विध्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचितः मानव धर्म प्रणेता
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STUDY PACKAGE

Subject : Mathematics

Topic : Binomial Theorem

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2. Short Revision
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4. Assertion & Reason
5. Que. from Compt. Exams
6. 34 Yrs. Que. from IIT-JEE
7. 10 Yrs. Que. from AIEEE

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Binomial Theorem

1. Binomial Expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression. For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

2. Statement of Binomial theorem :

If $a, b \in R$ and $n \in N$, then ;

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n$

or $\quad (a + b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$

Now, putting $a = 1$ and $b = x$ in the binomial theorem -

or $\quad (1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$

$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

Solved Example # 1: **Expand the following binomials :**

(i) $\mathbf{(x - 3)^5}$ **(ii)** $\left(1 - \frac{3x^2}{2}\right)^4$

Solution.

(i) $(x - 3)^5 = {}^5C_0x^5 + {}^5C_1x^4 (-3)^1 + {}^5C_2\, x^3 (-3)^2 + {}^5C_3\, x^2 (-3)^3 + {}^5C_4\, x (-3)^4 + {}^5C_5 (-3)^5$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

(ii) $\left(1 - \frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1 \left(-\frac{3x^2}{2}\right) + {}^4C_2 \left(-\frac{3x^2}{2}\right)^2 + {}^4C_3 \left(-\frac{3x^2}{2}\right)^3 + {}^4C_4 \left(-\frac{3x^2}{2}\right)^4$

$$= 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8$$

Solved Example # 2: **Expand the binomial** $\left(\frac{2x}{3} + \frac{3y}{2}\right)^{20}$ **up to four terms**

Solution.

$\left(\frac{2x}{3} + \frac{3y}{2}\right)^{20} = {}^{20}C_0 \left(\frac{2x}{3}\right)^{20} + {}^{20}C_1 \left(\frac{2x}{3}\right)^{19} \left(\frac{3y}{2}\right) + {}^{20}C_2 \left(\frac{2x}{3}\right)^{18} \left(\frac{3y}{2}\right)^2 + {}^{20}C_3 \left(\frac{2x}{3}\right)^{17} \left(\frac{3y}{2}\right)^3 + \ldots.$

$= \left(\frac{2x}{3}\right)^{20} + 20.\left(\frac{2}{3}\right)^{18} x^{19}y + 190 . \left(\frac{2}{3}\right)^{16} x^{18}y^2 + 1140 \left(\frac{2}{3}\right)^{14} x^{17}y^3 + \ldots..$

Self practice problems

1. Write the first three terms in the expansion of $\left(2 - \frac{y}{3}\right)^6$.

2. Expand the binomial $\left(\frac{x^2}{3} + \frac{3}{x}\right)^5$.

Ans. (1) $64 - 64y + \frac{80}{3}y^2$ (2) $\frac{x^{10}}{243} + \frac{5}{27}x^7 + \frac{10}{3}x^4 + 30x + \frac{135}{x^2} + \frac{243}{x^5}$.

3. Properties of Binomial Theorem :

(i) The number of terms in the expansion is $n + 1$.

(ii) The sum of the indices of x and y in each term is n.

(iii) The binomial coefficients (${}^nC_0$, ${}^nC_1$ ....${}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\because {}^nC_r = {}^nC_{n-r}\}$

Solved Example # 3: **The number of dissimilar terms in the expansion of** $\mathbf{(1 - 3x + 3x^2 - x^3)^{20}}$ **is**

(A) 21 **(B) 31** **(C) 41** **(D) 61**

Solution. $(1 - 3x + 3x^2 - x^3)^{20} = [(1 - x)^3]^{20} = (1 - x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is **61.**

4. Some important terms in the expansion of $(x + y)^n$:

(i) General term :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \ldots\ldots\ldots.+ {}^nC_r\, x^{n-r} y^r + \ldots\ldots\ldots.+ {}^nC_n\, x^0 y^n$

$(r + 1)$th term is called general term.

$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Solved Example # 4

Find **(i)** **28th term of** $\mathbf{(5x + 8y)^{30}}$ **(ii)** **7th term of** $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

Solution. (i) $T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}\,(8y)^{27}$

$= \mathbf{\dfrac{30\,!}{3\,!\,27\,!}\,(5x)^3\,.\,(8y)^{27}}$ **Ans.**

(ii) 7th term of $\left(\dfrac{4x}{5} - \dfrac{5}{2x}\right)^9$

$T_{6+1} = {}^9C_6 \left(\dfrac{4x}{5}\right)^{9-6}\left(-\dfrac{5}{2x}\right)^6$

$= \dfrac{9!}{3\,!\,6\,!}\left(\dfrac{4x}{5}\right)^3\left(\dfrac{5}{2x}\right)^6 \qquad = \mathbf{\dfrac{10500}{x^3}}$ **Ans.**

Solved Example # 5 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution. The general term in the expansion of $\left(9^{1/4} + 8^{1/6}\right)^{1000}$ is

$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r \qquad = {}^{1000}C_r\; 3^{\frac{1000-r}{2}}\; 2^{\frac{r}{2}}$

The above term will be rational if exponent of 3 and 2 are integres

It means $\dfrac{1000-r}{2}$ and $\dfrac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is **501 Ans.**

(ii) Middle term (s) :

(a) If n is even, there is only one middle term, which is $\left(\dfrac{n+2}{2}\right)$th term.

(b) If n is odd, there are two middle terms, which are $\left(\dfrac{n+1}{2}\right)$th and $\left(\dfrac{n+1}{2}+1\right)$th terms.

Solved Example # 6 : Find the middle term(s) in the expansion of

(i) $\left(1 - \dfrac{x^2}{2}\right)^{14}$ **(ii)** $\left(3a - \dfrac{a^3}{6}\right)^9$

Solution. (i) $\left(1 - \dfrac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\dfrac{14+2}{2}\right)$th term.

It means T_8 is middle term

$T_8 = {}^{14}C_7\left(-\dfrac{x^2}{2}\right)^7 = \mathbf{-\dfrac{429}{16}\,x^{14}}$**. Ans.**

(ii) $\left(3a - \dfrac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\dfrac{9+1}{2}\right)$th & $\left(\dfrac{9+1}{2}+1\right)$th.

It means T_5 & T_6 are middle terms

$T_5 = {}^9C_4\,(3a)^{9-4}\left(-\dfrac{a^3}{6}\right)^4 = \mathbf{\dfrac{189}{8}\,a^{17}}$ **Ans.**

$T_6 = {}^9C_5\,(3a)^{9-5}\left(-\dfrac{a^3}{6}\right)^5 = \mathbf{-\dfrac{21}{16}\,a^{19}}$**. Ans.**

(iii) Term containing specified powers of x in $\left(ax^{\alpha} \pm \dfrac{b}{x^{\beta}}\right)^n$

Solved Example # 7: Find the coefficient of x^{32} and x^{-17} in $\left(x^4 - \dfrac{1}{x^3}\right)^{15}$.

Solution.: Let (r + 1)th term contains x^m

$T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(-\dfrac{1}{x^3}\right)^r$

$= {}^{15}C_r\,x^{60-7r}\,(-1)^r$

(i) for x^{32}, $60 - 7r = 32$

$\Rightarrow \quad 7r = 28 \quad \Rightarrow \quad r = 4. \quad (T_5)$

$T_5 = {}^{15}C_4\,x^{32}\,(-1)^4$

Hence, coefficient of x^{32} is **1365 Ans.**

(ii) for x^{-17}, $60 - 7r = -17$

$\Rightarrow \quad r = 11 \quad (T_{12})$
$T_{12} = {}^{15}C_{11}\, x^{-17}\, (-1)^{11}$
Hence, coefficient of x^{-17} is **– 1365** **Ans.**

(iv) Numerically greatest term in the expansion of $(x + y)^n$, $n \in N$

Let T_r and T_{r+1} be the rth and (r + 1)th terms respectively

$T_r = {}^nC_{r-1}\, x^{n-(r-1)}\, y^{r-1}$
$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Now, $$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\frac{x^{n-r}\, y^r}{x^{n-r+1}y^{r-1}}\right| = \frac{n-r+1}{r}\;.\;\left|\frac{y}{x}\right|$$

Consider $$\left|\frac{T_{r+1}}{T_r}\right| \geq 1$$

$$\left(\frac{n-r+1}{r}\right)\left|\frac{y}{x}\right| \geq 1$$

$$\frac{n+1}{r} - 1 \geq \left|\frac{x}{y}\right|$$

$$r \leq \frac{n+1}{1+\left|\frac{x}{y}\right|}$$

Case - I When $\dfrac{n+1}{1+\left|\frac{x}{y}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ (r = 1, 2, 3 ...., m – 1)
i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$
i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ (r = m + 1, m + 2, ...n)
i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ...$T_{n+1} < T_n$

Conclusion : When $\dfrac{n+1}{1+\left|\frac{x}{y}\right|}$ is an integer, equal to m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II When $\dfrac{n+1}{1+\left|\frac{x}{y}\right|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \dfrac{n+1}{1+\left|\frac{x}{y}\right|}$ (r = 1, 2, 3,........, m–1, m)
i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \dfrac{n+1}{1+\left|\frac{x}{y}\right|}$ (r = m + 1, m + 2, ...n)
i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ..., $T_{n+1} < T_n$

Conclusion : When $\dfrac{n+1}{1+\left|\frac{x}{y}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Solved Example # 8 Find the numerically greatest term in the expansion of $(3 - 5x)^{15}$ when $x = \dfrac{1}{5}$.

Solution. Let r^{th} and $(r + 1)^{th}$ be two consecutive terms in the expansion of $(3 - 5x)^{15}$

$T_{r+1} \geq T_r$
${}^{15}C_r\, 3^{15-r}\, (|-5x|)^r \geq {}^{15}C_{r-1}\, 3^{15-(r-1)}\, (|-5x|)^{r-1}$

$$\frac{(15)!}{(15-r)!\,r!}\;|-5x| \geq \frac{3.\,(15)!}{(16-r)!\,(r-1)!}$$

$$5\;.\;\frac{1}{5}\;(16 - r) \geq 3r$$

$16 - r \geq 3r$
$4r \leq 16$
$r \leq 4$

Explanation: For $r \leq 4$, $T_{r+1} \geq T_r$ $\Rightarrow$ $T_2 > T_1$
$T_3 > T_2$
$T_4 > T_3$
$T_5 = T_4$ For $r > 5$, $T_{r+1} < T_r$
$T_6 < T_5$ $T_7 < T_6$

and so on Hence, T_4 and T_5 are numerically greatest terms and both are equal.

Self practice problems :

3. Find the term independent of x in $\left(x^2 - \frac{3}{x}\right)^9$

4. The sum of all rational terms in the expansion of $(3^{1/5} + 2^{1/3})^{15}$ is
(A) 60 (B) 59 (C) 95 (D) 105

5. Find the coefficient of x^{-1} in $(1 + 3x^2 + x^4)\left(1+\frac{1}{x}\right)^8$

6. Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

7. Find the numerically greatest term in the expansion of $(7 - 5x)^{11}$ where $x = \frac{2}{3}$.

Ans. (3) 28.3^7 (4) B (5) 232
(6) ${}^{6n}C_{3n} \cdot x^{3n}$ (7) $T_4 = \frac{440}{9} \times 7^8 \times 5^3$.

5. Multinomial Theorem:

As we know the Binomial Theorem –

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r \; x^{n-r} \, y^r$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} \; x^{n-r} \, y^r$$

putting $n - r = r_1$, $r = r_2$

therefore, $(x + y)^n = \sum_{r_1+r_2=n} \frac{n!}{r_1!\; r_2!} \; x^{r_1} \cdot y^{r_2}$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$

In the same fashion we can write the multinomial theorem

$$(x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!\, r_2! \ldots r_k!} \; x_1^{r_1} \cdot x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1 + x_2 + \ldots\ldots\ldots + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \ldots\ldots + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Solved Example # 9 Find the coeff. of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Solution. $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!\, r_2!\, r_3!\, r_4!} \; (a)^{r_1} \; (-b)^{r_2} \; (-c)^{r_3} \; (d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2$, $r_2 = 3$, $r_3 = 4$, $r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is

$\frac{(10)!}{2!\,3!\,4!\,1!} \; (-1)^3 \; (-1)^4 =$ **$-$ 12600 Ans.**

Solved Example # 10 In the expansion of $\left(1+x+\frac{7}{x}\right)^{11}$ find the term independent of x.

Solution.

$$\left(1+x+\frac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1!\, r_2!\, r_3!} \; (1)^{r_1} \; (x)^{r_2} \left(\frac{7}{x}\right)^{r_3}$$

The exponent 11 is to be divided among the base variables 1, x and $\frac{7}{x}$ in such a way so that we get x^0.

Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$\frac{(11)!}{(11)!}\,(7^0) + \frac{(11)!}{9!\,1!\,1!}\,7^1 + \frac{(11)!}{7!\,2!\,2!}\,7^2 + \frac{(11)!}{5!\,3!\,3!}\,7^3 + \frac{(11)!}{3!\,4!\,4!}\,7^4 + \frac{(11)!}{1!\,5!\,5!}\,7^5$$

$$= 1 + \frac{(11)!}{9!\,2!} \cdot \frac{2!}{1!\,1!}\,7^1 + \frac{(11)!}{7!\,4!} \cdot \frac{4!}{2!\,2!}\,7^2 + \frac{(11)!}{5!\,6!} \cdot \frac{6!}{3!\,3!}\,7^3$$

$$+ \frac{(11)!}{3!\,8!} \cdot \frac{8!}{4!\,4!}\,7^4 + \frac{(11)!}{1!\,10!} \cdot \frac{(10)!}{5!\,5!}\,7^5$$

$$= 1 + {}^{11}C_2 \cdot {}^2C_1 \cdot 7^1 + {}^{11}C_4 \cdot {}^4C_2 \cdot 7^2 + {}^{11}C_6 \cdot {}^6C_3 \cdot 7^3 + {}^{11}C_8 \cdot {}^8C_4 \cdot 7^4 + {}^{11}C_{10} \cdot {}^{10}C_5 \cdot 7^5$$

$$= \mathbf{1 + \sum_{r=1}^{5} {}^{11}C_{2r} \cdot {}^{2r}C_r \cdot 7^r} \qquad \textbf{Ans.}$$

Self practice problems :

8. The number of terms in the expansion of $(a + b + c + d + e + f)^n$ is
(A) ${}^{n+4}C_4$ (B) ${}^{n+3}C_n$ (C) ${}^{n+5}C_n$ (D) n + 1

9. Find the coefficient of $x^3 y^4 z^2$ in the expansion of $(2x - 3y + 4z)^9$

10. Find the coefficient of x^4 in $(1 + x - 2x^2)^7$

Ans. (8) C (9) $\dfrac{9!}{3!\,4!\,2!}$ (10) $2^3\,3^4\,4^2 - 91$

6. Application of Binomial Theorem :

(i) **If $(\sqrt{A}+B)^n$ = I + f where I and n are positive integers, n being odd and $0 < f < 1$ then $(I + f)\, f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.**
If n is an even integer, then $(I + f)(1 - f) = k^n$

Solved Example # 11

If n is positive integer, then prove that the integral part of $(7 + 4\sqrt{3})^n$ is an odd number.

Solution. Let $(7 + 4\sqrt{3})^n = I + f$ ....(i)

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now, $0 < 7 - 4\sqrt{3} < 1$

$0 < (7 - 4\sqrt{3})^n < 1$

Let $(7 - 4\sqrt{3})^n = f'$ ....(ii)

$\Rightarrow$ $0 < f' < 1$

Adding (i) and (ii)

$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= 2\,[{}^nC_0\, 7^n + {}^nC_2\, 7^{n-2}\,(4\sqrt{3})^2 + ...]$

$I + f + f'$ = even integer ($f + f'$ must be an integer)

$0 < f + f' < 2 \quad \Rightarrow \quad f + f' = 1$

$I + 1$ = even integer

therefore I is an odd integer.

Solved Example # 12

Show that the integer just above $(\sqrt{3} + 1)^{2n}$ is divisible by 2^{n+1} for all $n \in N$.

Solution. Let $(\sqrt{3} + 1)^{2n} = (4 + 2\sqrt{3})^n = 2^n\,(2 + \sqrt{3})^n = I + f$ ....(i)

where I and f are its integral & fractional parts respectively

$0 < f < 1$.

Now $0 < \sqrt{3} - 1 < 1$

$0 < (\sqrt{3} - 1)^{2n} < 1$

Let $(\sqrt{3} - 1)^{2n} = (4 - 2\sqrt{3})^n = 2^n\,(2 - \sqrt{3})^n = f'$. ....(ii)

$0 < f' < 1$

adding (i) and (ii)

$I + f + f' = (\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$

$= 2^n\,[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n]$

$= 2.2^n\,[{}^nC_0\, 2^n + {}^nC_2\, 2^{n-2}\,(\sqrt{3})^2 + ...]$

$I + f + f' = 2^{n+1}\,k$ (where k is a positive integer)

$0 < f + f' < 2 \quad \Rightarrow \quad f + f' = 1$

$I + 1 = 2^{n+1}\,k$.

$I + 1$ is the integer just above $(\sqrt{3} + 1)^{2n}$ and which is divisible by 2^{n+1}.

(ii) **Cheking divisibility**

Solved Example # 13: Show that $9^n + 7$ is divisible by 8, where n is a positive integer.

Solution. $9^n + 7 = (1 + 8)^n + 7$

$= {}^nC_0 + {}^nC_1 \,.\, 8 + {}^nC_2 \,.\, 8^2 + ... + {}^nC_n\, 8^n + 7$.

$= 8.\, C_1 + 8^2.\, C_2 + ... + C_n \,.\, 8^n + 8$.

$= 8\lambda$ where, λ is a positive integer, Hence, $9^n + 7$ is divisible by 8.

(iii) **Finding remainder**

Solved Example # 14

What is the remainder when 5^{99} is divided by 13.

Solution.: $5^{99} = 5.5^{98} = 5.\,(25)^{49}$

$= 5\,(26 - 1)^{49}$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - {}^{49}C_{49}\,(26)^0]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ...+ {}^{49}C_{48}\,(26)^1 - 1]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - 13] + 60$

$= 13\,(k) + 52 + 8$ (where k is a positive integer)

$= 13\,(k + 4) + 8$ Hence, remainder is 8. **Ans.**

(iv) **Finding last digit, last two digits and last there digits of the given number.**

Solved Example # 15 **Find the last two digits of the number $(17)^{10}$.**

Solution. $(17)^{10} = (289)^5$

$= (290 - 1)^5$

$= {}^5C_0\,(290)^5 - {}^5C_1\,(290)^4 + ... + {}^5C_4\,(290)^1 - {}^5C_5\,(290)^0$

$= {}^5C_0\,(290)^5 - {}^5C_1 \,.\, (290)^4 + ...{}^5C_3\,(290)^2 + 5 \times 290 - 1$

= A multiple of 1000 + 1449 Hence, last two digits are 49 **Ans.**

Note : We can also conclude that last three digits are 449.

(v) **Comparison between two numbers**

Solved Example # 16 : Which number is larger $(1.01)^{1000000}$ or 10,000 ?

Solution.: By Binomial Theorem

$(1.01)^{1000000} = (1 + 0.01)^{1000000}$

$= 1 + {}^{1000000}C_1\,(0.01)$ + other positive terms

$= 1 + 1000000 \times 0.01$ + other positive terms

$= 1 + 10000$ + other positive terms, Hence $(1.01)^{1000000} > 10{,}000$

Self practice problems :

11. If n is positive integer, prove that the integral part of $(5\sqrt{5} + 11)^{2n+1}$ is an even number.

12. If $(7 + 4\sqrt{3})^n = \alpha + \beta$, where α is a positive integer and β is a proper fraction then prove that $(1 - \beta)(\alpha + \beta) = 1$.

13. If n is a positive integer then show that $3^{2n+1} + 2^{n+2}$ is divisible by 7.

14. What is the remainder when 7^{103} is divided by 25 .

15. Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

16. Which number is larger $(1.2)^{4000}$ or 800

Ans. (14) 18 (15) 1, 01, 001 (16) $(1.2)^{4000}$.

7. Properties of Binomial Coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... + C_nx^n$ ......(1)

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

${}^nC_0 + {}^nC_1 + {}^nC_2 + ...+ {}^nC_n = 2^n$ ......(2)

or $\sum_{r=0}^{n} {}^nC_r = 2^n$

(2) Again putting x = –1 in (1), we get

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\,{}^nC_n = 0$ ......(3)

or $\sum_{r=0}^{n} (-1)^r\ {}^nC_r = 0$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

from (2) and (3)

${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{n-1}$

${}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

(4) Sum of two consecutive binomial coefficients

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

L.H.S. $= {}^nC_r + {}^nC_{r-1} = \dfrac{n!}{(n-r)!\,r!} + \dfrac{n!}{(n-r+1)!\,(r-1)!}$

$= \dfrac{n!}{(n-r)!\,(r-1)!}\left[\dfrac{1}{r} + \dfrac{1}{n-r+1}\right] = \dfrac{n!}{(n-r)!\,(r-1)!}\ \dfrac{(n+1)}{r(n-r+1)}$

$= \dfrac{(n+1)!}{(n-r+1)!\,r!} = {}^{n+1}C_r$ = R.H.S.

(5) Ratio of two consecutive binomial coefficients

$\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(6) ${}^nC_r = \dfrac{n}{r}\ {}^{n-1}C_{r-1} = \dfrac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = ... = \dfrac{n(n-1)(n-2).........(n-(r-1))}{r\,(r-1)(r-2).......2\,.1}$

Solved Example # 17

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + c_nx^n$, then show that

(i) $C_0 + 3C_1 + 3^2C_2 + ... + 3^n\,C_n = 4^n$.

(ii) $C_0 + 2C_1 + 3.\,C_2 + ... + (n + 1)\,C_n = 2^{n-1}\,(n + 2)$.

(iii) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} + ... + (-1)^n\,\dfrac{C_n}{n+1} = \dfrac{1}{n+1}$.

Solution. (i) $(1 + x)^n = C_0 + C_1\,x + C_2x^2 + ... + C_nx^n$

put x = 3

$C_0 + 3\,.\,C_1 + 3^2\,.\,C_2 + ... + 3^n\,.\,C_n = \mathbf{4^n}$

(ii) **<u>I Method : By Summation</u>**

L.H.S. $= {}^nC_0 + 2.\,{}^nC_1 + 3\,.\,{}^nC_2 + ... + (n + 1).\,{}^nC_n$.

$= \sum_{r=0}^{n}(r+1)\,.\,{}^nC_r$

$= \sum_{r=0}^{n} r.\,{}^nC_r + \sum_{r=0}^{n} {}^nC_r = n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r$

$= n\,.\,2^{n-1} + 2^n = 2^{n-1}\,(n + 2)$. RHS

<u>II Method : By Differentiation</u>

$(1 + x)^n = C_0 + C_xx + C_2x^2 + ... + C_nx^n$

Multiplying both sides by x,

$x(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_n\,x^{n+1}$.

Differentiating both sides

$(1 + x)^n + x\,n\,(1 + x)^{n-1} = C_0 + 2.\,C_1 + 3\,.\,C_2x^2 + ... + (n + 1)C_nx^n$.

putting x = 1, we get

$C_0 + 2.C_1 + 3\,.\,C_2 + ... + (n + 1)\,C_n = 2^n + n\,.\,2^{n-1}$

$C_0 + 2.C_1 + 3\,.\,C_2 + ... + (n + 1)\,C_n = 2^{n-1}\,(n + 2)$ **Proved**

(iii) **<u>I Method : By Summation</u>**

L.H.S. $= C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} + ... + (-1)^n.\ \dfrac{C_n}{n+1}$

$= \sum_{r=0}^{n} (-1)^r \cdot \dfrac{{}^nC_r}{r+1}$

$= \dfrac{1}{n+1} \sum_{r=0}^{n} (-1)^r \cdot {}^{n+1}C_{r+1} \qquad \left\{ \dfrac{n+1}{r+1} \cdot {}^nC_r = {}^{n+1}C_{r+1} \right\}$

$= \dfrac{1}{n+1} \left[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \ldots\ldots\ldots\ldots + (-1)^n \cdot {}^{n+1}C_{n+1} \right]$

$= \dfrac{1}{n+1} \left[- {}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \ldots\ldots\ldots + (-1)^n \cdot {}^{n+1}C_{n+1} + {}^{n+1}C_0 \right]$

$= \dfrac{1}{n+1} = \text{R.H.S.} \quad \left\{ -{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \ldots + (-1)^n \; {}^{n+1}C_{n+1} = 0 \right\}$

II Method : By Integration

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n x^n.$

Integrating both sides, with in the limits – 1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_{-1}^{0} = \left[C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \ldots\ldots + C_n \frac{x^{n+1}}{n+1} \right]_{-1}^{0}$$

$$\frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots\ldots + (-1)^{n+1} \frac{C_n}{n+1} \right]$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots\ldots\ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1} \qquad \textbf{Proved}$$

Solved Example # 18 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + \ldots\ldots\ldots\ldots + C_{n-2}C_n = {}^{2n}C_{n-2}$ **or** ${}^{2n}C_{n+2}$

(iii) $1.\ C_0^2 + 3.\ C_1^2 + 5.\ C_2^2 + \ldots\ldots\ldots + (2n + 1).\ C_n^2 = 2n.\ {}^{2n-1}C_n + {}^{2n}C_n$.

Solution. (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_n x^n.$ ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots\ldots + C_n x^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_nx^n)\,(C_0x^n + C_1x^{n-1} + \ldots\ldots\ldots + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^n,

$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots + C_n^2 = {}^{2n}C_n$

(ii) From the product of (i) and (ii) comparing coefficients of x^{n-2} or x^{n+2} both sides,

$C_0C_2 + C_1C_3 + C_2C_4 + \ldots\ldots\ldots + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$.

(iii) **I Method : By Summation**

L.H.S. $= 1.\ C_0^2 + 3.\ C_1^2 + 5.\ C_2^2 + \ldots\ldots\ldots\ldots + (2n + 1)\ C_n^2.$

$= \sum_{r=0}^{n} (2r+1)\ {}^nC_r^2$

$= \sum_{r=0}^{n} 2.r\ .\ ({}^nC_r)^2 + \sum_{r=0}^{n} ({}^nC_r)^2 \qquad = 2 \sum_{r=1}^{n} .\ n\ .\ {}^{n-1}C_{r-1}\ {}^nC_r + {}^{2n}C_n$

$(1 + x)^n = {}^nC_1 + {}^nC_4\, x + {}^nC_2\, x^2 + \ldots\ldots\ldots\ldots\ldots {}^nC_n\, x^n$ ....(i)

$(x + 1)^{n-1} = {}^{n-1}C_0\, x^{n-1} + {}^{n-1}C_1\, x^{n-2} + \ldots\ldots\ldots + {}^{n-1}C_{n-1}x^0$ ....(ii)

Multiplying (i) and (ii) and comparing coeffcients of x^n.

${}^{n-1}C_0\ .\ {}^nC_1 + {}^{n-1}C_1\ .\ {}^nC_2 + \ldots\ldots\ldots\ldots + {}^{n-1}C_{n-1}\ .\ {}^nC_n = {}^{2n-1}C_n$

$\sum_{r=0}^{n} {}^{n-1}C_{r-1}\ .\ {}^nC_r = {}^{2n-1}C_n$

Hence, required summation is

$2n.\ {}^{2n-1}C_n + {}^{2n}C_n = \text{R.H.S.}$

II Method : By Differentiation

$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + \ldots\ldots\ldots\ldots\ldots + C_n\, x^{2n}$

Multiplying both sides by x

$x(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots\ldots\ldots\ldots + C_nx^{2n+1}.$

Differentiating both sides

$x\ .\ n\ (1 + x^2)^{n-1}\ .\ 2x + (1 + x^2)^n = C_0 + 3.\ C_1x^2 + 5.\ C_2\, x^4 + \ldots\ldots\ldots + (2n + 1)\ C_n\, x^{2n}$ ....(i)

$(x^2 + 1)^n = C_0\, x^{2n} + C_1\, x^{2n-2} + C_2\, x^{2n-4} + \ldots\ldots\ldots + C_n$ ....(ii)

Multiplying (i) & (ii)

$(C_0 + 3C_1x^2 + 5C_2x^4 + \ldots\ldots\ldots + (2n + 1)\ C_n\, x^{2n})\ (C_0\, x^{2n} + C_1x^{2n-2} + \ldots\ldots\ldots\ldots + C_n)$

$= 2n\, x^2\, (1 + x^2)^{2n-1} + (1 + x^2)^{2n}$

comparing coefficient of x^{2n},

$C_0^2 + 3C_1^2 + 5C_2^2 + \ldots\ldots\ldots + (2n + 1)\ C_n^2 = 2n\ .\ {}^{2n-1}C_{n-1} + {}^{2n}C_n.$

$C_0^2 + 3C_1^2 + 5C_2^2 + \ldots\ldots\ldots + (2n + 1)\ C_n^2 = 2n\ .\ {}^{2n-1}C_n + {}^{2n}C_n.$ **Proved**

Solved Example # 19

Find the summation of the following series –

(i) ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots\ldots\ldots + {}^nC_m$ **(ii)** ${}^nC_3 + 2\ .\ {}^{n+1}C_3 + 3.\ {}^{n+2}C_3 + \ldots\ldots\ldots + n\ .\ {}^{2n-1}C_3$

Solution. **(i) I Method : Using property, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$**

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots\ldots\ldots + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + \ldots\ldots\ldots\ldots\ldots + {}^nC_m \quad \{\because {}^mC_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + \ldots\ldots\ldots\ldots + {}^{n}C_m$

$= {}^{m+3}C_{m+1} + \ldots\ldots\ldots + {}^{n}C_m \qquad = {}^{n}C_{m+1} + {}^{n}C_m = \mathbf{{}^{n+1}C_{m+1}}$ **Ans.**

II Method

${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots\ldots + {}^{n}C_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + \ldots\ldots\ldots + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + \ldots\ldots\ldots\ldots + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1}-1\right]}{x} \qquad = \frac{(1+x)^{n+1}-(1+x)^m}{x}$$

x^m : S (coefficient of x^m in S)

$$x^m : \frac{(1+x)^{n+1}-(1+x)^m}{x}$$

Hence, required summation of the series is $\mathbf{{}^{n+1}C_{m+1}}$ **Ans.**

(ii)

${}^{n}C_3 + 2\,.\,{}^{n+1}C_3 + 3\,.\,{}^{n+2}C_3 + \ldots\ldots\ldots + n\,.\,{}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2\,.\,(1 + x)^{n+1} + 3\,.\,(1 + x)^{n+2} + \ldots\ldots\ldots + n\,.\,(1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2\,.\,(1 + x)^{n+1} + 3.\,(1 + x)^{n+2} + \ldots\ldots\ldots + n\,(1 + x)^{2n-1}$ ....(i)

$(1 + x)S = \qquad (1 + x)^{n+1} + 2\,(1 + x)^{n+2} + \ldots\ldots\ldots\ldots + (n - 1)\,(1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$- xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + \ldots\ldots\ldots\ldots + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$$= \frac{(1+x)^n\left[(1+x)^n-1\right]}{x} - n\,(1 + x)^{2n}$$

$$S = \frac{-(1+x)^{2n}+(1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

x^3 : S (coefficient of x^3 in S)

$$x^3 : \frac{-(1+x)^{2n}+(1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $\mathbf{-{}^{2n}C_5 + {}^{n}C_5 + n\,.\,{}^{2n}C_4}$ **Ans.**

Self practice problems :

17. Prove the following

(i) $C_0 + 3C_1 + 5C_2 + \ldots\ldots\ldots\ldots + (2n + 1)\,C_n = 2^n\,(n + 1)$

(ii) $4C_0 + \dfrac{4^2}{2}\,.\,C_1 + \dfrac{4^3}{3}\,C_2 + \ldots\ldots\ldots\ldots + \dfrac{4^{n+1}}{n+1}\,C_n = \dfrac{5^{n+1}-1}{n+1}$

(iii) ${}^{n}C_0\,.\,{}^{n+1}C_n + {}^{n}C_1\,.\,{}^{n}C_{n-1} + {}^{n}C_2\,.\,{}^{n-1}C_{n-2} + \ldots\ldots\ldots + {}^{n}C_n\,.\,{}^{1}C_0 = 2^{n-1}\,(n + 2)$

(iv) ${}^{2}C_2 + {}^{3}C_2 + \ldots\ldots\ldots + {}^{n}C_2 = {}^{n+1}C_3$

8. Binomial Theorem For Negative Integer Or Fractional Indices

If $n \in R$ then,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!}\,x^2 + \frac{n(n-1)(n-2)}{3\,!}\,x^3 + \ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots.(n-r+1)}{r\,!}\,x^r + \ldots\ldots\ldots\ldots\ldots\ldots \infty.$$

Remarks:(i) The above expansion is valid for any rational number other than a whole number if $|\,x\,| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^{n}C_r$ cannot be used to denote the coefficient of the general term.

(iii) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x + y)^n = \begin{cases} x^n\left(1+\dfrac{y}{x}\right)^n = x^n\left\{1+n\,.\,\dfrac{y}{x}+\dfrac{n\,(n-1)}{2\,!}\left(\dfrac{y}{x}\right)^2+\ldots..\right\} & \text{if } \left|\dfrac{y}{x}\right|<1 \\[2ex] y^n\left(1+\dfrac{x}{y}\right)^n = y^n\left\{1+n\,.\,\dfrac{x}{y}+\dfrac{n\,(n-1)}{2\,!}\left(\dfrac{x}{y}\right)^2+\ldots..\right\} & \text{if } \left|\dfrac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \dfrac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r\,!}\,x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots + (-1)^r\,x^r + \ldots\ldots\ldots\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^5 + \ldots\ldots\ldots + x^r + \ldots\ldots\ldots\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots + (-1)^r\,(r + 1)\,x^r + \ldots\ldots\ldots\infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots + (r + 1)x^r + \ldots\ldots\ldots\ \infty$

Solved Example # 20: Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Soltion.: (r + 1)th term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots\ldots(-n-r+1)}{r\,!}\,(-x)^r$$

$$= (-1)^r \frac{n(n+1)(n+2)......(n+r-1)}{r!} (-x)^r$$

$$= \frac{n(n+1)(n+2)......(n+r-1)}{r!} x^r$$

$$= \frac{(n-1)!\, n(n+1)......(n+r-1)}{(n-1)!\, r!} x^r$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\,r!} = {}^{n+r-1}C_r$ **Proved**

Solved Example # 21: If x is so small such that its square and higher powers may be neglected then find the value of $\boldsymbol{\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}}$

Solution. $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

$$= \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} \qquad = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right) \qquad = \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)$$

$$= 1-\frac{x}{8}-\frac{19}{12}x \qquad = \mathbf{1-\frac{41}{24}x} \quad \textbf{Ans.}$$

Self practice problems :

18. Find the possible set of values of x for which expansion of $(3-2x)^{1/2}$ is valid in ascending powers of x.

19. If $y = \frac{3}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + ...$, then find the value of $y^2 + 2y$

20. The coefficient of x^{100} in $\frac{3-5x}{(1-x)^2}$ is

(A) 100 (B) –57 (C) –197 (D) 53

Ans. (18) $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ (19) 4 (20) C

Short Revision

BINOMIAL EXPONENTIAL & LOGARITHMIC SERIES

1. BINOMIAL THEOREM : The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM** .
If $x, y \in R$ and $n \in N$, then ;

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2}y^2 + ... + {}^nC_r\, x^{n-r}y^r + ... + {}^nC_n y^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r.$$

This theorem can be proved by Induction .

OBSERVATIONS :

(i) The number of terms in the expansion is $(n+1)$ i.e. one or more than the index .
(ii) The sum of the indices of x & y in each term is n .
(iii) The binomial coefficients of the terms ${}^nC_0, {}^nC_1$ .... **equidistant** from the beginning and the end are equal.

2. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE :

(i) General term   **(ii)** Middle term
(iii) Term independent of *x* &   **(iv)** Numerically greatest term

(i) The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by ;
$T_{r+1} = {}^nC_r\, x^{n-r} \cdot y^r$

(ii) The middle term(s) is the expansion of $(x+y)^n$ is (are) :
(a) If n is even , there is only one middle term which is given by ;
$T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$
(b) If n is odd , there are two middle terms which are :
$T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(iii) Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

(iv) To find the Numerically greatest term is the expansion of $(1+x)^n$, $n \in N$ find $\dfrac{T_{r+1}}{T_r} = \dfrac{{}^nC_r x^r}{{}^nC_{r-1}x^{r-1}} = \dfrac{n-r+1}{r}\, x$. Put the absolute value of x & find the value of r Consistent with the inequality $\dfrac{T_{r+1}}{T_r} > 1$.
Note that the Numerically greatest term in the expansion of $(1-x)^n$, $x > 0$, $n \in N$ is the same as the greatest term in $(1+x)^n$.

3. If $\left(\sqrt{A}+B\right)^n = I + f$, where I & n are positive integers, n being odd and $0 < f < 1$, then $(I+f) \cdot f = K^n$ where $A - B^2 = K > 0$ & $\sqrt{A} - B < 1$.
If n is an even integer, then $(I+f)(1-f) = K^n$.

4. BINOMIAL COEFFICIENTS : **(i)** $C_0 + C_1 + C_2 + ... + C_n = 2^n$
(ii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$
(iii) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{n!\, n!}$
(iv) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + ... + C_{n-r}.C_n = \dfrac{(2n)!}{(n+r)(n-r)!}$

REMEMBER : **(i)** $(2n)! = 2^n \cdot n!\, [1.\, 3.\, 5 ...... (2n-1)]$

5. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1+x)^n = 1 + n\,x + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + ...\infty$ Provided $|x| < 1$.

Note : (i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. $(n+1)$ & the coefficient of successive terms are :
${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3 ..... {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the Coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).
(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - .... \infty$ **(b)** $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ... \infty$
(c) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$ **(d)** $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \infty$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convinient to expand in powers of $\dfrac{1}{x}$, which then will be small.

6. APPROXIMATIONS : $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{1.2}x^2 + \dfrac{n(n-1)(n-2)}{1.2.3}x^3 .....$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1+x)^n = 1 + nx$, approximately.

This is an approximate value of $(1+x)^n$.

7. EXPONENTIAL SERIES :

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$; where x may be any real or complex & $e = \underset{n \to \infty}{\text{Limit}} \left(1 + \frac{1}{n}\right)^n$

(ii) $a^x = 1 + \frac{x}{1!} \ln a + \frac{x^2}{2!} \ln^2 a + \frac{x^3}{3!} \ln^3 a + ...\infty$ where $a > 0$

Note : **(a)** $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

(b) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(c) $e + e^{-1} = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\infty\right)$ **(d)** $e - e^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ...\infty\right)$

(e) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm**.

8. LOGARITHMIC SERIES :

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$ where $-1 < x \le 1$

(ii) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$ where $-1 \le x < 1$

(iii) $\ln \frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right)$ $|x| < 1$

REMEMBER : **(a)** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\infty = \ln 2$ **(b)** $e^{\ln x} = x$

(c) $\ln 2 = 0.693$ **(d)** $\ln 10 = 2.303$

EXERCISE - 1

Q.1 Find the coefficients : (i) x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ (ii) x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

(iii) Find the relation between a & b, so that these coefficients are equal.

Q.2 If the coefficients of $(2r+4)^{th}$, $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, find r.

Q.3 If the coefficients of the r^{th}, $(r+1)^{th}$ & $(r+2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, find r.

Q.4 Find the term independent of x in the expansion of (a) $\left[\sqrt{\frac{x}{3}} + \frac{\sqrt{3}}{2x^2}\right]^{10}$ (b) $\left[\frac{1}{2}x^{1/3} + x^{-1/5}\right]^8$

Q.5 Find the sum of the series $\sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ...\text{up to m terms}\right]$

Q.6 If the coefficients of 2^{nd}, 3^{rd} & 4^{th} terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

Q.7 Given that $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, find the values of :
(i) $a_0 + a_1 + a_2 + ... + a_{2n}$; (ii) $a_0 - a_1 + a_2 - a_3 ..... + a_{2n}$; (iii) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2$

Q.8 If a, b, c & d are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, $n \in N$, prove that $\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$.

Q.9 Find the value of x for which the fourth term in the expansion, $\left(5^{\frac{2}{5}\log_5 \sqrt{4^x + 44}} + \frac{1}{5^{\log_5 \sqrt[3]{2^{x-1} + 7}}}\right)^8$ is 336.

Q.10 Prove that : ${}^{n-1}C_r + {}^{n-2}C_r + {}^{n-3}C_r + ... + {}^rC_r = {}^nC_{r+1}$.

Q.11 (a) Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

(b) Show that ${}^{2n-2}C_{n-2} + 2 \cdot {}^{2n-2}C_{n-1} + {}^{2n-2}C_n > \frac{4n}{n+1}$, $n \in N$, $n > 2$

Q.12 In the expansion of $\left(1 + x + \frac{7}{x}\right)^{11}$ find the term not containing x.

Q.13 Show that coefficient of x^5 in the expansion of $(1+x^2)^5 \cdot (1+x)^4$ is 60.

Q.14 Find the coefficient of x^4 in the expansion of :
(i) $(1 + x + x^2 + x^3)^{11}$ (ii) $(2 - x + 3x^2)^6$

Q.15 Find numerically the greatest term in the expansion of :
(i) $(2 + 3x)^9$ when $x = \frac{3}{2}$ (ii) $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

Q.16 Given $s_n = 1 + q + q^2 + \ldots + q^n$ & $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$, $q \neq 1$, prove that ${}^{n+1}C_1 + {}^{n+1}C_2 . s_1 + {}^{n+1}C_3 . s_2 + \ldots + {}^{n+1}C_{n+1} . s_n = 2^n . S_n$.

Q.17 Prove that the ratio of the coefficient of x^{10} in $(1 - x^2)^{10}$ & the term independent of x in $\left(x - \frac{2}{x}\right)^{10}$ is 1 : 32 .

Q.18 Find the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$.

Q.19 In the expansion of the expression $(x + a)^{15}$, if the eleventh term is the geometric mean of the eighth and twelfth terms , which term in the expansion is the greatest ?

Q.20 Let $(1+x^2)^2 . (1+x)^n = \sum_{K=0}^{n+4} a_K . x^K$. If a_1, a_2 & a_3 are in AP, find n.

Q.21 If the coefficient of a^{r-1}, a^r , a^{r+1} in the expansion of $(1 + a)^n$ are in arithmetic progression, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

Q.22 If ${}^nJ_r = \frac{(1-x^n)(1-x^{n-1})(1-x^{n-2})\ldots\ldots(1-x^{n-r+1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^r)}$, prove that ${}^nJ_{n-r} = {}^nJ_r$.

Q.23 Prove that $\sum_{K=0}^{n} {}^nC_K \sin Kx . \cos(n-K)x = 2^{n-1} \sin nx$.

Q.24 The expressions $1 + x$, $1+x + x^2$, $1 + x + x^2 + x^3$,............ $1 + x + x^2 + \ldots\ldots + x^n$ are multiplied together and the terms of the product thus obtained are arranged in increasing powers of x in the form of $a_0 + a_1x + a_2x^2 + \ldots\ldots$, then,
(a) how many terms are there in the product.
(b) show that the coefficients of the terms in the product, equidistant from the beginning and end are equal.
(c) show that the sum of the odd coefficients = the sum of the even coefficients = $\frac{(n+1)!}{2}$

Q.25 Find the coeff. of
(a) x^6 in the expansion of $(ax^2 + bx + c)^9$.
(b) $x^2 y^3 z^4$ in the expansion of $(ax - by + cz)^9$.
(c) $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$.

Q.26 If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ & $a_k = 1$ for all $k \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$.

Q.27 If $P_k(x) = \sum_{i=0}^{i=k-1} x^i$ then prove that, $\sum_{k=1}^{n} {}^nC_k P_k(x) = 2^{n-1} . P_n\left(\frac{1+x}{2}\right)$

Q.28 Find the coefficient of x^r in the expression of :
$(x + 3)^{n-1} + (x + 3)^{n-2}(x + 2) + (x + 3)^{n-3}(x + 2)^2 + \ldots + (x + 2)^{n-1}$

Q.29(a) Find the index n of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$.
(b) For which positive values of x is the fourth term in the expansion of $(5 + 3x)^{10}$ is the greatest.

Q.30 Prove that $\frac{(72)!}{(36!)^2} - 1$ is divisible by 73.

Q.31 If the 3rd, 4th, 5th & 6th terms in the expansion of $(x + y)^n$ be respectively a , b , c & d then prove that $\frac{b^2 - ac}{c^2 - bd} = \frac{5a}{3c}$.

Q.32 Find x for which the $(k + 1)^{th}$ term of the expansion of $(x + y)^n$ is the greatest if $x + y = 1$ and $x > 0$, $y > 0$.

Q.33 If x is so small that its square and higher powers may be neglected, prove that :

(i) $\frac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{(4 + x)^{1/2}} = 1 - \left(\frac{41}{24}\right)x$ (ii) $\frac{\left(1 - \frac{3x}{7}\right)^{1/3} + \left(1 - \frac{3x}{5}\right)^{-5}}{\left(1 + \frac{x}{2}\right)^{1/3} + \left(1 - \frac{7x}{3}\right)^{1/7}} = 1 + \left(\frac{10}{7}\right)x + \left(\frac{1}{12}\right)x$ or $1 + \left(\frac{127}{84}\right)x$

Q.34 (a) If $x = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \ldots\ldots \infty$ then prove that $x^2 + 2x - 2 = 0$.

(b) If $y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots$ then find the value of $y^2 + 2y$.

Q.35 If $p = q$ nearly and $n > 1$, show that $\frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} = \left(\frac{p}{q}\right)^{1/n}$.

EXERCISE - 2

Q.1 Show that the integral part in each of the following is odd. $n \in N$

(A) $\left(5+2\sqrt{6}\right)^n$ (B) $\left(8+3\sqrt{7}\right)^n$ (C) $\left(6+\sqrt{35}\right)^n$

Q.2 Show that the integral part in each of the following is even. $n \in N$

(A) $\left(3\sqrt{3}+5\right)^{2n+1}$ (B) $\left(5\sqrt{5}+11\right)^{2n+1}$

Q.3 If $\left(7+4\sqrt{3}\right)^n = p+\beta$ where n & p are positive integers and β is a proper fraction show that $(1-\beta)(p+\beta)=1$.

Q.4 If x denotes $\left(2+\sqrt{3}\right)^n$, $n \in N$ & [x] the integral part of x then find the value of : $x - x^2 + x[x]$.

Q.5 If $P = \left(8+3\sqrt{7}\right)^n$ and $f = P - [P]$, where [ ] denotes greatest integer function.
Prove that : $P(1-f) = 1$ $(n \in N)$

Q.6 If $\left(6\sqrt{6}+14\right)^{2n+1} = N$ & F be the fractional part of N, prove that $NF = 20^{2n+1}$ $(n \in N)$

Q.7 Prove that if p is a prime number greater than 2, then the difference $\left[\left(2+\sqrt{5}\right)^p\right] - 2^{p+1}$ is divisible by p, where [ ] denotes greatest integer.

Q.8 Prove that the integer next above $\left(\sqrt{3}+1\right)^{2n}$ contains 2^{n+1} as factor $(n \in N)$

Q.9 Let I denotes the integral part & F the proper fractional part of $\left(3+\sqrt{5}\right)^n$ where $n \in N$ and if ρ denotes the rational part and σ the irrational part of the same, show that

$$\rho = \frac{1}{2}(I+1) \text{ and } \sigma = \frac{1}{2}(I+2F-1).$$

Q.10 Prove that $\frac{{}^{2n}C_n}{n+1}$ is an integer, $\forall\, n \in N$.

EXERCISE - 3

(NOT IN THE SYLLABUS OF IIT-JEE)

PROBLEMS ON EXPONENTIAL & LOGARITHMIC SERIES

For Q.1 TO Q.15, Prove That :

Q.1 $\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots\right)^2 - \left(1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots\right)^2 = 1$

Q.2 $\frac{e-1}{e+1} = \left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots\right) \div \left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots\right)$

Q.3 $\frac{e^2-1}{e^2+1} = \left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots\right) \div \left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots\right)$

Q.4 $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots\ldots\infty = \left(\frac{3}{2}\right)e$

Q.5 $\frac{1}{1.3}+\frac{1}{1.2.3.5}+\frac{1}{1.2.3.4.5.7}+\ldots\ldots\infty = \frac{1}{e}$

Q.6 $1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\frac{1+2+2^2+2^3}{4!}+\ldots\ldots = e^2 - e$

Q.7 $1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\ldots\ldots = 5e$

Q 8. $\frac{2}{1!}+\frac{3}{2!}+\frac{6}{3!}+\frac{11}{4!}+\frac{18}{5!}+\ldots\ldots = 3(e-1)$

Q.9 $\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\ldots\ldots\infty = 1-\log_e 2$

Q 10. $1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\ldots\ldots\infty = \log_e 3$

Q.11 $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots = \frac{1}{2}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots = \ln 2$

Q.12 $\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\ldots = \ln 3 - \ln 2$

Q 13. $\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\ldots = \left(\frac{1}{2}\right)\ln 2$

Q.14 $\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{4}\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\frac{1}{6}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)-\ldots\ldots = \ln\sqrt{2}$

Q.15 If $y = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ where $|x|<1$, then prove that $x = y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\ldots$

EXERCISE - 4

If $C_0, C_1, C_2, \ldots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$, then prove the following :

Q.1 $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \frac{(2n)!}{n!\, n!}$ Q.2 $C_0 C_1 + C_1 C_2 + C_2 C_3 + ... + C_{n-1} C_n = \frac{(2n)!}{(n+1)!\,(n-1)!}$

Q.3 $C_1 + 2C_2 + 3C_3 + ... + n \,.\, C_n = n \,.\, 2^{n-1}$

Q.4 $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$

Q.5 $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n = (n+1)\, 2^n$

Q.6 $(C_0+C_1)(C_1+C_2)(C_2+C_3) \,.....\, (C_{n-1}+C_n) = \frac{C_0 . C_1 . C_2 .... C_{n-1} (n+1)^n}{n!}$

Q.7 $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{n.C_n}{C_{n-1}} = \frac{n(n+1)}{2}$ Q 8. $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Q.9 $2 \,.\, C_o + \frac{2^2 . C_1}{2} + \frac{2^3 . C_2}{3} + \frac{2^4 . C_3}{4} + ... \frac{2^{n+1} . C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

Q.10 $C_oC_r + C_1C_{r+1} + C_2C_{r+2} + ... + C_{n-r}\, C_n = \frac{2n!}{(n-r)!\,(n+r)!}$

Q.11 $C_o - \frac{C_1}{2} + \frac{C_2}{3} - ...... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

Q.12 $C_o - C_1 + C_2 - C_3 + ... + (-1)^r . C_r = \frac{(-1)^r\,(n-1)!}{r!\,.\,(n-r-1)!}$

Q.13 $C_o - 2C_1 + 3C_2 - 4C_3 + ... + (-1)^n (n+1)\, C_n = 0$

Q.14 $C_o^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2 = 0$ or $(-1)^{n/2}\, C_{n/2}$ according as n is odd or even.

Q.15 If n is an integer greater than 1 , show that ;
$a - {}^nC_1(a-1) + {}^nC_2(a-2) - ..... + (-1)^n (a-n) = 0$

Q.16 $(n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 . C_5 + ... = n\,(n+1)2^{n-3}$

Q.17 $1 \,.\, C_o^2 + 3 \,.\, C_1^2 + 5 \,.\, C_2^2 + ... + (2n+1)\, C_n^2 = \frac{(n+1)\,(2n)!}{n!\, n!}$

Q.18 If $a_0 , a_1 , a_2 , ...$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x , then prove that : (i) $a_0\, a_1 - a_1\, a_2 + a_2\, a_3 - .... = 0$.
(ii) $a_0a_2 - a_1a_3 + a_2a_4 - ..... + a_{2n-2}\, a_{2n} = a_{n+1}$ or a_{n-1}.
(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + ...$; $E_2 = a_1 + a_4 + a_7 + ...$ & $E_3 = a_2 + a_5 + a_8 + ...$

Q.19 Prove that : $\sum_{r=0}^{n-2} \left({}^nC_r \,.\, {}^nC_{r+2}\right) = \frac{(2n)!}{(n-2)!\,(n+2)!}$

Q.20 If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n\, x^n$, then show that the sum of the products of the C_i's taken two at a time , represented by $\sum\sum_{0 \le i < j \le n} C_i\, C_j$ is equal to $2^{2n-1} - \frac{2n!}{2(n!)^2}$.

Q.21 $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + ... + \sqrt{C_n} \le 2^{n-1} + \frac{n-1}{2}$

Q.22 $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + ... + \sqrt{C_n} \le \left[n\left(2^n - 1\right)\right]^{1/2}$ for $n \ge 2$.

EXERCISE - 5

Q.1 If $(1+x)^{15} = C_0 + C_1 .\, x + C_2 .\, x^2 + ... + C_{15} .\, x^{15}$, then find the value of :
$C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$

Q.2 If $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np} .\, x^{np}$, then find the value of :
$a_1 + 2a_2 + 3a_3 + ... + np \,.\, a_{np}$

Q.3 $1^2 .\, C_0 + 2^2 .\, C_1 + 3^2 .\, C_2 + 4^2 .\, C_3 + ... + (n+1)^2\, C_n = 2^{n-2}\,(n+1)\,(n+4)$.

Q.4 $\sum_{r=0}^{n} r^2 . C_r = n\,(n+1)\, 2^{n-2}$

Q.5 Given p+q = 1 , show that $\sum_{r=0}^{n} r^2 . {}^nC_r . p^r . q^{n-r} = np\left[(n-1)\,p + 1\right]$

Q.6 Show that $\sum_{r=0}^{n} C_r \left(2r - n\right)^2 = n . 2^n$ where C_r denotes the combinatorial coeff. in the expansion of $(1 + x)^n$.

Q.7 $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \frac{C_3}{4}x^3 + ... + \frac{C_n}{n+1} . x^n = \frac{(1+x)^{n+1} - 1}{(n+1)x}$

Q.8 Prove that , $2 \,.\, C_0 + \frac{2^2}{2} . C_1 + \frac{2^3}{3} . C_2 + ... + \frac{2^{11}}{11} . C_{10} = \frac{3^{11} - 1}{11}$

Q.9 If $(1+x)^n = \sum_{r=0}^{n} C_r \,.\, x^r$ then prove that ;

$$\frac{2^2.C_0}{1.2}+\frac{2^3.C_1}{2.3}+\frac{2^4.C_2}{3.4}+......+\frac{2^{n+2}.C_n}{(n+1)\,(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)\,(n+2)}$$

Q.10 $\frac{C_0}{1}+\frac{C_2}{3}+\frac{C_4}{5}+........=\frac{2^n}{n+1}$

Q.11 $\frac{C_0}{1}-\frac{C_1}{5}+\frac{C_2}{9}-\frac{C_3}{13}+........(-1)^n\frac{C_n}{4n+1}=\frac{4^n.\,n!}{1.5.9.13.....(4n-3)\,(4n+1)}$

Q.12 $\frac{C_0}{2}+\frac{C_1}{3}+\frac{C_2}{4}+\frac{C_3}{5}+........+\frac{C_n}{n+2}=\frac{1+n.2^{n+1}}{(n+1)\,(n+2)}$

Q.13 $\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+.......+(-1)^n.\frac{C_n}{n+2}=\frac{1}{(n+1)\,(n+2)}$

Q.14 $\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+.......+(-1)^{n-1}.\frac{C_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.......+\frac{1}{n}$

Q.15 If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then show that :

$$C_1(1-x)-\frac{C_2}{2}(1-x)^2+\frac{C_3}{3}(1-x)^3-....+(-1)^{n-1}\frac{1}{n}(1-x)^n=(1-x)+\frac{1}{2}(1-x^2)+\frac{1}{3}(1-x^3)+......+\frac{1}{n}(1-x^n)$$

Q.16 Prove that , $\frac{1}{2}\,{}^nC_1-\frac{2}{3}\,{}^nC_2+\frac{3}{4}\,{}^nC_3-\frac{4}{5}\,{}^nC_4+.....+\frac{(-1)^{n+1}\,n}{n+1}.\,{}^nC_n=\frac{1}{n+1}$

Q.17 If $n\in N$; show that $\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-......+(-1)^n\frac{{}^nC_n}{x+n}=\frac{n!}{x\,(x+1)\,(x+2)\,....\,(x+n)}$

Q.18 Prove that , $({}^{2n}C_1)^2+2.({}^{2n}C_2)^2+3.({}^{2n}C_3)^2+...+2n.({}^{2n}C_{2n})^2=\frac{(4n-1)!}{[(2n-1)!]^2}$

Q.19 If $(1+x+x^2)^n=\sum_{r=0}^{2n}a_r\,x^r$, $n\in N$, then prove that

$$(r+1)\,a_{r+1}=(n-r)\,a_r+(2n-r+1)\,a_{r-1}. \qquad (0<r<2n)$$

Q.20 Prove that the sum to $(n+1)$ terms of $\frac{C_0}{n(n+1)}-\frac{C_1}{(n+1)\,(n+2)}+\frac{C_2}{(n+2)\,(n+3)}-.........$ equals $\int_0^1 x^{n-1}.(1-x)^{n+1}.dx$ & evaluate the integral.

EXERCISE - 6

Q.1 The sum of the rational terms in the expansion of $\left(\sqrt{2}+3^{1/5}\right)^{10}$ is____. [JEE '97, 2]

Q.2 If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals [JEE'98, 2]

(A) $(n-1)a_n$ (B) $n\,a_n$ (C) $n\,a_n/2$ (D) None of these

Q.3 Find the sum of the series $\frac{3}{1!}+\frac{5}{2!}+\frac{9}{3!}+\frac{15}{4!}+\frac{23}{5!}+........\infty$ [REE '98, 6]

Q.4 If in the expansion of $(1+x)^m(1-x)^n$, the co-efficients of x and x^2 are 3 and -6 respectively, then m is : [JEE '99, 2 (Out of 200)]

(A) 6 (B) 9 (C) 12 (D) 24

Q.5(i) For $2\le r\le n$, $\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}=$

(A) $\binom{n+1}{r-1}$ (B) $2\binom{n+1}{r+1}$ (C) $2\binom{n+2}{r}$ (D) $\binom{n+2}{r}$

(ii) In the binomial expansion of $(a-b)^n$, $n\ge 5$, the sum of the 5^{th} and 6^{th} terms is zero . Then $\frac{a}{b}$ equals: [JEE '2000 (Screening) , 1 + 1]

(A) $\frac{n-5}{6}$ (B) $\frac{n-4}{5}$ (C) $\frac{5}{n-4}$ (D) $\frac{6}{n-5}$

Q.6 For any positive integers m, n (with $n\ge m$) , let $\binom{n}{m}={}^nC_m$. Prove that

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+........+\binom{m}{m}=\binom{n+1}{m+1}$$

Hence or otherwise prove that ,

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+........+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}.$$

[JEE '2000 (Mains), 6]

Q.7 Find the largest co-efficient in the expansion of $(1 + x)^n$, given that the sum of co-efficients of the terms in its expansion is 4096. [REE '2000 (Mains)]

Q.8 In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of the 5th and 6th terms is zero. Then $\frac{a}{b}$ equals

(A) $\frac{n-5}{6}$ (B) $\frac{n-4}{5}$ (C) $\frac{5}{n-4}$ (D) $\frac{6}{n-5}$

[JEE '2001 (Screening), 3]

Q.9 Find the coeffcient of x^{49} in the polynomial [REE '2001 (Mains) , 3]

$$\left(x - \frac{C_1}{C_0}\right)\left(x - 2^2 \cdot \frac{C_2}{C_1}\right)\left(x - 3^2 \cdot \frac{C_3}{C_2}\right) \ldots\ldots\ldots\ldots \left(x - 50^2 \cdot \frac{C_{50}}{C_{49}}\right)$$ where $C_r = {}^{50}C_r$.

Q.10 The sum $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if $P < q$) is maximum when m is

(A) 5 (B) 10 (C) 15 (D) 20

Q.11(a) Coefficient of t^{24} in the expansion of $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(A) ${}^{12}C_6 + 2$ (B) ${}^{12}C_6 + 1$ (C) ${}^{12}C_6$ (D) none

[JEE 2003, Screening 3 out of 60]

(b) Prove that : $2^K . \binom{n}{0}\binom{n}{K} - 2^{K-1}\binom{n}{1}\binom{n-1}{K-1} + 2^{K-2}\binom{n}{2}\binom{n-2}{K-2} \ldots\ldots (-1)^K\binom{n}{K}\binom{n-K}{0} = \binom{n}{K}$.

[JEE 2003, Mains-2 out of 60]

Q.12 ${}^{n-1}C_r = (K^2 - 3).{}^nC_{r+1}$, if $K \in$

(A) $[-\sqrt{3}, \sqrt{3}]$ (B) $(-\infty, -2)$ (C) $(2, \infty)$ (D) $(\sqrt{3}, 2]$

[JEE 2004 (Screening)]

Q.13 The value of $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} \ldots\ldots + \binom{30}{20}\binom{30}{30}$ *is, where* $\binom{n}{r} = {}^nC_r$.

(A) $\binom{30}{10}$ *(B)* $\binom{30}{15}$ *(C)* $\binom{60}{30}$ *(D)* $\binom{31}{10}$

[JEE 2005 (Screening)]

EXERCISE - 7

Part : (A) Only one correct option

1. In the expansion of $\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{15}$, the 11th term is a:

(A) positive integer (B) positive irrational number (C) negative integer (D) negative irrational number.

2. If the second term of the expansion $\left[a^{1/13} + \frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$ then the value of $\frac{{}^nC_3}{{}^nC_2}$ is:

(A) 4 (B) 3 (C) 12 (D) 6

3. The value of, $\frac{18^3 + 7^3 + 3.18.7.25}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64}$ is :

(A) 1 (B) 2 (C) 3 (D) none

4. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P+Q}{Q}\right)^5 =$

(A) 9 (B) 27 (C) 81 (D) none of these

5. If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, then the greatest term in the expansion for x = 1/2 is :

(A) 4^{th} (B) 5^{th} (C) 6^{th} (D) none of these

6. Find numerically the greatest term in the expansion of $(2 + 3x)^9$, when x = 3/2.

(A) ${}^9C_6 . 2^9 . (3/2)^{12}$ (B) ${}^9C_3 . 2^9 . (3/2)^6$ (C) ${}^9C_5 . 2^9 . (3/2)^{10}$ (D) ${}^9C_4 . 2^9 . (3/2)^8$

7. The numbers of terms in the expansion of $\left(a^3 + \frac{1}{a^3} + 1\right)^{100}$ is

(A) 201 (B) 300 (C) 200 (D) ${}^{100}C_3$

8. The coefficient of x^{10} in the expansion of $(1 + x^2 - x^3)^8$ is

(A) 476 (B) 496 (C) 506 (D) 528

9. $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3) \ldots\ldots (1 + x + x^2 + \ldots\ldots + x^{100})$ when written in the ascending power of x then the highest exponent of x is

(A) 505 (B) 5050 (C) 100 (D) 50

10. If $x = \left(7 + 4\sqrt{3}\right)^{2n} = [x] + f$, then $x\,(1 - f) =$
(A) 2 (B) 0 (C) 1 (D) 2520

11. The remainder when 2^{2003} is divided by 17 is
(A) 1 (B) 2 (C) 8 (D) none of these

12. The last two digits of the number 3^{400} are:
(A) 81 (B) 43 (C) 29 (D) 01

13. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \ldots\ldots\ldots + \binom{50}{49}\binom{50}{50}$ is, where ${}^nC_r = \binom{n}{r}$
(A) $\binom{100}{50}$ (B) $\binom{100}{51}$ (C) $\binom{50}{25}$ (D) $\binom{50}{25}^2$

14. The value of the expression $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{K=0}^{10} (-1)^K \dfrac{{}^{10}C_K}{2^K}\right)$ is –
(A) 2^{10} (B) 2^{20} (C) 1 (D) 2^5

15. If $|x| < 1$, then the co-efficient of x^n in the expansion of $(1 + x + x^2 + x^3 + \ldots\ldots)^2$ is
(A) n (B) $n - 1$ (C) $n + 2$ (D) $n + 1$

16. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ is :
(A) 1 (B) 2 (C) 3 (D) 4

17. Number of elements in set of value of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ is satisfied
(A) 4 elements (B) 5 elements (C) 7 elements (D) 10 elements

18. The co-efficient of x^5 in the expansion of, $(1 + x)^{21} + (1 + x)^{22} + \ldots\ldots + (1 + x)^{30}$ is :
(A) ${}^{51}C_5$ (B) ${}^{9}C_5$ (C) ${}^{31}C_6 - {}^{21}C_6$ (D) ${}^{30}C_5 + {}^{20}C_5$

19. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{10}x^{10}$, then $(a_0 - a_2 + a_4 + a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is equal to
(A) 3^{10} (B) 2^{10} (C) 2^9 (D) none of these

20. The value of $\sum_{r=1}^{10} r\,.\,\dfrac{{}^nC_r}{{}^nC_{r-1}}$ is equal to
(A) $5\,(2n - 9)$ (B) $10\,n$ (C) $9\,(n - 4)$ (D) none of these

21. If $C_0, C_1, C_2, \ldots\ldots\ldots C_n$ are the Binomial coefficients in the expansion of $(1 + x)^n$. n being even, then $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots\ldots\ldots + (C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_{n-1})$ is equal to
(A) $n.\ 2^n$ (B) $n.\ 2^{n-1}$ (C) $n.\ 2^{n-2}$ (D) $n.\ 2^{n-3}$

22. If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_{40}x^{40}$, then $a_0 + a_2 + a_4 \ldots\ldots + a_{38}$ equals
(A) $2^{19}\,(2^{30} + 1)$ (B) $2^{19}\,(2^{20} - 1)$ (C) $2^{20}\,(2^{19} - 1)$ (D) none of these

23. Co-efficient of x^{15} in $(1 + x + x^3 + x^4)^n$ is :
(A) $\sum_{r=0}^{5} {}^nC_{5-r}.\ {}^nC_{3r}$ (B) $\sum_{r=0}^{5} {}^nC_{5r}$ (C) $\sum_{r=0}^{5} {}^nC_{3r}$ (D) $\sum_{r=0}^{3} {}^nC_{3-r}.\ {}^nC_{5r}$

24. The sum of the coefficients of all the integral powers of x in the expansion of $\left(1 + 2\sqrt{x}\right)^{40}$ is
(A) $3^{40} + 1$ (B) $3^{40} - 1$ (C) $\dfrac{1}{2}(3^{40} - 1)$ (D) $\dfrac{1}{2}(3^{40} + 1)$

25. If $\{x\}$ denotes the fractional part of 'x', then $\left\{\dfrac{3^{1001}}{82}\right\} =$
(A) 9/82 (B) 81/82 (C) 3/82 (D) 1/82

26. The coefficient of the term independent of x in the expansion of $\left(\dfrac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \dfrac{x-1}{x - x^{\frac{1}{2}}}\right)^{10}$ is
(A) 70 (B) 112 (C) 105 (D) 210

27. The coefficient of x^n in polynomial $(x + {}^{2n+1}C_0)\,(x + {}^{2n+1}C_1)\,(x + {}^{2n+1}C_2)\ldots\ldots.(x + {}^{2n+1}C_n)$ is
(A) 2^{n+1} (B) $2^{2n+1} - 1$ (C) 2^{2n} (D) none of these

28. In the expansion of $(1 + x)^n\,(1 + y)^n\,(1 + z)^n$, the sum of the co-efficients of the terms of degree 'r' is :
(A) ${}^{n^3}C_r$ (B) ${}^{n}C_{r^3}$ (C) ${}^{3n}C_r$ (D) $3\,.\,{}^{2n}C_r$

29. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\,{}^rC_p\,2^p\right)$ is equal to
(A) $4^n - 3^n + 1$ (B) $4^n - 3^n - 1$ (C) $4^n - 3^n + 2$ (D) $4^n - 3^n$

30. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots. + C_n x^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0 \leq i < j \leq n}{\Sigma\,\Sigma}\, C_i\, C_j$ is equal to
(A) $2^{2n+1} - \dfrac{2n\,!}{2\,(n\,!)}$ (B) $2^{2n-1} - \dfrac{2n!}{2(n!)^2}$ (C) $2^{2n-1} - \dfrac{2n\,!}{2(n\,!)}$ (D) $2^{2n+1} - \dfrac{2n!}{2\,(n!)^2}$

Part : (B) May have more than one options correct

31. In the expansion of $(x + y + z)^{25}$
(A) every term is of the form ${}^{25}C_r.\ {}^rC_k.\ x^{25-r}.\ y^{r-k}.\ z^k$

(B) the coefficient of $x^8\, y^9\, z^9$ is 0 (C) the number of terms is 325
(D) none of these

32. $7^9 + 9^7$ is divisible by
(A) 16 (B) 24 (C) 64 (D) 72

EXERCISE - 8

1. Find the value of ' x ' for which the fourth term in the expansion, $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336.

2. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7th term from the begining to the 7th term from the end is 1 : 6 ; find n.

3. Find the terms independent of 'x' in the expansion of the expression, $(1 + x + 2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

4. If in the expansion of $(1-x)^{2n-1}$, the co-efficient of x^r is denoted by a_r, then prove that $a_{r-1} + a_{2n-r} = 0$.

5. Show that the term independent of x in the expansion of $\left(1+x+\frac{6}{x}\right)^{10}$ is, $1+\sum_{r=1}^{5} {}^{10}C_{2r}\,{}^{2r}C_r\,6^r$.

6. Find the coefficient of $a^5\, b^4\, c^7$ in the expansion of $(bc + ca + ab)^8$.

7. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then calculate a_1, a_2, a_4.

8. If $\left(3\sqrt{3}+5\right)^n = p + f$, where p is an integer and f is a proper fraction then find the value of $\left(3\sqrt{3}-5\right)^n$, $n \in N$.

9. Write down the binomial expansion of $(1 + x)^{n+1}$, when x = 8. Deduce that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

10. Prove that $53^{53} - 33^{33}$ is divisible by 10.

11. Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

If $C_0, C_1, C_2, ..., C_n$ are the combinatorial co-efficients in the expansion of $(1 + x)^n$, $n \in N$, then prove the followings: (Q. No. 12 - 14)

12. $2.\,C_o+\frac{2^2.C_1}{2}+\frac{2^3.C_2}{3}+\frac{2^4.C_3}{4}+......\frac{2^{n+1}.C_n}{n+1}=\frac{3^{n+1}-1}{n+1}$

13. $\frac{C_1}{C_0}+2.\frac{C_2}{C_1}+3\frac{C_3}{C_2}+........+n\frac{C_n}{C_{n-1}}=\frac{n(n+1)}{2}$

14. $1^2.\,C_0 + 2^2.\,C_1 + 3^2.\,C_2 + 4^2.\,C_3 + ... + (n+1)^2\,C_n = 2^{n-2}\,(n+1)\,(n+4)$.

15. Assuming ' x ' to be so small that x^2 and higher powers of ' x ' can be neglected, show that, $\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ is approximately equal to, $1-\frac{305}{96}x$.

16. If $\sum_{r=0}^{n}(-1)^r.\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+........\text{ to m terms}\right]=k\left(1-\frac{1}{2^{mn}}\right)$, then find the value of k.

17. Find the coefficient of x^{50} in the expression:
$(1 + x)^{1000} + 2x.\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + ... + 1001\,x^{1000}$

18. Given $s_n = 1 + q + q^2 + ... + q^n$ & $S_n=1+\frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+....+\left(\frac{q+1}{2}\right)^n$, $q \neq 1$,
prove that ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 + ... + {}^{n+1}C_{n+1}.s_n = 2^n.\,S_n$.

19. Show that if the greatest term in the expansion of $(1 + x)^{2n}$ has also the greatest co-efficient, then ' x ' lies between, $\frac{n}{n+1}$ & $\frac{n+1}{n}$.

20. Find the remainder when $32^{32^{32}}$ is divided by 7.

21. If $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np}.\,x^{np}$, then find the value of :
$a_1 + 2a_2 + 3a_3 + ... + np.\,a_{np}$.

22. Prove that, $({}^{2n}C_1)^2 + 2.\,({}^{2n}C_2)^2 + 3.\,({}^{2n}C_3)^2 + ... + 2n.\,({}^{2n}C_{2n})^2=\frac{(4n-1)\,!}{\{(2n-1)\,!\}^2}$

23. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n\,x^n$, then show that:

$$C_1(1-x)-\frac{C_2}{2}(1-x)^2+\frac{C_3}{3}(1-x)^3-........+(-1)^{n-1}\frac{1}{n}(1-x)^n$$
$$=(1-x)+\frac{1}{2}(1-x^2)+\frac{1}{3}(1-x^3)+........+\frac{1}{n}(1-x^n)$$

24. Prove that $\sum_{r=0}^{n} r^2\,{}^nC_r\,p^r\,q^{n-r} = npq + n^2p^2$ if p + q = 1.

25. If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ & $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$.

26. If $a_0, a_1, a_2, \ldots$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then prove that :
(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 - \ldots = 0$ (ii) $a_0 a_2 - a_1 a_3 + a_2 a_4 - \ldots + a_{2n-2} a_{2n} = a_{n+1}$
(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + \ldots$; $E_2 = a_1 + a_4 + a_7 + \ldots$ & $E_3 = a_2 + a_5 + a_8 + \ldots$

27. If $(1 + x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots$, then prove that :
(a) $p_0 - p_2 + p_4 - \ldots = 2^{n/2} \cos \frac{n\pi}{4}$ (b) $p_1 - p_3 + p_5 - \ldots = 2^{n/2} \sin \frac{n\pi}{4}$

28. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\sum\sum_{0 \le i < j \le n} C_i C_j$ is equal to $2^{2n-1} - \frac{2n!}{2(n!)^2}$.

ANSWER KEY EXERCISE - 1

Q 1. **(i)** ${}^{11}C_5 \frac{a^6}{b^5}$ **(ii)** ${}^{11}C_6 \frac{a^5}{b^6}$ **(iii)** ab = 1 **Q 2.** r = 6 **Q 3.** r = 5 or 9 **Q 4.** (a) $\frac{5}{12}$ (b) T_6 **=7**

Q 5. $\frac{(2^{mn}-1)}{(2^n-1)(2^{mn})}$ **Q 7. (i)** 3^n **(ii)** 1, **(iii)** a_n **Q 9.** x = 0 or 1 **Q 10.** x = 0 or 2

Q 11. **(a)**101^{50} (Prove that $101^{50} - 99^{50} = 100^{50}$ + some +ive qty) **Q 12.** $1 + \sum_{k=1}^{5} {}^{11}C_{2k} \,.\, {}^{2k}C_k\ 7^k$

Q 14. **(i)** 990 **(ii)** 3660 **Q 15. (i)** $T_7 = \frac{7.3^{13}}{2}$ **(ii)** 455 x 3^{12} **Q 18.** $\frac{17}{54}$

Q.19 T_8 **Q.20** n = 2 or 3 or 4 **Q.24 (a)** $\frac{n^2+n+2}{2}$

Q 25. **(a)** $84b^6c^3 + 630ab^4c^4 + 756a^2b^2c^5 + 84a^3c^6$ **; (b)** $-1260 \,.\, a^2b^3c^4$ **; (c)** -12600

Q 28. ${}^nC_r (3^{n-r} - 2^{n-r})$ **Q 29.** (a) n = 12 (b) $\frac{5}{8} < x < \frac{20}{21}$ **Q.32** $\frac{n-k}{n}$

Q 34. **(a)** Hint : Add 1 to both sides & compare the RHS series with the expansion $(1+y)^n$ to get n & y **(b)** 4

EXERCISE - 2

Q.4 1

EXERCISE - 5

Q 1. divide expansion of $(1+x)^{15}$ both sides by x & diff. w.r.t.x , put x = 1 to get 212993

Q 2. Differentiate the given expn. & put x = 1 to get the result $\frac{np}{2}(p+1)^n$

Q 9. Integrate the expn. of $(1 + x)^n$. Determine the value of constant of integration by putting x = 0. Integrate the result again between 0 & 2 to get the result.

Q 10. Consider $\frac{1}{2}[(1+x)^n + (1-x)^n] = C_0 + C_2x^2 + C_4x^4 + \ldots$ Integrate between 0 & 1.

Q 12. Multiply both sides by x the expn. $(1+x)^n$. Integrate both sides between 0 & 1.

Q 14. Note that $\frac{(1-x)^n - 1}{x} = -C_1 + C_2x - C_3x^2 + \ldots + C_n . x^{n-1}$. Integrate between 1 & 0

Q 20. $\frac{(n-1)!\,(n+1)!}{(2n+1)!}$

EXERCISE - 6

Q.1 41 **Q.2** C **Q.3** 4e – 3 **Q.4** C **Q.5 (i)** D **(ii)** B
Q.7 ${}^{12}C_6$ **Q.8** B **Q.9** – 22100 **Q.10** C
Q.11 (a) A **Q.12** D **Q.13** A

EXERCISE - 7

1. B	**2.** A	**3.** A	**4.** D	**5.** B	**6.** A	**7.** A	**8.** A	**9.** B	**10.** C
11. C	**12.** D	**13.** B	**14.** C	**15.** D	**16.** B	**17.** C	**18.** C	**19.** B	**20.** A
21. B	**22.** B	**23.** A	**24.** D	**25.** C	**26.** D	**27.** C	**28.** C	**29.** D	**30.** B
31. AB	**32.** AC								

EXERCISE - 8

1. x = 0 or 1 **2.** n = 9 **3.** $\frac{17}{54}$ **6.** 280 **7.** $a_1 = 20,\ a_2 = 210,\ a_4 = 8085$

8. 1 – f, if n is even and f, if n is odd **11.** 101^{50} **16.** $\frac{1}{2^n - 1}$ **17.** ${}^{1002}C_{50}$

20. 4 **21.** $\frac{np}{2}(p+1)^n$