Q.4 Using LMVT prove that : **(a)** $\tan x > x$ in $\left(0, \frac{\pi}{2}\right)$, **(b)** $\sin x < x$ for $x > 0$

Q.5 Prove that if f is differentiable on [a, b] and if f (a) = f (b) = 0 then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

Q.6 For what value of a, m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$ satisfy the hypothesis of the mean value theorem for the interval [0, 2].

Q.7 Suppose that on the interval [–2, 4] the function f is differentiable, $f(-2) = 1$ and $|f'(x)| \le 5$. Find the bounding functions of f on [–2, 4], using LMVT.

Q.8 Let f, g be differentiable on R and suppose that f (0) = g (0) and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$.

Q.9 Let f be continuous on [a, b] and differentiable on (a, b). If f (a) = a and f (b) = b, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

Q.10 Let f (x) and g (x) be differentiable functions such that $f'(x)\, g(x) \ne f(x)\, g'(x)$ for any real x. Show that between any two real solutions of f (x) = 0, there is at least one real solution of g (x) = 0.

Q.11 Let f defined on [0, 1] be a twice differentiable function such that, $|f''(x)| \le 1$ for all $x \in [0, 1]$
If f (0) = f (1), then show that, $|f'(x)| < 1$ for all $x \in [0, 1]$

Q.12 f (x) and g (x) are differentiable functions for $0 \le x \le 2$ such that f (0) = 5, g (0) = 0, f (2) = 8, g (2) = 1. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3 g'(c)$.

Q.13 If f, ϕ, ψ are continuous in [a, b] and derivable in]a, b[then show that there is a value of c lying between a & b such that,
$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \Psi(a) & \Psi(b) & \Psi'(c) \end{vmatrix} = 0$$

Q.14 Show that exactly two real values of x satisfy the equation $x^2 = x \sin x + \cos x$.

Q.15 Let $a > 0$ and f be continuous in [–a, a]. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that f (0) = 0.

Q.16 Let a, b, c be three real number such that a < b < c, f (x) is continuous in [a, c] and differentiable in (a, c). Also f ' (x) is strictly increasing in (a, c). Prove that
$$(c - b)\, f(a) + (b - a)\, f(c) > (c - a)\, f(b)$$

Q.17 Use the mean value theorem to prove, $\frac{x-1}{x} < \ln x < x - 1,\ \forall\ x > 1$

Q.18 Use mean value theorem to evaluate, $\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$.

Q.19 Using L.M.V.T. or otherwise prove that difference of square root of two consecutive natural numbers greater than N^2 is less than $\frac{1}{2N}$.

Q.20 Prove the inequality $e^x > (1 + x)$ using LMVT for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater.

EXERCISE–8

Q.1 If $f(x) = \frac{x}{\sin x}$ & $g(x) = \frac{x}{\tan x}$, where $0 < x \le 1$, then in this interval :
(A) both f (x) & g (x) are increasing functions
(B) both f (x) & g (x) are decreasing functions
(C) f (x) is an increasing function
(D) g (x) is an increasing function
[JEE '97 (Scr), 2]

Q.2 Let a + b = 4 , where a < 2 and let g (x) be a differentiable function . If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as (b – a) increases. [JEE '97, 5]

Q.3(a) Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then :
(A) h is increasing whenever f is increasing
(B) h is increasing whenever f is decreasing
(C) h is decreasing whenever f is decreasing
(D) nothing can be said in general.

(b) $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f :
(A) does not exist because f is unbounded
(B) is not attained even though f is bounded
(C) is equal to 1
(D) is equal to –1.
[JEE '98, 2 + 2]

Q.4(a) For all $x \in (0, 1)$:
(A) $e^x < 1 + x$
(B) $\log_e(1 + x) < x$
(C) $\sin x > x$
(D) $\log_e x > x$

(b) Consider the following statements S and R :
S : Both sin x & cos x are decreasing functions in the interval $(\pi/2, \pi)$.
R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

Which of the following is true ?
(A) both S and R are wrong
(B) both S and R are correct, but R is not the correct explanation for S
(C) S is correct and R is the correct explanation for S (D) S is correct and R is wrong.

(c) Let $f(x) = \int e^x (x-1)(x-2)\, dx$ then f decreases in the interval :
(A) $(-\infty, 2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, +\infty)$
[JEE 2000 (Scr.) 1+1+1 out of 35]

Q.5(a) If $f(x) = xe^{x(1-x)}$, then f(x) is
(A) increasing on $\left(-\frac{1}{2}, 1\right)$ (B) decreasing on R
(C) increasing on R (D) decreasing on $\left[-\frac{1}{2}, 1\right]$

(b) Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it. [JEE 2001, 1 + 5]

Q.6 The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is
(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{2}$ (D) π
[JEE 2002 (Screening), 3]

Q.7(a) Using the relation $2(1 - \cos x) < x^2$, $x \ne 0$ or otherwise, prove that $\sin(\tan x) \ge x$, $\forall\ x \in \left[0, \frac{\pi}{4}\right]$.

(b) Let $f : [0, 4] \to R$ be a differentiable function.
(i) Show that there exist a, b $\in [0, 4]$, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$
(ii) Show that there exist α, β with $0 < \alpha < \beta < 2$ such that

$$\int_0^4 f(t)\, dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2))$$ [JEE 2003 (Mains), 4 + 4 out of 60]

Q.8(a) Let $f(x) = \begin{cases} x^\alpha \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if α =
(A) –2 (B) –1 (C) 0 (D) $\frac{1}{2}$

(b) If *f* is a strictly increasing function, then $\lim\limits_{x \to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is equal to
(A) 0 (B) 1 (C) –1 (D) 2
[JEE 2004 (Scr)]

Q.9 If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem, prove that at least one root of p(x) lies between $(45^{1/100}, 46)$. [JEE 2004, 2 out of 60]

Q.10(a) If $f(x)$ is a twice differentiable function and given that f(1) = 1, f(2) = 4, f(3) = 9, then
(A) $f''(x) = 2$, for $\forall\ x \in (1, 3)$ (B) $f''(x) = f'(x) = 2$, for some $x \in (2, 3)$
(C) $f''(x) = 3$, for $\forall\ x \in (2, 3)$ (D) $f''(x) = 2$, for some $x \in (1, 3)$
[JEE 2005 (Scr), 3]

(b) f (x) is differentiable function and g (x) is a double differentiable function such that $|f(x)| \le 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c) \cdot g''(c) < 0$.
[JEE 2005 (Mains), 6]

EXERCISE–9

Only one correct options

1. The function $\frac{|x-1|}{x^2}$ is monotonically decreasing for
(A) $(2, \infty)$ (B) $(0, 1)$ (C) $(0, 1) \cup (2, \infty)$ (D) $(-\infty, \infty)$

2. The values of p for which the function $f(x) = \left(\frac{\sqrt{p+4}}{1-p} - 1\right) x^5 - 3x + \ln 5$ decreases for all real x is
(A) $(-\infty, \infty)$ (B) $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$
(C) $\left[-3, \frac{5-\sqrt{27}}{2}\right] \cup (2, \infty)$ (D) $[1, \infty)$

3. The set of all x for which $\ell n(1 + x) \le x$ is equal to
(A) $x > 0$ (B) $x > -1$ (C) $-1 < x < 0$ (D) null set

4. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function in the set of real numbers R. Then a & b satisfy the condition:
(A) $a^2 - 3b - 15 > 0$ (B) $a^2 - 3b + 15 < 0$ (C) $a^2 - 3b - 15 < 0$ (D) $a > 0$ & $b > 0$

5. If $f(x) = a^{\{a^{|x|}\text{sgn}x\}}$; $g(x) = a^{[a^{|x|}\text{sgn}x]}$ for $a > 1$, $a \ne 1$ and $x \in R$, where { } & [ ] denote the fractional part and integral part functions respectively, then which of the following statements holds good for the function h(x), where $(\ell n\, a)\, h(x) = (\ell n\, f(x) + \ell n\, g(x))$.
(A) 'h' is even and increasing (B) 'h' is odd and decreasing
(C) 'h' is even and decreasing (D) 'h' is odd and increasing

6. If $f : [1, 10] \to [1, 10]$ is an non-decreasing function and $g : [1, 10] \to [1, 10]$ is a non-increasing function. Let $h(x) = f(g(x))$ with $h(1) = 1$, then $h(2)$
(A) lies in (1, 2) (B) is more than 2 (C) is equal to 1 (D) is not defined

7. Let f be a differentiable function of x, $\forall x \in R$. If $f(1) = -4$ and $f'(x) \ge 2$ $\forall x \in [1, 6]$, then minimum value of f(6) is
(A) 6 (B) 2 (C) 4 (D) none of these

8. For what values of a does the curve $f(x) = x(a^2 - 2a - 2) + \cos x$ is always strictly monotonic $\forall x \in R$.
(A) $a \in R$ (B) $|a| < \sqrt{2}$
(C) $1 - \sqrt{2} < a < 1 + \sqrt{2}$ (D) $|a| < \sqrt{2} - 1$

9. If $f(x) = \dfrac{x^2}{2 - 2\cos x}$; $g(x) = \dfrac{x^2}{6x - 6\sin x}$ where $0 < x < 1$, then
(A) both 'f' and 'g' are increasing functions (B) 'f' is decreasing & 'g' is increasing function
(C) 'f' is increasing & 'g' is decreasing function (D) both 'f' & 'g' are decreasing function

One or more than one correct options :

10. The set of values of a for which the function $f(x) = x^2 + ax + 1$ is an increasing function on [1, 2] is I_1 and decreasing in [1, 2] is I_2, then :
(A) $I_1 : a \in (2, \infty)$ (B) $I_2 : a \in (-\infty, -4)$
(C) $I_2 : a \in (-\infty, -4]$ (D) $I_1 ; a \in [-2, \infty)$

11. If f is an even function then
(A) f^2 increases on (a, b) (B) f cannot be monotonic
(C) f^2 need not increases on (a, b) (D) f has inverse

12. Let $g(x) = 2f(x/2) + f(1 - x)$ and $f''(x) < 0$ in $0 \le x \le 1$ then g(x) :
(A) decreases in $\left[0, \dfrac{2}{3}\right]$ (B) decreases $\left[\dfrac{2}{3}, 1\right]$
(C) increases in $\left[0, \dfrac{2}{3}\right]$ (D) increases in $\left[\dfrac{2}{3}, 1\right]$

13. On which of the following intervals, the function $x^{100} + \sin x - 1$ is strictly increasing
(A) $(-1, 1)$ (B) $[0, 1]$ (C) $[\pi/2, \pi]$ (D) $[0, \pi/2]$

14. The function $y = \dfrac{2x-1}{x-2}$ $(x \ne 2)$:
(A) is its own inverse (B) decreases for all values of x
(C) has a graph entirely above x-axis (D) is bound for all x.

15. Let f and g be two functions defined on an interval I such that $f(x) \ge 0$ and $g(x) \le 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then
(A) the product function fg is strictly increasing on I
(B) the product function fg is strictly decreasing on I
(C) fog(x) is monotonically increasing on I (D) fog (x) is monotonically decreasing on I

EXERCISE–10

1. Let $f(x) = \begin{cases} \max(x, x^2) & x \ge 0 \\ \min(x, x^2 - 2) & x < 0 \end{cases}$. Draw the graph of f(x) and hence comment on the nature of monotonic behaviour at x = –1, 0, 1.

2. Let $f(x) = \begin{cases} x^2 & x \ge 0 \\ x|a| & x < 0 \end{cases}$. Find possible values of a such that f(x) is monotonically increasing at x = 0.

3. Find the relation between the constants a, b, c & d so that the function,
$f(x) = [a \sin x + b \cos x]@[c \sin x + d \cos x]$ is always increasing.

4. Find the intervals of monotonocity for the following functions :
$$f(x) = \frac{1 - x + x^2}{1 + x + x^2}$$

5. If p, q, r be real, locate the intervals in which, $f(x) = \begin{vmatrix} x + p^2 & pq & pr \\ pq & x + q^2 & qr \\ pr & qr & x + r^2 \end{vmatrix}$,
(a) increase (b) decrease

6. Find the values of 'a' for which the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases for all real values of x.

7. Prove that for $0 < x < 1$, the inequality, $x < -\ell n(1-x) < x(1-x)^{-1}$.

8. For $x \in \left(0, \frac{\pi}{2}\right)$ identify which is greater (2sinx + tanx) Or (3x) hence find $\lim\limits_{x \to 0}\left[\frac{3x}{2\sin x + \tan x}\right]$.

9. Prove that $f(x) = \frac{\sin x}{x}$ is monotonically decreasing function for $x \in \left(0, \frac{\pi}{2}\right)$. Hence prove that

(i) for $x \in \left(0, \frac{\pi}{6}\right)$, $x \operatorname{cosec} x < \frac{\pi}{3}$ (ii) $\frac{\sin x}{x} < \frac{\sin(\sin x)}{\sin x}$

10. A (0, 1), B $\left(\frac{\pi}{2}, 1\right)$ are two points on the graph given by y = 2 sinx + cos2x. Prove that there exists a point P on the curve between A & B such that tangent at P is parallel to AB. Find the co-ordinates of P.

11. Using Rolle's theorem prove that the equation $3x^2 + px - 1 = 0$ has at least one real root in the interval $x \in (-1, 1)$.

12. Show that $xe^x = 2$ has one & only one root between 0 & 1.

13. Find the interval in which the following function is increasing or decreasing :

$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ in $[0, \pi]$.

14. Show that the derivative of the function $f(x) = \begin{cases} x\sin\frac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ vanishes on an infinite set of points of the interval (0, 1).

15. Assume that f is continuous on [a, b] a > 0 and differentiable in (a, b) such that $\frac{f(a)}{a} = \frac{f(b)}{b}$. Prove that there exists $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(x_0)}{x_0}$.

16. Find the greatest & least value of $f(x) = \sin^{-1}\frac{x}{\sqrt{x^2+1}} - \ell n\, x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$.

17. Show that, $1 + x\,\ell n\left[x + \sqrt{x^2+1}\right] \geq \sqrt{1+x^2}$ for all $x \geq 0$.

18. Prove the inequality, $\frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$ for $0 < x_1 < x_2 < \frac{\pi}{2}$.

19. A function f is differentiable in the interval $0 \leq x \leq 5$ such that f(0) = 4 & f(5) = – 1. If $g(x) = \frac{f(x)}{x+1}$, then prove that there exists some $c \in (0, 5)$ such that $g'(x) = -\frac{5}{6}$.

20. Prove that for all $x \in R$ $e^x + \sqrt{1+e^{2x}} \geq (1+x) + \sqrt{2+2x+x^2}$.

21. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall x \in \left(0, \frac{\pi}{2}\right)$ and g(x) = f(sin x) + f(cos x), then find the interval in which g(x) is increasing and decreasing.

22. If $ax^2 + (b/x) \geq c$ for all positive x where a > 0 and b > 0 then show that $27ab^2 \geq 4c^3$.

23. Prove that for $0 \leq p \leq 1$ & for any a > 0, b > 0 the inequality $(a+b)^p \leq a^p + b^p$.

24. Find the greatest and the least values of the function f(x) defined as below.
f(x) = minimum of $\{3t^4 - 8t^3 - 6t^2 + 24t \; ; \; 1 \leq t \leq x\}$, $1 \leq x < 2$.

maximum of $\left\{3t + \frac{1}{4}\sin^2\pi t + 2 \; ; \; 2 \leq t \leq x\right\}$, $2 \leq x \leq 4$.

25. If a > b > 0, with the aid of Largrange's formula prove the validity of the inequailities
$nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$, if n > 1. Also prove that the inequalities of the opposite sense if 0 < n < 1.

MAXIMA - MINIMA

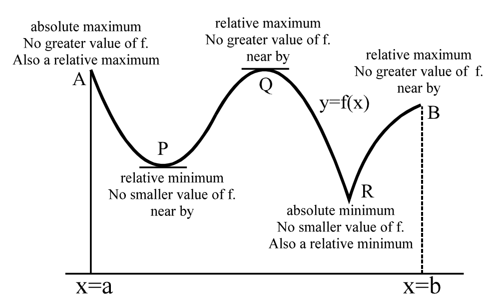
FUNCTIONS OF A SINGLE VARIABLE

HOW MAXIMA & MINIMA ARE CLASSIFIED

1. A function f(x) is said to have a maximum at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x = a. Symbolically

$\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a$ gives maxima for a sufficiently small positive h.

Similarly, a function f(x) is said to have a minimum



value at x = b if f(b) is least than every other value assumed by f(x) in the immediate neighbourhood at x = b. Symbolically if

$$\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b \text{ gives minima for a sufficiently small positive h.}$$

Note that :

(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.

(v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

2. A NECESSARY CONDITION FOR MAXIMUM & MINIMUM :

If f(x) is a maximum or minimum at x = c & if $f'(c)$ exists then $f'(c) = 0$.

Note :

(i) The set of values of x for which $f'(x) = 0$ are often called as stationary points or critical points. The rate of change of function is zero at a stationary point.

(ii) In case $f'(c)$ does not exist f(c) may be a maximum or a minimum & in this case left hand and right hand derivatives are of opposite signs.

(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b] are f(a) or f(b) or are given by the values of x for which $f'(x) = 0$.

(iv) Critical points are those where $\frac{dy}{dx} = 0$, if it exists , or it fails to exist either by virtue of a vertical tangent or by virtue of a geometrical sharp corner but not because of discontinuity of function.

3. SUFFICIENT CONDITION FOR EXTREME VALUES :

$$\left.\begin{matrix} f'(c-h) > 0 \\ f'(c+h) < 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local maxima, where } f'(c) = 0.$$

Similarly $\left.\begin{matrix} f'(c-h) < 0 \\ f'(c+h) > 0 \end{matrix}\right] \Rightarrow x = c$ is a point of local minima, where $f'(c) = 0$.

h is a sufficiently small positive quantity

Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(c) is not an extreme value of f.

4. USE OF SECOND ORDER DERIVATIVE IN ASCERTAINING THE MAXIMA OR MINIMA:

(a) f(c) is a minimum value of the function f, if $f'(c) = 0$ & $f''(c) > 0$.

(b) f(c) is a maximum value of the function f, $f'(c) = 0$ & $f''(c) < 0$.

Note : if $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaning the maxima or minima.

5. SUMMARY–WORKING RULE :

FIRST : When possible , draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

SECOND : Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. his may require some algebraic manipulations.

THIRD : If y = f (x) is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

FOURTH : Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow$ y is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow$ y is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) If $\frac{dy}{dx}$ is $\left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right] \Rightarrow$ a maximum occurs at $x = x_0$.

But if dy/dx changes sign from negative to zero to positive as x advances through x_o there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called **INFLECTION POINTS**.

FIFTH : If the function y = f (x) is defined for only a limited range of values $a \le x \le b$ then examine x = a & x = b for possible extreme values.

SIXTH : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Important Note :

– Given a fixed point $A(x_1, y_1)$ and a moving point P(x, f (x)) on the curve y = f(x). Then AP will be maximum or minimum if it is normal to the curve at P.

– If the sum of two positive numbers x and y is constant than their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then

$$xy = \frac{1}{4}\left[(x + y)^2 - (x - y)^2 \right]$$

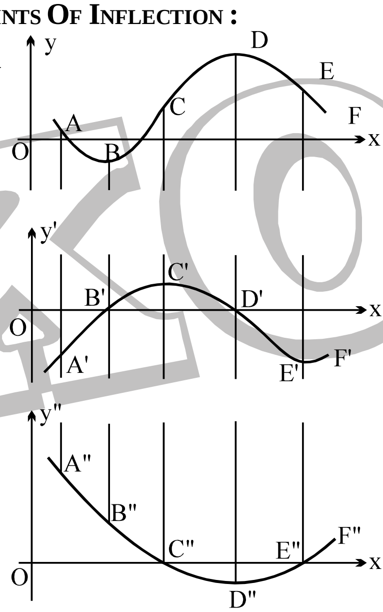
– If the product of two positive numbers is constant then their sum is least if they are equal. i.e. $(x + y)^2 = (x - y)^2 + 4xy$

6. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

☞ Volume of a cuboid = lbh. ☞ Surface area of a cuboid = $2(lb + bh + hl)$.
☞ Volume of a prism = area of the base x height.
☞ Lateral surface of a prism = perimeter of the base x height.
☞ Total surface of a prism = lateral surface + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).
☞ Volume of a pyramid = $\frac{1}{3}$ area of the base x height.
☞ Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).
☞ Volume of a cone = $\frac{1}{3}\pi r^2 h$. ☞ Curved surface of a cylinder = $2\pi rh$.
☞ Total surface of a cylinder = $2\pi rh + 2\pi r^2$.
☞ Volume of a sphere = $\frac{4}{3}\pi r^3$. ☞ Surface area of a sphere = $4\pi r^2$.
☞ Area of a circular sector = $\frac{1}{2} r^2\theta$, when θ is in radians.

7. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION :

The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:



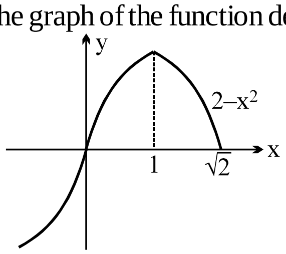
(i) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ & $\frac{d^2y}{dx^2}$ changes sign.

Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist. For example, consider the graph of the function defined as,

$$f(x) = \begin{cases} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{cases}$$



Note that the graph exhibits two critical points one is a point of local maximum & the other a point of inflection.

EXERCISE–11

Q.1 A cubic f(x) vanishes at $x = -2$ & has relative minimum/maximum at $x = -1$ and $x = \frac{1}{3}$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, find the cubic $f(x)$.

Q.2 Investigate for maxima & minima for the function, $f(x) = \int_{1}^{x} [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$

Q.3 Find the maximum & minimum value for the function ;
(a) $y = x + \sin 2x$, $0 \le x \le 2\pi$ (b) $y = 2\cos 2x - \cos 4x$, $0 \le x \le \pi$

Q.4 Suppose f(x) is real valued polynomial function of degree 6 satisfying the following conditions ;

(a) f has minimum value at x = 0 and 2 (b) f has maximum value at x = 1

(c) for all x, $\underset{x \to 0}{\text{Limit}} \frac{1}{x} \ln \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$. Determine f (x).

Q.5 Find the maximum perimeter of a triangle on a given base 'a' and having the given vertical angle α.

Q.6 The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

Q.7 The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's'. If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum.

Q.8 For a given curved surface of a right circular cone when the volume is maximum, prove that the semi vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

Q.9 Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

Q.10 A statue 4 metres high sits on a column 5.6 metres high. How far from the column must a man, whose eye level is 1.6 metres from the ground, stand in order to have the most favourable view of statue.

Q.11 By the post office regulations, the combined length & girth of a parcel must not exceed 3 metre. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post.

Q.12 A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end. If the area of the rectangular portion is to be maximum, find the length of its sides. Use : $\pi \approx \frac{22}{7}$.

Q.13 A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semicircle. The semicircular portion is fitted with coloured glass while the rectangular part is fitted with clean glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light?

Q.14 A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

Q.15 Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

Q.16 A trapezium ABCD is inscribed into a semicircle of radius l so that the base AD of the trapezium is a diameter and the vertices B & C lie on the circumference. Find the base angle θ of the trapezium ABCD which has the greatest perimeter.

Q.17 If $y = \frac{ax+b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a & b and show that the turning value is a maximum.

Q.18 Prove that among all triangles with a given perimeter, the equilateral triangle has the maximum area.

Q.19 A sheet of poster has its area 18 m². The margin at the top & bottom are 75 cms and at the sides 50 cms. What are the dimensions of the poster if the area of the printed space is maximum?

Q.20 A perpendicular is drawn from the centre to a tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.

Q.21 Consider the function, $F(x) = \int_{-1}^{x}(t^2-t)\,dt,\ x \in R$.

(a) Find the x and y intercept of F if they exist.
(b) Derivatives F ' (x) and F " (x).
(c) The intervals on which F is an increasing and the invervals on which F is decreasing.
(d) Relative maximum and minimum points.
(e) Any inflection point.

Q.22 A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

Q.23 What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

Q.24 The flower bed is to be in the shape of a circular sector of radius r & central angle θ. If the area is fixed & perimeter is minimum, find r and θ.

Q.25 The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and varable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

EXERCISE–12

Q.1 The mass of a cell culture at time t is given by, $M(t) = \dfrac{3}{1+4e^{-t}}$

(a) Find $\lim\limits_{t\to-\infty} M(t)$ and $\lim\limits_{t\to\infty} M(t)$ (b) Show that $\dfrac{dM}{dt} = \dfrac{1}{3}M(3-M)$

(c) Find the maximum rate of growth of M and also the vlaue of t at which occurs.

Q.2 Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length l of the median drawn to its lateral side.

Q.3 From a fixed point A on the circumference of a circle of radius 'a', let the perpendicular AY fall on the tangent at a point P on the circle, prove that the greatest area which the ΔAPY can have is $3\sqrt{3}\,\dfrac{a^2}{8}$ sq. units.

Q.4 Given two points A $(-2, 0)$ & B $(0, 4)$ and a line $y = x$. Find the co-ordinates of a point M on this line so that the perimeter of the Δ AMB is least.

Q.5 A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum, the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi+2)$.

Q.6 Depending on the values of $p \in R$, find the value of 'a' for which the equation $x^3 + 2px^2 + p = a$ has three distinct real roots.

Q.7 Show that for each $a > 0$ the function $e^{-ax}.\ x^{a^2}$ has a maximum value say F(a), and that F(x) has a minimum value, $e^{-e/2}$.

Q.8 For $a > 0$, find the minimum value of the integral $\displaystyle\int_0^{1/a}(a^3+4x-a^5x^2)e^{ax}\,dx$.

Q.9 Find the maximum value of the integral $\displaystyle\int_{-1}^{1}|x-a|e^x\,dx$ where $|a| \le 1$.

Q.10 Consider the function $f(x) = \begin{cases} \sqrt{x}\,\ln x & \text{when } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$

(a) Find whether f is continuous at $x = 0$ or not. (b) Find the minima and maxima if they exist.

(c) Does $f'(0)$? Find $\lim\limits_{x\to0} f'(x)$.

(d) Find the inflection points of the graph of $y = f(x)$..

Q.11 Consider the function $y = f(x) = \ln(1 + \sin x)$ with $-2\pi \le x \le 2\pi$. Find

(a) the zeroes of $f(x)$ (b) inflection points if any on the graph

(c) local maxima and minima of $f(x)$ (d) asymptotes of the graph

(e) sketch the graph of $f(x)$ and compute the value of the definite integral $\displaystyle\int_{-\pi/2}^{\pi/2} f(x)\,dx$.

Q.12 A right circular cone is to be circumscribed about a sphere of a given radius. Find the ratio of the altitude of the cone to the radius of the sphere, if the cone is of least possible volume.

Q.13 Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$.

Q.14 Find the set of value of m for the cubic $x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

Q.15 Let A $(p^2, -p)$, B (q^2, q), C $(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram AFDE is drawn with vertices D, E & F on the line segments BC, CA & AB respectively. Using calculus, show that maximum area of such a parallelogram is : $\dfrac{1}{4}(p+q)(q+r)(p-r)$.

Q.16 A cylinder is obtained by revolving a rectangle about the x–axis, the base of the rectangle lying on the x–axis and the entire rectangle lying in the region between the curve $y = \dfrac{x}{x^2+1}$ & the x–axis. Find the maximum possible volume of the cylinder.

Q.17 For what values of 'a' does the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have a positive point of maximum.

Q.18 Among all regular triangular prism with volume V, find the prism with the least sum of lengths of all edges. How long is the side of the base of that prism?

Q.19 A segment of a line with its extremities on AB and AC bisects a triangle ABC with sides a, b, c into two equal areas. Find the length of the shortest segment.

Q.20 What is the radius of the smallest circular disk large enough to cover every acute isosceles triangle of a given perimeter L?

Q.21 Find the magnitude of the vertex angle 'α' of an isosceles triangle of the given area 'A' such that the radius 'r' of the circle inscribed into the triangle is the maximum.

Q.22 Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

Q.23 The function f (x) defined for all real numbers x has the following properties
(i) $f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find
(a) the intervals on which *f* is increasing and decreasing and any local maximum or minimum values.
(b) the intervals on which the graph *f* is concave down and concave up.
(c) the function f (x) and plot its graph.

Q.24 Find the minimum value of $|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$ for all real x.

Q.25 Use calculus to prove the inequality, $\sin x \ge \frac{2x}{\pi}$ in $0 \le x \le \frac{\pi}{2}$.

You may use the inequality to prove that, $\cos x \le 1 - \frac{x^2}{\pi}$ in $0 \le x \le \frac{\pi}{2}$

EXERCISE–13

Q.1 A conical vessel is to be prepared out of a circular sheet of gold of unit radius. How much sectorial area is to be removed from the sheet so that the vessel has maximum volume. [REE '97, 6]

Q.2(a) The number of values of x where the function $f(x) = \cos x + \cos\left(\sqrt{2}\,x\right)$ attains its maximum is :
(A) 0 (B) 1 (C) 2 (D) infinite

(b) Suppose f(x) is a function satisfying the following conditions :

(i) $f(0) = 2, f(1) = 1$ (ii) f has a minimum value at $x = \frac{5}{2}$ and

(iii) for all x $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b & the function f(x).

Q.3 Find the points on the curve $ax^2 + 2bxy + ay^2 = c$; $c > b > a > 0$, whose distance from the origin is minimum. [REE '98, 6]

Q.4 The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$ has a local minimum at x =
(A) 0 (B) 1 (C) 2 (D) 3

Q.5 Find the co-ordinates of all the points P on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N the foot of the perpendicular from O to the tangent at P. [JEE '99, 10 out of 200]

Q.6 Find the normals to the ellipse $(x^2/9) + (y^2/4) = 1$ which are farthest from its centre. [REE '99, 6]

Q.7 Find the point on the straight line, $y = 2x + 11$ which is nearest to the circle,
$16(x^2 + y^2) + 32x - 8y - 50 = 0$. [REE 2000 Mains, 3 out of 100]

Q.8 Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \le 2 \\ 1 & \text{for } x = 0 \end{cases}$. Then at x = 0, ' f ' has :
(A) a local maximum (B) no local maximum
(C) a local minimum (D) no extremum.

Q.9 Find the area of the right angled triangle of least area that can be drawn so as to circumscribe a rectangle of sides 'a' and 'b', the right angles of the triangle coinciding with one of the angles of the rectangle.

Q.10(a) Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let m(b) be the minimum value of f(x). As b varies, the range of m (b) is
(A) [0, 1] (B) $\left(0, \frac{1}{2}\right]$ (C) $\left[\frac{1}{2}, 1\right]$ (D) (0, 1]

(b) The maximum value of $(\cos\alpha_1)\cdot(\cos\alpha_2)$.......... $(\cos\alpha_n)$, under the restrictions
$0 \le \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \le \frac{\pi}{2}$ and $\cot\alpha_1 \cdot \cot\alpha_2 \ldots\ldots \cot\alpha_n = 1$ is
(A) $\frac{1}{2^{n/2}}$ (B) $\frac{1}{2^n}$ (C) $\frac{1}{2n}$ (D) 1

Q.11(a) If $a_1, a_2, \ldots\ldots, a_n$ are positive real numbers whose product is a fixed number e, the minimum value of $a_1 + a_2 + a_3 + \ldots\ldots + a_{n-1} + 2a_n$ is
(A) $n(2e)^{1/n}$ (B) $(n+1)e^{1/n}$ (C) $2ne^{1/n}$ (D) $(n+1)(2e)^{1/n}$

(b) A straight line L with negative slope passes through the point (8,2) and cuts the positive coordinates axes at points P and Q. Find the absolute minimum value of OP + OQ, as L varies, where O is the origin.

Q.12(a) Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.

(b) For a circle $x^2 + y^2 = r^2$, find the value of 'r' for which the area enclosed by the tangents drawn from the point P(6, 8) to the circle and the chord of contact is maximum. [JEE-03, Mains-2 out of 60]

Q.13(a) Let $f(x) = x^3 + bx^2 + cx + d$, $0 < b^2 < c$. Then f
(A) is bounded (B) has a local maxima
(C) has a local minima (D) is strictly increasing [JEE 2004 (Scr.)]

(b) Prove that $\sin x + 2x \ge \frac{3x\cdot(x+1)}{\pi}$ $\forall\, x \in \left[0, \frac{\pi}{2}\right]$. (Justify the inequality, if any used).

Q.14 If P(x) be a polynomial of degree 3 satisfying P(–1) = 10, P(1) = – 6 and P(x) has maximum at x = – 1 and P'(x) has minima at x = 1. Find the distance between the local maximum and local minimum of the curve. [JEE 2005 (Mains), 4 out of 60]

Q.15(a) If f (x) is cubic polynomial which has local maximum at x = – 1. If f (2) = 18, f (1) = – 1 and f '(x) has local maxima at x = 0, then

(A) the distance between (–1, 2) and (a, f (a)), where x = a is the point of local minima is $2\sqrt{5}$.

(B) f (x) is increasing for $x \in [1, 2\sqrt{5}]$

(C) f (x) has local minima at x = 1

(D) the value of f(0) = 5

(b) $f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [1, 3]$ then g(x) has

(A) local maxima at x = 1 + ln 2 and local minima at x = e

(B) local maxima at x = 1 and local minima at x = 2 (C) no local maxima

(D) no local minima

(c) If f (x) is twice differentiable function such that f (a) = 0, f (b) = 2, f (c) = – 1, f (d) = 2, f (e) = 0, where a < b < c < d < e, then find the minimum number of zeros of $g(x) = (f'(x))^2 + f(x).f''(x)$ in the interval [a, e]. [JEE 2006, 6]

EXERCISE–14

Only one correct option

1. The greatest value of f(x) = $(x + 1)^{1/3} - (x - 1)^{1/3}$ on [0, 1] is:
(A) 1 (B) 2 (C) 3 (D) $2^{1/3}$

2. The function 'f' is defined by f(x) = $x^p (1 - x)^q$ for all x ∈ R, where p,q are positive integers, has a maximum value, for x equal to:
(A) $\frac{pq}{p+q}$ (B) 1 (C) 0 (D) $\frac{p}{p+q}$

3. The co–ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line y = x – 4 is
(A) (2, 1) (B) (– 2, 1) (C) (– 2, –1) (D) none

4. Tangents are drawn to $x^2 + y^2 = 16$ from the point P(0, h). These tangents meet the x-axis at A and B. If the area of triangle PAB is minimum, then
(A) h = $12\sqrt{2}$ (B) h = $6\sqrt{2}$ (C) h = $8\sqrt{2}$ (D) h= $4\sqrt{2}$

5. A function f is such that f′(2) = f″(2) = 0 and f has a local maximum of – 17 at x = 2, then f(x) may be
(A) f(x) = – 17 – $(x – 2)^n$ n ∈ N n ≥ 4 (B) f(x) = – 17 – $(x – 2)^n$ n ≥ 3
(C) f(x) = – 17 + $(x – 2)^n$ n ≥ 3 (D) f(x) – 171 $(x – 2)^n$ n ≥ 4

6. $f(x) = \begin{cases} \tan^{-1} x, & |x| < \frac{\pi}{4} \\ \frac{\pi}{2} - |x|, & |x| \ge \frac{\pi}{4} \end{cases}$, then
(A) f(x) has no point of local maxima (B) f(x) has only one point of local maxima
(C) f(x) has exactly two points of local maxima (D) f(x) has exactly two points of local minimas

7. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5 & ,x \le 1 \\ -2x + \log_2(b^2 - 2) & ,x > 1 \end{cases}$ the set of values of b for which f(x) have greatest value at x = 1 is given by :
(A) 1 ≤ b ≤ 2 (B) b = {1, 2}
(C) b ∈ (–∞, – 1) (D) $\left[-\sqrt{130}, \sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$

8. A tangent to the curve y = 1 – x^2 is drawn so that the abscissa x_0 of the point of tangency belongs to the interval (0, 1]. The tangent at x_0 meets the x–axis and y–axis at A & B respectively. The minimum area of the triangle OAB, where O is the origin is
(A) $\frac{2\sqrt{3}}{9}$ (B) $\frac{4\sqrt{3}}{9}$ (C) $\frac{2\sqrt{2}}{9}$ (D) none

9. The lower corner of a leaf in a book is folded over so as to just reach the inner edge of the page. The fraction of width folded over if the area of the folded part is minimum is:
(A) 5/8 (B) 2/3 (C) 3/4 (D) 4/5

10. {a_1, a_2,....., a_4,......} is a progression where $a_n = \frac{n^2}{n^3+200}$. The largest term of this progression is:
(A) a_6 (B) a_7 (C) a_8 (D) none

11. If $f(x) = \frac{(\sin^{-1} x + \tan^{-1} x)}{\pi} + 2\sqrt{x}$ then the range of f(x) is

(A) $\left[\frac{3}{4}, \frac{5}{4}\right]$ (B) $\left[0, \frac{11}{4}\right]$ (C) $\left[\frac{3}{4}, \frac{7}{4}\right]$ (D) $\left[\frac{7}{4}, \frac{11}{4}\right]$

12. Let $f(x) = \sin \frac{\{x\}}{a} + \cos \frac{\{x\}}{a}$ where $a > 0$ and $\{\,.\,\}$ denotes the fractional part function. Then the set of values of a for which f can attain its maximum values is

(A) $\left(0, \frac{4}{\pi}\right)$ (B) $\left(\frac{4}{\pi}, \infty\right)$ (C) $(0, \infty)$ (D) none of these

13. A and B are the points (2, 0) and (0, 2) respectively. The coordinates of the point P on the line $2x + 3y + 1 = 0$ are

(A) (7, –5) if |PA – PB| is maximum (B) $\left(\frac{1}{5}, \frac{1}{5}\right)$ if |PA – PB| is maximum

(C) (7, –5) if |PA – PB| is minimum (D) $\left(\frac{1}{5}, \frac{1}{5}\right)$ if |PA – PB| is minimum

14. The maximum area of the rectangle whose sides pass through the angular points of a given rectangle of sides a and b is

(A) 2 (ab) (B) $\frac{1}{2}(a + b)^2$ (C) $\frac{1}{2}(a^2 + b^2)$ (D) none of these

15. Number of solution(s) satisfying the equation, $3x^2 - 2x^3 = \log_2 (x^2 + 1) - \log_2 x$ is:
(A) 1 (B) 2 (C) 3 (D) none

16. Least value of the function, $f(x) = 2^{x^2} - 1 + \frac{2}{2^{x^2}+1}$ is:

(A) 0 (B) 3/2 (C) 2/3 (D) 1

17. A straight line through the point (h, k) where h > 0 and k > 0, makes positive intercepts on the coordinate axes. Then the minimum length of the line intercepted between the coordinate axes is
(A) $(h^{2/3} + k^{2/3})^{3/2}$ (B) $(h^{3/2} + k^{3/2})^{2/3}$ (C) $(h^{2/3} - k^{2/3})^{3/2}$ (D) $(h^{3/2} - k^{3/2})^{2/3}$

18. The value of a for which the function $f(x) = (4a - 3)(x + \log 5) + 2(a - 7) \cot \frac{x}{2} \sin^2 \frac{x}{2}$ does not posses critical points is
(A) $(-\infty, -4/3)$ (B) $(-\infty, -1)$ (C) $[1, \infty)$ (D) $(2, \infty)$

19. The minimum value of $\left(1+\frac{1}{\sin^n \alpha}\right)\left(1+\frac{1}{\cos^n \alpha}\right)$ is

(A) 1 (B) 2 (C) $(1 + 2^{n/2})^2$ (D) None of these

20. The altitude of a right circular cone of minimum volume circumscribed about a sphere of radius r is
(A) 2 r (B) 3 r (C) 5 r (D) none of these

One or more than one correct options

21. Let $f(x) = 40/(3x^4 + 8x^3 - 18x^2 + 60)$, consider the following statement about f(x).
(A) f(x) has local minima at x = 0 (B) f(x) has local maxima at x = 0
(C) absolute maximum value of f(x) is not defined (D) f(x) is local maxima at x = – 3, x = 1

22. Maximum and minimum values of the function,

$f(x) = \frac{2-x}{\pi} \cos \pi (x + 3) + \frac{1}{\pi^2} \sin \pi (x + 3) \quad 0 < x < 4$ occur at :

(A) x = 1 (B) x = 2 (C) x = 3 (D) $x = \pi$

23. If $\lim_{x\to a} f(x) = \lim_{x\to a} [f(x)]$ ([.] denotes the greater integer function) and f(x) is non-constant continuous function, then

(A) $\lim_{x\to a} f(x)$ is integer (B) $\lim_{x\to a} f(x)$ is non-integer
(C) f (x) has local maximum at x = a (D) f (x) has local minima at x = a

24. If the derivative of an odd cubic polynomial vanishes at two different values of 'x' then
(A) coefficient of x^3 & x in the polynomial must be same in sign
(B) coefficient of x^3 & x in the polynomial must be different in sign
(C) the values of 'x' where derivative vanishes are closer to origin as compared to the respective roots on either side of origin.
(D) the values of 'x' where derivative vanishes are far from origin as compared to the respective roots on either side of origin.

25. Let $f(x) = \ln (2x - x^2) + \sin \frac{\pi x}{2}$. Then

(A) graph of f is symmetrical about the line x = 1 (B) graph of f is symmetrical about the line x = 2
(C) maximum value of f is 1 (D) minimum value of f does not exist

26. The curve $y = \frac{x+1}{x^2+1}$ has:

(A) x = 1, the point of inflection (B) $x = -2 + \sqrt{3}$, the point of inflection

(C) x = – 1, the point of minimum (D) $x = -2 - \sqrt{3}$, the point of inflection

27. If the function y = f (x) is represented as,
$x = \phi(t) = t^3 - 5t^2 - 20t + 7$
$y = \psi(t) = 4t^3 - 3t^2 - 18t + 3 \ (-2 < t < 2)$, then:
(A) $y_{max} = 12$ (B) $y_{max} = 14$ (C) $y_{min} = -67/4$ (D) $y_{min} = -69/4$

28. The maximum and minimum values of $y = \frac{ax^2 + 2bx + c}{Ax^2 + 2Bx + C}$ are those for which

(A) $ax^2 + 2bx + c - y\,(Ax^2 + 2Bx + C)$ is equal to zero
(B) $ax^2 + 2\,bx + c - y\,(Ax^2 + 2Bx + C)$ is a perfect square
(C) $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \neq 0$
(D) $ax^2 + 2bx + c - y\,(Ax^2 + 2\,Bx + C)$ is not a perfect square

29. f(x) is cubic polynomial which has local maximum at $x = -1$, If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minima at $x = 0$, then **[IIT – 2006, (5, –1)]**

(A) the distance between point of maxima and minima is $2\sqrt{5}$.
(B) f(x) is increasing for $x \in [1, 2\sqrt{5})$
(C) f(x) has local minima at $x = 1$
(D) the value of $f(0) = 5$

EXERCISE–15

1. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

2. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length ℓ of the median drawn to its lateral side.

3. Find the set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a + 2)\,x^2 + (a - 1)\,x + 2$ possess a negative point of inflection.

4. The fuel charges for running a train are proportional to the square of the speed generated in m.p.h. & costs Rs. 48/- per hour at 16 mph. What is the most economical speed if the fixed charges i.e. salaries etc. amount to Rs. 300/- per hour.

5. The three sides of a trapezium are equal each being 6 cms long, find the area of the trapezium when it is maximum.

6. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft? Assume that one side of the rectangle lies on the base of the triangle.

7. A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

8. Find the point on the curve, $4\,x^2 + a^2 y^2 = 4\,a^2$, $4 < a^2 < 8$, that is farthest from the point $(0, -2)$.

9. A cone is circumscribed about a sphere of radius ' r '. Show that the volume of the cone is minimum when its semi – vertical angle is, $\sin^{-1}\left(\frac{1}{3}\right)$.

10. Find the values of 'a' for which the function $f(x) = \frac{a}{3}\,x^3 + (a + 2)\,x^2 + (a - 1)\,x + 2$ possess a negative point of minimum.

11. A figure is bounded by the curves, $y = x^2 + 1$, $y = 0$, $x = 0$ & $x = 1$. At what point (a, b), a tangent should be drawn to the curve, $y = x^2 + 1$ for it to cut off a trapezium of the greatest area from the figure.

12. Prove that the least perimeter of an isosceles triangle in which a circle of radius ' r ' can be inscribed is $6\,r\sqrt{3}$.

13. Find the polynomial f (x) of degree 6, which satisfies $\underset{x \to 0}{\text{Limit}} \left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e^2$ and has local maximum at $x = 1$ and local minimum at $x = 0$ & 2.

14. Two towns located on the same side of the river agree to construct a pumping station and filteration plant at the river's edge, to be used jointly to supply the towns with water. If the distance of the two towns from the river are 'a' & 'b' and the distance between them is 'c', show that the pipe lines joining them to the pumping station is atleast as great as $\sqrt{c^2 + 4ab}$.

15. Find the co-ordinates of all the points P on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N the foot of the perpendicular from O to the tangent at P. **[IIT – 1999, 10]**

16. If p(x) be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and p(x) has maxima at $x = -1$ and $p'(x)$ has minima at $x = 1$. Find the distance between the local maxima and local minima of the curve. **[IIT – 2005]**

TANGENT & NORMAL

EXERCISE - 1

Q.1 $2\sqrt{3}\ x - y = 2\left(\sqrt{3}-1\right)$ or $2\sqrt{3}\ x + y = 2\left(\sqrt{3}+1\right)$ **Q.2** (0, 1)

Q.3 x = 1 when t = 1, $m \to \infty$; 5x – 4y = 1 if $t \neq 1$, m = 1/3]

Q.7 T : x – 2y = 0 ; N : 2x + y = 0 **Q.8** $x + 2y = \pi/2$ & $x + 2y = -3\pi/2$

Q.9 (a) n = – 2 **Q.12** a = 1 **Q.14** $-\dfrac{1}{x+2}$; x – 4y = 2 **Q.16** a = – 1/2 ; b = – 3/4 ; c = 3

Q.20 $2e^{-x/2}$ **Q.22 (b)** a – b = a′ – b′ **Q.23** $\theta = \tan^{-1}\dfrac{2}{C}$ **Q.25** $\dfrac{m\sqrt{m}}{\sqrt{2}}$

EXERCISE - 2

Q.1 $1/9\pi$ m/min **Q.2 (i)** 6 km/h **(ii)** 2 km/hr **Q.3** (4 , 11) & (– 4, – 31/3)

Q.4 $3/8\pi$ cm/min **Q.5** $1 + 36\pi$ cu. cm/sec **Q.6** $1/48\pi$ cm/s **Q.7** 0.05 cm/sec

Q.8 $\dfrac{\sqrt{2}}{4\pi}$ cm/s **Q.9** $200\pi r^3/(r+5)^2$ km²/h **Q.10** $\dfrac{66}{7}$ **Q.11** $\dfrac{1}{4}$ cm/sec.

Q.12 (a) $-\dfrac{1}{24\pi}$ m/min., (b) $-\dfrac{5}{288\pi}$ m/min. **Q.14** (a) $r = (1+t)^{1/4}$, (b) t = 80 **Q.15 (a)** 5.02, **(b)** $\dfrac{80}{27}$

EXERCISE - 3

Q.1 $\theta = \tan^{-1}\left|\dfrac{4\sqrt{2}}{7}\right|$ **Q.2** $\sqrt{2}\ x + y - 2\sqrt{2} = 0$ or $\sqrt{2}\ x - y - 2\sqrt{2} = 0$

Q.3 D **Q.4** D **Q.5** D

EXERCISE - 4

1.	B	**2.**	B	**3.**	D	**4.**	B	**5.**	B	**6.**	A	**7.**	B	**8.**	A
9.	B	**10.**	B	**11.**	B	**12.**	B	**13.**	A	**14.**	C	**15.**	B	**16.**	ABD
17.	AC	**18.**	AB	**19.**	ABC	**20.**	AC	**21.**	CD	**22.**	BD				

EXERCISE - 5

1. a = 1, b = 1, c = 0 **2.** (9/4, 3/8) **3.** $\dfrac{8b}{27}$

5. (i) $\dfrac{\pi}{2}$ at (0, 0); $\tan^{-1}\left(\dfrac{1}{2}\right)$ at (8, 16), (8, – 16) (ii) $\pi/3$ (iii) $\tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$ at $\left(\sqrt{2}, 2\right)$, $\left(-\sqrt{2}, 2\right)$

6. (i) – 2 cm/min (ii) 2 cm²/min. **7.** (4 , 11) & (– 4, – 31/3)

11. 2x + y = 0, x = 2y **12.** $\pm\dfrac{c}{\sqrt{2}}$ **14.** $y = x - 5x^3$ **16.** $a \in \left(-\dfrac{13}{4}, 3\right)$

17. $25y^2 + 4x^2 = 4x^2y^2$ **19.** $t = \dfrac{H}{k}$ **20.** y = 2

MONOTONOCITY

EXERCISE - 6

Q.1 **(a)** I in $(2, \infty)$ & D in $(-\infty, 2)$ **(b)** I in $(1, \infty)$ & D in $(-\infty, 0) \cup (0, 1)$
(c) I in (0, 2) & D in $(-\infty, 0) \cup (2, \infty)$

(d) I for $x > \dfrac{1}{2}$ or $-\dfrac{1}{2} < x < 0$ & D for $x < -\dfrac{1}{2}$ or $0 < x < \dfrac{1}{2}$

Q.2 $(-2, 0) \cup (2, \infty)$

Q.3 **(a)** I in $[0, 3\pi/4) \cup (7\pi/4, 2\pi]$ & D in $(3\pi/4, 7\pi/4)$
(b) I in $[0, \pi/6) \cup (\pi/2, 5\pi/6) \cup (3\pi/2, 2\pi]$ & D in $(\pi/6, \pi/2) \cup (5\pi/6, 3\pi/2)$]

Q.5 continuous but not diff. at $x = 1$ Q.6 $a < -\left(2+\sqrt{5}\right)$ or $a > \sqrt{5}$

Q.7 **(a)** $(\pi/6)+(1/2)\ln 3$, $(\pi/3)-(1/2)\ln 3$, **(b)** least value is equal to $(1/e)^{1/e}$, no greatest value, **(c)** 2 & -10

Q.8 $[1, \infty)$ **Q.10** $a \in (-\infty, -3] \cup [1, \infty)$ **Q.11** $[-7, -1) \cup [2, 3]$

Q.12 increasing in $x \in (\pi/2, 2\pi/3)$ & decreasing in $[0, \pi/2) \cup (2\pi/3, \pi]$

Q.13 $0 \le a \le \frac{3}{2}$ **Q.14** $\uparrow$ in $(3, \infty)$ and $\downarrow$ in $(1, 3)$ **Q.15** $(6, \infty)$ **Q.16** $a \ge 0$

Q.17 **(a)** $(-\infty, 0]$; **(b)** $\uparrow$ in $\left(1, \frac{5}{3}\right)$ and $\downarrow$ in $(-\infty, 1) \cup \left(\frac{5}{3}, \infty\right) - \{-3\}$; **(c)** $x = \frac{5}{3}$;

(d) removable discont. at $x = -3$ (missing point) and non removable discont. at $x = 1$ (infinite type)

(e) -2

Q.24 $(-1, 0) \cup (0, \infty)$ **Q.25** $(b-a)^3/4$

EXERCISE - 7

Q.1 $c = \frac{mb + na}{m + n}$ which lies between a & b **Q.6** $a = 3$, $b = 4$ and $m = 1$

Q.7 $y = -5x - 9$ and $y = 5x + 11$ **Q.18** 0

EXERCISE - 8

Q.1 C **Q.3 (a)** A, C ; **(b)** D **Q.4** **(a)** B ; **(b)** D ; **(c)** C

Q.5 (a) A, **(b)** $\cos\left(\frac{1}{3}\cos^{-1} p\right)$ **Q.6** A **Q.8** **(a)** D ; **(b)** C **Q.10 (a)** D

EXERCISE - 9

1.	C	**2.**	B	**3.**	B	**4.**	C	**5.**	D	**6.**	C	**7.**	A
8.	C	**9.**	C	**10.**	CD	**11.**	BC	**12.**	BC	**13.**	ABD	**14.**	AB
15.	ABD												

EXERCISE - 10

1. Neither increasing nor decreasing at $x = -1$, increasing at $x = 0, 1$.

2. $a \ne 0$ **3.** $ad > bc$

4. I in $(-\infty, -1] \cup [1, \infty)$ & D in $[-1, 1]$

5. (a) $x < -\frac{2}{3}(p^2 + q^2 + r^2)$, $x > 0$ (b) $\left(-\frac{2}{3}(p^2 + q^2 + r^2), 0\right)$

6. $(-\infty, -3]$ **8.** $2\sin x + \tan x > 3x$, limit = 0

10. $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ **13.** increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$

16. $(\pi/6) + (1/2)\ \ell n\ 3$, $(\pi/3) - (1/2)\ \ell n\ 3$

21. increasing when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, decreasing when $x \in \left(0, \frac{\pi}{4}\right)$.

23. Prove that for $0 \le p \le 1$ & for any $a > 0$, $b > 0$ the inequality $(a + b)^p \le a^p + b^p$.

24. greatest = 14, least = 8

* *

MAXIMA - MINIMA

EXERCISE - 11

Q.1 $f(x) = x^3 + x^2 - x + 2$ **Q.2** max. at $x = 1$; $f(1) = 0$, min. at $x = 7/5$; $f(7/5) = -108/3125$

Q.3 **(a)** Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value = 0

(b) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and

Max value = 3/2 , Min at $x = \pi/2$, Min value = -3

Q.4 $f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$ **Q.5** $P_{max} = a\left(1 + \operatorname{cosec}\frac{\alpha}{2}\right)$ **Q.6** $75\sqrt{3}$ sq. units

Q.7 $r = \sqrt{\frac{2A}{\pi+4}}$, $s = \sqrt{\frac{2A}{\pi+4}}$ **Q.9** $3x + 4y - 9 = 0$; $3x - 4y + 9 = 0$

Q.10 $4\sqrt{2}$ m **Q.11** $1/\pi$ cu m

Q.12 110 ', 70 ' **Q.13** $6/(6 + \pi)$ **Q.14** side 10', height 10'

Q.15 32 sq. units **Q.16** $\theta = 60^0$ **Q.17** $a = 1, b = 0$

Q.19 width $2\sqrt{3}$ m, length $3\sqrt{3}$ m **Q.20** $|a - b|$

Q.21 **(a)** (–1, 0), (0, 5/6) ; **(b)** $F'(x) = (x^2 - x)$, $F''(x) = 2x - 1$, **(c)** increasing $(-\infty, 0) \cup (1, \infty)$, decreasing (0, 1) ; **(d)** (0, 5/6) ; (1, 2/3) ; **(e)** x = 1/2

Q.22 **(a)** $x = y = \frac{d}{\sqrt{2}}$, **(b)** $x = \frac{d}{\sqrt{3}}$, $y = \sqrt{\frac{2}{3}}\,d$ **Q.23** $6' \times 18'$

Q.24 $r = \sqrt{A}$, $\theta = 2$ radians **Q.25** $\frac{4}{3\sqrt{3}}$

EXERCISE - 12

Q.1 **(a)** 0, 3, **(c)** $\frac{3}{4}$, $t = \ln 4$ **Q.2** cos A = 0.8 **Q.4** (0 , 0)

Q.6 $p < a < \frac{32p^3}{27} + p$ if $p > 0$; $\frac{32p^3}{27} + p < a < p$ if $p < 0$ **Q.8** 4 when $a = \sqrt{2}$

Q.9 Maximum value is $(e + e^{-1})$ when $a = -1$

Q.10 **(a)** f is continuous at x = 0 ; **(b)** $-\frac{2}{e}$; **(c)** does not exist, does not exist ; **(d)** pt. of inflection x = 1

Q.11 (a) $x = -2\pi, -\pi, 0, \pi, 2\pi$, (b) no inflection point, (c) maxima at $x = \frac{\pi}{2}$ and $-\frac{3\pi}{2}$ and no minima,

(d) $x = \frac{3\pi}{2}$ and $x = -\frac{\pi}{2}$, (e) $-\pi \ln 2$

Q.12 4 **Q.13** (0 , 2) & max. distance = 4 **Q.14** $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$ **Q.16** $\frac{\pi}{4}$

Q.17 $(-\infty, -3) \cup (3, 29/7)$ **Q.18** $H = x = \left(\frac{4V}{\sqrt{3}}\right)^{1/3}$ **Q.19** $\sqrt{\frac{(c+a-b)(a+b-c)}{2}}$

Q.20 L/4 **Q.21** $\frac{\pi}{3}$

Q.23 **(a)** increasing in (0, 2) and decreasing in $(-\infty, 0) \cup (2, \infty)$, local min. value = 0 and local max. value = 2

(b) concave up for $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$ and concave down in $(2-\sqrt{2}), (2+\sqrt{2})$

(c) $f(x) = \frac{1}{2}e^{2\cdot x}\cdot x^2$ **Q.24** $2\sqrt{2} - 1$

EXERCISE - 13

Q.1 $\pi\left(1 - \sqrt{\frac{2}{3}}\right)$ sq. units **Q.2** **(a)** B, **(b)** $a = \frac{1}{4}$; $b = -\frac{5}{4}$; $f(x) = \frac{1}{4}(x^2 - 5x + 8)$

Q.3 $\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$ & $\left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$ **Q.4** **(a)** B, D,

Q.5 $\pm\frac{a^2}{\sqrt{a^2+b^2}}$, $\pm\frac{b^2}{\sqrt{a^2+b^2}}$ **Q.6** $\pm\sqrt{3}\,x \pm \sqrt{2}\,y = \sqrt{5}$ **Q.7** (–9/2 , 2)

Q.8 A **Q.9** 2ab **Q.10** (a) D ; **(b)** A **Q.11** **(a)** A ; **(b)** 18 **Q.12** **(a)** (2, 1) ; **(b)** 5

Q.13 **(a)** D **Q.14** $4\sqrt{65}$ **Q.15** (a) B, C; (b) A, B, (c) 6 solutions

EXERCISE - 14

1. B	**2.** D	**3.** A	**4.** D	**5.** A	**6.** C	**7.** D	**8.** B
9. B	**10.** B	**11.** B	**12.** A	**13.** A	**14.** B	**15.** A	**16.** D
17. A	**18.** A	**19.** C	**20.** D	**21.** ACD	**22.** AC	**23.** AD	**24.** BC
25. ACD	**26.** ABD	**27.** BD	**28.** BC		**29.** BC		

EXERCISE - 15

1. 32 sq. units **2.** cos A = 0.8 **3.** $(-\infty, -2) \cup (0, \infty)$ **4.** 40 mph

5. $27\sqrt{3}$ sq. cms **6.** $6' \times 18'$ **7.** side 10', height 10' **8.** (0, 2)

10. $(1, \infty)$ **11.** $\left(\frac{1}{2}, \frac{5}{4}\right)$ **13.** $f(x) = 2x^4 - \frac{12}{5}x^5 + \frac{2}{3}x^6$

15. $\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}$ **16.** $4\sqrt{65}$