विध्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचितः मानव धर्म प्रणेता
सद्‌गुरु श्री रणछोड़दासजी महाराज

STUDY PACKAGE

Subject : Mathematics
Topic : Sequence & Progression

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Properties & Solution of Triangle

1. Sine Rule:

In any triangle ABC, the sines of the angles are proportional to the opposite sides i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Example : In any ΔABC, prove that $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

Solution. $\because$ We have to prove $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

$\because$ From **sine rule**, we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (let)}$$

$\Rightarrow$ a = k sinA, b = k sinB and c = k sinC

$\because$ L.H.S. $= \frac{a+b}{c}$

$$= \frac{k(\sin A + \sin B)}{k \sin C} = \frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

= R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Example : In any ΔABC, prove that

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Solution. $\because$ We have to prove that

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

$\because$ from **sine rule**, we know that

a = k sinA, b = k sinB and c = k sinC

$\therefore$ $(b^2 - c^2) \cot A = k^2 (\sin^2 B - \sin^2 C) \cot A$

$\because$ $\sin^2 B - \sin^2 C = \sin (B + C) \sin (B - C)$

$\therefore$ $(b^2 - c^2) \cot A = k^2 \sin (B + C) \sin (B - C) \cot A$ $\quad\because\quad B + C = \pi - A$

$\therefore$ $(b^2 - c^2) \cot A = k^2 \sin A \sin (B - C) \frac{\cos A}{\sin A}$ $\quad\because\quad \cos A = -\cos(B + C)$

$= -k^2 \sin (B - C) \cos (B + C)$

$= -\frac{k^2}{2} [2\sin (B - C) \cos (B + C)]$

$\Rightarrow$ $(b^2 - c^2) \cot A = -\frac{k^2}{2} [\sin 2B - \sin 2C]$ ....(i)

Similarly $(c^2 - a^2) \cot B = -\frac{k^2}{2} [\sin 2C - \sin 2A]$ ....(ii)

and $(a^2 - b^2) \cot C = -\frac{k^2}{2} [\sin 2A - \sin 2B]$ ....(iii)

adding equations (i), (ii) and (iii), we get

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$ **Hence Proved**

Self Practice Problems

In any ΔABC, prove that

1. $a \sin\left(\frac{A}{2} + B\right) = (b + c) \sin\left(\frac{A}{2}\right)$.

2. $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$

3. $\frac{c}{a-b} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$.

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2. Cosine Formula:

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 + 2bc \cos (B + C)$

(ii) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ **(iii)** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Example : In a triangle ABC if a = 13, b = 8 and c = 7, then find sin A.

Solution. $\because$ $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{64+49-169}{2.8.7}$

$\Rightarrow$ $\cos A = -\dfrac{1}{2}$ $\Rightarrow$ $A = \dfrac{2\pi}{3}$

$\therefore$ $\sin A = \sin \dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$ **Ans.**

***Example :** In a ΔABC, prove that $a(b \cos C - c \cos B) = b^2 - c^2$

Solution. $\because$ We have to prove $a (b \cos C - c \cos B) = b^2 - c^2$.

$\because$ from **cosine rule** we know that

$\cos C = \dfrac{a^2+b^2-c^2}{2ab}$ & $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

$\therefore$ L.H.S. $= a\left\{ b\left(\dfrac{a^2+b^2-c^2}{2ab}\right) - c\left(\dfrac{a^2+c^2-b^2}{2ac}\right)\right\}$

$= \dfrac{a^2+b^2-c^2}{2} - \dfrac{(a^2+c^2-b^2)}{2}$

$= (b^2 - c^2)$ = R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Example : If in a ΔABC, $\angle A = 60°$ then find the value of $\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right)$.

Solution. $\because$ $\angle A = 60°$

$\therefore$ $\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right) = \left(\dfrac{c+a+b}{c}\right)\left(\dfrac{b+c-a}{b}\right)$

$= \dfrac{(b+c)^2-a^2}{bc}$

$= \dfrac{(b^2+c^2-a^2)+2bc}{bc}$

$= \dfrac{b^2+c^2-a^2}{bc} + 2$

$= 2\left(\dfrac{b^2+c^2-a^2}{2bc}\right) + 2$

$= 2\cos A + 2$ $\because$ $\angle A = 60°$ $\Rightarrow$ $\cos A = \dfrac{1}{2}$

$\therefore$ $\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right) = 3$ **Ans.**

Self Practice Problems :

1. The sides of a triangle ABC are a, b, $\sqrt{a^2+ab+b^2}$, then prove that the greatest angle is 120°.

2. In a triangle ABC prove that $a(\cos B + \cos C) = 2(b + c) \sin^2 \dfrac{A}{2}$.

3. Projection Formula:

(i) $a = b \cos C + c \cos B$ **(ii)** $b = c \cos A + a \cos C$ **(iii)** $c = a \cos B + b \cos A$

Example : In a triangle ABC prove that $a(b \cos C - c \cos B) = b^2 - c^2$

Solution. $\because$ L.H.S. $= a (b \cos C - c \cos B)$

$= b (a \cos C) - c (a \cos B)$ ....(i)

$\because$ From **projection rule**, we know that

$b = a \cos C + c \cos A$ $\Rightarrow$ $a \cos C = b - c \cos A$

& $c = a \cos B + b \cos A$ $\Rightarrow$ $a \cos B = c - b \cos A$

Put values of a cosC and a cosB in equation (i), we get

L.H.S. $= b (b - c\cos A) - c(c - b \cos A)$

$= b^2 - bc \cos A - c^2 + bc \cos A$

$= b^2 - c^2$

= R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Note: We have also proved $a (b \cos C - c\cos B) = b^2 - c^2$ by using **cosine – rule** in solved ***Example.**

Example : In a ΔABC prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

Solution. $\because$ L.H.S. $= (b + c) \cos A\ (c + a) \cos B + (a + B) \cos C$

= b cos A + c cos A + c cos B + a cos B + a cos C + b cos C
= (b cos A + a cos B) + (c cos A + a cos C) + (c cos B + b cos C)
= a + b + c
= R.H.S.
Hence L.H.S. = R.H.S. **Proved**

Self Practice Problems

In a ΔABC, prove that

1. $2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)$ = a + b + c.

2. $\frac{\cos B}{\cos C} = \frac{c-b\cos A}{b-c\cos A}$.

3. $\frac{\cos A}{c\cos B+b\cos C} + \frac{\cos B}{a\cos C+c\cos A} + \frac{\cos C}{a\cos B+b\cos A} = \frac{a^2+b^2+c^2}{2abc}$.

4. Napier's Analogy - tangent rule:

(i) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ **(ii)** $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

(iii) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

Example : Find the unknown elements of the ΔABC in which a = $\sqrt{3}$ + 1, b = $\sqrt{3}$ – 1, C = 60°.

Solution. $\because$ a = $\sqrt{3}$ + 1, b = $\sqrt{3}$ – 1, C = 60°

$\because$ A + B + C = 180°

$\therefore$ A + B = 120° ....(i)

$\because$ From law of tangent, we know that

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$= \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30^\circ$$

$$= \frac{2}{2\sqrt{3}}\cot 30^\circ$$

$\Rightarrow$ $\tan\left(\frac{A-B}{2}\right) = 1$

$\therefore$ $\frac{A-B}{2} = \frac{\pi}{4}$ = 45°

$\Rightarrow$ A – B = 90° ....(ii)

From equation (i) and (ii), we get
A = 105° and B = 15°
Now,

$\because$ From **sine-rule,** we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore$ $c = \frac{a\sin C}{\sin A} = \frac{(\sqrt{3}+1)\sin 60^\circ}{\sin 105^\circ}$

$$= \frac{(\sqrt{3}+1)\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}} \qquad \because \quad \sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\Rightarrow$ c = $\sqrt{6}$

$\therefore$ c = $\sqrt{6}$, A = 105°, B = 15° **Ans.**

Self Practice Problem

1. In a ΔABC if b = 3, c = 5 and cos (B – C) = $\frac{7}{25}$, then find the value of $\tan\frac{A}{2}$.

Ans. $\frac{1}{3}$

2. If in a ΔABC, we define $x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$, $y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$ and $z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$ then show that $x + y + z = -xyz$.

5. Trigonometric Functions of Half Angles:

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ is semi perimetre of triangle.

(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

6. Area of Triangle (Δ)

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

Example : In a ΔABC if a, b, c are in A.P. then find the value of $\tan\frac{A}{2}.\tan\frac{C}{2}$.

Solution. $\because$ $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$ and $\tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

$\therefore$ $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{\Delta^2}{s^2(s-a)(s-c)}$ $\quad\because\ \Delta^2 = s(s-a)(s-b)(s-c)$

$\therefore$ $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{s-b}{s} = 1 - \frac{b}{s}$ ....(i)

$\because$ it is given that a, b, c are in A.P.

$\Rightarrow$ $2b = a + c$

$\because$ $s = \frac{a+b+c}{2} = \frac{3b}{2}$

$\therefore$ $\frac{b}{s} = \frac{2}{3}$ put in equation (i)

$\therefore$ $\tan\frac{A}{2}.\tan\frac{C}{2} = 1 - \frac{2}{3}$

$\Rightarrow$ $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{1}{3}$ **Ans.**

Example : In a ΔABC if $b\sin C(b\cos C + c\cos B) = 42$, then find the area of the ΔABC.

Solution. $\because$ $b\sin C(b\cos C + c\cos B) = 42$ ....(i) given

$\because$ From **projection rule**, we know that

$a = b\cos C + c\cos B$ put in (i), we get

$ab\sin C = 42$ ....(ii)

$\because$ $\Delta = \frac{1}{2}ab\sin C$

$\therefore$ $\Delta = 21$ sq. unit **Ans.**

Example : In any ΔABC prove that $(a + b + c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$.

Solution. $\because$ L.H.S. $= (a + b + c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)$

$\because$ $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\therefore$ L.H.S. $= (a + b + c)\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right]$

$= 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$

$$= 2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right] \qquad \because \quad 2s = a + b + c$$

$$\therefore \quad 2s - b - a = c$$

$$= 2\sqrt{s(s-c)}\left[\frac{c}{\sqrt{(s-a)(s-b)}}\right]$$

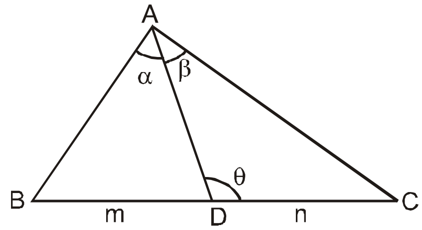
$$= 2c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \qquad \because \quad \cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= 2c\cot\frac{C}{2}$$

= R.H.S.

Hence L.H.S. = R.H.S. **Proved**

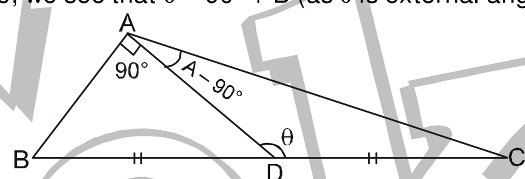
7. m - n Rule:



$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot B - m\cot C$$

Example : If the median AD of a triangle ABC is perpendicular to AB, prove that tan A + 2tan B = 0.

Solution. From the figure, we see that $\theta = 90° + B$ (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

$(1 + 1)\cot(90 + B) = 1.\cot 90° - 1.\cot(A - 90°)$

$\Rightarrow \quad -2\tan B = \cot(90° - A)$

$\Rightarrow \quad -2\tan B = \tan A$

$\Rightarrow \quad \tan A + 2\tan B = 0$ **Hence proved.**

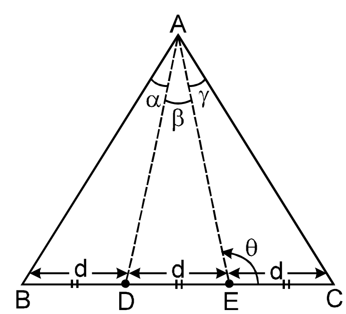
Example : The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that

$$4\left(1+\frac{1}{t_2^2}\right) = \left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right).$$

Solution. Let point D and E divides the base BC into three equal parts i.e. BD = DE = DC = d (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex.

$\Rightarrow \quad t_1 = \tan\alpha$, $t_2 = \tan\beta$ and $t_3 = \tan\gamma$

Now in ΔABC



$\because \quad$ BE : EC = 2d : d = 2 : 1

$\therefore \quad$ from **m-n rule**, we get

$(2 + 1)\cot\theta = 2\cot(\alpha + \beta) - \cot\gamma$

$\Rightarrow \quad 3\cot\theta = 2\cot(\alpha + \beta) - \cot\gamma \qquad$ ....(i)

again

$\because \quad$ in ΔADC

$\because \quad$ DE : EC = x : x = 1 : 1

$\therefore \quad$ if we apply **m-n rule** in ΔADC, we get

$(1 + 1)\cot\theta = 1.\cot\beta - 1\cot\gamma$

$2\cot\theta = \cot\beta - \cot\gamma \qquad$ ....(ii)

from (i) and (ii), we get

$$\frac{3\cot\theta}{2\cot\theta} = \frac{2\cot(\alpha+\beta) - \cot\gamma}{\cot\beta - \cot\gamma}$$

$\Rightarrow \quad 3\cot\beta - 3\cot\gamma = 4\cot(\alpha + \beta) - 2\cot\gamma$

$\Rightarrow \quad 3\cot\beta - \cot\gamma = 4\cot(\alpha + \beta)$

$\Rightarrow \quad 3\cot\beta - \cot\gamma = 4\left\{\dfrac{\cot\alpha.\cot\beta - 1}{\cot\beta + \cot\alpha}\right\}$

$\Rightarrow \quad 3\cot^2\beta + 3\cot\alpha\cot\beta - \cot\beta\cot\gamma - \cot\alpha\cot\gamma = 4\cot\alpha\cot\beta - 4$

$\Rightarrow \quad 4 + 3\cot^2\beta = \cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\alpha\cot\gamma$

$\Rightarrow \quad 4 + 4\cot^2\beta = \cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma + \cot^2\beta$

$\Rightarrow \quad 4(1 + \cot^2\beta) = (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma)$

$\Rightarrow \quad 4\left(1+\dfrac{1}{\tan^2\beta}\right) = \left(\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}\right)\left(\dfrac{1}{\tan\beta}+\dfrac{1}{\tan\gamma}\right)$

$$\Rightarrow \quad 4\left(1+\frac{1}{t_2^2}\right) = \left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)$$ **Hence proved**

Self Practice Problems :

1. In a ΔABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides angle A into the angles of $30°$ and $45°$. Prove that the side BC is of length 2 units.

8. Radius of Circumcirlce :

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

Example : In a ΔABC prove that $\sin A + \sin B + \sin C = \frac{s}{R}$

Solution. In a ΔABC, we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$\therefore \quad \sin A = \frac{a}{2R}$, $\sin B = \frac{b}{2R}$ and $\sin C = \frac{c}{2R}$.

$\therefore \quad \sin A + \sin B + \sin C = \frac{a+b+c}{2R}$ $\quad \because \quad a + b + c = 2s$

$= \frac{2s}{2R}$ $\quad \Rightarrow \quad \sin A + \sin B + \sin C = \frac{s}{R}$.

Example : In a ΔABC if a = 13 cm, b = 14 cm and c = 15 cm, then find its circumradius.

Solution. $\because \quad R = \frac{abc}{4\Delta}$ ....(i)

$\because \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because \quad s = \frac{a+b+c}{2} = 21$ cm

$\therefore \quad \Delta = \sqrt{21.8.7.6} = \sqrt{7^2.4^2.3^2}$

$\Rightarrow \quad \Delta = 84$ cm^2

$\therefore \quad R = \frac{13.14.15}{4.84} = \frac{65}{8}$ cm

$\therefore \quad R = \frac{65}{8}$ cm.

Example : In a ΔABC prove that $s = 4R \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$.

Solution. In a ΔABC,

$\because \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ and $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ and $R = \frac{abc}{4\Delta}$

$\because \quad$ R.H.S. $= 4R \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$.

$= \frac{abc}{\Delta} . s\sqrt{\frac{s(s-a)(s-b)(s-c)}{(abc)^2}}$ $\quad \because \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= s$

$=$ L.H.S.

Hence R.H.L = L.H.S. proved

Example : In a ΔABC, prove that $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$.

Solution. $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$

$\because \quad$ L.H.S. $= \left(\frac{1}{s-a}+\frac{1}{s-b}\right) + \left(\frac{1}{s-c}-\frac{1}{s}\right)$

$= \frac{2s-a-b}{(s-a)(s-b)} + \frac{(s-s+c)}{s(s-c)}$ $\quad \because \quad 2s = a + b + c$

$= \frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}$

$= c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right] = c\left[\frac{2s^2-s(a+b+c)+ab}{\Delta^2}\right]$

$\therefore$ L.H.S. $= c\left[\dfrac{2s^2 - s(2s) + ab}{\Delta^2}\right] = \dfrac{abc}{\Delta^2} = \dfrac{4R\Delta}{\Delta^2} = \dfrac{4R}{\Delta}$ $\quad\because\quad R = \dfrac{abc}{4\Delta}$

$\Rightarrow \quad abc = 4R\Delta$

$\therefore$ L.H.S. $= \dfrac{4R}{\Delta}$

Self Practice Problems :

In a ΔABC, prove the followings :

1. $a \cot A + b \cot B + \cos C = 2(R + r)$.

2. $4\left(\dfrac{s}{a} - 1\right)\left(\dfrac{s}{b} - 1\right)\left(\dfrac{s}{c} - 1\right) = \dfrac{r}{R}$.

3. If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the incircle, then prove that $\dfrac{\alpha\beta y}{\alpha + \beta + y} = r^2$

9. Radius of The Incircle :

(i) $r = \dfrac{\Delta}{s}$

(ii) $r = (s - a)\tan\dfrac{A}{2} = (s - b)\tan\dfrac{B}{2} = (s - c)\tan\dfrac{C}{2}$

(iii) $r = \dfrac{a \sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(iv) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

10. Radius of The Ex- Circles :

(i) $r_1 = \dfrac{\Delta}{s - a}$; $r_2 = \dfrac{\Delta}{s - b}$; $r_3 = \dfrac{\Delta}{s - c}$

(ii) $r_1 = s\tan\dfrac{A}{2}$; $r_2 = s\tan\dfrac{B}{2}$; $r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(iv) $r_1 = 4R\sin\dfrac{A}{2}\,.\cos\dfrac{B}{2}\,.\cos\dfrac{C}{2}$

Example : In a ΔABC, prove that $r_1 + r_2 + r_3 - r = 4R = 2a\,\text{cosec}A$

Solution. $\because$ L.H.S $= r_1 + r_2 + r_3 - r$

$= \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} - \dfrac{\Delta}{s}$

$= \Delta\left(\dfrac{1}{s-a} + \dfrac{1}{s-b}\right) + \Delta\left(\dfrac{1}{s-c} - \dfrac{1}{s}\right)$

$= \Delta\left[\left(\dfrac{s-b+s-a}{(s-a)(s-b)}\right) + \left(\dfrac{s-s+c}{s(s-c)}\right)\right]$

$= \Delta\left[\dfrac{c}{(s-a)(s-b)} + \dfrac{c}{s(s-c)}\right]$

$= c\Delta\left[\dfrac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$

$= c\Delta\left[\dfrac{2s^2 - s(a+b+c) + ab}{\Delta^2}\right]$ $\quad\because\quad a + b + c = 2s$

$= \dfrac{abc}{\Delta}$ $\quad\because\quad R = \dfrac{abc}{4\Delta}$

$= 4R = 2a\text{cosec}A$ $\quad\because\quad \dfrac{a}{\sin A} = 2R = a\text{cosec}A$

$=$ R.H.S.

Hence L.H.S. $=$ R.H.S. **proved**

Example : If the area of a ΔABC is 96 sq. unit and the radius of the escribed circles are respectively 8, 12 and 24. Find the perimeter of ΔABC.

Solution. $\because$ $\Delta = 96$ sq. unit

$r_1 = 8$, $r_2 = 12$ and $r_3 = 24$

$\because$ $r_1 = \dfrac{\Delta}{s-a}$ $\Rightarrow$ $s - a = 12$ ....(i)

$\because$ $r_2 = \dfrac{\Delta}{s-b}$ $\Rightarrow$ $s - b = 8$ ....(ii)

$\because$ $r_3 = \dfrac{\Delta}{s-c}$ $\Rightarrow$ $s - c = 4$ ....(iii)

$\therefore$ adding equations (i), (ii) & (iii), we get

$3s - (a + b + c) = 24$

$s = 24$

$\therefore$ perimeter of $\Delta ABC = 2s = 48$ unit.

Self Practice Problems

In a ΔABC prove that

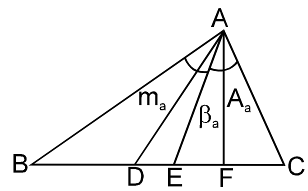
1. $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

2. $rr_1 + rr_2 + rr_3 = ab + bc + ca - s^2$

3. If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC, then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

4. $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.

11. Length of Angle Bisectors, Medians & Altitudes :



(i) Length of an angle bisector from the angle A = $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$;

(ii) Length of median from the angle A = $m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

& (iii) Length of altitude from the angle A = $A_a = \dfrac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$

Example : AD is a median of the ΔABC. If AE and AF are medians of the triangles ABD and ADC respectively, and $AD = m_1$, $AE = m_2$, $AF = m_3$, then prove that $m_2^2 + m_3^2 - 2m_1^2 = \dfrac{a^2}{8}$.

Solution. $\because$ In ΔABC

$AD^2 = \dfrac{1}{4}(2b^2 + 2c^2 - a^2) = m_1^2$ ....(i)

$\because$ In ΔABD, $AE^2 = m_2^2 = \dfrac{1}{4}\left(2c^2 + 2AD^2 - \dfrac{a^2}{4}\right)$ ....(ii)

Similarly in ΔADC, $AF^2 = m_3^2 = \dfrac{1}{4}\left(2AD^2 + 2b^2 - \dfrac{a^2}{4}\right)$ ....(iii)

by adding equations (ii) and (iii), we get

$\because$ $m_2^2 + m_3^2 = \dfrac{1}{4}\left(4AD^2 + 2b^2 + 2c^2 - \dfrac{a^2}{2}\right)$

$= AD^2 + \dfrac{1}{4}\left(2b^2 + 2c^2 - \dfrac{a^2}{2}\right)$

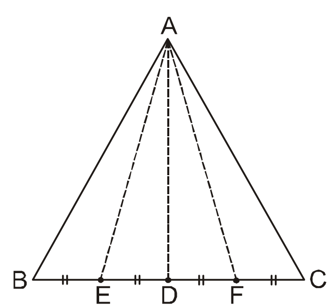
$= AD^2 + \dfrac{1}{4}\left(2b^2 + 2c^2 - a^2 + \dfrac{a^2}{2}\right)$

$= AD^2 + \dfrac{1}{4}(2b^2 + 2c^2 - a^2) + \dfrac{a^2}{8}$

$= AD^2 + AD^2 + \dfrac{a^2}{8}$

$= 2AD^2 + \dfrac{a^2}{8}$ $\because$ $AD^2 = m_1^2$

$= 2m_1^2 + \dfrac{a^2}{8}$



$\therefore$ $m_2^2 + m_3^2 - 2m_1^2 = \dfrac{a^2}{8}$ **Hence Proved**

Self Practice Problem :

3. In a ΔABC a = 5, b = 4, c = 3. 'G' is the centroid of triangle, then find circumradius of ΔGAB.

Ans. $\frac{5}{12}\sqrt{13}$

12. The Distances of The Special Points from Vertices and Sides of Triangle:

(i) Circumcentre (O) : $OA = R$ & $O_a = R\cos A$

(ii) Incentre (I) : $IA = r\,\text{cosec}\frac{A}{2}$ & $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\,\text{cosec}\frac{A}{2}$ & $I_{1a} = r_1$

(iv) Orthocentre (H) : $HA = 2R\cos A$ & $H_a = 2R\cos B\cos C$

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ & $G_a = \frac{2\Delta}{3a}$

Example : If x, y and z are respectively the distances of the vertices of the ΔABC from its orthocentre, then prove that

(i) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$ (ii) $x\ \ y+z=2(R+r)$

Solution. $\because$ $x = 2R\cos A,\ y = 2R\cos B,\ z = 2R\cos C$ and

and $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$

$\therefore$ $\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \tan A+\tan B+\tan C$ ....(i)

& $\frac{abc}{xyz} = \tan A.\ \tan B.\ \tan C$ ....(ii)

$\because$ We know that in a ΔABC **$\Sigma\tan A = \Pi\tan A$**

$\therefore$ From equations (i) and (ii), we get

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

$\because$ $x + y + z = 2R(\cos A+\cos B+\cos C)$

$\because$ in a ΔABC $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $x+y+z = 2R\left(1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right)$

$= 2\left(R+4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right)$ $\because$ $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $x+y+z = 2(R+r)$

Self Practice Problems

1. If I be the incentre of ΔABC, then prove that IA . IB . IC = abc $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

2. If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

13. Orthocentre and Pedal Triangle:

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are $a\cos A = R\sin 2A$,
$b\cos B = R\sin 2B$ and
$c\cos C = R\sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

14. Excentral Triangle:

The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle.

(i) Δ ABC is the pedal triangle of the $\Delta\ I_1I_2I_3$. (ii) Its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ & $\frac{\pi}{2}-\frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ & $4R\cos\frac{C}{2}$.

(iv) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I of Δ ABC is the orthocentre of the excentral $\Delta\, I_1I_2I_3$.

15. Distance Between Special Points :

(i) Distance between circumcentre and orthocentre

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$$

(ii) Distance between circumcentre and incentre

$$OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$$

(iii) Distance between circumcentre and centroid

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

Example : In I is the incentre and I_1, I_2, I_3 are the centres of escribed circles of the ΔABC, prove that
(i) $II_1 . II_2 . II_3 = 16R^2r$ (ii) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

Solution. (i) $\because$ We know that

$$II_1 = a\sec\frac{A}{2},\ II_2 = b\sec\frac{B}{2} \quad \text{and} \quad II_3 = c\sec\frac{C}{2}$$

$\because$ $I_1I_2 = c.\ \mathrm{cosec}\frac{C}{2}$, $I_2I_3 = a\ \mathrm{cosec}\frac{A}{2}$ and $I_3I_1 = b\ \mathrm{cosec}\frac{B}{2}$

$\because$ $II_1 . II_2 . II_3 = abc\sec\frac{A}{2}\sec\frac{B}{2}.\sec\frac{C}{2}$ ....(i)

$\because$ $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$

$\therefore$ equation (i) becomes

$\because$ $II_1 . II_2 . II_3 = (2R\sin A)(2R\sin B)(2R\sin C)\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$

$$= 8R^3 . \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$$

$= 64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ $\quad\because\quad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $II_1 . II_2 . II_3 = 16R^2r$ **Hence Proved**

(ii) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

$\because$ $II_1^2 + I_2I_3^2 = a^2\sec^2\frac{A}{2} + a^2\mathrm{cosec}^2\frac{A}{2} = \dfrac{a^2}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$

$\because$ $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$ $\quad\therefore\quad II_1^2 + I_2I_3^2 = \dfrac{16R^2\sin^2\frac{A}{2}.\cos^2\frac{A}{2}}{\sin^2\frac{A}{2}.\cos^2\frac{A}{2}} = 16R^2$

Similarly we can prove $II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 = 16R^2$

Hence $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

Self Practice Problem :

1. In a ΔABC, if b = 2 cm, c = $\sqrt{3}$ cm and $\angle A = \frac{\pi}{6}$, then find distance between its circumcentre and incentre.

Ans. $\sqrt{2-\sqrt{3}}$ cm