fo/u fopkjr Hkh# tu] ugh vkjEHk dke] foifr n[k NkM rjr e/;e eu dj ';keA
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jfprh ekuo /ke i.krk
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STUDY PACKAGE

Subject : Mathematics

Topic : FUNCTIONS

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Functions

A. Definition :

Function is a special case of relation, from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write f: A $\rightarrow$ B. We read it as "f is a function from A to B".

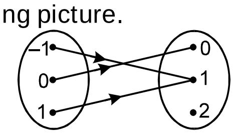
Set 'A' is called **domain** of f and set 'B' is called **co-domain** of f.

For example, let A $\equiv$ {–1, 0, 1} and B $\equiv$ {0, 1, 2}. Then A × B $\equiv$ {(–1, 0), (–1, 1), (–1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)}

Now, " f : A $\rightarrow$ B defined by f(x) = x^2 " is the function such that

f $\equiv$ {(–1, 1), (0, 0), (1, 1)}

f can also be show diagramatically by following picture.



Every function say f : A $\rightarrow$ B satisfies the following conditions:

(a) f $\subseteq$ A x B, (b) $\forall$ a $\in$ A $\Rightarrow$ (a, f(a)) $\in$ f and (c) (a, b) $\in$ f & (a, c) $\in$ f $\Rightarrow$ b = c

Illustration # 1: (i) Which of the following correspondences can be called a function ?

(A) f(x) = x^3 ; {–1, 0, 1} $\rightarrow$ {0, 1, 2, 3}

(B) f(x) = $\pm\sqrt{x}$; {0, 1, 4} $\rightarrow$ {–2, –1, 0, 1, 2}

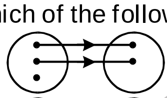
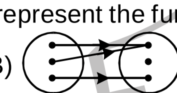
(C) f(x) = $\sqrt{x}$; {0, 1, 4} $\rightarrow$ {–2, –1, 0, 1, 2}

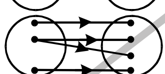
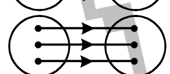
(D) f(x) = $-\sqrt{x}$; {0, 1, 4} $\rightarrow$ {–2, –1, 0, 1, 2}

Solution: f(x) in (C) & (D) are functions as definition of function is satisfied. while in case of (A) the given relation is not a function, as f(–1) $\notin$ codomain. Hence definition of function is not satisfied.

While in case of (B), the given relation is not a function, as f(1) = ± 1 and f(4) = ± 2 i.e. element 1 as well as 4 in domain is related with two elements of codomain. Hence definition of function is not satisfied.

(ii) Which of the following pictorial diagrams represent the function

(A)  (B) 

(C)  (D) 

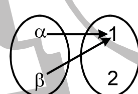
Solution: B & D. In (A) one element of domain has no image, while in (C) one element of domain has two images in codomain

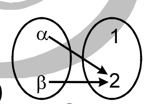
Assignment: 1. Let g(x) be a function defined on [–1, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) & **(**x,g(x)**)** is $\sqrt{3}/4$ sq. units, then the function g(x) may be.

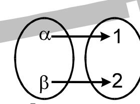
(A) g(x)= $\pm\sqrt{(1-x^2)}$ (B*) g(x) = $\sqrt{(1-x^2)}$ (C*) g(x) = $-\sqrt{(1-x^2)}$ (D) g(x) = $\sqrt{(1+x^2)}$

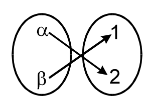
2. Represent all possible functions defined from {α, β} to {1, 2}

Answer (1) B

(2) (i) 

(ii) 

(iii) 

(iv) 

B. Domain, Co-domain & Range of a Function :

Let f: A $\rightarrow$ B, then the set A is known as the domain of f & the set B is known as co–domain of f. If a member 'a' of A is associated to the member 'b' of B, then 'b' is called **the f-image** of 'a' and we write b = f (a). Further 'a' is called **a pre-image** of 'b'. The set {f(a): $\forall$ a $\in$A} is called **the range** of f and is denoted by f(A). Clearly f(A) $\subseteq$ B.

Sometimes if only definition of f (x) is given (domain and codomain are not mentioned), then domain is set of those values of ' x' for which f (x) is defined, while codomain is considered to be (– ∞, ∞)

A function whose domain and range both are sets of real numbers is called **a real function**. Conventionally the word **"FUNCTION"** is used only as the meaning of real function.

Illustration # 2 : Find the domain of following functions :

(i) f(x) = $\sqrt{x^2-5}$ (ii) $\sin^{-1}(2x-1)$

Solution :(i) f(x) = $\sqrt{x^2-5}$ is real iff $x^2 - 5 \geq 0$ $\Rightarrow$ $|x| \geq \sqrt{5}$ $\Rightarrow$ $x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}\,] \cup [\,\sqrt{5}, \infty)$

(ii) $-1 \leq 2x - 1 \leq +1$ $\therefore$ domain is x $\in$ [0, 1]

Algebraic Operations on Functions :

If f & g are real valued functions of x with domain set A and B respectively, then both f & g are defined in A $\cap$ B. Now we define f + g, f – g, (f · g) & (f /g) as follows:

(i) (f ± g) (x) = f(x) ± g(x)

(ii) (f.g) (x) = f(x). g(x)

— domain in each case is A $\cap$ B

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is {x | x $\in$ A $\cap$ B such that g(x) $\neq$ 0}.

Note :

☞ For domain of $\phi(x) = \{f(x)\}^{g(x)}$, conventionally, the conditions are f(x) > 0 and g(x) must be defined.

☞ For domain of $\phi(x) = {}^{f(x)}C_{g(x)}$ or $\phi(x) = {}^{f(x)}P_{g(x)}$ conditions of domain are f(x) $\geq$ g(x) and f(x) $\in$ N and g(x) $\in$ W

Illustration # 3: Find the domain of following functions :

(i) $f(x) = \sqrt{\sin x} - \sqrt{16 - x^2}$ (ii) $f(x) = \dfrac{3}{\sqrt{4 - x^2}}\log(x^3 - x)$ (iii) $f(x) = x^{\cos^{-1}x}$

Solution: (i) $\sqrt{\sin x}$ is real iff $\sin x \ge 0 \Leftrightarrow x \in [2n\pi, 2n\pi + \pi]$, $n \in I$.

$\sqrt{16 - x^2}$ is real iff $16 - x^2 \ge 0 \Leftrightarrow -4 \le x \le 4$.

Thus the domain of the given function is $\{x : x \in [2n\pi, 2n\pi + \pi], n \in I\} \cap [-4, 4] = [-4, -\pi] \cup [0, \pi]$.

(ii) Domain of $\sqrt{4 - x^2}$ is $[-2, 2]$ but $\sqrt{4 - x^2} = 0$ for $x = \pm 2$ $\Rightarrow$ $x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x - 1)(x + 1) > 0$.

$\therefore$ domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is $\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2) =$
$(-1, 0) \cup (1, 2)$.

(iii) $x > 0$ and $-1 \le x \le 1$ $\therefore$ domain is $(0, 1]$

Assignment :

3. Find the domain of following functions.

(i) $f(x) = \dfrac{1}{\log(2 - x)} + \sqrt{x + 1}$ (ii) $f(x) = \sqrt{1 - x} - \sin^{-1}\dfrac{2x - 1}{3}$

Ans. (i) $[-1, 1) \cup (1, 2)$ (ii) $[-1, 1]$

Methods of determining range :

(i) **Representing x in terms of y**

Definition of the function is usually represented as y (i.e. f(x) which is dependent variable) in terms of an expression of x (which is independent variable). To find range rewrite given definition so as to represent x in terms of an expression of y and thus obtain range (possible values of y).

If $y = f(x)$ $\Leftrightarrow$ $x = g(y)$, then domain of g(y) represents possible values of y, i.e. range of f(x).

Illustration # 4: Find the range of $f(x) = \dfrac{x^2 + x + 1}{x^2 + x - 1}$

Solution $f(x) = \dfrac{x^2 + x + 1}{x^2 + x - 1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$y = \dfrac{x^2 + x + 1}{x^2 + x - 1}$ $\Rightarrow$ $yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow$ $(y - 1)x^2 + (y - 1)x - y - 1 = 0$

If $y = 1$, then the above equation reduces to $-2 = 0$. Which is not true.

Further if $y \ne 1$, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if
$(y - 1)^2 - 4(y - 1)(-y - 1) \ge 0$ i.e. if $y \le -3/5$ or $y \ge 1$ but $y \ne 1$

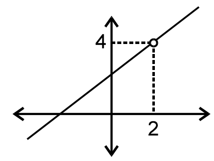
Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

(ii) **Graphical Method :** Values covered on y-axis by the graph of function is range

Illustration # 5: Find the range of $f(x) = \dfrac{x^2 - 4}{x - 2}$

Solution $f(x) = \dfrac{x^2 - 4}{x - 2}$ $= x + 2;\ x \ne 2$

$\therefore$ graph of f(x) would be



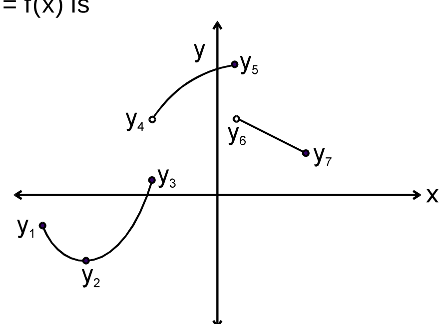
Thus the range of $f(x)$ is $R - \{4\}$

(iii) **Using Monotonocity/Maxima-Minima**

(a) **Continuous function:** If $y = f(x)$ is continuous in its domain then range of f(x) is $y \in$ [min f(x), max. f(x)]

(b) **Sectionally continuous function:** In case of sectionally continuous functions, range will be union of [min f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.

Let graph of function $y = f(x)$ is



Then range of above sectionally continuous function is $[y_2, y_3] \cup (y_4, y_5] \cup (y_6, y_7]$

Note : ☞ In case of monotonic functions minimum and maximum values lie at end points of interval.

Illustration # 6: Find the range of following functions :

(i) $y = \ell n\,(2x - x^2)$ (ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Solution : (i) **Step – 1**

Using maxima-minima, we have $2x - x^2 \in (-\infty, 1]$

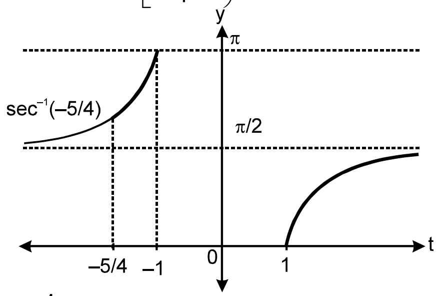
Step – 2 For log to be defined accepted values are $2x - x^2 \in (0, 1]$ {i.e. domain (0, 1]}

Now, using monotonocity $\ell n\,(2x - x^2) \in (-\infty, 0]$

$\therefore$ range is $(-\infty, 0]$ **Ans.**

(ii) $y = \sec^{-1}(x^2 + 3x + 1)$ Let $t = x^2 + 3x + 1$ for $x \in R$

then $t \in \left[-\frac{5}{4}, \infty\right)$ but $y = \sec^{-1}(t)$ $\Rightarrow$ $t \in \left[-\frac{5}{4}, -1\right] \cup [1, \infty)$



from graph range is $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\sec^{-1}\left(-\frac{5}{4}\right), \pi\right]$

Assignment:

4. Find domain and range of following functions.

(i) $y = x^3$ (ii) $y = \frac{x^2 - 2x + 5}{x^2 + 2x + 5}$

Answer (i) domain R; range R (ii) domain R ; range $\left[\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right]$

(iii) $y = \frac{1}{\sqrt{x^2 - x}}$ **Answer** domain R – [0, 1] ; range $(0, \infty)$

(iv) $y = \cot^{-1}(2x - x^2)$ **Answer** domain R ; range $\left[\frac{\pi}{4}, \pi\right)$

(v) $y = \ell n \sin^{-1}\left(x^2 + x + \frac{3}{4}\right)$ **Answer** domain $x \in \left[\frac{-2-\sqrt{5}}{4}, \frac{-2+\sqrt{5}}{4}\right]$; range $\left[\ell n\frac{\pi}{6}, \ell n\frac{\pi}{2}\right]$

C. Classification of Functions :

Functions can be classified as :

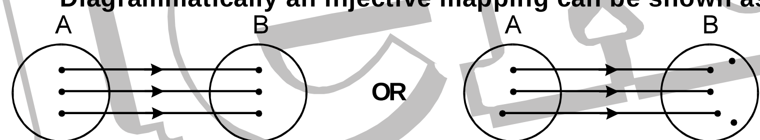
(i) One – One Function (Injective Mapping) and Many – One Function:

One – One Function :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as

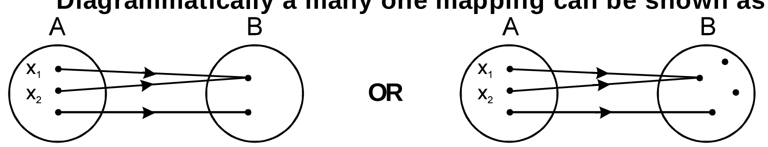


Many – One function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one iff there exists atleast two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



Note : ☞ If a function is one–one, it cannot be many–one and vice versa.

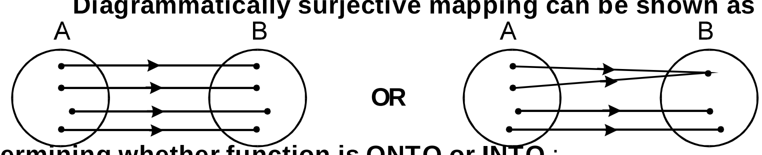
Methods of determining whether function is ONE-ONE or MANY-ONE :

(a) If $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$, then function is ONE-ONE otherwise MANY-ONE. (b) If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE-ONE. (c) If either $f'(x) \geq 0$, $\forall$ x $\in$ complete domain or $f'(x) \leq 0$ $\forall$ x $\in$ complete domain, where equality can hold at discrete point(s) only, then function is ONE-ONE, otherwise MANY-ONE.

(ii) Onto function (Surjective mapping) and Into function : Onto function :

If the function $f : A \to B$ is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall$ b $\in$ B, there exists some a $\in$ A such that f (a) = b.

Diagrammatically surjective mapping can be shown as



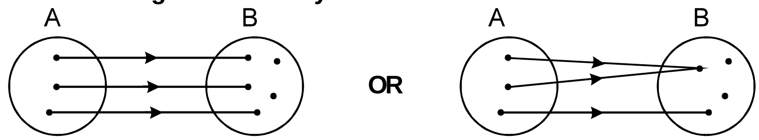
Method of determining whether function is ONTO or INTO :

Find the range of given function. If range $\equiv$ co–domain, then f(x) is onto, otherwise into

Into function :

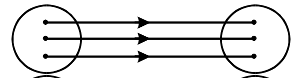
If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as

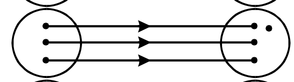


Note : ☞ If a function is onto, it cannot be into and vice versa.
Thus a function can be one of these four types:

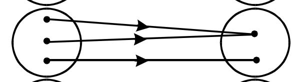
(a) one–one onto (injective & surjective)



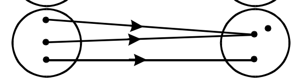
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : ☞ If f is both injective & surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

☞ If a set A contains 'n' distinct elements then the number of different functions defined from $A \to A$ is n^n and out of which $n!$ are one one.

Illustration # 7 (i) Find whether $f(x) = x + \cos x$ is one-one.

Solution The domain of $f(x)$ is R. $f'(x) = 1 - \sin x$.

$\therefore$ $f'(x) \ge 0\ \forall\ x \in$ complete domain and equality holds at discrete points only

$\therefore$ $f(x)$ is strictly increasing on R. Hence $f(x)$ is one-one.

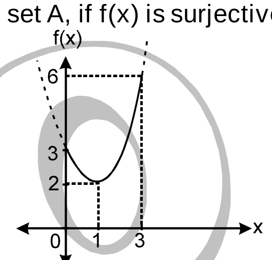
(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$; $R \to R$ is ONTO or INTO

Solution As codomain $\equiv$ range, therefore given function is ONTO

(iii) $f(x) = x^2 - 2x + 3$; $[0, 3] \to A$. Find whether $f(x)$ is injective or not. Also find the set A, if $f(x)$ is surjective.

Solution $f'(x) = 2(x - 1)$; $0 \le x \le 3$

$\therefore$ $f'(x) = \begin{cases} -\text{ve} & ; \quad 0 \le x < 1 \\ +\text{ve} & ; \quad 1 < x < 3 \end{cases}$



$\therefore$ $f(x)$ is not monotonic. Hence it is not injective.

For $f(x)$ to be surjective, A should be equal to its range. By graph range is [2, 6]

$\therefore$ $A \equiv [2, 6]$

Assignment:

5. For each of the following functions find whether it is one-one or many-one and also into or onto

(i) $f(x) = 2 \tan x$; $(\pi/2, 3\pi/2) \to R$

Answer one-one onto

(ii) $f(x) = \dfrac{1}{1+x^2}$; $(-\infty, 0) \to R$

Answer one-one into

(iii) $f(x) = x^2 + \ell n\, x$

Answer one-one onto

D. Various Types of Functions :

(i) **Polynomial Function :** If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ where n is a **non negative integer** and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \ne 0$, then f is called a polynomial function of degree n.

Note : ☞ There are two polynomial functions, satisfying the relation; $f(x).f(1/x) = f(x) + f(1/x)$, which are $f(x) = 1 \pm x^n$

(ii) **Algebraic Function :** y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x)\, y^n + P_1(x)\, y^{n-1} + \ldots\ldots + P_{n-1}(x)\, y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x)\ldots\ldots$ are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note : ☞ All polynomial functions are algebraic but not the converse.

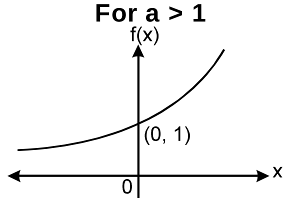
☞ A function that is not algebraic is called **Transcendental Function**.

(iii) **Fractional / Rational Function :** A rational function is a function of the form, $y = f(x) = \dfrac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials and $h(x) \ne 0$.

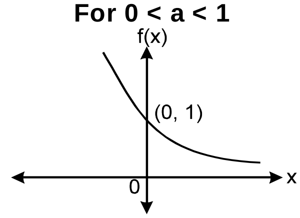
(iv) **Exponential Function :**

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \ne 1, x \in R)$ is called an exponential function. Graph of exponential function can be as follows :

Case - I
For a > 1

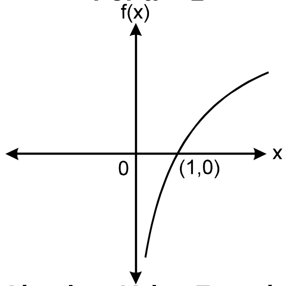


Case - II
For 0 < a < 1

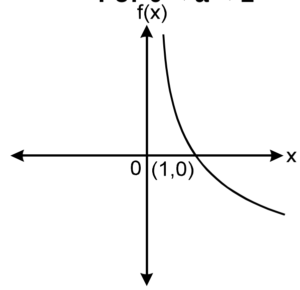


(v) **Logarithmic Function :** $f(x) = \log_a x$ is called logarithmic function where $a > 0$ and $a \neq 1$ and $x > 0$. Its graph can be as follows

Case- I
For a > 1

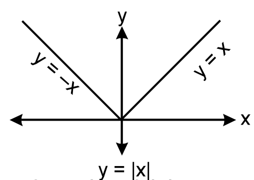


Case- II
For 0 < a < 1

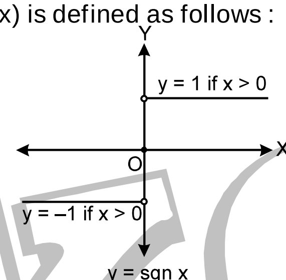


(vi) **Absolute Value Function / Modulus Function :**

The symbol of modulus function is $f(x) = |x|$ and is defined as: $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



(vi) **Signum Function :** A function f (x) = sgn (x) is defined as follows :



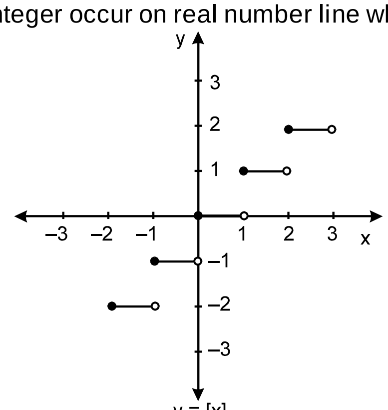
$$f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\text{sgn } x = \begin{cases} \dfrac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

Note : $\text{sgn } f(x) = \begin{cases} \dfrac{|f(x)|}{f(x)}; & f(x) \neq 0 \\ 0; & f(x) = 0 \end{cases}$

(vii) **Greatest Integer Function or Step Up Function :**
The function y = f (x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. For example :
for $-1 \leq x < 0$; $[x] = -1$; for $0 \leq x < 1$; $[x] = 0$
for $1 \leq x < 2$; $[x] = 1$; for $2 \leq x < 3$; $[x] = 2$ and so on.

Alternate Definition :
The greatest integer occur on real number line while moving L.H.S. of x (starting from x) is [x]

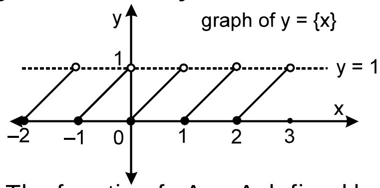


Properties of greatest integer function :

(a) $x - 1 < [x] \leq x$ (b) $[x \pm m] = [x] \pm m$ iff m is an integer.

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$ (d) $[x] + [-x] = \begin{cases} 0; & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

(viii) **Fractional Part Function:** It is defined as, $y = \{x\} = x - [x]$.
e.g. the fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.



(ix) **Identity function :** The function f : A → A defined by, f(x) = x ∀ x ∈ A is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.

(x) **Constant function :** A function f : A → B is said to be a constant function, if every element of A has the same f image in B. Thus f : A → B; f(x) = c, ∀ x ∈ A, c ∈ B is a constant function.

Illustration # 8 (i) Let {x} & [x] denote the fractional and integral part of a real number x respectively. Solve 4{x} = x + [x]

Solution As x = [x] + {x}

∴ Given equation ⇒ 4{x} = [x] + {x} + [x] ⇒ $\{x\} = \dfrac{2[x]}{3}$

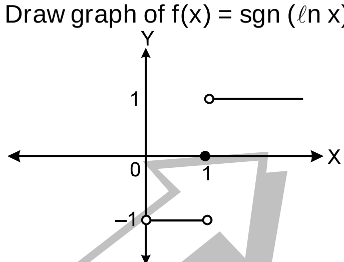
As [x] is always an integer and {x} ∈ [0, 1), possible values are

[x]	{x}	x = [x] + {x}
0	0	0
1	$\frac{2}{3}$	$\frac{5}{3}$

∴ There are two solution of given equation x = 0 and $x = \dfrac{5}{3}$

(ii) Draw graph of f(x) = sgn (ℓn x)

Solution



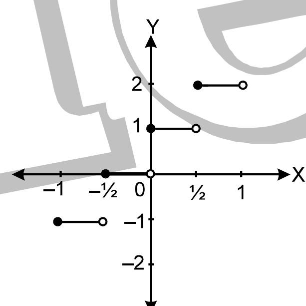
Assignment: 6. If f : R → R satisfying the conditions f(0) = 1, f(1) = 2 and f(x + 2) = 2f (x) + f(x + 1), then find f (6).

Answer 64

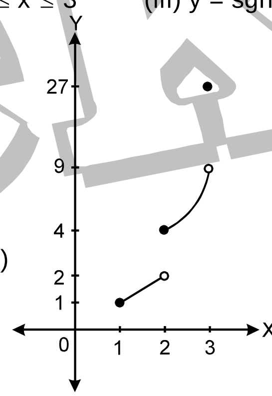
7. Draw the graph of following functions where [.] denotes greatest integer function

(i) y = [2 x] + 1 (ii) $y = x^{[x]}$, 1 ≤ x ≤ 3 (iii) y = sgn (x^2 – x)

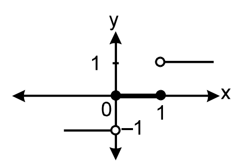
Answer (i)



(ii)



(iii)



E. **Odd & Even Functions :** (i) If f (–x) = f (x) for all x in the domain of 'f' then f is said to be an even function. If f (x) – f (–x) = 0 ⇒ f (x) is even. e.g. f (x) = cos x; g (x) = x^2 + 3.

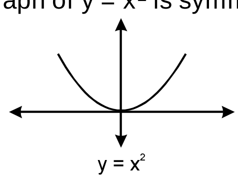
(ii) If f (–x) = –f (x) for all x in the domain of 'f' then f is said to be an odd function. If f (x) + f (–x) = 0 ⇒ f (x) is odd. e.g. f (x) = sin x; g (x) = x^3 + x.

Note : ☞ A function may neither be odd nor even. (e.g. f(x) = e^x , $\cos^{-1}x$)

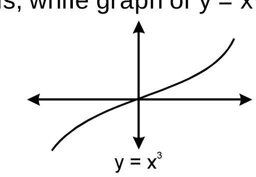
☞ If an odd function is defined at x = 0, then f(0) = 0

Properties of Even/Odd Function

(a) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin. For example graph of y = x^2 is symmetric about y-axis, while graph of y = x^3 is symmetric about origin



y = x^2



y = x^3

(b) All functions (whose domain is symmetrical about origin) can be expressed as the sum of an even & an odd function, as follows

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd}}$$

(c) The only function which is defined on the entire number line and is even & odd at the same time is f(x) = 0.

(d) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd. (e) If f(x) is even then f′(x) is odd but converse need not be true.

Illustration # 9: Show that $\log\left(x+\sqrt{x^2+1}\right)$ is an odd function.

Solution Let $f(x) = \log\left(x+\sqrt{x^2+1}\right)$. Then $f(-x) = \log\left(-x+\sqrt{(-x)^2+1}\right)$

$$= \log\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x} = \log\frac{1}{\sqrt{x^2+1}+x} - \log\left(x+\sqrt{x^2+1}\right) = -f(x)$$

Hence f(x) is an odd function.

Illustration # 10 Show that $a^x + a^{-x}$ is an even function.

Solution Let $f(x) = a^x + a^{-x}$ Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$.

Hence f(x) is an even function

Illustration # 11 Show that $\cos^{-1} x$ is neither odd nor even.

Solution Let $f(x) = \cos^{-1}x$. Then $f(-x) = \cos^{-1}(-x) = \pi - \cos^{-1} x$ which is neither equal to f(x) nor equal to f(–x).

Hence $\cos^{-1} x$ is neither odd nor even

Assignment: **8.** Determine whether following functions are even or odd?

(i) $\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ **Answer** Odd

(ii) $\log\left(\sqrt{x^2+1}-x\right)$ **Answer** Odd

(iii) $x\log\left(x+\sqrt{x^2+1}\right)$ **Answer** Even

(iv) $\sin^{-1} 2x\ \sqrt{1-x^2}$ **Answer** Odd

Even extension / Odd extension :

Let the defincition of the function f(x) is given only for $x \ge 0$. Even extension of this function implies to define the function for x < 0 assuming it to be even. In order to get even extension replace x by –x in the given defincition Similarly, odd extension implies to define the function for x < 0 assuming it to be odd. In order to get odd extension, multiply the definition of even extension by –1

Illustration # 12 What is even and odd extension of $f(x) = x^3 - 6x^2 + 5x - 11$; $x > 0$

Solution Even extension

$f(x) = -x^3 - 6x^2 + 5x - 11$; $x < 0$

Odd extension

$f(x) = x^3 + 6x^2 + 5x + 11$; $x < 0$

F. Periodic Function :

A function f(x) is called periodic with a period T if there exists a real number T > 0 such that for each x in the domain of f the numbers x – T and x + T are also in the domain of f and f(x) = f(x + T) for all x in the domain of 'f'. Domain of a periodic function is always unbounded. Graph of a periodic function with period T is repeated after every interval of 'T'.

e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π.

The **least positive period is called the principal or fundamental period** of f or simply the period of f.

Note : ☞ f (T) = f (0) = f (–T), where 'T' is the period.

☞ Inverse of a periodic function does not exist. ☞ Every constant function is always periodic, with no fundamental period.

Properties of Periodic Function

(a) If f(x) has a period T, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T then f (ax + b) has a period $\dfrac{T}{|a|}$.

(c) If f (x) has a period T_1 & g (x) also has a period T_2 then period of f(x) ± g(x) or f(x) . g(x) or $\dfrac{f(x)}{g(x)}$ is L.C.M. of T_1 & T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists f(x) ± g(x) or f(x) . g(x) or $\dfrac{f(x)}{g(x)}$ is aperiodic.

e.g. |sinx| has the period π, | cosx | also has the period π

∴ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\dfrac{\pi}{2}$.

Illustration # 13 Find period of following functions

(i) $f(x) = \sin\dfrac{x}{2} + \cos\dfrac{x}{3}$ (ii) $f(x) = \{x\} + \sin x$

(iii) $f(x) = \cos x \,.\, \cos 3x$ (iv) $f(x) = \sin\dfrac{3x}{2} - \cos\dfrac{x}{3} - \tan\dfrac{2x}{3}$

Solution **(i)** Period of $\sin\dfrac{x}{2}$ is 4π while period of $\cos\dfrac{x}{3}$ is 6π. Hence period of $\sin\dfrac{x}{2} + \cos\dfrac{x}{3}$ is 12π {L.C.M. of 4 & 6 is 12}

(ii) Period of sin x = 2p

Period of {x} = 1 but L.C.M. of 2π & 1 is not possible

∴ it is aperiodic

(iii) $f(x) = \cos x \cdot \cos 3x$

period of f(x) is L.C.M. of $\left(2\pi, \dfrac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period $= \dfrac{2\pi}{n}$, where $n \in N$. Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period $f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$

(iv) Period of f(x) is L.C.M. of $\dfrac{2\pi}{3/2}, \dfrac{2\pi}{1/3}, \dfrac{\pi}{3/2}$

$=$ L.C.M. of $\dfrac{4\pi}{3}, 6\pi, \dfrac{2\pi}{3} = 12\pi$

NOTE : **L.C.M. of** $\left(\dfrac{a}{b}, \dfrac{p}{q}, \dfrac{\ell}{m}\right) = \dfrac{\textbf{L.C.M.}(a,\ p,\ \ell)}{\textbf{H.C.F.}(b,\ q,\ m)}$

Assignment: **9.** Find the period of following function.

(i) $f(x) = \sin x + |\sin x|$ **Answer** 2π

(ii) $f(x) = \sqrt{3}\cos x - \sin\dfrac{x}{3}$ **Answer** 6π

(iii) $\sin\dfrac{2x}{5} - \cos\dfrac{3x}{7}$ **Answer** 70π

(iv) $f(x) = \sin^2 x + \cos^4 x$ **Answer** π

G. Composite Function :

Let $f: X \rightarrow Y_1$ and $g: Y_2 \rightarrow Z$ be two functions and the set $D = \{x \in X: f(x) \in Y_2\}$. If $D \neq \phi$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note : ☞ Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(x) \subseteq Y_2$.

Properties of Composite Functions :

(a) In general gof $\neq$ fog (i.e. not commutative)

(b) The composite of functions are associative i.e. if three functions f, g, h are such that fo (goh) & (fog) oh are defined, then fo (goh) = (fog) oh.

(c) If f and g both are one-one, then gof and fog would also be one-one.

(d) If f and g both are onto, then gof or fog may or may not be onto.

(e) The composite of two bijections is a bijection iff f & g are two bijections such that gof is defined, then gof is also a bijection only **when co-domain of f is equal to the domain of g**.

(f) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if

\# g is one-one, then T is the period of gof

\# g is also periodic with T′ as the period and the range of f is a sub-set of [0, T′], then T is the period of gof

Illustration # 14 Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$ (ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Solution (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

Since range of f is a subset of domain of g,

$\therefore$ domain of gof is $[-3, \infty)$ {equal to the domain of f }

$gof(x) = g\{f(x)\} = g(\sqrt{x+3}) = 1 + (x+3) = x + 4$. Range of gof is $[1, \infty)$.

Further since range of g is a subset of domain of f,

$\therefore$ domain of fog is R {equal to the domain of g}

$fog(x) = f\{g(x)\} = f(1 + x^2) = \sqrt{x^2+4}$ Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

Since range of f is a subset of the domain of g,

$\therefore$ domain of gof is $[0, \infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is $[-1, \infty)$

Further since range of g is not a subset of the domain of f

i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is $\{x \in R$, the domain of g : $g(x) \in [0, \infty)$, the domain of f$\}$.

Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e. $D = \{x \in R: 0 \le x^2 - 1\} = \{x \in R: x \le -1 \text{ or } x \ge 1\} = (-\infty, -1] \cup [1, \infty)$

$fog(x) = f\{g(x)\} = f(x^2 - 1) = \sqrt{x^2-1}$ Its range is $[0, \infty)$.

(iii) Let $f(x) = e^x$; $R^+ \rightarrow R$ and $g(x) = \sin^{-1} x$; $[-1, 1] \rightarrow \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. Find domain and range of fog (x)

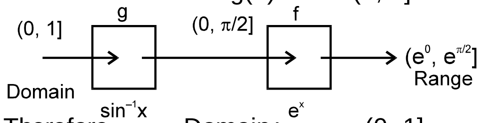
Solution

Domain of f(x) : $(0, \infty)$ Range of g(x) : $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

values in range of g(x) which are accepted by f(x) are $\left(0, \dfrac{\pi}{2}\right]$

$\Rightarrow$ $0 < g(x) \le \dfrac{\pi}{2}$ $0 < \sin^{-1}x \le \dfrac{\pi}{2}$ $0 < x \le 1$

Hence domain of fog(x) is $x \in (0, 1]$



Therefore Domain : $(0, 1]$
Range : $(1, e^{\pi/2}]$

Example of composite function of non-uniformly defined functions :

Illustration # 15 If $f(x) = |\,|x-3|-2\,|$ $\quad 0 \le x \le 4$

$g(x) = 4 - |2 - x|$ $\quad -1 \le x \le 3$

then find fog(x) and draw rough sketch of fog(x).

Solution $f(x) = |\,|x-3|-2|$ $\quad 0 \le x \le 4$

$$= \begin{cases} |x-1| & 0 \le x < 3 \\ |x-5| & 3 \le x \le 4 \end{cases}$$

$$= \begin{cases} 1-x & 0 \le x < 1 \\ x-1 & 1 \le x < 3 \\ 5-x & 3 \le x \le 4 \end{cases}$$

$g(x) = 4 - |2 - x|$ $\quad -1 \le x \le 3$

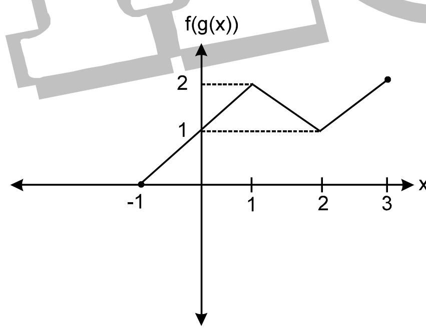
$$= \begin{cases} 4-(2-x) & -1 \le x < 2 \\ 4-(x-2) & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$$

$$\therefore \quad fog(x) = \begin{cases} 1-g(x) & 0 \le g(x) < 1 \\ g(x)-1 & 1 \le g(x) < 3 \\ 5-g(x) & 3 \le g(x) \le 4 \end{cases}$$

$$= \begin{cases} 1-(2+x) & 0 \le 2+x < 1 & \text{and} & -1 \le x < 2 \\ 2+x-1 & 1 \le 2+x < 3 & \text{and} & -1 \le x < 2 \\ 5-(2+x) & 3 \le 2+x \le 4 & \text{and} & -1 \le x < 2 \\ 1-6+x & 0 \le 6-x < 1 & \text{and} & 2 \le x \le 3 \\ 6-x-1 & 1 \le 6-x \le 3 & \text{and} & 2 \le x \le 3 \\ 5-6+x & 3 \le 6-x \le 4 & \text{and} & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} -1-x & -2 \le x < 1 & \text{and} & -1 \le x < 2 \\ 1+x & -1 \le x < 1 & \text{and} & -1 \le x < x \\ 3-x & 1 \le x \le 2 & \text{and} & -1 \le x < 2 \\ x-5 & -6 \le -x < -5 & \text{and} & 2 \le x \le 3 \\ 5-x & -5 \le -x < -3 & \text{and} & 2 \le x \le 3 \\ x-1 & -3 \le -x \le -2 & \text{and} & 2 \le x \le 3 \end{cases}$$



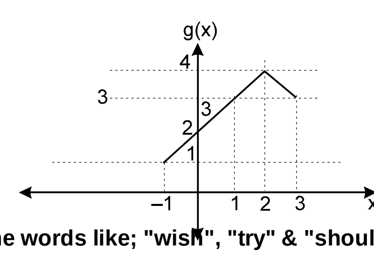
$$= \begin{cases} -1-x & -2 \le x < -1 & \text{and} & -1 \le x < 2 \\ 1+x & -1 \le x < 1 & \text{and} & -1 \le x < 2 \\ 3-x & 1 \le x \le 2 & \text{and} & -1 \le x < 2 \\ x-5 & 5 < x \le 6 & \text{and} & 2 \le x \le 3 \\ 5-x & 3 < x \le 5 & \text{and} & 2 \le x \le 3 \\ x-1 & 2 \le x \le 3 & \text{and} & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} 1+x & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$$

Alternate method for finding fog

$$g(x) = \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$$

graph of g(x) is



$\therefore \quad fog(x) = \begin{cases} 1-g(x) & 0 \le g(x) < 1 \\ g(x)-1 & 1 \le g(x) < 3 \\ 5-g(x) & 3 \le g(x) \le 4 \end{cases}$

$= \begin{cases} 1-g(x) & \text{for no value} \\ g(x)-1 & -1 \le x < 1 \\ 5-g(x) & 1 \le x \le 3 \end{cases} \quad = \begin{cases} 2+x-1 & -1 \le x < 1 \\ 5-(2+x) & 1 \le x < 2 \\ 5-(6-x) & 2 \le x \le 3 \end{cases} \quad = \begin{cases} x+1 & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$

Assignment: **10.** Define fog(x) and gof(x). Also their Domain & Range.

(i) $f(x) = [x]$, $g(x) = \sin x$ (ii) $f(x) = \tan x$, $x \in (-\pi/2, \pi/2)$; $g(x) = \sqrt{1-x^2}$

Answer (i) gof = sin [x]
domain : R range { sin a : a ∈ I}
fog = [sin x] domain : R range : {–1, 0, 1}

Answer (ii) gof = $\sqrt{1-\tan^2 x}$

domain : $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ range : [0, 1]

fog = tan $\sqrt{1-x^2}$ domain : [–1, 1] range [0, tan 1]

11. Let $f(x) = e^x : R^+ \to R$ and $g(x) = x^2 - x : R \to R$. Find domain and range of fog (x) & gof (x)

Answer
fog (x) — Domain : $(-\infty, 0) \cup (1, \infty)$ — Range : $[1, \infty)$
gof f(x) — Domain : $(0, \infty)$ — Range : $\left[-\frac{1}{4}, \infty\right)$

H. Inverse of a Function :

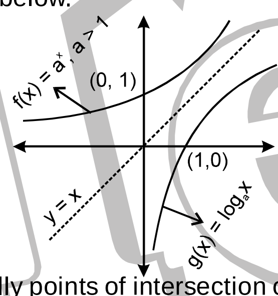
Let $f : A \to B$ be a function. Then f is invertible iff there is a function $g : B \to A$ such that go f is an identity function on A and fog is an identity function on B. Then g is called inverse of f and is denoted by f^{-1}.

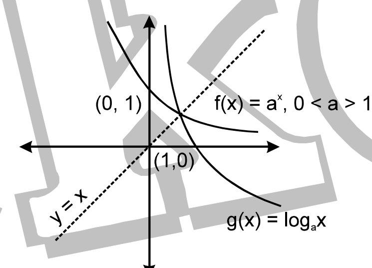
For a function to be invertible it must be bijective

Note : ☞ The inverse of a bijection is unique. ☞ Inverse of an even function is not defined.

Properties of Inverse Function :

(a) The graphs of f & g are the mirror images of each other in the line y = x. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.





(b) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also.

(c) In general fog(x) and gof(x) are not equal but if they are equal then in majority of cases either f and g are inverse of each other or atleast one of f and g is an identity function.

(d) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

(e) If f(x) and g are inverse function of each other then $f'(g(x)) = \frac{1}{g'(x)}$

Illustration # 16 (i) Determine whether $f(x) = \frac{2x+3}{4}$; $R \to R$, is invertible or not? If so find it.

Solution: As given function is one-one and onto, therefore it is invertible. $y = \frac{2x+3}{4}$

$\Rightarrow \quad x = \frac{4y-3}{2} \qquad \therefore \quad f^{-1}(x) = \frac{4x-3}{2}$

(ii) Is the function $f(x) = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ invertible?

Solution: Domain of f is [–1, 1] and f is continuous

$$f'(x) = \frac{2\left(1-2x^2\right)}{\left|1-2x^2\right|\sqrt{1-x^2}} = \begin{cases} \dfrac{2}{\sqrt{1-x^2}} & \text{if } \dfrac{-1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}} \\ \dfrac{-2}{\sqrt{1-x^2}} & \text{if } x < \dfrac{-1}{\sqrt{2}} \text{ or } x > \dfrac{1}{\sqrt{2}} \end{cases}$$

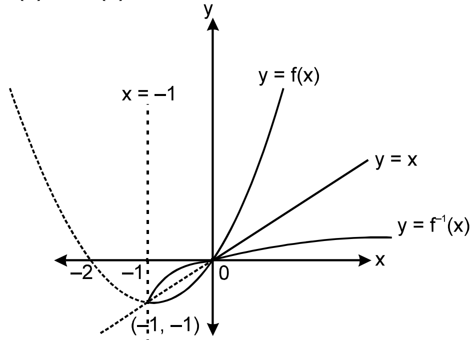
$\therefore$ f(x) is increasing in $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and is decreasing in each of the intervals

$\left(-1, \frac{-1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, 1\right)$

$\therefore$ f(x) is not one-one, so is not invertible.

(iii) Let $f(x) = x^2 + 2x$; $x \geq -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

Solution



$f(x) = f^{-1}(x)$ is equilavent to solving y = f(x) and y = x

$\Rightarrow \quad x^2 + 2x = x \quad \Rightarrow \quad x(x + 1) = 0 \quad \Rightarrow \quad x = 0, -1$

Hence two solution for $f(x) = f^{-1}(x)$

(iv) If $y = f(x) = x^2 - 3x + 1$, $x \geq 2$. Find the value of $g'(1)$ where g is inverse of f

Solution $y = 1 \quad \Rightarrow \quad x^2 - 3x + 1 = 1 \quad \Rightarrow \quad x(x - 3) = 0 \quad \Rightarrow \quad x = 0, 3$

But $x \geq 2 \quad \therefore \quad x = 3$

Now $g(f(x)) = x$

Differentiating both sides w.r.t. x

$\Rightarrow \quad g'(f(x)).\, f'(x) = 1 \quad \Rightarrow \quad g'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow \quad g'(f(3)) = \dfrac{1}{f'(3)} \quad \Rightarrow \quad g'(1) = = \dfrac{1}{6-3}$ (As $f'(x) = 2x - 3$) $= \dfrac{1}{3}$

<u>Alternate Method</u>

$y = x^2 - 3x + 1$

$x^2 - 3x + 1 - y = 0$

$$x = \frac{3 \pm \sqrt{9 - 4(1-y)}}{2}$$

$$= \frac{3 \pm \sqrt{5 + 4y}}{2}$$

$x \geq 2$

$$x = \frac{3 + \sqrt{5+4y}}{2}$$

$$g(x) = \frac{3 + \sqrt{5+4y}}{2}$$

$$g'(x) = 0 + \frac{1}{x\sqrt{5+4x}}\, x \qquad g'(1) = \frac{1}{\sqrt{5+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Assignment: **12.** Determine $f^{-1}(x)$, if given function is invertible

(i) $f : (-\infty, -1) \to (-\infty, -2)$ defined $f(x) = -(x + 1)^2 - 2$

(ii) $f : \left[\dfrac{\pi}{6}, \dfrac{7\pi}{6}\right] \to [-1, 1]$ defined by $f(x) = \sin\left(x + \dfrac{\pi}{3}\right)$

Answer (i) $-1 + \sqrt{-x-2}$ (ii) $\dfrac{2\pi}{3} - \sin^{-1}x$

I. Equal or Identical Function :

Two functions f & g are said to be identical (or equal) iff :

(i) The domain of f $\equiv$ the domain of g. (ii) The range of f $\equiv$ the range of g and

(iii) f(x) = g(x), for every x belonging to their common domain. e.g. $f(x) = \dfrac{1}{x}$ & $g(x) = \dfrac{x}{x^2}$ are identical functions.

But f(x) = x and $g(x) = \dfrac{x^2}{x}$ are not identical functions.

Illustration # 17 Examine whether following pair of functions are identical or not

(i) $f(x) = \dfrac{x^2}{x}$ & g(x) = x **Answer** No, as domain of f(x) is R – {0} while domain of g(x) is R

(ii) $f(x) = \sin^2x + \cos^2x$ & $g(x) = \sec^2x - \tan^2x$

Answer No, as domain are not same. Domain of f(x) is R while that of g(x) is $R - \left\{(2n+1)\dfrac{\pi}{2};\ n \in I\right\}$

Assignment: **13.** Examine whether following pair of functions are identical or not

(i) f(x) = sgn (x) & $g(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(ii) $f(x) = \sin^{-1}x + \cos^{-1}x$ & $g(x) = \dfrac{\pi}{2}$ **Answer** (i) Yes (ii) No

J. General : If x, y are independent variables, then:

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$. (ii) $f(xy) = f(x).\,f(y) \Rightarrow f(x) = x^n,\ n \in R$

(iii) $f(x + y) = f(x).\,f(y) \Rightarrow f(x) = a^{kx}$. (iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

(v) $f(x)\,.\,f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right) \Rightarrow f(x) = 1 \pm x^n$ where $n \in N$

Illustration # 18 If f(x) is a polynomial function satisfying $f(x)\,.\,f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and f(2) = 9, then find f (3)

Solution $f(x) = 1 \pm x^n$ As f(2) = 9 $\therefore$ $f(x) = 1 + x^3$

Hence $f(3) = 1 + 3^3 = 28$

Assignment: 14. If f(x) is a polynomial function satisfying $f(x)\,.\,f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and f(3) = –8, then find f(4) **Answer** – 15

15. If f(x + y) = f(x) . f(y) for all real x, y and f(0) ≠ 0 then prove that the function, $g(x) = \dfrac{f(x)}{1+f^2(x)}$ is an even function

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