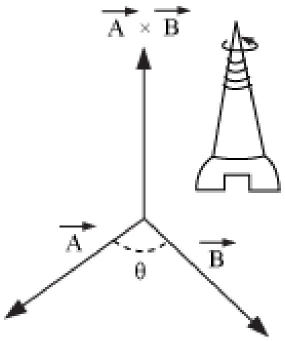
System of particles and rotational motion

- The magnitude of the vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitude of the vectors $\vec{A}$ and $\vec{B}$ and sine of the smaller angle between them.



$$\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$$

- The cross product of the two vectors is at right angles to both the vectors and points in the direction in which a right-handed screw will advance.
- Properties of vector product:
  - The cross product of a vector with itself is a null vector.
  - The cross product of two vectors does not obey commutative law. That is,

    $$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

  - The cross product of vectors obeys the distributive law. That is,

    $$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

- If the vectors $\vec{A}$and $\vec{B}$ represent the two adjacent sides of a parallelogram, the magnitude of cross product of $\vec{A}$ and $\vec{B}$ will represent the area of the parallelogram.

Types of motion

- **Translational Motion**
  - Motion of a rigid body along a staright line path.
  - All the particles of the body move together i.e, they have the same velocity at any instant of time.

- **Rotational Motion**
  - Motion of rigid body about a fix point.
  - Every particle of the body moves in concentric circles about the fix point.

- **Combination of translational and rotational motion**
  - Motion of body in which body rotates while moving.
  - Motion of a tyer on the road.

- Centre of mass of a body is a point at which the whole mass of the body is supposes to be concentrated.
- Position of centre of mass of discreet distribution of mass

$$X = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum m_ix_i}{\sum m_i}$$
$$Y = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum m_iy_i}{\sum m_i}$$

- Position of centre of mass of continuous distribution of mass

$$\vec{R} = \frac{1}{M}\int \vec{r}\, dm$$

For a system of n particles of masses $m_1, m_2, m_3, \ldots, m_n$,

- Eqaution of motion of the system is given by,

$$M\vec{A} = \vec{F}_{\text{ext}} \quad (\text{ when, } \vec{F}_{\text{int}} = 0\text{ })$$

- Centre of mass of a system of particles moves as if all the mass of the system is concentrated at the centre of the mass and all the external forces acting on the system are applied directly at this point.
- Total linear momentum of the syastem remains constant, when no external force acts on the system.

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\left(m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n\right)$$
$$\text{If, } \vec{F}_{ext} = 0$$
$$\frac{d}{dt}\left(m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n\right) = 0$$
$$m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n = \text{constant}$$

- Centre of mass of an isolated system moves with constant velocity.

- **Angular Velocity** – It is defined as the ratio of angular displacement to the time taken by the object to undergo the displacement. It is denoted by ω_{av}.

$$\omega_{av} = \frac{\Delta\theta}{\Delta t}$$

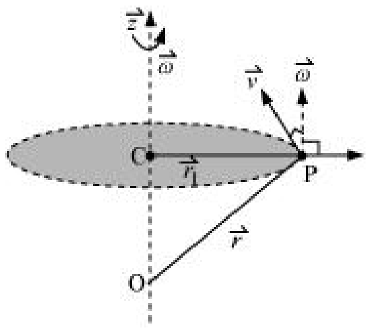
- **Instantaneous angular velocity** – It is defined as the limiting value of the average angular velocity of the object in a small time interval as the time interval approaches zero. It is denoted by ω.

$$\omega = \underset{\Delta t \to 0}{Lt} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- **Relation between Linear and Angular Velocities**

$v_i = \omega r_i$ (v_i= Linear velocity, w= Angular velocity)

v_i acts tangentially and $w$acts perpendicularly.



$$\vec{v} = \vec{\omega} \times \vec{r} \quad \left(\left|\vec{v}\right| = \omega r_{\perp}\right)$$

- **Angular acceleration**: It is defined as the ratio of change in angular velocity of the object to the time taken to undergo the change in angular velocity.

$$\alpha = \frac{d\omega}{dt}$$

- **Moment of force** $torque$

$$\bar{\tau} = \bar{r} \times \bar{F}$$
$$|\bar{\tau}| = rF\sin\theta = rF_{\perp} = r_{\perp}F \quad (\text{unit} \rightarrow \text{kg m}^2\ \text{s}^{-2})$$

- **Angular momentum**

$$\bar{l} = \bar{r} \times \bar{p} \quad (\bar{p} = \text{linear momentum})$$

$$\frac{d\bar{l}}{dt} = \bar{\tau}_{\text{ext}}$$

  - For a rigid body, $\bar{L} = \sum \bar{l}_i = \sum \bar{r}_i \times \bar{p}_i$ and $\frac{\mathrm{d}\bar{L}}{\mathrm{d}t} = \sum \bar{\tau}_i$

$$\bar{\tau} = \bar{\tau}_{\text{ext}} + \bar{\tau}_{\text{int}}$$

  - Total angular momentum is conserved if total external torque is zero on a system.

- Mechanical equilibrium
  - In mechanical equilibrium, total external force is zero and total external torque is zero.

- Translational equilibrium
  - When a body is in translational equilibrium, it will be either at rest ($v = 0$) or in uniform motion.
  - The body will have zero linear acceleration.

- In equilibrium, potential energy of the body is constant $maximumorminimum$.
- Rotational equilibrium
  - A body is in rotational equilibrium, when algebraic sum of moments of all the forces acting on the body about a fixed point is zero.
  - Angular acceleration of the body in rotational equilibrium will be zero.
- At centre of gravity, total gravitational torque is zero.

Moment of inertia

- Moment of inertia of a body about a given axis is the sum of the products of masses of all the particles of the body and squares of their respective perpendicular distance from the axis of rotation.

$$I = \sum_{i=1}^{n} m_i r_1^2$$

- K.E. of rotation of body $= \frac{1}{2} I\omega^2$
- Mass (m) of the body is an analogue of moment of inertia (I) of the body in rotational motion.

Radius of gyration

- Radius of gyration of a body about a given axis is the perpendicular distance of a point P from the axis, where if whole mass of the body were concentrated, then the body shall have the same moment of inertia as it has with the actual distribution of mass. This distance is represented by K.
- The radius of gyration of a body about an axis is equal to the root mean square distance of the various particles constituting the body from the axis of rotation.

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_n^2}{n}}$$

Theorem of Perpendicular Axes

- The moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

$$I_Z = I_X + I_Y$$

Theorem of Parallel Axes

- The moment of inertia of a body about any axis is equal to the sum of the moments of inertia of the body about the parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

$$I_Z = I_Z + Ma^2$$

- Moment of inertia of a ring of mass M and radius R about its diameter is given by $I = \frac{1}{2}MR^2$.
- Moment of inertia of a ring of mass M and radius R about a tangent in its plane is given by $I = \frac{3}{2}MR^2$.
- Moment of inertia of a ring of mass M and radius R about a tangent perpendicular to its plane is given by $I = 2\ MR^2$.
- Moment of inertia of a disc of mass M and radius R about an axis passing through its centre and perpendicular to its plane is given by $I = \frac{1}{2}MR^2$.

- Moment of inertia of a solid cylinder of mass *M*, height *l* and radius *R* about its geometrical axis is given by $I = \frac{1}{2} MR^2$.
- Moment of inertia of a solid cylinder of mass *M*, height *l* and radius *R* about a transverse $perpendicular axis$ passing through its centre is given by $I = \frac{MR^2}{4} + \frac{Ml^2}{12}$.

- **Kinematics equations of motion**

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$
$$\theta_0 = \text{initial angular displacement}$$
$$\omega_0 = \text{initial angular velocity}$$

Dynamics of rotational motion

- Work done, $dW = \bar{\tau} . d\bar{\theta}$
- Power, $P = \tau\omega$
- Angular acceleration (α) α Torque (τ)

$$\text{Angular acceleration}(\alpha) \propto \frac{1}{I}$$

Therefore, $\tau = I\alpha$

- Angular momentum of a rigid body

$$\overline{L_z} = I\omega\,\hat{k}$$

- Angular speed increases when distance from the rotational axis decreases.
- For any particle, the angular momentum vector and the angular velocity vector are not necessarily parallel.

- If moment of inertia *I* does not change with time, then

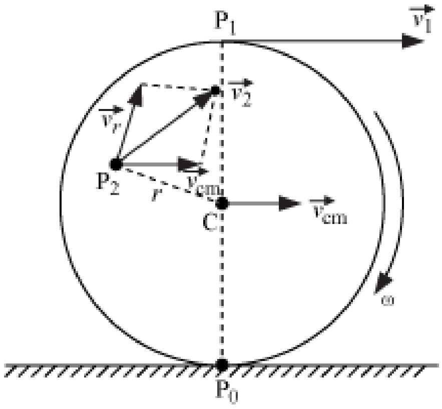
$$\boxed{I\frac{d\omega}{dt} = \tau}$$
$$\boxed{I\alpha = \tau}$$

- **Principle of Conservation of Angular Momentum**

According to this law, the angular momentum of a system remains constant if the net external torque on it is zero.

- **Applications of Conservation of Angular Momentum**
  - Increase in the angular velocity of a planet around the Sun as it comes near to it
  - A diver jumping from a springboard and performing somersaults in air
  - Change in the angular velocity of a person $sitting on a rotating chair$ on folding of arms

- Rolling motion is a combination of translational motion and rotational motion.



- For a disc to roll without slipping, the essential condition is $\boxed{\vec{v}_{cm} = R\omega}$.

- Kinetic energy or rolling motion:

$$K = \frac{1}{2} I\omega^2 + \frac{1}{2} m v_{cm}^2$$

- If a body is rolling without slipping on an inclined plane, then its velocity is given by

$$v = \sqrt{\frac{2gh}{1+\frac{k^2}{r^2}}}$$

- If a body is rolling without slipping on an inclined plane, then its acceleration is given by

$$a = \frac{g\sin\theta}{1+\frac{k^2}{r^2}}$$

The given table shows the values of *v* and *a* for rigid bodies of different shapes.

	Body	*k*	v	*a*
1	Ring or hollow cylinder	*r*	$\sqrt{gh}$	$\frac{1}{2} g \sin\theta$
2	Disc or solid cylinder	$\frac{r}{\sqrt{2}}$	$\sqrt{\frac{4}{3} gh}$	$\frac{2}{3} g \sin\theta$

3	Solid sphere	$\sqrt{\frac{2}{5}}r$	$\sqrt{\frac{10}{7}gh}$	$\frac{5}{7}g\sin\theta$