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STUDY PACKAGE

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Topic : Diffrential Equations

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Differential Equation

1. Introduction :

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation:

1.1 **Ordinary Differential Equation :** If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary.

for example $\frac{dy}{dx} + \frac{dz}{dx} = y + z$,

$\frac{dy}{dx} + xy = \sin x$, $\frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x$,

$k\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$, $y = x\frac{dy}{dx} + k\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

1.2 **Partial differential equation :** If the dependent variables depend on two or more independent variables, then it is known as partial differential equation

for example $y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y} = ax$, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

2. Order and Degree of a Differential Equation:

2.1 **Order :** Order is the highest differential appearing in a differential equation.

2.2 **Degree :**

It is determined by the degree of the highest order derivative present in it after the differential equation is cleared of radicals and fractions so far as the derivatives are concerned.

$$f_1(x, y)\left[\frac{d^my}{dx^m}\right]^{n_1} + f_2(x, y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^{n_2} + \ldots\ldots\ f_k(x, y)\left[\frac{dy}{dx}\right]^{n_k} = 0$$

The above differential equation has the order m and degree n_1.

Example :

Find the order & degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $y = e^{\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\right)}$

(iii) $\sin\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\right) = y$ (iv) $ey''' - xy'' + y = 0$

Solution.

(i) $\left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^6$ $\therefore$ order = 2, degree = 4

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \ell ny$ $\therefore$ order = 2, degree = 1

(iii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin^{-1} y$ $\therefore$ order = 2, degree = 1

(iv) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$ $\because$ equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

Self Practice Problems :

1. Find order and degree of the following differential equations.

(i) $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$ **Ans.** order = 1, degree = 2

(ii) $e^{\left(\frac{dy}{dx}-\frac{d^3y}{dx^3}\right)} = \ell n\left(\frac{d^5t}{dx^5}+1\right)$ **Ans.** order = 5, degree = not applicable.

(iii) $\left[\left(\frac{dy}{dx}\right)^{1/2} + y\right]^2 = \frac{d^2y}{dx^2}$ **Ans.** order = 2, degree = 2

3. Formation of Differential Equation:

Differential equation corresponding to a family of curve will have :
(a) Order exactly same as number of essential arbitrary constants in the equation of curve.
(b) no arbitrary constant present in it.
The differential equation corresponding to a family of curve can be obtained by using the following steps:
(a) Identify the number of essential arbitrary constants in equation of curve.

NOTE : If arbitrary constants appear in addition, subtraction, multiplication or division, then we can club them to reduce into one new arbitrary constant.

(b) Differentiate the equation of curve till the required order.
(c) Eliminate the arbitrary constant from the equation of curve and additional equation obtained in step (b) above.

Example :
Form a differential equation of family of straight lines passing through origin.
Sol. Family of straight lines passing through origin is y = mx where'm' is parameter.
Differentiating w.r.t. x

$$\frac{dy}{dx} = m$$

Eliminating 'm' from both equations

$$\frac{dy}{dx} = \frac{y}{x}$$

which is the required differential equation.

Example :
Form a differential equation of family of circles touching x-axis at the origin ?
Sol. Equation of family of circles touching x-axis at the origin is
$x^2 + y^2 + \lambda y = 0$ ....(i) where λ is parameter

$2x + 2y\frac{dy}{dx} + \lambda\frac{dy}{dx} = 0$ ....(ii)

Eliminating 'λ' from (i) and (ii)

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

which is required differential equation.

Self Practice Problems :

1. Obtain a differential equation of the family of curves y = a sin (bx + c) where a and c being arbitrary constant.

Ans. $\frac{d^2y}{dx^2} + b^2y = 0$

2. Show the differential equation of the system of parabolas $y^2 = 4a(x - b)$ is given by

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

3. Form a differential equation of family of parabolas with focus origin and axis of symmetry along the x-axis.

Ans. $y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$

4. Solution of a Differential Equation:

Finding the dependent variable from the differential equation is called solving or integrating it. The solution or the integral of a differential equation is, therefore, a relation between dependent and independent variables (free from derivatives) such that it satisfies the given differential equation
NOTE : The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

There can be three types of solution of a differential equation:
(i) General solution (or complete integral or complete primitive) : A relation in x and y satisfying a given differential equation and involving exactly same number of arbitrary constants as order of differential equation.
(ii) Particular Solution : A solution obtained by assigning values to one or more than one arbitrary constant of general solution.
(iii) Singular Solution : It is not obtainable from general solution. Geometrically, **General solution** acts as an envelope to **singular solution**.

5. Differential Equation of First Order and First Degree :

A differential equation of first order and first degree is of the type

$$\frac{dy}{dx} + f(x, y) = 0, \text{ which can also be written as :}$$

Mdx + Ndy = 0, where M and N are functions of x and y.

6. Elementary Types of First Order and First Degree Differential Equations :

6.1 **Variables separable :** If the differential equation can be put in the form, $f(x)\,dx = \phi(y)\,dy$ we say that variables are separable and solution can be obtained by integrating each side separately.

A general solution of this will be $\int f(x)\,dx = \int \phi(y)\,dy + c$, where c is an arbitrary constant

Example : Solve the differential equation

$$(1 + x)\, y\, dx = (y - 1)\, x\, dy$$

Solution. The equation can be written as -

$$\left(\frac{1+x}{x}\right) dx = \left(\frac{y-1}{y}\right) dy$$

$$\int\left(\frac{1}{x}+1\right) dx = \int\left(1-\frac{1}{y}\right) dy$$

$\ell n\, x + x = y - \ell n y + c$

$\ell n y + \ell n x = y - x + c$

$xy = ce^{y-x}$

Example : Solve : $\frac{dy}{dx} = (e^x + 1)(1 + y^2)$

Solution. The equation can be written as-

$$\frac{dy}{1+y^2} = (e^x + 1)dx$$

Integrating both sides,

$\tan^{-1} y = e^x + x + c.$

Example : Solve : $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

Solution. The equation can be written as -

$$y - ay^2 = (x + a)\frac{dy}{dx}$$

$$\frac{dx}{x+a} = \frac{dy}{y-ay^2}$$

$$\frac{dx}{x+a} = \frac{1}{y(1-ay)}\, dy$$

$$\frac{dx}{x+a} = \left(\frac{1}{y}+\frac{a}{1-ay}\right) dy$$

Integrating both sides,

$\ell n\,(x + a) = \ell n\, y - \ell n\,(1 - ay) + \ell n\, c$

$$\ell n\,(x + a) = \ell n\left(\frac{cy}{1-ay}\right)$$

$cy = (x + a)(1 - ay)$

where 'c' is an arbitrary constant.

6.1.1 Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials:

If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $x\,dx + y\,dy = r\,dr$ (ii) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$ (iii) $x\,dy - y\,dx = r^2d\theta$

If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $x\,dx - y\,dy = r\,dr$ (ii) $x\,dy - y\,dx = r^2\sec\theta\, d\theta$.

Example : Solve the differential equation $xdx + ydy = x\,(xdy - ydx)$

Solution. Taking $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$

$2x\,dx + 2ydy = 2rdr$

$xdx + ydy = rdr$ ....(i)

$$\frac{y}{x} = \tan\theta$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta\,.\,\frac{d\theta}{dx}$$

$xdy - y\,dx = x^2\sec^2\theta\,.\,d\theta$

$xdy - ydx = r^2\,d\theta$ ....(ii)

Using (i) & (ii) in the given differential equation then it becomes

$r\,dr = r\cos\theta.\ r^2\,d\theta$

$$\frac{dr}{r^2} = \cos\theta\, d\theta$$

$$-\frac{1}{r} = \sin\theta + \lambda$$

$$-\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda$$

$$\frac{y+1}{\sqrt{x^2+y^2}} = c \quad \text{where} - \lambda' = c$$

$$(y + 1)^2 = c(x^2 + y^2)$$

6.1.2 Equations Reducible to the Variables Separable form : If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is $\frac{dy}{dx} = f(ax + by + c)$ a, b $\neq$ 0. To solve this, put ax + by + c = t.

Example : Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution. Putting 4x + y + 1 = t

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Given equation becomes

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{t^2+x} = dx \qquad \text{(Variables are separated)}$$

Integrating both sides,

$$\int \frac{dt}{4+t^2} = \int dx$$

$$\Rightarrow \quad \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \Rightarrow \quad \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Example : Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Solution. $\frac{dy}{dx} = \sin(x + y)$

putting x + y = t

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\therefore \quad \frac{dt}{dx} - 1 = \sin t \quad \Rightarrow \quad \frac{dt}{dx} = 1 + \sin t \quad \Rightarrow \quad \frac{dt}{1+\sin t} = dx$$

Integrating both sides,

$$\int \frac{dt}{1+\sin t} = \int dx$$

$$\int \frac{1-\sin t}{\cos^2 t}\, dt = x + c$$

$$\int (\sec^2 t - \sec t \tan t)\, dt = x + c$$

$$\tan t - \sec t = x + c$$

$$-\frac{1-\sin t}{\cos t} = x + c$$

$$-\frac{\cos\frac{t}{2} - \sin\frac{t}{2}}{\cos\frac{t}{2} + \sin\frac{t}{2}} = x + c$$

$$-\tan\left(\frac{\pi}{4} - \frac{t}{2}\right) = x + c$$

$$\tan\left(\frac{\pi}{4}-\frac{x+y}{2}\right)+x+c=0$$

Self Practice Problems :

1. Solve the differential equation

$x^2 y\,\frac{dy}{dx} = (x+1)(y+1)$ **Ans.** $y\,\ell n(y+1) = \ell n x - \frac{1}{x} + c$

2. Solve the differential equation

$\frac{xdx+ydy}{\sqrt{x^2+y^2}} = \frac{ydx-xdy}{x^2}$ **Ans.** $\sqrt{x^2+y^2} + \frac{y}{x} = c$

3. Solve : $\frac{dy}{dx} = e^{x+y} + x^2e^y$ **Ans.** $-\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$

4. Solve : $xy\,\frac{dy}{dx} = 1 + x + y + xy$ **Ans.** $y = x + \ell n\,|x(1+y)| + c$

5. Solve $\frac{dy}{dx} = 1 + e^{x-y}$ **Ans.** $e^{y-x} = x + c$

6. $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$ **Ans.** $\log\left|\tan\frac{x+y}{2}+1\right| = x + c$

7. $\frac{dy}{dx} = x\tan(y-x) + 1$ **Ans.** $\sin(y-x) = e^{x+c}$

6.2 Homogeneous Differential Equations :

A differential equation of the from $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y, and of the same degree, is called homogeneous differential equaiton and can be solved easily by putting y = vx.

Example : Solve $2\frac{y}{x} + \left(\left(\frac{y}{x}\right)^2 - 1\right)\frac{dy}{dx}$

Solution. Putting y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$2v + (v^2-1)\left(v + x\frac{dv}{dx}\right) = 0$$

$$v + x\frac{dv}{dx} = -\frac{2v}{v^2-1}$$

$$x\frac{dv}{dx} = \frac{-v(1+v^2)}{v^2-1}$$

$$\int\frac{v^2-1}{v(1+v^2)}\,dv = -\int\frac{dx}{x}$$

$$\int\left(\frac{2v}{1+v^2}-\frac{1}{v}\right)dv = -\ell n\,x + c$$

$$\ell n(1+v^2) - \ell n\,v = -\ell n\,x + c$$

$$\ell n\left|\frac{1+v^2}{v}.\,x\right| = c$$

$$\ell n\left|\frac{x^2+y^2}{y}\right| = c$$

$x^2 + y^2 = yc'$ where $c' = e^c$

Example : Solve : $(x^2 - y^2)\,dx + 2xydy = 0$ given that y = 1 when x = 1

Solution. $\frac{dy}{dx} = -\frac{x^2-y^2}{2xy}$

$y = vx$

$\frac{dy}{dx} = v + \frac{dv}{dx}$ $\quad\therefore\quad$ $v + x\frac{dv}{dx} = -\frac{1-v^2}{2v}$

$\int \frac{2v}{1+v^2}\,dv = -\int \frac{dx}{x}$

$\ell n\,(1 + v^2) = -\,\ell nx + c$

at $\quad x = 1,\ y = 1 \quad\therefore\quad v = 1$

$\ell n\,2 = c$

$\therefore\quad \ell n\left\{\left(1+\frac{y^2}{x^2}\right).\,x\right\} = \ell n_2$

$x^2 + y^2 = 2x$

6.2.1 Equations Reducible to the Homogeneous form

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ ....(1)

can be made homogeneous (in new variables X and Y) by substituting $x = X + h$ and $y = Y + k$, where h and k are constants, we get $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$. ....(2)

Now, h and k are chosen such that $ah + bk + c = 0$, and $Ah + Bk + C = 0$; the differential equation can now be solved by putting $Y = vX$.

Example : Solve the differential equation $\frac{dy}{dx} = \frac{x+2y-5}{2x+xy-4}$

Solution. Let $\quad x = Y + h, \qquad y = Y + k$

$\frac{dy}{dx} = \frac{dy}{dY}.\frac{dY}{dX}.\frac{dX}{dx}$

$= 1.\ \frac{dY}{dX}.\ 1.$

$= \frac{dY}{dX} \quad\therefore\quad \frac{dY}{dX} = \frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4}$

$= \frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$

h & k are such that $\quad h + 2k - 5 = 0$ & $2h + k - 4 = 0$

$h = 1,\ k = 2$

$\therefore\quad \frac{dY}{dX} = \frac{X+2Y}{2x+Y}$ which is homogeneous differential equation.

Now, substituting $Y = vX$

$\frac{dY}{dX} = v + X\,\frac{dv}{dX}$

$\therefore\quad X\,\frac{dv}{dX} = \frac{1+2v}{2+v} - v$

$\int \frac{2+v}{1-v^2}\,dv = \int \frac{dx}{X}$

$\int \left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right) dv = \ell n\,X + c$

$\frac{1}{2}\ \ell n\,(v + 1) - \frac{3}{2}\ \ell n\,(1 - v) = \ell n\,X + c$

$\ell n\left|\frac{v+1}{(1-v)^3}\right| = \ell nX^2 + 2c$

$\frac{(Y+Y)}{(X-Y)^3}\ \frac{X^2}{X^2} = e^{2c}$

$X + Y = c'(X - Y)^3 \qquad$ where $e^{2c} = c^1$

$x - 1 + y - 2 = c'\,(-1- y + 2)^3$

$x + y - 3 = c'\,(x - y + 1)^3$

Special case :

(A) In equation (1) if $aB = Ab$, then the substitution $ax + by = v$ will reduce it to the form in which variables are separable.

Example : Solve $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Solution. Putting u = 2x + 3y

$$\frac{du}{dx} = 2 + 3 \cdot \frac{dy}{dx}$$

$$\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u-1}{2u-5}$$

$$\frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$$

$$\int \frac{2u-5}{7u-13}\, dx = \int dx$$

$$\Rightarrow \quad \frac{2}{7}\int 1.du - \frac{9}{7}\int \frac{1}{7u-13}\,.\, du = x + c$$

$$\Rightarrow \quad \frac{2}{7}u - \frac{9}{7} \cdot \frac{1}{7}\,\ell n\,(7u - 13) = x + c$$

$$\Rightarrow \quad 4x + 6y - \frac{9}{7}\,\ell n\,(14x + 21y - 13) = 7x + 7c$$

$$\Rightarrow \quad -3x + 6y - \frac{9}{7}\,\ell n\,(14x + 21y - 13) = c'$$

(B) In equation (1), if b + A = 0, then by a simple cross multiplication equation (1) becomes an **exact differential equation**.

Example : Solve $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Solution. Cross multiplying,

$$2xdy + y\,dy - dy = xdx - 2ydx + 5dx$$
$$2\,(xdy + y\,dx) + ydy - dy = xdx + 5\,dx$$
$$2\,d(xy) + y\,dy - dy = xdx + 5dx$$

On integrating,

$$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

$\Rightarrow \quad x^2 - 4xy - y^2 + 10x + 2y = c'$ where $c' = -2c$

(C) If the homogeneous equation is of the form :
$yf(xy)\,dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

Self Practice Problems :

Solve the following differential equations

1. $\left(x\frac{dy}{dx} - y\right)\tan^{-1}\frac{y}{x} = x$ given that y = 0 at x = 1 **Ans.** $\sqrt{x^2+y^2} = e^{\frac{y}{x}\tan^{-1}\frac{y}{x}}$

2. $x\frac{dy}{dx} = y - x\tan\frac{y}{x}$ **Ans.** $x \sin\frac{y}{x} = C$

3. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y+3}$ **Ans.** $x + y = c\,(x - y + 6)^3$

4. $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ **Ans.** $3(2y - x) + \log(3x + 7y + 4) = C$

5. $\frac{dy}{dx} = \frac{3x+2y-5}{3y-2x+5}$ **Ans.** $3x^2 + 4xy - 3y^2 - 10x - 10y = C$

6.3 Exact Differential Equation :

The differential equation $M + N\frac{dy}{dx} = 0$ ....(1)

Where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation (without any subsequent multiplication, elimination etc.) of an equation of the form f(x, y) = c

e.g. $y^2\,dy + x\,dx + \frac{dx}{x} = 0$ is an exact differential equation.

NOTE : (i) The necessary condition for (1) to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) For finding the solution of Exact differential equation, following exact differentials must be remembered :

(a) $x dy + y dx = d(xy)$ (b) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$ (c) $2(x\,dx + y\,dy) = d\,(x^2 + y^2)$

(d) $\frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$ (e) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ (f) $\frac{xdy + ydx}{xy} = d(\ln xy)$

(g) $\frac{xdy + ydx}{x^2 y^2} = d\left(-\frac{1}{xy}\right)$

Example : Solve : $y\,dx + x\,dy = \frac{xdy - ydx}{x^2 + y^2}$

Solution. $ydx + xdy = \frac{xdy - ydx}{x^2 + y^2}$

$d(xy) = d(\tan^{-1} y/x)$

Integrating both sides -

$xy = \tan^{-1} y/x + c$

Example : Solve : $(2x\,\ell ny)\,dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

Solution. The given equation can be written as -

$\ell ny\,(2x)\,dx + x^2\left(\frac{dy}{y}\right) + 3y^2\,dy = 0$

$\Rightarrow \quad \ell ny\; d(x^2) + x^2\, d(\ell ny) + d(y^3) = 0$

$\Rightarrow \quad d(x^2\,\ell ny) + d(y^3) = 0$

Now integrating each term, we get

$x^2\,\ell ny + y^3 = c$

Self Practice Problems :

1. Solve : $xdy + ydx + xy\,e^y\,dy = 0$ **Ans.** $\ell n(xy) + e^y = c$

2. Solve : $ye^{-x/y}\,dx - (xe^{-x/y} + y^3)\,dy = 0$ **Ans.** $2e^{-x/y} + y^2 = c$

6.4 Linear Differential Equation :

When the dependent variable and its derivative occur in the first degree only and are not multiplied together, the differential equation is called linear

The m^{th} order linear differential equation is of the form.

$$P_0(x)\frac{d^m y}{dx^m} + P_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \ldots\ldots\ldots\ldots + P_{m-1}(x)\frac{dy}{dx} + P_m(x)\,y = \phi(x),$$

where $P_0(x), P_1(x) \ldots\ldots\ldots P_m(x)$ are called the coefficients of the differential equation.

NOTE : $\frac{dy}{dx} + y^2 \sin x = \ln x$ is not a Linear differential equation.

Linear differential equations of first order :

The differential equation $\frac{dy}{dx} + Py = Q$, is linear in y.

where P and Q are functions of x.

Integrating Factor (I.F.) : - It is an expression which when multiplied to a differential equation converts it into an exact form.

I.F for linear differential equation = $e^{\int P dx}$ (constant of integration will not be considered)

$\therefore$ after multiplying above equation by I.F it becomes;

$$\frac{dy}{dx}\cdot e^{\int Pdx} + Py\,.\,e^{\int Pdx} = Q.\,e^{\int Pdx}$$

$$\Rightarrow \quad \frac{d}{dx}\left(y.e^{\int Pdx}\right) = Q.e^{\int Pdx}$$

$$\Rightarrow \quad y.e^{\int Pdx} = \int Q.e^{\int Pdx} + C.$$

NOTE : Some times differential equation becomes linear if x is taken as the dependent variable and y as independent variable. The differential equation has then the following form :

$$\frac{dx}{dy} + P_1 x = Q_1.$$

where P_1 and Q_1 are functions of y.

The I.F. now is $e^{\int P_1 dy}$

Example : Solve

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3}\,y = \frac{\sin^2 x}{1+x^3}$$

Solution. $\frac{dy}{dx} + Py = Q$

$P = \frac{3x^2}{1+x^3}$

$IF = e^{\int P.dx} = e^{\int \frac{3x^2}{1+x^3}dx} = e^{\ell n(1+x^3)} = 1 + x^3$

$\therefore$ General solution is

$y(IF) = \int Q(IF).dx + c$

$y\,(1 + x^3) = \int \frac{\sin^2 x}{1+x^3}\,(1 + x^3)\,dx + c$

$y(1 + x^3) = \int \frac{1-\cos 2x}{2}\,dx + c$

$y(1 + x^3) = \frac{1}{2}\,x - \frac{\sin 2x}{4} + c$

Example : Solve : $x\,\ell nx\,\frac{dy}{dx} + y = 2\,\ell n\,x$

Solution. $\frac{dy}{dx} + \frac{1}{x\ell nx}\,y = \frac{2}{x}$

$P = \frac{1}{x\ell nx}$, $Q = \frac{2}{x}$

$IF = e^{\int P.dx} = e^{\int \frac{1}{x\ell nx}dx} = e^{\ell n(\ell nx)} = \ell n\,x$

$\therefore$ General solution is

$y.\,(\ell n\,x) = \int \frac{2}{x}.\ell nx.dx + c$

$y\,(\ell n\,x) = (\ell n\,x)^2 + c$

Example : Solve the differential equation

$t\,(1 + t^2)\,dx = (x + xt^2 - t^2)\,dt$ and it given that $x = -\pi/4$ at $t = 1$

Solution. $t\,(1 + t^2)\,dx = [x\,(1 + t^2) - t^2]\,dt$

$\frac{dx}{dt} = \frac{x}{t} - \frac{t}{(1+t^2)}$

$\frac{dx}{dt} - \frac{x}{t} = -\frac{t}{1+t^2}$

which is linear in $\frac{dx}{dt}$

Here, $P = -\frac{1}{t}$, $Q = -\frac{t}{1+t^2}$

$IF = e^{-\int \frac{1}{t}dt} = e^{-\ell nt} = \frac{1}{t}$

$\therefore$ General solution is -

$x \cdot \frac{1}{t} = \int \frac{1}{t}.\left(-\frac{t}{1+t^2}\right)dt + c$

$\frac{x}{t} = -\tan^{-1} t + c$

putting $x = -\pi/4$, $t = 1$

$-\pi/4 = -\pi/4 + c \Rightarrow c = 0$

$\therefore$ $x = -t\tan^{-1} t$

Equations reducible to linear form

6.4.1 By change of variable.

Example : Solve : $y \sin x \dfrac{dy}{dx} = \cos x (\sin x - y^2)$

Solution. The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$$2y \frac{dy}{dx} = \frac{dz}{dx}$$

differential equation becomes

$$\frac{\sin x}{2} \frac{dz}{dx} + \cos x . z = \sin x \cos x$$

$$\frac{dz}{dx} + 2 \cot x . z = 2 \cos x \quad \text{which is linear in } \frac{dz}{dx}$$

IF $= e^{\int 2\cot x\,dx} = e^{2\ell n \sin x} = \sin^2 x$ $\quad\therefore\quad$ General solution is -

$$z. \sin^2 x = \int 2\cos x . \sin^2 x . dx + c$$

$$y^2 \sin^2 x = \frac{2}{3} \sin^3 x + c$$

6.4.2 Bernoulli's equation :

Equations of the form $\dfrac{dy}{dx} + Py = Q.y^n$, $n \neq 0$ and $n \neq 1$

where P and Q are functions of x, is called Bernoulli's equation and can be made linear in v by dividing by y^n and putting $y^{-n+1} = v$. Now its solution can be obtained as in (v).

e.g. : $2 \sin x \dfrac{dy}{dx} - y \cos x = xy^3 e^x$.

Example : Solve : $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{y^2}{x^2}$ (Bernoulli's equation)

Solution. Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = \frac{1}{x^2} \qquad ..... (1)$$

Putting $\dfrac{1}{y} = t$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore$ differential equation (1) becomes,

$$-\frac{dt}{dx} - \frac{t}{x} = \frac{1}{x^2}$$

$$\frac{dt}{dx} + \frac{t}{x} = -\frac{1}{x^2} \quad \text{which is linear differential equation in } \frac{dt}{dx}$$

IF $= e^{\int \frac{1}{x} dx} = e^{\ell n x} = x$ $\quad\therefore\quad$ General solution is -

$$t. x = \int -\frac{1}{x^2} \cdot x \, dx + c$$

$$tx = -\ell n x + c$$

$$\frac{x}{y} = - \ell n x + c$$

Self Practice Problems :

1. Solve : $x (x^2 + 1) \dfrac{dy}{dx} = y (1 - x^2) + x^2 \ell n x$ **Ans.** $\left(\dfrac{x^2+1}{x}\right) y = x \ell n x - x + c$

2. Solve : $(x + 2y^3) \dfrac{dy}{dx} = y$ **Ans.** $x = y (c + y^2)$

3. Solve : $x \dfrac{dy}{dx} + y = y^2 \log x$ **Ans.** $y (1 + cx + \log x) = 1$

4. Solve the differential equation

$xy^2 \left(\dfrac{dy}{dx}\right) - 2y^3 = 2x^3$ given $y = 1$ at $x = 1$ **Ans.** $y^3 + 2x^3 = 3x^6$

7. Clairaut's Equation :

The differential equation

$y = px + f(p)$, ...(10), where $p = \dfrac{dy}{dx}$

is known as Clairout's Equation.
To solve (10), differentiate it w.r.t. x, which gives

either $\dfrac{dp}{dx} = 0 \Rightarrow p = c$ ....(11)

or $x + f'(p) = 0$ ....(12)

NOTE : (i) If p is eliminated between (10) and (11), the solution obtained is a general solution of (10)
(ii) If p is eliminated between (10) and (12), then solution obtained does not contain any arbitrary constant and is not particular solution of (10). This solution is called singular solution of (10).

Example : Solve : $y = mx + m - m^3$ where, $m = \dfrac{dy}{dx}$

Solution. $y = mx + m - m^3$ ..... (i)
The given equation is in clairaut's form.
Now, differentiating wrt. x -

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\frac{dm}{dx}(x + 1 - 3m^2) = 0$$

$\dfrac{dm}{dx} = 0 \Rightarrow m = c$ ..... (ii)

or $x + 1 - 3m^2 = 0 \Rightarrow m^2 = \dfrac{x+1}{3}$ ..... (iii)

Eliminating 'm' between (i) & (ii) is called the general solution of the given equation.
$y = cx + c - c^3$ where, 'c' is an arbitrary constant.
Again, eliminating 'm' between (i) & (iii) is called singular solution of the given equation.

$$y = m(x + 1 - m^2)$$

$$y = \left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right)$$

$$y = \left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1)$$

$$y = 2\left(\frac{x+1}{3}\right)^{3/2}$$

$$y^2 = \frac{4}{27}(x+1)^3$$

$$27y^2 = 4(x+1)^3$$

Self Practice Problems :

1. Solve the differential equation

$Y = mx + 2/m$ where, $m = \dfrac{dy}{dx}$

Ans. General solution : $y = cx + 2/c$ where c is an arbitrary constant
Singular solution : $y^2 = 8x$

2. Solve : $\sin px \cos y = \cos px \sin y + p$ where $p = \dfrac{dy}{dx}$

Ans. General solution : $y = cx - \sin^{-1}(c)$ where c is an arbitrary constant.

Singular solution : $y = \sqrt{x^2-1} - \sin^{-1}\sqrt{\dfrac{x^2-1}{x^2}}$

8 Orthogonal Trajectory :

An orthogonal trajectory of a given system of curves is defined to be a curve which cuts every member of a given family of curve at right angle.

Steps to find orthogonal trajectory :

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where 'c' is an arbitrary constant.

(ii) Differentate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation obtained in (iii).
Hence solution obtained in (iv) is the required orthogonal trajectory.

Example : Find the orthogonal trajectory of family of straight lines passing through the origin.

Solution. Family of straight lines passing through the origin is -

$y = mx$ ..... (i)

where 'm' is an arbitrary constant.

Differentiating wrt x

$$\frac{dy}{dx} = m \quad ..... (ii)$$

Eliminate 'm' from (i) & (ii)

$$y = \frac{dy}{dx}x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$y = -\frac{dx}{dy}x$$

$x\,dy + y\,dy = 0$

Integrating each term,

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$\Rightarrow$ $x^2 + y^2 = 2c$

which is the required orthogonal trajectory.

Example : Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

Solution. $y^2 = 4ax$ ..... (i)

$$2y\frac{dy}{dx} = 4a \quad ..... (ii)$$

Eliminating 'a' from (i) & (ii)

$$y^2 = 2y\frac{dy}{dx}x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$y = 2\left(-\frac{dx}{dy}\right)x$$

$2\,x\,dx + y\,dy = 0$

Integrating each term,

$$x^2 + \frac{y^2}{2} = c$$

$2x^2 + y^2 = 2c$

which is the required orthogonal trajectories.

Self Practice Problems :

1. Find the orthogonal trajectory of family of circles concentric at (a, 0)
Ans. $y = c(x - a)$ where c is an arbitrary constant.

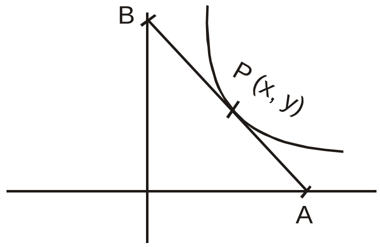
2. Find the orthogonal trajectory of family of circles touching x – axis at the origin.
Ans. $x^2 + y^2 = cx$ where c is an arbitrary constant.

3. Find the orthogonal trajectory of the family of rectangular hyperbola $xy = c^2$
Ans. $x^2 - y^2 = k$ where k is an arbitrary constant.

Geometrical application of differential equation :

Example : Find the curves for which the portion of the tangent included between the co-ordinate axes is bisected at the point of contact.

Solution. Let P (x, y) be any point on the curve.
Equation of tangent at P (x, y) is -



$Y - y = m (X - x)$ where $m = \frac{dy}{dx}$ is slope of the tangent at P (x, y).

Co-ordinates of $A\left(\frac{mx - y}{m}, 0\right)$ & B (0, y–mx)

P is the middle point of A & B

$\therefore \quad \frac{mx - y}{m} = 2x$

$\Rightarrow \quad mx - y = 2mx$

$\Rightarrow \quad mx = -y$

$\Rightarrow \quad \frac{dy}{dx} x = -y$

$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow \quad \ell nx + \ell ny = \ell nc$

$\therefore \quad xy = c$

Example : Show that $(4x + 3y + 1) dx + (3x + 2y + 1) dy = 0$ represents a hyperbola having as asymptotes the lines $x + y = 0$ and $2x + y + 1 = 0$

Solution. $(4x + 3y + 1) dx + (3x + 2y + 1) dy = 0$

$4xdx + 3 (y dx + x dy) + dx + 2y dy + dy = 0$

Integrating each term,

$2x^2 + 3 xy + x + y^2 + y + c = 0$

$2x^2 + 3xy + y^2 + x + y + c = 0$

which is the equation of hyperbola when $x^2 > ab$ & $\Delta \neq 0$.

Now, combined equation of its asymptotes is -

$2x^2 + 3xy + y^2 + x + y + \lambda = 0$

which is pair of straight lines

$\therefore \quad \Delta = 0$

$\Rightarrow \quad 2.1\, \lambda + 2 \,.\, \frac{1}{2}.\frac{1}{2}.\frac{3}{2} - 2 \,.\, \frac{1}{4} - 1 \,.\, \frac{1}{4} - \lambda\, \frac{9}{4} = 0$

$\Rightarrow \quad \lambda = 0$

$\therefore \quad 2x^2 + 3xy + y^2 + x + y = 0$

$(x + y) (2x + y) + (x + y) = 0$

$(x + y) (2x + y + 1) = 0$

$x + y = 0 \quad$ or $\quad 2x + y + 1 = 0$

Example : The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1)

Solution. Let P (x, y) be any point on the curve

Equation of tangent at 'P' is -

$Y - y = m (X - x)$

$mX - Y + y - mx = 0$

Now,

$$\left(\frac{y - mx}{\sqrt{1 + m^2}}\right) = x$$

$y^2 + m^2x^2 - 2mxy = x^2 (1 + m^2)$

$\frac{y^2 - x^2}{2xy} = \frac{dy}{dx}$ which is homogeneous equation

Putting $y = vx$

$\frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore \quad v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v}$

$x \frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$

$$\int \frac{2v}{v^2+1}\,dv = -\int \frac{dx}{x}$$

$\ell n\,(v^2 + 1) = -\ell n\, x + \ell n\, c$

$$x\left(\frac{y^2}{x^2}+1\right) = c$$

Curve is passing through (1, 1)

$\therefore$ $c = 2$

$x^2 + y^2 - 2x = 0$

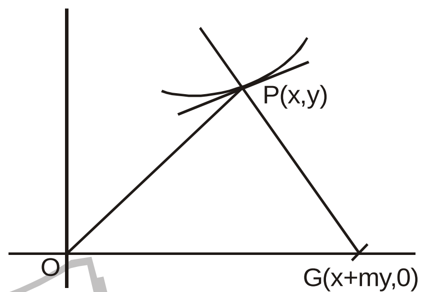
Example : Find the nature of the curve for which the length of the normal at a point 'P' is equal to the radius vector of the point 'P'.

Solution. Let the equation of the curve be y = f(x). P(x, y) be any point on the curve.

Slope of the tanget at P(x, y) is $\frac{dy}{dx} = m$

$\therefore$ Slope of the normal at P is

$$m' = -\frac{1}{m}$$



Equation of the normal at 'P'

$$Y - y = -\frac{1}{m}(X - x)$$

Co-ordinates of G (x + my, 0)

Now, $OP^2 = PG^2$

$x^2 + y^2 = m^2y^2 + y^2$

$$m = \pm\frac{x}{y}$$

$$\frac{dy}{dx} = \pm\frac{x}{y}$$

Taking as the sign

$$\frac{dy}{dx} = \frac{x}{y}$$

$y\,.\,dy = x\,.\,dx$

$$\frac{y^2}{2} = \frac{x^2}{2} + \lambda$$

$x^2 - y^2 = -2\lambda$

$x^2 - y^2 = c$ (Rectangular hyperbola)

Again taking as –ve sign

$$\frac{dy}{dx} = -\frac{x}{y}$$

$y\,dy = -x\,dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \lambda'$$

$x^2 + y^2 = 2\lambda'$

$x^2 + y^2 = c'$ (circle)