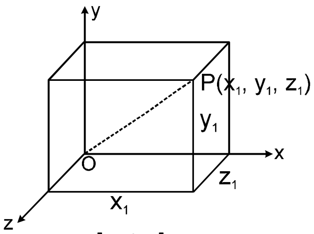
ADDRESS: R-1, Opp. Raiway Track,
New Corner Glass Building, Zone-2, M.P. NAGAR, Bhopal
☎ : (0755) 32 00 000, 98930 58881, www.tekoclasses.com

THREE DIMENSIONAL GEOMETRY

Coordinate of a point in space

There are infinite number of points in space. We want to identify each and every point of space with the help of three mutually perpendicular coordinate axes OX, OY and OZ.
Three mutually perpendicular lines OX, OY, OZ are considered as the three axes.

The plane formed with the help of x and y axes is called x-y plane, similarly y & z axes form y-z plane and z and x axes form z - x plane.
Consider any point P in the space, Drop a perpendicular from that point to x -y plane, then the algebraic length of this perpendicular is considered as z-coordinate and from foot of the perpendicular drop perpendiculars to x and y axes. These algebraic lengths of perpendiculars are considered as y and x coordinates respectively.



1. Vector representation of a point in space

If coordinate of a point P in space is (x, y, z) then the position vector of the point P with respect to the same origin is $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Distance formula

Distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Vector method We know that if position vector of two points A and B are given as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ then

$$AB = |\overrightarrow{OB} - \overrightarrow{OA}|$$

$\Rightarrow$ $AB = |(x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)|$ $\Rightarrow AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

3. Distance of a point P from coordinate axes

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then

$$PA = \sqrt{y^2 + z^2}\ ,\ PB = \sqrt{z^2 + x^2}\ ,\ PC = \sqrt{x^2 + y^2}$$

Example : Show that the points (0, 7, 10), (– 1, 6, 6) and (– 4, 9, 6) form a right angled isosceles triangle.

Solution: Let $A \equiv (0, 7, 10)$, $B \equiv (-1, 6, 6)$, $C \equiv (-4, 9, 6)$

$AB^2 = (0 + 1)^2 + (7 - 6)^2 + (10 - 6)^2 = 18$ $\quad\therefore\quad AB = 3\sqrt{2}$

Similarly $\therefore$ $BC = 3\sqrt{2}$,

& $AC = 6$

Clearly $AB^2 + BC^2 = AC^2$ $\quad\therefore\quad \angle ABC = 90°$

Also $AB = BC$

Hence ΔABC is right angled isosceles.

Example : Show by using distance formula that the points (4, 5, –5), (0, –11, 3) and (2, –3, –1) are collinear.

Solution Let $A \equiv (4, 5, -5)$, $B \equiv (0, -11, 3)$, $C \equiv (2, -3, -1)$.

$$AB = \sqrt{(4-0)^2 + (5+11)^2 + (-5-3)^2} = \sqrt{336} = \sqrt{4 \times 84} = 2\sqrt{84}$$

$$BC = \sqrt{(0-2)^2 + (-11+3)^2 + (3+1)^2} = \sqrt{84} \qquad AC = \sqrt{(4-2)^2 + (5+3)^2 + (-5+1)^2} = \sqrt{84}$$

$BC + AC = AB$

Hence points A, B, C are collinear and C lies between A and B.

Example : Find the locus of a point which moves such that the sum of its distances from points $A(0, 0, -\alpha)$ and $B(0, 0, \alpha)$ is constant.

Solution. Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$\therefore$ $\sqrt{(x-0)^2 + (y-0)^2 + (z+\alpha)^2} + \sqrt{(x-0)^2 + (y-0)^2 + (z-\alpha)^2} = 2a$

$\Rightarrow$ $\frac{1}{2}$ $= 2a - \sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow$ $x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha - 4a\sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow$ $4z\alpha - 4a^2 = -4a\sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow$ $\frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$

or, $\quad x^2 + y^2 + z^2\left(1-\dfrac{\alpha^2}{a^2}\right) = a^2 - \alpha^2$ or, $\dfrac{x^2+y^2}{a^2-\alpha^2}+\dfrac{z^2}{a^2} = 1$ This is the required locus.

Self practice problems :

1. One of the vertices of a cuboid is (1, 2, 3) and the edges from this vertex are along the +ve x-axis, +ve y-axis and +z axis respectively and are of length 2, 3, 2 respectively find out the vertices.
Ans. (1, 2, 5), (3, 2, 5), (3, 2, 3), (1, – 1, 3), (1, – 1, 5), (3, – 1, 5), (3, – 1, 3).

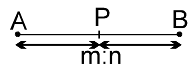
2. Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

3. Find the locus of point P if $AP^2 - BP^2 = 18$, where $A \equiv (1, 2, -3)$ and $B \equiv (3, -2, 1)$
Ans. $2x - 4y + 4z - 9 = 0$

4. Section Formula

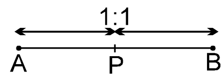
If point P divides the distance between the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio of m : n, then coordinates of P are given as

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$$

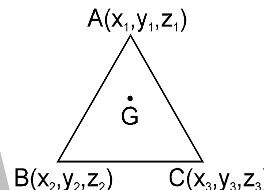


Note :- Mid point

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$



5. Centroid of a triangle



$$G \equiv \left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$

6. Incentre of triangle ABC: $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c},\dfrac{ay_1+by_2+cy_3}{a+b+c},\dfrac{az_1+bz_2+cz_3}{a+b+c}\right)$

Where AB = c, BC = a, CA = b

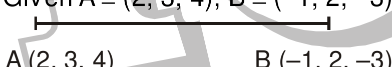
7. Centroid of a tetrahedron

A (x_1, y_1, z_1) B (x_2, y_2, z_2) C (x_3, y_3, z_3) and D (x_4, y_4, z_4) are the vertices of a tetrahedron then coordinate of its centroid (G) is given as

$$\left(\frac{\sum x_i}{4},\frac{\sum y_i}{4},\frac{\sum z_i}{4}\right)$$

Example : Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Solution: Given $A \equiv (2, 3, 4)$, $B \equiv (-1, 2, -3)$, $C \equiv (-4, 1, -10)$.



Let C divide AB internally in the ratio k : 1, then

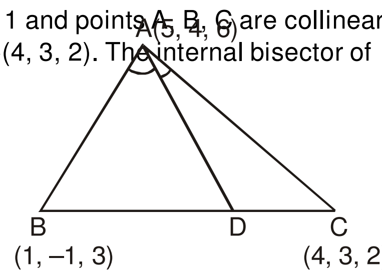
$$C \equiv \left(\frac{-k+2}{k+1},\frac{2k+3}{k+1},\frac{-3k+4}{k+1}\right)$$

$\therefore \quad \dfrac{-k+2}{k+1} = -4 \quad \Rightarrow \quad 3k = -6 \Rightarrow \quad k = -2$

For this value of k, $\dfrac{2k+3}{k+1} = 1$, and $\dfrac{-3k+4}{k+1} = -10$

Since k < 0, therefore k divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

Example : The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of $\angle$BAC meets BC in D. Find AD.



Solution $\quad AB = \sqrt{4^2+5^2+3^2} = 5\sqrt{2}$

$AC = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}$

Since AD is the internal bisector of $\angle$BAC

$\therefore \quad \dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{5}{3}$

$\therefore \quad$ D divides BC internally in the ratio 5 : 3

$\therefore \quad D \equiv \left(\dfrac{5\times4+3\times1}{5+3},\dfrac{5\times3+3(-1)}{5+3},\dfrac{5\times2+3\times3}{5+3}\right)$ or, $D = \left(\dfrac{23}{8},\dfrac{12}{8},\dfrac{19}{8}\right)$

$\therefore \quad AD = \sqrt{\left(5-\dfrac{23}{8}\right)^2+\left(4-\dfrac{12}{8}\right)^2+\left(6-\dfrac{19}{8}\right)^2}$

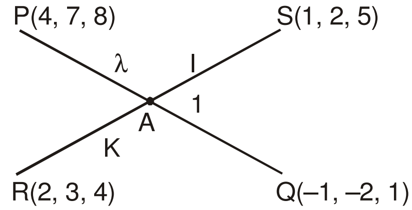
$= \dfrac{\sqrt{1530}}{8}$ unit

Example :

If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1, 2, 5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Solution

Let the lines PQ and RS intersect at point A.



Let A divide PQ in the ratio λ : 1, then

$$A \equiv \left(\frac{-\lambda+4}{\lambda+1}, \frac{-2\lambda+7}{\lambda+1}, \frac{\lambda+8}{\lambda+1}\right). \qquad \text{.... (1)}$$

Let A divide RS in the ratio k : 1, then

$$A \equiv \left(\frac{k+2}{k+1}, \frac{2k+3}{k+1}, \frac{5k+4}{k+1}\right) \qquad \text{..... (2)}$$

From (1) and (2), we have,

$$\frac{-\lambda+4}{\lambda+1} = \frac{k+2}{k+1} \qquad \text{..... (3)}$$

$$\frac{-2\lambda+7}{\lambda+1} = \frac{2k+3}{k+1} \qquad \text{..... (4)}$$

$$\frac{\lambda+8}{\lambda+1} = \frac{5k+4}{k+1} \qquad \text{..... (5)}$$

From (3), $-\lambda k - \lambda + 4k + 4 = \lambda k + 2\lambda + k + 2$

or $\quad 2\lambda k + 3\lambda - 3k - 2 = 0$ ..... (6)

From (4), $-2\lambda k - 2\lambda + 7k + 7 = 2\lambda k + 3\lambda + 2k + 3$

or $\quad 4\lambda k + 5\lambda - 5k - 4 = 0$ ..... (7)

Multiplying equation (6) by 2, and subtracting from equation (7), we get

$-\lambda + k = 0$ or, $\lambda = k$

Putting $\lambda = k$ in equation (6), we get

$2\lambda^2 + 3\lambda - 3\lambda - 2 = 0$

or, $\lambda = \pm 1$.

But $\lambda \neq -1$, as the co-ordinates of P would then be underfined and in this case PQ || RS, which is not true.

$\therefore \quad \lambda = 1 = k$.

Clearly λ k = 1 satisfies eqn. (5).

Hence our assumption is correct

$$\therefore \quad A \equiv \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{1+8}{2}\right) \quad \text{or,} \quad A \equiv \left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right).$$

Self practice problems:

1. Find the ratio in which xy plane divides the line joining the points A (1, 2, 3) and B (2, 3, 6).
Ans. – 1 : 2

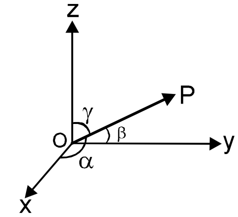
2. Find the co-ordinates of the foot of perpendicular drawn from the point A(1, 2, 1) to the line joining the point B(1, 4, 6) and C(5, 4, 4).
Ans. (3, 4, 5)

3. Two vertices of a triangle are (4, –6, 3) and (2, –2, 1) and its centroid is $\left(\frac{8}{3}, -1, 2\right)$. Find the third vertex.
Ans. (2, 5, 2)

4. If centroid of the tetrahedron OABC, where co-ordinates of A, B, C are (a, 2, 3), (1, b, 2) and (2, 1, c) respectively be (1, 2, 3), then find the distance of point (a, b, c) from the origin.
Ans. $\sqrt{107}$

5. Show that $\left(-\frac{1}{2}, 2, 0\right)$ is the circumcentre of the triangle whose vertices are A (1, 1, 0), B (1, 2, 1) and C (– 2, 2, –1) and hence find its orthocentre. **Ans.** (1, 11, 0)

8. Direction Cosines And Direction Ratios



(i) Direction cosines: Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n).
Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios: Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.
If a, b, c, are the direction ratios of any line L then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L then $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\left(\ell = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

or $\quad \ell = \dfrac{-a}{\sqrt{a^2+b^2+c^2}}$, $m = \dfrac{-b}{\sqrt{a^2+b^2+c^2}}$, $n = \dfrac{-c}{\sqrt{a^2+b^2+c^2}}$

(v) If OP = r, when O is the origin and the direction cosines of OP are ℓ, m, n then the coordinates of P are (ℓr, mr, nr).

If direction cosines of the line AB are ℓ, m,n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1) then the coordinates of B is given as $(x_1 + r\ell, y_1 + rm, z_1 + rn)$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are,

$a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \dfrac{x_2 - x_1}{|PQ|}$,

$m = \dfrac{y_2 - y_1}{|PQ|}$ and $n = \dfrac{z_2 - z_1}{|PQ|}$

(vii) Direction cosines of axes**:** Since the positive x–axis makes angles 0º, 90º, 90º with axes of x, y and z respectively. Therefore
Direction cosines of x–axis are (1, 0, 0)
Direction cosines of y–axis are (0, 1, 0)
Direction cosines of z–axis are (0, 0, 1)

Example : If a line makes angles α, β, γ with the co-ordinate axes, prove that $\sin^2\alpha + \sin^2\beta + \sin\gamma^2 = 2$.

Solution Since a line makes angles α, β, γ with the co-ordinate axes,
hence $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are its direction cosines

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$

Example : Find the direction cosines ℓ, m, n of a line which are connected by the relations $\ell + m + n = 0$, $2mn + 2m\ell - n\ell = 0$

Solution Given, $\ell + m + n = 0$ ..... (1)

$2mn + 2m\ell - n\ell = 0$ ..... (2)

From (1), $n = -(\ell + m)$.

Putting $n = -(\ell + m)$ in equation (2), we get,

$-2m(\ell + m) + 2m\ell + (\ell + m)\ell = 0$

or, $\quad -2m\ell - 2m^2 + 2m\ell + \ell^2 + m\ell = 0$

or, $\quad \ell^2 + m\ell - 2m^2 = 0$

or, $\quad \left(\dfrac{\ell}{m}\right)^2 + \left(\dfrac{\ell}{m}\right) - 2 = 0 \quad$ [dividing by m^2]

or $\quad \dfrac{\ell}{m} = \dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm 3}{2} = 1, -2$

Case I. when $\dfrac{\ell}{m} = 1$: In this case $m = \ell$

From (1), $2\ell + n = 0 \quad \Rightarrow \quad n = -2\ell$

$\therefore \quad \ell : m : n = 1 : 1 : -2$

$\therefore$ Direction ratios of the line are 1, 1, – 2

$\therefore$ Direction cosines are

$$\pm\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \pm\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \pm\frac{-2}{\sqrt{1^2+1^2+(-2)^2}}$$

or, $\quad \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}$ or $-\dfrac{1}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}$

Case II. When $\dfrac{\ell}{m} = -2$: In this case $\ell = -2m$

From (1), $-2m + m + n = 0 \quad \Rightarrow \quad n = m$

$\therefore \quad \ell : m : n = -2m : m : m$

$= -2 : 1 : 1$

$\therefore$ Direction ratios of the line are – 2, 1, 1.

$\therefore$ Direction cosines are

$$\frac{-2}{\sqrt{(-2)^2+1^2+1^2}}, \frac{-1}{\sqrt{(-2)^2+1^2+1^2}}, \frac{-1}{\sqrt{(-2)^2+1^2+1^2}} \quad \text{or,} \quad \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}.$$

Self practice problems:

1. Find the direction cosine of a line lying in the xy plane and making angle 30° with x-axis.

Ans. $m = \pm \frac{1}{2}, \ell = \frac{\sqrt{3}}{2}, n = 0$

2. A line makes an angle of 60° with each of x and y axes, find the angle which this line makes with z-axis. **Ans.** 45°

3. A plane intersects the co-ordinates axes at point A(a, 0, 0), B(0, b, 0), C(0, 0, c) O is origin. Find the direction ratio of the line joining the vertex B to the centroid of face AOC.

Ans. $\frac{a}{3}, -b, \frac{c}{3}$

4. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

9. Angle Between Two Line Segments:

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively then we can consider two vectors parallel to the lines as $a_1i + b_1j + c_1k$ and $a_2i + b_2j + c_2k$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(i) The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ **(ii)** The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Two parallel lines have same direction cosines i.e. $\ell_1 = \ell_2$, $m_1 = m_2$, $n_1 = n_2$

Example : What is the angle between the lines whose direction cosines are $-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}; -\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$?

Solution Let θ be the required angle, then

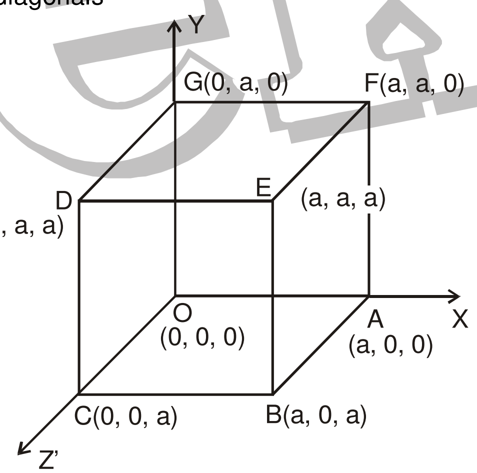
$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2} \quad \Rightarrow \quad \theta = 120°,$

Example : Find the angle between any two diagonals of a cube.

Solution The cube has four diagonals



OE, AD, CF and GB

The direction ratios of OE are

a, a, a or, 1, 1, 1

$\therefore$ its direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Direction ratios of AD are – a, a, a. or, – 1, 1, 1.

$\therefore$ its direction cosines are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Similarly, direction cosines of CF and GB respectively are

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

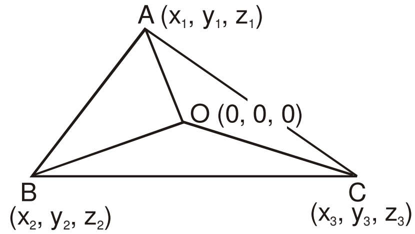
We take any two diagonals, say OE and AD

Let θ be the acute angle between them, then

$\cos\theta = \left|\left(\frac{1}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{3}}\right)\right| = \frac{1}{3}$ or, $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

Example : If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular.

Solution: Let OABC be the tetrahedron where O is the origin and co-ordinates of A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, x_3) respectively.



Let $OA \perp BC$ and $OB \perp CA$.

We have to prove that

$OC \perp BA$.

Now, direction ratios of OA are $x_1 - 0$, $y_1 - 0$, $z_1 - 0$ or, x_1, y_1, z_1

direction ratios of BC are $(x_3 - x_2)$, $(y_3 - y_2)$, $(z_3 - z_2)$.

$\because$ $OA \perp BC$.

$\therefore$ $x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$ ..... (1)

Similarly,

$\because$ $OB \perp CA$

$\therefore$ $x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$ ..... (2)

Adding equations (1) and (2), we get

$x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ [$\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2)$, $(y_1 - y_2)$, $(z_1 - z_2)$]

Self practice problems:

1. Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$ **Ans.** 60°

2. P (6, 3, 2)
Q (5, 1, 4)
R (3, 3, 5)
are vertices of a Δ find $\angle Q$. **Ans.** 90°

3. Show that the direction cosines of a line which is perpendicular to the lines having directions cosines $\ell_1\ m_1\ n_1$ and $\ell_2\ m_2\ n_2$ respectively are proportional to
$m_1n_2 - m_2n_1$, $n_1\ell_2 - n_2\ell_1$, $\ell_1m_2 - \ell_2m_1$

10. Projection of a line segment on a line

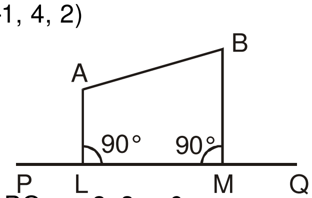
(i) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is $\left|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)\right|$

(ii) Vector form: projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $\vec{a}\,.\,\hat{b} = \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{b}|}$

In the above case we can consider $\overrightarrow{PQ}$ as $(x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$ in place of $\vec{a}$ and $\ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ in place of $\vec{b}$. **(iii)** $\ell\,|\vec{r}|$, $m\,|\vec{r}|$ & $n\,|\vec{r}|$ are the projection of $\vec{r}$ in OX, OY & OZ axes. **(iv)** $\vec{r} = |\vec{r}|\,(\ell\,\hat{i} + m\,\hat{j} + n\,\hat{k})$

Solved Example : Find the projection of the line joining (1, 2, 3) and (–1, 4, 2) on the line having direction ratios 2, 3, – 6.

Solution Let $A \equiv (1, 2, 3)$, $B \equiv (-1, 4, 2)$



Direction ratios of the given line PQ are 2, 3, – 6

$\sqrt{2^2 + 3^2 + (-6)^2} = 7$ $\therefore$ direction cosines of PQ are

$\dfrac{2}{7}, \dfrac{3}{7}, -\dfrac{6}{7}$

Projection of AB on PQ

$= \ell\,(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$= \dfrac{2}{7}\,(-1 - 1) + \dfrac{3}{7}\,(4 - 2) - \dfrac{6}{7}\,(2 - 3) \qquad = \dfrac{-4 + 6 + 6}{7} = \dfrac{8}{7}$

Self practice problems:

1. A (6, 3, 2), B (5, 1, 1,), C(3, –1, 3) D (0, 2, 5)
Find the projection of line segment AB on CD line. **Ans.** 5/7

2. The projections of a directed line segment on co-ordinate axes are – 2, 3, – 6. Find its length and direction cosines. **Ans.** 13 ; $\frac{12}{13}, \frac{4}{13}, \frac{3}{13}$

3. Find the projection of the line segment joining (2, – 1, 3) and (4, 2, 5) on a line which makes equal acute angles with co-ordinate axes. **Ans.** $\frac{7}{\sqrt{3}}$

A PLANE

If line joining any two points on a surface lies completely on it then the surface is a plane.

OR

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

11. Equation Of A Plane

(i) Normal form of the equation of a plane is $\ell x + my + nz = p$, where, ℓ,m n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(ii) General form: $ax + by + cz + d = 0$ is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.

(iii) The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(iv) Plane through three points: The equation of the plane through three non–collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

(v) Intercept Form: The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(vi) Vector form: The equation of a plane passing through a point having position vector $\vec{a}$ & normal to vector $\vec{n}$ is $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

Note: **(a)** Vector equation of a plane normal to unitvector $\hat{n}$ and at a distance d from the origin is $\vec{r} \cdot \vec{n} = d$

(b) **Coordinate planes (i)** Equation of yz–plane is x = 0 **(ii)** Equation of xz–plane is y = 0 **(iii)** Equation of xy–plane is z = 0

(c) **Planes parallel to the axes:**
If a = 0, the plane is parallel to x–axis i.e. equation of the plane parallel to the x–axis is by + cz + d = 0.
Similarly, equation of planes parallel to y–axis and parallel to z–axis are ax + cz +d = 0 and ax + by + d = 0 respectively.

(d) **Plane through origin:** Equation of plane passing through origin is ax + by + cz = 0.

(e) **Transformation of the equation of a plane to the normal form:** To reduce any equation ax + by + cz – d = 0 to the normal form, first write the constant term on the right hand side and make it positive, then divide each term by $\sqrt{a^2+b^2+c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2+b^2+c^2}} + \frac{by}{\pm\sqrt{a^2+b^2+c^2}} + \frac{cz}{\pm\sqrt{a^2+b^2+c^2}} = \frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

Where (+) sign is to be taken if d > 0 and (–) sign is to be taken if d < 0.

(f) Any plane parallel to the given plane ax + by + cz + d = 0 is ax + by + cz + λ = 0.
Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given as $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

(g) **Equation of a plane passing through a given point & parallel to the given vectors:**
The equation of a plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ & $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.
or $\vec{r}.(\vec{b}\times\vec{c}) = \vec{a}.(\vec{b}\times\vec{c})$ (non parametric form)

(h) A plane ax + by + cz + d = 0 divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$

(i) The xy–plane divides the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\frac{z_1}{z_2}$. Similarly yz–plane in $-\frac{x_1}{x_2}$ and zx–plane in $-\frac{y_1}{y_2}$

(j) Coplanarity of four points

The points $A(x_1\, y_1\, z_1)$, $B(x_2\, y_2\, z_2)$ $C(x_3\, y_3\, z_3)$ and $D(x_4\, y_4\, z_4)$ are coplaner then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

very similar in vector method the points A $(\vec{r}_1)$, B$(\vec{r}_2)$, C$(\vec{r}_3)$ and D$(\vec{r}_4)$ are coplanar if $[\vec{r}_4 - \vec{r}_1, \vec{r}_4 - \vec{r}_2, \vec{r}_4 - \vec{r}_3] = 0$

Example : Find the equation of the plane upon which the length of normal from origin is 10 and direction ratios of this normal are 3, 2, 6.

Solution If p be the length of perpendicular from origin to the plane and ℓ, m, n be the direction cosines of this normal, then its equation is

$\ell x + my + nz = p$ ..... (1)

Here p = 10; Direction ratios of normal to the plane are 3, 2, 6

$\sqrt{3^2 + 2^2 + 6^2} = 7$ $\therefore$ Direction cosines of normal to the required plane are

$\ell = \frac{3}{7}$, $m = \frac{2}{7}$, $n = \frac{6}{7}$

Putting the values of ℓ, m, n, p in (1), equation of required plane is

$\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$ or, $3x + 2y + 6z = 70$

Example : Show that the points (0, – 1, 0), (2, 1, – 1), (1, 1, 1), (3, 3, 0) are coplanar.

Solution Let $A \equiv (0, -1, 0)$, $B \equiv (2, 1, -1)$, $C \equiv (1, 1, 1)$ and $D \equiv (3, 3, 0)$

Equation of a plane through A (0, – 1, 0) is

$a (x - 0) + b (y + 1) + c (z - 0) = 0$

or, $ax + by + cz + b = 0$ ..... (1)

If plane (1) passes through B (2, 1, – 1) and C (1, 1, 1)

Then $2a + 2b - c = 0$ ..... (2)

and $a + 2b + c = 0$ ..... (3)

From (2) and (3), we have

$$\frac{a}{2+2} = \frac{b}{-1-2} = \frac{c}{4-2}$$

or, $\frac{a}{4} = \frac{b}{-3} = \frac{c}{2} = k$ (say)

Putting the value of a, b, c, in (1), equation of required plane is

$4kx - 3k(y + 1) + 2kz = 0$

or, $4x - 3y + 2z - 3 = 0$ ..... (2)

Clearly point D (3, 3, 0) lies on plane (2)

Thus points D lies on the plane passing through A, B, C and hence points A, B, C and D are coplanar.

Example : If P be any point on the plane $\ell x + my + nz = p$ and Q be a point on the line OP such that OP . OQ = p^2, show that the locus of the point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Solution Let $P \equiv (\alpha, \beta, \gamma)$, $Q \equiv (x_1, y_1, z_1)$

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have

$\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k$ (say) ..... (1)

As P (α, β, γ) lies on the plane $\ell x + my + nz = p$,

$\ell\alpha + m\beta + n\gamma = p$ or $k(x_1 + my_1 + nz_1) = p$ ..... (2)

Given, OP . OQ = p^2

$\therefore$ $\sqrt{\alpha^2 + \beta^2 + \gamma^2}\, \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$

or, $\sqrt{k^2(x_1^2 + y_1^2 + z_1^2)}\, \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$

or, $k(x_1^2 + y_1^2 + z_1^2) = p^2$ ..... (3)

On dividing (2) by (3), we get,

$$\frac{\ell x_1 + my_1 + nz_1}{x_1^2 + y_1^2 + z_1^2} = \frac{1}{p}$$

or, $p(\ell x_1 + my_1 + nz_1) = x_1^2 + y_1^2 + z_1^2$

Hence the locus of point Q is

$p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Example : A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the co-ordinate axes in A, B and C. If the planes through A, B and C parallel to the planes x = 0, y = 0, z = 0 intersect in Q, find the locus of Q.

Solution Given plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \text{..... (1)}$$

Let P ≡ (h, k, ℓ)

Then $\frac{h}{a}+\frac{k}{b}+\frac{\ell}{c} = 1$ ..... (2)

$$OP = \sqrt{h^2+k^2+\ell^2}$$

Direction cosines of OP are $\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$

∴ Equation of the plane through P and normal to OP is

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x+\frac{k}{\sqrt{h^2+k^2+\ell^2}}y+\frac{\ell}{\sqrt{h^2+k^2+\ell^2}}=\sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = (h^2 + k^2 + \ell^2)$

∴ $A \equiv \left(\frac{h^2+k^2+\ell^2}{h},\ 0, 0\right)$, $B \equiv \left(0, \frac{h^2+k^2+\ell^2}{k}, 0\right)$,

$C \equiv \left(0, 0, \frac{h^2+k^2+\ell^2}{\ell}\right)$

Let Q ≡ (α, β, γ), then

$$\alpha = \frac{h^2+k^2+\ell^2}{h},\ \beta = \frac{h^2+k^2+\ell^2}{k},\ \gamma = \frac{h^2+k^2+\ell^2}{\ell} \qquad \text{..... (3)}$$

Now $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{h^2+k^2+\ell^2}{(h^2+k^2+\ell^2)^2}=\frac{1}{(h^2+k^2+\ell^2)}$ ..... (4)

From (3), $h = \frac{h^2+k^2+\ell^2}{\alpha}$

∴ $\frac{h}{a}=\frac{h^2+k^2+\ell^2}{a\alpha}$

Similarly $\frac{k}{b}=\frac{h^2+k^2+\ell^2}{b\beta}$ and $\frac{\ell}{c}=\frac{h^2+k^2+\ell^2}{c\gamma}$

∴ $\frac{h^2+k^2+\ell^2}{a\alpha}+\frac{h^2+k^2+\ell^2}{b\beta}+\frac{h^2+k^2+\ell^2}{c\gamma}=\frac{h}{a}+\frac{k}{b}+\frac{\ell}{c} = 1$ [from (2)]

or, $\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+\ell^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ [from (4)]

∴ Required locus of Q (α, β, γ) is

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}.$$

Self practice problems :

1. Check wether this point are coplanar if yes find the equation of plane containing them
A ≡ (1, 1, 1)
B ≡ (0, – 1, 0)
C ≡ (2, 1, –1)
D ≡ (3, 3, 0) **Ans.** yes, 4x – 3y + 2z = 3

2. Find the plane passing through point (– 3, – 3, 1) and perpendicular to the line joining the points (2, 6, 1) and (1, 3, 0). **Ans.** x + 3y + z + 11 = 0

3. Find the equation of plane parallel to x + 5y – 4z + 5 = 0 and cutting intercepts on the axes whose rum is 150. **Ans.** $x + 5y - 4z = \frac{3000}{19}$

4. Find the equation of plane passing through (2, 2, 1) and (9, 3, 6) and perpendicular to the plane x + 3y + 3z = 8. **Ans.** 3x + 4y – 5z = 9

5. Find the equation of the plane | | to $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}$ and passing through (1, 1, 2).
Ans. x + y – 2z + 2 = 0

6. Find the equation of the plane passing through the point (1, 1, – 1) and perpendicular to the planes x + 2y + 3z – 7 = 0 and 2x – 3y + 4z = 0. **Ans.** 17x + 2y – 7z = 26

12. Sides of a plane:

A plane divides the three dimensional space in two equal parts. Two points A (x_1 y_1 z_1) and B (x_2 y_2 z_2) are on the same side of the plane ax + by + cz + d = 0 if $ax_1 + by_1 + cz_1 + d$ and

$ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

Example : Show that the points (1, 2, 3) and (2, – 1, 4) lie on opposite sides of the plane x + 4y + z – 3 = 0.

Solution Since the numbers 1+ 4 × 2 + 3 – 3 = 9 and 2 – 4 + 4 – 3 = – 1 are of opposite sign., the points are on opposite sides of the plane.

13. A Plane & A Point

(i) Distance of the point (x′, y′, z′) from the plane ax + by + cz+ d = 0 is given by $\dfrac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}$.

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}\,.\,\vec{n} = d$ is given by $p = \dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$.

(iii) The coordinates of the foot of perpendicular from the point (x, y, z) to the plane ax + by + cz + d = 0 aregain by $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c} = -\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(iv) **To find image of a point w.r.t. a plane.**

Let P (x_1, y_1, z_1) is a given point and ax + by + cz + d = 0 is given plane Let (x′, y′, z′) is the image point. then

(a) $x' - x_1 = \lambda a,\quad y' - y_1 = \lambda b,\quad z' - z_1 = \lambda c$

$\Rightarrow\quad x' = \lambda a + x_1,\quad y' = \lambda b + y_1,\quad z' = \lambda c + z_1$

(b) $a\left(\dfrac{x'+x_1}{2}\right) + b\left(\dfrac{y'+y_1}{2}\right) + c\left(\dfrac{z'+z_1}{2}\right) = 0$

from (i) put the values of x′, y′, z′ in (ii) and get the values of λ and resubtitute in (i) to get (x′ y′ z′).

The coordinate of the image of point (x_1, y_1, z_1) w.r.t the plane ax + by + cz + d = 0 are given by $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c} = -2\,\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(v) The distance between two parallel planes ax + by + cx + d = 0 and ax + by + cx + d′ = 0 is given by $\dfrac{|d-d'|}{\sqrt{a^2+b^2+c^2}}$

Example : Find the image of the point P (3, 5, 7) in the plane 2x + y + z = 16.

Solution Given plane is 2x + y + z = 16 ..... (1)

P ≡ (3, 5, 7)

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R

then PQ ⊥ plane (1)

Let R ≡ (2r + 3, r + 5, r + 7)

Since R lies on plane (1)

∴ 2(2r + 3) + r + 5 + r + 7 = 0 or, 6r + 18 = 0 ∴ r = – 3

∴ R ≡ (– 3, 2, 4)

Let Q ≡ (α, β, γ)

Since R is the middle point of PQ

$\therefore\quad -3 = \dfrac{\alpha+3}{2} \Rightarrow \alpha = -9$

$2 = \dfrac{\beta+5}{2} \Rightarrow \beta = -1$

$4 = \dfrac{\gamma+7}{2} \Rightarrow \gamma = 1 \quad \therefore \quad Q = (-9, -1, 1).$

Example : Find the distance between the planes 2x – y + 2z = 4 and 6x – 3y + 6z = 2.

Solution Given planes are

2x – y + 2z – 4 = 0 ..... (1)

and 6x – 3y + 6z – 2 = 0 ..... (2)

We find that $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ Hence planes (1) and (2) are parallel.

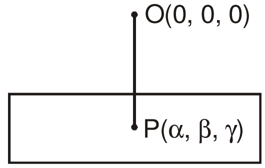
Plane (2) may be written as $2x - y + 2z - \dfrac{2}{3} = 0$ ..... (3)

∴ Required distance between the planes

$$= \frac{\left|4 - \frac{2}{3}\right|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{10}{3.3} = \frac{10}{9}$$

Example : A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$

Solution Let the equation of the variable plane be



$\ell x + my + nz + d = 0$ ..... (1)

Plane passes through the fixed point (a, b, c)

$\therefore$ $\ell a + mb + nc + d = 0$ ..... (2)

Let P (α, β, γ) be the foot of perpendicular from origin to plane (1).

Direction ratios of OP are

$\alpha - 0, \beta - 0, \gamma - 0$ i.e. α, β, γ

From equation (1), it is clear that the direction ratios of normal to the plane i.e. OP are ℓ, m, n ;

α, β, γ and ℓ, m, n are the direction ratios of the same line OP

$\therefore$ $\dfrac{\alpha}{\ell} = \dfrac{\beta}{m} = \dfrac{\gamma}{n} = \dfrac{1}{k}$ (say)

$\therefore$ $\ell = k\alpha,\ m = k\beta,\ n = k\gamma$ ..... (3)

Putting the values of

ℓ, m, n in equation (2), we get

$ka\alpha + kb\beta + kc\gamma + d = 0$ ..... (4)

Since α, β, γ lies in plane (1)

$\therefore$ $\ell\alpha + m\beta + n\gamma + d = 0$ ..... (5)

Putting the values of ℓ, m, n from (3) in (5), we get

$k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ..... (6)

or $k\alpha^2 + k\beta^2 + k\gamma^2 - ka\alpha - kb\beta - kc\gamma = 0$

[putting the value of d from (4) in (6)]

or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of perpendicular P (α, β, γ) is

$x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (7)

Self practice problems:

1. Find the intercepts of the plane 3x + 4y – 7z = 84 on the axes. Also find the length of perpendicular from origin to this line and direction cosines of this normal.

Ans. a = 28, b = 21, c = – 12, p = $\dfrac{1}{\sqrt{74}}$; $\dfrac{3}{\sqrt{74}}, \dfrac{4}{\sqrt{74}}, \dfrac{-7}{\sqrt{74}}$

2. Find : (i) perpendicular distance
(ii) foot of perpendicular
(iii) image of (1, 0, 2) in the plane 2x + y + z = 5

Ans. (i) $\dfrac{1}{\sqrt{6}}$ (ii) $\left(\dfrac{4}{3}, \dfrac{1}{6}, \dfrac{13}{6}\right)$ (iii) $\left(\dfrac{5}{3}, \dfrac{1}{3}, \dfrac{7}{3}\right)$

14. Angle Between Two Planes:

(i) Consider two planes ax + by + cz + d = 0 and a′x + b′y + c′z + d′ = 0. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a′, b′, c′) respectively, hence θ, the angle between them, is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}$$

Planes are perpendicular if aa′ + bb′ + cc′ = 0 and planes are parallel if $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

(ii) The angle θ between the planes $\vec{r}\cdot\vec{n} = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by, $\cos\theta = \dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|.|\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$.

15. Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(ii) Equation of bisector of the angle containing origin: First make both the constant terms positive. Then the positive sign in $\dfrac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$ gives the bisector of the angle which contains the origin.

(iii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2 + b_1b_2 + c_1c_2 > 0$ $\Rightarrow$ origin lies on obtuse angle

$a_1a_2 + b_1b_2 + c_1c_2 < 0 \quad \Rightarrow \quad$ origin lies in acute angle

16. Family of Planes

(i) Any plane passing through the line of intersection of non–parallel planes or equation of the plane through the given line in serval form.
$a_1x + b_1y + c_1z + d_1 = 0 \quad \& \quad a_2x + b_2y + c_2z + d_2 = 0$ is
$a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\,.\,\vec{n}_1 = d_1$ & $\vec{r}\,.\,\vec{n}_2 = d_2$ is $\vec{r}\,.\,(n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

Example : The plane $x - y - z = 4$ is rotated through $90°$ about its line of intersection with the plane $x + y + 2z = 4$. Find its equation in the new position.

Solution Given planes are
$x - y - z = 4$ ..... (1)
and $x + y + 2z = 4$ ..... (2)
Since the required plane passes through the line of intersection of planes (1) and (2)
$\therefore$ its equation may be taken as
$x + y + 2z - 4 + k(x - y - z - 4) = 0$
or $(1 + k)x + (1 - k)y + (2 - k)z - 4 - 4k = 0$ ..... (3)
Since planes (1) and (3) are mutually perpendicular,
$\therefore$ $(1 + k) - (1 - k) - (2 - k) = 0$
or, $1 + k - 1 + k - 2 + k = 0$ or, $k = \dfrac{2}{3}$
Putting $k = \dfrac{2}{3}$ in equation (3), we get,
$5x + y + 4z = 20$
This is the equation of the required plane.

Example : Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Solution Given planes are
$x + y + z - 6 = 0$ ..... (1)
and $2x + 3y + 4z + 5 = 0$ ..... (2)
Given point is P (1, 1, 1).
Equation of any plane through the line of intersection of planes (1) and (2) is
$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0$ ..... (3)
If plane (3) passes through point P, then
$1 + 1 + 1 - 6 + k(2 + 3 + 4 + 5) = 0$ or, $k = \dfrac{3}{14}$
From (1) required plane is
$20x + 23y + 26z - 69 = 0$

Example : Find the planes bisecting the angles between planes
$2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$.
Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Solution Given planes are
$-2x - y - 2z + 9 = 0$ ..... (1)
and $3x - 4y + 12z + 13 = 0$ ..... (2)
Equations of bisecting planes are

$$\frac{-2x - y - 2z + 9}{\sqrt{(-2)^2 + (-1)^2 + (-2)^2}} = \pm \frac{3x - 4y + 12z + 13}{\sqrt{3^2 + (-4)^2 + (12)^2}}$$

or, $13[-2x - y - 2z + 9] = \pm 3(3x - 4y + 12z + 13)$
or, $35x + y + 62z = 78$, ... (3) [Taking +ve sign]
and $17x + 25y - 10z = 156$ ..... (4) [Taking – ve sign]
Now $a_1a_2 + b_1b_2 + c_1c_2 = (-2)(3) + (-1)(-4) + (-2)(12)$
$= -6 + 4 - 24 = -26 < 0$
$\therefore$ Bisector of acute angle is given by $35x + y + 62z = 78$
$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Example : If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Solution Given planes are
$x - cy - bz = 0$ ..... (1)
$cx - y + az = 0$ ..... (2)
$bx + ay - z = 0$ ..... (3)
Equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as
$x - cy - bz + \lambda(cx - y + az) = 0$
or, $x(1 + \lambda c) - y(c + \lambda) + z(-b + a\lambda) = 0$ ..... (4)
If planes (3) and (4) are the same, then equations (3) and (4) will be identical.

$$\therefore \quad \underset{(i)}{\frac{1 + c\lambda}{b}} = \underset{(ii)}{\frac{-(c + \lambda)}{a}} = \underset{(iii)}{\frac{-b + a\lambda}{-1}}$$

From (i) and (ii), $a + ac\lambda = -bc - b\lambda$

or, $\lambda = -\dfrac{(a+bc)}{(ac+b)}$ ..... (5)

From (ii) and (iii),

$c + \lambda = -ab + a^2\lambda$ or $\lambda = \dfrac{(ab+c)}{1-a^2}$ ..... (6)

From (5) and (6), we have,

$$\frac{-(a+bc)}{ac+b} = \frac{-(ab+c)}{(1-a^2)}.$$

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

or, $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$

or, $a^2 + b^2 + c^2 + 2abc = 1$.

Self practice problems:

1. A tetrahedron has vertices at O(0, 0, 0), A(1, 2, 1), B(2, 1, 3) and C(–1, 1, 2). Prove that the angle between the faces OAB and ABC will be $\cos^{-1}\left(\dfrac{19}{35}\right)$.

2. Find the equation of plane passing through the line of intersection of the planes $4x - 5y - 4z = 1$ and $2x + y + 2z = 8$ and the point (2, 1, 3). **Ans.** $32x - 5y + 8z - 83 = 0$, $\lambda = \dfrac{10}{3}$

3. Find the equations of the planes bisecting the angles between the planes $x + 2y + 2z - 3 = 0$, $3x + 4y + 12z + 1 = 0$ and sepecify the plane which bisects the acute angle between them. **Ans.** $2x + 7y - 5z = 21$, $11x + 19y + 31z = 18$; $2x + 7y - 5z = 21$

4. Show that the origin lies in the acute angle between the planes $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$

5. Prove that the planes $12x - 15y + 16z - 28 = 0$, $6x + 6y - 7z - 8 = 0$ and $2x + 35y - 39z + 12 = 0$ have a common line of intersection.

17. Area of a triangle:

Let A (x_1, y_1, z_1), B (x_2, y_2, z_2), C (x_3, y_3, z_3) be the vertices of a triangle, then $\Delta = \sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)}$

where $\Delta_x = \dfrac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_y = \dfrac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Vector Method – From two vector $\vec{AB}$ and $\vec{AC}$. Then area is given by

$$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\begin{vmatrix} i & j & k \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Example : Through a point P (h, k, ℓ) a plane is drawn at right angles to OP to meet the co-ordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\dfrac{p^5}{2hk\ell}$.

Solution $OP = \sqrt{h^2 + k^2 + \ell^2} = p$

Direction cosines of OP are

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$ ..... (1)

$\therefore$ $A \equiv \left(\dfrac{p^2}{h}, 0, 0\right)$, $B \equiv \left(0, \dfrac{p^2}{k}, 0\right)$, $C \equiv \left(0, 0, \dfrac{p^2}{\ell}\right)$

Now area of ΔABC, $\Delta = A^2_{xy} + A^2_{yz} + A^2_{zx}$

Now A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $A_{yz} = \frac{1}{2}\frac{p^4}{|k\ell|}$ and $A_{zx} = \frac{1}{2}\frac{p^4}{|\ell h|}$

$\therefore \quad \Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2\ell^2} + \frac{1}{4}\frac{p^8}{\ell^2h^2}$

$= \frac{p^8}{4h^2k^2\ell^2}(\ell^2 + k^2 + h^2) = \frac{p^{10}}{4h^2k^2\ell^2}$ or, $\Delta = \frac{p^5}{2hk\ell}$.

18. Volume Of A Tetrahedron:

Volume of a tetrahedron with vertices A (x_1, y_1, z_1), B(x_2, y_2, z_2), C (x_3, y_3, z_3) and D (x_4, y_4, z_4) is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

A LINE

19. Equation Of A Line

(i) A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + +c_2z + d_2 = 0$. This form is also known as non–symmetrical form.

(ii) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$. This form is called symmetric form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(iii) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar.

(iv) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(v) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(vi) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \Leftrightarrow \quad \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

Note: Straight lines parallel to co-ordinate axes:

	Straight lines	Equation		Straight lines	Equation
(i)	Through origin	y = mx, z = nx	**(v)**	Parallel to x–axis	y = p, z = q
(ii)	x–axis	y = 0, z = 0	**(vi)**	Parallel to y–axis	x = h, z = q
(iii)	y–axis	x = 0, z = 0	**(vii)**	Parallel to z–axis	x = h, y = p
(iv)	z–axis	x = 0, y = 0			

Example : Find the equation of the line through the points (3, 4, –7) and (1, – 1, 6) in vector form as well as in cartesian form.

Solution Let A ≡ (3, 4, – 7), B ≡ (1, – 1, 6)

Now $\vec{a} = \vec{OA} = 3\vec{i} + 4\vec{j} - 7\vec{k}$,

$= \vec{b} = \vec{OB} = \vec{i} - \vec{j} + 6\vec{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = 3\vec{i} + 4\vec{j} - 7\vec{k} + t(-2\vec{i} - 5\vec{j} + 13\vec{k})$ ..... (1)

Equation in cartesian form :

Equation of AB is $\frac{x-3}{3-1} = \frac{y-4}{4+1} = \frac{z+7}{-7-6}$ or, $\frac{x-3}{2} = \frac{y-4}{5} = \frac{z+7}{-13}$

Example : Find the co-ordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which is at a distance of 3 units from point (1, –2, 3).

Solution Given line is $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ ..... (1)

Let $P \equiv (1, -2, 3)$ Direction ratios of line (1) are 2, 3, 6

$\therefore$ Direction cosines of line (1) are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

Equation of line (1) may be written as

$$\frac{x-1}{\frac{2}{7}} = \frac{y+2}{\frac{3}{7}} = \frac{z-3}{\frac{6}{7}} \quad ..... (2)$$

Co-ordinates of any point on line (2) may be taken as

$$\left(\frac{2}{7}r+1, \frac{3}{7}r-2, \frac{6}{7}r+3\right)$$

Let $Q \equiv \left(\frac{2}{7}r-1,\ \frac{3}{7}r-2,\ \frac{6}{7}r+3\right)$

Distance of Q from P = | r |

According to question | r | = 3 $\therefore$ r = ± 3 Putting the value of r, we have

$$Q \equiv \left(-\frac{1}{7}, -\frac{5}{7}, \frac{39}{7}\right) \quad \text{or} \quad Q \equiv \left(-\frac{13}{7}, -\frac{23}{7}, \frac{3}{7}\right)$$

Example : Find the equation of the line drawn through point (1, 0, 2) to meet at right angles the line $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$

Solution Given line is

$$\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1} \quad ..... (1)$$

Let $P \equiv (1, 0, 2)$

Co-ordinates of any point on line (1) may be taken as

$Q \equiv (3r - 1, -2r + 2, -r - 1)$

Direction ratios of PQ are $3r - 2, -2r + 2, -r - 3$

Direction ratios of line AB are $3, -2, -1$

Since PQ $\perp$ AB

$\therefore$ $3(3r - 2) - 2(-2r + 2) - 1(-r - 3) = 0$

$\Rightarrow$ $9r - 6 + 4r - 4 + r + 3 = 0$ $\Rightarrow$ $14r = 7$ $\Rightarrow$ $r = \frac{1}{2}$

Therefore, direction ratios of PQ are

$-\frac{1}{2}, 1, -\frac{7}{2}$ or, $-1, 2, -7$

Equation of line PQ is

$$\frac{x-1}{-1} = \frac{y=0}{2} = \frac{z-2}{-7} \quad \text{or,} \quad \frac{x-1}{1} = \frac{y}{-2} = \frac{z-2}{7}$$

Example : Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find also the point of intersection of these lines.

Solution Given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ ..... (1)

and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ ..... (2)

Any point on line (1) is P $(2r + 1, 3r + 2, 4r + 3)$

and any point on line (2) is Q $(5\lambda + 4, 2\lambda + 1, \lambda)$

Lines (1) and (2) will intersect if P and Q coincide for some value of λ and r.

$\therefore$ $2r + 1 = 5\lambda + 4$ $\Rightarrow$ $2r - 5\lambda = 3$ ..... (1)

$3r + 2 + 2\lambda + 1$ $\Rightarrow$ $3r - 2\lambda = -1$ ..... (2)

$4r + 3 = \lambda$ $\Rightarrow$ $4r - \lambda = -3$ ..... (3)

Solving (1) and (2), we get $r = -1$, $\lambda = -1$

Clearly these values of r and λ satisfy eqn. (3)

Now $P \equiv (-1, -1, -1)$ Hence lines (1) and (2) intersect at $(-1, -1, -1)$.

Self practice problems:

1. Find the equation of the line passing through point (1, 0, 2) having direction ratio 3, – 1, 5. Prove that this line passes through (4, – 1, 7). **Ans.** $\frac{x-1}{3} - \frac{y}{-1} = \frac{z-2}{5}$

2. Find the equation of the line parallel to line $\frac{x-2}{3} = \frac{y+1}{1} = \frac{z-7}{9}$ and passing through the point (3, 0, 5).

Ans. $\frac{x-3}{3}=\frac{y}{1}=\frac{z-5}{3}$

3. Find the coordinates of the point when the line through (3, 4, 1) and (5, 1, 6) crosses the xy plane.

Ans. $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

20. Reduction Of Non-Symmetrical Form To Symmetrical Form:

Let equation of the line in non–symmetrical form be $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$. To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

(i) **Direction ratios:** Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes. So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by cross–multiplication as $\frac{\ell}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}$

Alternative method The vector $\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i\,(b_1c_2 - b_2c_1) + j\,(c_1a_2 - c_2a_1) + k\,(a_1b_2 - a_2b_1)$ will be parallel to the line of intersection of the two given planes. hence $\ell : m : n = (b_1c_2 - b_2c_1) : (c_1a_2 - c_2a_1) : (a_1b_2 - a_2b_1)$

(ii) Point on the line – Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non–zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy plane, so it intersect it. Let it intersect xy–plane in $(x_1, y_1, 0)$. Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$.Solving these,we get a point on the line. Then its equation becomes.

$$\frac{x-x_1}{b_1c_2-b_2c_1}=\frac{y-y_1}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1} \text{ or } \frac{x-\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}}{b_1c_2-b_2c_1}=\frac{y-\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}$$

Note: If $\ell \neq 0$, take a point on yz–plane as $(0, y_1, z_1)$ and if $m \neq 0$, take a point on xz–plane as $(x_1, 0, z_1)$.

Alternative method

If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ Put z = 0 in both the equations and solve the equations $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ otherwise Put y = 0 and solve the equations $a_1x + c_1z + d_1 = 0$ and $a_2x + c_2z + d_2 = 0$

Example : Find the equation of the line of intersection of planes 4x + 4y – 5z = 12, 8x + 12y – 13z = 32 in the symmetric form.

Solution Given planes are $4x + 4y - 5z - 12 = 0$ ..... (1)

and $8x + 12y - 13z - 32 = 0$ ..... (2)

Let ℓ, m, n be the direction ratios of the line of intersection :

then $4\ell + 4m - 5n = 0$ ..... (3)

and $8\ell + 12m - 13n = 0$

$\therefore$ $\frac{\ell}{-52+60}=\frac{m}{-40+52}=\frac{n}{48-32}$ or, $\frac{\ell}{8}=\frac{m}{12}=\frac{n}{16}$ or, $\frac{\ell}{2}=\frac{m}{3}=\frac{n}{4}$

Hence direction ratios of line of intersection are 2, 3, 4.

Here 4 ≠ 0, therefore line of intersection is not parallel to xy-plane.

Let the line of intersection meet the xy-plane at P (α, β, 0).

Then P lies on planes (1) and (2)

$\therefore$ $4\alpha + 4\beta + 12 = 0$

or, $\alpha + \beta - 3 = 0$ ..... (5)

and $8\alpha + 12\beta - 32 = 0$

or, $2\alpha + 3\beta - 8 = 0$ ..... (6)

Solving (5) and (6), we get

$\frac{\alpha}{-8+9}=\frac{\beta}{-6+8}=\frac{1}{3-2}$ or, $\frac{\alpha}{1}=\frac{\beta}{2}=\frac{1}{1}$

$\therefore$ $\alpha = 1, \beta = 2$

Hence equation of line of intersection in symmetrical form is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-0}{4}$.

Example : Find the angle between the lines x – 3y – 4 = 0, 4y – z + 5 = 0 and x + 3y – 11 =0, 2y – z + 6 = 0.

Solution Given lines are

$\left.\begin{array}{l} x-3y-4=0 \\ 4y-z+5=0 \end{array}\right\}$ ..... (1)

and $\left.\begin{array}{l} x+3y-11=0 \\ 2y-z+6=0 \end{array}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_1, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes
$x - 3y - 4 = 0$ and $4y - z + 5 = 0$

$\therefore$ $\ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3)

and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get,

$$\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0} \quad \text{or,} \quad \frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4} = k \text{ (let).}$$

Since line (2) is perpendicular to the normals of each of the planes
$x + 3y - 11 = 0$ and $2y - z + 6 = 0$,

$\therefore$ $\ell_2 + 3m_2 = 0$ ..... (5)

and $2m_2 - n_2 = 0$ ..... (6)

$\therefore$ $\ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$

and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

$\therefore$ $\frac{\ell_2}{-3} = \frac{m_3}{1} = \frac{n_2}{2} = t$ (let).

If θ be the angle between lines (1) and (2), then

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$
$= (3k)(-3t) + (k)(t) + (4k)(2t)$
$= -9kt + kt + 8kt = 0$ $\therefore$ $\theta = 90°$.

Self practice problems:

1. Find the equation of the line of intersection of the plane
$4x + 4y - 5z = 12$
$8x + 12y - 13z = 32$ **Ans.** $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-0}{4}$

2. Show that the angle between the two lines defined by the equations $x = y$ and $xy + yz + zx = 0$ is $\cos^{-1}\left(\frac{1}{3}\right)$

3. Prove that the three planes $2x + y - 4z - 17 = 0$, $3x + 2y - 2z - 25 = 0$, $2x - 4y + 3z + 25 = 0$ intersect at a point and find its co-ordinates. **Ans.** $(3, 7, -1)$

21. Foot, Length And Equation Of Perpendicular From A Point To A Line:

(i) Cartesian form: Let equation of the line be $\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n} = r$ (say) ....(i)

and A (α, β, γ) be the point.

Any point on line (i) is P $(\ell r + a, mr + b, nr + c)$ .... (ii)

If it is the foot of the perpendicular from A on the line, then AP is perpendicular to the line. So $\ell(\ell r + a - \alpha) + m(mr + b - \beta) + n(nr + c - \gamma) = 0$ i.e. $r = (\alpha - a)\ell + (\beta - b)m + (\gamma - c)n$ since $\ell^2 + m^2 + n^2 = 1$. Putting this value of r in (ii), we get the foot of perpendicular from point A on the given line. Since foot of perpendicular P is known, then the length of perpendicular is given by AP $= \sqrt{(\ell r + a - \alpha)^2 + (mr + b - \beta)^2 + (nr + c - \gamma)^2}$ the equation of perpendicular is given by $\frac{x-\alpha}{\ell r + a - \alpha} = \frac{y-\beta}{mr + b - \beta} = \frac{z-\gamma}{nr + c - \gamma}$ **(ii)** Vector Form: Equation of a line passing through a point having position vector $\vec{\alpha}$ and perpendicular to the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is parallel to $\vec{b}_1 \times \vec{b}_2$. So the vector equation of such a line is $\vec{r} = \vec{\alpha} + \lambda(\vec{b}_1 \times \vec{b}_2)$. Position vector $\vec{\beta}$ of the image of a point $\vec{\alpha}$ in a straight line $\vec{r} = \vec{a} + \lambda\vec{b}$ is given by $\vec{\beta} = 2\vec{a} - \left[\frac{2(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right]\vec{b} - \vec{\alpha}$. Position vector of the foot of the perpendicular on line is $\vec{f} = \vec{a} - \left[\frac{(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right]\vec{b}$. The equation of the perpendicular is $\vec{r} = \vec{\alpha} + \mu \left[(\vec{a}-\vec{\alpha}) - \left(\frac{(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right)\vec{b}\right]$.

22. To find image of a point w. r. t a line

Let $L \equiv \frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$ is a given line

Let (x', y', z') is the image of the point P (x_1, y_1, z_1) with respect to the line L. Then

(i) $a(x_1 - x') + b(y_1 - y') + c(z_1 - z') = 0$

(ii) $\dfrac{\frac{x_1+x'}{2}-x_2}{a} = \dfrac{\frac{y_1+y'}{2}-y_2}{b} = \dfrac{\frac{z_1+z'}{2}-z_2}{c} = \lambda$

from (ii) get the value of x', y', z' in terms of λ as
$x' = 2a\lambda + 2x_2 - x_1$, $y' = 2b\alpha - 2y_2 - y_1$,
$z' = 2c\lambda + 2z_2 - z_1$
now put the values of x', y', z' in (i) get λ and resubtitute the value of λ to to get $(x'\ y'\ z')$.

Example : Find the length of the perpendicular from P (2, – 3, 1) to the line $\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z+2}{-1}$.

Solution Given line is $\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z+2}{-1}$ ..... (1)

$P \equiv (2, -3, 1)$
Co-ordinates of any point on line (1) may be taken as
$Q \equiv (2r - 1, 3r + 3, - r - 2)$
Direction ratios of PQ are $2r - 3, 3r + 6, - r - 3$
Direction ratios of AB are $2, 3, - 1$
Since $PQ \perp AB$
$\therefore$ $2 (2r - 3) + 3 (3r + 6) - 1 (- r - 3) = 0$

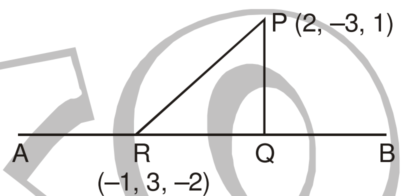
or, $14r + 15 = 0$ $\therefore$ $r = \dfrac{-15}{14}$

$\therefore$ $Q \equiv \left(\dfrac{-22}{7}, \dfrac{-3}{14}, \dfrac{-13}{14}\right)$ $\therefore$ $PQ = \sqrt{\dfrac{531}{14}}$ units.

Second method : Given line is

$\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z+2}{-1}$

$P \equiv (2, -3, 1)$



Direction ratios of line (1) are $\dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}, -\dfrac{1}{\sqrt{14}}$

RQ = length of projection of RP on AB

$= \left|\dfrac{2}{\sqrt{14}}(2+1) + \dfrac{3}{\sqrt{14}}(-3-3) - \dfrac{1}{\sqrt{4}}(1+2)\right| = \dfrac{15}{\sqrt{14}}$

$PR^2 = 3^2 + 6^2 + 3^2 = 54$

$\therefore$ $PQ = \sqrt{PR^2 - RQ^2} = \sqrt{54 - \dfrac{225}{14}} = \sqrt{\dfrac{531}{14}}$

Self practice problems:

1. Find the length and foot of perpendicular drawn from point (2, –1, 5) to the line $\dfrac{x-11}{10} = \dfrac{y+2}{-4} = \dfrac{z+8}{-11}$.
Also find the image of the point in the line. **Ans.** $\sqrt{14}$, $N \equiv (1, 2, 3)$, $I \equiv (0, 5, 1)$

2. Find the image of the point (1, 6, 3) in the line $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3}$. **Ans.** (1, 0 , 7)

3. Find the foot and hence the length of perpendicular from (5, 7, 3) to the line $\dfrac{x-15}{3} = \dfrac{y-29}{8} = \dfrac{z-5}{-5}$.
Find also the equation of the perpendicular. **Ans.** (9, 13, 15) ; 14 ; $\dfrac{x-5}{2} = \dfrac{y-7}{3} = \dfrac{z-3}{6}$

23. Angle Between A Plane And A Line:

(i) If θ is the angle between line $\dfrac{x-x_1}{\ell} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\,\sqrt{\ell^2+m^2+n^2}}\right].$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\dfrac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\dfrac{\ell}{a} = \dfrac{m}{b} = \dfrac{n}{c}$ $\vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $a\ell + bm + cn = 0$ $\vec{b}\cdot\vec{n} = 0$

24. Condition For A Line To Lie In A Plane

(i) Cartesian form: Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form: Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}\cdot\vec{n} = d$ if $\vec{b}\cdot\vec{n} = 0$ & $\vec{a}\cdot\vec{n} = d$

25. Coplanar Lines:

(i) If the given lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$, then condition for intersection/coplanarity is $\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$ & plane containing the above two lines is $\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$

(ii) Condition of coplanarity if both the lines are in general form Let the lines be

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \quad \&$$
$$\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Alternative method: get vector along the line of shortest distance as $\begin{vmatrix} i & j & k \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix}$

Now get unit vector along this vector

$\hat{u} = \ell i + mj + nk$

Let $\vec{v} = (\alpha - \alpha')\,\hat{i} + (B - B')\,\hat{j} + (y - y')$

S. D. $= u.\,v$

Example : Find the distance of the point (1, 0, – 3) from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

Solution Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ..... (2)

Equation of a line passing through the point Q(1, 0, – 3) and parallel to line (2) is

$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r.$ ..... (3)

Co-ordinates of point on line (3) may be taken as

P $(2r + 1, 3r, -6r - 3)$

If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),

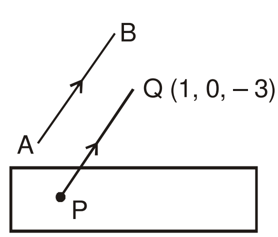
$\therefore$ $(2r + 1) - (3r) - (-6r - 3) = 9$

$r = 1$

or, $P \equiv (3, 3, -9)$

Distance between points Q (1, 0, – 3) and P (3, 3, – 9)

$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(3))^2} = \sqrt{4+9+36} = 7.$



Example: Find the equation of the plane passing through (1, 2, 0) which contains the line $\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$.

Solution Equation of any plane passing through (1, 2, 0) may be taken as

$a(x-1) + b(y-2) + c(z-0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point $(-3, 1, 2)$ on line (2) lies in plane (1)

$\therefore$ $a(-3-1) + b(1-2) + c(2-0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get,

$\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13} = k$ (say). ..... (5)

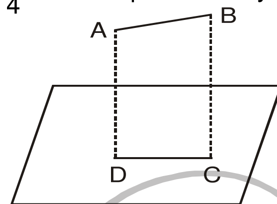
Substituting the values of a, b and c in equation (1), we get,

$6(x-1) + 2(y-2) + 13(z-0) = 0.$

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Example : Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane $x + 2y + z = 9$.

Solution



Let the given line AB be

$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ ..... (1)

Given plane is

$x + 2y + z = 9$ ..... (2)

Let DC be the projection of AB on plane (2)

Clearly plane ABCD is perpendicular to plane (2).

Equation of any plane through AB may be taken as (this plane passes through the point (1, – 1, 3) on line AB)

$a(x-1) + b(y+1) + c(z-3) = 0$ ..... (3)

where $2a - b + 4c = 0$ ..... (4)

[$\because$ normal to plane (3) is perpendicular to line (1)]

Since plane (3) is perpendicular to plane (2),

$\therefore$ $a + 2b + c = 0$ ..... (5)

Solving equations (4) & (5), we get,

$\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$.

Substituting these values of a, b and c in equation (3), we get

$9(x-1) - 2(y+1) - 5(z-3) = 0$

or, $9x - 2y - 5z + 4 = 0$ ...... (6)

Since projection DC of AB on plane (2) is the line of intersection of plane ABCD and plane (2), therefore equation of DC will be

and $\left.\begin{array}{ll} 9x - 2y - 5z + 4 = 0 & \text{.....(i)} \\ x + 2y + z - 9 = 0 & \text{.....(ii)} \end{array}\right\}$ ..... (7)

Let ℓ, m, n be the direction ratios of the line of intersection of planes (i) and (ii)

$\therefore$ $9\ell - 2m - 5n = 0$ ..... (8)

and $\ell + 2m + n = 0$ ..... (9)

$\therefore$ $\frac{\ell}{-2+10} = \frac{m}{-5-9} = \frac{n}{18+2}$

Example : Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+2}{2}$ are coplanar. Also find the equation of the plane containing them.

Solution Given lines are

$\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1} = r$ (say) ..... (1)

and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2} = R$ (say) ..... (2)

If possible, let lines (1) and (2) intersect at P.

Any point on line (1) may be taken as

$(2r + 3, -3r - 1, r - 2) = P$ (let).

Any point on line (2) may be taken as

$(-3R + 7, R, 2R - 7) = P$ (let).

$\therefore$ $2r + 3 = -3R + 7$

or, $2r + 3R = 4$ ..... (3)

Also $-3r - 1 = R$

or, $-3r - R = 1$ ..... (4)

and $\quad r - 2 = 2R - 7$

or, $\quad r - 2R = -5$. ..... (5)

Solving equations (3) and (4), we get,

$\quad r = -1,\ R = 2$

Clearly $r = -1$, $R = 2$ satisfies equation (5).

Hence lines (1) and (2) intersect. $\therefore$ lines (1) and (2) are coplanar.

Equation of the plane containing lines (1) and (2) is

$$\begin{vmatrix} x-3 & y+1 & z+2 \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0$$

or, $\quad (x - 3)(-6 - 1) - (y + 1)(4 + 3) + (z + 2)(2 - 9) = 0$

or, $\quad -7(x - 3) - 7(y + 1) - 7(z + 2) = 0$

or, $\quad x - 3 + y + 1 + z + 2 = 0$ or, $\quad x + y + z = 0$.

Self practice problems:

1. Find the values of a and b for which the line $\dfrac{x-2}{a} = \dfrac{y+3}{4} = \dfrac{z-6}{-2}$ is perpendicular to the plane $3x - 2y + bz + 10 = 0$. **Ans.** $a = 3,\ b = -2$

2. Prove that the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{3}$ and $\dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-4}{5}$ are coplanar. Also find the equation of the plane in which they lie. **Ans.** $x - 2y + z = 0$

3. Find the plane containing the line $\dfrac{x-2}{2} = \dfrac{y-3}{3} = \dfrac{z-4}{5}$ and parallel to the line $\dfrac{x+1}{1} = \dfrac{y-1}{-2} = \dfrac{-z+1}{1}$

Ans. $13x + 3y - 72 - 7 = 0$

4. Show that the line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ & $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ are intersecting each other. Find thire intersection and the plane containing the line. **Ans.** $(-1, -1, -1)$ & $5x - 8y + 11z - 2 = 0$

5. Show that the lines $\vec{r} = (-\hat{i} - 3\hat{j} - 5\hat{k}) + \lambda(-3\hat{i} - 5\hat{j} - 7\hat{k})$ & $\vec{r}\ (2\hat{i} + 4\hat{j} + 6\hat{k}) + \mu(\hat{i} + 4\hat{j} + 7\hat{k})$ are coplanar and find the plane containing the line. **Ans.** $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

26. Skew Lines:

(i) The straight lines which are not parallel and non–coplanar i.e. non–intersecting are called skew lines. If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

(ii) Shortest distance: Suppose the equation of the lines are

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ and } \frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$$

$$\text{S.D.} = \frac{(\alpha-\alpha')(mn'-m'n) + (\beta-\beta')(n\ell-n'\ell) + (\gamma-\gamma')(\ell m'-\ell' m)}{\sqrt{\sum (mn'-m'n)^2}}$$

$$= \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \div \sqrt{\sum (mn'-m'n)^2}$$

(iii) Vector Form: For lines $\vec{a}_1 + \lambda\vec{b}_1$ & $\vec{a}_2 + \lambda\vec{b}_2$ to be skew

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \neq 0 \quad \text{or} \quad [\vec{b}_1\ \vec{b}_2\ (\vec{a}_2 - \vec{a}_1)] \neq 0.$$

(iv) Shortest distance between the two parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ &

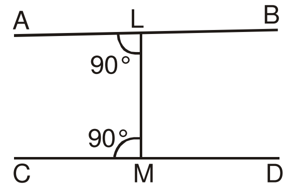
$$\vec{r} = \vec{a}_2 + \mu\vec{b} \text{ is } d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \times \vec{b}}{|\vec{b}|} \right|.$$

Example : Find the shortest distance and the vector equation of the line of shortest distance between the lines given by

$$\vec{r} = 3\vec{r} + 8\vec{j} + 3\vec{k} + \lambda\left(3\vec{r} - \vec{j} + \vec{k}\right) \quad \text{and} \quad \vec{r} = -3\vec{r} - 7\vec{j} + 6\vec{k} + \mu\left(-3\vec{i} + 2\vec{j} + 4\vec{k}\right)$$

Solution Given lines are

$$\vec{r} = 3\vec{r} + 8\vec{j} + 3\vec{k} + \lambda\left(3\vec{i} - \vec{j} + \vec{k}\right) \quad \text{..... (1)}$$



and $\quad \vec{r} = -3\vec{r} - 7\vec{j} + 6\vec{k} + \mu\left(-3\vec{i} + 2\vec{j} + 4\vec{k}\right)$ ..... (2)

Equation of lines (1) and (2) in cartesian form is

AB : $\dfrac{x-3}{3} = \dfrac{y-8}{-1} = \dfrac{z-3}{1} = \lambda$

and CD : $\left(\dfrac{1}{3} \right) = \mu$

Let $\quad L \equiv (3\lambda + 3, -\lambda + 8, \lambda + 3)$
and $\quad M \equiv (-3\mu - 3, 2\mu - 7, 4\mu + 6)$
Direction ratios of LM are
$\quad 3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$.

Since $\quad LM \perp AB$
$\therefore \quad 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$
or, $\quad 11\lambda + 7\mu = 0$ ..... (5)

Again $LM \perp CD$
$\therefore \quad -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$
or, $\quad -7\lambda - 29\mu - 0$ ..... (6)
Solving (5) and (6), we get $\quad \lambda = 0, \mu = 0$
$\therefore \quad L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2}$

$\quad = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is

$$\vec{r} = 3\vec{i} + 8\vec{j} + 3\vec{k} + t\left(6\vec{i} + 15\vec{j} - 3\vec{k}\right)$$

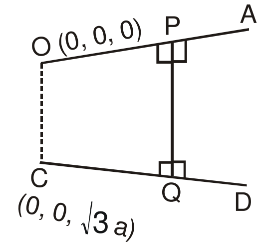
Note : Cartesian equation of LM is $\dfrac{x-3}{6} = \dfrac{y-8}{15} = \dfrac{z-3}{-3}$.

Example : Prove that the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0$, $x + z = 0$, $x + y = 0$, $x + y + z = \sqrt{3}\,a$ is $\sqrt{2}\,a$.

Solution Given planes are $\quad y + z = 0$ ..... (i) $\quad x + z = 0$ ..... (ii)

$x + y = 0$ ..... (iii) $\quad x + y + z = \sqrt{3}\,a$ ..... (iv)

Clearly planes (i), (ii) and (iii) meet at O(0, 0, 0)
Let the tetrahedron be OABC
Let the equation to one of the pair of opposite edges OA and BC be



$\quad y + z = 0, x + z = 0$ ..... (1)

$\quad x + y = 0, x + y + z = \sqrt{3}\,a$ ..... (2)

equation (1) and (2) can be expressed in symmetrical form as

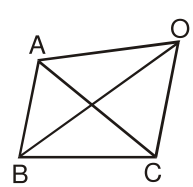
$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{-1} \qquad ..... (3)$$

and, $\quad \dfrac{x-0}{1} = \dfrac{y-0}{-1} = \dfrac{z-\sqrt{3}\,a}{0}$ ..... (4)

d. r. of OA and BC are (1, – 1) and (1, – 1, 0).
Let PQ be the shortest distance between OA and BC having direction cosine (ℓ, m, n)
$\therefore \quad$ PQ is perpendicular to both OA and BC.
$\therefore \quad \ell + m - n = 0$
and $\quad \ell - m = 0$
Solving (5) and (6), we get,

$$\frac{\ell}{1} = \frac{m}{1} = \frac{n}{2} = k \text{ (say)}$$

also, $\quad \ell^2 + m^2 + n^2 = 1$

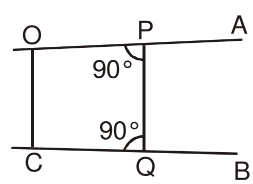


$\therefore \quad k^2 + k^2 + 4k^2 = 1 \;\Rightarrow\; k = \dfrac{1}{\sqrt{6}}$

$\therefore \quad \ell = \dfrac{1}{\sqrt{6}}, \; m = \dfrac{1}{\sqrt{6}}, \; n = \dfrac{2}{\sqrt{6}}$

Shortest distance between OA and BC
i.e. $\quad$ PQ = The length of projection of OC on PQ
$\quad = |(x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)\,n|$

$$= \left| 0 \cdot \frac{1}{\sqrt{6}} + 0 \cdot \frac{1}{\sqrt{6}} + \sqrt{3}\,a \cdot \frac{2}{\sqrt{6}} \right| = \sqrt{2}\,a.$$



Self practice problems:

1. Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$. Find also its equation. **Ans.** $\frac{1}{\sqrt{6}}$, $6x - y = 10 - 3y = 6z - 25$

2. Prove that the shortest distance between the diagonals of a rectangular parallelopiped whose sides are a, b, c and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

27. Sphere

Sphere: General equation of a sphere is given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ $(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.

Example : Find the equation of the sphere having centre at (1, 2, 3) and touching the plane $x + 2y + 3z = 0$.

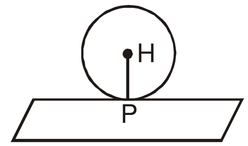
Solution: Given plane is $x + 2y + 3z = 0$ ..... (1)

Let H be the centre of the required sphere.

Given $H \equiv (1, 2, 3)$

Radius of the sphere,

HP = length of perpendicular from H to plane (1)



$$= \frac{|1+2\times2+3\times3|}{\sqrt{14}} = \sqrt{14}$$

Equation of the required sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$

or $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$

Example : Find the equation of the sphere if it touches the plane $\vec{r}\cdot(2\vec{i}-2\vec{j}-\vec{k}) = 0$ and the position vector of its centre is $3\vec{i}+6\vec{j}-4\vec{k}$

Solution Given plane is $\vec{r}\cdot(2\vec{i}-2\vec{j}-\vec{k}) = 0$ ..... (1)

Let H be the centre of the sphere, then

$$\vec{OH} = 3\vec{i}+6\vec{j}-4\vec{k} = \vec{c} \text{ (say)}$$

Radius of the sphere = length of perpendicular from H ot plane (1)

$$= \frac{|\vec{c}\cdot(2\vec{i}-2\vec{j}-\vec{k})|}{|2\vec{i}-2\vec{j}-\vec{k}|}$$

$$= \frac{|(3\vec{i}+6\vec{j}-4\vec{k})\cdot(2\vec{i}-2\vec{j}-\vec{k})|}{|2\vec{i}-2\vec{j}-\vec{k}|}$$

$$= \frac{|6-12+4|}{3} = \frac{2}{3} = a \text{ (say)}$$

Equation of the required sphere is

$$|\vec{r}-\vec{c}| = a$$

or $|x\vec{i}+y\vec{j}+z\vec{k}-(3\vec{i}+6\vec{j}-4\vec{k})| = \frac{2}{3}$ or $|(x-3)\vec{i} + (y-6)\vec{j} + (z+4)\vec{k}|^2 = \frac{4}{9}$

or $(x-3)^2 + (y-6)^2 + (z+4)^2 = \frac{4}{9}$ or $9(x^2 + y^2 + z^2 - 6x - 12y + 8z + 61) = 4$

or $9x^2 + 9y^2 + 9z^2 - 54x - 108y + 72z + 545 = 0$

Example : Find the equation of the sphere passing through the points (3, 0, 0), (0, – 1, 0), (0, 0, – 2) and whose centre lies on the plane $3x + 2y + 4z = 1$

Solution Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ..... (1)

Let $A \equiv (3, 0, 0)$, $B \equiv (0, -1, 0)$, $C \equiv (0, 0, -2)$

Since sphere (1) passes through A, B and C,

$\therefore$ $9 + 6u + d = 0$ ..... (2)

$1 - 2v + d = 0$ ..... (3)

$4 - 4w + d = 0$ ..... (4)

Since centre $(-u, -v, -w)$ of the sphere lies on plane

$3x + 2y + 4z = 1$

$\therefore$ $-3u - 2v - 4w = 1$ ..... (5)

(2) – (3) $\Rightarrow 6u + 2v = -8$ ..... (6)

(3) – (4) $\Rightarrow -2v + 4w = 3$ ..... (7)

From (6), $u = \frac{-2v-8}{6}$ ..... (8)

From (7), $4w = 3 + 2v$ ..... (9)

Putting the values of u, v and w in (5), we get

$$\frac{2v+8}{2} - 2v - 3 - 2v = 1$$

$\Rightarrow \quad 2v + 8 - 4v - 6 - 4v = 2 \Rightarrow \quad v = 0$

From (8), $\quad u = \frac{0-8}{6} = -\frac{4}{3}$

From (9), $4w = 3 \quad \therefore \quad w = \frac{3}{4}$

From (3), $d = 2v - 1 = 0 - 1 = -1$

From (1), equation of required sphere is

$$x^2 + y^2 + z^2 - \frac{0-8}{6} - \frac{8}{3}\ x + \frac{3}{2} z - 1 = 0$$

or $\quad 6x^2 + 6y^2 + 6z^2 - 16x + 9z - 6 = 0$

Example :

Find the equation of the sphere with the points (1, 2, 2) and (2, 3, 4) as the extremities of a diameter. Find the co-ordinates of its centre.

Solution

Let $A \equiv (1, 2, 2)$, $B \equiv (2, 3, 4)$

Equation of the sphere having (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Here $x_1 = 1$, $x_2 = 2$, $y_1 = 2$, $y_2 = 3$, $z_1 = 2$, $z_2 = 4$

$\therefore$ required equation of the sphere is

$$(x - 1)(x - 2) + (y - 2)(y - 3) + (z - 2)(z - 4) = 0$$

or $\quad x^2 + y^2 + z^2 - 3x - 5y - 6z + 16 = 0$

Centre of the sphere is middle point of AB

$\therefore \quad$ Centre is $\left(\frac{3}{2}, \frac{5}{2}, 3\right)$

Self practice problems:

1. Find the value of k for which the plane $x + y + z = \sqrt{3}k$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

Ans. $\sqrt{3} \pm 3$

2. Find the equation to the sphere passing through (1, – 3, 4), (1, – 5, 2) and (1, – 3, 0) which has its centre in the plane $x + y + z = 0$

Ans. $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$

3. Find the equation of the sphere having centre on the line $2x - 3y = 0$, $5y + 2z = 0$ and passing through the points (0, – 2, – 4) and (2, – 1, – 1).

Ans. $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$

4. Find the centre and radius of the circle in which the plane $3x + 2y - z - 7\sqrt{14} = 0$ intersects the sphere $x^2 + y^2 + z^2 = 81$.

Ans. $4\sqrt{2}$ units

5. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$