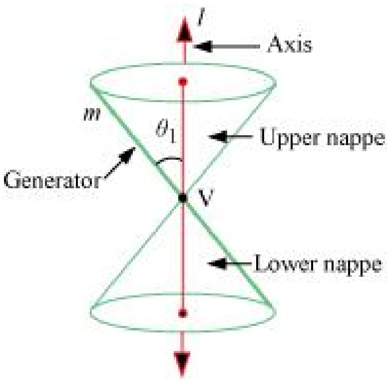
Conic Sections

- **Conic sections**

Conic sections or conics are the curves that are obtained by intersecting a plane with a double-napped right circular cone. Circles, ellipses, parabolas and hyperbolas are examples of conic sections.

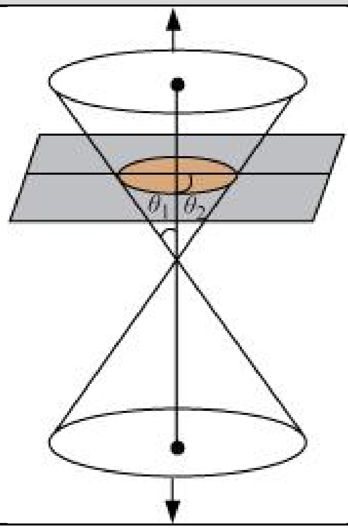
A double-napped cone can be obtained by rotating a line (let us say *m*) about a fixed vertical line (let us say *l*).

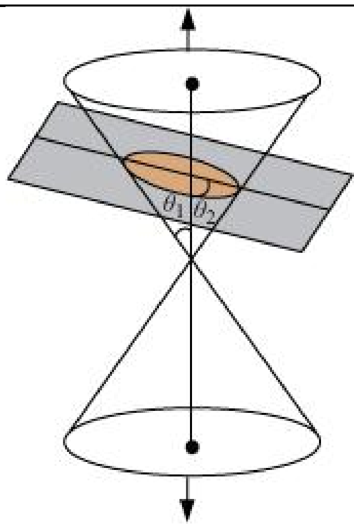
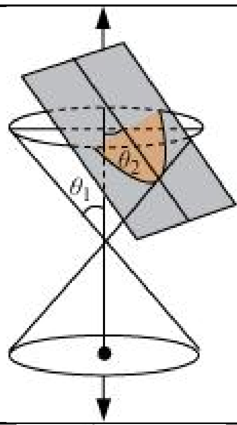
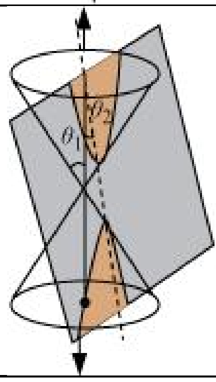


Here, the fixed line *l* is called the axis of the cone and *m* is called the generator of the cone. The intersection (V) of *l* and *m* is called the vertex of the cone.

Different conics formed by intersecting a plane and a double-napped cone:

If θ_1 is the angle between the axis and the generator and θ_2 is the angle between the plane and the axis, then, for different conditions of θ_1 and θ_2, we get different conics, which are described with the help of a table as shown below.

Condition	Conic Formed	Figure
$\theta_2 = 90°$ (Only one nappe of the cone is entirely cut by the plane)	A circle	

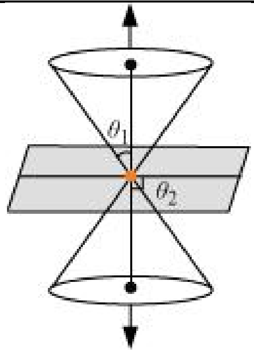
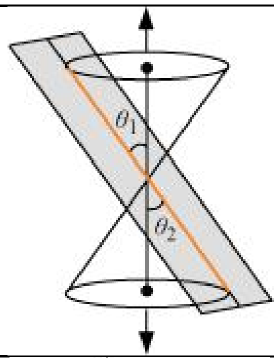
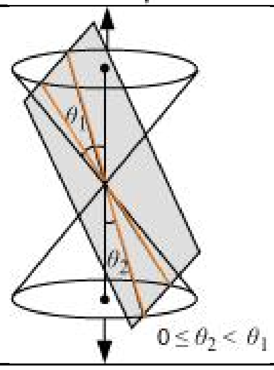
$\theta_1 < \theta_2 < 90°$ (Only one nappe of the cone is entirely cut by the plane)	An ellipse	
$\theta_1 = \theta_2$ (Only one nappe of the cone is entirely cut by the plane)	A parabola	
$0 < \theta_2 < \theta_1$ (Both the nappes of the cone are entirely cut by the plane)	A hyperbola	

- **Degenerated conics**

The conics obtained by cutting a plane with a double-napped cone at its vertex are known as degenerating conic sections.

If θ_1 is the angle between the axis and the generator and θ_2 is the angle between the plane and the axis, then, for different conditions of θ_1 and θ_2, we get different conics, which are described with the help of a table as shown below.

Condition	Conic Formed	Figure

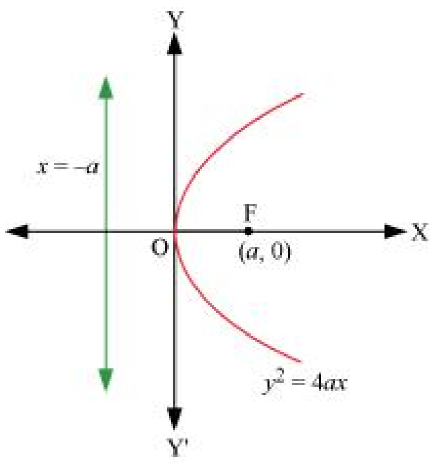
$\theta_2 = 90°$	A point	
$\theta_1 = \theta_2$	A straight line	
$0 \leq \theta_2 < \theta_1$	A hyperbola	

- **Parabola**

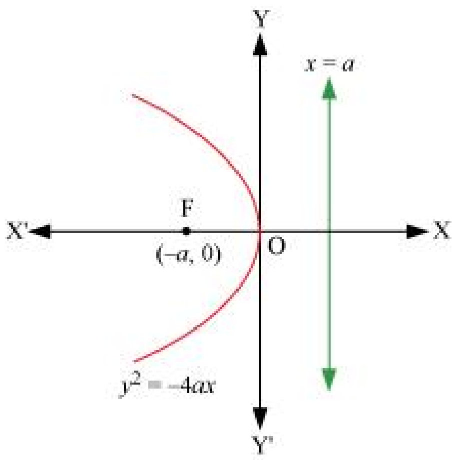
A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line in the plane).

  - The fixed line is called the directrix.
  - The fixed point F is called the focus.
  - The line through the focus and perpendicular to the directrix is called the axis of the parabola.
  - The point of intersection of parabola with the axis is called the vertex of the parabola.
  - The line segment that is perpendicular to the axis of the parabola through the focus and whose end points lie on the parabola is called the latus rectum of the parabola.

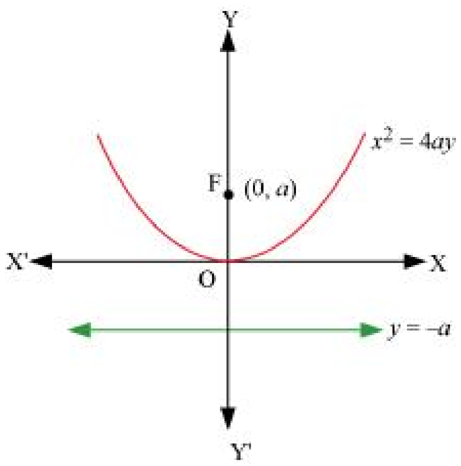
- **Standard equations of parabola**



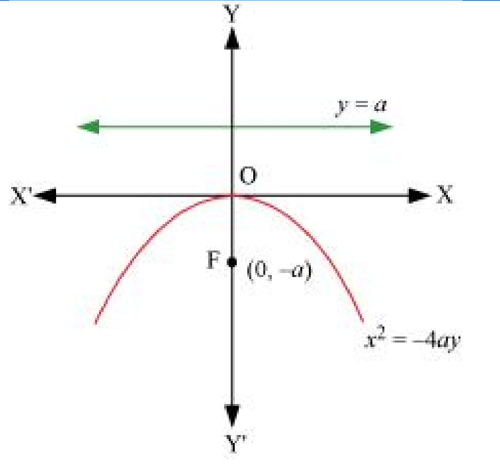
Open Towards	Right
Standard Equation	$y^2 = 4ax$, $a > 0$
Coordinates of Focus	$(a, 0)$
Coordinates of Vertex	$(0, 0)$
Equation of Directrix	$x = -a$
Length of Latus Rectum	$4a$
Axis of Parabola	$y = 0$



Open Towards	Left
Standard Equation	$y^2 = -4ax$, $a > 0$
Coordinates of Focus	$(-a, 0)$
Coordinates of Vertex	$(0, 0)$
Equation of Directrix	$x = a$
Length of Latus Rectum	$4a$
Axis of Parabola	$y = 0$



Open Towards	Upward
Standard Equation	$x^2 = 4ay$, $a > 0$
Coordinates of Focus	$(0, a)$
Coordinates of Vertex	$(0, 0)$
Equation of Directrix	$y = -a$
Length of Latus Rectum	$4a$
Axis of Parabola	$x = 0$



Open Towards	Downward
Standard Equation	$x^2 = -4ay$, $a > 0$
Coordinates of Focus	$(0, -a)$
Coordinates of Vertex	(0, 0)
Equation of Directrix	$y = a$
Length of Latus Rectum	$4a$
Axis of Parabola	$x = 0$

Example: Consider a parabola $x^2 = -16y$.
Comparing the parabola with $x^2 = -4ay$
Here, $4a = 16 \Rightarrow a = 4$

- Coordinates of vertex = (0, 0)
- Coordinates of focus = (0, –4)
- Equation of directrix is $y = 4$
- Length of latus rectum = $4 \times 4 = 16$
- Equation of axis is $x = 0$

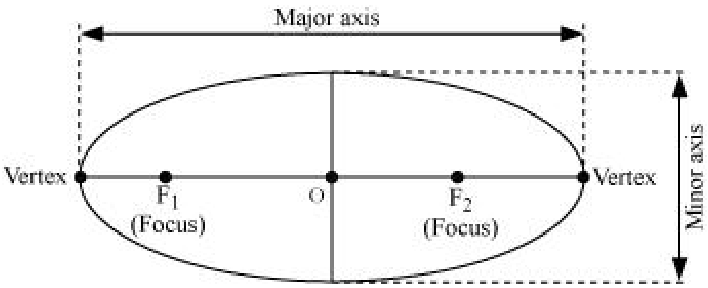
If the fixed point lies on the fixed line, then the set of points in the plane that are equidistant from the fixed point and the fixed line is a straight line through the fixed point and perpendicular to the fixed line. We call this straight line the degenerate case of parabola.

• Ellipse

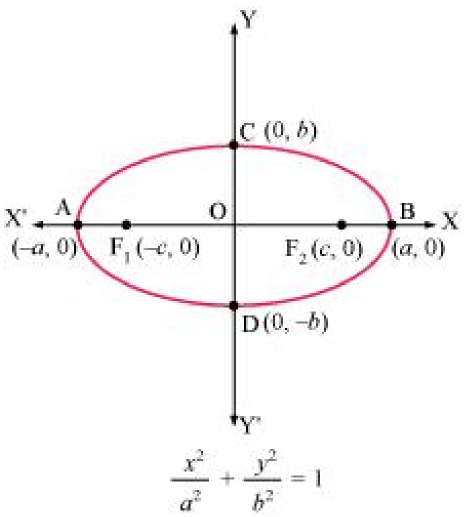
An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.

- The two fixed points are called the foci.
- The constant, which is the sum of the distances of a point on the ellipse from the two fixed points, is always greater than the distance between the two fixed points.
- The mid-point of the line segment joining the foci is called the centre of the ellipse.
- The line segment through the foci of the ellipse is called the major axis and the line segment through the centre and perpendicular to the major axis is called the minor axis.
- The end points of the major axis are called the vertices of the ellipse.
- The eccentricity of the ellipse is the ratio of the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
- An ellipse is symmetric with respect to both the coordinate axes.
- The line segment that is perpendicular to the major axis of the ellipse through the

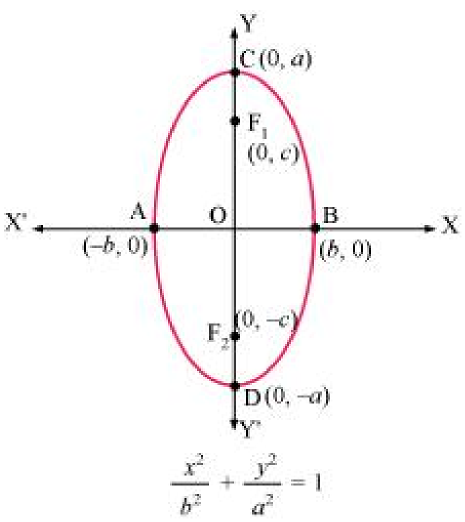
focus and whose end points lie on the ellipse is called the latus rectum of the ellipse.



- **Standard equations of ellipse:**



Standard Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$
Centre	(0, 0)
Vertex	$(\pm a, 0)$
End points of minor axis	$(0, \pm b)$
Foci	$(\pm c, 0)$
Length of major axis	$2a$ along x-axis
Length of minor Axis	$2b$ along y-axis
Length between foci	$2c$ along x-axis
Relation between a, b and c	$a^2 = b^2 + c^2$
Length of latus rectum	$\frac{2b^2}{a}$
Eccentricity ($e < 1$)	$\frac{c}{a}$



Standard Equation	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$
Centre	(0, 0)
Vertex	$(0, \pm a)$
End points of minor axis	$(\pm b, 0)$
Foci	$(0, \pm c)$
Length of major axis	$2a$ along y-axis
Length of minor Axis	$2b$ along x-axis
Length between foci	$2c$ along y-axis
Relation between a, b and c	$a^2 = b^2 + c^2$
Length of latus rectum	$\frac{2b^2}{a}$
Eccentricity ($e < 1$)	$\frac{c}{a}$

Example: Consider the ellipse $9x^2 + 4y^2 = 36$.

$9x^2 + 4y^2 = 36$ can be rewritten as $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Comparing with standard form of ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Here $a = 3$ and $b = 2$

$a^2 = b^2 + c^2$
$\Rightarrow c^2 = (3)^2 - (2)^2 = 9 - 4 = 5$
$\Rightarrow c = \sqrt{5}$

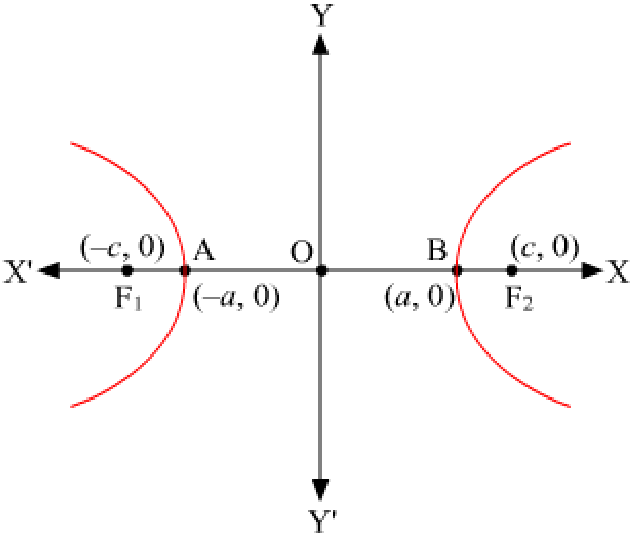
- Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{5}}{3}$
- Coordinates of the centre = (0, 0)
- Coordinates of the vertices = $(0, \pm a) = (0, \pm 3)$
- Coordinates of the foci = $(0, \pm c) = \left(0, \pm\sqrt{5}\right)$
- Length of the major axis = $2a = 2 \times 3 = 6$
- Length of the minor axis = $2b = 2 \times 2 = 4$

- **Hyperbola**

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.

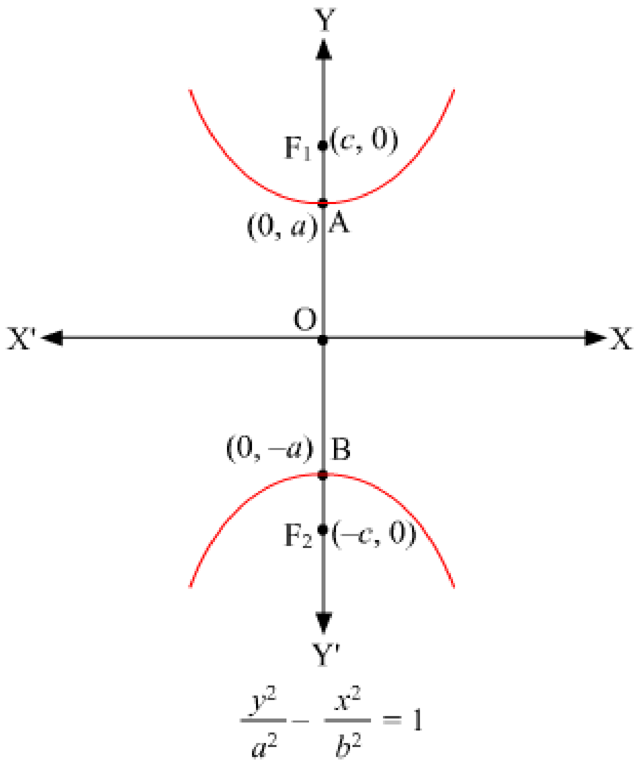
- The two fixed points are called the foci.
- The constant, which is the difference of the distances of a point on the hyperbola from the two fixed points, is always less than the distance between the two fixed points.
- The mid-point of the line segment joining the foci is called the centre of the hyperbola.
- The line through the foci is called the transverse axis and the line through the centre and perpendicular to the transverse axis is called the conjugate axis.
- The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola.
- The eccentricity of the hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.
- A hyperbola is symmetric with respect to both the coordinate axes.
- The line segment that is perpendicular to the transverse axis through the focus and whose end points lie on the hyperbola is called the latus rectum of the hyperbola.

- **Standard equations of hyperbola:**



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Standard Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Centre	(0, 0)
Vertices	($\pm a$, 0)
Foci	($\pm c$, 0)
Conjugate axis	y-axis
Transverse axis	x-axis
Length of conjugate axis	$2b$
Length of transverse axis	$2a$
Length between foci	$2c$
Relation between a, b and c	$c^2 = a^2 + b^2$
Length of latus rectum	$\frac{2b^2}{a}$
Eccentricity (e >1)	$\frac{c}{a}$



Standard Equation	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Centre	(0, 0)
Vertices	$(0, \pm a)$
Foci	$(0, \pm c)$
Conjugate axis	x-axis
Transverse axis	y-axis
Length of conjugate axis	$2b$
Length of transverse axis	$2a$
Length between foci	$2c$
Relation between a, b and c	$c^2 = a^2 + b^2$
Length of latus rectum	$\frac{2b^2}{a}$
Eccentricity (e >1)	$\frac{c}{a}$

Example: Consider the hyperbola $16x^2 - 9y^2 = 144$.

$16x^2 - 9y^2 = 144$ can be rewritten as $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Comparing with standard form of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Here, $a = 3$ and $b = 4$.

Now, $c^2 = a^2 + b^2$

$\therefore c = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

- Eccentricity $= \frac{c}{a} = \frac{5}{3}$
- Coordinates of the centre = (0, 0)
- Coordinates of the vertices = $(\pm a, 0) = (\pm 3, 0)$

- Coordinates of foci = ($\pm c$, 0) = (± 5, 0)
- Length of transverse axis = $2a = 2 \times 3 = 6$
- Length of conjugate axis = $2b = 2 \times 4 = 8$
- Distance between foci = $2c = 2 \times 5 = 10$
- Length of latus rectum $\frac{2b^2}{a} = \frac{2 \times 16}{3} = \frac{32}{3}$

A hyperbola having equal lengths of both the axes i.e., transverse and conjugate ($a = b$) is called an equilateral hyperbola.