SHORT REVISION

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE DEFINITIONS:

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **Differential Equation**.

2. A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be **Partial** if there are two or more independent variables. We are concerned with ordinary differential equations only. eg. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation.

3. Finding the unknown function is called **Solving Or Integrating** the differential equation. The solution of the differential equation is also called its **Primitive**, because the differential equation can be regarded as a relation derived from it.

4. The order of a differential equation is the order of the highest differential coefficient occuring in it.

5. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ... = 0 \text{ is order } m \text{ \& degree } p.$$

Note that in the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't apply.

6. FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

☞ Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

☞ Eliminate the arbitrary constants.

The eliminant is the required differential equation. Consider forming a differential equation for $y^2 = 4a(x + b)$ where a and b are arbitary constant.

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

7. GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **General Solution (Or Complete Integral Or Complete Primitive)**. A solution obtainable from the general solution by giving particular values to the constants is called a **Particular Solution**.

Note that the general solution of a differential equation of the n^{th} order contains 'n' & only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = Ae^{x+B}$ are not independent since the equation can be written as $y = Ae^B. e^x = C e^x$. Similarly the solution $y = A\sin x + B\cos(x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

8. Elementary Types Of First Order & First Degree Differential Equations .

TYPE–1. VARIABLES SEPARABLE : If the differential equation can be expressed as ; $f(x)dx + g(y)dy = 0$ then this is said to be variable – separable type.

A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$; where c is the arbitrary constant . consider the example $(dy/dx) = e^{x-y} + x^2. e^{-y}$.

Note : Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.
If $x = r\cos\theta$; $y = r\sin\theta$ then,
(i) $x\,dx + y\,dy = r\,dr$ **(ii)** $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ **(iii)** $x\,dy - y\,dx = r^2 d\theta$

If $x = r\sec\theta$ & $y = r\tan\theta$ then $x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

TYPE–2 : $\frac{dy}{dx} = f(ax + by + c)$, $b \neq 0$.

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x + y)^2 \frac{dy}{dx} = a^2$.

TYPE–3. HOMOGENEOUS EQUATIONS :

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where $f(x, y)$ & $\phi(x, y)$ are homogeneous functions of x & y, and of the same degree, is called **HOMOGENEOUS** . This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable.

Consider $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$.

TYPE–4. EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM :

If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1b_2 - a_2b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3. If

(i) $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1x + b_1y$ transforms the differential equation to an equation with variables separable. and

(ii) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting d(xy) for $x\,dy + y\,dx$ & integrating term by term yields the result easily.

Consider $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$; $\frac{dy}{dx} = \frac{2x + 3y - 1}{4x + 6y - 5}$ & $\frac{dy}{dx} = \frac{2x - y + 1}{6x - 5y + 4}$

(iii) In an equation of the form : $yf(xy)\,dx + xg(xy)dy = 0$ the variables can be separated by the substitution $xy = v$.

IMPORTANT NOTE :

(a) The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number $t\,(\neq 0)$, we have $f(tx, ty) = t^n f(x, y)$.
For e.g. $f(x, y) = ax^{2/3} + hx^{1/3} . y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3 .

(b) A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous if $f(x, y)$ is a homogeneous function of degree zero i.e. $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x & y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

LINEAR DIFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together .

The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_n(x) . y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differental equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

TYPE – 5. LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x . To solve such an equation multiply both sides by $e^{\int P dx}$.

NOTE : (1) The factor $e^{\int P\,dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I. F.

(2) It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of y and the I. F.

(3) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ;

$(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential equation.

TYPE–6. EQUATIONS REDUCIBLE TO LINEAR FORM :

The equation $\frac{dy}{dx} + py = Q\,.\,y^n$ where P & Q functions of x, is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in **Type–5**. Consider the example $(x^3y^2 + xy)\,dx = dy$.

The equation $\frac{dy}{dx} + Py = Q\,.\,y^n$ is called **BERNOULI's EQUATION**.

9. TRAJECTORIES :

Suppose we are given the family of plane curves.

$\Phi\,(x, y, a) = 0$

depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family ; if in particular $\alpha = \pi/2$, then it is called an ***orthogonal trajectory***.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it be of the form

$F\,(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x,\ y,\ -\frac{1}{y'}\right) = 0$$

The general integral of this equation

$\Phi_1\,(x, y, C) = 0$

gives the family of orthogonal trajectories.

Note : Following exact differentials must be remembered :

(i) $x\,dy + y\,dx = d(xy)$ **(ii)** $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$ **(iii)** $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(iv) $\frac{x\,dy + y\,dx}{xy} = d(\ln xy)$ **(v)** $\frac{dx + dy}{x + y} = d(\ln(x+y))$ **(vi)** $\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$

(vii) $\frac{y\,dx - x\,dy}{xy} = d\left(\ln\frac{x}{y}\right)$ **(viii)** $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ **(ix)** $\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

(x) $\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$ **(xi)** $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$ **(xii)** $d\left(\frac{e^x}{y}\right) = \frac{y\,e^x\,dx - e^x\,dy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right) = \frac{x\,e^y\,dy - e^y\,dx}{x^2}$

EXERCISE–I

[FORMATION & TYPE – 1 & TYPE – 2]

Q.1 State the order and degree of the following differential equations :

(i) $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$ (ii) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Q.2 Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where a & b are arbitary constants.

Q.3 Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitary constants.

Q.4 Form the differential equation of the family of curves represented by, $c\,(y + c)^2 = x^3$; where c is any arbitrary constant.

Q.5 $\dfrac{\ell n\,(\sec x + \tan x)}{\cos x}\,dx = \dfrac{\ell n\,(\sec y + \tan y)}{\cos y}\,dy$ Q.6 $(1 - x^2)\,(1 - y)\,dx = xy\,(1 + y)\,dy$

Q.7 $\dfrac{dy}{dx} + \dfrac{\sqrt{(x^2 - 1)(y^2 - 1)}}{xy} = 0$ Q.8 $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

Q.9 $\dfrac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{1 + x^2 - y^2}{x^2 - y^2}}$ Q.10 $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$ Q.11 $\dfrac{dy}{dx} = \dfrac{x\,(2\ln x + 1)}{\sin y + y\cos y}$

Q.12 It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at $t = 0$, the mass of the radius was m_0 and during time t_0 α % of the original mass of radium decay.

Q.13 $\dfrac{dy}{dx} + \sin\dfrac{x + y}{2} = \sin\dfrac{x - y}{2}$ Q.14 $\sin x \,.\, \dfrac{dy}{dx} = y \,.\, \ell ny$ if $y = e$, when $x = \dfrac{\pi}{2}$

Q.15 $e^{(dy/dx)} = x + 1$ given that when $x = 0$, $y = 3$

Q.16 A normal is drawn at a point $P(x , y)$ of a curve. It meets the x– axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\dfrac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through (0, k).

Q.17 Find the curve for which the sum of the lengths of the tangent and subtangent at any of its point is proportional to the product of the co–ordinates of the point of tangency, the proportionality factor is equal to k.

Q.18 Obtain the differential equation associated with the primitive,
$y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^{x}$, where c_1 , c_2 , c_3 are arbitrary constants .

Q.19 A curve is such that the length of the polar radius of any point on the curve is equal to the length of the tangent drawn at this point . Form the differential equation and solve it to find the equation of the curve.

Q.20 Find the curve $y = f(x)$ where $f(x) \geq 0$, $f(0) = 0$, bounding a curvilinear trapezoid with the base [0, x] whose area is proportional to $(n + 1)^{th}$ power of $f(x)$. It is known that $f(1) = 1$.

EXERCISE–II

[TYPE–3 & TYPE–4]

Q.1 $\dfrac{dy}{dx} = \dfrac{x^2 + xy}{x^2 + y^2}$

Q.2 Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

Q.3 The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x–axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.

Q.4 The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1).

Q.5 Find the equation of the curve intersecting with the x- axis at the point x = 1 and for which the length of the subnormal at any point of the curve is equal to the arthemetic mean of the co-ordinates of this point $(y - x)^2(x + 2y) = 1$.

Q.6 Use the substitution $y^2 = a - x$ to reduce the equation $y^3 . \dfrac{dy}{dx} + x + y^2 = 0$ to homogeneous form and hence solve it.

Q.7 Find the isogonal trajectories for the family of rectangular hyperbolas $x^2 - y^2 = a^2$ which makes with it an angle of 45°.

Q.8 $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$

Q.9 Show that every homogeneous differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{g(x, y)}$ where f and g are homogeneous function of the same degree can be converted into variable separable by the substitution
$x = r\cos\theta$ and $y = r\sin\theta$.

Q.10 $\left[x\cos\dfrac{y}{x} + y\sin\dfrac{y}{x}\right] y - \left[y\sin\dfrac{y}{x} - x\cos\dfrac{y}{x}\right] x\dfrac{dy}{dx} = 0$

Q.11 Find the curve for which any tangent intersects the y–axis at the point equidistant from the point of tangency and the origin.

Q.12 $(x - y)\,dy = (x + y + 1)\,dx$ Q.13 $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$ Q.14 $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

Q.15 $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$ Q.16 $\frac{dy}{dx} = \frac{2(y+2)^2}{(x + y - 1)^2}$ Q.17 $\frac{dy}{dx} + \frac{\cos x\,(3\cos y - 7\sin x - 3)}{\sin y\,(3\sin x - 7\cos y + 7)} = 0$

Q.18 Show that $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ represents a hyperbola having an asymptotes, $x + y = 0$ & $2x + y + 1 = 0$.

Q.19 If the normal drawn to a curve at any point P intersects the x-axis at G and the perpendicular from P on the x-axis meets at N, such that the sum of the lengths of PG and NG is proportional to the abscissa of the point P, the constant of proportionality being k. Form the differential equation and solve it to show that the equation of the curve is, $y^2 = c x^{\frac{1}{k}} - \frac{k^2 x^2}{2k - 1}$ or $y^2 = \frac{k^2 x^2}{2k + 1} - c x^{-\frac{1}{k}}$, where c is any arbitrary constant.

Q.20 Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point,

$$\sqrt{x^2 + y^2} = c\,e^{\pm \tan^{-1}\frac{y}{x}}$$

EXERCISE–III

[TYPE–5 & TYPE–6]

Q.1 $(x + \tan y)\,dy = \sin 2y\,dx$

Q.2 Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which pass through the origin is $y = 2(e^x - x - 1)$.

Q.3 $\frac{dy}{dx} + \frac{x}{1 + x^2}\,y = \frac{1}{2x(1 + x^2)}$ Q.4 $(1 - x^2)\frac{dy}{dx} + 2xy = x(1 - x^2)^{1/2}$

Q.5 Find the curve such that the area of the trapezium formed by the co–ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa .

Q.6 $x(x - 1)\frac{dy}{dx} - (x - 2)\,y = x^3(2x - 1)$ Q.7 $(1 + y + x^2 y)\,dx + (x + x^3)\,dy = 0$

Q.8 Find the curve possessing the property that the intercept, the tangent at any point of a curve cuts off on the y–axis is equal to the square of the abscissa of the point of tangency.

Q.9 $\sin x\,\frac{dy}{dx} + 3y = \cos x$ Q.10 $x(x^2 + 1)\frac{dy}{dx} = y(1 - x^2) + x^3 . \ln x$

Q.11 $x\frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$ Q.12 $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

Q.13 Find the curve such that the area of the rectangle constructed on the abscissa of any point and the initial ordinate of the tangent at this point is equal to a^2. (Initial ordinate means y intercept of the tangent).

Q.14 Let the function $\ln f(x)$ is defined where $f(x)$ exists for $x \ge 2$ & k is fixed positive real number, prove that if $\frac{d}{dx}(x . f(x)) \le -k f(x)$ then $f(x) \le A x^{-1-k}$ where A is independent of x.

Q.15 Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t - x)\,dt$ where $x \in (-\pi/2, \pi/2)$

Q.16 $y - x\,Dy = b(1 + x^2 Dy)$

Q.17 Integrate $(1 + x^2)\frac{dy}{dx} + 2yx - 4x^2 = 0$ and obtain the cubic curve satisfying this equation and passing through the origin.

Q.18 If y_1 & y_2 be solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x alone, and $y_2 = y_1 z$, then prove that

$$z = 1 + a\,e^{-\int \frac{Q}{y_1}dx}, \text{ 'a' being an arbitrary constant .}$$

Q.19 $\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$ Q.20 $\frac{dy}{dx} + xy = y^2 e^{x^2/2} . \sin x$

Q.21 $2\frac{dy}{dx} - y\sec x = y^3 \tan x$ Q.22 $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$

Q.23 $y(2xy + e^x)\,dx - e^x\,dy = 0$

Q.24 Find the curve for which the area of the triangle formed by the x–axis, the tangent line and radius vector of the point of tangency is equal to a^2.

Q.25 A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min, and the mixture is pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

EXERCISE–IV

(GENERAL – CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION)

Q 1. $(x - y^2)\,dx + 2xy\,dy = 0$

Q 2. $(x^3 + y^2 + 2)\,dx + 2y\,dy = 0$

Q 3. $x\frac{dy}{dx} + y\ln y = xye^x$

Q 4. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)\,e^x \sec y$

Q 5. $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$

Q 6. $\left(\frac{dy}{dx}\right)^2 - (x+y)\frac{dy}{dx} + xy = 0$

Q 7. $\frac{dy}{dx} = \frac{y^2 - x}{2y\,(x+1)}$

Q 8. $(1 - xy + x^2y^2)\,dx = x^2\,dy$

Q 9. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Q 10. $y\,y'\sin x = \cos x\,(\sin x - y^2)$

EXERCISE–V (MISCELLANEOUS)

Q.1 $\frac{dy}{dx} - y\,ln\,2 = 2^{\sin x}.(\cos x - 1)\,ln\,2$, y being bounded when $x \to +\infty$.

Q.2 $\frac{dy}{dx} = y + \int_0^1 y\,dx$ given $y = 1$, where $x = 0$

Q.3 Given two curves $y = f(x)$ passing through the points (0, 1) & $y = \int_{-\infty}^{x} f(t)\,dt$ passing through the points (0, 1/2). The tangents drawn to both curves at the points with equal abscissas intersect on the x – axis. Find the curve f(x).

Q.4 Consider the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$

(i) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in terms of u(x) and v(x).

(ii) If α and β are constants such that the linear combinations $\alpha \cdot u(x) + \beta \cdot v(x)$ is a solution of the given equation, find the relation between α and β.

(iii) If w(x) is the third particular solution different from u(x) and v(x) then find the ratio $\frac{v(x) - u(x)}{w(x) - u(x)}$.

Q.5 $x^3\frac{dy}{dx} = y^3 + y^2\sqrt{y^2 - x^2}$

Q.6 Find the curve which passes through the point (2, 0) such that the segment of the tangent between the point of tangency & the y – axis has a constant length equal to 2 .

Q.7 $x\,dy + y\,dx + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$

Q.8 $\frac{y\,dx - x\,dy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$, given that $y = 2$ when $x = 1$

Q.9 Find the equation of the curve passing through the orgin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

Q.10 Find the continuous function which satisfies the relation, $\int_0^x t\,f(x-t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$, for all real number x.

Q.11 $(x^2 + y^2 + a^2)\,y\frac{dy}{dx} + x\,(x^2 + y^2 - a^2) = 0$

Q.12 $(1 - x^2)^2\,dy + \left(y\sqrt{1-x^2} - x - \sqrt{1-x^2}\right)dx = 0$

Q.13 $3x^2y^2 + \cos(xy) - xy\sin(xy) + \frac{dy}{dx}\{2x^3y - x^2\sin(xy)\} = 0$.

Q.14 Find the integral curve of the differential equation, $x\,(1 - x\,ln\,y).\frac{dy}{dx} + y = 0$ which passes through $\left(1, \frac{1}{e}\right)$.

Q.15 Find all the curves possessing the following property ; the segment of the tangent between the point of tangency & the x-axis is bisected at the point of intersection with the y-axis.

Q.16 $y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy) = 0$

Q.17 A perpendicular drawn from any point P of the curve on the x–axis meets the x–axis at A. Length of the perpendicular from A on the tangent line at P is equal to 'a'. If this curve cuts the y-axis orthogonally, find the equation to all possible curves, expressing the answer explicitly.

Q.18 Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.
(i) $y = ax^2$ (ii) $\cos y = a\,e^{-x}$ (iii) $x^k + y^k = a^k$

Q.19 A curve passing through (1, 0) such that the ratio of the square of the intercept cut by any tangent off the y-axis to the subnormal is equal to the ratio of the product of the co-ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

Q.20 A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water ?

Q.21 $(2x^2 + 3y^2 - 7)x\,dx = (3x^2 + 2y^2 - 8)y\,dy$

Q.22 Find the curve such that the segment of the tangent at any point contained between the x-axis and the straight line $y = ax + b$ is bisected by the point of tangency.

Q.23 Find the curve such that the ratio of the distance between the normal at any of its point and the origin to the distance between the same normal and the point (a, b) is equal to the constant k. Interpret the curve. (k > 0)

Q.24 Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, $y = mx$ then prove that the slopes of these two curves at P_1 and P_2 are equal.

Q.25 Find the curve for which the portion of y-axis cut-off between the origin and the tangent varies as cube of the absissa of the point of contact.

EXERCISE–VI

(PROBLEMS ASKED IN JEE & REE)

Q.1 Determine the equation of the curve passing through the origin in the form $y = f(x)$, which satisfies the differential equation $\frac{dy}{dx} = \sin(10x + 6y)$. [JEE '96 , 5]

Q.2 Solve the differential equation ; $\cos^2 x\,\frac{dy}{dx} - (\tan 2x)\,y = \cos^4 x$, $|x| < \frac{\pi}{4}$, when $y(\pi/6) = 3\sqrt{3}/8$.

Q.3 Solve the diff. equation ; $y\cos\frac{y}{x}(x\,dy - y\,dx) + x\sin\frac{y}{x}(x\,dy + y\,dx) = 0$, when $y(1) = \frac{\pi}{2}$

Q.4 Let u(x) & v(x) satisfy the differential equations $\frac{du}{dx} + p(x)\,u = f(x)$ & $\frac{dv}{dx} + p(x)\,v = g(x)$ where p(x), f(x) & g(x) are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$, prove that any point (x, y) where $x > x_1$ does not satisfy the equations $y = u(x)$ & $y = v(x)$. [JEE '97 , 5]

Q.5(i) The order of the differential equation whose general solution is given by
$y = (C_1 + C_2)\cos(x + C_3) - C_4\,e^{x + C_5}$ where C_1, C_2, C_3, C_4, C_5 are arbitrary constants, is
(A) 5 (B) 4 (C) 3 (D) 2

(ii) A curve C has the property that if the tangent drawn at any point P on C meets the coordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve.

Q.6 Solve the differential equation $(1 + \tan y)(dx - dy) + 2x\,dy = 0$ [REE '98 , 6]

Q.7(a) A soluton of the differential equation, $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is :
(A) y = 2 (B) y = 2x (C) y = 2x – 4 (D) $y = 2x^2 - 4$

(b) The differential equation representing the family of curves, $y^2 = 2c\left(x + \sqrt{c}\right)$, where c is a positive parameter, is of :
(A) order 1 (B) order 2 (C) degree 3 (D) degree 4

(c) A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x–axis. Determine the equation of the curve. [JEE '99, 2 + 3 + 10, out of 200]

Q.8 Solve the differential equation, $(x^2 + 4y^2 + 4xy)\, dy = (2x + 4y + 1)\, dx$. [REE '99, 6]

Q.9 A country has a food deficit of 10 % . Its population grows continuously at a rate of 3 %. Its annual food production every year is 4 % more than that of the last year . Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after 'n' years, where 'n' is the smallest integer bigger than or equal to ,

$$\frac{\ell n 10 - \ell n 9}{\ell n (1.04) - 0.03} .$$ [JEE '2000 (Mains) 10]

Q.10 A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm^2 cross sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $V(t) = 0.6\sqrt{2gh(t)}$, where V(t) and h(t) are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank. [JEE '2001 (Mains) 10]

Q.11 Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to $\frac{x^4 + 2xy - 1}{1 + x^2}$. [REE '2001 (Mains) 3]

Q.12 Let f(x), $x \geq 0$, be a nonnegative continuous function, and let $F(x) = \int_0^x f(t)dt$, $x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$. [JEE 2001 (Mains) 5 out of 100]

Q.13(a) A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0). Find the time after which the cone is empty.

(b) If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$. [JEE 2003, (Mains) 4 + 4]

Q.14(a) If $\left(\frac{2 + \sin x}{1 + y}\right)\frac{dy}{dx} = -\cos x$, $y(0) = 1$, then $y\left(\frac{\pi}{2}\right) =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ [JEE 2004 (Scr.)]

(b) A curve passes through (2, 0) and the slope of tangent at point P (x, y) equals $\frac{(x+1)^2 + y - 3}{(x+1)}$. Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant.

Q.15(a) The solution of primitive integral equation $(x^2 + y^2)dy = xy\, dx$, is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is

(A) $\sqrt{2(e^2 - 1)}$ (B) $\sqrt{2(e^2 + 1)}$ (C) $\sqrt{3}\, e$ (D) $\sqrt{\frac{e^2 + 1}{2}}$

(b) For the primitive integral equation $ydx + y^2dy = xdy$; $x \in R$, $y > 0$, $y = y(x)$, $y(1) = 1$, then $y(-3)$ is

(A) 3 (B) 2 (C) 1 (D) 5 [JEE 2005 (Scr.)]

(c) If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the x–axis is of length 1. Find the equation of the curve. [JEE 2005 (Mains)]

Q.16 A tangent drawn to the curve, $y = f(x)$ at P(x, y) cuts the x-axis at A and B respectively such that BP : AP = 3 : 1, given that $f(1) = 1$, then

(A) equation of the curve is $x\frac{dy}{dx} - 3y = 0$ (B) equation of curve is $x\frac{dy}{dx} + 3y = 0$

(C) curve passes through (2, 1/8) (C) normal at (1, 1) is $x + 3y = 4$ [JEE 2006, 5]

ANSWER KEY

EXERCISE–I

Q 1. (i) order 2 & degree 3 **(ii)** order 2 & degree 2

Q 2. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

Q 3. $[1 + (y')^2]\,.y''' - 3y'(y'')^2 = 0$

Q 4. $12y\,(y')^2 = x[8(y')^3 - 27]$

Q 5. $ln^2(\sec x + \tan x) - ln^2(\sec y + \tan y) = c$

Q 6. $ln\, x\,(1 - y)^2 = c - \frac{1}{2}y^2 - 2y + \frac{1}{2}x^2$

Q 7. $\sqrt{x^2-1} - \sec^{-1}x + \sqrt{y^2-1} = c$

Q 8. $y = c\,(1 - ay)\,(x + a)$

Q 9. $\sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = \dfrac{c\,(x+y)}{\sqrt{x^2-y^2}}$

Q 10. $\ln\left[1 + \tan\dfrac{x+y}{2}\right] = x + c$

Q 11. $y \sin y = x^2 \ln x + c$

Q 12. $m = m_0\, e^{-kt}$ where $k = -\dfrac{1}{t_0}\ln\left(1 - \dfrac{\alpha}{100}\right)$

Q 13. $\ln\left|\tan\dfrac{y}{4}\right| = c - 2\sin\dfrac{x}{2}$

Q.14 $y = e^{\tan(x/2)}$

Q 15. $y = (x+1)\,.\,\ln(x+1) - x + 3$

Q 16. $x^2 + y^2 = k^2$

Q 17. $y = \dfrac{1}{k}\,\ell n\left|c\left(k^2x^2 - 1\right)\right|$

Q 18. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

Q 19. $y = kx$ or $xy = c$

Q 20. $y = x^{1/n}$

EXERCISE–II

Q.1 $c\,(x-y)^{2/3}\,(x^2+xy+y^2)^{1/6} = \exp\left[\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x+2y}{x\sqrt{3}}\right]$ where $\exp x \equiv e^x$

Q.2 $\dfrac{y^2 \pm y\sqrt{y^2-x^2}}{x^2} = \ell n\left|\left(y \pm \sqrt{y^2-x^2}\right)\cdot\dfrac{c^2}{x^3}\right|$, where same sign has to be taken.

Q 4. $x^2 + y^2 - 2x = 0$

Q 5. $(x-y)^2\,(x+2y) = 1$

Q 6. $\dfrac{1}{2}\,ln\left|x^2 + a^2\right| - \tan^{-1}\left(\dfrac{a}{x}\right) = c$, where $a = x + y^2$

Q 7. $x^2 - y^2 + 2xy = c\ ;\ x^2 - y^2 - 2xy = c$

Q 8. $y^2 - x^2 = c\,(y^2 + x^2)^2$

Q 10. $xy\cos\dfrac{y}{x} = c$

Q 11. $x^2 + y^2 = cx$

Q 12. $\arctan\dfrac{2y+1}{2x+1} = ln\,c\sqrt{x^2+y^2+x+y+\dfrac{1}{2}}$

Q 13. $(x+y-2) = c\,(y-x)^3$

Q 14. $\tan^{-1}\dfrac{y+3}{x+2} + ln\,c\sqrt{(y+3)^2+(x+2)^2} = 0$

Q 15. $x + y + \dfrac{4}{3} = ce^{3(x-2y)}$

Q 16. $e^{-2\tan^{-1}\frac{y+2}{x-3}} = c\,.\,(y+2)$

Q 17. $(\cos y - \sin x - 1)^2\ (\cos y + \sin x - 1)^5 = c$

EXERCISE–III

Q 1. $x\sqrt{\cot y} = c + \sqrt{\tan y}$

Q 2. $y = 2\,(e^x - x - 1)$

Q 3. $y\sqrt{1+x^2} = c + \dfrac{1}{2}ln\left[\tan\dfrac{1}{2}\arctan x\right]$ Another form is $y\sqrt{1+x^2} = c + \dfrac{1}{2}\,ln\,\dfrac{\sqrt{1+x^2}-1}{x}$

Q 4. $y = c\,(1-x^2) + \sqrt{1-x^2}$ **Q 5.** $y = cx^2 \pm x$ **Q 6.** $y\,(x-1) = x^2\,(x^2 - x + c)$ **Q 7.** $xy = c - \text{arc}\tan x$

Q 8. $y = cx - x^2$

Q 9. $\left(\dfrac{1}{3} + y\right)\tan^3\dfrac{x}{2} = c + 2\tan\dfrac{x}{2} - x$

Q 10. $4\,(x^2+1)\,y + x^3\,(1 - 2\,ln x) = cx$

Q 11. $y = cx + x\,ln\tan x$

Q 12. $x = ce^{-\arctan y} + \text{arc}\tan y - 1$

Q 13. $y = cx \pm \dfrac{a^2}{2x}$

Q.15 $\cos x - 1$ **Q 16.** $y(1+bx) = b + cx$ **Q 17.** $3y\,(1+x^2) = 4x^3$ **Q 19.** $x = lny\left(cx^2 + \dfrac{1}{2}\right)$

Q 20. $e^{-x^2/2} = y\,(c + \cos x)$

Q 21. $\dfrac{1}{y^2} = -1 + (c+x)\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$

Q 22. $x^3y^{-3} = 3\,\text{Sin}x + c$ **Q 23.** $y^{-1}e^x = c - x^2$ **Q 24.** $x = cy \pm \dfrac{a^2}{y}$ **Q 25.** $27\dfrac{7}{9}$ minutes

EXERCISE–IV

Q 1. $y^2 + x\, ln\, ax = 0$ **Q 2.** $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$ **Q 3.** $x\, ln\, y = e^x (x - 1) + c$

Q 4. $\sin y = (e^x + c)(1 + x)$ **Q 5.** $cx^2 + 2xe^{-y} = 1$ **Q 6.** $y = ce^x$; $y = c + \frac{x^2}{2}$

Q 7. $y^2 = -1 + (x+1)\ \ell n \frac{c}{x+1}$ or $x + (x+1)\ ln \frac{c}{x+1}$ **Q 8.** $y = \frac{1}{x}\tan\left(\ell n\, |cx|\right)$

Q 9. $e^y = c \,.\, \exp(-e^x) + e^x - 1$ **Q 10.** $y^2 = \frac{2}{3}\sin x + \frac{c}{\sin^2 x}$

EXERCISE–V

Q.1 $y = 2^{\sin x}$ **Q.2** $y = \frac{1}{3-e}(2e^x - e + 1)$ **Q.3** $f(x) = e^{2x}$

Q.5 (i) $y = u(x) + K(u(x) - v(x))$ where K is any constant ; (ii) $\alpha + \beta = 1$; (iii) constant

Q.5 $xy = c\left(y + \sqrt{y^2 - x^2}\right)$ **Q.6** $y = \pm\left[\sqrt{4 - x^2} + 2\,\ell n \frac{2 - \sqrt{4 - x^2}}{x}\right]$ **Q.7** $xy + \tan^{-1}\frac{y}{x} = c$

Q.8 $\frac{\sin^{-1}x}{2} + \frac{y}{x-y} = \frac{\pi}{4} - 2$ **Q.9** $y^2 = 2x + 1 - e^{2x}$ **Q.10** $f(x) = e^x - \cos x$

Q.11 $(x^2 + y^2)^2 + 2a^2(y^2 - x^2) = c$ **Q.12** $y = \frac{x}{\sqrt{1-x^2}} + c\ e^{-\frac{x}{\sqrt{1-x^2}}}$

Q.13 $x(x^2y^2 + \cos xy) = c$ **Q.14** $x(e\,y + ln\, y + 1) = 1$

Q.15 $y^2 = cx$ **Q.16** $x^2y^2(y^2 - x^2) = c$ **Q.17** $y = \pm a \,.\, \frac{e^{x/a} + e^{-x/a}}{2}$ & $y = \pm a$

Q.18 (i) $x^2 + 2y^2 = c$, (ii) $\sin y = ce^{-x}$, (iii) $y = cx$ if $k = 2$ and $\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{c^{k-2}}$ if $k \neq 2$

Q.19 $x = e^{2\sqrt{y/x}}$; $x = e^{-2\sqrt{y/x}}$ **Q.20** $T = \log_{4/3} 2$ hrs from the start

Q.21 $(x^2 - y^2 - 1)^5 = c(x^2 + y^2 - 3)$ **Q.22** $y^2 - a\,xy - by = c$

Q.23 $(k+1)x^2 + (k+1)y^2 - 2kax - 2kby = c$
or $(k-1)x^2 + (k-1)y^2 - 2kax - 2kby = c$ both represents a circle. **Q.25** $2y + Kx^3 = cx$

EXERCISE–VI

Q.1 $y = \frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4 - 3\tan 4x}\right) - \frac{5x}{3}$ **Q.2** $y = \frac{1}{2}\tan 2x \,.\, \cos^2 x$ **Q.3** $xy\sin\frac{y}{x} = \frac{\pi}{2}$

Q.5 **(i)** C **(ii)** $xy = 1\ (x > 0, y > 0)$ **Q.6** $x\, e^y(\cos y + \sin y) = e^y \sin y + C$

Q.7 **(a)** C **(b)** A, C **(c)** $x^2 + y^2 - 2x = 0$

Q.8 $y = ln\left((x + 2y)^2 + 4(x + 2y) + 2\right) - \frac{3}{2\sqrt{2}}\ ln\left(\frac{x + 2y + 2 - \sqrt{2}}{x + 2y + 2 + \sqrt{2}}\right) + c$

Q.10 $\frac{7\pi \times 10^5}{135\sqrt{g}}$ sec. **Q.11** $y = (x - 2\tan^{-1}x)(1 + x^2)$

Q.13 (a) $T = \frac{H}{k}$ **Q.14** (a) C ; (b) $y = x^2 - 2x$, area $= \frac{4}{3}$ sq. units

Q.15 **(a)** C; **(b)** A; **(c)** $\sqrt{1 - y^2} + ln\left|\frac{1 - \sqrt{1 - y^2}}{y}\right| = \pm x + c$ **Q.16** B, C

EXERCISE–VII

Only one correct option

1. The degree of differential equation satisfying the relation

$$\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right) \text{ is :}$$

(A) 1 (B) 2 (C) 3 (D) none of these

2. If p and q are order and degree of differential equation $y\frac{dy}{dx} + x^3\left(\frac{d^2y}{dx^2}\right) + xy = \cos x$, then

(A) p < q (B) p = q (C) p > q (D) none of these

3. The differential equation for all the straight lines which are at a unit distance from the origin is

(A) $\left(y - x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$ (B) $\left(y + x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

(C) $\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$ (D) $\left(y + x\frac{dy}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2$

4. The differential equation obtained on eliminating A and B from = y = A cos (ωt) + B sin (ωt) is

(A) $y'' + y' = 0$ (B) $y'' - \omega^2 y = 0$ (C) $y'' = -\omega^2 y$ (D) $y'' + y = 0$

5. The differential equation whose solution is $(x - h)^2 + (y - k)^2 = a^2$ is (a is a constant)

(A) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\frac{d^2y}{dx^2}$ (B) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(C) $\left[1+\left(\frac{dy}{dx}\right)\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$ (D) none of these

6. The differential equation of all circles which pass through the origin and whose centres lie on y-axis is

(A) $(x^2 - y^2)\frac{dy}{dx} - 2xy = 0$ (B) $(x^2 - y^2)\frac{dy}{dx} + 2xy = 0$

(C) $(x^2 - y^2)\frac{dy}{dx} - xy = 0$ (D) $(x^2 - y^2)\frac{dy}{dx} + xy = 0$

7. If $\frac{dy}{dx} = e^{-2y}$ and y = 0 when x = 5, the value of x for y = 3 is

(A) e^5 (B) $e^6 + 1$ (C) $\frac{e^6+9}{2}$ (D) $\log_e 6$

8. If $\phi(x) = \phi'(x)$ and $\phi(1) = 2$, then $\phi(3)$ equals

(A) e^2 (B) $2e^2$ (C) $3e^2$ (D) $2e^3$

9. If integrating factor of $x(1 - x^2)\,dy + (2x^2 y - y - ax^3)\,dx = 0$ is $e^{\int p\,.\,dx}$, then P is equal to

(A) $\frac{2x^2 - ax^3}{x(1-x^2)}$ (B) $(2x^2 - 1)$ (C) $\frac{2x^2-1}{ax^3}$ (D) $\frac{(2x^2-1)}{x(1-x^2)}$

10. If $\frac{dy}{dx} = 1 + x + y + xy$ and $y(-1) = 0$, then function y is

(A) $e^{(1-x)^2/2}$ (B) $e^{(1+x)^2/2} - 1$ (C) $\log_e(1 + x) - 1$ (D) $1 + x$

11. Integral curve satisfying $y' = \frac{x^2+y^2}{x^2-y^2}$, y(1) = 2, has the slope at the point (1, 2) of the curve, equal to

(A) $-\frac{5}{3}$ (B) -1 (C) 1 (D) $\frac{5}{3}$

12. The solution of $\frac{dv}{dt} + \frac{k}{m}v = -g$ is

(A) $v = ce^{-\frac{k}{m}t} - \frac{mg}{k}$ (B) $v = c - \frac{mg}{k}e^{-\frac{k}{m}t}$ (C) $v\,e^{-\frac{k}{m}t} = c - \frac{mg}{k}$ (D) $v\,e^{\frac{k}{m}t} = c - \frac{mg}{k}$

13. The solution of the differential equation $\sqrt{a+x}\,\frac{dy}{dx} + xy = 0$ is

(A) $y = Ae^{2/3}(2a - x)\sqrt{x+a}$ (B) $y = Ae^{-2/3}(a - x)\sqrt{x+a}$
(C) $y = Ae^{2/3}(2a + x)\sqrt{x+a}$ (D) $y = Ae^{-2/3}(2a - x)\sqrt{x+a}$
Where A is an arbitrary constant.

14. If $y_1(x)$ and $y_2(x)$ are two solutions of $\frac{dy}{dx} + y(x)\,y = r(x)$ then $y_1(x) + y_2(x)$ is solution of :
(A) $\frac{dy}{dx} + f(x)\,y = 0$ (B) $\frac{dy}{dx} + 2f(x)\,y = r(x)$
(C) $\frac{dy}{dx} + f(x)\,y = 2\,r(x)$ (D) $\frac{dy}{dx} + 2f(x)\,y = 2r(x)$

15. The differential equation of all 'Simple Harmonic Motions' of given period $\frac{2\pi}{n}$ is
(A) $\frac{d^2x}{dt^2} + nx = 0$ (B) $\frac{d^2x}{dt^2} + n^2x = 0$ (C) $\frac{d^2x}{dt^2} - n^2x = 0$ (D) $\frac{d^2x}{dt^2} + \frac{1}{n^2}x = 0.$

16. If $\sqrt{(x^2+y^2)} = ae^{\tan^{-1}(y/x)}$, $a > 0$. Then $y''(0)$, equals
(A) $\frac{a}{2}e^{\pi/2}$ (B) $ae^{\pi/2}$ (C) $-\frac{2}{a}e^{-\pi/2}$ (D) $\frac{a}{2}e^{-\pi/2}$

17. The function $f(\theta) = \frac{d}{d\theta}\int_0^{\theta}\frac{dx}{1-\cos\theta\cos x}$ satisfies the differential equation
(A) $\frac{df}{d\theta} + 2f(\theta)\cot\theta = 0$ (B) $\frac{df}{d\theta} - 2f(\theta)\cot\theta = 0$ (C) $\frac{df}{d\theta} + 2f(\theta) = 0$ (D) $\frac{df}{d\theta} - 2f(\theta) = 0$

18. The solution of the differential equation $y_1 y_3 = 3y_2^2$ is
(A) $x = A_1y^2 + A_2y + A_3$ (B) $x = A_1y + A_2$
(C) $x = A_1y^2 + A_2y$ (D) none of these

19. The solution of $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$ is
(A) $\frac{x}{y} + e^{x^3} = C$ (B) $\frac{x}{y} - e^{x^3} = 0$ (C) $-\frac{x}{y} + e^{x^3} = C$ (D) none of these

20. The solution of the differential equation
$(x^2\sin^3 y - y^2\cos x)\,dx + (x^3\cos y\sin^2 y - 2y\sin x)\,dy = 0$ is
(A) $x^3\sin^3 y = 3y^2\sin x + C$ (B) $x^3\sin^3 y + 3y^2\sin x = C$
(C) $x^2\sin^3 y + y^3\sin x = C$ (D) $2x^2\sin y + y^2\sin x = C$

One or more than one options correct

21. The differential equation of the curve for which the initial ordinate of any tangent is equal to the corresponding subnormal
(A) is linear (B) is homogeneous (C) has separable variables (D) is none of these

22. The solution of $x^2y_1^2 + xy\,y_1 - 6y^2 = 0$ are
(A) $y = Cx^2$ (B) $x^2y = C$ (C) $\frac{1}{2}\log y = C + \log x$ (D) $x^3y = C$

23. The orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2 = a/x$ are
(A) $9a(y + c) = 4x^3$ (B) $y + C = \frac{-2}{3\sqrt{a}}x^{3/2}$ (C) $y + C = \frac{2}{3\sqrt{a}}x^{3/2}$ (D) none of these

24. The solution of $\left(\frac{dy}{dx}\right)(x^2y^3 + xy) = 1$ is
(A) $1/x = 2 - y^2 + Ce^{-y^2}/2$ (B) the solution of an equation which is reducible to linear equation.
(C) $2/x = 1 - y^2 + e^{-y}/2$ (D) $\frac{1-2x}{x} = -y^2 + Ce^{-y^2}/2$

EXERCISE–VIII

1. Solve : $\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1+x^2-y^2}{x^2-y^2}}$

2. Solve :

(a) $x^2\,dy + y(x + y)\,dx = 0$, given that $y = 1$, when $x = 1$

(b) $\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right] y - \left[y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right] x\ \frac{dy}{dx} = 0$

3. Find the equation of the curve satisfying $\frac{dy}{dx} = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$ and passing through (1, –1).

4. Find the solution of the differential equation $\frac{d^3y}{dx^3} = 8\ \frac{d^2y}{dx^2}$ satisfying $y(0) = \frac{1}{8}$, $y_1(0) = 0$ and $y_2(0) = 1$.

5. **Solve :** (i) $(x + 3y^2)\ \frac{dy}{dx} = y,\ y > 0$ (ii) $(1 + y + x^2y)\,dx + (x + x^3)\,dy = 0$

(iii) $\frac{dy}{dx} = y\tan x - 2\sin x$ (iv) $(1 + x^2)\ \frac{dy}{dx} + 2xy = \cos x$

6. Solve : (i) $y(x^2y + e^x)\,dx = e^x\,dy$ (ii) $x\ \frac{dy}{dx} + y = x^2y^4$

(iii) $2y\sin x\,dy + (y^2\cos x + 2x)\,dx = 0$

7. Solve the following differential equations. $3\ \frac{dy}{dx} + \frac{2y}{x+1} = \frac{x^3}{y^2}$

8. Find the curve $y = f(x)$ where $f(x) \ge 0$, $f(0) = 0$, bounding a curvilinear trapezoid with the base $[0, x]$ whose area is propostinal to $(n + 1)^{th}$ power of $f(x)$. It is known that $f(1) = 1$

9. Find the nature of the curve for which the length of the normal at the point P is equal to the radius vector of the point P.

10. A particle, P, starts from origin and moves along positive direction of y-axis. Another particle, Q, follows P i.e. it's velocity is always directed towards P, in such a way that the distance between P and Q remains constant. If Q starts from (2, 0), find the equation of the path traced by Q. Assume that they start moving at the same instant.

11. Let c_1 and c_2 be two integral curves of the differential equation $\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$. A line passing through origin meets c_1 at $P(x_1, y_1)$ and c_2 at $Q(x_2, y_2)$. If $c_1 : y = f(x)$ and $c_2 : y = g(x)$ prove that $f'(x_1) = g'(x_2)$.

12. Find the integral curve of the differential equation $x(1 - xy)\ \frac{dy}{dx} + y = 0$ which passes through (1, 1/e).

13. Show that the integral curves of the equation $(1 - x^2)\ \frac{dy}{dx} + xy = ax$ are ellipses and hyperbolas, with the centres at the point (0, a) and the axes parallel to the co-ordinate axes, each curve having one constant axis whose length is equal to 2.

14. If y_1 & y_2 be solutions of the differential equation $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x alone, and $y_2 = y_1 z$, then prove that $z = 1 + a e^{-\int \frac{Q}{y_1} dx}$, 'a' being an arbitrary constant.

15. Find the curve for which the sum of the lengths of the tangent and subtangent at any of its point is proportional to the product of the co–ordinates of the point of tangency, the proportionality factor is equal to k.

16. Find all the curves possessing the following property; the segment of the tangent between the point of tangency & the x–axis is bisected at the point of intersection with the y–axis .

17. A curve passing through (1 , 0) such that the ratio of the square of the intercept cut by any tangent off the y–axis to the subnormal is equal to the ratio of the product of the co–ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

18. A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water , their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released , the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water ?

19. A curve $y = f(x)$ passes through the point P (1 ,1). The normal to the curve at P is ; $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y–axis, the curve & the normal to the curve at P. **[IIT - 1996, 5]**

20. Let u (x) & v (x) satisfy the differential equations $\frac{du}{dx}$ + p (x) u = f (x) & $\frac{dv}{dx}$ + p (x) v = g (x) where p (x) , f (x) & g (x) are continuous functions. If u (x_1) > v (x_1) for some x_1 and f (x) > g (x) for all x > x_1 , prove that any point (x , y) where x > x_1 does not satisfy the equations y = u (x) & y = v (x).

21. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x – axis. Determine the equation of the curve. **[IIT - 1999, 10]**

22. A country has a food deficit of 10 %. Its population grows continuously at a rate of 3 % per year. Its annual food production every year is 4 % more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self–sufficient in food after ' n ' years , where ' n ' is the smallest integer bigger than or equal to, $\frac{\ell n 10 - \ell n 9}{\ell n (1.04) - 0.03}$.
[IIT - 2000 (Mains) 10]

23. An inverted cone of height H and radius R is pointed at bottom. It is filled with a volatile liquid completely. If the rate of evaporation is directly proportional to the surface area of the liquid in contact with air (constant of proportionality k > 0) , find the time in which whole liquid evaporates.
[IIT - 2003 (Mains) 4]

EXERCISE–VII

1. A	**2.** C	**3.** C	**4.** C
5. B	**6.** A	**7.** C	**8.** B
9. D	**10.** B	**11.** A	**12.** A
13. A	**14.** C	**15.** B	**16.** C
17. A	**18.** A	**19.** A	**20.** A
21. AB	**22.** ACD	**23.** ABC	**24.** ABD

EXERCISE–VIII

1. $\sqrt{x^2 - y^2} + \sqrt{1 + x^2 - y^2} = \frac{c(x+y)}{\sqrt{x^2 - y^2}}$

2. (a) $3x^2y = 2x + y$ (b) $xy \cos\left(\frac{y}{x}\right) = c$

3. x + y = 0 **4.** $64y = (e^{8x} - 8x) + 7$

5. (i) $\frac{x}{y} = 3y + c$ (ii) xy = c – arc tanx
(iii) y = cosx + c secx (iv) $y(1 + x^2) = c + \sin x$.

6. (i) $\frac{1}{y} e^x = -\frac{x^3}{3} + c$ (ii) $\frac{1}{y^3} = 3x^2 + cx^3$
(iii) $y^2 \sin x = -x^2 + c$

7. $y^3 (x + 1)^2 = \frac{x^6}{6} + \frac{2}{5} x^5 + \frac{1}{4} x^4 + c$

8. $y = x^{1/n}$ **9.** Rectangular hyperbola or circle.

10. $y = 2\ell n\, x - 2\ell n\,(2 - \sqrt{4 - x^2}) - \sqrt{4 - x^2}$

12. $x(ey + \ell ny + 1) = 1$ **15.** $y = \frac{1}{k} \ell n \left|c\left(k^2x^2 - 1\right)\right|$

16. $y^2 = cx$ **17.** $x = e^{2\sqrt{y/x}}$; $x = e^{-2\sqrt{y/x}}$

18. $T = \log_{4/3} 2$ hrs from the start

19. $e^{a(x-1)} \frac{1}{a}\left[a - \frac{1}{2} + e^{-a}\right]$, sq. unit

21. (c) $x^2 + y^2 - 2x = 0$

23. t = H/k

For 38 Years Que. from IIT-JEE(Advanced) &
14 Years Que. from AIEEE (JEE Main)
we distributed a book in class room