fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
i#"k flg ldYi dj] lgr foifr vud] ^cuk^ u NkM /;; dk] j?kcj jk[k VdAA
jfpr% ekuo /ke i.krk
lnx# Jh j.kNkMnklth egkjkt

STUDY PACKAGE

Subject : Mathematics

Topic : LIMITS

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Limit

1. Limit of a function f(x) is said to exist as, $x \to a$ when,

$\underset{h \to 0^+}{\text{Limit}}\ f(a - h) = \underset{h \to 0^+}{\text{Limit}}\ f(a + h)$ = some finite value M.

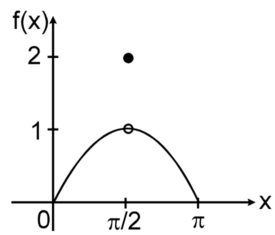
(Left hand limit) (Right hand limit)

Note that we are not interested in knowing about what happens at x = a. Also note that if L.H.L. & R.H.L. are both tending towards '∞' or '$-\infty$' then it is said to be infinite limit.

Remember, $\underset{x \to a}{\text{Limit}} \Rightarrow x \neq a$

Solved Example # 1

Find $\underset{x \to \pi/2}{\text{Limit}}\ f(x)$

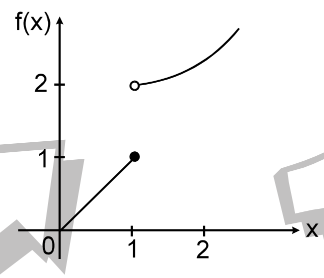


Solution.

Here $\underset{x \to \pi/2}{\text{Limit}}\ f(x) = 1$

Solved Example # 2

Find $\underset{x \to 1}{\text{Limit}}\ f(x)$



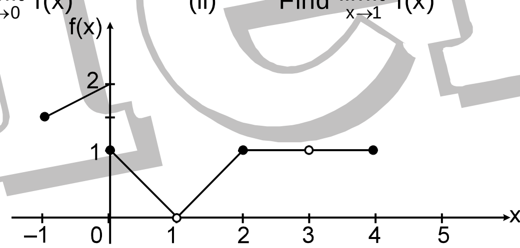
Solution.

Left handed limit = 1 Right handed limit = 2

Hence $\underset{x \to 1}{\text{Limit}}\ f(x)$ = does not exist.

Solved Example # 3

(i) Find $\underset{x \to 0}{\text{limit}}\ f(x)$ (ii) Find $\underset{x \to 1}{\text{limit}}\ f(x)$ (iii) Find $\underset{x \to 3}{\text{limit}}\ f(x)$



Solution.

(i) $\underset{x \to 0}{\text{limit}}\ f(x)$ = does not exists

because left handed limit $\neq$ right handed limit

(ii) $\underset{x \to 1}{\text{limit}}\ f(x) = 0$ (iii) $\underset{x \to 3}{\text{limit}}\ f(x) = 1$

2. Indeterminant Forms:

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, \infty^o, 0^o$, and 1^∞.

Solved Example # 4

Which of the following limits are forming indeterminant from also indicate the form

(i) $\lim\limits_{x \to 0} \frac{1}{x}$ (ii) $\lim\limits_{x \to 0} \frac{1-x}{1-x^2}$

(iii) $\lim\limits_{x \to 0} x\ \ell n\ x$ (iv) $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

(v) $\lim_{x\to 0} (\sin x)^x$ (vi) $\lim_{x\to 0} (\ell n\, x)^x$

(vii) $\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}}$ (viii) $\lim_{x\to 0} (1)^{1/x}$

Solution

(i) No (ii) Yes $\frac{0}{0}$ from

(iii) Yes $0 \times \infty$ form (iv) Yes $(\infty - \infty)$ form

(v) Yes, $(0)^o$ form (vi) Yes $(\infty)^o$ form

(vii) Yes $(1)^\infty$ form (viii)

NOTE :

(i) ' 0 ' doesn't means exact zero but represent a value approaching towards zero similary to ' 1 ' and infinity.

(ii) $\infty + \infty = \infty$

(iii) $\infty \times \infty = \infty$

(iv) $(a/\infty) = 0$ if a is finite

(v) $\frac{a}{0}$ is not defined for any $a \in R$.

(vi) $a\,b = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

3. Method of Removing Indeterminancy

To evaluate a limit, we must always put the value where ' x ' is approaching to in the function. If we get a determinate form, then that value becomes the limit otherwise if an indeterminant form comes. Then apply one of the following methods:

(i) Factorisation (ii) Rationalisation or double rationalisation

(iii) Substitution (iv) Using standard limits

(v) Expansions of functions.

1. Factorization method :-

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaing expression.

Solved Example # 5

$$\underset{x\to 2}{\text{limit}}\ \frac{x^6 - 24x - 16}{x^3 + 2x - 12}$$

Solution.

$$\underset{x\to 2}{\text{limit}}\ \frac{x^6 - 24x - 16}{x^3 + 2x - 12} = \underset{x\to 2}{\text{limit}}\ \frac{(x-2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 8)}{(x^2 + 2x + 6)(x-2)}$$

$$= \frac{168}{14} = \mathbf{12}$$

2. Rationalization /Double Rationalization.

We can rationalize the irrational expression by multiplying with their conjugates to remove the indeterminancy.

Solved Example # 6 $\underset{x\to 1}{\text{limit}}\ \frac{4-\sqrt{5x+1}}{2-\sqrt{3x+1}}$.

Solution. $\underset{x\to 1}{\text{limit}}\ \frac{4-\sqrt{5x+1}}{2-\sqrt{3x+1}}$

$$= \underset{x\to 1}{\text{limit}}\ \frac{(4-\sqrt{5x+1})(2+\sqrt{3x+1})(4+\sqrt{5x+1})}{(2-\sqrt{3x+1})(4+\sqrt{5x+1})(2+\sqrt{3x+1})}$$

$$= \underset{x\to 1}{\text{limit}}\ \frac{(15-5x)}{(3-3x)} \times \frac{2+\sqrt{3x+1}}{4+\sqrt{5x+1}} = \frac{5}{6}$$

Solved Example # 7

Evaluate : (i) $\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2\,(2x-3)}{x^3 - 3x^2 + 2x}\right]$ (ii) $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(iii) $\lim_{x\to 1} \frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2 + x - 3}$

Solution (i) We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2\,(2x-3)}{x^3 - 3x^2 + 2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2\,(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x(x-1) - 2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] \qquad = \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

(ii) The given limit taken the form $\frac{0}{0}$ when $x \to 0$. Rationalising the numerator, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right] \qquad = \lim_{x\to 0}\left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{2}{2} = 1$$

(iii) We have

$$\lim_{x\to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}\right] = \lim_{x\to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)(x-1)}\right]$$

$$= \lim_{x\to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}\right]$$

$$= \lim_{x\to 1}\left[\frac{2x-3}{(2x+3)\left(\sqrt{x}+1\right)}\right] \qquad = \frac{-1}{(5)(2)} = \frac{-1}{10}$$

4. Fundamental Theorems on Limits:

Let $\underset{x\to a}{\text{Limit}}\ f(x) = \ell$ & $\underset{x\to a}{\text{Limit}}\ g(x) = m$. If ℓ & m exists then:

(i) $\underset{x\to a}{\text{Limit}}\ \{f(x) \pm g(x)\} = \ell \pm m$ (ii) $\underset{x\to a}{\text{Limit}}\ \{f(x).\ g(x)\} = \ell.m$

(iii) $\underset{x\to a}{\text{Limit}}\ \frac{f(x)}{g(x)} = \frac{\ell}{m}$, provided $m \neq 0$

(iv) $\underset{x\to a}{\text{Limit}}\ k f(x) = k\ \underset{x\to a}{\text{Limit}}\ f(x)$; where k is a constant.

(v) $\underset{x\to a}{\text{Limit}}\ f[g(x)] = f\left(\underset{x\to a}{\text{Limit}}\ g(x)\right) = f(m)$; provided f is continuous at g (x) = m.

Solved Example # 8 Evaluate

(i) $\lim_{x\to 2}(x+2)$ (ii) $\lim_{x\to 2}x(x-1)$ (iii) $\lim_{x\to 2}\frac{x^2+4}{x+2}$ (iv) $\lim_{x\to 0}\cos(\sin x)$

(v) $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-1}$ (vi) $\lim_{x\to 1}\frac{x^2+3x+2}{x^2-1}$

Solution (i) x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\lim_{x\to 2}(x+2) = 2 + 2 = 4$

(ii) Again x(x – 1) being a polynomial in x, its limit as $x \to 2$ is given by

$$\lim_{x\to 2}x(x-1) = 2(2-1) = 2$$

(iii) By (II) above, we have $\lim_{x\to 2}\frac{x^2+4}{x+2} = \frac{(2)^2+4}{2+2} = 2$

(iv) $\lim_{x\to 0}\cos(\sin x) = \cos\left(\lim_{x\to 0}\sin x\right) = \cos 0 = 1$

(v) Note that for x = 1 both the numerator and the denominator of the given fraction vanish. Therefore by (III) above, we have $\lim_{x\to 1}\frac{x^2-3x+2}{x^2-1} = \lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x+1)} = \lim_{x\to 1}\frac{x-2}{x+1} = -\frac{1}{2}$

(vi) Note that for x = 1, the numerator of the given expression is a non-zero constant 6 and the denominator is zero. Therefore, the given limit is of the form $\frac{6}{0}$. Hence, by (IV) above, we conclude that $\lim_{x\to 1}\frac{x^2+3x+2}{x^2-1}$ does not exist

5. Standard Limits:

(a) $\underset{x\to 0}{\text{Limit}}\ \frac{\sin x}{x} = 1 = \underset{x\to 0}{\text{Limit}}\ \frac{\tan x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\tan^{-1}x}{x} = \underset{x\to 0}{\text{Limit}}\ \frac{\sin^{-1}x}{x}$

[Where x is measured in radians]

(b) $\underset{x\to 0}{\text{Limit}}\ (1+x)^{1/x} = e$; $\underset{x\to\infty}{\text{Limit}}\left(1+\frac{1}{x}\right)^x = e$

(c) $\underset{x\to 0}{\text{Limit}} \dfrac{e^x-1}{x} = 1;$ $\quad \underset{x\to 0}{\text{Limit}} \dfrac{a^x-1}{x} = \log_e a,\ a > 0$

(d) $\underset{x\to 0}{\text{Limit}} \dfrac{\ell n(1+x)}{x} = 1$ $\quad$ (e) $\underset{x\to a}{\text{Limit}} \dfrac{x^n-a^n}{x-a} = na^{n-1}.$

Solved Example # 9: Find $\underset{x\to 0}{\text{Limit}} \dfrac{\sin 2x}{x}$

Solution. $\underset{x\to 0}{\text{Limit}} \dfrac{\sin 2x}{x} \Rightarrow \underset{x\to 0}{\text{Limit}} \dfrac{\sin 2x}{2x}.\ 2 = 2$

Solved Example # 10: $\underset{x\to 0}{\text{Limit}} \dfrac{e^{3x}-1}{x/2}$

Solution. $\underset{x\to 0}{\text{Limit}} \dfrac{e^{3x}-1}{x/2}$ $\quad \underset{x\to 0}{\text{Limit}}\ 2\times 3\ \dfrac{e^{3x}-1}{3x} = -\ 6.$

Solved Example # 11 $\underset{x\to 0}{\text{Limit}} \dfrac{\tan x - \sin x}{x^3}$

Solution. $\underset{x\to 0}{\text{Limit}} \dfrac{\tan x - \sin x}{x^3}$

$$= \underset{x\to 0}{\text{Limit}} \frac{\tan x(1-\cos x)}{x^3}$$

$$= \underset{x\to 0}{\text{Limit}} \frac{\tan x\ .\ 2\sin^2\frac{x}{2}}{x^3} = \underset{x\to 0}{\text{Limit}} \frac{\tan x}{x}\ .\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1.$$

Solved Example # 12 Compute $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

Solution We have $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0}\left[\dfrac{\sin 2x}{2x}.\dfrac{2x}{3x}.\dfrac{3x}{\sin 3x}\right]$

$$= \left[\lim_{2x\to 0}\frac{\sin 2x}{2x}\right].\ \frac{2}{3}\ .\left[\lim_{3x\to 0}\frac{3x}{\sin 2x}\right],\ x \neq 0$$

$$= 1\ .\ \frac{2}{3} + \left[\lim_{3x\to 0}\frac{\sin 3x}{3x}\right] = \frac{2}{3}\times 1 = \frac{2}{3}$$

Solved Example # 13 Evaluate $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$

Solution $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x = e^{\lim\limits_{x\to\infty}\frac{2}{x}.x} = e^2.$

Solved Example # 14 Compute (i) $\lim\limits_{x\to 3}\dfrac{e^x-e^3}{x-3}$ $\quad$ (ii) $\lim\limits_{x\to 0}\dfrac{x(e^x-1)}{1-\cos x}$

Solution (i) Put $y = x - 3$. So, as $x \to 3$, $y \to 0$. Thus

$$\lim_{x\to 3}\frac{e^x-e^3}{x-3} = \lim_{y\to 0}\frac{e^{3+y}-e^3}{y}$$

$$= \lim_{y\to 0}\frac{e^3\ .\ e^y - e^3}{y}$$

$$= e^3 \lim_{y\to 0}\frac{e^y-1}{y} = e^3\ .\ 1 = e^3$$

(ii) We have

(ii) $$\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x} = \lim_{x\to 0}\frac{x(e^x-1)}{2\sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\ .\ \lim_{x\to 0}\left[\frac{e^x-1}{x}.\frac{x^2}{\sin^2\frac{x}{2}}\right] = 2.$$

Solved Example # 15 Evaluate $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4}$

Solution **(First Method)**

The given expression is of the form

$$\frac{x^3-(2)^3}{x^2-(2)^2} = \frac{x^3-(2)^3}{x-2} \div \frac{x^2-(2)^2}{x-2}$$

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$$\Rightarrow \quad \lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{x^3-(2)^3}{x-2} \div \lim_{x\to 2} \frac{x^2-(2)^2}{x-2}$$

$$= 3(2^2) \div 2(2^1) \qquad \left(\text{using } \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right)$$

$$= 12 \div 4 = 3$$

(Second Method)

The numerator and denominator have a common factor $(x-2)$. Cancelling this factor, we obtain

$$\frac{x^3-8}{x^2-4} = \frac{x^2+2x+4}{x+2} \quad \Rightarrow \quad \lim_{x\to 2} \frac{x^3-8}{x^2-4} = \lim_{x\to 2} \frac{x^2+2x+4}{x+2}$$

$$= \frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4} = 3$$

Note : Since $x \to 2$, $x-2$ is not zero, so the cancellation of the factor $x-2$ in the above example is carried out.

6. Use of Subsitution in Solving Limit Problems

Sometimes in solving limit problem we convert $\lim_{x\to a} f(x)$ by subtituting $x = a + h$ or $x = a - h$ as $\lim_{h\to 0} f(a+h)$ or $\lim_{h\to 0} f(a-h)$ according as need of the problem.

Solved Example # 16 $\quad \underset{x\to\pi/4}{\text{limit}} \dfrac{1-\tan x}{1-\sqrt{2}\sin x}$

Solution. Put $x = \dfrac{\pi}{4} + h \quad \because \quad x \to \dfrac{\pi}{4} \Rightarrow h \to 0$

$$\underset{h\to 0}{\text{limit}} \frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)}$$

$$= \underset{h\to 0}{\text{limit}} = \frac{1-\frac{1-\tan h}{1-\tan h}}{1-\sin h-\cos h}$$

$$= \underset{h\to 0}{\text{limit}} \frac{\frac{-2\tan h}{1-\tan h}}{2\sin^2\frac{h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$= \underset{h\to 0}{\text{limit}} \frac{-2\tan h}{2\sin^2\frac{h}{2}\left[2\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tan h)}$$

$$= \underset{h\to 0}{\text{limit}} \frac{-2\frac{\tan h}{h}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]} \frac{1}{(1-\tan h)} = \frac{-2}{-1} = 2.$$

7. Limit When $x \to \infty$

Since $x \to \infty \Rightarrow \dfrac{1}{x} \to 0$ hence in this type of problem we express most of the part of expression in terms of $\dfrac{1}{x}$ and apply $\dfrac{1}{x} \to 0$. We can see this approch in the given solve examples.

Solved Example # 17 $\quad \underset{x\to\infty}{\text{limit}}\ x \sin\dfrac{1}{x}$

Solution. $\underset{x\to\infty}{\text{limit}}\ x \sin\dfrac{1}{x}$

$$= \underset{x\to\infty}{\text{limit}} \frac{\sin 1/x}{1/x} = 1$$

Solved Example # 18 $\quad \underset{x\to\infty}{\text{limit}} \dfrac{x-2}{2x-3}$

Solution. $\underset{x\to\infty}{\text{limit}} \dfrac{x-2}{2x-3}$

$$\underset{x\to\infty}{\text{limit}} \frac{1-2/x}{2-3/x} = \frac{1}{2}.$$

Solved Example # 19 $\quad \underset{x\to\infty}{\text{limit}} \dfrac{x^2-4x+5}{3x^2-x^3+2}$

Solution. $\lim\limits_{x\to\infty} \dfrac{x^2-4x+5}{3x^2-x^3+2}$

$$= \lim\limits_{x\to\infty} \frac{\frac{1}{x}-\frac{4}{x^2}+\frac{5}{x^3}}{\frac{3}{x}-1+\frac{2}{x^3}} = 0$$

Solved Example # 20 $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Solution. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Put $x = \dfrac{-1}{t}$ $\quad x \to -\infty \quad t \to 0^+$

$$= \lim\limits_{x\to 0^+} \frac{\sqrt{3+2t^2}.\frac{1}{\sqrt{t^2}}}{\frac{1-2t}{t}}$$

$$= \lim\limits_{x\to 0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)} \frac{t}{|t|} = \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$

8. Limits Using Expansion

(i) $a^x = 1+\dfrac{x\ln a}{1!}+\dfrac{x^2\ln^2 a}{2!}+\dfrac{x^3\ln^3 a}{3!}+\ldots\ldots\ldots a>0$ (ii) $e^x = 1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\ldots$

(iii) $\ln(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots\ldots\ldots$ for $-1<x\le 1$ (iv) $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots\ldots$

(v) $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots\ldots$ (vi) $\tan x = x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots\ldots$

(vii) $\tan^{-1}x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots.$ (viii) $\sin^{-1}x = x+\dfrac{1^2}{3!}x^3+\dfrac{1^2.3^2}{5!}x^5+\dfrac{1^2.3^2.5^2}{7!}x^7+\ldots\ldots$

(ix) $\sec^{-1}x = 1+\dfrac{x^2}{2!}+\dfrac{5x^4}{4!}+\dfrac{61x^6}{6!}+\ldots\ldots$

(x) for $|x| < 1$, $n \in R$ $(1+x)^n = 1+nx+\dfrac{n(n-1)}{1.2}x^2+\dfrac{n(n-1)(n-2)}{1.2.3}x^3+\ldots\ldots\ldots\infty$

Solved Example # 21 $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2}$

Solution. $\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2}$

$$= \lim\limits_{x\to 0} \frac{\left(1+x+\frac{x^2}{2!}\ldots\ldots\right)-1-x}{x^2} = \frac{1}{2}$$

Solved Example # 22 $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$

Solution. $\lim\limits_{x\to 0} \dfrac{\tan x-\sin x}{x^3}$

$$= \lim\limits_{x\to 0} \frac{\left(x+\frac{x^3}{3}\ldots\ldots\right)-\left(x-\frac{x^3}{3!}\ldots\ldots\right)}{x^3} = \frac{1}{3}+\frac{1}{6} = \frac{1}{2}.$$

Solved Example # 23 $\lim\limits_{x\to 0} \dfrac{(7+x)^{1/3}-2}{x-1}$

Solution. Put $x \to 1+h$ $\lim\limits_{h\to 0} \dfrac{(8+h)^{1/3}-2}{h}$

$$\lim\limits_{h\to 0} \frac{2.\left(1+\frac{h}{8}\right)^{1/3}-2}{h}$$

$$= \lim\limits_{h\to 0} \frac{2\left\{1+\frac{1}{3}.\frac{h}{8}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{h}{8}\right)^2}{1.2}+\ldots\ldots-1\right\}}{h}$$

$$= \lim_{h \to 0} 2 \times \frac{1}{24} = \frac{1}{12}$$

Solved Example # 24 $\displaystyle \lim_{x \to 0} \frac{\ell n(1+x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$

$$\lim_{x \to 0} \frac{\ell n(1+x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} .....\right) - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) + \frac{x^2}{2}}{x^3 . \frac{\tan x}{x} . \frac{\sin x}{x}} \qquad = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

9. Limits of form 1^∞, 0^0, ∞^0

All these forms can be convered into $\frac{0}{0}$ form in the following ways

(i) if $x \to 1$, $y \to \infty$, then $z = (x)^y$

$\Rightarrow \quad \ell n\, z = y\, \ell n\, x \quad \Rightarrow \quad \ell n\, z = \dfrac{\ell nx}{(1/y)}$

Since $y \to \infty$ hence $\frac{1}{y} \to 0$ and $x \to 1$ hence $\ell nx \to 0$

(ii) If $x \to 0$, $y \to 0$, then $z = x^y$ $\quad \Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{y}{1/\ell ny} = \dfrac{0}{0}$ form

(iii) If $x \to \infty$, $y \to 0$, then $z = x^y$ $\quad \Rightarrow \quad \ell n\, z = y\, \ell n\, x$

$\Rightarrow \quad \ell n\, z = \dfrac{y}{1/\ell nx} = \dfrac{0}{0}$ form

also for $(1)^\infty$ type of problems we can use following rules.

(i) $\displaystyle \lim_{x\to 0} (1 + x)^{1/x} = e$ (ii) $\displaystyle \lim_{x\to a} [f(x)]^{g(x)}$

where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$$= \lim_{x\to a} [1+f(x)-1]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)} \qquad = e^{\lim_{x\to a}[f(x)-1]\, g(x)}$$

Solved Example # 25 $\displaystyle \lim_{x\to\infty} \left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Solution. Since it is in the form of 1^∞

$$\lim_{x\to\infty} \left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2} = e^{\lim_{x\to\infty} \left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)} = e^{-8}$$

Solved Example # 26: $\displaystyle \lim_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Solution $= e^{\lim_{x\to\frac{\pi}{4}} (\tan x-1)\tan 2x}$

$$= e^{\lim_{x\to\frac{\pi}{4}} (\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$$

$$= e^{2\times\frac{\tan \pi/4}{-1(1+\tan \pi/4)}} \qquad = e^{-1} \quad = \frac{1}{e}$$

Solved Example # 27 Evaluate $\displaystyle \lim_{x\to a} \left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$.

Solution. $\displaystyle \lim_{x\to a} \left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

put $x = a + h$ $\qquad \Rightarrow \quad \displaystyle \lim_{h\to 0} \left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)}$

$\Rightarrow \quad \displaystyle \lim_{h\to 0} \left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)} \qquad \Rightarrow \quad e^{\lim_{h\to 0} -\cot\frac{\pi h}{2a} . \left(1+\frac{h}{a+h}-1\right)}$

$\Rightarrow \quad e^{\lim_{h\to 0} -\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right) . \frac{\frac{2a}{\pi}}{a+h}} \qquad = e^{-2/\pi}$

Solved Example # 28: $\lim\limits_{x\to 0^+} x^x$

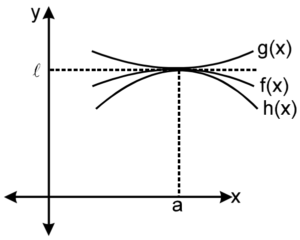
Solution. $y = \lim\limits_{x\to 0} x^x$

$\ell n\, y = \lim\limits_{x\to 0} x\, \ell n\, x$

$= \lim\limits_{x\to 0} -\dfrac{\ell n \frac{1}{x}}{\frac{1}{x}} = 0 \because \quad \dfrac{1}{x} \to \infty \qquad y = 1$

10. Sandwich Theorem or Squeeze Play Theorem:

If $f(x) \le g(x) \le h(x)\ \forall\ x$ & $\underset{x\to a}{\text{Limit}}\ f(x) = \ell = \underset{x\to a}{\text{Limit}}\ h(x)$ then $\underset{x\to a}{\text{Limit}}\ g(x) = \ell$.



Solved Example # 29: Evaluate $\lim\limits_{n\to\infty} \dfrac{[x]+[2x]+[3x]+....+[nx]}{n^2}$

Where [ ] denotes the greatest integer function.

Solution.

We know that, $x - 1 < [x] \le x$

$\Rightarrow \quad 2x - 1 < [2x] \le 2x$

$\Rightarrow \quad 3x - 1 < [3x] \le 3x$

....

....

$\Rightarrow \quad nx - 1 < [nx] \le nx$

$\therefore \quad (x + 2x + 3x + ... + nx) - n < [x] + [2x] + ... + [nx] \le (x + 2x + ... + nx)$

$\Rightarrow \quad \dfrac{xn(n+1)}{2} - n < \sum\limits_{r=1}^{n} [r\,x] \le \dfrac{x.n(n+1)}{2}$

Thus, $\lim\limits_{n\to\infty} \dfrac{[x]+[2x]+....+[nx]}{n^2}$

$\Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{x}{2}\left(1+\dfrac{1}{n}\right) - \dfrac{1}{n} < \lim\limits_{n\to\infty} \dfrac{[x]+[2x]+....+[nx]}{n^2} \le \lim\limits_{n\to\infty} \dfrac{x}{2}\left(1+\dfrac{1}{n}\right)$

$\Rightarrow \quad \dfrac{x}{2} < \lim\limits_{n\to\infty} \dfrac{[x]+[2x]+....+[nx]}{n^2} \le \dfrac{x}{2} \quad \Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{[x]+[2x]+....+[nx]}{n^2} = \dfrac{x}{2}$

Aliter We know that $[x] = x - \{x\}$

$\sum\limits_{r=1}^{n} r\,x = [x] + [2x] + ... + nx - [nx]$

$= (x + 2x + 3x + ... + nx) - (\{x\} + \{2x\} + ... + \{nx\})$

$= \dfrac{xn(n+1)}{2} - (\{x\} + \{2x\} + .. + \{nx\})$

$\therefore \quad \dfrac{1}{n^2} \sum\limits_{r=1}^{n} [r\,x] = \dfrac{x}{2}\left(1+\dfrac{1}{n}\right) - \dfrac{\{x\}+\{2x\}+....+\{nx\}}{n^2}$

Since, $0 \le \{rx\} < 1$, $\quad \therefore \quad 0 \le \sum\limits_{r=1}^{n} [r\,x] < n$

$\Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n} [rx]}{n^2} = 0 \qquad \therefore \quad \lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n} [rx]}{n^2} = \lim\limits_{n\to\infty} \dfrac{x}{2}\left(1+\dfrac{1}{n}\right) - \lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n} \{rx\}}{n^2}$

$\lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n} [rx]}{n^2} = \dfrac{x}{2}$

Solved Example # 30 $\underset{x\to 0}{\text{limit}}\ x \sin \dfrac{1}{x}$

Solution. $\underset{x\to 0}{\text{limit}}\ x \sin \dfrac{1}{x}$

$= 0 \times (\text{some value in } [-1, 1]) = 0$

11. Some Important Notes :

(i) $\lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$

As $x \to \infty$, ℓn x increnes much slower than any (+ve) power of x where e^x increases much faster than (+ve) power of x

(iii) $\text{Limit}_{n\to\infty} (1 - h)^n = 0$ & $\text{Limit}_{n\to\infty} (1 + h)^n \to \infty$, where h > 0.

(iv) If $\text{Limit}_{x\to a} f(x) = A > 0$ & $\text{Limit}_{x\to a} \phi(x) = B$ (a finite quantity) then;

$\text{Limit}_{x\to a} [f(x)]^{\phi(x)} = e^z$ where $z = \text{Limit}_{x\to a} \phi(x). \ln[f(x)] = e^{B\ln A} = A^B$

Solved Example # 31 $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x}$

Solution. $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x} = 0$

Short Revesion (LIMIT)

THINGS TO REMEMBER :

1. Limit of a function f(x) is said to exist as, $x\to a$ when

$$\text{Limit}_{x\to a^-} f(x) = \text{Limit}_{x\to a^+} f(x) = \text{finite quantity.}$$

2. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\text{Limit}_{x\to a} f(x) = l$ & $\text{Limit}_{x\to a} g(x) = m$. If l & m exists then :

(i) $\text{Limit}_{x\to a} f(x) \pm g(x) = l \pm m$ **(ii)** $\text{Limit}_{x\to a} f(x)\cdot g(x) = l.\, m$

(iii) $\text{Limit}_{x\to a} \dfrac{f(x)}{g(g)} = \dfrac{\ell}{m}$, provided $m \neq 0$

(iv) $\text{Limit}_{x\to a} k\, f(x) = k\, \text{Limit}_{x\to a} f(x)$; where k is a constant.

(v) $\text{Limit}_{x\to a} f[g(x)] = f\left(\text{Limit}_{x\to a} g(x)\right) = f(m)$; provided f is continuous at g (x) = m.

For example $\text{Limit}_{x\to a} l\,n(f(x) = l\,n\left[\text{Limit}_{x\to a} f(x)\right] l\,n\, l\ (l > 0)$.

REMEMBER

$$\text{Limit}_{x\to a} \Rightarrow x \neq a$$

3. STANDARD LIMITS :

(a) $\text{Limit}_{x\to 0} \dfrac{\sin x}{x} = 1 = \text{Limit}_{x\to 0} \dfrac{\tan x}{x} = \text{Limit}_{x\to 0} \dfrac{\tan^{-1} x}{x} = \text{Limit}_{x\to 0} \dfrac{\sin^{-1} x}{x}$

[Where x is measured in radians]

(b) $\text{Limit}_{x\to 0} (1 + x)^{1/x} = e = \text{Limit}_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$ note however there $\text{Limit}_{\substack{h\to 0\\ n\to\infty}} (1 - h)^n = 0$

and $\text{Limit}_{\substack{h\to 0\\ n\to\infty}} (1 + h)^n \to \infty$

(c) If $\text{Limit}_{x\to a} f(x) = 1$ and $\text{Limit}_{x\to a} \phi(x) = \infty$, then ;

$$\text{Limit}_{x\to a} [f(x)]^{\phi(x)} = e^{\text{Limit}_{x\to a} \phi(x)[f(x)-1]}$$

(d) If $\text{Limit}_{x\to a} f(x) = A > 0$ & $\text{Limit}_{x\to a} \phi(x) = B$ (a finite quantity) then ;

$\text{Limit}_{x\to a} [f(x)]^{\phi(x)} = e^z$ where $z = \text{Limit}_{x\to a} \phi(x). \ln[f(x)] = e^{B\ln A} = A^B$

(e) $\text{Limit}_{x\to 0} \dfrac{a^x - 1}{x} = 1n\, a\ (a > 0)$. In particular $\text{Limit}_{x\to 0} \dfrac{e^x - 1}{x} = 1$

(f) $\text{Limit}_{x\to a} \dfrac{x^n - a^n}{x - a} = n\, a^{n-1}$

4. SQUEEZE PLAY THEOREM : If $f(x) \le g(x) \le h(x)\ \forall\ x$ **&** $\text{Limit}_{x\to a} f(x) = l = \text{Limit}_{x\to a} h(x)$ then $\text{Limit}_{x\to a} g(x) = l$.

5. INDETERMINANT FORMS : $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0 \times \infty$, 0° , ∞° , $\infty - \infty$ and 1^∞

Note : (i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementry algebra. **(ii)** $\infty + \infty = \infty$ **(iii)** $\infty \times \infty = \infty$

(iv) $(a/\infty) = 0$ if a is finite **(v)** $\dfrac{a}{0}$ is not defined , if $a \neq 0$.

(vi) a b = 0 , if & only if a = 0 or b = 0 and a & b are finite.

6. The following strategies should be born in mind for evaluating the limits:

(a) Factorisation **(b)** Rationalisation or double rationalisation

(c) Use of trigonometric transformation ;

appropriate substitution and using standard limits

(d) Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for} -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots.$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots..$

(vii) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots.$

(viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots.$

(ix) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots\ldots$

EXERCISE–1

Q 1. $\lim\limits_{x\to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$

Q 2. $\lim\limits_{x\to 1} \frac{\sqrt[13]{x} - \sqrt[7]{x}}{\sqrt[5]{x} - \sqrt[3]{x}}$

Q3. $\lim\limits_{x\to 1} \frac{x^2 - x\ln x + \ln x - 1}{x - 1}$

Q 4. $\lim\limits_{x\to 1}\left(\frac{p}{1-x^p} - \frac{q}{1-x^q}\right) p, q \in N$

Q 5. $\lim\limits_{x\to\infty} \frac{2\sqrt{x} + 3x^{1/3} + 5x^{1/5}}{\sqrt{3x-2} + (2x-3)^{1/3}}$

Q 6. $\lim\limits_{x\to \frac{3\pi}{4}} \frac{1 + \sqrt[3]{\tan x}}{1 - 2\cos^2 x}$

Q 7. (a) $\lim\limits_{x\to 0} \tan^{-1}\frac{a}{x^2}$ where $a \in R$

(b) Plot the graph of the function $f(x) = \lim\limits_{t\to 0}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right)$

Q 8. $\lim\limits_{x\to 1} \frac{\left[\sum\limits_{K=1}^{100} x^k\right] - 100}{x - 1}$

Q 9. Find the sum of an infinite geometric series whose first term is the limit of the function

$f(x) = \frac{\tan x - \sin x}{\sin^3 x}$ as $x \to 0$ and whose common ratio is the limit of the function

$g(x) = \frac{1 - \sqrt{x}}{(\cos^{-1} x)^2}$ as $x \to 1$. (Use of series expansion or L' Hospital's rule is not allowed.)

Q 10. $\lim\limits_{x\to\infty} (x - \ln\cosh x)$ where $\cosh t = \frac{e^t + e^{-t}}{2}$.

Q 11. $\lim\limits_{x\to\frac{\pi}{2}} \cos^{-1}[\cot x]$ where [ ] denotes greatest integer function

Q 12. $\lim\limits_{x\to\frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2}\sin x}$

Q 13. $\lim\limits_{x\to 0} [\ln(1 + \sin^2 x).\cot(\ln^2(1 + x))]$

Q 14. $\lim\limits_{x\to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

Q 15. $\lim\limits_{\theta\to\frac{\pi}{4}} \frac{\sqrt{2} - \cos\theta - \sin\theta}{(4\theta - \pi)^2}$

Q 16. $\lim\limits_{x\to\frac{\pi}{2}} \frac{2^{-\cos x} - 1}{x(x - \frac{\pi}{2})}$

Q 17. If $\lim\limits_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ is finite then find the value of 'a' & the limit.

Q 18. $\lim\limits_{x\to 0} \frac{8}{x^8}\left[1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$

Q 19. $\lim\limits_{x\to 1} \frac{(\ln(1+x) - \ln 2)(3.4^{x-1} - 3x)}{[(7+x)^{\frac{1}{3}} - (1+3x)^{\frac{1}{2}}].\sin(x-1)}$

Q 20. Using Sandwich theorem to evaluate $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots\ldots. + \frac{1}{\sqrt{n^2+2n}}\right)$

Q21. Given $f(x) = \lim\limits_{n\to\infty} \tan^{-1}(nx)$; $g(x) = \lim\limits_{n\to\infty} \sin^{2n} x$ and $\sin(h(x)) = \frac{1}{2}[\cos\pi(g(x)) + \cos(2f(x))]$
Find the domain and range of h (x).

Q 22. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\sqrt{1 - \sqrt{\sin 2x}}}{\pi - 4x}$

Q 23. $\lim\limits_{x\to 3} \frac{(x^3 + 27)\ln(x - 2)}{x^2 - 9}$

Q 24. $\lim\limits_{x\to 2} \frac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x - 2}$

Q 25. $\lim\limits_{x\to 0} \frac{27^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}}$

Q 26. Let $f(x) = \dfrac{x}{\sin x},\ x > 0$ and $g(x) = x + 3, \quad x < 1$
$= 2 - x, \quad x \le 0$ $\qquad = x^2 - 2x - 2, \ 1 \le x < 2$
$\qquad = x - 5, \quad x \ge 2$
find LHL and RHL of $g(f(x))$ at $x = 0$ and hence find $\displaystyle\lim_{x\to 0} g(f(x))$.

Q 27. Let $P_n = a^{P_{n-1}} - 1, \ \forall\ n = 2, 3, ...$ and Let $P_1 = a^x - 1$ where $a \in R^+$ then evaluate $\displaystyle\lim_{x\to 0} \frac{P_n}{x}$.

Q 28. $\displaystyle\lim_{x\to -\infty} \frac{(3x^4 + 2x^2)\sin\frac{1}{x} + |x|^3 + 5}{|x|^3 + |x|^2 + |x| + 1}$

Q.29 If $f(x) = \ln \operatorname{cosec}(x\pi) \quad 0 < x < 1$
$\quad = \ln \sin(2x\pi) \quad 1 < x < 3/2$ and $g(x) = \dfrac{2^{f(x)} + 1}{3^{f(x)} + 1}$ then
find $\tan^{-1}(g(1^-))$ and $\sec^{-1}(g(1^+))$.

Q.30 At the end-points and the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the two triangles thus formed tends to 4 as the arc AB decreases indefinitely.

EXERCISE–2

Q 1. $\displaystyle\lim_{x\to\infty} \left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$ **Q 2.** $\displaystyle\lim_{x\to\infty} \left(\frac{x+c}{x-c}\right)^x = 4$ then find c **Q 3.** $\displaystyle\lim_{x\to 0} \left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}$

Q 4. $\displaystyle\lim_{x\to 0} \left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$ **Q.5** $\displaystyle\lim_{x\to\infty} x^2 \sin\left(\ell n \sqrt{\cos\frac{\pi}{x}}\right)$

Q 6. $\displaystyle\lim_{x\to\infty} \left[\cos\left(2\pi\left(\frac{x}{1+x}\right)^a\right)\right]^{x^2} \quad a \in R$ **Q 7.** $\displaystyle\lim_{x\to 1} \left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$

Q 8. $\displaystyle\lim_{x\to 0} \left(\frac{x - 1 + \cos x}{x}\right)^{\frac{1}{x}}$ **Q 9.** $\displaystyle\lim_{x\to\infty} \left(\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + a_3^{\frac{1}{x}} + ... + a_n^{\frac{1}{x}}}{n}\right)^{nx}$ where $a_1, a_2, a_3, ...a_n > 0$

Q 10. Let $f(x) = \dfrac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,.\,(1-\{x\})}$ then find $\displaystyle\lim_{x\to 0^+} f(x)$ and $\displaystyle\lim_{x\to 0^-} f(x)$, where {x} denotes the fractional part function.

Q 11. Find the values of a, b & c so that $\displaystyle\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x.\sin x} = 2$

Q 12. $\displaystyle\lim_{x\to a} \frac{1}{(a^2 - x^2)^2}\left(\frac{a^2 + x^2}{ax} - 2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)\right)$ where a is an odd integer

Q 13. $\displaystyle\lim_{x\to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$ **Q 14.** $\displaystyle\lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1} \quad x\ \varepsilon\ R$

Q 15. $\displaystyle\lim_{n\to\infty} \frac{[1.x] + [2.x] + [3.x] + ... + [n.x]}{n^2}$, Where [.] denotes the greatest integer function.

Q 16. Without using series expansion or L'Hospital's rule evaluate, $\displaystyle\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos \pi x}$

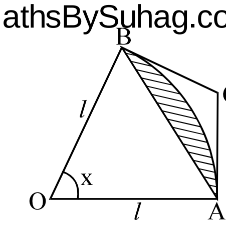
Q 17. $\displaystyle\lim_{y\to 0} \left[\operatorname*{Limit}_{x\to\infty} \frac{\exp\left(x\ln\left(1+\frac{ay}{x}\right)\right) - \exp\left(x\ln\left(1+\frac{by}{x}\right)\right)}{y}\right]$

Q 18. If s_n be the sum of n terms of the series, sin x + sin 2x + sin 3x +..... + sin nx then show that $\displaystyle\operatorname*{Limit}_{n\to\infty} \frac{s_1 + s_2 + ... + s_n}{n} = \frac{1}{2}\cot\frac{x}{2} \quad (x \ne 2k\pi,\ k \in I)$

Q 19. $\displaystyle\lim_{x\to 0} \left[\frac{\ell n\,(1+x)^{1+x}}{x^2} - \frac{1}{x}\right]$

Q 20. Let $P_n = \dfrac{2^3-1}{2^3+1}.\dfrac{3^3-1}{3^3+1}.\dfrac{4^3-1}{4^3+1}.......\dfrac{n^3-1}{n^3+1}$. Evaluate $\displaystyle\lim_{n\to\infty} P_n$

Q 21. A circular arc of radius 1 subtends an angle of x radians, $0 < x < \dfrac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute :



(a) T(x) (b) S(x) & (c) the limit of $\frac{T(x)}{S(x)}$ as $x \to 0$.

Q 22. (a) $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$ **(b)** $\lim\limits_{x\to\infty} \left[\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right]$

Q 23. If $f(n, \theta) = \prod\limits_{r=1}^{n}\left(1-\tan^2\frac{\theta}{2^r}\right)$, then compute $\lim\limits_{n\to\infty} f(n,\theta)$

Q 24. Let $l = \lim\limits_{x\to a} \frac{x^x - a^x}{x - a}$ & $m = \lim\limits_{x\to a} \frac{a^x - x^a}{x - a}$ where $a > 0$. If $l = m$ then find the value of 'a'.

Q 25. $\lim\limits_{x\to\infty}\left(\frac{\cosh\frac{\pi}{x}}{\cos\frac{\pi}{x}}\right)^{x^2}$ where $\cosh t = \frac{e^t + e^{-t}}{2}$ **Q 26.** $\lim\limits_{x\to 0} \frac{2(\tan x - \sin x) - x^3}{x^5}$

Q 27. Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

Q 28. Using Sandwich theorem, evaluate

(a) $\lim\limits_{n\to\infty} \frac{1}{1+n^2} + \frac{2}{2+n^2} + \dots\dots\dots + \frac{n}{n+n^2}$ **(b)** $\lim\limits_{n\to\infty} (a^n + b^n)^{\frac{1}{n}}$, $0 < a < b$

Q 29. Find a & b if : **(i)** $\lim\limits_{x\to\infty}\left[\frac{x^2+1}{x+1} - ax - b\right] = 0$ **(ii)** $\lim\limits_{x\to-\infty}\left[\sqrt{x^2-x+1} - ax - b\right] = 0$

Q 30. Show that $\lim\limits_{h\to 0} \frac{(\sin(x+h))^{x+h} - (\sin x)^x}{h} = (\sin x)^x\,[\,x\cot x + l\,n \sin x\,]$

EXERCISE–3

Q.1 $\lim\limits_{x\to 0}\left[\frac{1+5x^2}{1+3x^2}\right]^{\frac{1}{x^2}} =$ ______. [IIT'96, 1]

Q.2 $\lim\limits_{x\to 1} \frac{\sqrt{1-\cos 2(x-1)}}{x-1}$ [IIT '98, 2]

(A) exits and it equals $\sqrt{2}$
(B) exits and it equals - $\sqrt{2}$
(C) does not exist because $x - 1 \to 0$
(D) does not exist because left hand limit is not equal to right hand limit.

Q.3 $\lim\limits_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ is : [JEE '99, 2 (out of 200)]

(A) 2 (B) – 2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Q.4 For $x \in R$, $\lim\limits_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x =$ [JEE 2000, Screening]

(A) e (B) e^{-1} (C) e^{-5} (D) e^5

Q.5 $\lim\limits_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$ equals [JEE 2001, Screening]

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) 1

Q.6 Evaluate $\lim\limits_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$, $a > 0$. [REE 2001, 3 out of 100]

Q.7 The integer n for which $\lim\limits_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is

(A) 1 (B) 2 (C) 3 (D) 4

[JEE 2002 (screening), 3]

Q.8 If $\lim\limits_{x\to 0} \frac{\sin(nx)[(a-n)nx - \tan x]}{x^2} = 0$ $(n > 0)$ then the value of 'a' is equal to

(A) $\frac{1}{n}$ (B) n^2+1 (C) $\frac{n^2+1}{n}$ (D) None

[JEE 2003 (screening)]

Q.9 Find the value of $\lim\limits_{n\to\infty}\left[\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n\right]$. [JEE ' 2004, 2 out of 60]

EXERCISE–4

1. $\text{Limit}_{n\to\infty}\frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}$ = (A) 5 (B) 3 (C) 1 (D) zero

2. $\text{Limit}_{x\to-1}\frac{\cos 2-\cos 2x}{x^2-|x|}$ =

(A) 2 cos 2 (B) − 2 cos 2 (C) 2 sin 2 (D) − 2 sin 2

3. The value of $\text{Limit}_{x\to 0}\frac{1}{x}\sqrt{\frac{1-\cos 2x}{2}}$ is:

(A) 1 (B) − 1 (C) 0 (D) none

4. $\text{Limit}_{x\to 0}\sin^{-1}(\sec x)$.

(A) is equal to $\pi/2$ (B) is equal to 1 (C) is equal to zero (D) none of these

5. $\text{Limit}_{x\to 5}\frac{x^2-9x+20}{x-[x]}$ where [x] is the greatest integer not greater than x:

(A) is equal to 1 (B) 0 (C) 4 (D) none

6. $\text{Limit}_{x\to-\pi}\frac{|x+\pi|}{\sin x}$:

(A) is equal to − 1 (B) is equal to 1 (C) is equal to π (D) does not exist

7. $\text{Limit}_{x\to 3}\frac{(x^3+27)\ln(x-2)}{(x^2-9)}$ =

(A) − 8 (B) 8 (C) 9 (D) − 9

8. $\text{Limit}_{x\to 1}\frac{\sum_{k=1}^{100}x^k-100}{x-1}$ =

(A) 0 (B) 5050 (C) 4550 (D) − 5050

9. $\text{Limit}_{x\to\infty}\left(\sqrt{(x+a)(x+b)}-x\right)$ =

(A) $\sqrt{ab}$ (B) $\frac{a+b}{2}$ (C) ab (D) none

10. $\text{Limit}_{x\to\infty}\frac{x^3 . \sin\frac{1}{x}+x+1}{x^2+x+1}$ =

(A) 0 (B) 1/2 (C) 1 (D) none

11. $\text{Limit}_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+3)!}$, $n\in N$ =

(A) 0 (B) 1 (C) 2 (D) − 1

12. $\text{Limit}_{x\to 0}|x|^{\sin x}$ =

(A) 0 (B) 1 (C) − 1 (D) none of these

13. $\text{Limit}_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$ =

(A) 1 (B) 2 (C) e^2 (D) e

14. The values of a and b such that $\lim\limits_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$ are

(A) $\frac{5}{2},\frac{3}{2}$ (B) $\frac{5}{2},-\frac{3}{2}$ (C) $-\frac{5}{2},-\frac{3}{2}$ (D) $-\frac{5}{2},\frac{3}{2}$

15. $\text{Limit}_{x\to 0}\frac{2\left(\sqrt{3}\sin\left(\frac{\pi}{6}+x\right)-\cos\left(\frac{\pi}{6}+x\right)\right)}{x\sqrt{3}\left(\sqrt{3}\cos x-\sin x\right)}$ =

(A) − 1/3 (B) 2/3 (C) 4/3 (D) − 4/3

16. If $f(x)=\begin{cases}x-1, & x\ge 1\\ 2x^2-2, & x<1\end{cases}$, $g(x)=\begin{cases}x+1, & x>0\\ -x^2+1, & x\le 0\end{cases}$ and $h(x)=|x|$

then find $\lim\limits_{x\to 0}f(g(h(x)))$

(A) 1 (B) 0 (C) –1 (D) does not exists

17. $\underset{x\to 1}{\text{Limit}}\ (1 - x + [x - 1] + [1 - x])$ = where [x] denotes greatest integer function.

(A) 0 (B) 1 (C) –1 (D) does not exist

18. $\underset{x\to 0}{\text{Lt}}\left[\dfrac{\sin[x-3]}{[x-3]}\right]$, where [.] denotes greatest integer function is :

(A) 0 (B) 1 (C) does not exist (D) sin 1

19. Let f (x) = $\begin{cases} x\sin\left(\dfrac{1}{x}\right)+\sin\left(\dfrac{1}{x^2}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$, then $\lim\limits_{x\to\infty}$ f(x) equals

(A) 0 (B) – 1/2 (C) 1 (D) none of these.

20. $\lim\limits_{x\to a^-}\left(\dfrac{|x|^3}{a}-\left[\dfrac{x}{a}\right]^3\right)$ (a > 0), where [x] denotes the greatest integer less than or equal to x is

(A) $a^2 + 1$ (B) $a^2 - 1$ (C) a^2 (D) $-a^2$

21. Let α, β be the roots of $ax^2 + bx + c = 0$, where 1 < α < β. Then $\underset{x\to x_0}{\text{Lim}}\ \dfrac{|ax^2+bx+c|}{ax^2+bx+c} = 1$ then which of the following statements is incorrect

(A) a > 0 and $x_0 < 1$ (B) a > 0 and $x_0 > \beta$
(C) a < 0 and $\alpha < x_0 < \beta$ (D) a < 0 and $x_0 < 1$

22. $\underset{n\to\infty}{\text{Limit}}\ \dfrac{1n+2(n-1)+3(n-2)+\ldots\ldots+n.1}{1^2+2^2+3^2+\ldots\ldots+n^2}$ has the value :

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{4}$ (D) 1

23. $\underset{x\to 0}{\text{Lim}}\left[(1-e^x)\dfrac{\sin x}{|x|}\right]$ is (where [·] represents greatest integral part function)

(A) – 1 (B) 1 (C) 0 (D) does not exist

24. If $\ell = \lim\limits_{x\to\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})$ and m = $\lim\limits_{x\to-\infty}[\sin\sqrt{x+1}-\sin\sqrt{x}]$ where [.] denotes the greatest integer function then :

(A) ℓ = m = 0 (B) ℓ = 0 ; m is undefined
(C) ℓ, m both do not exist (D) ℓ = 0, m ≠ 0 (although m exist)

25. If f(x) = $\sum\limits_{\lambda=1}^{n}\left(x-\dfrac{1}{\lambda}\right)\left(x-\dfrac{1}{\lambda+1}\right)$ then $\lim\limits_{n\to\infty}$ f(0) is.

(A) 1 (B) –1 (C) 2 (D) None

26. The limit $\lim\limits_{\theta\to 0}\left(\left[\dfrac{n\sin\theta}{\theta}\right]+\left[\dfrac{n\tan\theta}{\theta}\right]\right)$, where [x] is the greatest integer function and n ∈ I, is

(A) 2n (B) 2n + 1 (C) 2n – 1 (D) does not exist

27. The limit $\underset{x\to\infty}{\text{Limit}}\ x - x^2\,\ell n\left(1+\dfrac{1}{x}\right)$ is equal to :

(A) 1/2 (B) 3/2 (C) 1/3 (D) 1

28. $\lim\limits_{x\to\pi/2}\left[\dfrac{x-\frac{\pi}{2}}{\cos x}\right]$ is : (where [.] represents greatest integer function.

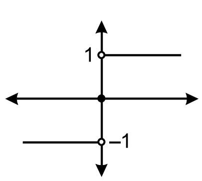
(A) – 1 (B) 0 (C) – 2 (D) does not exist

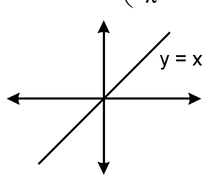
29. If f(x) = sin x , x ≠ nπ , n = 0 , ± 1 , ± 2 , ± 3, ,....
= 2 , otherwise and
g(x) = $x^2 + 1$, x ≠ 0 , 2
= 4 , x = 0
= 5 , x = 2

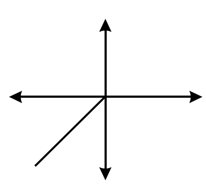
then $\underset{x\to 0}{\text{Limit}}$ g [f(x)] is :

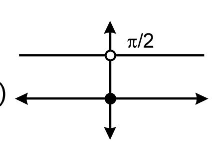
(A) 1 (B) 0 (C) 4 (D) does not exists

30. The graph of the function f(x) = $\underset{t\to 0}{\text{Lim}}\left(\dfrac{2x}{\pi}\cot^{-1}\dfrac{x}{t^2}\right)$, is

(A) 

(B) 

(C) 

(D) 

31. The value of $\underset{x\to 0}{\text{Limit}}\ \dfrac{\cos(\sin x)-\cos x}{x^4}$ is equal to:

(A) 1/5 (B) 1/6 (C) 1/4 (D) 1/2

32. $\underset{x\to\infty}{\text{Limit}} \dfrac{e^x\left(\left(2^{x^n}\right)^{\frac{1}{e^x}}-\left(3^{x^n}\right)^{\frac{1}{e^x}}\right)}{x^n}$, $n \in N$ is equal to :

(A) 0 (B) ln(2/3) (C) ln(3/2) (D) none

33. $\underset{y\to 0}{\text{Limit}}\left(\underset{x\to\infty}{\text{Limit}}\dfrac{\exp\left(x\ln\left(1+\frac{ay}{x}\right)\right)-\exp\left(x\ln\left(1+\frac{by}{x}\right)\right)}{y}\right) =$

(A) a + b (B) a – b (C) b – a (D) – (a + b)

EXERCISE–5

1. Evaluate the following limits, where [.] represents greatest integer function and { . } represents fractional part function

(i) $\lim\limits_{x\to\frac{\pi}{2}} [\sin x]$ (ii) $\lim\limits_{x\to 2}\left\{\dfrac{x}{2}\right\}$ (iii) $\lim\limits_{x\to\pi} \text{sgn}\,[\tan x]$

2. If $f(x) = \begin{cases} x^2+2, & x \ge 2 \\ 1-x, & x<2\end{cases}$ and $g(x) = \begin{cases} 2x, & x>1 \\ 3-x, & x \le 1\end{cases}$, evaluate $\lim\limits_{x\to 1} f(g(x))$.

3. Evaluate each of the following limits, if exists

(i) $\lim\limits_{x\to 2}\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ (ii) $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a \ne 0$

4. Evaluate the following limits, if exists

(i) $\lim\limits_{x\to 0}\dfrac{\tan 3x-2x}{3x-\sin^2 x}$ (ii) $\lim\limits_{x\to a}\dfrac{(x+2)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$ (iii) $\lim\limits_{x\to 0}\dfrac{x\left(e^{2+x}-e^2\right)}{1-\cos x}$

5. Evaluate the following limits, if exist :

(i) $\lim\limits_{x\to\infty}\sqrt{x^2+x-1}-x$ (ii) $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2}+\dfrac{2}{x^2}+....+\dfrac{x}{x^2}\right)$

(iii) $\underset{x\to\infty}{\text{Limit}}\left\{\cos\left(\sqrt{x+1}\right)-\cos\left(\sqrt{x}\right)\right\}$ (iv) $\lim\limits_{x\to\infty}\sqrt{x^2-8x}+x$

6. Evaluate the following limits using expansions : (i) $\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x-\dfrac{\tan^2 x}{2}}{x^3}$

(ii) If $\lim\limits_{x\to 0}\dfrac{a+b\sin x-\cos x+ce^x}{x^3}$ exists, then find values of a, b, c. Also find the limit

7. Evaluate $\lim\limits_{x\to\infty}\dfrac{[1.\,2x]+[2\,.\,3x]+.....+[n\,.\,(n+1)\,x]}{n^3}$ where [.] denotes greatest integer function

8. If $f(x) = \underset{n\to\infty}{\text{Lim}}\dfrac{x^{2n}-1}{x^{2n}+1}$, find range of f(x).

9. Evaluate the following limits

(i) $\underset{x\to 1}{\text{Limit}}\dfrac{(\ln(1+x)-\ln 2)(3.4^{x-1}-3x)}{[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}].\sin(x-1)}$ (ii) $\underset{x\to 4}{\text{Limit}}\dfrac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-4}$, $\alpha \in \left(0,\dfrac{\pi}{2}\right)$

10. Evaluate the following limits

(i) $\underset{x\to\infty}{\text{Limit}}\; x^3\left\{\sqrt{x^2+\sqrt{1+x^4}}\; - \;x\sqrt{2}\right\}$ (ii) $\underset{x\to-\infty}{\text{Limit}}\dfrac{x^5\tan\left(\dfrac{1}{\pi x^2}\right)+3|x|^2+7}{|x|^3+7|x|+8}$

11. Evaluate the following limits (i) $\underset{x\to 0}{\text{Limit}}\left[\sin^2\left(\dfrac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$

(ii) $\underset{x\to\infty}{\text{Limit}}\left(\dfrac{a_1^{1/x}+a_2^{1/x}+a_3^{1/x}+.......+a_n^{1/x}}{n}\right)^{nx}$, where $a_1, a_2, a_3, ..., a_n > 0$.

12. Find the values of a & b so that: (i) $\underset{x\to 0}{\text{Limit}}\dfrac{(1+ax\,\sin x)-(b\cos x)}{x^4}$ may find to a definite limit.

(ii) $\lim\limits_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right) = 4$

13. Find the limits using expansion : $\underset{x\to 0}{\text{Limit}}\left[\dfrac{\ell n(1+x)^{(1+x)}}{x^2}-\dfrac{1}{x}\right]$

14. Let $f(x) = \dfrac{\sin^{-1}(1-\{x\})\,.\,\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,.\,(1-\{x\})}$ then find $\underset{x\to 0^+}{\text{Limit}}\ f(x)$ and $\underset{x\to 0^-}{\text{Limit}}\ f(x)$, where {.} denotes the fractional part function.

15. Let $f(x) = \underset{m\to\infty}{\text{Limit}}\left\{\underset{n\to\infty}{\text{Limit}}\left(\cos^{2m}(n!\pi x)\right)\right\}$ where $x \in R$. Prove that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational}\cdot \end{cases}$$

16. Evaluate $\underset{x\to 0^+}{\text{Limit}}\left\{\underset{n\to\infty}{\text{Limit}}\left(\dfrac{[1^2(\sin x)^x]+[2^2(\sin x)^x]+.......+[n^2(\sin x)^x]}{n^3}\right)\right\}$,

where [.] denotes the greatest integer function.

17. Evaluate the following limits

(i) $\underset{n\to\infty}{\text{Limit}}\ \cos\dfrac{x}{2}\ \cos\dfrac{x}{4}\ \cos\dfrac{x}{8}\,........\ \cos\dfrac{x}{2^n}$

(ii) $\underset{n\to\infty}{\text{Limit}}\ \dfrac{1}{2}\tan\dfrac{x}{2} + \dfrac{1}{2^2}\tan\dfrac{x}{2^2} + \dfrac{1}{2^3}\tan\dfrac{x}{2^3} + ... + \dfrac{1}{2^n}\tan\dfrac{x}{2^n}$.

(iii) $\underset{x\to\infty}{\text{Limit}}\ \log_{x-1}(x).\log_x(x+1).\log_{x+1}(x+2).\log_{x+2}(x+3).....\log_k(x^5)$; where $k = x^5 - 1$.

(iv) Let $P_n = \dfrac{2^3-1}{2^3+1}\cdot\dfrac{3^3-1}{3^3+1}\cdot\dfrac{4^3-1}{4^3+1}\,.........\,\dfrac{n^3-1}{n^3+1}$. Prove that $\underset{n\to\infty}{\text{Limit}}\ P_n = \dfrac{2}{3}$.

ANSWER

EXERCISE–1

Q 1. 3 **Q 2.** $\frac{45}{91}$ **Q 3.** 2 **Q 4.** $\frac{p-q}{2}$ **Q 5.** $\frac{2}{\sqrt{3}}$ **Q.6** $-\frac{1}{3}$

Q 7. (a) $\pi/2$ if $a > 0$; 0 if $a = 0$ and $-\pi/2$ if $a < 0$(b) $f(x) = |x|$

Q 8. 5050 **Q 9.** $a = \frac{1}{2}$; $r = \frac{1}{4}$; $S = \frac{2}{3}$ **Q 10.** $\ln 2$ **Q 11.** does not exist **Q 12.** 2

Q 13. 1 **Q 14.** $\frac{3}{2}$ **Q 15.** $\frac{1}{16\sqrt{2}}$ **Q 16.** $\frac{2\ln 2}{\pi}$

Q 17. a = 2 ; limit = 1 **Q 18.** $\frac{1}{32}$ **Q 19.** $-\frac{9}{4}\ln\frac{4}{e}$ **Q 20.** 2

Q 21. Domain, $x \in R$, Range, $x = \frac{n\pi}{2}$, $n \in I$ **Q 22.** does not exist **Q 23.** 9

Q 24. $\cos^2\alpha\ \ln\text{Cos}\alpha + \text{Sin}^2\alpha\ \ln\text{Sin}\alpha$ **Q 25.** $8\sqrt{2}(\ln 3)^2$ **Q 26.** –3, –3, –3

Q 27. $(\ln a)^n$ **Q 28.** –2 **Q 29.** 0, 0 **Q. 30** 4

EXERCISE–2

Q 1. e^{-8} **Q 2.** c = ln2 **Q 3.** $e^{-\frac{1}{2}}$ **Q 4.** e^{-a^2/b^2} **Q.5** $-\frac{\pi^2}{4}$ **Q 6.** $e^{-2\pi^2 a^2}$ **Q 7.** e^{-1}

Q 8. $e^{-1/2}$ **Q 9.** $(a_1.a_2.a_3....a_n)$ **Q 10.** $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$ **Q 11.** a = c = 1, b = 2 **Q 12.** $\frac{\pi^2 a^2 + 4}{16a^4}$

Q 13. $\frac{2}{3}$ **Q 14.** f(x) when $|x| > 1$; g(x) when $|x| < 1$; $\frac{g(x)+f(x)}{2}$ when x = 1 & not defined when x = -1

Q 15. $\frac{x}{2}$ **Q 16.** $-\frac{1}{\pi^2}$ **Q 17.** a - b **Q 19.** 1/2 **Q 20.** $\frac{2}{3}$

Q 21. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit $= \frac{3}{2}$

Q 22. (a) 1 **(b)** $\frac{1}{2}$ **Q 23.** $\frac{\theta}{\tan\theta}$ **Q 24.** $a = e^2$ **Q 25.** e^{π^2}

Q 26. $\frac{1}{4}$ **Q 28.** (a) 1/2, (b) b **Q 29. (i)** a =1, b = –1 **(ii)** a = –1 , b = $\frac{1}{2}$

EXERCISE–3

Q 1. e^2 **Q 2.** D **Q 3.** C **Q 4.** C **Q5.** B

Q6. $\ln a$ **Q7.** C **Q8.** C **Q.9** $1-\frac{2}{\pi}$

EXERCISE–4

1. D	**2.** C	**3.** D	**4.** D	**5.** D	**6.** D	**7.** C	**8.** B	**9.** B	**10.** C
11. A	**12.** B	**13.** C	**14.** C	**15.** C	**16.** B	**17.** C	**18.** C	**19.** C	**20.** C
21. D	**22.** A	**23.** A	**24.** B	**25.** A	**26.** C	**27.** A	**28.** C	**29.** A	**30.** C
31. B	**32.** B	**33.** B							

EXERCISE–5

1. (i) 0 (ii) Limit does not exists (iii) Limit does not exists

2. 6 **3.** (i) (– 8) (ii) $\frac{2}{3\sqrt{3}}$ **4.** (i) 1/3 (ii) $\frac{5}{2}(a + 2)^{3/2}$ (iii) $2e^2$

5. (i) 1/2 (ii) 1/2 (iii) zero (iv) ∞ **6.** (i) $\frac{1}{3}$ (ii)a = 2, b = 1, c = –1 and value = $-\frac{1}{3}$

7. $\frac{x}{3}$ **8.** {–1, 0, 1} ***9.*** *(i)* $-\frac{9}{4}\ln\frac{4}{e}$ *(ii)* $\cos^4 a \ln(\cos a) - \sin^4 a \ln(\sin a)$

10. (i) $\frac{1}{4\sqrt{2}}$ (ii) $-\frac{1}{\pi}$ **11.** (i) $e^{-\frac{a^2}{b^2}}$ (ii) $(a_1 a_2 a_3 \ldots\ldots a_n)$

12. (i) $a = -\frac{1}{2}$, b = 1 (ii) a = 2, b $\in$ R, c = 5, d $\in$ R **13.** $\frac{1}{2}$

14. $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$ **16.** $\frac{1}{3}$ **17.** (i) $\frac{\sin x}{x}$ (ii) $\frac{1}{x} - \cot x$ (iii) 5