Chapter 10
Motion in a Non-Inertial Reference Frame

The laws of physics are only valid in inertial reference frames. However, it is not always easy to express the motion of interest in an inertial reference frame. Consider for example the motion of a book laying on top of a table. In a reference frame that is fixed with respect to the Earth, the motion is simple: if the book is at rest, it will remain at rest (here we assume that the surface of the table is horizontal). However, we do know that the earth frame is not an inertial frame. In order to describe the motion of the book in an inertial frame, we need to take into account the rotation of the Earth around its axis, the rotation of the Earth around the sun, the rotation of our solar system around the center of our galaxy, etc. etc. The motion of the book will all of a sudden be a lot more complicated!

For many experiments, the effect of the Earth not being an inertial reference frame is too small to be observed. Other effects, such as the tides, can only be explained if we take into consideration the non-inertial nature of the reference frame of the Earth and apply the laws of physics in n inertial frame.

Rotating Coordinate Systems

Consider the two coordinate systems shown in Figure 1. The non-primed coordinates are the coordinates in the rotating frame, and the primed coordinates are the coordinates in the fixed coordinate system.

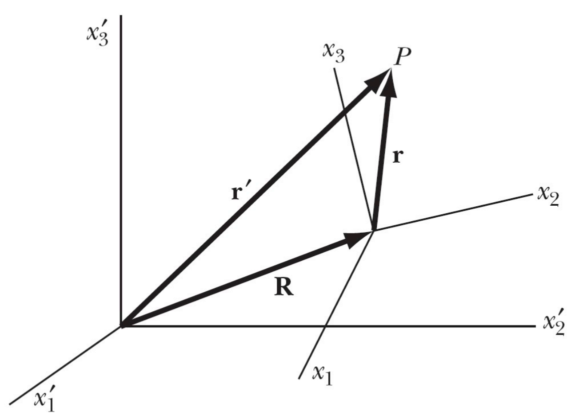


Figure 1. Fixed (primed) and rotating (non-primed) coordinate systems. The vector R specifies the origin of the rotating coordinate system in the fixed coordinate system.

Consider the motion of a point P. In the fixed coordinate system, the position of P is specified by the position vector r' and in the rotating coordinate system, its position is specified by the position vector r. As can be seen from Figure 1, these two vectors are related:

$$\bar{r}' = \bar{r} + \bar{R}$$

Consider what happens when the rotating coordinate system rotates by an infinitesimal angle $d\theta$. If point P is at rest in the rotating coordinate system, we will see the position of P in our fixed coordinate system change:

$$\left(d\bar{r}\right)_{fixed} = d\bar{\theta} \times \bar{r}$$

It the rotation occurs during a period dt, we can rewrite the previous equation as

$$\left(\frac{d\bar{r}}{dt}\right)_{fixed} = \frac{d\bar{\theta}}{dt} \times \bar{r} = \bar{\omega} \times \bar{r}$$

To derive this relation we have assume that point P remains are test in the rotating coordinate system. If point P is moving with respect to the rotating coordinate system, we need to add this contribution to the expression of the velocity of P in the fixed coordinate system:

$$\left(\frac{d\bar{r}}{dt}\right)_{fixed} = \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{r}$$

This relation is valid for any vector, not just the position vector. If instead of the position vector we use the angular velocity vector we find that

$$\left(\frac{d\bar{\omega}}{dt}\right)_{fixed} = \left(\frac{d\bar{\omega}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{\omega} = \left(\frac{d\bar{\omega}}{dt}\right)_{rotating}$$

This relation shows that the angular acceleration is the same in both reference frames.

In order to determine the velocity of point P in the fixed coordinate frame in terms of the velocity of point P in the rotating coordinate system, we have to go back to the correlation between the position vectors shown in Figure 1. By differentiating the vectors with respect to time we obtain the following relation:

$$\left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{fixed}$$

Using our expression for the velocity of P in the fixed coordinate system we find that

$$\left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega} \times \bar{r}$$

This equation can also be rewritten as

$$v_f = \left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} + \left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r} = V + v_r + \bar{\omega}\times\bar{r}$$

where

$$\bar{v}_f = \left(\frac{d\bar{r}'}{dt}\right)_{fixed} = \text{velocity of P in fixed frame}$$

$$\bar{V} = \left(\frac{d\bar{R}}{dt}\right)_{fixed} = \text{linear velocity of the origin of the rotating frame}$$

$$\bar{v}_r = \left(\frac{d\bar{r}}{dt}\right)_{rotating} = \text{velocity of P in rotating frame}$$

$$\bar{\omega}\times\bar{r} = \text{velocity of P due to the rotation of the axes}$$

"Newton's Law" in Rotating Reference Frames

Consider the situation in which an external force F is acting on P. Only on the fixed reference frame can we used Newton's second law to determine the corresponding acceleration of P:

$$\bar{a}_f = \left(\frac{d\bar{v}_f}{dt}\right)_{fixed} = \frac{\bar{F}}{m}$$

Another expression for the acceleration of P can be obtained by differentiating the velocity-relation obtained in the previous section with respect to time:

$$\begin{aligned}\bar{a}_f &= \left(\frac{d\bar{v}_f}{dt}\right)_{fixed} = \left(\frac{d\bar{V}}{dt}\right)_{fixed} + \left(\frac{d\bar{v}_r}{dt}\right)_{fixed} + \left(\frac{d\bar{\omega}}{dt}\right)_{fixed}\times\bar{r} + \bar{\omega}\times\left(\frac{d\bar{r}}{dt}\right)_{fixed} = \\ &= \left(\frac{d\bar{V}}{dt}\right)_{fixed} + \left\{\left(\frac{d\bar{v}_r}{dt}\right)_{rotating} + \bar{\omega}\times\bar{v}_r\right\} + \left(\frac{d\bar{\omega}}{dt}\right)_{fixed}\times\bar{r} + \bar{\omega}\times\left\{\left(\frac{d\bar{r}}{dt}\right)_{rotating} + \bar{\omega}\times\bar{r}\right\} = \\ &= \left(\frac{d\bar{V}}{dt}\right)_{fixed} + \left(\frac{d\bar{v}_r}{dt}\right)_{rotating} + 2\bar{\omega}\times\bar{v}_r + \dot{\bar{\omega}}\times\bar{r} + \bar{\omega}\times\left\{\bar{\omega}\times\bar{r}\right\}\end{aligned}$$

An observer in the rotating reference frame will observe an acceleration

$$\bar{a}_r = \left(\frac{d\bar{v}}{dt} \right)_{rotating}$$

This acceleration is certainly not equal to F/m, but the previous relations can be used to express the acceleration in the rotating reference frame in terms of the acceleration in the fixed reference frame:

$$\bar{a}_r = \bar{a}_f - \left(\frac{d\bar{V}}{dt} \right)_{fixed} - 2\bar{\omega} \times \bar{v}_r - \dot{\bar{\omega}} \times \bar{r} - \bar{\omega} \times \{ \bar{\omega} \times \bar{r} \}$$

This relation immediately shows what has been repeated already many times: the acceleration of an object at P will be the same in two reference frames, only if one frame does not rotate with respect to the other frame (that is ω = 0 rad/s and $d\omega/dt$ = 0 rad/s^2) and if the frames do not accelerate with respect to the other frame.

In order to explore the implication of the relation between the acceleration of P in the rotating and in the fixed coordinate frames, we assume for the moment that the origin of the rotating reference frame is not accelerating with respect to the origin of the fixed reference frame ($dV/dt = 0$), and that the axis of the rotating reference frame are rotating with a constant angular velocity ($d\omega/dt$ = 0 rad/s^2). Under these assumptions, we find that the acceleration of P in the rotating reference frame is equal to

$$\bar{a}_r = \bar{a}_f - 2\bar{\omega} \times \bar{v}_r - \bar{\omega} \times \{ \bar{\omega} \times \bar{r} \}$$

The second and the third terms on the right-hand side are non-inertial terms that are introduced to correct the real force F in order to be able to use Newton-like laws in the rotating frame:

$$\bar{F}_{eff} = m\bar{a}_f - 2m\bar{\omega} \times \bar{v}_r - m\bar{\omega} \times \{ \bar{\omega} \times \bar{r} \}$$

Using this effective force, an observer in the rotating frame will be able to determine the acceleration in the rotating frame by dividing this effective force by the mass of the object.

The second term on the right-hand side of the expression of the effective force is called the **Coriolis force**, and the last term on the right-hand side is called the **centripetal force**. Both of these forces are however not real forces; they are introduced in order to be able to use an equation similar to Newton's second law in non-inertial reference frames. When we try to describe an object on the surface of the earth, we need to take the effects of these artificial forces into consideration. In the next two sections we will focus on these two forces in some detail.

<u>The Centripetal Force</u>

The surface of the earth is a non-inertial reference frame. The biggest deviation from good "inertial" behavior is due to the rotation of the earth around its axis. In the current discussion we will thus ignore the motion of the earth around the sun, the motion of the solar system in our in our galaxy, etc. etc.

Consider a pendulum at rest in our rotating reference frame, which is at rest with respect to the surface of the earth. Since the pendulum is at rest in this rotating reference frame, its velocity v_r in this frame is zero. The effective force seen by the pendulum is thus equal to

$$\bar{F}_{eff} = m\bar{a}_f - m\bar{\omega}\times\{\bar{\omega}\times\bar{r}\} = m\{\bar{g}_0 - \bar{\omega}\times\{\bar{\omega}\times\bar{r}\}\}$$

The direction of g_0 in this equation is directly towards the center of the earth, while the direction of the non-inertial correction term is radially outwards (see Figure 2). If the angle between the position vector r and the rotation axis is equal to q, we find the magnitude of the correction term is equal to

$$\left|\bar{\omega}\times\{\bar{\omega}\times\bar{r}\}\right| = \omega^2 r\sin\theta$$

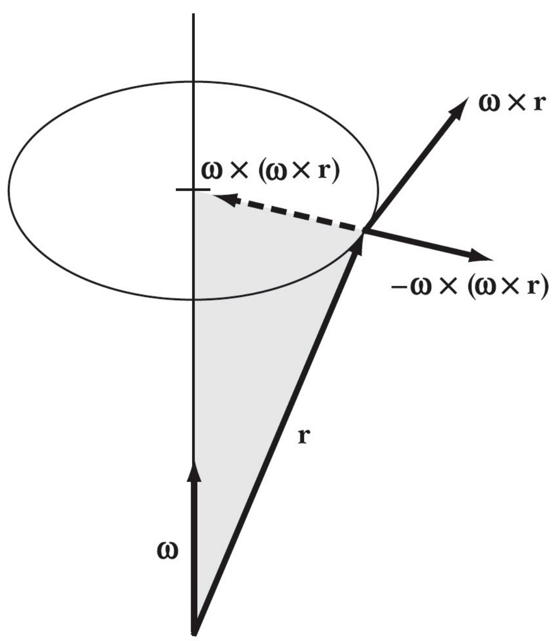


Figure 2. Direction of the centripetal correction term.

The effect of this correction is that the equilibrium position of the pendulum (the position in which the arm of the pendulum is parallel to the direction of the net force) is changed, and the arm of the pendulum no longer points towards the center of the earth (see Figure 3). The direction of the gravitational acceleration, as measured by an observer in the rotating reference frame, is thus equal to

$$\bar{g} = \bar{g}_0 - \bar{\omega} \times \{\bar{\omega} \times \bar{r}\}$$

The centripetal correction changes both the magnitude of the observed acceleration and its direction. The angle between the direction of g_0 and the direction of g can be found easily (see Figure 4):

$$\Delta\theta = \theta - \alpha = \theta - \mathrm{atan}\left(\frac{g_0 \sin\theta - \omega^2 R \sin\theta}{g_0 \cos\theta}\right) = \theta - \mathrm{atan}\left(\left(1 - \frac{\omega^2}{g_0}\right)\tan\theta\right)$$

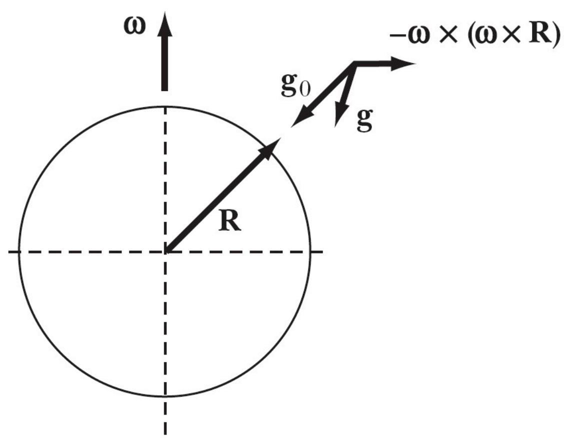


Figure 3. Effect of the centripetal term on a pendulum located on the surface of the earth.

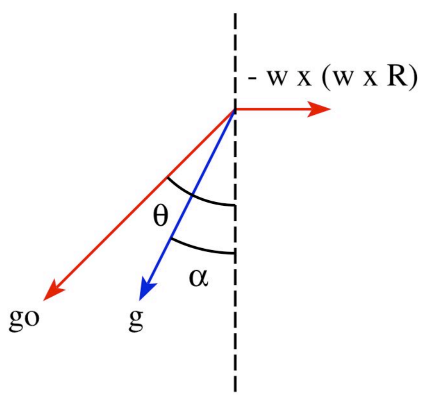


Figure 4. Direction of net gravitational acceleration.

The same result could have been obtained if we had solved this problem in a non-rotating frame. Consider a simple pendulum of mass m attached a string. There are two forces acting on this mass: the tension T in the string and the gravitational force F_g. An observer in the inertial frame will observe that mass m carries out circular motion, with a radius $R \sin\theta$, and know thus that there must be a net force acting on it, pointing towards the rotation axis. This force must have a magnitude of

$$F_r = m\frac{v^2}{R\sin\theta} = m\frac{\left(\dfrac{2\pi R\sin\theta}{T}\right)^2}{R\sin\theta} = m\omega^2 R\sin\theta$$

This force must be generated by the component of the tension and the gravitational force in this direction. We must thus require that (see Figure 5):

$$mg_0\sin\theta - T\sin\alpha = m\omega^2 R\sin\theta$$

or

$$T\sin\alpha = mg_0\sin\theta - m\omega^2 R\sin\theta$$

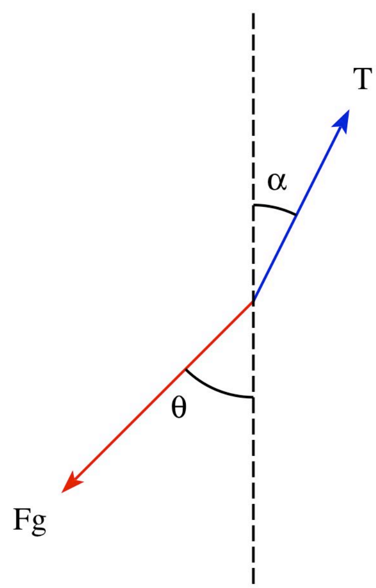


Figure 5. Pendulum in inertial frame.

The net force in the direction perpendicular to the plan of rotation must be zero, and we must thus require that

$$T\cos\alpha = mg\cos\theta$$

Combing these two equations we obtain the following relation between the angles:

$$\tan\alpha = \frac{T\sin\alpha}{T\cos\alpha} = \frac{mg_0\sin\theta - m\omega^2 R\sin\theta}{mg_0\cos\theta} = \tan\left(\left(1-\frac{\omega^2}{g_0}\right)\tan\theta\right)$$

which is the same results we obtained previously.

Coriolis Force

The Coriolis force is responsible for the deflection of objects moving in a rotating coordinate system. The force is proportional to the vector product of the angular velocity vector of the rotating coordinate system (as measured by an observer in a fixed coordinate system) and the velocity vector of the object in the rotating coordinate frame:

$$\bar{F}_{Coriolis} = -2m\left(\bar{\omega}\times\bar{v}_r\right)$$

The effect of the Coriolis force on the motion of an object is illustrated in Figure 6. Note that the deflection depends on the *z* component of the angular velocity vector, which is perpendicular to the surface of the earth. The *z* component reaches a maximum value at the North pole, and is zero at the equator.

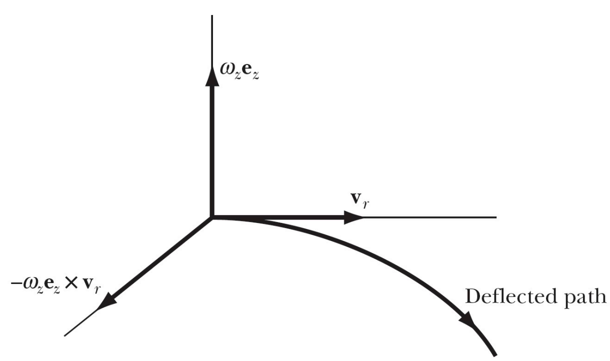


Figure 6. Deflection of a moving object as a result of the Coriolis force.

As a result of the Coriolis force, air flowing from West to East towards a region of low pressure will be deflected to the South on the Northern hemisphere. Air approaching the low from the East will be deflected to the North. On the Northern hemisphere we will thus expect that the air is flowing counter clockwise around an area of low pressure; in the same manner we can show that air flows clockwise around an area of high pressure. A lot about the weather can be understood on the basis of these observations. See for example the forecast map shown Figure 7. The position of the high pressure system over Michigan will bring cold air from Canada to Rochester (since the circulation around the high is in the clockwise direction). We thus expect the winds to be from the North. Once the high passes Rochester, the wind should come from the

South, bringing us higher temperatures. The low in the South of the USA will pull in moisture from the gulf of Mexico and rain and thunder can be expected in the region in front of the low (since this is the region where moisture of the golf of Mexico will go as a result of the counter-clockwise flow around the low).

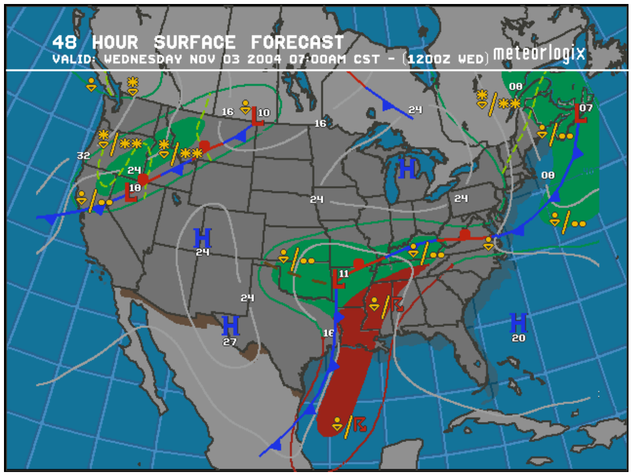
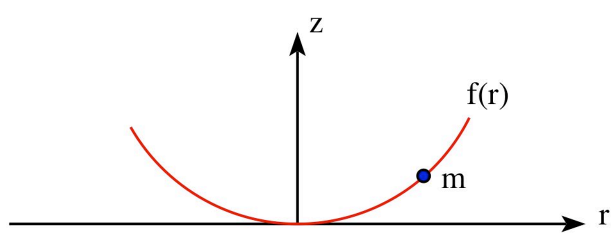


Figure 7. Forty-eight hour forecast map for Wednesday November 3, 2004, at 1200 Z (http://www.aopa.org/members/wx/focpage.cfm?sfcmap=0700d485).

Example: Problem 10.6

A bucket of water is set spinning about its symmetry axis. Determine the shape of the water in the bucket.



Consider a small mass *m* on the surface of the water. From Eq. (10.25) in our text book we get

$$\overline{F}_{eff} = \overline{F} - m\ddot{\overline{R}}_f - m\dot{\overline{\omega}} \times \overline{r} - m\overline{\omega} \times \left(\overline{\omega} \times \overline{r}\right) - 2m\overline{\omega} \times \overline{v}_r$$

In the rotating frame, the mass is at rest; thus

$$\bar{F}_{eff} = 0$$

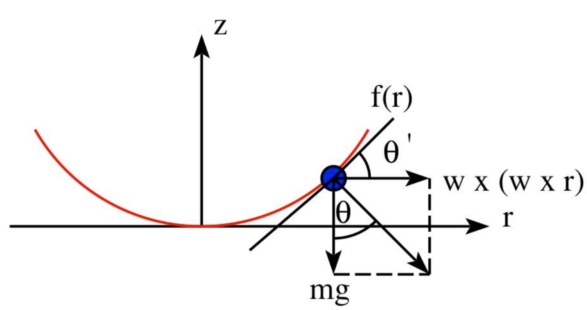
The force F will consist of gravity and the force due to the pressure gradient, which is normal to the surface in equilibrium. Since

$$\ddot{\bar{R}}_f = \dot{\bar{\omega}} = \bar{v}_r = 0$$

we now have

$$0 = m\bar{g} + \bar{F}_p - m\bar{\omega} \times (\bar{\omega} \times \bar{r})$$

where F_p is due to the pressure gradient.



Since $F_{eff} = 0$, the sum of the gravitational and centrifugal forces must also be normal to the surface. Thus $\theta' = \theta$.

$$\tan\theta' = \tan\theta = \frac{\omega^2 r}{g}$$

but

$$\tan\theta' = \frac{dz}{dr}$$

Thus

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}$$