Chapter 8
Central-Force Motion

In this Chapter we will use the theory we have discussed in Chapter 6 and 7 and apply it to very important problems in physics, in which we study the motion of two-body systems on which central force are acting. We will encounter important examples from astronomy and from nuclear physics.

Two-Body Systems with a Central Force

Consider the motion of two objects that are effected by a force acting along the line connecting the centers of the objects. To specify the state of the system, we must specify six coordinates (for example, the (x, y, z) coordinates of their centers). The Lagrangian for this system is given by

$$L = \frac{1}{2}m_1\left|\dot{\bar{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\bar{r}}_2\right|^2 - U\left(\bar{r}_1 - \bar{r}_2\right)$$

Note: here we have assumed that the potential depends on the position vector between the two objects. This is not the only way to describe the system; we can for example also specify the position of the center-of-mass, R, and the three components of the relative position vector r. In this case, we choose a coordinate system such that the center-of-mass is at rest, and located at the origin. This requires that

$$\bar{R} = \frac{m_1}{m_1 + m_2}\bar{r}_1 + \frac{m_2}{m_1 + m_2}\bar{r}_2 = 0$$

The relative position vector is defined as

$$\bar{r} = \bar{r}_1 - \bar{r}_2$$

The position vectors of the two masses can be expressed in terms of the relative position vector:

$$\bar{r}_1 = \frac{m_2}{m_1 + m_2}\bar{r}$$

$$\bar{r}_2 = \frac{m_1}{m_1 + m_2}\bar{r}$$

The Lagrangian can now be rewritten as

$$L = \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\bar{r}}|^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2 |\dot{\bar{r}}|^2 - U(\bar{r}) = \frac{1}{2}\mu|\dot{\bar{r}}|^2 - U(\bar{r})$$

where μ is the reduced mass of the system:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Two-Body Systems with a Central Force: Conserved Quantities

Since we have assumed that the potential U depends only on the relative position between the two objects, the system poses spherical symmetry. As we have seen in Chapter 7, this type of symmetry implies that the angular momentum of the system is conserved. As a result, the momentum and position vector will lay in a plane, perpendicular to the angular momentum vector, which is fixed in space. The three-dimensional problem is thus reduced to a two-dimensional problem. We can express the Lagrangian in terms of the radial distance r and the polar angle θ:

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

The generalized momenta for this Lagrangian are

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$$

The Lagrange equations can be used to determine the derivative of these momenta with respect to time:

$$\dot{p}_r = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0$$

The last equation tells us that the generalized momentum p_θ is constant:

$$l = \mu r^2 \dot{\theta} = \text{constant}$$

The constant l is related to the areal velocity. Consider the situation in Figure 1. During the time interval dt, the radius vector sweeps an area dA where
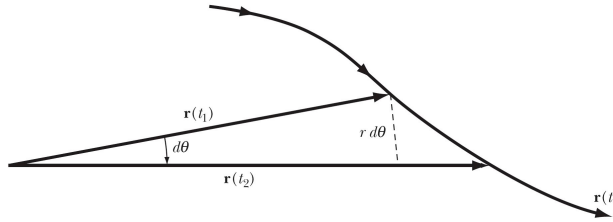
$$dA = \frac{1}{2} r^2 d\theta$$

Figure 1. Calculation of the areal velocity.

The areal velocity, dA/dt, is thus equal to

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{l}{\mu r^2} = \frac{l}{2\mu} = \text{constant}$$

This result is also known as **Kepler's Second Law**.

The Lagrangian for the two-body system does not depend explicitly on time. In Chapter 7 we showed that in that case, the energy of the system is conserved. The total energy E of the system is equal to

$$E = T + U = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\left(\frac{l}{\mu r^2}\right)^2\right) + U(r) =$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

Two-Body Systems with a Central Force: Equations of Motion

If the potential energy is specified, we can use the expression for the total energy E to determine dr/dt:

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2}}$$

This equation can be used to find the time t as function of r:

$$t = \int dt = \pm \int \frac{1}{\sqrt{\frac{2}{\mu}(E - U(r)) - \frac{l^2}{\mu^2 r^2}}} dr$$

However, in many cases, the shape of the trajectory, $\theta(r)$, is more important than the time dependence. We can express the change in the polar angle in terms of the change in the radial distance:

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr} dr = \frac{\dot{\theta}}{\dot{r}} dr$$

Integrating both sides we obtain the following orbital equation

$$\theta(r) = \int \frac{\dot{\theta}}{\dot{r}} dr = \pm \int \frac{\frac{l}{\mu r^2}}{\sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2}}} dr = \pm \int \frac{l}{r^2 \sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}} dr$$

The extremes of the orbit can be found in general by requiring that $dr/dt = 0$, or

$$\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2} = \frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right) = 0$$

In general, this equation has two solutions, and the orbit is confined between a minimum and maximum value of r. Under certain conditions, there is only a single solution, and in that case the orbit is circular. Using the orbital equation we can determine the change in the polar angle when the radius changes from r_{min} to r_{max}. During one period, the polar angle will change by

$$\Delta\theta = 2 \int_{r_{min}}^{r_{max}} \frac{l}{r^2 \sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}} dr$$

If the change in the polar angle is a rational fraction of 2π then after a number of complete orbits, the system will have returned to its original position. In this case, the orbit is closed. In all other cases, the orbit is open.

The orbital motion is specified above in terms of the potential U. Another approach to study the equations of motion is to start from the Lagrange equations. In this case we obtain an equation of motion that includes the force F instead of the potential U:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)$$

This version of the equations of motion is useful when we can measure the orbit and want to find the force that produces this orbit.

Example: Problem 8.8

Investigate the motion of a particle repelled by a force center according to the law $F(r) = kr$. Show that the orbit can only be hyperbolic.

The general expression for $\theta(r)$ is [see Eq. (8.17) in the text book]

$$\theta(r) = \int \frac{\left(\ell/r^2\right)dr}{\sqrt{2\mu\left[E - U - \dfrac{\ell^2}{2\mu r^2}\right]}} \tag{8.8.1}$$

where

$$U = -\int kr\, dr = -kr^2/2$$

in the present case. Substituting $x = r^2$ and $dx = 2rdr$ into (8.8.1), we have

$$\theta(r) = \frac{1}{2}\int \frac{dx}{x\sqrt{\dfrac{\mu k}{\ell^2}x^2 + \dfrac{2\mu E}{\ell^2}x - 1}} \tag{8.8.2}$$

Using Eq. (E.10b), Appendix E,

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left[\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right] \tag{8.8.3}$$

and expressing again in terms of r, we find

$$\theta(r) = \frac{1}{2}\sin^{-1}\left[\frac{\left[\frac{\mu E}{\ell^2}r^2 - 1\right]}{r^2\sqrt{\frac{\mu^2 E^2}{\ell^4} + \frac{\mu k}{\ell^2}}}\right] + \theta_0 \tag{8.8.4}$$

or,

$$\sin 2(\theta - \theta_0) = \frac{1}{\sqrt{1 + \frac{\ell^2 k}{\mu E^2}}} - \frac{1}{r^2}\frac{\ell^2/\mu E}{\sqrt{1 + \frac{\ell^2 k}{\mu E^2}}} \tag{8.8.5}$$

In order to interpret this result, we set

$$\left.\begin{array}{c}\sqrt{1 + \dfrac{\ell^2 k}{\mu E^2}} \equiv \varepsilon' \\[2em] \dfrac{\ell^2}{\mu E} \equiv \alpha'\end{array}\right] \tag{8.8.6}$$

and specifying $\theta_0 = \pi/4$, (8.8.5) becomes

$$\frac{\alpha'}{r^2} = 1 + \varepsilon'\cos 2\theta \tag{8.8.7}$$

or,

$$\alpha' = r^2 + \varepsilon' r^2\left(\cos^2\theta - \sin^2\theta\right) \tag{8.8.8}$$

Rewriting (8.8.8) in x-y coordinates, we find

$$\alpha' = x^2 + y^2 + \varepsilon'\left(x^2 - y^2\right) \tag{8.8.9}$$

or,

$$1 = \frac{x^2}{\frac{\alpha'}{1+\varepsilon'}} + \frac{y^2}{\frac{\alpha'}{1-\varepsilon'}} \tag{8.8.10}$$

Since $\alpha' > 0$, $\varepsilon' > 1$ from the definition, (8.8.10) is equivalent to

$$1 = \frac{x^2}{\frac{\alpha'}{1+\varepsilon'}} + \frac{y^2}{\frac{\alpha'}{|1-\varepsilon'|}} \tag{8.8.11}$$

which is the equation of a hyperbola.

Solving the Orbital Equation

The orbital equation can only be solved analytically for certain force laws. Consider for example the gravitational force. The corresponding potential is $-k/r$ and the polar angle θ is thus equal to

$$\theta(r) = \pm \int \frac{l/r^2}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} \, dr$$

Consider the change of variables from r to $u = l/r$:

$$\theta(r) = \pm \int \frac{(l/r)^2/l}{\sqrt{2\mu\left(E + \frac{k}{l}\frac{l}{r} - \frac{1}{2\mu}\left(\frac{l}{r}\right)^2\right)}} \, d\left(\frac{l}{u}\right) = \pm \int \frac{u^2/l}{\sqrt{2\mu\left(E + \frac{k}{l}u - \frac{1}{2\mu}u^2\right)}} \frac{l}{u^2} \, du =$$

$$= \pm \int \frac{1}{\sqrt{2\mu\left(E + \frac{k}{l}u - \frac{1}{2\mu}u^2\right)}} \, du$$

The integral can be solved using one of the integrals found in Appendix E (see E8.c):

$$\theta(r) \;=\; \pm \int \frac{1}{\sqrt{-u^2 + 2\mu \dfrac{k}{l} u + 2\mu E}}\, du = \pm \sin^{-1}\left(\frac{-2u + 2\mu \dfrac{k}{l}}{\sqrt{\left(2\mu \dfrac{k}{l}\right)^2 + 8\mu E}}\right) + C =$$

$$= \pm \sin^{-1}\left(\frac{\mu \dfrac{k}{l} - u}{\sqrt{\left(\mu \dfrac{k}{l}\right)^2 + 2\mu E}}\right) + C = \pm \sin^{-1}\left(\frac{\mu \dfrac{k}{l} - \dfrac{l}{r}}{\sqrt{\left(\mu \dfrac{k}{l}\right)^2 + 2\mu E}}\right) + C =$$

$$= \pm \sin^{-1}\left(\frac{\mu k - \dfrac{l^2}{r}}{\sqrt{(\mu k)^2 + 2\mu l^2 E}}\right) + C$$

This equation can be rewritten as

$$\sin\left(\theta + \text{constant}\right) = \frac{\mu k - \dfrac{l^2}{r}}{\sqrt{(\mu k)^2 + 2\mu l^2 E}}$$

We can always choose our reference position such that the constant is equal to $\pi/2$ and we thus find the following solution:

$$\cos\left(\theta\right) = \frac{\mu k - \dfrac{l^2}{r}}{\sqrt{(\mu k)^2 + 2\mu l^2 E}}$$

We can rewrite this expression such that we can determine the distance r as function of the polar angle:

$$r = \frac{l^2}{\mu k - \sqrt{(\mu k)^2 + 2\mu l^2 E}\,\cos(\theta)} = \frac{l^2}{\mu k \left(1 - \sqrt{1 + \dfrac{2}{\mu}\dfrac{l^2}{k^2} E}\,\cos(\theta)\right)}$$

Since $\cos\theta$ varies between -1 and +1, we see that the minimum (the **pericenter**) and the maximum (the **apocenter**) positions are

$$r_{min} = \frac{l^2}{\mu k \left(1 + \sqrt{1 + \frac{2}{\mu} \frac{l^2}{k^2} E} \right)}$$

$$r_{max} = \frac{l^2}{\mu k \left(1 - \sqrt{1 + \frac{2}{\mu} \frac{l^2}{k^2} E} \right)}$$

The equation for the orbit is in general expressed in terms of the **eccentricity** ε and the **latus rectum** 2α:

$$\varepsilon = \sqrt{1 + \frac{2}{\mu} \frac{l^2}{k^2} E}$$

$$\alpha = \frac{l^2}{\mu k}$$

The possible orbits are usually parameterized in terms of the eccentricity, and examples are shown Figure 2.
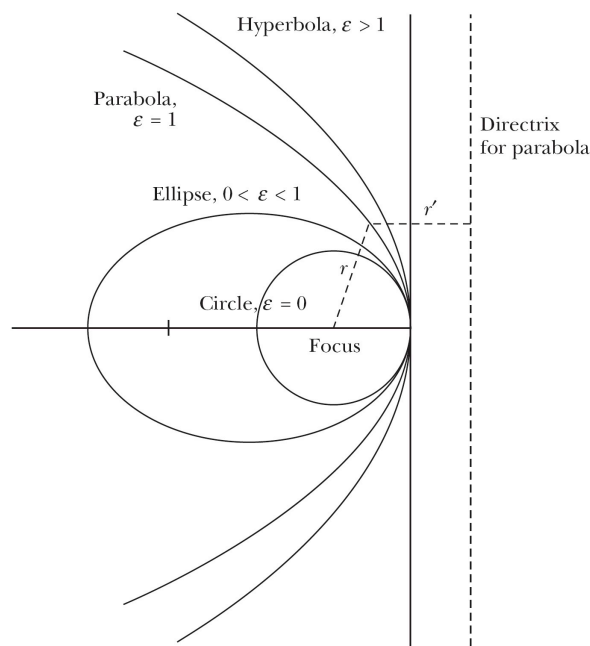

Figure 2. Possible orbits in the gravitational field.

The period of the orbital motion can be found by integrating the expression for dt over one complete period:

$$\tau = \int dt = \frac{2\mu}{l}\int dA = \frac{2\mu}{l}(\pi ab) = \frac{2\mu}{l}\left(\pi \frac{k}{2|E|}\frac{l}{\sqrt{2\mu|E|}}\right) = \pi k \sqrt{\frac{\mu}{2}}|E|^{-3/2}$$

When we take the square of this equation we get Kepler's third law:

$$\tau^2 = \pi^2 k^2 \frac{\mu}{2}|E|^{-3} = \pi^2 k^2 \frac{\mu}{2}\left(\frac{2a}{k}\right)^3 = \frac{4\pi^2\mu}{k}a^3$$

The Centripetal Force and Potential

In the previous discussion it appears as if the potential U is modified by the term $l^2/(2\mu r^2)$. This term depends only on the position r since l is constant, and it is interpreted as a potential energy. The force associated with this potential energy is

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 = \mu\frac{\left(r\dot{\theta}\right)^2}{r}$$

This force is often called the **centripetal force** (although it is not a real force), and the potential is called the **centripetal potential**. This potential is a fictitious potential and it represents the effect of the angular momentum about the origin. Figure 3 shows an example of the real potential, due to the gravitational force in this case, and the centripetal potential. The effective potential is the sum of these two potentials and has a characteristic dip where the potential energy has a minimum. The result of this dip is that there are certain energies for which the orbit is bound (has a minimum and maximum distance). These turning points are called the **apsidal distances** of the orbit.
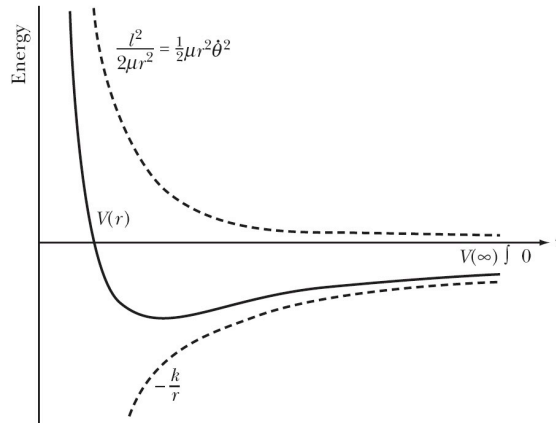

Figure 3. The effective potential for the gravitational force when the system has an angular momentum l.

We also note that at small distances the force becomes repulsive.

Example: Problem 8.22

Discuss the motion of a particle moving in an attractive central-force field described by $F(r) = -k/r^3$. Sketch some of the orbits for different values of the total energy.

For the given force

$$F(r) = -\frac{k}{r^3}$$

the potential is

$$U(r) = -\frac{k}{2r^2} \tag{8.22.1}$$

and the effective potential is

$$V(r) = \frac{1}{2}\left[\frac{\ell^2}{\mu} - k\right]\frac{1}{r^2} \tag{8.22.2}$$

The equation of the orbit is [cf. Eq. (8.20) in the text book]

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2}\left(-ku^3\right) \tag{8.22.3}$$

or,

$$\frac{d^2u}{d\theta^2} + \left[1 - \frac{\mu k}{\ell^2}\right]u = 0 \tag{8.22.4}$$

Let us consider the motion for various values of ℓ.

i) $\ell^2 = \mu k$:

In this case the effective potential $V(r)$ vanishes and the orbit equation is

$$\frac{d^2 u}{d\theta^2} = 0 \qquad (8.22.5)$$

with the solution

$$u = \frac{1}{r} = A\theta + B \qquad (8.226)$$

and the particle spirals towards the force center.

ii) $\ell^2 > \mu k$:

In this case the effective potential is positive and decreases monotonically with increasing r. For any value of the total energy E, the particle will approach the force center and will undergo a reversal of its motion at $r = r_0$; the particle will then proceed again to an infinite distance. Setting

$$1 - \mu k / \ell^2 \equiv \beta^2 > 0$$

equation (8.22.4) becomes

$$\frac{d^2 u}{d\theta^2} + \beta^2 u = 0 \qquad (8.22.7)$$

with the solution

$$u = \frac{1}{r} = A \cos(\beta\theta - \delta) \qquad (8.22.8)$$

Since the minimum value of u is zero, this solution corresponds to unbounded motion, as expected from the form of the effective potential $V(r)$.

iii) $\ell^2 < \mu k$:

For this case we set

$$\mu k / \ell^2 - 1 \equiv G^2 > 0$$

and the orbit equation becomes

$$\frac{d^2 u}{d\theta^2} - G^2 u = 0 \tag{8.22.9}$$

with the solution

$$u = \frac{1}{r} = A \cosh\left(\beta\theta - \delta\right) \tag{8.22.10}$$

so that the particle spirals in towards the force center.

Orbital Motion
The understanding of orbital dynamics is very important for space travel. The orbit in which a spaceship travels is determined by the energy of the spaceship. When we change the energy of the ship, we will change the orbit from for example a spherical orbit to an elliptical orbit. By changing the velocity at the appropriate point, we can control the orientation of the new orbit.

The Hofman transfer represents the path of minimum energy expenditure to move from one solar-based orbit to another. Consider travel from earth to mars (see Figure 4). The goal is to get our spaceship in an orbit that has apsidal distances that correspond to the distance between the earth and the sun and between mars and the sun. This requires that

$$r_1 = a(1 - \varepsilon)$$

and

$$r_2 = a(1 + \varepsilon)$$

The eccentricity of such an orbit is thus equal to

$$\varepsilon = \frac{r_2 - r_1}{2a}$$

The total energy of an orbit with a major axis of $a = (r_1 + r_2)/2$ is equal to

$$E = \frac{k}{2a} = \frac{k}{(r_1 + r_2)}$$

Since the space ship starts from a circular orbit with a major axis $a = r_1$, its initial energy is equal to
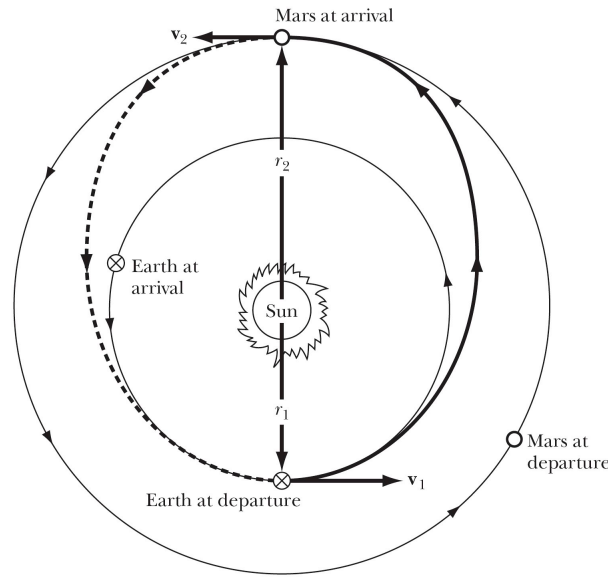
$$E = -\frac{k}{2r_1}$$

Figure 4. The Hofman transfer to travel from earth to mars.

The increase in the total energy is thus equal to

$$\Delta E = -\frac{k}{(r_1 + r_2)} - \left(-\frac{k}{2r_1}\right) = \left(\frac{k}{2}\right)\left\{\frac{1}{r_1} - \frac{2}{(r_1 + r_2)}\right\} = \left(\frac{k}{2}\right)\frac{r_2 - r_1}{r_1(r_1 + r_2)}$$

This energy must be provided by the thrust of the engines that increase the velocity of the space ship (note: the potential energy does not change at the moment of burn, assuming the thrusters are only fired for a short period of time).

   The problem with the Hofman transfer mechanism is that the conditions have to be just right, and only of the planets are in the proper position will the transfer work. There are many other ways to travel between earth and mars. Many of these require less time than the time required for the Hofman transfer, but they require more fuel (see Figure 5).
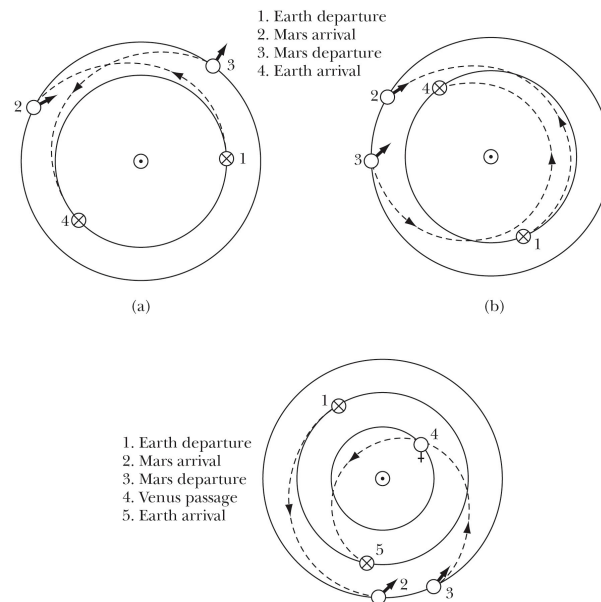
Figure 5. Different ways to get from earth to mars.

SECTIONS 8.9 AND 8.10 WILL BE SKIPPED!