$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;
$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

10. n^{th} ROOTS OF UNITY :

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$ **&**

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots. + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

(iii) $(1 - \alpha_1)(1 - \alpha_2) \ldots\ldots (1 - \alpha_{n-1}) = n$ **&**
$(1 + \alpha_1)(1 + \alpha_2) \ldots\ldots. (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(iv) $1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots\ldots\ldots \alpha_{n-1} = 1$ or -1 according as n is odd or even.

11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots.. + \cos n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\dfrac{n+1}{2}\right)\theta.$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots.. + \sin n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\dfrac{n+1}{2}\right)\theta.$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(A) If z_1 & z_2 are two complex numbers then the complex number $z = \dfrac{nz_1 + mz_2}{m + n}$ divides the joins of z_1 & z_2 in the ratio m : n.

Note:(i) If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively, then :

(a) Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$:

(b) Orthocentre of the $\Delta ABC =$
$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \quad \textbf{OR} \quad \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(c) Incentre of the $\Delta ABC = (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(d) Circumcentre of the ΔABC = :
$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(B) $amp(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x–axis.

(C) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(D) The equation of a line joining z_1 & z_2 is given by ;
$z = z_1 + t(z_1 - z_2)$ where t is a perameter.

(E) $z = z_1(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to oz_1.

(F) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as
$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$ This is also the condition for three complex numbers to be collinear.

(G) Complex equation of a straight line through two given points z_1 & z_2 can be written as $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$, which on manipulating takes the form as $\bar{\alpha} z + \alpha \bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(H) The equation of circle having centre z_0 & radius ρ is :
$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form

$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$, r is real centre $-\alpha$ & radius $\sqrt{\alpha\bar{\alpha} - r}$.

Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.

(I) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is :

(i) $\arg\dfrac{z - z_2}{z - z_1} = \pm\dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

(J) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number

$\frac{z_3-z_1}{z_3-z_2}\cdot\frac{z_4-z_2}{z_4-z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real $\Rightarrow \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$

13.(a) Reflection points for a straight line :
Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if ; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(b) Inverse points w.r.t. a circle :
Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :
(i) the point O, P, Q are collinear and on the same side of O. **(ii)** OP . OQ = ρ^2.
Note that the two points z_1 & z_2 will be the inverse points w.r.t. the circle $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$ if and only if $z_1\bar{z}_2+\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$.

14. PTOLEMY'S THEOREM : It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides. i.e. $|z_1-z_3|\,|z_2-z_4| = |z_1-z_2|\,|z_3-z_4| + |z_1-z_4|\,|z_2-z_3|$.

15. LOGARITHM OF A COMPLEX QUANTITY :

(i) $\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi+\frac{\pi}{2}\right)}$, $n \in I$.

VERY ELEMENTARY EXERCISE

Q.1 Simplify and express the result in the form of $a + bi$

(a) $\left(\frac{1+2i}{2+i}\right)^2$ (b) $-i(9+6i)(2-i)^{-1}$ (c) $\left(\frac{4i^3-i}{2i+1}\right)^2$ (d) $\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$ (e) $\frac{(2+i)^2}{2-i}-\frac{(2-i)^2}{2+i}$

Q.2 Given that $x, y \in R$, solve : (a) $(x + 2y) + i(2x - 3y) = 5 - 4i$ (b) $(x + iy) + (7 - 5i) = 9 + 4i$
(c) $x^2 - y^2 - i(2x + y) = 2i$ (d) $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$
(e) $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

Q.3 Find the square root of : (a) $9 + 40i$ (b) $-11 - 60i$ (c) $50i$

Q.4 (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$
(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$

Q.5 Among the complex numbers z satisfying the condition $\left|z + 3 - \sqrt{3}\,i\right| = \sqrt{3}$, find the number having the least positive argument.

Q.6 Solve the following equations over C and express the result in the form a + ib, a, b $\in$ R.
(a) $ix^2 - 3x - 2i = 0$ (b) $2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$

Q.7 Locate the points representing the complex number z on the Argand plane:
(a) $|z+1-2i| = \sqrt{7}$; **(b)** $|z-1|^2 + |z+1|^2 = 4$; **(c)** $\left|\frac{z-3}{z+3}\right| = 3$; **(d)** $|z-3| = |z-6|$

Q.8 If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'.

Q.9 For what real values of x & y are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex?

Q.10 Find the modulus, argument and the principal argument of the complex numbers.
(i) $6(\cos 310° - i\sin 310°)$ (ii) $-2(\cos 30° + i\sin 30°)$ (iii) $\frac{2+i}{4i+(1+i)^2}$

Q.11 If $(x + iy)^{1/3} = a + bi$; prove that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

Q.12(a) If $\frac{a+ib}{c+id} = p + qi$, prove that $p^2 + q^2 = \frac{a^2+b^2}{c^2+d^2}$.
(b) Let z_1, z_2, z_3 be the complex numbers such that
$z_1 + z_2 + z_3 = z_1z_2 + z_2z_3 + z_3z_1 = 0$. Prove that $|z_1| = |z_2| = |z_3|$.

Q.13 Let z be a complex number such that $z \in c\backslash R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$.

Q.14 Prove the identity, $|1-z_1\bar{z}_2|^2 - |z_1-z_2|^2 = \left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$

Q.15 For any two complex numbers, prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$. Also give the geometrical interpretation of this identity.

Q.16 (a) Find all non–zero complex numbers Z satisfying $\overline{Z} = iZ^2$.
(b) If the complex numbers $z_1, z_2, ...z_n$ lie on the unit circle $|z| = 1$ then show that $|z_1 + z_2 + ...+z_n| = |z_1^{-1} + z_2^{-1} + ...+z_n^{-1}|$.

Q.17 Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z - 4| + |z + 4| = 16$.

Q.18 If ω is an imaginary cube root of unity then prove that :
(a) $(1 + \omega - \omega^2)^3 - (1 - \omega + \omega^2)^3 = 0$ (b) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$
(c) If ω is the cube root of unity, Find the value of, $(1 + 5\omega^2 + \omega^4)(1 + 5\omega^4 + \omega^2)(5\omega^3 + \omega + \omega^2)$.

Q.19 If ω is a cube root of unity, prove that ; (i) $(1 + \omega - \omega^2)^3 - (1 - \omega + \omega^2)^3$
(ii) $\dfrac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} = \omega^2$ (iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = 9$

Q.20 If $x = a + b$; $y = a\omega + b\omega^2$; $z = a\omega^2 + b\omega$, show that
(i) $xyz = a^3 + b^3$ (ii) $x^2 + y^2 + z^2 = 6ab$ (iii) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$

Q.21 If $(w \neq 1)$ is a cube root of unity then $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} =$
(A) 0 (B) 1 (C) i (D) w

Q.22(a) $(1 + w)^7 = A + Bw$ where w is the imaginary cube root of a unity and $A, B \in R$, find the ordered pair (A, B).
(b) The value of the expression ;
$1.(2 - w)(2 - w^2) + 2.(3 - w)(3 - w^2) + ... + (n - 1).(n - w)(n - w^2)$, where w is an imaginary cube root of unity is ________.

Q.23 If $n \in N$, prove that $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2}+1} . \cos\dfrac{n\pi}{4}$.

Q.24 Show that the sum $\sum_{k=1}^{2n}\left(\sin\dfrac{2\pi k}{2n+1} - i\cos\dfrac{2\pi k}{2n+1}\right)$ simplifies to a pure imaginary number.

Q.25 If $x = \cos\theta + i\sin\theta$ & $1 + \sqrt{1 - a^2} = na$, prove that $1 + a\cos\theta = \dfrac{a}{2n}(1 + nx)\left(1 + \dfrac{n}{x}\right)$.

Q.26 The number t is real and not an integral multiple of π/2. The complex number x_1 and x_2 are the roots of the equation, $\tan^2(t) \cdot x^2 + \tan(t) \cdot x + 1 = 0$
Show that $(x_1)^n + (x_2)^n = 2\left(\cos\dfrac{2n\pi}{3}\right)\cot^n(t)$.

EXERCISE-1

Q.1 Simplify and express the result in the form of a + bi :
(a) $-i(9 + 6i)(2 - i)^{-1}$ (b) $\left(\dfrac{4i^3 - i}{2i + 1}\right)^2$ (c) $\dfrac{3 + 2i}{2 - 5i} + \dfrac{3 - 2i}{2 + 5i}$
(d) $\dfrac{(2 + i)^2}{2 - i} - \dfrac{(2 - i)^2}{2 + i}$ (e) $\sqrt{i} + \sqrt{-i}$

Q.2 Find the modulus , argument and the principal argument of the complex numbers.
(i) $z = 1 + \cos\left(\dfrac{10\pi}{9}\right) + i\sin\left(\dfrac{10\pi}{9}\right)$ (ii) $(\tan 1 - i)^2$
(iii) $z = \dfrac{\sqrt{5 + 12i} + \sqrt{5 - 12i}}{\sqrt{5 + 12i} - \sqrt{5 - 12i}}$ (iv) $\dfrac{i - 1}{i\left(1 - \cos\dfrac{2\pi}{5}\right) + \sin\dfrac{2\pi}{5}}$

Q.3 Given that $x, y \in R$, solve :
(a) $(x + 2y) + i(2x - 3y) = 5 - 4i$ (b) $\dfrac{x}{1 + 2i} + \dfrac{y}{3 + 2i} = \dfrac{5 + 6i}{8i - 1}$
(c) $x^2 - y^2 - i(2x + y) = 2i$ (d) $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$
(e) $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

Q.4(a) Let Z is complex satisfying the equation, $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$.

Suppose the equation has a real root, then find the value of *m*.

(b) a, b, c are real numbers in the polynomial, $P(Z) = 2Z^4 + aZ^3 + bZ^2 + cZ + 3$
If two roots of the equation $P(Z) = 0$ are 2 and i, then find the value of 'a'.

Q.5(a) Find the real values of x & y for which $z_1 = 9y^2 - 4 - 10\,i\,x$ and $z_2 = 8y^2 - 20\,i$ are conjugate complex of each other.

(b) Find the value of $x^4 - x^3 + x^2 + 3x - 5$ if $x = 2 + 3i$

Q.6 Solve the following for z : (a) $z^2 - (3 - 2\,i)z = (5i - 5)$ (b) $|z| + z = 2 + i$

Q.7(a) If $iZ^3 + Z^2 - Z + i = 0$, then show that $|Z| = 1$.

(b) Let z_1 and z_2 be two complex numbers such that $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, find $|z_1|$.

(c) Let $z_1 = 10 + 6i$ & $z_2 = 4 + 6i$. If z is any complex number such that the argument of, $\dfrac{z - z_1}{z - z_2}$ is $\dfrac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

Q.8 Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]\ldots\ldots\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right] \text{ is equal to } \left(1 - \frac{1}{2^{2^n}}\right)(1+i) \text{ where } n \geq 2.$$

Q.9 Let a & b be complex numbers (which may be real) and let,
$Z = z^3 + (a + b + 3i)\,z^2 + (ab + 3\,ia + 2\,ib - 2)\,z + 2\,abi - 2a$.
(i) Show that Z is divisible by, $z + b + i$. (ii) Find all complex numbers z for which $Z = 0$.
(iii) Find all purely imaginary numbers a & b when $z = 1 + i$ and Z is a real number.

Q.10 Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ (b) $\operatorname{Re}\left(\dfrac{z + 2i}{iz + 2}\right) \leq 4$ $(z \neq 2i)$

(c) $\operatorname{Arg}(z + i) - \operatorname{Arg}(z - i) = \pi/2$ (d) $\operatorname{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$.

Q.11 Prove that the complex numbers z_1 and z_2 and the origin form an isosceles triangle with vertical angle $2\pi/3$ if $z_1^2 + z_2^2 + z_1 z_2 = 0$.

Q.12 P is a point on the Aragand diagram. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin & P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2 \cdot \cos 2\theta = Z_1 \cdot Z_3 \cos^2\theta$.

Q.13 Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundry.

Q.14 If $a\,\text{CiS}\,\alpha$, $b\,\text{CiS}\,\beta$, $c\,\text{CiS}\,\gamma$ represent three distinct collinear points in an Argand's plane, then prove the following :
(i) $\Sigma\, ab \sin(\alpha - \beta) = 0$.
(ii) $(a\,\text{CiS}\,\alpha)\sqrt{b^2 + c^2 - 2bc\cos(\beta - \gamma)} \pm (b\,\text{CiS}\,\beta)\sqrt{a^2 + c^2 - 2ac\cos(\alpha - \gamma)}$
$\mp (c\,\text{CiS}\,\gamma)\sqrt{a^2 + b^2 - 2ab\cos(\alpha - \beta)} = 0$.

Q.15 Find all real values of the parameter a for which the equation
$(a - 1)z^4 - 4z^2 + a + 2 = 0$ has only pure imaginary roots.

Q.16 Let $A \equiv z_1$; $B \equiv z_2$; $C \equiv z_3$ are three complex numbers denoting the vertices of an acute angled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that

$$z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$$

hence show that the ΔABC is a right angled triangle $\Leftrightarrow z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1 = 0$

Q.17 If the complex number P(w) lies on the standard unit circle in an Argand's plane and $z = (aw + b)(w - c)^{-1}$ then, find the locus of z and interpret it. Given a, b, c are real.

Q.18(a) Without expanding the determinant at any stage, find $K \in R$ such that

$$\begin{vmatrix} 4i & 8+i & 4+3i \\ -8+i & 16i & i \\ -4+Ki & i & 8i \end{vmatrix} \text{ has purely imaginary value.}$$

(b) If A, B and C are the angles of a triangle

$$D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$$ where $i = \sqrt{-1}$ then find the value of D.

Q.19 If w is an imaginary cube root of unity then prove that :
(a) $(1 - w + w^2)(1 - w^2 + w^4)(1 - w^4 + w^8)$ ..... to 2n factors $= 2^{2n}$.
(b) If w is a complex cube root of unity, find the value of
$(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ ..... to n factors .

Q.20 Prove that $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right)$. Hence deduce that

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5 + i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5 = 0$$

Q.21 If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2$ then prove that :
(a) $\Sigma\cos 2\alpha = 0 = \Sigma\sin 2\alpha$ (b) $\Sigma\sin(\alpha + \beta) = 0 = \Sigma\cos(\alpha + \beta)$ (c) $\Sigma\sin^2\alpha = \Sigma\cos^2\alpha = 3/2$
(d) $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$ (e) $\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$
(f) $\cos^3(\theta+\alpha)+\cos^3(\theta+\beta)+\cos^3(\theta+\gamma) = 3\cos(\theta+\alpha)\,.\,\cos(\theta+\beta)\,.\,\cos(\theta+\gamma)$ where $\theta \in R$.

Q.22 Resolve $Z^5 + 1$ into linear & quadratic factors with real coefficients. Deduce that : $4\cdot\sin\frac{\pi}{10}\cdot\cos\frac{\pi}{5} = 1$.

Q.23 If $x = 1+ i\sqrt{3}$; $y = 1 - i\sqrt{3}$ & $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

Q.24 If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

Q.25(a) Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by $A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1 - i)z + (1 + i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

(b) For all real numbers x, let the mapping $f(x) = \frac{1}{x - i}$, where $i = \sqrt{-1}$. If there exist real number a, b, c and d for which $f(a)$, $f(b)$, $f(c)$ and $f(d)$ form a square on the complex plane. Find the area of the square.

EXERCISE-2

Q.1 If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p , q , r are the moduli of non–zero complex numbers u, v, w respectively,

prove that, $\arg\frac{w}{v} = \arg\left(\frac{w-u}{v-u}\right)^2$.

Q.2 The equation $x^3 = 9 + 46i$ where $i = \sqrt{-1}$ has a solution of the form $a + bi$ where a and b are integers. Find the value of $(a^3 + b^3)$.

Q.3 Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the co-ordinate axes in the Argand's plane. Further show that $|z| = \sqrt{\frac{-\text{Im}(z)}{2\text{Re}(z)\text{Im}(z)+1}}$

Q.4 If ω is the fifth root of 2 and $x = \omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.

Q.5 Prove that, with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p , p′, q , q′ are all real.
(i) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.
(ii) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$.
State whether these equal roots are real or complex.

Q.6 If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, ... z_7$, find the value of

(a) $\sum_{r=1}^{7}\text{Re}(Z_r)$ and (b) $\sum_{r=1}^{7}\text{Im}(Z_r)$

Q.7 Find the roots of the equation $Z^n = (Z + 1)^n$ and show that the points which represent them are collinear on the complex plane. Hence show that these roots are also the roots of the equation

$$\left(2\sin\frac{m\pi}{n}\right)^2\bar{Z}^2 + \left(2\sin\frac{m\pi}{n}\right)^2\bar{Z} + 1 = 0.$$

Q.8 Dividing f(z) by z – i, we get the remainder i and dividing it by z + i, we get the remainder

$1 + i$. Find the remainder upon the division of $f(z)$ by $z^2 + 1$.

Q.9 Let z_1 & z_2 be any two arbitrary complex numbers then prove that :

$$|z_1 + z_2| \ \geq \ \frac{1}{2}\left(|z_1| + |z_2|\right)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|.$$

Q.10 If Z_r, r = 1, 2, 3, ... 2m, $m \varepsilon N$ are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + \dots\dots\dots\dots + Z + 1 = 0$ then prove that $\sum\limits_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$

Q.11 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_n x^n$ $(n \in N)$, prove that :

(a) $C_0 + C_4 + C_8 + \dots = \frac{1}{2}\left[2^{n-1} + 2^{n/2} \cos\frac{n\pi}{4}\right]$ (b) $C_1 + C_5 + C_9 + \dots = \frac{1}{2}\left[2^{n-1} + 2^{n/2} \sin\frac{n\pi}{4}\right]$

(c) $C_2 + C_6 + C_{10} + \dots = \frac{1}{2}\left[2^{n-1} - 2^{n/2} \cos\frac{n\pi}{4}\right]$ (d) $C_3 + C_7 + C_{11} + \dots = \frac{1}{2}\left[2^{n-1} - 2^{n/2} \sin\frac{n\pi}{4}\right]$

(e) $C_0 + C_3 + C_6 + C_9 + \dots\dots = \frac{1}{3}\left[2^n + 2 \cos\frac{n\pi}{3}\right]$

Q.12 Let z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that :
(i) $2z_2 = (1 + i)\, z_1 + (1 - i)z_3$ & (ii) $2z_4 = (1 - i)\, z_1 + (1 + i)\, z_3$

Q.13 Show that all the roots of the equation $\left(\frac{1 + ix}{1 - ix}\right)^n = \frac{1 + ia}{1 - ia}$ $a \in R$ are real and distinct.

Q.14 Prove that:

(a) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \dots + {}^nC_n \cos (n + 1)\, x = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos\left(\frac{n+2}{2}\right) x$

(b) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \dots + {}^nC_n \sin (n + 1)\, x = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin\left(\frac{n+2}{2}\right) x$

(c) $\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{6\pi}{2n+1}\right) + \dots + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2}$ When $n \in N$.

Q.15 Show that all roots of the equation $a_0z^n + a_1z^{n-1} + \dots\dots + a_{n-1}z + a_n = n$,
where $|a_i| \leq 1$, i = 0, 1, 2, ... , n lie outside the circle with centre at the origin and radius $\frac{n-1}{n}$.

Q.16 The points A, B, C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that
$(z_2 - z_3)^2 = 4\,(z_3 - z_1)\,(z_1 - z_2) \sin^2 \frac{\alpha}{2}$.

Q.17 Show that the equation $\frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + \dots\dots + \frac{A_n^2}{x - a_n} = k$ has no imaginary root, given that:
$a_1, a_2, a_3 \dots a_n$ & $A_1, A_2, A_3 \dots A_n$, k are all real numbers.

Q.18 Let a, b, c be distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$. Find the value of k.

Q.19 Let α, β be fixed complex numbers and z is a variable complex number such that,
$|z - \alpha|^2 + |z - \beta|^2 = k$.
Find out the limits for 'k' such that the locus of z is a circle. Find also the centre and radius of the circle.

Q.20 C is the complex number. $f : C \to R$ is defined by $f(z) = |z^3 - z + 2|$. What is the maximum value of f on the unit circle $|z| = 1$?

Q.21 Let $f(x) = \log_{\cos 3x}(\cos 2ix)$ if $x \neq 0$ and $f(0) = K$ (where $i = \sqrt{-1}$) is continuous at x = 0 then find the value of K. Use of L Hospital's rule or series expansion not allowed.

Q.22 If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with a, b, c > 0 ; $2b^2 > 4ac > b^2$;
$z_1 \in$ third quadrant ; $z_2 \in$ second quadrant in the argand's plane then, show that

$$\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$$

Q.23 Find the set of points on the argand plane for which the real part of the complex number $(1 + i)\, z^2$ is positive where $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$.

Q.24 If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer. Find N.

Q.25 If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ $(a, b, c, d \in R)$ has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$. Find the value of 'b'.

EXERCISE-3

Q.1 Evaluate: $\displaystyle\sum_{p=1}^{32} (3p + 2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$. [REE '97, 6]

Q.2(a) Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the co–efficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and OA = OB, where O is the origin . Prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$. [JEE '97 , 5]

(b) Prove that $\displaystyle\sum_{k=1}^{n-1} (n - k)\, \cos\frac{2k\pi}{n} = -\frac{n}{2}$ where $n \geq 3$ is an integer . [JEE '97, 5]

Q.3(a) If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals

(A) 128ω (B) -128ω (C) $128\omega^2$ (D) $-128\omega^2$

(b) The value of the sum $\displaystyle\sum_{n=1}^{13}\left(i^n + i^{n+1}\right)$, where $i = \sqrt{-1}$, equals

(A) i (B) $i - 1$ (C) $-i$ (D) 0 [JEE' 98, 2 + 2]

Q.4 Find all the roots of the equation $(3z - 1)^4 + (z - 2)^4 = 0$ in the simplified form of $a + ib$. [REE '98, 6]

Q.5(a) If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to :

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

(b) For complex numbers z & ω, prove that, $|z|^2\,\omega - |\omega|^2\, z = z - \omega$ if and only if, $z = \omega$ or $z\bar{\omega} = 1$ [JEE '99, 2 + 10 (out of 200)]

Q.6 If $\alpha = e^{\frac{2\pi i}{7}}$ and $f(x) = A_0 + \displaystyle\sum_{k=1}^{20} A_k x^k$, then find the value of, $f(x) + f(\alpha x) + ... + f(\alpha^6 x)$ independent of α . [REE '99, 6]

Q.7(a) If z_1 , z_2 , z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is :

(A) equal to 1 (B) less than 1 (C) greater than 3 (D) equal to 3

(b) If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$

(A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

[JEE 2000 (Screening) 1 + 1 out of 35]

Q.8 Given , $z = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1}$, 'n' a positive integer, find the equation whose roots are, $\alpha = z + z^3 + ... + z^{2n-1}$ & $\beta = z^2 + z^4 + ... + z^{2n}$.

[REE 2000 (Mains) 3 out of 100]

Q.9(a) The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

(A) of area zero (B) right-angled isosceles

(C) equilateral (D) obtuse – angled isosceles

(b) Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form
(A) 4k + 1 (B) 4k + 2 (C) 4k + 3 (D) 4k
[JEE 2001 (Scr) 1 + 1 out of 35]

Q.10 Find all those roots of the equation $z^{12} - 56z^6 - 512 = 0$ whose imaginary part is positive.
[REE 2000, 3 out of 100]

Q.11(a) Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is
(A) 3ω (B) $3\omega(\omega - 1)$ (C) $3\omega^2$ (D) $3\omega(1 - \omega)$

(b) For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
(A) 0 (B) 2 (C) 7 (D) 17
[JEE 2002 (Scr) 3+3]

(c) Let a complex number α , $\alpha \neq 1$, be a root of the equation
$z^{p+q} - z^p - z^q + 1 = 0$ where p, q are distinct primes.
Show that either $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} = 0$, but not both together.
[JEE 2002, (5)]

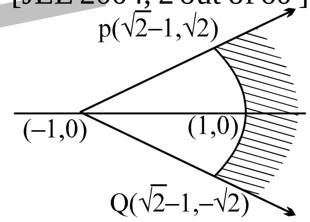
Q.12(a) If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.

(b) Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$.
[JEE-03, 2 + 2 out of 60]

Q.13(a) ω is an imaginary cube root of unity. If $(1 + \omega^2)^m = (1 + \omega^4)^m$, then least positive integral value of m is
(A) 6 (B) 5 (C) 4 (D) 3
[JEE 2004 (Scr)]

(b) Find centre and radius of the circle determined by all complex numbers z = x + i y satisfying $\left|\frac{(z-\alpha)}{(z-\beta)}\right| = k$,
where $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex and $k \neq 1$. [JEE 2004, 2 out of 60]

Q.14(a) The locus of z which lies in shaded region is best represented by



(A) $z : |z + 1| > 2, |\arg(z + 1)| < \pi/4$
(B) $z : |z - 1| > 2, |\arg(z - 1)| < \pi/4$
(C) $z : |z + 1| < 2, |\arg(z + 1)| < \pi/2$
(D) $z : |z - 1| < 2, |\arg(z - 1)| < \pi/2$

(b) If a, b, c are integers not all equal and w is a cube root of unity ($w \neq 1$), then the minimum value of $|a + bw + cw^2|$ is
(A) 0 (B) 1 (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2}$
[JEE 2005 (Scr), 3 + 3]

(c) If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Find the other vertices of square.
[JEE 2005 (Mains), 4]

Q.15 If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\frac{w - \bar{w}z}{1 - z}$ is purely real, then the set of values of z is
(A) $\{z : |z| = 1\}$ (B) $\{z : z = \bar{z}\}$ (C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$
[JEE 2006, 3]

ANSWER KEY
VERY ELEMENTARY EXERCISE

Q.1 **(a)** $\frac{7}{25} + \frac{24}{25}$ i; **(b)** $\frac{21}{5} - \frac{12}{5}$ i; **(c)** 3 + 4i; **(d)** $-\frac{8}{29}$ + 0i; **(e)** $\frac{22}{5}$ i

Q.2 **(a)** x =1, y = 2; **(b)** (2, 9); **(c)** (–2 , 2) or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$; **(d)** (1 ,1) $\left(0, \frac{5}{2}\right)$ **(e)** x = K, y = $\frac{3K}{2}$, K∈R

Q.3 **(a)** $\pm(5+4i)$; **(b)** $\pm(5-6i)$ **(c)** $\pm 5(1+i)$ **Q.4** **(a)** -160 ; **(b)** $-(77+108\,i)$

Q.5 $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$ **Q.6** **(a)** $-i, -2i$ **(b)** $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$

Q.7 **(a)** on a circle of radius $\sqrt{7}$ with centre $(-1, 2)$ **;** **(b)** on a unit circle with centre at origin
(c) on a circle with centre $(-15/4, 0)$ & radius $9/4$ **; (d)** a straight line

Q.8 $a = b = 2-\sqrt{3}$; **Q.9** $x = 1,\ y = -4$ or $x = -1,\ y = -4$

Q.10 **(i)** Modulus = 6 , Arg $= 2k\pi + \frac{5\pi}{18}$ $(K \in I)$, Principal Arg $= \frac{5\pi}{18}$ $(K \in I)$

(ii) Modulus = 2 , Arg $= 2k\pi + \frac{7\pi}{6}$, Principal Arg $= -\frac{5\pi}{6}$

(iii) Modulus $= \frac{\sqrt{5}}{6}$, Arg $= 2k\pi - \tan^{-1}2$ $(K \in I)$, Principal Arg $= -\tan^{-1}2$

Q.16 **(a)** $\frac{\sqrt{3}}{2}-\frac{i}{2}, -\frac{\sqrt{3}}{2}-\frac{i}{2}, i$; **Q.17** $\frac{x^2}{64}+\frac{y^2}{48}=1$; **Q.18** **(c)** 64 ; **Q.21** A

Q.22 **(a)** (1, 1) ; **(b)** $\left[\frac{n(n+1)}{2}\right]^2 - n$

EXERCISE-1

Q.1 (a) $\frac{21}{5}-\frac{12}{5}i$ **(b)** $3+4i$ **(c)** $-\frac{8}{29}+0i$ **(d)** $\frac{22}{5}i$ **(e)** $\pm\sqrt{2}+0i$ or $0\pm\sqrt{2}i$

Q.2 **(i)** Principal Arg $z = -\frac{4\pi}{9}$; $|z| = 2\cos\frac{4\pi}{9}$; Arg $z = 2k\pi - \frac{4\pi}{9}$ $k \in I$

(ii) Modulus = $\sec^2 1$, Arg $= 2n\pi + (2-\pi)$, Principal Arg $= (2-\pi)$

(iii) Principal value of Agr $z = -\frac{\pi}{2}$ & $|z| = \frac{3}{2}$; Principal value of Arg $z = \frac{\pi}{2}$ & $|z| = \frac{2}{3}$

(iv) Modulus $= \frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}$, Arg $z = 2n\pi + \frac{11\pi}{20}$, Principal Arg $= \frac{11\pi}{20}$

Q.3(a) $x = 1, y = 2$; **(b)** $x = 1$ & $y = 2$; **(c)** $(-2, 2)$ or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$; **(d)** $(1, 1)$ $\left(0, \frac{5}{2}\right)$; **(e)** $x = K$, $y = \frac{3K}{2}$ $K \in R$

Q.4 **(a)** 2, **(b)** $-11/2$ **Q.5** **(a)** $[(-2, 2) ; (-2, -2)]$ **(b)** $-(77+108\,i)$

Q.6 **(a)** $z = (2+i)$ or $(1-3i)$; **(b)** $z = \frac{3+4i}{4}$

Q.7 **(b)** 2

Q.9 **(ii)** $z = -(b+i)$; $-2i, -a$ **(iii)** $\left(-\frac{2ti}{3t+5}, ti\right)$ where $t \in R - \left\{-\frac{5}{3}\right\}$

Q.10 **(a)** The region between the co encentric circles with centre at $(0, 2)$ & radii 1 & 3 units

(b) region outside or on the circle with centre $\frac{1}{2}+2i$ and radius $\frac{1}{2}$.

(c) semi circle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$ **(d)** a ray emanating from the point $(3+4i)$ directed away from the origin & having equation $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$

Q.15 $[-3, -2]$ **Q.17** $(1-c^2)\,|z|^2 - 2(a+bc)\,(\text{Re } z) + a^2 - b^2 = 0$

Q.18 (a) K = 3 **, (b)** -4 **Q.19** **(b)** one if n is even ; $-w^2$ if n is odd

Q.22 $(Z+1)\,(Z^2 - 2Z\cos 36° + 1)\,(Z^2 - 2Z\cos 108° + 1)$ **Q.24** 4

Q.25 **(a)** $\pi - 2$; **(b)** 1/2

EXERCISE-2

Q.2 35 **Q.6** (a) $-\frac{7}{2}$, (b) zero **Q.8** $\frac{iz}{2}+\frac{1}{2}+i$ **Q.18** $-\omega$ or $-\omega^2$

Q.19 $k > \frac{1}{2}|\alpha-\beta|^2$ **Q.20** $|f(z)|$ is maximum when $z = \omega$, where ω is the cube root unity and $|f(z)| = \sqrt{13}$

Q.21 $K = -\frac{4}{9}$

Q.23 required set is constituted by the angles without their boundaries, whose sides are the straight lines $y = (\sqrt{2}-1)\ x$ and $y + (\sqrt{2}+1)\ x = 0$ containing the x – axis

Q.24 198 **Q.25** 51

EXERCISE-3

Q.1 $48(1 - i)$ **Q.3 (a)** D **(b)** B

Q.4 **Z =** $\dfrac{(29+20\sqrt{2})+i(\pm 15+25\sqrt{2})}{82}$, $\dfrac{(29-20\sqrt{2})+i(\pm 15-25\sqrt{2})}{82}$ **Q.5 (a)** C

Q.6 $7A_0 + 7A_7x^7 + 7A_{14}x^{14}$ **Q.7 (a)** A **(b)** A **Q.8** $z^2 + z + \dfrac{\sin^2 n\theta}{\sin^2 \theta} = 0$, where $\theta = \dfrac{2\pi}{2n+1}$

Q.9 (a) C, **(b)** D **Q.10** $\pm 1 + i\sqrt{3}$, $\dfrac{(\pm\sqrt{3}+i)}{\sqrt{2}}$, $\sqrt{2}\,i$ **Q.11 (a)** B ; **(b)** B

Q.13 (a) D ; **(b)** Centre $\equiv \dfrac{k^2\beta-\alpha}{k^2-1}$, Radius $= \dfrac{1}{(k^2-1)}\sqrt{|\alpha-k^2\beta|^2-(k^2.|\beta|^2-|\alpha|^2)(k^2-1)}$

Q.14 (a) A, **(b)** B, **(c)** $z_2 = -\sqrt{3}\,i$; $z_3 = (1-\sqrt{3})+i$; $z_4 = (1+\sqrt{3})-i$ **Q.15** D

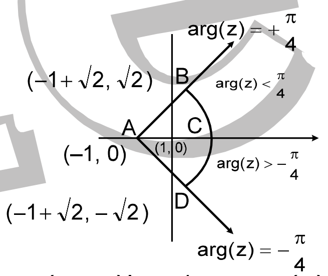
EXERCISE-4

Part : (A) Only one correct option

1. If |z| = 1 and $\omega = \dfrac{z-1}{z+1}$ (where $z \neq -1$), the Re(ω) is **[IIT – 2003, 3]**

(A) 0 (B) $-\dfrac{1}{|z+1|^2}$ (C) $\left|\dfrac{z}{z+1}\right|.\dfrac{1}{|z+1|^2}$ (D) $\dfrac{\sqrt{2}}{|z+1|^2}$

2. The locus of z which lies in shaded region (excluding the boundaries) is best represented by



[IIT – 2005, 3]

(A) z : |z + 1| > 2 and |arg (z + 1)| < π/4 (B) z : |z – 1| > 2 and |arg (z – 1)| < π/4
(C) z : |z + 1| < 2 and |arg (z + 1)| < π/2 (D) z : |z – 1| < 2 and |arg (z + 1)| < π/2

3. If w = α, + iβ, where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\left(\dfrac{w-\overline{w}z}{1-z}\right)$ is purely real, then the set of values of z is **[IIT – 2006, (3, –1)]**

(A) {z : |z| = 1} (B) {z : z = $\overline{z}$} (C) {z : z $\neq$ 1} (D) {z : |z| = 1, z $\neq$1}

4. If $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$, then b is equal to

(A) $\sqrt{3}$ (B) $\sqrt{2}$ (C) 1 (D) none of these

5. If Re$\left(\dfrac{z-8i}{z+6}\right)$ = 0, then z lies on the curve

(A) $x^2 + y^2 + 6x - 8y = 0$ (B) $4x - 3y + 24 = 0$ (C) 4ab (D) none of these

6. If n_1, n_2 are positive integers then : $(1+i)^{n_1} + (1+i^3)^{n_1} + (1-i^5)^{n_2} + (1-i^7)^{n_2}$ is a real number if and only if

(A) $n_1 = n_2 + 1$ (B) $n_1 + 1 = n_2$
(C) $n_1 = n_2$ (D) n_1, n_2 are any two positive integers

7. The three vertices of a triangle are represented by the complex numbers, 0, z_1 and z_2. If the triangle is equilateral, then

(A) $z_1^2 - z_2^2 = z_1z_2$ (B) $z_2^2 - z_1^2 = z_1 z_2$ (C) $z_1^2 + z_2^2 = z_1z_2$ (D) $z_1^2 + z_2^2 + z_1z_2 = 0$

8. If $x^2 - x + 1 = 0$ then the value of $\displaystyle\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$ is

(A) 8 (B) 10 (C) 12 (D) none of these

9. If α is nonreal and $\alpha = \sqrt[5]{1}$ then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to
(A) 4 (B) 2 (C) 1 (D) none of these

10. If z = x + iy and $z^{1/3}$ = a – ib then $\frac{x}{a} - \frac{y}{b} = k\left(a^2 - b^2\right)$ where k =
(A) 1 (B) 2 (C) 3 (D) 4

11. $\left[\frac{-1+i\sqrt{3}}{2}\right]^6 + \left[\frac{-1-i\sqrt{3}}{2}\right]^6 + \left[\frac{-1+i\sqrt{3}}{2}\right]^5 + \left[\frac{-1-i\sqrt{3}}{2}\right]^5$ is equal to :
(A) 1 (B) – 1 (C) 2 (D) none

12. Expressed in the form r (cos θ + i sin θ), – 2 + 2i becomes :
(A) $2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ (B) $2\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$
(C) $2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$ (D) $\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

13. The number of solutions of the equation in z, $z\bar{z}$ - (3 + i) z - (3 - i) $\bar{z}$ - 6 = 0 is :
(A) 0 (B) 1 (C) 2 (D) infinite

14. If |z| = max {|z – 1|, |z + 1|} then
(A) $|z + \bar{z}| = \frac{1}{2}$ (B) $z + \bar{z} = 1$ (C) $|z + \bar{z}| = 1$ (D) none of these

15. If P, P′ represent the complex number z_1 and its additive inverse respectively then the complex equation of the circle with PP′ as a diameter is
(A) $\frac{z}{z_1} = \left(\frac{z_1}{z}\right)$ (B) $z\bar{z} + z_1\bar{z}_1 = 0$ (C) $z\bar{z}_1 + \bar{z}z_1 = 0$ (D) none of these

16. The points $z_1 = 3 + \sqrt{3}\,i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is :
(A) $z = \frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$ (B) z = 5 + 5 i
(C) z = – 1 – i (D) none

17. The expression $\left[\frac{1+i\tan\alpha}{1-i\tan\alpha}\right]^n - \frac{1+i\tan n\alpha}{1-i\tan n\alpha}$ when simplified reduces to :
(A) zero (B) 2 sin n α (C) 2 cos n α (D) none

18. All roots of the equation, $(1 + z)^6 + z^6 = 0$:
(A) lie on a unit circle with centre at the origin (B) lie on a unit circle with centre at (– 1, 0)
(C) lie on the vertices of a regular polygon with centre at the origin (D) are collinear

19. Points z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to z_2 ($z_3 \neq z_1$) is represented by :
(A) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$ (B) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 - z_2)$
(C) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_2 - z_1)$ (D) none of these

20. If z = x + i y then the equation of a straight line Ax + By + C = 0 where A, B, C $\in$ R, can be written on the complex plane in the form $\bar{a}z + a\bar{z} + 2C = 0$ where 'a' is equal to :
(A) $\frac{(A+iB)}{2}$ (B) $\frac{A-iB}{2}$ (C) A + i B (D) none

21. The points of intersection of the two curves $|z - 3| = 2$ and $|z| = 2$ in an argand plane are:
(A) $\frac{1}{2}\left(7 \pm i\sqrt{3}\right)$ (B) $\frac{1}{2}\left(3 \pm i\sqrt{7}\right)$ (C) $\frac{3}{2} \pm i\sqrt{\frac{7}{2}}$ (D) $\frac{7}{2} \pm i\sqrt{\frac{3}{2}}$

22. The equation of the radical axis of the two circles represented by the equations, $|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is :
(A) $3iz - 3i\bar{z} - 2 = 0$ (B) $3iz - 3i\bar{z} + 2 = 0$ (C) $iz - i\bar{z} + 1 = 0$ (D) $2iz - 2i\bar{z} + 3 = 0$

23. If $\prod_{p=1}^{r} e^{ip\theta} = 1$ where Π denotes the continued product, then the most general value of θ is :
(A) $\frac{2n\pi}{r(r-1)}$ (B) $\frac{2n\pi}{r(r+1)}$ (C) $\frac{4n\pi}{r(r-1)}$ (D) $\frac{4n\pi}{r(r+1)}$

24. The set of values of a $\in$ R for which $x^2 + i(a - 1)x + 5 = 0$ will have a pair of conjugate imaginary roots is
(A) R (B) {1} (C) {a| $a^2 - 2a + 21 > 0$} (D) none of these

25. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$ then $|z_1 + z_2 + z_3|$
(A) is less than 6 (B) is more than 3
(C) is less than 12 (D) lies between 6 and 12

26. If $z_1, z_2, z_3, \ldots\ldots, z_n$ lie on the circle $|z| = 2$, then the value of

$$E = |z_1 + z_2 + \ldots.. + z_n| - 4\left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right| \text{ is}$$

(A) 0 (B) n (C) –n (D) none of these

Part : (B) May have more than one options correct

27. If z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$, then
(A) $3 \le |z_1 - 2z_2| \le 5$ (B) $1 \le |z_1 + z_2| \le 3$
(C) $|z_1 - 3z_2| \ge 5$ (D) $|z_1 - z_2| \ge 1$

28. If z_1, z_2, z_3, z_4 are root of the equation $a_0z^4 + z_1z^3 + z_2z^2 + z_3z + z_4 = 0$, where a_0, a_1, a_2, a_3 and a_4 are real, then
(A) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of the equation (B) z_1 is equal to at least one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$
(C) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of the equation (D) none of these

29. If $a^3 + b^3 + 6\,abc = 8\,c^3$ & ω is a cube root of unity then :
(A) a, c, b are in A.P. (B) a, c, b are in H.P.
(C) $a + b\omega - 2\,c\omega^2 = 0$ (D) $a + b\omega^2 - 2\,c\omega = 0$

30. The points z_1, z_2, z_3 on the complex plane are the vertices of an equilateral triangle if and only if :
(A) $\Sigma\,(z_1 - z_2)\,(z_2 - z_3) = 0$ (B) $z_1^2 + z_2^2 + z_3^2 = 2\,(z_1z_2 + z_2z_3 + z_3z_1)$
(C) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ (D) $2\,(z_1^2 + z_2^2 + z_3^2) = z_1z_2 + z_2z_3 + z_3z_1$

31. If $|z_1 + z_2| = |z_1 - z_2|$ then
(A) $|\text{amp } z_1 - \text{amp } z_2| = \dfrac{\pi}{2}$ (B) $|\text{ amp } z_1 - \text{amp}_2| = \pi$
(C) $\dfrac{z_1}{z_2}$ is purely real (D) $\dfrac{z_1}{z_2}$ is purely imaginary

EXERCISE-5

1. Given that $x, y \in R$, solve : $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

2. If α & β are any two complex numbers, prove that :

$$\left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|.$$

3. If α, β are the numbers between 0 and 1, such that the points $z_1 = \alpha + i$, $z_2 = 1 + \beta i$ and $z_3 = 0$ form an equilateral triangle, then find α and β.

4. ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represent the complex numbers 1 + i and 2 - i respectively, then find the complex number corresponding to A.

5. Show that the sum of the p^{th} powers of n^{th} roots of unity :
(a) is zero, when p is not a multiple of n. (b) is equal to n, when p is a multiple of n.

6. If $(1 + x)^n = p_0 + p_1x + p_2x^2 + p_3x^3 + \ldots\ldots.$, then prove that :
(a) $p_0 - p_2 + p_4 - \ldots\ldots. = 2^{n/2}\cos\dfrac{n\pi}{4}$ (b) $p_1 - p_3 + p_5 - \ldots\ldots. = 2^{n/2}\sin\dfrac{n\pi}{4}$

7. Prove that, $\log_e\left(\dfrac{1}{1 - e^{i\theta}}\right) = \log_e\left(\dfrac{1}{2}\,\text{cosec}\,\dfrac{\theta}{2}\right) + i\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

8. If $i^{i^{\ldots\ldots\infty}} = A + i\,B$, principal values only being considered, prove that
(a) $\tan\dfrac{1}{2}\pi A = \dfrac{B}{A}$ (b) $A^2 + B^2 = e^{-\pi B}$

9. Prove that the roots of the equation, $(x - 1)^n = x^n$ are $\dfrac{1}{2}\left(1 + i\cot\dfrac{r\pi}{n}\right)$, where $r = 0, 1, 2, \ldots\ldots. (n - 1)$ & $n \in N$.

10. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2$ then prove that :
(a) $\Sigma\cos 2\alpha = 0 = \Sigma\sin 2\alpha$ (b) $\Sigma\sin(\alpha + \beta) = 0 = \Sigma\cos(\alpha + \beta)$
(c) $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$ (d) $\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$
(e) $\Sigma\sin^2\alpha = \Sigma\cos^2\alpha = 3/2$
(f) $\cos^3(\theta + \alpha) + \cos^3(\theta + \beta) + \cos^3(\theta + \gamma) = 3\cos(\theta + \alpha).\cos(\theta + \beta).\cos(\theta + \gamma)$ where $\theta \in R$.

11. If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then find the value of $\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1}$.

12. Given that, $|z - 1| = 1$, where ' z ' is a point on the argand plane. Show that $\frac{z-2}{z} = i \tan(\arg z)$.

13. P is a point on the Argand diagram. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin & P, Q & R are represented by the complex numbers Z_1, Z_2 & Z_3 respectively, show that : $Z_2^2 \cos 2\theta = Z_1 Z_3 \cos^2\theta$.

14. Find an expression for $\tan 7\theta$ in terms of $\tan\theta$, using complex numbers. By considering $\tan 7\theta = 0$, show that $x = \tan^2(3\pi/7)$ satisfies the cubic equation $x^3 - 21x^2 + 35x - 7 = 0$.

15. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ $(n \in N)$, prove that : $C_2 + C_6 + C_{10} + ... = \frac{1}{2}\left[2^{n-1} - 2^{n/2}\cos\frac{n\pi}{4}\right]$

16. Prove that : $\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{6\pi}{2n+1}\right) + ... + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2}$ When $n \in N$.

17. Show that all the roots of the equation $a_1z^3 + a_2z^2 + a_3z + a_4 = 3$, where $|a_i| \le 1$, i = 1, 2, 3, 4 lie outside the circle with centre origin and radius 2/3.

18. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n} = -\frac{n}{2}$, where $n \ge 3$ is an integer

19. Show that the equation $\frac{A_1^2}{x - a_1} + \frac{A_2^2}{x - a_2} + ... + \frac{A_n^2}{x - a_n} = k$ has no imaginary root, given that : $a_1, a_2, a_3 .... a_n$ & $A_1, A_2, A_3 .... A_n$, k are all real numbers.

20. Let z_1, z_2, z_3 be three distinct complex numbers satisfying, ½z_1-1½ = ½z_2-1½ = ½z_3-1½. Let A, B & C be the points represented in the Argand plane corresponding to z_1, z_2 and z_3 resp. Prove that $z_1 + z_2 + z_3 = 3$ if and only if D ABC is an equilateral triangle.

21. Let α, β be fixed complex numbers and z is a variable complex number such that,
$|z - \alpha|^2 + |z - \beta|^2 = k$.
Find out the limits for 'k' such that the locus of z is a circle. Find also the centre and radius of the circle.

22. If 1, $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ are the n, n^{th} roots of unity, then prove that
$(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) .... (1 - \alpha_{n-1}) = n$.
Hence prove that $\sin\frac{\pi}{n} . \sin\frac{2\pi}{n} . \sin\frac{3\pi}{n} .... \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

23. Find the real values of the parameter 'a' for which at least one complex number
$z = x + iy$ satisfies both the equality $|z - ai| = a + 4$ and the inequality $|z - 2| < 1$.

24. Prove that, with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$; where p, p′, q, q′ are all real.
(a) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.
(b) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$.
State whether these equal roots are real or complex.

25. The points A, B, C depict the complex numbers z_1, z_2, z_3 respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that

$(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$.

26. If z_1, z_2 & z_3 are the affixes of three points A, B & C respectively and satisfy the condition $|z_1 - z_2| = |z_1| + |z_2|$ and $|(2 - i)z_1 + iz_3| = |z_1| + |(1 - i)z_1 + iz_3|$ then prove that Δ ABC in a right angled.

27. If 1, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $x^5 - 1 = 0$, then prove that

$$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} = \omega.$$

28. If one the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Find the other vertices of the square. **[IIT – 2005, 4]**

EXERCISE-4

1.	A	**2.**	C	**3.**	D	**4.**	A
5.	A	**6.**	D	**7.**	C	**8.**	A
9.	A	**11.**	D	**12.**	A	**13.**	B
14.	D	**15.**	D	**16.**	A	**17.**	B
18.	A	**19.**	D	**20.**	C	**21.**	C
22.	B	**23.**	B	**24.**	D	**25.**	B
26.	C	**27.**	A	**28.**	ABCD	**29.**	AB
30.	ACD	**31.**	AC	**10.**	AD		

EXERCISE-5

1. $x = K,\ y = \frac{3K}{2}\ \ K \in R$

3. $2-\sqrt{3}, 2-\sqrt{3}$

4. $3 - \frac{i}{2}$ or $1 - \frac{3}{2}i$

11. $3\,\omega^2$

21. $k > \frac{1}{2}\left|\alpha - \beta\right|^2$

23. $\left(-\frac{21}{10}, -\frac{5}{6}\right)$

28. $-i\sqrt{3}$, $1 - \sqrt{3} + i$, $1 + \sqrt{3} - i$

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we distributed a book in class room