Permutations and Combinations

- **Fundamental Principle of Counting:** If an event occurs in m different ways, following which another event occurs in n different ways, then the total number of occurrence of the events in the given order is $m \times n$. This is called the fundamental principle of counting.

Example:

Find the number of 5-letter words, with or without meaning, which can be formed out of the letters of the word MATHS, where the repetition of digits is not allowed.

Solution:

There are as many words as there are ways of filling 5 vacant places $\square\square\square\square\square$ by the 5 letters.

The first place can be filled with any of the 5 letters in 5 different ways, following which the second place can be filled with any of the remaining 4 letters in 4 different ways, following which the third place can be filled in 3 different ways, following which the fourth place can be in 2 different ways, following which the fifth place can be filled in 1 way.

Thus, the number of ways in which the 5 places can be filled, by the multiplication principle, is $5 \times 4 \times 3 \times 2 \times 1 = 120$.

Note: If repetition of letters had been allowed, then the required number of words would be $5 \times 5 \times 5 \times 5 \times 5 = 3125$.

- **Factorial notation:** The notation $n!$ represents the product of the first n natural numbers, i.e.,

$n! = n \times (n-1) \times (n-2) \times \ldots\ldots \times 5 \times 4 \times 3 \times 2 \times 1$

$0! = 1$

Example:

$$\frac{12!}{8!} = \frac{12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1}$$

$$=12\times11\times10\times9$$

$$=11880$$

- **Permutation when all objects are distinct:** A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

The number of permutations of n different things taken r at a time, when

  - repetition is not allowed, is ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.
  - repetition is allowed, is n^r, where $0 \le r \le n$.

Example 1: Twenty five students are participating in a competition. In how many ways, can the first three prizes be won in such a way that a prize cannot be shared by more than one student?

Solution: The total number of ways in which first three prizes can be won is the number of arrangements of 25 different things taken 3 at a time.
So, required number of ways = ${}^{25}P_3$

$$= \frac{25!}{(25-3)!}$$

$$= \frac{25!}{22!}$$

$$= \frac{25 \times 24 \times 23 \times 22!}{22!}$$

$$= 25 \times 24 \times 23 = 13800$$

Example 2: Find the total number of four digit numbers that can be formed by using the digits 0, 2, 5, and 6?

Solution: A four digit number has four places i.e., units, tens, hundreds and thousands. Units, tens and hundreds place can be filled with either 0, 2, 5, or 6 where as thousands place can be filled with 2, 5 or 6 only.

Number of ways to fill the units place = 4
Number of ways to fill the tens place = 4
Number of ways to fill the hundreds place = 4
Number of ways to fill the thousands place = 3 ways.
∴ Total number of four digit numbers = 4 × 4 × 4 × 3 = 192

- **Concept of permutations when all objects are not distinct**
    - The number of permutations of n objects, when p objects are of the same kind and the rest are all different, is $\frac{n!}{p!}$.
        - In general, the number of permutations of n objects, when p_1 objects are of one kind, p_2 are of the second kind , ... , p_k are of the k^{th} kind and the rest, if any, are of different kinds, is $\frac{n!}{p_1!\, p_2! \ldots\ldots p_k!}$.

Example: Find the number of permutations of the letters of the word ARRANGEMENT.

Solution: Here, there are 11 objects (letters) of which there are 2A's, 2R's, 2N's, 2E's and the rest are all different.
∴ Required number of arrangements

$$= \frac{11!}{2!\,2!\,2!\,2!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 2}{2 \times 2 \times 2 \times 2}$$

$$= 2494800$$

- **Combinations:** The number of combinations of n different things taken r at a time is denoted by ${}^{n}C_{r}$, which is given by

$$^nC_r = \frac{n!}{r!(n-r)!}, \text{ where } 0 \le r \le n.$$

In particular, $^nC_0 = {}^nC_n = 1$

Example 1: A box contains 8 red bulbs and 5 blue bulbs. Determine the number of ways in which 4 red and 2 blue bulbs can be selected.

Solution:

It is given that a box contains 8 red bulbs and 5 blue bulbs.

Now, 4 red bulbs can be selected from 8 red bulbs in $^8C_4$ number of ways, and 2 blue bulbs can be selected from 4 blue bulbs in $^4C_2$ number of ways.

Hence, 4 red bulbs and 2 blue bulbs can be selected from a box containing 8 red bulbs and 4 blue bulbs in $^8C_4 \times {}^4C_2$ number of ways.

Now,

$^8C_4 \times {}^4C_2$

$= \frac{8!}{4! \times (8-4)!} \times \frac{4!}{2!(4-2)!}$

$= \frac{8!}{4! \times 4!} \times \frac{4!}{2!2!}$

$= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}$

$= 70 \times 6$

$= 420$

Thus, the number of ways of selecting the bulbs is 420.

- $^nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = {}^nC_r$

In other words, selecting r objects out of n objects is the same as rejecting $(n-r)$ objects.

- $^nC_a = {}^nC_b \Rightarrow a = b$ or $a = n$, i.e., $n = a + b$

Example 2: If $^{19}C_{3r} = {}^{19}C_{2r+4}$, then find the value of r.

Solution:

$^{19}C_{3r} = {}^{19}C_{2r+4}$

$\Rightarrow 3r + (2r + 4) = 19$ or $3r = 2r + 4$

$\Rightarrow 3r + (2r + 4) = 19$ $\Rightarrow r = 4$

$\Rightarrow 5r + 4 = 19$

$\Rightarrow 5r = 19 - 4 = 15$

$\Rightarrow r = 3$

$\therefore$ The value of r is either 3 or 4.

- $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

Example 3: If $^nC_{r-1} + {}^nC_r + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2} = {}^{n+a}C_{r+(a-1)}$, then find the value of a.

Solution:

$^{n}C_{r-1} + {}^{n}C_{r} + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$

$= {}^{n+1}C_{r} + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$ $\quad (\because {}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r})$

$= {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2}$

$= {}^{n+3}C_{r+2}$

$\therefore {}^{n+3}C_{r+2} = {}^{n+a}C_{r+(a-1)}$

$\Rightarrow a = 3$