fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
i#"k flg ldYi dj] lgr foifr vud] ^cuk^ u NkM /;; dk] j?kcj jk[k VdAA

jfpr% ekuo /keZ i.krk
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STUDY PACKAGE

Subject : Mathematics
Topic : Continuity & Diffrentiability

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Continuity

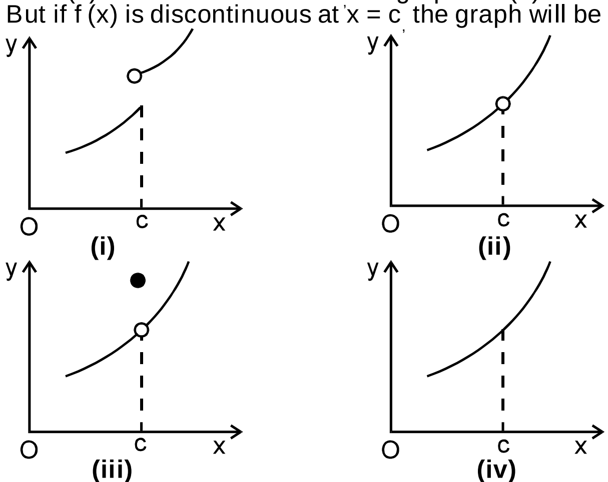
1. A function f(x) is said to be continuous at x = c,

if $\underset{x \to c}{\text{Limit}}$ f(x) = f(c). Symbolically f is continuous at

x = c if $\underset{h \to 0}{\text{Limit}}$ f(c – h) = $\underset{h \to 0}{\text{Limit}}$ f(c+h) = f(c).

i.e. LHL at x = c = RHL at x = c equals value of 'f' at x = c.

If a function f (x) is continuous at x = c the graph of f (x) at the corresponding point { c f (c) } will not be broken. But if f (x) is discontinuous at x = c the graph will be broken at the corresponding point.



((i), (ii) and (iii) are discontinuous at x = c)
((iv) is continuous at x = c)

A function f can be discontinuous due to any of the following three reasons:

(i) $\underset{x \to c}{\text{Limit}}$ f(x) does not exist i.e. $\underset{x \to c^-}{\text{Limit}}$ f(x) ≠ $\underset{x \to c^+}{\text{Limit}}$ f (x) **[figure (i)]**

(ii) f (x) is not defined at x = c **[figure (ii)]**

(iii) $\underset{x \to c}{\text{Limit}}$ f(x) ≠ f (c) **[figure (iii)]**

Geometrically, the graph of the function will exhibit a break at x= c.

Solved Example # 1 Find whether f(x) is continuous or not at x = 1

f(x) = sin $\frac{\pi x}{2}$; x < 1
= [x] x ≥ 1

Solution $f(x) = \begin{cases} \sin\frac{\pi x}{2} & \forall \quad x < 1 \\ [x] & \forall \quad x \ge 1 \end{cases}$ for continuity at x = 1, we determine, f(1), $\lim\limits_{x \to 1^-}$ f(x) and $\lim\limits_{x \to 1^+}$ f(x).

Now, f(1) = [1] = 1

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1$$

and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} [x] = 1$

so $f(1) = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$ ∴ f(x) is continuous at x = 1

Self practice problems :

1. If possible find value of λ for which f(x) is continuous at x = $\frac{\pi}{2}$

$$f(x) = \frac{1-\sin x}{1+\cos 2x}, \quad x < \frac{\pi}{2}$$
$$= \lambda \quad x = \frac{\pi}{2}$$
$$= \frac{\sqrt{2x-\pi}}{\sqrt{4+\sqrt{2x-\pi}}-2} \quad x > \frac{\pi}{2}$$

Answer discontinuous

2. Find the values of a and b such that the function

$$f(x) = x + a\sqrt{2}\sin x \quad ; \quad 0 \le x < \frac{\pi}{4}$$
$$= 2x\cot x + b \quad \frac{\pi}{4} \le x \le \frac{\pi}{2}$$

$= a \cos 2x - b \sin x \qquad \frac{\pi}{2} < x \le \pi$ is continuous at $\frac{\pi}{4}$ and $\frac{\pi}{2}$

Answer $a = \frac{\pi}{6}$, $b = \frac{-\pi}{12}$

3. If $f(x) = (1+ax)^{\frac{1}{x}} \qquad x < 0$

$= b \qquad x = 0$

$= \frac{(x+c)^{\frac{1}{3}} - 1}{x} \qquad x > 0$

The find the values of a, b, c, f(x) is continuous at x = 0 **Answer** $a = -\ln 3$, $b = \frac{1}{3}$, $c = 1$

2. Types of Discontinuity :

(a) Removable Discontinuity

In case $\underset{x\to c}{\text{Limit}}$ f(x) exists but is not equal to f(c) then the function is said to have a removable discontinuity. In this case we can redefine the function such that $\underset{x\to c}{\text{Limit}}$ f(x) = f(c) & make it continuous at x = c.

Removable type of discontinuity can be further classified as :

(i) Missing Point Discontinuity :

Where $\underset{x\to a}{\text{Limit}}$ f(x) exists finitely but f(a) is not defined.

e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1.

(ii) Isolated Point Discontinuity:

Where $\underset{x\to a}{\text{Limit}}$ f(x) exists & f(a) also exists but;

$\underset{x\to a}{\text{Limit}}$ f(x) ≠ f(a). e.g. $f(x) = \frac{x^2-16}{x-4}$, x ≠ 4 & f (4) = 9 has a break at x = 4.

(b) Irremovable Discontinuity: In case $\underset{x\to c}{\text{Limit}}$ f(x) does not exist then it is not possible to make the function continuous by redefining it. However if both the limits (i.e. L.H. L. & R.H.L.) are finite, then discontinuity is said to be of first kind otherwise it is non–removable discontinuity of second kind.

Irremovable type of discontinuity can be further classified as:

(i) Finite discontinuity e.g. f(x) = x – [x] at all integral x.

(ii) Infinite discontinuity e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at x = 4.

(iii) Oscillatory discontinuity e.g. $f(x) = \sin\frac{1}{x}$ at x = 0.

In all these cases the value of f (a) of the function at x = a (point of discontinuity) may or may not exist but $\underset{x\to a}{\text{Limit}}$ does not exist.

(c) Discontinuity of Ist kind

If L.H.L. and R.H.L both exist finitely then discontinuity is said to be of Ist kind

(d) Discontinuity of IInd kind

If either L.H.L. or R.H.L does not exist then discontinuity is said to be of IInd kind

(e) Point functions defined at single point only are to be treated as discontinuous.

eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

Solved Example # 2

If $f(x) = x \qquad x < 1$

$= x^2 \qquad x > 1$

then check if f(x) is continuous at x = 1 or not if not, then comment on the type of discontinuity.

Solution $f(x) = \begin{cases} x & \forall \ x < 1 \\ x^2 & \forall \ x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \text{finite}$

and f(1) is not defined.

So f(x) is discontinuous at x = 1 and this discontinuity is removable missing point discontinuity

Self practice problems :

4. $f(x) = x, \qquad x < 1$

$= x^2 \qquad x > 1$

$= 2 \qquad x = 1$ which type of discontinuity is there **Answer** isolated point discontinuity

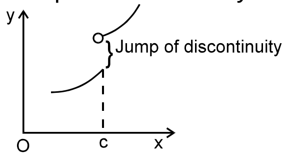
5. $f(x) = x \ ; \quad x < 1$

$= 2x \qquad 1 \le x$ Find which type of discontinuity it is.

Answer non removable of Ist kind

3. Jump of discontinuity

In case of non–removable discontinuity of the first kind the non–negative difference between the value of the RHL at x = c & LHL at x = c is called, the Jump of discontinuity.
Jump of discontinuity = | RHL – LHL |



NOTE : A function having a finite number of jumps in a given interval is called a Piece Wise Continuous or Sectionally Continuous function in this interval. For e.g. {x} , [x]

Solved Example # 3 f(x) = $\cos^{-1}\{\cot x\}$ $x < \frac{\pi}{2}$

= $\pi[x] - 1$ $x \geq \frac{\pi}{2}$ Find jump of discontinuity.

Ans. $= \frac{\pi}{2} - 1$

Sol. $f(x) = \begin{cases} \cos^{-1}\{\cot x\} & \text{if } x < \frac{\pi}{2} \\ \pi[x] - 1 & \text{if } x \geq \frac{\pi}{2} \end{cases}$

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \cos^{-1}\{\cot x\}$$
$$= \cos^{-1}\{0^+\}$$
$$= \cot^{-1} 0 = \frac{\pi}{2}$$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \pi[x] - 1 = \pi - 1$$

$\therefore$ jump of discontinuity $= \pi - 1 - \frac{\pi}{2}$

$$= \frac{\pi}{2} - 1$$

4. Continuity in an Interval :

(a) A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in$(a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if:
  (i) f is continuous in the open interval (a, b) &
  (ii) f is right continuous at 'a' i.e. $\text{Limit}_{x \to a^+} f(x) = f(a)$ = a finite quantity.
  (iii) f is left continuous at 'b' i.e. $\text{Limit}_{x \to b^-} f(x) = f(b)$ = a finite quantity.

(c) All Polynomials, Trigonometrical functions, Exponential and Logarithmic functions are continuous in their domains.

(d) Continuity of {f(x)} and [f(x)] should be checked at all points where f(x) becomes integer.

(e) Continuity of sgn (f(x)) should be checked at the points where f(x) = 0 (if f(x) is constanly equal to 0 when $x \to a$ then x = a is not a point of discontinuity)

(f) Continuity of a function should be checked at the points where definition of a function changes.

Solved Example # 5 If f(x) $= [\sin \pi x]$ $0 \leq x < 1$

$= \text{Sgn}\left(x - \frac{5}{4}\right)\left\{x - \frac{2}{3}\right\}$ $1 \leq x \leq 2$, where { . } represents fractional function

then comment on the continuity of function in the interval [0, 2].

Solution(i) Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2

(ii) For $[\sin \pi x]$, continuity should be checked at all values of x at which $\sin \pi x \in I$

i.e. $x = 0, \frac{1}{2}$

(iii) For sgn $\left(x - \frac{5}{4}\right)\left\{x - \frac{2}{3}\right\}$, continuity should be checked when $x - \frac{5}{4} = 0$ (as sgn (x) is discontinuous at x = 0) i.e. $x = \frac{5}{4}$ and when $x - \frac{2}{3} \in I$

i.e. $x = \frac{5}{3}$ (as {x} is discontinuous when $x \in I$)

$\therefore$ overall discontinuity should be checked at $x = 0, \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$ and 2

check the discontinuity your self.

Answer discontinuous at $x = \frac{1}{2}, 1\ \frac{5}{4}, \frac{5}{3}$

Self practice problems : **6.** If $f(x) = \text{sgn}\left(\left\{x - \frac{1}{2}\right\}\right)$ [ln x] $1 < x \le 3$

$= \{x^2\}$ $3 < x \le 3.5$

Find the point where the continuity of f(x) should be checked.

Ans. $\{ 1, \frac{3}{2}, \frac{5}{2}, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5 \}$

5. If f & g are two functions which are continuous at x = c then the functions defined by: $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

Note : (i) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ may be continuous but sum or difference function $\phi(x) = f(x) \pm g(x)$ will necessarily be discontinuous at x = a. e.g.

$$f(x) = x \ \& \ g(x) = \begin{bmatrix} \sin\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{bmatrix}$$

(ii) If f (x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = g(x) = \begin{bmatrix} 1 & x \ge 0 \\ -1 & x < 0 \end{bmatrix}$$

Solved Example # 6 If f(x) = [sin(x–1)] – {sin(x–1)}

Comment on continuity of f(x) at $x = \frac{\pi}{2} + 1$

Solution f(x) = [sin (x – 1)] – {sin (x – 1)}

Let g(x) = [sin (x – 1)] + {sin (x – 1} = sin (x – 1)

which is continuous at $x = \frac{\pi}{2} + 1$

as [sin (x – 1)] and { sin (x – 1)} both are discontinuous at $x = \frac{\pi}{2} + 1$

$\therefore$ At most one of f(x) or g(x) can be continuous at $x = \frac{\pi}{2} + 1$

As g(x) is continuous at $x = \frac{\pi}{2} + 1$, there fore, f(x) must be discontinuous

Alternatively, check the continuity of f(x) by evaluling $\lim_{x \to \frac{\pi}{2}^+} f(x)$ and $f\left(\frac{\pi}{2} + 1\right)$

6. Continuity of Composite Function :

If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c. eg. $f(x) = \frac{x\sin x}{x^2 + 2}$ & $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\frac{x\sin x}{x^2 + 2}\right|$ will also be continuous at x = 0.

Solved Example # 7 If $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x).

Sol. $f(x) = \frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain
and f is not defined at x = 1 $\therefore$ f is discontinuous at x = 1

$g(x) = \frac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and fog is not defined at x = 2

$\therefore$ g is discontinuous at x = 2

Now fog (x) will be discontinuous at

(i) x = 2 (point of discontinuity of g(x))

(ii) g(x) = 1 (when g(x) = point of discontinuity of f(x))

if g(x) = 1

$\Rightarrow$ $\frac{1}{x-2} = 1$ $\Rightarrow$ x = 3

$\therefore$ discontinuity of fog(x) should be checked at x = 2 and x = 3

at x = 2

$$fog(x) = \frac{\frac{1}{x-2} + 1}{\frac{1}{x-2} - 1}$$ fog (2) is not defined

$$\lim_{x\to 2} \text{fog}(x) = \lim_{x\to 2} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x\to 2} \frac{1+x-2}{1-x+2} = 1$$

$\therefore$ fog (x) is discontinuous at x = 2 and it is removable discontinuity at x = 3

fog (3) = not defined

$$\lim_{x\to 3^+} \text{fog}(x) = \lim_{x\to 3^+} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

$$\lim_{x\to 3^-} \text{fog}(x) = \lim_{x\to 3^-} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

$\therefore$ fog (x) is discontinuous at x = 3 and it is non removable discontinuity of IInd kind.

Self practice problems :

7. $f(x) = \begin{cases} 1+x^3 , & x < 0 \\ x^2 - 1 , & x \ge 0 \end{cases}$ $\quad g(x) = \begin{cases} (x-1)^{\frac{1}{3}} , & x < 0 \\ (x+1)^{\frac{1}{2}} , & x \ge 0 \end{cases}$

Then defined fog (x) and comment the continuity of gof(x) at x = 1

Ans. [fog(x) = x, x $\in$ R gof(x) is discontinous at x = 0, 1]

7. Intermediate Value Theorem :

A function f which is continuous in $[a, b]$ possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open inetrval (a, b).

Solved Example # 7 Given that a > b > c > d then prove that the equation (x – a) (x – c) + 2(x – b) (x – d) = 0 will have real and distinct roots.

Solution (x – a) (x – c) + 2 (x – b) (x – d) = 0

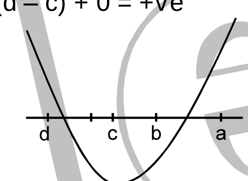
f(x) = (x – a) (x – c) + 2 (x – b) (x – d)

f(a) = (a – a) (a – c) + 2 (a – b) (a – d) = + ve

f(b) = (b – a) (b – c) + 0 = –ve

f(c) = 0 + 2 (c – b) (c – d) = –ve

f(d) = (d – a) (d – c) + 0 = +ve



hence (x – a) (x – c) + 2(x – b) (x – d) = 0

have real and distinct root

Self practice problems :

8. f(x) = xe^x – 2 then show that f(x) = 0 has exactly one root in the interval (0, 1).

Solved Example # 8

Let $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2 x}$, then find $f\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Ans. [Discontinous, removable discontinuity of Isolated type]

Sol. Let $f(x) = \lim_{n\to\infty} \frac{1}{1+n\sin^2 x}$

$$f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n\,.\,\sin^2\frac{\pi}{4}}$$

$$= \lim_{n\to\infty} \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$$

Now $\quad f(0) = \lim_{n\to\infty} \frac{1}{n\,.\,\sin^2(0)+1}$

$$= \frac{1}{1+0} = 1$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty}\frac{1}{1+n\sin^2 x}\right]$$

$$= \left[\frac{1}{1+\infty}\right]$$

{here sin^{2}x is very small quantity but not zero and very small quantity when multiplied with ∞ becomes ∞} $\therefore$ f(x) is not continuous at x = 0

Self practice problems :

9. $f(x) = \lim\limits_{n\to\infty} (1 + x)^n$

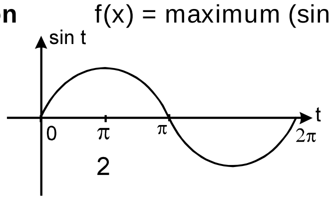
Comment on the continuity of f(x) at 0 and explain $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

Ans. Discontinous (non-removable)

Solved Example # 9

f(x) = maximum (sin t, $0 \le t \le x$), $0 \le x \le 2\pi$ discuss the continuity of this function at x= $\frac{\pi}{2}$

Solution f(x) = maximum (sin t, $0 \le t \le x$), $0 \le x \le 2\pi$



if $x \in \left[0, \frac{\pi}{2}\right]$, sin t is increasing function

Hence if $t \in [0, x]$, sin t will attain its maximum value at t = x.

$\therefore$ f(x) = sin x if $x \in \left[0, \frac{\pi}{2}\right]$

if $x \in \left(\frac{\pi}{2}, 2\pi\right]$ and $t \in [0, x]$

then sin t will attain its maximum value when t = $\frac{\pi}{2}$

$\therefore$ $f(x) = \sin\frac{\pi}{2} = 1$ if $x \in \left(\frac{\pi}{2}, 2\pi\right]$

$$\therefore \quad f(x) = \begin{cases} \sin x \quad , & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 1 \quad , & \text{if } x \in \left(\frac{\pi}{2}, 2\pi\right] \end{cases}$$

Now $f\left(\frac{\pi}{2}\right) = 1$

$$\lim_{x\to\frac{\pi}{2}^{+}} f(x) = \lim_{x\to\frac{\pi}{2}^{+}} \sin x = 1$$

$$\lim_{x\to\frac{\pi}{2}^{+}} f(x) = \lim_{x\to\frac{\pi}{2}^{+}} 1 = 1$$

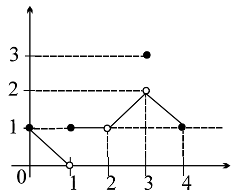
as f(x) = L.H.S. = R.H.S. $\therefore$ f(x) is continuous at x = $\frac{\pi}{2}$

Short Revesion (CONTINUITY)

THINGS TO REMEMBER :

1. A function f(x) is said to be continuous at x = c, if $\underset{x\to c}{\text{Limit}}\ f(x) = f(c)$. Symbolically f is continuous at x = c if $\underset{h\to 0}{\text{Limit}}\ f(c - h) = \underset{h\to 0}{\text{Limit}}\ f(c+h) = f(c)$.
i.e. LHL at x = c = RHL at x = c equals Value of 'f' at x = c.
It should be noted that continuity of a function at x = a is meaningful only if the function is defined in the immediate neighbourhood of x = a, not necessarily at x = a.

2. Reasons of discontinuity:



(i) $\underset{x\to c}{\text{Limit}}\ f(x)$ does not exist
i.e. $\underset{x\to c^-}{\text{Limit}}\ f(x) \neq \underset{x\to c^+}{\text{Limit}}\ f(x)$

(ii) f(x) is not defined at x= c

(iii) $\underset{x\to c}{\text{Limit}}\ f(x) \neq f(c)$

Geometrically, the graph of the function will exhibit a break at x= c. The graph as shown is discontinuous at x = 1 , 2 and 3.

3. Types of Discontinuities :

Type - 1: (Removable type of discontinuities)

In case $\underset{x\to c}{\text{Limit}}\ f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\underset{x\to c}{\text{Limit}}\ f(x) = f(c)$ & make it continuous at x= c. Removable type of discontinuity can be further classified as :

(a) **MISSING POINT DISCONTINUITY :** Where $\underset{x\to a}{\text{Limit}}\ f(x)$ exists finitely but f(a) is not defined.
e.g. $f(x) = \dfrac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 , and $f(x) = \dfrac{\sin x}{x}$ has a missing point discontinuity at x = 0

(b) **ISOLATED POINT DISCONTINUITY :** Where $\underset{x\to a}{\text{Limit}}\ f(x)$ exists & f(a) also exists but ; $\underset{x\to a}{\text{Limit}} \neq f(a)$.
e.g. $f(x) = \dfrac{x^2-16}{x-4}$, $x \neq 4$ & f (4) = 9 has an isolated point discontinuity at x = 4.
Similarly $f(x) = [x] + [-x] = \begin{bmatrix} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{bmatrix}$ has an isolated point discontinuity at all x ∈ I.

Type-2: (Non - Removable type of discontinuities)

In case $\underset{x\to c}{\text{Limit}}\ f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such discontinuities are known as non - removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

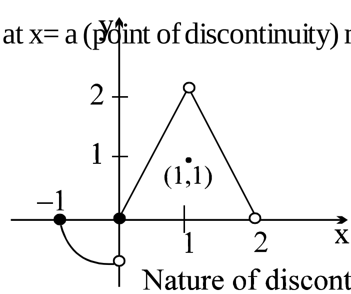
(a) Finite discontinuity e.g. f(x) = x – [x] at all integral x ; $f(x) = \tan^{-1}\dfrac{1}{x}$ at x = 0 and $f(x) = \dfrac{1}{1+2^{\frac{1}{x}}}$ at x = 0
(note that $f(0^+) = 0$; $f(0^-) = 1$)

(b) Infinite discontinuity e.g. $f(x) = \dfrac{1}{x-4}$ or $g(x) = \dfrac{1}{(x-4)^2}$ at x = 4 ; $f(x) = 2^{\tan x}$ at $x = \dfrac{\pi}{2}$ and $f(x) = \dfrac{\cos x}{x}$ at x = 0.

(c) Oscillatory discontinuity e.g. $f(x) = \sin\dfrac{1}{x}$ at x = 0.

In all these cases the value of f(a) of the function at x= a (point of discontinuity) may or may not exist but $\underset{x\to a}{\text{Limit}}$ does not exist.

Note: From the adjacent graph note that
– f is continuous at x = – 1
– f has isolated discontinuity at x = 1
– f has missing point discontinuity at x = 2
– f has non removable (finite type) discontinuity at the origin.



Nature of discontinuity

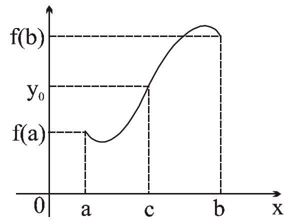
4. In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at x = c & LHL at x = c is called **THE JUMP OF DISCONTINUITY.** A function having a finite number of jumps in a given interval I is called a **PIECE WISE CONTINUOUS** or **SECTIONALLY CONTINUOUS** function in this interval.

5. All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

6. If f & g are two functions that are continuous at x= c then the functions defined by : $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x= c. Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x= c.

7. <u>The intermediate value theorem:</u>

Suppose f(x) is continuous on an interval I , and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b) , their exists a number c between a and b such that $f(c) = y_0$.



NOTE VERY CAREFULLY THAT :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = x \ \& \ g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(b) If f(x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(c) Point functions are to be treated as discontinuous. eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(d) A Continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c. eg. $f(x) = \frac{x\sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0 , hence the composite (gof) (x) $= \left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0 .

7. CONTINUITY IN AN INTERVAL :

(a) A function f is said to be continuous in (a , b) if f is continuous at each & every point $\in$(a , b).

(b) A function f is said to be continuous in a closed interval$[a, b]$ if :

(i) f is continuous in the open interval (a , b) &

(ii) f is right continuous at **'a'** i.e. $\underset{x\to a^+}{\text{Limit}} f(x) = f(a)$ = a finite quantity.

(iii) f is left continuous at **'b'** i.e. $\underset{x\to b^-}{\text{Limit}} f(x) = f(b)$ = a finite quantity.

Note that a function f which is continuous in$[a, b]$ possesses the following properties :

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a , b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open inetrval (a , b).

8. SINGLE POINT CONTINUITY:

Functions which are continuous only at one point are said to exhibit single point continuity e.g. $f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$ and $g(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$ are both continuous only at x = 0.

<u>EXERCISE–1</u>

Q 1. Let $f(x) = \begin{cases} \dfrac{\ln\cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \\ \dfrac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0), if not then indicate the nature of discontinuity.

Q 2. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \dfrac{f(x)}{x-3} & , x \neq 3 \\ K & , x = 3 \end{cases}$ then

(a) find all zeros of $f(x)$ (b) find the value of K that makes h continuous at x = 3
(c) using the value of K found in (b), determine whether h is an even function.

Q 3. Let $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots\ldots\ldots\ldots + \frac{x^2}{(1+x^2)^{n-1}}$

and $y(x) = \underset{n\to\infty}{\text{Limit}}\, y_n(x)$

Discuss the continuity of $y_n(x)$ ($n = 1, 2, 3$..........n) and $y(x)$ at $x = 0$

Q 4. Draw the graph of the function $f(x) = x - |x - x^2|$, $-1 \le x \le 1$ & discuss the continuity or discontinuity of f in the interval $-1 \le x \le 1$.

Q 5. Let $f(x) = \begin{cases} \dfrac{1-\sin \pi x}{1+\cos 2\pi x} & , \ x < \frac{1}{2} \\ p & , \ x = \frac{1}{2} \\ \dfrac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2} & , \ x > \frac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at $x = 1/2$.

Q 6. Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate $h(g(2))$ (b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

Q 7. Let $f(x) = \begin{cases} 1+x & , \ 0 \le x \le 2 \\ 3-x & , \ 2 < x \le 3 \end{cases}$. Determine the form of $g(x) = f[f(x)]$ & hence find the point of discontinuity of g, if any.

Q 8. Let $[x]$ denote the greatest integer function & $f(x)$ be defined in a neighbourhood of 2 by

$$f(x) = \begin{cases} \dfrac{\left(\exp\{(x+2)\ell n 4\}\right)^{\frac{[x+1]}{4}} - 16}{4^x - 16} & , x < 2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)} & , x > 2 \end{cases}.$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

Q 9. The function $f(x) = \begin{cases} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if} \quad 0 < x < \frac{\pi}{2} \\ b+2 & \text{if} \quad x = \frac{\pi}{2} \\ \left(1+|\cos x|\right)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if} \quad \frac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$.

Q.10 Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

Use squeeze play theorem to prove that f is continuous at x = 0.

Q.11 Let $f(x) = x + 2$, $\quad -4 \le x \le 0$
$\qquad\quad = 2 - x^2$, $\quad 0 < x \le 4$

then find $f(f(x))$, domain of $f(f(x))$ and also comment upon the continuity of $f(f(x))$.

Q 12. Let $f(x) = \begin{cases} 1 + x^3 & , \ x < 0 \\ x^2 - 1 & , \ x \ge 0 \end{cases}$; $g(x) = \begin{cases} (x-1)^{1/3} & , \ x < 0 \\ (x+1)^{1/2} & , \ x \ge 0 \end{cases}$. Discuss the continuity of $g(f(x))$.

Q.13 Determine a & b so that f is continuous at $x = \dfrac{\pi}{2}$. $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & \text{if} \quad x < \frac{\pi}{2} \\ a & \text{if} \quad x = \frac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi - 2x)^2} & \text{if} \quad x > \frac{\pi}{2} \end{cases}$

Q.14 Determine the values of a, b & c for which the function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{for} \quad x < 0 \\ c & \text{for} \quad x = 0 \\ \dfrac{\left(x+bx^2\right)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for} \quad x > 0 \end{cases}$

is continuous at x = 0.

Q.15 If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \ne 0)$ is cont. at x = 0. Find A & B. Also find f(0). Do not use series expansion or L' Hospital's rule.

Q.16 Discuss the continuity of the function 'f' defined as follows : $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{for} \quad 0 \le x \le 2 \\ \dfrac{3}{x+1} & \text{for} \quad 2 < x \le 4 \\ \dfrac{x+1}{x-5} & \text{for} \quad 4 < x \le 6 \end{cases}$ and draw the graph of the function for $x \in [0, 6]$. Also indicate the nature of discontinuities if any.

Q.17 If $f(x) = x + \{-x\} + [x]$, where $[x]$ is the integral part & $\{x\}$ is the fractional part of x. Discuss the continuity of f in $[-2, 2]$.

Q.18 Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax - b & \text{for } x \le 1 \\ 3x & \text{for } 1 < x < 2 \\ bx^2 - a & \text{for } x \ge 2 \end{cases}$ is continuous at x = 1 but discontinuous at x = 2.

Q.19 Prove that the inverse of the discontinuous function $y = (1 + x^2)\,\text{sgn}\,x$ is a continuous function.

Q.20 Let $g(x) = \lim_{n\to\infty} \dfrac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$, $x \ne 1$ and $g(1) = \lim_{x\to 1} \dfrac{\sin^2(\pi\cdot 2^x)}{\ln\left(\sec(\pi\cdot 2^x)\right)}$ be a continuous function at x = 1, find the value of $4\,g(1) + 2\,f(1) - h(1)$. Assume that f (x) and h (x) are continuous at x = 1.

Q.21 If g : [a, b] onto [a, b] is continous show that there is some $c \in [a, b]$ such that g (c) = c.

Q.22 The function $f(x) = \left(\dfrac{2 + \cos x}{x^3 \sin x} - \dfrac{3}{x^4}\right)$ is not defined at x = 0. How should the function be defined at x = 0 to make it continuous at x = 0. Use of expansion of trigonometric functions and L′ Hospital's rule is not allowed.

Q.23 $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for x > 0

$= \dfrac{\ln(1 + x + x^2) + \ln(1 - x + x^2)}{\sec x - \cos x}$ for x < 0, if f is continuous at x = 0, find 'a'

now if $g(x) = \ln\left(2 - \dfrac{x}{a}\right)\cdot\cot(x - a)$ for $x \ne a$, $a \ne 0$, $a > 0$. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

Q.24 (a) Let f(x + y) = f(x) + f(y) for all x , y & if the function f(x) is continuous at x = 0 , then show that f(x) is continuous at all x.

(b) If $f(x\cdot y) = f(x)\cdot f(y)$ for all x , y and f(x) is continuous at x = 1. Prove that f(x) is continuous for all x except at x = 0. Given $f(1) \ne 0$.

Q.25 Given $f(x) = \sum_{r=1}^{n} \tan\left(\dfrac{x}{2^r}\right) \sec\left(\dfrac{x}{2^{r-1}}\right)$; r , n ∈ N

$$g(x) = \underset{n\to\infty}{\text{Limit}} \frac{\ell n\left(f(x) + \tan\frac{x}{2^n}\right) - \left(f(x) + \tan\frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1 + \left(f(x) + \tan\frac{x}{2^n}\right)^n}$$

$= k$ for $x = \dfrac{\pi}{4}$ and the domain of g (x) is (0 , π/2).

where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that g (x) is continuous at x = π/4. Also state the points of discontinuity of g (x) in (0 , π/4) , if any.

Q.26 Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \dfrac{f(x)}{g(x)}$ where h is a function such that

(a) it is continuous every where except when x = – 1, (b) $\lim_{x\to\infty} h(x) = \infty$ and (c) $\lim_{x\to -1} h(x) = \dfrac{1}{2}$.

Find $\lim_{x\to 0} \left(3h(x) + f(x) - 2g(x)\right)$

Q.27 Let f be continuous on the interval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \dfrac{1}{2}\right]$ such that $f(c) = f\left(c + \dfrac{1}{2}\right)$

Q.28 Consider the function $g(x) = \begin{cases} \dfrac{1 - a^x + x a^x \ell n a}{a^x x^2} & \text{for } x < 0 \\[2ex] \dfrac{2^x a^x - x \ell n 2 - x \ell n a - 1}{x^2} & \text{for } x > 0 \end{cases}$ where a > 0.

Without using , L 'Hospital's rule or power series , find the value of 'a' & 'g(0)' so that the function g(x) is continuous at x = 0.

Q.29 Let $f(x) = \begin{cases} \dfrac{\left(\frac{\pi}{2} - \sin^{-1}\left(1 - \{x\}^2\right)\right).\sin^{-1}\left(1 - \{x\}\right)}{\sqrt{2}\left(\{x\} - \{x\}^3\right)} & \text{for } x \ne 0 \\[2ex] \dfrac{\pi}{2} & \text{for } x = 0 \end{cases}$ where {x} is the fractional part of x.

Consider another function g(x) ; such that

$g(x) = f(x)$ for $x \ge 0$

$= 2\sqrt{2}\, f(x)$ for $x < 0$ Discuss the continuity of the functions f(x) & g(x) at x = 0.

Q.30 Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x-5|\,[x] & \text{for } x > 1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where [x] is the greatest integer not greater than x. Also draw the graph.

EXERCISE–2

(OBJECTIVE QUESTIONS)

Q 1. State whether True or False.

(a). If $f(x) = \dfrac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$ for $x \neq \dfrac{\pi}{4}$, then the value which can be given to f(x) at $x = \dfrac{\pi}{4}$ so that the function becomes continuous every where in $(0,\pi/2)$ is 1/4.

(b) The function f, defined by $f(x) = \dfrac{1}{1+2^{\tan x}}$ is continuous for real x.

(c) $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{1+n\sin^2 \pi x}$ is continuous at x = 1.

(d) The function $f(x) = \begin{cases} 2x+1 & \text{if } -3<x<-2 \\ x-1 & \text{if } -2\le x<0 \\ x+2 & \text{if } 0\le x<1 \end{cases}$ is continuous everywhere in (–3 , 1).

(e) The function defined by $f(x) = \dfrac{x}{|x|+2x^2}$ for $x \neq 0$ & f(0) = 1 is continuous at x = 0.

(f) The function $f(x) = 2^{-2^{1/(1-x)}}$ if $x \neq 1$ & f(1) = 1 is not continuous at x = 1.

(g) The function $f(x) = 2x\sqrt{(x^3-1)} + 5\sqrt{x}\,\sqrt{(1-x^4)} + 7x^2\sqrt{(x-1)} + 3x + 2$ is continuous at x = 1.

(h) There exists a continuous function f: $[0, 1] \longrightarrow [0, 10]$, but there exists no continuous function g : $[0, 1] \longrightarrow (0, 10)$.

Q 2. Fill in the blanks

(a) Given $f(x) = \dfrac{1-\cos(cx)}{x\sin x}$, $x \neq 0$ & $f(0) = \dfrac{1}{2}$. If f is continuous at x = 0, then the value of c is _______.

(b). The function $f(x) = \dfrac{1}{\ln|x|}$ has non removable discontinuity at x = ______ & removable discontinuity at x = ______ respectively.

(c). If f(x) is continuous in [0 , 1] & f(x) = 1 for all rational numbers in [0 , 1] then $f\left(\dfrac{1}{\sqrt{2}}\right)$ = ______.

(d). The values of 'a' & 'b' so that the function $f(x) = \begin{cases} x + a\sqrt{2}\,\sin x & , \ 0 \le x < \frac{\pi}{4} \\ 2x\,\cot x + b & , \ \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\,\cos 2x - b\sin x & , \ \frac{\pi}{2} < x \le \pi \end{cases}$

is continuous for $0 \le x \le \pi$ are ________ & ________.

(e). If $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$ is continuous at $x = \dfrac{\pi}{4}$ then $f\left(\dfrac{\pi}{4}\right)$ = ______.

Q3. Indicate the correct alternative(s):

(a) The function defined as $f(x) = \lim\limits_{n\to\infty} \dfrac{\cos \pi x - x^{2n}\sin(x-1)}{1 + x^{2n+1} - x^{2n}}$

(A) is discontinuous at x = 1 because $f(1^+) \neq f(1^-)$
(B) is discontinuous at x = 1 because f(1) is not defined
(C) is discontinuous at x = 1 because $f(1^+) = f(1^-) \neq f(1)$
(D) is continuous at x = 1

(b) Let 'f' be a continuous function on R. If $f(1/4^n) = (\sin e^n)\,e^{-n^2} + \dfrac{n^2}{n^2+1}$ then f(0) is :

(A) not unique
(B) 1
(C) data sufficient to find f(0)
(D) data insufficient to find f(0)

(c) Indicate all correct alternatives if, $f(x) = \dfrac{x}{2} - 1$, then on the interval $[0, \pi]$

(A) $\tan(f(x))$ & $\dfrac{1}{f(x)}$ are both continuous
(B) $\tan(f(x))$ & $\dfrac{1}{f(x)}$ are both discontinuous
(C) $\tan(f(x))$ & $f^{-1}(x)$ are both continuous
(D) $\tan(f(x))$ is continuous but $\dfrac{1}{f(x)}$ is not

(d) 'f' is a continuous function on the real line. Given that $x^2 + (f(x) - 2)\,x - \sqrt{3}\cdot f(x) + 2\sqrt{3} - 3 = 0$. Then the value of $f(\sqrt{3})$

(A) can not be determined (B) is $2(1-\sqrt{3})$

(C) is zero (D) is $\frac{2\left(\sqrt{3}-2\right)}{\sqrt{3}}$

(e) If $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$, where sgn ( ) is the signum function, then f(x)
(A) is continuous over its domain (B) has a missing point discontinuity
(C) has isolated point discontinuity (D) has irremovable discontinuity.

(f) Let $g(x) = \tan^{-1}|x| - \cot^{-1}|x|$, $f(x) = \frac{[x]}{[x+1]}\{x\}$, $h(x) = |g(f(x))|$ where {x} denotes fractional part and [x] denotes the integral part then which of the following holds good?
(A) h is continuous at x = 0 (B) h is discontinuous at x = 0
(C) $h(0^-) = \frac{\pi}{2}$ (D) $h(0^+) = -\frac{\pi}{2}$

(g) Consider $f(x) = \underset{n\to\infty}{\text{Limit}} \frac{x^n - \sin x^n}{x^n + \sin x^n}$ for $x > 0$, $x \neq 1$
$f(1) = 0$
then
(A) f is continuous at x = 1 (B) f has a finite discontinuity at x = 1
(C) f has an infinite or oscillatory discontinuity at x = 1.
(D) f has a removable type of discontinuity at x = 1.

(h) Given $f(x) = \frac{\left[\{|x|\}\right] e^{x^2}\left\{\left[x+\{x\}\right]\right\}}{\left(e^{\frac{1}{x^2}}-1\right)\text{sgn}(\sin x)}$ for $x \neq 0$
$= 0$ for $x = 0$
where {x} is the fractional part function; [x] is the step up function and sgn(x) is the signum function of x then, f(x)
(A) is continuous at x = 0 (B) is discontinuous at x = 0
(C) has a removable discontinuity at x = 0 (D) has an irremovable discontinuity at x = 0

(i) Consider $f(x) = \begin{cases} x[x]^2 \log_{(1+x)} 2 & \text{for } -1 < x < 0 \\ \dfrac{ln\left(e^{x^2} + 2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{cases}$
where [*] & {*} are the greatest integer function & fractional part function respectively, then
(A) $f(0) = ln2 \Rightarrow$ f is continuous at x = 0 (B) $f(0) = 2 \Rightarrow$ f is continuous at x = 0
(C) $f(0) = e^2 \Rightarrow$ f is continuous at x = 0 (D) f has an irremovable discontinuity at x = 0

(j) Consider $f(x) = \frac{\sqrt{1+x}-\sqrt{1-x}}{\{x\}}$ $x \neq 0$

$g(x) = \cos 2x$ $-\frac{\pi}{4} < x < 0$

$h(x) = \begin{cases} \frac{1}{\sqrt{2}} f(g(x)) & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ f(x) & \text{for } x > 0 \end{cases}$

then, which of the following holds good.
where {x} denotes fractional part function.
(A) 'h' is continuous at x = 0 (B) 'h' is discontinuous at x = 0
(C) f(g(x)) is an even function (D) f(x) is an even function

(k) The function $f(x) = [x].\cos\frac{2x-1}{2}\pi$, where [•] denotes the greatest integer function, is discontinuous at
(A) all x (B) all integer points
(C) no x (D) x which is not an integer

EXERCISE–3

Q.1 Let $f(x) = [x]\sin\frac{\pi}{[x+1]}$, where [•] denotes the greatest integer function. The domain of f is_____ & the points of discontinuity of f in the domain are ______. [JEE '96, 2]

Q.2 Let f (x) be a continuous function defined for $1 \le x \le 3$. If f (x) takes rational values for all x and f (2) = 10, then f (1.5) = _______. [JEE '97, 2]

Q.3 The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at :
(A) all integers (B) all integers except 0 & 1
(C) all integers except 0 (D) all integers except 1
[JEE '99, 2 (out of 200)]

Q.4 Determine the constants a, b & c for which the function $f(x) = \begin{cases} (1+ax)^{1/x} & \text{for } x < 0 \\ b & \text{for } x = 0 \\ \dfrac{(x+c)^{1/3} - 1}{(x+1)^{1/2} - 1} & \text{for } x > 0 \end{cases}$ is continuous at x = 0. [REE '99, 6]

Q.5 Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{e^{1/(x-1)} - 2}{e^{1/(x-1)} + 2}, & x \ne 1 \\ 1, & x = 1 \end{cases}$$

at x = 1. [REE 2001 (Mains) , 3 out of 100]

Short Revesion (DIFFERENTIABILITY)

THINGS TO REMEMBER :

1. Right hand & Left hand Derivatives ;

By definition : $f'(a) = \underset{h\to 0}{\text{Limit}} \dfrac{f(a+h) - f(a)}{h}$ if it exist

(i) The right hand derivative of f ′ at x = a denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \underset{h\to 0^+}{\text{Limit}} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists & is finite.

(ii) The left hand derivative : of f at x = a denoted by $f'(a^+)$ is defined by :

$$f'(a^-) = \underset{h\to 0^+}{\text{Limit}} \frac{f(a-h) - f(a)}{-h},$$

Provided the limit exists & is finite.

We also write $f'(a^+) = f'_+(a)$ & $f'(a^-) = f'_-(a)$.

* This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.

(iii) **Derivability & Continuity :**

(a) If f ′(a) exists then f(x) is derivable at x= a ⇒ f(x) is continuous at x = a.

(b) If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \underset{h\to 0}{\text{Limit}} \dfrac{f(x+h) - f(x)}{h}$ exists.

Also $f(x+h) - f(x) = \dfrac{f(x+h) - f(x)}{h} . h \; [h \ne 0]$

Therefore :

$$\underset{h\to 0}{\text{Limit}} [f(x+h) - f(x)] = \underset{h\to 0}{\text{Limit}} \frac{f(x+h) - f(x)}{h} . h = f'(x).0 = 0$$

Therefore $\underset{h\to 0}{\text{Limit}} [f(x+h) - f(x)] = 0 \Rightarrow \underset{h\to 0}{\text{Limit}} f(x+h) = f(x) \Rightarrow$ f is continuous at x.

Note : If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true :

" IF f IS CONTINUOUS AT x , THEN f IS DERIVABLE AT x " IS NOT TRUE.

e.g. the functions $f(x) = |x|$ & $g(x) = x \sin\dfrac{1}{x}$; $x \ne 0$ & $g(0) = 0$ are continuous at x = 0 but not derivable at x = 0.

NOTE CAREFULLY :

(a) Let $f'_+(a) = p$ & $f'_-(a) = q$ where p & q are finite then :

(i) p = q ⇒ f is derivable at x = a ⇒ f is continuous at x = a.

(ii) p ≠ q ⇒ f is not derivable at x = a.

It is very important to note that f may be still continuous at x = a.

In short, for a function f :

Differentiability ⇒ Continuity ; Continuity $\not\Rightarrow$ derivability ;
Non derivibality $\not\Rightarrow$ discontinuous ; But discontinuity ⇒ Non derivability

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

3. **DERIVABILITY OVER AN INTERVAL :** f (x) is said to be derivable over an interval if it is derivable at each & every point of the interval f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &

(ii) for any point c such that a < c < b, $f'(c+)$ & $f'(c-)$ exist & are equal.

NOTE :1. If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) − g(x) , f(x).g(x) will also be derivable at x = a & if g (a) ≠ 0 then the function f(x)/g(x) will also be derivable at x = a.

2. If f(x) is differentiable at x = a & g(x) is not differentiable at x = a , then the product function F(x) = f(x)· g(x) can still be differentiable at x = a e.g. f(x) = x & g(x) = $|x|$.

3. If f(x) & g(x) both are not differentiable at x = a then the product function ; F(x) = f(x)· g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$.

4. If f(x) & g(x) both are non-deri. at x = a then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

5. If f(x) is derivable at x = a $\not\Rightarrow$ $f'(x)$ is continuous at x = a.

e.g. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

6. **A surprising result :** Suppose that the function f (x) and g (x) defined in the interval (x_1, x_2) containing the point x_0, and if f is differentiable at $x = x_0$ with $f(x_0) = 0$ together with g is continuous as $x = x_0$ then the function F (x) = f (x) · g (x) is differentiable at $x = x_0$

e.g. F (x) = sinx · $x^{2/3}$ is differentiable at x = 0.

EXERCISE–4

Q.1 Discuss the continuity & differentiability of the function f(x) = sinx + sin$|x|$, x ∈ R. Draw a rough sketch of the graph of f(x).

Q.2 Examine the continuity and differentiability of f(x) = $|x| + |x-1| + |x-2|$ x ∈ R. Also draw the graph of f(x).

Q.3 Given a function *f* (x) defined for all real x, and is such that

$f(x + h) - f(x) < 6h^2$ for all real *h* and *x*. Show that *f* (x) is constant.

Q.4 A function f is defined as follows : $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+\sin x & \text{for } 0 \le x < \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < +\infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = π/2.

Q.5 Examine the origin for continuity & derrivability in the case of the function f defined by $f(x) = x \tan^{-1}(1/x)$, x ≠ 0 and f(0) = 0.

Q.6 Let f (0) = 0 and f ' (0) = 1. For a positive integer k, show that

$$\lim_{x\to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + ...f\left(\frac{x}{k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + ...+ \frac{1}{k}$$

Q.7 Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$; x ≠ 0 , f(0) = 0, test the continuity & differentiability at x = 0

Q.8 If f(x)= $|x-1|.([x]-[-x])$, then find $f'(1^+)$ & $f'(1^-)$ where [x] denotes greatest integer function.

Q.9 If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \ge 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

Q.10 Let f(x) be defined in the interval [-2, 2] such that $f(x) = \begin{cases} -1 & , -2 \le x \le 0 \\ x - 1 & , 0 < x \le 2 \end{cases}$ &

g(x) = f($|x|$) + $|f(x)|$. Test the differentiability of g(x) in (– 2, 2).

Q.11 Given $f(x) = \cos^{-1}\left(\text{sgn}\left(\frac{2[x]}{3x-[x]}\right)\right)$ where sgn (.) denotes the signum function & [.] denotes the greatest integer function. Discuss the continuity & differentiability of f (x) at x = ± 1.

Q.12 Examine for continuity & differentiability the points x = 1 & x = 2, the function f defined by $f(x) = \begin{cases} x[x] & , 0 \le x < 2 \\ (x-1)[x] & , 2 \le x \le 3 \end{cases}$ where [x] = greatest integer less than or equal to x.

Q.13 $f(x) = x.\left(\frac{e^{[x]+|x|}-2}{[x]+|x|}\right)$, x ≠ 0 & f(0) = –1 where [x] denotes greatest integer less than or equal to x. Test the differentiability of f(x) at x = 0.

Q.14 Discuss the continuity & the derivability in [0 , 2] of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \ge 1 \\ \sin\frac{\pi x}{2} & \text{for } x < 1 \end{cases}$

where [ ] denote greatest integer function .

Q.15 If f(x) = –1 + $|x - 1|$, –1 ≤ x ≤ 3 ; g(x) = 2 – $|x + 1|$, – 2 ≤ x ≤ 2 , then calculate (fog) (x) & (gof) (x). Draw their graph. Discuss the continuity of (fog)(x) at x = –1 & the differentiability of (gof) (x) at x = 1.

Q.16 The function : $f(x) = \begin{cases} ax(x-1)+b & \text{when } x<1 \\ x-1 & \text{when } 1 \le x \le 3 \\ px^2+qx+2 & \text{when } x>3 \end{cases}$

Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x (ii) f ' (1) does not exist (iii) f '(x) is continuous at x = 3

Q.17 Examine the function, $f(x) = x.\dfrac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}$, $x \neq 0$ $(a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin.

Q.18 Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function.

$f(x) = x\sin\dfrac{1}{x}\sin\dfrac{1}{x\sin\frac{1}{x}}$ where $x \neq 0$, $x \neq 1/r\pi$ & $f(0) = f(1/r\pi) = 0$, r = 1, 2, 3,.........

Q.19 $f(x) = \begin{cases} 1-x & , \ (0 \le x \le 1) \\ x+2 & , \ (1<x<2) \\ 4-x & , \ (2 \le x \le 4) \end{cases}$ Discuss the continuity & differentiability of y = f [f(x)] for $0 \le x \le 4$.

Q.20 Consider the function, $f(x) = \begin{cases} x^2\left|\cos\dfrac{\pi}{2x}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Show that $f'(0)$ exists and find its value (b) Show that $f'\left(\dfrac{1}{3}\right)$ does not exist.

(c) For what values of x, $f'(x)$ fails to exist.

Q.21 Discuss the continuity & the derivability of 'f' where f (x) = degree of $(u^{x^2} + u^2 + 2u - 3)$ at $x = \sqrt{2}$.

Q.22 Let $f(x)$ be a function defined on (–a, a) with a > 0. Assume that $f(x)$ is continuous at x = 0 and $\lim\limits_{x\to 0}\dfrac{f(x) - f(kx)}{x} = \alpha$, where $k \in (0, 1)$ then compute $f'(0^+)$ and $f'(0^-)$, and comment upon the differentiability of f at x = 0.

Q.23 A function f : R → R satisfies the equation f(x + y) = f(x). f(y) for all x, y in R & f(x) ≠ 0 for any x in R. Let the function be differentiable at x = 0 & f′(0) = 2. Show that f′(x) = 2f(x) for all x in R. Hence determine f(x).

Q.24 Let f(x) be a real valued function not identically zero satisfies the equation, $f(x + y^n) = f(x) + (f(y))^n$ for all real x & y and $f'(0) \ge 0$ where n (> 1) is an odd natural number. Find f(10).

Q.25 A function f : R → R where R is a set of real numbers satisfies the equation

$f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)+f(0)}{3}$ for all x ,y in R. If the function is differentiable at x = 0 then show that it is differentiable for all x in R.

EXERCISE–5

Fill in the blanks :

Q.1 If f(x) is derivable at x = 3 & f ′(3) = 2 , then $\lim\limits_{h\to 0}\dfrac{f(3+h^2) - f((3-h^2)}{2h^2}$ = ________.

Q.2 If f(x) = $|\sin x|$ & g(x) = x^3 then f[g(x)] is _______ & _______ at x = 0. (State continuity and derivability)

Q.3 Let f(x) be a function satisfying the condition f(– x) = f(x) for all real x. If f ′(0) exists, then its value is _______.

Q.4 For the function $f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$, the derivative from the right, $f'(0^+)$ = ______ & the derivative from the left, $f'(0^-)$ = ________.

Q.5 The number of points at which the function f(x) = max. {a – x, a + x, b}, – ∞ < x < ∞, 0 < a < b cannot be differentiable is _______.

Select the correct alternative : (only one is correct)

Q.6 Let $f(x) = \dfrac{|x|}{\sin x}$ for $x \neq 0$ & f(0) = 1 then ,

(A) f(x) is conti. & diff. at x = 0 (B) f(x) is continuous & not derivable at x = 0
(C) f(x) is discont. & not diff. at x = 0 (D) none

Q.7 Given $f(x) = \begin{cases} \log_a \left(a\left|[x]+[-x]\right|\right)^x \left(\dfrac{a^{\frac{2}{\left(\frac{[x]+[-x]}{|x|}\right)}} - 5}{3 + a^{\frac{1}{|x|}}} \right) & \text{for } |x| \neq 0 \ ; \ a > 1 \\ 0 & \text{for } x = 0 \end{cases}$ where [ ] represents the integral part function, then :

(A) f is continuous but not differentiable at $x = 0$ (B) f is cont. & diff. at $x = 0$
(C) the differentiability of 'f' at $x = 0$ depends on the value of a
(D) f is cont. & diff. at $x = 0$ and for $a = e$ only.

Q.8 For what triplets of real numbers (a, b, c) with $a \neq 0$ the function

$f(x) = \begin{cases} x & x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$ is differentiable for all real x ?

(A) $\{(a, 1-2a, a) \mid a \in R, a \neq 0\}$ (B) $\{(a, 1-2a, c) \mid a, c \in R, a \neq 0\}$
(C) $\{(a, b, c) \mid a, b, c \in R, a + b + c = 1\}$ (D) $\{(a, 1-2a, 0) \mid a \in R, a \neq 0\}$

Q.9 A function f defined as $f(x) = x[x]$ for $-1 \le x \le 3$ where [x] defines the greatest integer $\le x$ is :
(A) conti. at all points in the domain of f but non-derivable at a finite number of points
(B) discontinuous at all points & hence non-derivable at all points in the domain of f
(C) discont. at a finite number of points but not derivable at all points in the domain of f
(D) discont. & also non-derivable at a finite number of points of f.

Q.10 [x] denotes the greatest integer less than or equal to x. If $f(x) = [x]\,[\sin \pi x]$ in $(-1,1)$ then f(x) is :
(A) cont. at $x = 0$ (B) cont. in $(-1, 0)$
(C) differentiable in $(-1,1)$ (D) none

Q.11 A function $f(x) = x[1 + (1/3) \sin (\ln x^2)]$, $x \neq 0$. [ ] = integral part $f(0) = 0$. Then the function :
(A) is cont. at $x = 0$ (B) is monotonic
(C) is derivable at $x = 0$ (D) can not be defined for $x < -1$

Q.12 The function f(x) is defined as follows $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ x^3 - x + 1 & \text{if } x > 1 \end{cases}$ then f(x) is :
(A) derivable & cont. at $x = 0$ (B) derivable at $x = 1$ but not cont. at $x = 1$
(C) neither derivable nor cont. at $x = 1$ (D) not derivable at $x = 0$ but cont. at $x = 1$

Q.13 If $f(x) = \begin{cases} x + \{x\} + x \sin\{x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ where {x} denotes the fractional part function, then :
(A) 'f' is cont. & diff. at $x = 0$ (B) 'f' is cont. but not diff. at $x = 0$
(C) 'f' is cont. & diff. at $x = 2$ (D) none of these

Q.14 The set of all points where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable is :
(A) $(-\infty, \infty)$ (B) $[0, \infty)$ (C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$ (E) none

Select the correct alternative : (More than one are correct)

Q.15 If $f(x) = |2x+1| + |x-2|$ then f(x) is :
(A) cont. at all the points (B) conti. at $x = 2$ but not differentiable at $x = -1/2$
(C) discontinuous at $x = -1/2$ & $x = 2$ (D) not derivable at $x = -1/2$ & $x = 2$

Q.16 $f(x) = |[x]x|$ in $-1 \le x \le 2$, where [x] is greatest integer $\le x$ then f(x) is :
(A) cont. at $x = 0$ (B) discont. $x = 0$ (C) not diff. at $x = 2$ (D) diff. at $x = 2$

Q.17 $f(x) = 1 + x.[\cos x]$ in $0 < x \le \pi/2$, where [ ] denotes greatest integer function then,
(A) It is continuous in $0 < x < \pi/2$ (B) It is differentiable in $0 < x < \pi/2$
(C) Its maximum value is 2 (D) It is not differentiable in $0 < x < \pi/2$

Q.18 $f(x) = (\sin^{-1}x)^2 . \cos (1/x)$ if $x \neq 0$; $f(0) = 0$, f(x) is :
(A) cont. no where in $-1 \le x \le 1$ (B) cont. every where in $-1 \le x \le 1$
(C) differentiable no where in $-1 \le x \le 1$ (D) differentiable everywhere in $-1 < x < 1$

Q.19 $f(x) = |x| + |\sin x|$ in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. It is :
(A) Conti. no where (B) Conti. every where
(C) Differentiable no where (D) Differentiable everywhere except at $x = 0$

Q.20 If $f(x) = 3(2x+3)^{2/3} + 2x+3$ then,
(A) f(x) is cont. but not diff. at $x = -3/2$ (B) f(x) is diff. at $x = 0$
(C) f(x) is cont. at $x = 0$ (D) f(x) is diff. but not cont. at $x = -3/2$

Q.21 If $f(x) = 2 + |\sin^{-1} x|$, it is :
(A) continuous no where (B) continuous everywhere in its domain
(C) differentiable no where in its domain (D) Not differentiable at $x = 0$

Q.22 If $f(x) = x^2 . \sin (1/x)$, $x \neq 0$ and $f(0) = 0$ then,
(A) f(x) is continuous at $x = 0$ (B) f(x) is derivable at $x = 0$
(C) $f'(x)$ is continuous at $x = 0$ (D) $f''(x)$ is not derivable at $x = 0$

Q.23 A function which is continuous & not differentiable at $x = 0$ is :
(A) $f(x) = x$ for $x < 0$ & $f(x) = x^2$ for $x \ge 0$ (B) $g(x) = x$ for $x < 0$ & $g(x) = 2x$ for $x \ge 0$
(C) $h(x) = x|x|$ $x \in R$ (D) $K(x) = 1 + |x|$, $x \in R$

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Q.24 If $\sin^{-1}x + |y| = 2y$ then y as a function of x is :
(A) defined for $-1 \le x \le 1$ (B) continuous at x = 0
(C) differentiable for all x (D) such that $\frac{dy}{dx} = \frac{1}{3\sqrt{1-x^2}}$ for $-1 < x < 0$

Q.25 Let f(x) = Cosx & $H(x) = \begin{cases} \text{Min}[f(t)/0 \le t \le x] & \text{for } 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{for } \frac{\pi}{2} < x \le 3 \end{cases}$, then
(A) H (x) is cont. & deri. in [0, 3] (B) H(x) is cont. but not deri. at $x = \pi/2$
(C) H(x) is neither cont. nor deri. at $x = \pi/2$ (D) Max. value of H(x) in [0,3] is 1

EXERCISE–6

Q.1 Determine the values of x for which the following function fails to be continuous or differentiable
$f(x) = \begin{cases} 1-x & , \quad x < 1 \\ (1-x)(2-x) & , \quad 1 \le x \le 2 \\ 3-x & , \quad x > 2 \end{cases}$. Justify your answer. [JEE'97, 5]

Q.2 Let $h(x) = \min\{x, x^2\}$, for every real number of x. Then :
(A) h is cont. for all x (B) h is diff. for all x
(C) $h'(x) = 1$, for all x > 1 (D) h is not diff. at two values of x. [JEE'98, 2]

Q.3 Discuss the continuity & differentiability of the function $f(x) = \begin{cases} 2 + \sqrt{(1-x^2)} & , \quad |x| \le 1 \\ 2e^{(1-x)^2} & , \quad |x| > 1 \end{cases}$.
[REE '98, 6]

Q.4 The function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at :
(A) – 1 (B) 0 (C) 1 (D) 2
[JEE '99, 2 (out of 200)]

Q.5 Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is
(A) onto if f is onto (B) one one if f is one one
(C) continuous if f is continuous (D) differentiable if f is differentiable.
[JEE 2000, Screening, 1 out of 35]

Q.6 Discuss the continuity and differentiability of the function,
$$f(x) = \begin{cases} \frac{x}{1+|x|} & , \quad |x| \ge 1 \\ \frac{x}{1-|x|} & , \quad |x| < 1 \end{cases}.$$ [REE, 2000 (3)]

Q.7 [JEE 2001 (Screening)]
(a) Let $f : R \to R$ be a function defined by , $f(x) = \max[x, x^3]$. The set of all points where f (x) is NOT differentiable is :
(A) {– 1 , 1} (B) {– 1 , 0} (C) {0 , 1} (D) {– 1 , 0 , 1}
(b) The left hand derivative of , $f(x) = [x] \sin(\pi x)$ at x = k , k an integer is :
(A) $(-1)^k (k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$
(C) $(-1)^k k\pi$ (D) $(-1)^{k-1} k\pi$
(c) Which of the following functions is differentiable at x = 0 ?
(A) $\cos(|x|) + |x|$ (B) $\cos(|x|) - |x|$
(C) $\sin(|x|) + |x|$ (D) $\sin(|x|) - |x|$

Q.8 Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$.
[JEE 2001, (mains) 5 out of 100]

Q.9 The domain of the derivative of the function
$$f(x) = \begin{cases} \tan^{-1}x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$ is
(A) R – {0} (B) R – {1} (C) R – {–1} (D) R – {–1, 1}
[JEE 2002 (Screening), 3]

Q.10 Let f: R $\to$ R be such that $f(1) = 3$ and $f'(1) = 6$. The $\lim\limits_{x \to 0} \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals
(A) 1 (B) $e^{1/2}$ (C) e^2 (D) e^3
[JEE 2002 (Screening), 3]

Q.11 $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0? Justify your answer. [JEE 2002, 5 out of 60]

Q.12 If a function $f : [-2a , 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at x = a is 0 then find the left hand derivative at x = – a.
[JEE 2003, Mains-2 out of 60]

Q.13**(a)** The function given by $y = \big|\,|x| - 1\big|$ is differentiable for all real numbers except the points

(A) {0, 1, –1} (B) ± 1 (C) 1 (D) – 1

(b) If f (x) is a continuous and differentiable function and $f\left(\dfrac{1}{n}\right) = 0$, $\forall$ n ≥ 1 and n ∈ I, then

(A) f (x) = 0, x ∈ (0, 1] (B) f (0) = 0, f ' (0) = 0
(C) f '(x) = 0 = f "(x), x ∈ (0, 1] (D) f (0) = 0 and f ' (0) need not to be zero
[JEE 2005 (Screening), 3 + 3]

(c) If $|f(x_1) - f(x_2)|$ £ $(x_1 - x_2)^2$, for all x_1, x_2 Î R. Find the equation of tangent to the curve y = f (x) at the point (1, 2). [JEE 2005 (Mains), 2]

Q.14 If $f(x) = \min. (1, x^2, x^3)$, then

(A) f (x) is continuous ∀ x ∈ R (B) $f'(x) > 0$, □∀ x > 1
(C) f(x) is not differentiable but continuous ∀ x ∈ R
(D) f(x) is not differentiable for two values of x
[JEE 2006, 5 (–1)]

EXERCISE–7(Continuity)

Part : (A) Only one correct option

1. The value of f (0), so that the function, $f(x) = \dfrac{\sqrt{(a^2 - ax + x^2)} - \sqrt{(a^2 + ax + x^2)}}{\sqrt{(a+x)} - \sqrt{(a-x)}}$ (a > 0) becomes continuous for all x, is given by :

(A) $a\sqrt{a}$ (B) $\sqrt{a}$ (C) $-\sqrt{a}$ (D) $-a\sqrt{a}$

2. The value of R which makes $f(x) = \begin{cases} \sin(1/x) & , \ x \ne 0 \\ R & , \ x = 0 \end{cases}$ continuous at x = 0 is:

(A) 8 (B) 1 (C) – 1 (D) None of these

3. A function f(x) is defined as below $f(x) = \dfrac{\cos(\sin x) - \cos x}{x^2}$, x ≠ 0 and f(0) = a
f(x) is continuous at x = 0 if a equals

(A) 0 (B) 4 (C) 5 (D) 6

4. Let $f(x) = (\sin x)^{\frac{1}{\pi - 2x}}$, $x \ne \dfrac{\pi}{2}$. If f(x) is continuous at $x = \dfrac{\pi}{2}$ then $f\left(\dfrac{\pi}{2}\right)$ is

(A) e (B) 1 (C) 0 (D) none of these

5. $f(x) = \begin{cases} \dfrac{\sqrt{(1+px)} - \sqrt{(1-px)}}{x} & , -1 \le x < 0 \\ \dfrac{2x+1}{x-2} & , 0 \le x \le 1 \end{cases}$ is continuous in the interval [– 1, 1], then ' p' is equal to:

(A) – 1 (B) – 1/2 (C) 1/2 (D) 1

6. Let $f(x) = \left|\left(x + \dfrac{1}{2}\right)[x]\right|$ when – 2 ≤ x ≤ 2. where [.] represents greatest integer function. Then

(A) f(x) is continuous at x = 2 (B) f(x) is continuous at x = 1
(C) f(x) is continuous at x = – 1 (D) f(x) is discontinuous at x = 0

7. The set of all points for which

$f(x) = \dfrac{|x-3|}{|x-2|} + \dfrac{1}{[1+x]}$ where [.] represents greatest integer function is continuous is

(A) R (B) R – [– 1, 0]
(C) R – $(\{2\} \cup [-1, 0))$ (D) R – {(–1, 0) ∪ n, n ∈ I}

8. The function $f(x) = [x] \cos\left[\dfrac{(2x-1)}{2}\right]\pi$, ([.] denotes the greatest integer function) is dicontinuous at:

(A) all x (B) x = n/2, n ∈ I – {1} (C) no x (D) x which is not an integer

9. Let [x] denote the integral part of x ∈ R and g(x) = x – [x]. Let f(x) be any continuous function with f(0) = f(1) then the function h(x) = f(g(x)) :

(A) has finitely many discontinuities (B) is continuous on R
(C) is discontinuous at some x = c (D) is a constant function.

10. The function f(x) is defined by $f(x) = \begin{cases} \log_{(4x-3)}(x^2 - 2x + 5) & \text{if } \frac{3}{4} < x < 1 \ \& \ x > 1 \\ 4 & \text{if } x = 1 \end{cases}$

(A) is continuous at x = 1

(B) is discontinuous at x = 1 since $f(1^+)$ does not exist though $f(1^-)$ exists
(C) is discontinuous at x = 1 since $f(1^-)$ does not exist though $f(1^+)$ exists
(D) is discontinuous since neither $f(1^-)$ nor $f(1^+)$ exists.

11. Let $f(x) = \frac{1-\sin x}{(\pi-2x)^2} \cdot \frac{\ell n(\sin x)}{\ell n\left(1+\pi^2-4\pi x+4x^2\right)}$ $x \neq \frac{\pi}{2}$. The value of $f\left(\frac{\pi}{2}\right)$ so that the function is continuous at $x = \pi/2$ is:
(A) 1/16 (B) 1/32 (C) – 1/64 (D) 1/128

12. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$ then:
(A) f(x) is discontinuous for all x
(B) discontinuous for all x except at x = 0
(C) discontinuous for all x except at x = 1 or – 1
(D) none of these

13. Let $f(x) = [x^2] - [x]^2$, where [.] denotes the greatest integer function. Then
(A) f(x) is discontinuous for all integeral values of x
(B) f(x) is discontinuous only at x = 0, 1
(C) f(x) is continuous only at x = 1
(D) none of these

14. Let f(x) be a continuous function defined for $1 \le x \le 3$. If f(x) takes rational values for all x and f(2) = 10 then the value of f(1.5) is
(A) 7.5 (B) 10 (C) 8 (D) none of these

15. Let f(x) = Sgn(x) and $g(x) = x(x^2 - 5x + 6)$. The function f(g(x)) is discontinuous at
(A) infinitely many points
(B) exactly one point
(C) exactly three points
(D) no point

16. The function $f(x) = \left[x^2\left[\frac{1}{x^2}\right]\right]$, $x \ge 0$, is [.] represents the greatest integer less than or equal to x
(A) continuous at x = 1
(B) continuous at x = – 1
(C) discontinuous at infinitely many points
(D) continuous at x = – 1

17. The function f defined by $f(x) = \lim_{t\to\infty} \cdot \left\{\frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1}\right\}$ is
(A) everywhere continuous
(B) discontinuous at all integer values of x
(C) continuous at x = 0
(D) none of these

18. If [x] and {x} represent integral and fractional parts of a real number x, and $f(x) = \frac{a^{2|x|+\{x\}} - 1}{2[x] + \{x\}}$, $x \neq 0$, $f(0) = \log_e a$, where a > 0, $a \neq 1$, then
(A) f(x) is continuous at x = 0
(B) f(x) has a removable discontinuity at x = 0
(C) $\lim_{x\to 0}$ f(x) does not exist
(D) none of these

Part : (B) May have more than one options correct

19. If $f(x) = \sqrt{x}$ and g(x) = x – 1, then
(A) fog is continuous on $[0, \infty)$
(B) gof is continuous on $[0, \infty)$
(C) fog is continuous on $[1, \infty)$
(D) none of these

20. The function $f(x) = \begin{cases} x^m \sin\frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$ is continuous at x = 0 if
(A) $m \ge 0$ (B) m > 0 (C) m < 1 (D) $m \ge 1$

21. Let $f(x) = \frac{1}{[\sin x]}$ ([.] denotes the greatest integer function) then
(A) domain of f(x) is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \{2n\pi + \pi/2\}$
(B) f(x) is continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$
(C) f(x) is continuous at $x = 2n\pi + \pi/2$
(D) f(x) has the period 2π

22. Let $f(x) = [x] + \sqrt{x - [x]}$, where [x] denotes the greatest integer function. Then
(A) f(x) is continuous on R^+
(B) f(x) is continuous on R
(C) f(x) is continuous on R – I
(D) discontinuous at x = 1

23. Let f(x) and g(x) be defined by f(x) = [x] and $g(x) = \begin{cases} 0, & x \in I \\ x^2, & x \in R - I \end{cases}$ (where [.] denotes the greatest integer function) then
(A) $\lim_{x\to 1}$ g(x) exists, but g is not continuous at x = 1
(B) $\lim_{x\to 1}$ f(x) does not exist and f is not continuous at x = 1
(C) gof is continuous for all x
(D) fog is continuous for all x

24. Which of the following function(s) defined below has/have single point continuity.
(A) $f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$
(B) $g(x) = \begin{cases} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{cases}$
(C) $h(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$
(D) $k(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$

EXERCISE–8

1. Discuss the continuity of the function, f (x) at x = 3, if

$f(x) = \begin{cases} x\,[x] & , \text{if } \; 0 \le x < 3 \\ (x-1)\,[x] & , \text{if } \; 3 \le x \le 4 \end{cases}$ where [.] denotes greatest integer function.

2. Find the values of ' a ' & ' b ' so that the function, $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & , x < \pi/2 \\ a & , x = \pi/2 \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & , x > \pi/2 \end{cases}$ is continuous at $x = \pi/2$.

3. Discuss the continuity of the function, $f(x) = \begin{cases} \dfrac{e^x-1}{\ell n(1+2x)} & , x \ne 0 \\ 7 & , x = 0 \end{cases}$ at x = 0. If discontinuous, find the nature of discontinuity ?

4. If f(x) = x + {– x} + [x], where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [–2, 2]. Also find nature of each discontinuity.

5. Let $f(x) = \begin{cases} 1+x & , 0 \le x \le 2 \\ 3-x & , 2 < x \le 3 \end{cases}$. Determine the form of g(x) = f (f(x)) & hence find the point of discontinuity of g, if any.

6. Examine the continuity at x = 0 of the sum function of the infinite series:

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots\ldots\ldots\ldots\infty .$$

7. If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \ne 0)$ is continuous at x = 0. Find A & B. Also find f (0).

8. Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x) = \begin{cases} \dfrac{\exp\left\{(x+2)\dfrac{1}{4}[x+1]\ln 4\right\} - 16}{4^x - 16} & , x < 2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)} & , x > 2 \end{cases}.$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

9. Discuss the continuity of the function $f(x) = \underset{n\to\infty}{\text{Limit}} \dfrac{(1+\sin x)^n + \ln x}{2+(1+\sin x)^n}$.

10. Let f(x + y) = f(x) + f(y) for all x, y and if the function f(x) is continuous at x = 0, then show that f(x) is continuous at all x.

11. If $f(x \cdot y) = f(x) \cdot f(y)$ for all x, y and f(x) is continuous at x = 1. Prove that f(x) is continuous for all x except at x = 0. Given $f(1) \ne 0$.

12. If $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x)+2f(y)}{3}$ $\forall$ x, y $\in$ R and f(x) is continuous at x = 0. Prove that f (x) is continuous for all x $\in$ R.

13. If f(x) = sin x and $g(x) = \begin{cases} \max^m \{f(t) ; 0 \le t \le x, 0 \le x \le 2 \\ 3x-4 \quad ; \; x > 2 \end{cases}$, then discuss the continuity of g(x) $\forall$ x $\ge$ 0 .

Que. From Compt. Exams
(Already given with Function)

Limit Lollypop Sheet Given

Assertion & Reasons
(DOWNLOAD EXTRA FILE FOR
LIMIT ,CONTINUITY, DIFFRENTIABILITY)

for 34 Yrs. Que. of IIT-JEE
&
10 Yrs. Que. of AIEEE
we have distributed already a book

CONTINUITY

EXERCISE–1

Q1. $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define

Q 2. **(a)** –2, 2, 3 **(b)** K = 5 **(c)** even

Q 3. $y_n(x)$ is continuous at x = 0 for all n and y(x) is dicontinuous at x = 0

Q 4. f is cont. in $-1 \le x \le 1$ **Q 5.** P not possible.

Q 6. (a) $4 - 3\sqrt{2} + a$, (b) a = 3

Q 7. $g(x) = 2 + x$ for $0 \le x \le 1$, $2 - x$ for $1 < x \le 2$, $4 - x$ for $2 < x \le 3$,
g is discontinuous at x = 1 & x = 2

Q 8. A = 1 ; f(2) = 1/2 **Q 9.** a = 0 ; b = –1

Q.11 $f(f(x))$ is continuous and domain of $f(f(x))$ is $[-4, \sqrt{6}\,]$

Q 12. gof is dis-cont. at x = 0, 1 & -1

Q 13. a = 1/2, b = 4 **Q14.** a = – 3/2, $b \neq 0$, c = 1/2

Q 15. A = – 4 , B = 5, f(0) = 1 **Q 16.** discontinuous at x = 1, 4 & 5

Q 17. discontinuous at all integral values in [–2 , 2]

Q 18. locus (a, b) $\to$ x, y is y = x – 3 excluding the points where y = 3 intersects it.

Q 20. 5 **Q22.** $\frac{1}{60}$

Q 25. k = 0 ; $g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g (x) is continuous everywhere.

Q 26. g (x) = 4 (x + 1) and limit $= -\frac{39}{4}$

Q 28. $a = \frac{1}{\sqrt{2}}$, $g(0) = \frac{(\ell n 2)^2}{8}$

Q 29. $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ $\Rightarrow$ f is discont. at x = 0 ;
$g(0^+) = g(0^-) = g(0) = \pi/2 \Rightarrow$ g is cont. at x = 0

Q 30. the function f is continuous everywhere in [0 , 2] except for $x = 0, \frac{1}{2}$, 1 & 2.

EXERCISE–2

Q 1. **(a)** false; **(b)** false ; **(c)** false ; **(d)** false ; **(e)** false ; **(f)** true ; **(g)** false ; **(h)** true

Q 2. **(a)** c = ±1 ; **(b).** x ± 1, – 1 & x = 0 ; **(c).** 1 ; **(d).** $a = \frac{\pi}{6}$, $b = -\frac{\pi}{12}$ **(e).** 1/2

Q 3. **(a)** D **(b).** B, C **(c).** C, D **(d).** B **(e).** C **(f).** A **(g).** B **(h)** A **(i)** D **(j)** A **(k)** C

EXERCISE–3

Q.1 R – [–1, 0) ; discontinuous for all integral values in domain except at zero

Q.2 10 **Q.3** D **Q.4** $a = ln\frac{2}{3}$; $b = \frac{2}{3}$; c = 1

Q.5 Discontinuous at x = 1; $f(1^+) = 1$ and $f(1^-) = -1$

* *

DIFFERENTIABILITY

EXERCISE–4

Q 1. f(x) is conti. but not derivable at x = 0 **Q 2.** conti. $\forall$ $x \in R$, not diff. at x = 0,1 & 2

Q 4. conti. but not diff.at x = 0 ; diff. & conti. at x = $\pi/2$ **Q 5.** conti. but not diff. at x = 0

Q 7. f is cont. but not diff. at x = 0 **Q 8.** $f'(1^+) = 3$, $f'(1^-) = -1$

Q 9. a= 1/2 , b = 3/2 **Q 10.** not derivable at x = 0 & x = 1

Q 11. f is cont. & derivable at x = – 1 but f is neither cont. nor derivable at x = 1

Q 12. discontinuous & not derivable at x = 1, continuous but not derivable at x = 2

Q 13. not derivable at x = 0

Q 14. f is conti. at x = 1, 3/2 & disconti. at x = 2, f is not diff. at x = 1, 3/2, 2

Q15. $(fog)(x) = x+1$ for $-2 \le x \le -1$, $-(x+1)$ for $-1 < x \le 0$ & $x-1$ for $0 < x \le 2$.
(fog)(x) is cont. at x = –1, $(gof)(x) = x+1$ for $-1 \le x \le 1$ & $3-x$ for $1 < x \le 3$.
(gof)(x) is not differentiable at x = 1

Q 16. $a \ne 1, b = 0, p = \frac{1}{3}$ and $q = -1$

Q 17. If $a \in (0, 1)$ $f'(0^+) = -1$; $f'(0^-) = 1 \Rightarrow$ continuous but not derivable
a = 1 ; f(x) = 0 which is constant ⇒ continuous and derivable
If $a > 1$ $f'(0^-) = -1$; $f'(0^+) = 1 \Rightarrow$ continuous but not derivable

Q18. conti. in $0 \le x \le 1$ & not diff. at x = 0

Q.19 f is conti. but not diff. at x = 1, disconti. at x = 2 & x = 3. cont.& diff.at all other points

Q.20 (a) f '(0) = 0, (b) $f'\left(\frac{1^-}{3}\right) = -\frac{\pi}{2}$ and $f'\left(\frac{1^+}{3}\right) = \frac{\pi}{2}$, (c) $x = \frac{1}{2n+1}$ $n \in I$

Q.21 continuous but not derivable at $x = \sqrt{2}$ **Q.22** $f'(0) = \frac{\alpha}{1-k}$

Q.23 $f(x) = e^{2x}$ **Q.24** $f(x) = x \Rightarrow f(10) = 10$

EXERCISE–5

Q.1 2 **Q.2** conti. & diff. **Q.3** 0 **Q.4** $f'(0^+) = 0$, $f'(0^-) = 1$
Q.5 2 **Q.6** C **Q.7** B **Q.8** A
Q.9 D **Q.10** B **Q.11** A **Q.12** D
Q.13 D **Q.14** A **Q.15** A, B, D **Q.16** A, C
Q.17 A, B **Q.18** B, D **Q.19** B, D **Q.20** A, B, C
Q.21 B, D **Q.22** A, B , D **Q.23** A, B, D **Q.24** A, B, D
Q.25 A, D

EXERCISE–6

Q.1 f(x) is conti. & diff. at x = 1 ; f(x) is not conti. & not diff. at x = 2 **Q.2** A, C, D
Q.3 conti. but not derivable at x = 1, neither cont. nor deri. at x = –1 **Q.4** D
Q.5 C **Q.6** Discont. hence not deri. at x = 1 & –1. Cont. & deri. at x = 0
Q.7 **(a)** D, **(b)** A, **(c)** D **Q.9** D **Q.10** C **Q.11** a = 1; b = 0(gof)'(0) = 0
Q.12 $f'(a^-) = 0$ **Q.13** **(a)** A, **(b)** B, **(c)** y – 2 = 0 **Q.14** A, C

Continuity
EXERCISE– 7

1. C **2.** D **3.** A **4.** B **5.** B **6.** D **7.** D
8. B **9.** B **10.** D **11.** C **12.** C **13.** D **14.** B
15. C **16.** C **17.** B **18.** C **19.** BC **20.** BD
21. ABD **22.** ABC **23.** ABC **24.** BCD

EXERCISE– 8

1. continuous at x = 3 **2.** $a = \frac{1}{2}$, b = 4
3. Removable isolated point
4. discontinuous at all integral values in [– 2, 2]
5. $g(x) = 2 + x$; $0 \le x \le 1$,
$= 2 - x$; $1 < x \le 2$,
$= 4 - x$; $2 < x \le 3$,
g is discontinuous at x = 1 & x = 2
6. Discontinuous **7.** A = – 4, B = 5, f(0) = 1
8. A = 1; f(2) = 1/2
9. f (x) is discontinuous at natural multiples of π
13. continuous for all x ≥ 0 except at x = 2