Mathematical Reasoning

- **Mathematically Acceptable Statement**

A sentence is called a mathematically acceptable statement if it is either true or false, but not both.

We denote statements by small letters *p*, *q*, *r*,*s*, etc.

Example: Consider the following sentences:

- "Complete your homework."

It is not a statement as it is an order.

- "I can't wait to open my present!"

It is not a statement as it is an exclamatory sentence.

- "The value of iota is –1."

It is a statement. Since we know that $i = \sqrt{-1}$, the given sentence is false.

- "He is going to America."

It is not a statement as in this sentence we do not know who "he" is.

- "The number 0 is the smallest whole number."

It is a statement as this sentence is always true.

- "Do you have internet at your home?"

It is not a statement as it is a question.

- "The probability of getting a head when a coin is tossed is $\frac{1}{3}$"

It is a statement as this sentence is always false. The probability of getting a head when a coin is tossed is $\frac{1}{2}$.

Note:

- Sentences involving variable time such as today, tomorrow and yesterday are not statements.
- Sentences involving an exclamation, a question or an order are not statements.
- Sentences involving pronouns such as he or she, unless a particular person is referred to, are not statements.
- Sentences involving pronouns for variable places such as here and there are not statements.

- **Negations of Statements**
  - The denial of a statement is called the negation of the statement.
  - If *p* is a statement, then the negation of *p* is also a statement, and is denoted by ~*p*, and is read as "not *p*".
  - While writing the negation of a statement, phrases such as "It is not the case" or "It is false that" are used.

Example 1: Write the negation of the following statement:
p: The square root of every positive number is positive.
Solution:
The negation of the given statement can be written as:
The square root of every positive number is not positive.
Or
It is false that the square root of every positive number is positive.
Or
It is not the case that the square root of every positive number is positive.
Or
There exists a positive number whose square root is not positive.

- **Compound Statement**

A compound statement is one that is made up of two or more statements. In this case, each smaller statement is called the component of the compound statement. These component statements are joined by words such as "And" and "Or". These are called connectors or connecting words.

Example: The statement "27 is a multiple of 9 and it is even" is a compound statement.
Its component statements are:
p: "27 is a multiple of 9."
q: "27 is even."
Here, the connecting word is "And".

- **Connectives in Compound Statements**

A statement with "And" is not always a compound statement.

Example: Water can be prepared by the mixture of hydrogen and oxygen in a certain ratio.
This statement is not a compound statement.

- **Rules regarding the connector "And"**
  - The compound statement with the connector "And" is true if all its component statements are true.
  - The compound statement with the connector "And" is false if any/both of its component statements is/are false.

  Example: The compound statement "27 is a multiple of 9 and it is even" is false. The component statement "27 is a multiple of 9" is true; however, the other component statement “27 is even” is false.

- **Rules regarding the connector “Or”**
  - A compound statement with the connector “Or” is true when one component statement is true, or both the component statements are true.
  - A compound statement with the connector “Or” is false when both the component statements are false.

  Example: The compound statement “The equation $x^2 - 1 = 0$ is true for $x = 1$ or $x = -1$” is true. This is because both its component statements “The equation $x^2 - 1 = 0$ is true for $x = 1$” and “The equation $x^2 - 1 = 0$ is true for $x = -1$” are true.

- **Types of “Or”**
  - Exclusive “Or”: A compound statement with the connector “Or” in which either of the component statements may be true, but not both

  Example: A student can take home science or painting as his/her additional subject in class XI.

  - Inclusive “Or”: A compound statement with the connector “Or” in which either of the component statements or both may be true.

  Example: In an equilateral triangle, all the three sides are of equal length or all the three angles are of equal measure.

- **Quantifiers in Compound Statements**

Some statements may contain special phrases such as “There exists”, “For all”, “For every”. These are called quantifiers.

  - The quantifier for the statement “For every prime number x, $n \in \mathbb{N}$, x^n has exactly $(n + 1)$ factors” is “For every”. This statement is equivalent to “If A is the set of all prime numbers, then for any number x in the set A and $n \in \mathbb{N}$, x^n has exactly $(n + 1)$ factors”.
  - The quantifier for the statement “There exists a number which is divisible by 4 and 7, but not by 2” is “There exists”. This statement is equivalent to ”Out of all numbers that are divisible by both 4 and 7, there is at least one number which is not divisible by 2”.

- **Validity of “If-then” and “If and only if” statements**
  - In order to show that the statement “p and q” is true, the following steps are followed:

  Step 1: Show that statement p is true.

  Step 2: Show that statement q is true.

  - In order to show that the statement “p or q” is true, the following cases are to be considered:

Case 1: Assuming that *p* is false, show that *q* must be true

Case 2: Assuming that *q* is false, show that *p* must be true

- In order to prove the statement "if *p*, then *q*', we need to show that any one of the following cases is true.

Case 1: Assuming that *p* is true, prove that *q* must be true. (Direct method)

Case 2: Assuming that *q* is false, prove that *p* must be false. (Contrapositive method)

Example 1: Prove the following statement: "If *r* is irrational, then $\sqrt[3]{r}$ is irrational."
Solution: We shall prove the contrapositive of the given statement.

This means that we will prove if $\sqrt[3]{r}$ is rational, then *r* is rational.
Let $\sqrt[3]{r}$ be rational.
$\Rightarrow \sqrt[3]{r} = \frac{a}{b}$, for some integers *a* and *b*, and $b \neq 0$
$\Rightarrow r = \frac{a^3}{b^3}$
Now, since *a* and *b* are integers, a^3 and b^3 are also integers. Since $b \neq 0$, $b^3 \neq 0$.
Hence, *r* is rational.

Thus, if *r* is irrational, then $\sqrt[3]{r}$ is irrational.

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Thus, if r is irrational, then $\sqrt[3]{r}$ is irrational.