Chapter 2
Newtonian Mechanics – Single Particle

In this Chapter we will review what Newton's laws of mechanics tell us about the motion of a single particle. Newton's laws are only valid in suitable reference frames, and we will discuss what makes a reference frame a suitable reference frame. We will also review the various conservation laws you should have already encountered in your introductory physics course. In addition, we will discuss the limits of Newtonian mechanics.

Newton's Laws

The following three laws of Newtonian mechanics should have been discussed in your introductory physics course:

- **Newton's First Law: A body remains at rest or in uniform motion unless acted upon by a force.** This law does not tell us very much about the concept of force, except what we mean with zero force: if we see a body at rest or in uniform motion, we know that the force acting on the body is zero.

- **Newton's Second Law: A body acted upon by a force moves in such a manner that the time rate of change of its linear momentum equals the force.** Newton defined the linear momentum of a particle of mass m moving with a velocity v as mv. The second law can thus be used to define the force $F = d(mv)/dt$. This definition is of course only useful if the mass and the velocity of a particle are defined.

- **Newton's Third Law: If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.** This law is only true if the force acting between the bodies is directed along the line connecting the bodies (these forces are called central forces). Forces that are velocity dependent are in general non-central forces and do not satisfy Newton's Third Law. Conservation of linear momentum is a direct consequence of the third law. If $F_1 = -F_2$ then $d(m_1v_1)/dt = -d(m_2v_2)/dt$. This equation can be rewritten as $d(m_1v_1 + m_2v_2)/dt = 0$ or $m_1v_1 + m_2v_2 = $ constant.

Reference Systems

Newton's laws are only valid in an appropriate reference frame. We can use this requirement to define inertial reference systems: an inertial reference frame is a reference frame in which Newton's laws are valid. If Newton's laws are valid in one reference frame, they are also valid in any reference frame in uniform motion with respect to the first frame. In order to be able to describe a free particle (a particle on which no force is acting) in a reference frame, the reference frame must satisfy the following conditions:

- The equation of motion of the particle should be independent of the position of the origin of the coordinate system.
- The equation of motion of the particle should be independent of the orientation of the coordinate system.
- Time must be homogeneous (the velocity of a free particle must be constant).

Single-Particle Motion

If we know the force acting on the particle, we can use Newton's second law to describe its motion:

$$\frac{d\bar{v}}{dt} = \frac{1}{m}\bar{F}$$

Note: F must be the total force acting on the particle.

As long as the force is constant, we can usually obtain an analytical expression for the velocity and/or the trajectory of the particle. The situation becomes more complicated when the force is time dependent, velocity dependent, and/or position dependent. In those circumstances we may need to rely on numerical methods to predict the motion of the particle. The second law can be rewritten as

$$d\bar{v} = \bar{v}(t+dt) - \bar{v}(t) = \frac{dt}{m}\bar{F}$$

The velocity at time $t + dt$ can thus be determined from the velocity at time t using the following relation:

$$\bar{v}(t+dt) = \bar{v}(t) + \frac{dt}{m}\bar{F}$$

This relation can be used to determine the velocity as function of time, if we know 1) the velocity at one specific time and 2) the force F is known (**but does not need to be constant**). Once we know the velocity as function of time, we can determine the position as function of time:

$$\bar{r}(t+dt) = \bar{r}(t) + \bar{v}(t)dt$$

Let us illustrate the use of numerical methods by focusing on projectile motion. If the gravitational force is the only force present, we can express the motion of the particle analytically. By comparing the results of a numerical calculation with the analytical solution, we can study the limitations of the numerical approach. Since the gravitational force is acting in the

vertical direction (along the y axis), the force will have no effect on the motion in the horizontal direction (along the x axis):

$$v_x(t+dt) = v_x(t)$$

$$v_y(t+dt) = v_y(t) - g dt$$

$$x(t+dt) = x(t) + v_x(t) dt$$

$$y(t+dt) = y(t) + v_y(t) dt$$

Many different programs can be used to study the evolution of these equations. No matter what approach is being used, the most critical choice the user will have to make is the size of the step size dt. In the case of a constant force, the expressions for the velocity at time $t + dt$ are correct, independent of the step size dt. However, the expressions for the position at time $t + dt$ are only correct if the velocity is constant over the period between time t and time $t + dt$. This is a reasonable approximations if the step size dt is small, but for a large step size dt this is clearly a poor approximation (and large errors will result). A very small step size will increase the computing time and may lead to rounding errors.

On the Physics 235 homepage you can find an example of a study of projectile motion using Excel. Using the setup stored in the file *ProjectileMotion.xls* we can study the effect of the choice of the step size dt. Consider the case of projectile motion, starting at time $t = 0$ s at the origin of our coordinate system with a velocity of +700 m/s in the horizontal and +700 m/s in the vertical direction. Figure 1 shows a comparison between the results of the analytical calculation of the trajectory of the projectile (blue data points) and the results of the numerical calculation of the trajectory (red data points) with a time step of 10 s. The difference between these two are shown by the green data points (defined as $y_{numerical} - y_{analytical}$). There is clearly a significant difference between the numerical and the analytical calculation.

Figure 2 shows the results of the same calculation as shown in Figure 1, except that the time step was changed to 1 s. As a result, there are clearly more data points on the trajectory. However, the more important difference is the significant reduction of the difference between the analytical and the numerical calculations. Figure 3 shows another result of the projectile motion, now obtained with a step size of 0.25 s. The difference between the numerical and the analytical method is further reduced.

The differences between the numerical method and the analytical method at a horizontal distance of 100,000 m are shown in the following table for the step sizes used to generate Figures 1 - 3.

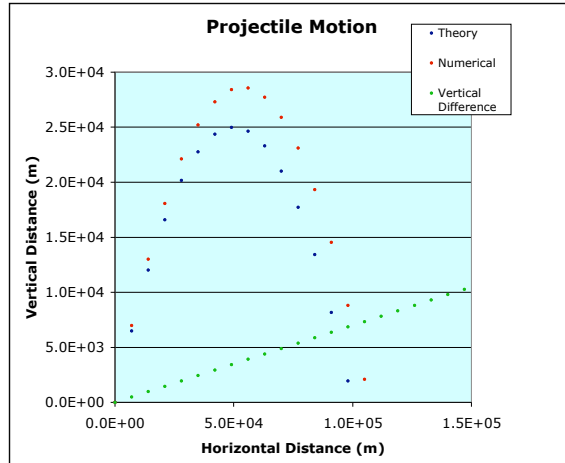| dt (s)	10	1	0.25
Difference (m)	7350	701	175

Figure 1. **Results of numerical and analytical calculations of projectile motion with dt = 10 s.**
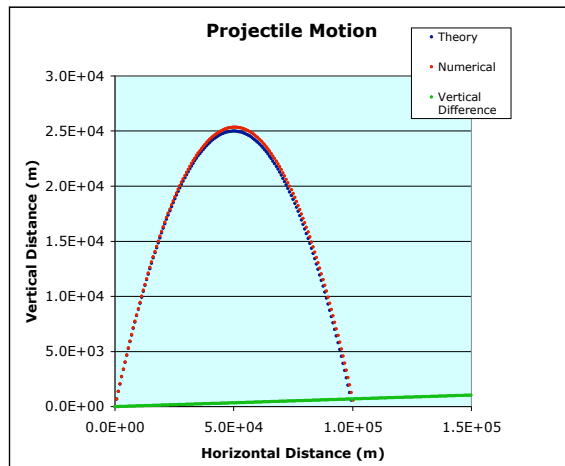

Figure 2. **Results of numerical and analytical calculations of projectile motion with dt = 1 s.**
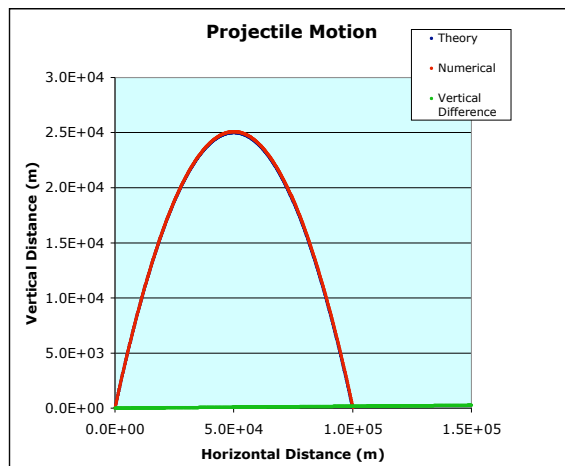

Figure 3. **Results of numerical and analytical calculations of projectile motion with dt = 0.25 s.**

Now let us consider what happens when besides the gravitational force, there is a drag force acting on the particle. The drag force is usually proportional to the power of the velocity and directed in a direction opposite to the direction of motion. The net force on the particle is this equal to

$$F_x = -kmv_x$$

$$F_y = -kmv_y - mg$$

Since the force is known, we can determine the velocity of the particle:

$$v_x(t+dt) = v_x(t) + \frac{dt}{m}F_x = v_x(t) + \frac{dt}{m}\left(-kmv_x(t)\right) = (1-kdt)v_x(t)$$

$$v_y(t+dt) = v_y(t) + \frac{dt}{m}F_y = v_y(t) + \frac{dt}{m}\left(-kmv_y(t) - mg\right) = (1-kdt)v_y(t) - gdt$$

The position of the particle can be found once we have determined the velocity as function of time:

$$x(t+dt) = x(t) + v_x(t)dt$$

$$y(t+dt) = y(t) + v_y(t)dt$$

In principle, we can still solve this problem analytically (see Example 2.5 in the text book) but we will use the same numerical approach as we used in our study of projectile motion without drag to study the trajectory for different values of k. The results of a calculation for $k = 0.01$ and with time steps of 0.25 s is shown in Figure 4.
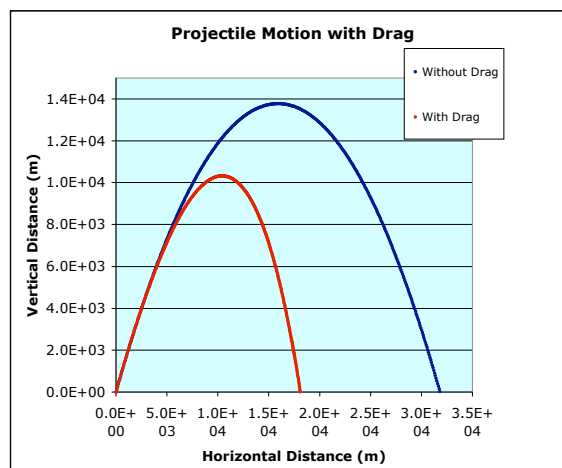

Figure 4. Projectile motion with drag.

Conservation Laws

Several important conservation laws are a direct consequence of Newton's laws of motion. These conservation laws can significantly reduce the effort required to solve certain mechanics problems. In this Section we will briefly discuss the most important conservation laws that we will use in classical mechanics.

- **The total linear momentum p of a particle is conserved when the total force on it is zero.**
  This law is a direct consequence of Newton's second law, which relates the change in the linear momentum of a particle to the force acting on it.

- **The angular momentum L of a particle subject to no torque is conserved.**
  This law is a direct consequence of the definition of angular momentum and torque. In fact one can argue that torque was defined such that it is equal to the rate of change of the angular momentum (dL/dt).

- **The total energy E of a particle in a conservative force field is constant in time**.
  The total energy E is defined as the sum of the kinetic energy T and the potential energy U. The potential energy U is defined by the force field only to within a constant. It has no absolute meaning, and only differences in the potential energy are physically meaningful. The force field is conservative if the line integral of the force between two points is path independent. In this case, we can write the force as the gradient of a scalar function, and this scalar function is the potential energy U:

$$\bar{F} = -\bar{\nabla}U$$

  We can show that the rate of energy change dE/dt will be zero if the potential energy U does not depend explicitly on time ($\partial U/\partial t = 0$).

These three conservation laws are the most important conservation laws in classical mechanics and we will use them in many different applications.

An important application of one of our conservation laws is the prediction of motion based on a potential energy curve $U(x)$. Consider for example, the potential energy curve shown in Figure 5. Since the total energy E is the sum of the potential energy U and the kinetic energy T, the total energy will always be larger or equal to the potential energy U. Looking at Figure 5, we can immediately draw some important conclusions:

- No particle can exist with a total energy E less than E_0.
- A particle with energy E_1 can only be present between x_a and x_b.
- A particle with energy E_4 can be present at any position.

Since the force is related to the derivative of the potential energy, the positions where the derivative is equal to 0 are the positions where the net for the on the particle is zero (these are the equilibrium positions).
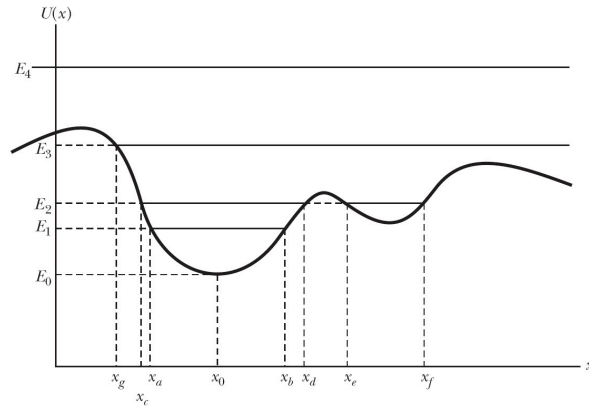

Figure 5. Potential energy $U(x)$ as function of position.

Consider the potential energy in the vicinity of an equilibrium position, and assume we have chosen our coordinate system such that the equilibrium position corresponds to $x = 0$. We can expand the potential around the equilibrium point:

$$U(x) = U_0 + x\left(\frac{dU}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3U}{dx^3}\right)_0 + \ldots$$

Since at the equilibrium point, the slope of $U(x)$ is zero ($dU/dx = 0$) and since we can define the potential to be zero at this equilibrium point, we can rewrite the expansion of U as

$$U(x) = \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3U}{dx^3}\right)_0 + \ldots$$

For small displacements with respect to the equilibrium position, x is small. As a result, the first non-zero term in the expansion will dominate the expansion:

$$U(x) = \frac{x^2}{2!}\left(\frac{d^2U}{dx^2}\right)_0$$

For the equilibrium to be **stable**, the potential energy on either side of the equilibrium point must be higher than the potential energy at the equilibrium point (which we defined to be 0). In order to achieve this we must require that

$$\left(\frac{d^2U}{dx^2}\right)_0 > 0$$

If this condition is not satisfied, the equilibrium is an **unstable** equilibrium.

Limitations of Newtonian Mechanics

Newtonian mechanics can be used to describe many every-day macroscopic phenomena. When we study macroscopic phenomena, we can measure both the position and the linear momentum of objects of interest with great precision. However, when we start to study microscopic object we discover that we can no longer measure the position and the linear momentum with great precision. In fact, the actual measurement may influence the state of the system. In this regime, our capability of determining the position and the linear momentum of an object are limited by the Heisenberg uncertainty principle, which states that

$$\Delta x \Delta p \geq 10^{-34} \text{ Js}$$

If we measure the position with infinite precision, the uncertainty in the linear momentum approaches infinity. In this regime, Newtonian mechanics can no longer be used, and we need **quantum mechanics** to describe microscopic systems.

The limitations of Newtonian mechanics also appear when we study motion with velocities close to the speed of light. In this regime, we need the **theory of relativity**. One fundamental assumption in the theory of relativity is that the speed of light is constant, the same in each reference frame. This is clearly inconsistent with Newtonian mechanics, and the rules that govern transformations of position and velocity between coordinate systems.

Another limitation of Newtonian mechanics becomes obvious when we try to describe systems with large numbers of particles. Even if we know all of the details of the interaction between the particles, is becomes very difficult to predict the properties of the system by carrying out calculations involving the each individual interaction between all the particles. Such systems can be described by theory of **statistical mechanics**, which relates the properties of microscopic interactions to the average macroscopic properties of the system.