SHORT REVISION

1. The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of order two .

Its value is given by : $D = a_1 b_2 - a_2 b_1$

2. The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three .

Its value can be found as : $D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ **OR**

$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ .... and so on .

In this manner we can expand a determinant in 6 ways using elements of ; R_1, R_2, R_3 or C_1, C_2, C_3 .

3. Following examples of short hand writing large expressions are :

(i) The lines : $a_1x + b_1y + c_1 = 0$........ (1)
$a_2x + b_2y + c_2 = 0$........ (2)
$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if , $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Condition for the consistency of three simultaneous linear equations in 2 variables.

(ii) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(iii) Area of a triangle whose vertices are (x_r, y_r) ; r = 1 , 2 , 3 is :

$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If D = 0 then the three points are collinear .

(iv) Equation of a straight line passsing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

4. **MINORS :** The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands . For example, the minor of a_1 in (Key Concept 2) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order two will have "4 minors" & a determinant of order three will have "9 minors" .

5. **COFACTOR :** If M_{ij} represents the minor of some typical element then the cofactor is defined as : $C_{ij} = (-1)^{i+j} . M_{ij}$; Where i & j denotes the row & column in which the particular element lies. Note that the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as : $D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ **OR** $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ & so on ....

6. **PROPERTIES OF DETERMINANTS : P–1 :** The value of a determinant remains unaltered , if the rows & columns are inter changed . e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$

D & D′ are transpose of each other . If D′ = – D then it is **SKEW SYMMETRIC** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero .

P– 2 : If any two rows (or columns) of a determinant be interchanged , the value of determinant is changed in sign only . e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

P– 3 : If a determinant has any two rows (or columns) identical , then its value is zero . e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that $D = 0$.

P– 4 : If all the elements of any row (or column) be multiplied by the same number , then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = KD$

P–5 : If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants . e.g.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P– 6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column). e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then $D' = D$.

Note : that while applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged .

P– 7 : If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant .

7. MULTIPLICATION OF TWO DETERMINANTS :

(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ x } \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$

Similarly two determinants of order three are multiplied.

(ii) If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ then , $D^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ where A_i , B_i , C_i are cofactors

PROOF : Consider $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{vmatrix}$

Note : $a_1A_2 + b_1B_2 + c_1C_2 = 0$ etc.

therefore , $D \text{ x } \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^3 \Rightarrow \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = D^2$ OR $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ CA_3 & B_3 & C_3 \end{vmatrix} = D^2$

8. SYSTEM OF LINEAR EQUATION (IN TWO VARIABLES) :

(i) Consistent Equations : Definite & unique solution . [intersecting lines]
(ii) Inconsistent Equation : No solution . [Parallel line]
(iii) Dependent equation : Infinite solutions . [Identical lines]

Let $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ then :

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent &

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are dependent

9. **CRAMER'S RULE :** **[SIMULTANEOUS EQUATIONS INVOLVING THREE UNKNOWNS]**

Let ,$a_1x + b_1y + c_1z = d_1$ ...(I) ; $a_2x + b_2y + c_2z = d_2$ ...(II) ; $a_3x + b_3y + c_3z = d_3$ ...(III)

Then, $x = \frac{D_1}{D}$, $Y = \frac{D_2}{D}$, $Z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

NOTE : (a) If $D \neq 0$ and alteast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solutions.

In case $\left.\begin{array}{l} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{array}\right\}$ represents these parallel planes then also

$D = D_1 = D_2 = D_3 = 0$ but the system is inconsistent.

(d) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

10. If x, y, z are not all zero, the condition for $a_1x + b_1y + c_1z = 0$; $a_2x + b_2y + c_2z = 0$ & $a_3x + b_3y + c_3z = 0$ to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **TRIVIAL SOLUTION**.

EXERCISE-1

Q 1. Without expanding the determinant prove that :

(a) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$ (c) $\begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix}$ is real

(d) $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$ (e) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

Q 2. Without expanding as far as possible, prove that :

(a) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$ (b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = [(x-y)(y-z)(z-x)(x+y+z)]$

Q 3. If $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = 0$ and x, y, z are all different then, prove that $xyz = -1$.

Q 4. Using properties of determinants or otherwise evaluate $\begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$.

Q 5. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

Q 6. If $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ and $D' = \begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$ then prove that $D' = 2D$.

Q 7. Prove that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4\,[(a+b)(b+c)(c+a)]$

Q 8. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$.

Q 9. Prove that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$.

Q 10. Show that the value of the determinant $\begin{vmatrix} \tan(A+P) & \tan(B+P) & \tan(C+P) \\ \tan(A+Q) & \tan(B+Q) & \tan(C+Q) \\ \tan(A+R) & \tan(B+R) & \tan(C+R) \end{vmatrix}$ vanishes for all values of A, B, C, P, Q & R where $A+B+C+P+Q+R=0$

Q 11. Factorise the determinant $\begin{vmatrix} bc & bc'+b'c & b'c' \\ ca & ca'+c'a & c'a' \\ ab & ab'+a'b & a'b' \end{vmatrix}$.

Q 12. Prove that $\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix} = -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$

Q 13. For a fixed positive integer n , if $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ then show that $\left[\dfrac{D}{(n!)^3} - 4\right]$ is divisible by n .

Q 14. Solve for x $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$.

Q 15. If $a+b+c=0$, solve for x : $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$.

Q 16. If $a^2+b^2+c^2=1$ then show that the value of the determinant

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\theta & ba(1-\cos\theta) & ca(1-\cos\theta) \\ ab(1-\cos\theta) & b^2+(c^2+a^2)\cos\theta & cb(1-\cos\theta) \\ ac(1-\cos\theta) & bc(1-\cos\theta) & c^2+(a^2+b^2)\cos\theta \end{vmatrix}$$ simplifies to $\cos^2\theta$.

Q 17. If $p+q+r=0$, prove that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Q 18. If a, b, c are all different & $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then prove that : $abc\,(ab+bc+ca) = a+b+c$.

Q 19. Show that

$$\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$$ is divisible by λ^2 and find the other factor.

Q 20. (a) Without expanding prove that $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$.

(b) $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

Q 21. Without expanding a determinant at any stage, show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax+B$ where A & B are determinants of order 3 not involving x .

Q 22. Prove that $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$.

Q 23. Solve $\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix} = 0$ where a, b, c are non zero and distinct .

Q 24. Solve for x : $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$.

Q 25. If $\begin{vmatrix} \frac{1}{a+x} & \frac{1}{b+x} & \frac{1}{c+x} \\ \frac{1}{a+y} & \frac{1}{b+y} & \frac{1}{c+y} \\ \frac{1}{a+z} & \frac{1}{b+z} & \frac{1}{c+z} \end{vmatrix} = \frac{P}{Q}$ where Q is the product of the denominator , prove that $P = (a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$

Q 26. If $D_r = \begin{vmatrix} 2^{r-1} & 2\left(3^{r-1}\right) & 4\left(5^{r-1}\right) \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$ then prove that $\sum_{r=1}^{n} D_r = 0$.

Q 27. If $2s = a+b+c$ then prove that $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$.

Q 28. In a Δ ABC, determine condition under which $\begin{vmatrix} \cot\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\tan\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \\ 1 & 1 & 1 \end{vmatrix} = 0$

Q 29. Show that $\begin{vmatrix} -b^2c^2 & ab\left(c^2+a^2\right) & ac\left(a^2+b^2\right) \\ ba\left(b^2+c^2\right) & -c^2a^2 & bc\left(a^2+b^2\right) \\ ca\left(b^2+c^2\right) & cb\left(c^2+a^2\right) & -a^2b^2 \end{vmatrix} = (a^2b^2+b^2c^2+c^2a^2)^3$.

Q 30. Prove that

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ -bc+ca+ab & bc-ca+ab & bc+ca-ab \\ (a+b)(a+c) & (b+c)(b+a) & (c+a)(c+b) \end{vmatrix} = 3.(b-c)(c-a)(a-b)(a+b+c)(ab+bc+ca)$$

Q 31. For all values of A, B, C & P, Q, R show that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$.

Q 32. Show that $\begin{vmatrix} a_1 l_1 + b_1 m_1 & a_1 l_2 + b_1 m_2 & a_1 l_3 + b_1 m_3 \\ a_2 l_1 + b_2 m_1 & a_2 l_2 + b_2 m_2 & a_2 l_3 + b_2 m_3 \\ a_3 l_1 + b_3 m_1 & a_3 l_2 + b_3 m_2 & a_3 l_3 + b_3 m_3 \end{vmatrix} = 0$.

Q 33. Prove that $\begin{vmatrix} (a_1 - b_1)^2 & (a_1 - b_2)^2 & (a_1 - b_3)^2 \\ (a_2 - b_1)^2 & (a_2 - b_2)^2 & (a_2 - b_3)^2 \\ (a_3 - b_1)^2 & (a_3 - b_2)^2 & (a_3 - b_3)^2 \end{vmatrix} = 2\,(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1)$

Q 34. Prove that $\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta + \gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) \\ \alpha\beta + \gamma\delta & \alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$.

Q 35. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$, then prove that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Q 36. Prove that

$$\begin{vmatrix} 1 & \cos^2(A-B) & \cos^2(A-C) \\ \cos^2(B-A) & 1 & \cos^2(B-C) \\ \cos^2(C-A) & \cos^2(C-B) & 1 \end{vmatrix} = 2\sin^2(A-B)\sin^2(B-C)\sin^2(C-A)$$

Q 37. If $ax_1^2 + by_1^2 + cz_1^2 = ax_2^2 + by_2^2 + cz_2^2 = ax_3^2 + by_3^2 + cz_3^2 = d$ and $ax_2x_3 + by_2y_3 + cz_2z_3 = ax_3x_1 + by_3y_1 + cz_3z_1 = ax_1x_2 + by_1y_2 + cz_1z_2 = f$, then prove that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left[\frac{d+2f}{abc}\right]^{1/2} \qquad (a, b, c \neq 0)$$

Q 38. If $(x_1-x_2)^2 + (y_1-y_2)^2 = a^2$, $(x_2-x_3)^2 + (y_2-y_3)^2 = b^2$ and $(x_3-x_1)^2 + (y_3-y_1)^2 = c^2$

prove that $4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

Q 39. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$.

Q 40. If $u = ax^2 + 2bxy + cy^2$, $u' = a'x^2 + 2b'xy + c'y^2$. Prove that

$$\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix} = -\frac{1}{y}\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix} .$$

EXERCISE-2

Q 1. Solve using Cramer's rule : $\frac{4}{x+5} + \frac{3}{y+7} = -1$ & $\frac{6}{x+5} - \frac{6}{y+7} = -5$.

Q 2. Solve the following using Cramer's rule and state whether consistent or not.

(a) $\begin{aligned} x + 2y + z &= 1 \\ 3x + y + z &= 6 \\ x + 2y &= 0 \end{aligned}$ (b) $\begin{aligned} x - 3y + z &= 2 \\ 3x + y + z &= 6 \\ 5x + y + 3z &= 3 \end{aligned}$ (c) $\begin{aligned} 7x - 7y + 5z &= 3 \\ 3x + y + 5z &= 7 \\ 2x + 3y + 5z &= 5 \end{aligned}$

Q 3. Solve the system of equations ; $\left.\begin{aligned} z + ay + a^2x + a^3 &= 0 \\ z + by + b^2x + b^3 &= 0 \\ z + cy + c^2x + c^3 &= 0 \end{aligned}\right]$

Q 4. For what value of K do the following system of equations possess a non trivial (i.e. not all zero) solution over the set of rationals Q ?
$x+Ky+3z=0$, $3x+Ky-2z=0$, $2x+3y-4z=0$.
For that value of K, find all the solutions of the system.

Q 5. Given $x=cy+bz$; $y=az+cx$; $z=bx+ay$ where x, y, z are not all zero, prove that $a^2+b^2+c^2+2abc=1$.

Q 6. Given $a=\dfrac{x}{y-z}$; $b=\dfrac{y}{z-x}$; $c=\dfrac{z}{x-y}$ where x, y, z are not all zero, prove that : $1+ab+bc+ca=0$.

Q 7. If $\sin q \neq \cos q$ and x, y, z satisfy the equations
$x\cos p - y\sin p + z = \cos q + 1$
$x\sin p + y\cos p + z = 1 - \sin q$
$x\cos(p+q) - y\sin(p+q) + z = 2$ then find the value of $x^2+y^2+z^2$.

Q 8. If A, B and C are the angles of a triangle then show that
$\sin 2A\cdot x + \sin C\cdot y + \sin B\cdot z = 0$
$\sin C\cdot x + \sin 2B\cdot y + \sin A\cdot z = 0$
$\sin B\cdot x + \sin A\cdot y + \sin 2C\cdot z = 0$ possess non-trivial solution.

Q 9. Investigate for what values of λ , μ the simultaneous equations $x+y+z=6$; $x+2y+3z=10$ & $x+2y+\lambda z=\mu$ have ; (a) A unique solution .
(b) An infinite number of solutions . (c) No solution .

Q 10. For what values of p, the equations : $x+y+z=1$; $x+2y+4z=p$ & $x+4y+10z=p^2$ have a solution ? Solve them completely in each case .

Q 11. Solve the equations : $Kx+2y-2z=1$, $4x+2Ky-z=2$, $6x+6y+Kz=3$ considering specially the case when $K=2$.

Q 12. Solve the system of equations :
$\alpha x+y+z=m$, $x+\alpha y+z=n$ and $x+y+\alpha z=p$

Q 13. Find all the values of t for which the system of equations ;
$(t-1)x+(3t+1)y+2tz=0$
$(t-1)x+(4t-2)y+(t+3)z=0$
$2x+(3t+1)y+3(t-1)z=0$ has non trivial solutions and in this context find the ratios of x : y : z , when t has the smallest of these values.

Q 14. Solve : $(b+c)(y+z)-ax=b-c$, $(c+a)(z+x)-by=c-a$ and $(a+b)(x+y)-cz=a-b$ where $a+b+c\neq 0$.

Q 15. If $bc+qr=ca+rp=ab+pq=-1$ show that $\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix}=0$.

Q 16. If x, y, z are not all zero & if $ax+by+cz=0$, $bx+cy+az=0$ & $cx+ay+bz=0$, then prove that $x:y:z=1:1:1$ **OR** $1:\omega:\omega^2$ **OR** $1:\omega^2:\omega$, where ω is one of the complex cube root of unity.

Q 17. If the following system of equations $(a-t)x+by+cz=0$, $bx+(c-t)y+az=0$ and $cx+ay+(b-t)z=0$ has non–trivial solutions for different values of t, then show that we can express product of these values of t in the form of determinant .

Q18. Show that the system of equations
$3x-y+4z=3$, $x+2y-3z=-2$ and $6x+5y+\lambda z=-3$
has atleast one solution for any real number λ. Find the set of solutions of $\lambda=-5$.

EXERCISE-3

Q.1 For what values of p & q, the system of equations $2x+py+6z=8$; $x+2y+qz=5$ & $x+y+3z=4$ has ; (i) no solution (ii) a unique solution (iii) infinitely many solutions

Q.2 (i) Let a, b, c positive numbers. The following system of equations in x, y & z.
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$; $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$; $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ has:
(A) no solution (B) unique solution
(C) infinitely many solutions (D) finitely many solutions

(ii) If $\omega(\neq 1)$ is a cube root of unity, then $\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$ equals :
(A) 0 (B) 1 (C) i (D) ω [IIT '95 , 1 + 1]

Q.3 Let $a > 0, d > 0$. Find the value of determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}.$$ [IIT '96 , 5]

Q.4 Find those values of c for which the equations :
$2x + 3y = 3$
$(c+2)x + (c+4)y = c+6$
$(c+2)^2 x + (c+4)^2 y = (c+6)^2$ are consistent .
Also solve above equations for these values of c . [REE '96 , 6]

Q.5 For what real values of k , the system of equations $x + 2y + z = 1$; $x + 3y + 4z = k$; $x + 5y + 10z = k^2$ has solution ? Find the solution in each case. [REE ' 97, 6]

Q.6 The parameter, on which the value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend upon is :
(A) a (B) p (C) d (D) x

Q.7 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then :
(A) $x = 3, y = 1$ (B) $x = 1, y = 3$ (C) $x = 0, y = 3$ (D) $x = 0, y = 0$

Q.8 (i) If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$ then $f(100)$ is equal to :
(A) 0 (B) 1 (C) 100 (D) -100

(ii) Let a, b, c, d be real numbers in G.P. If u, v, w satisfy the system of equations,
$u + 2v + 3w = 6$
$4u + 5v + 6w = 12$
$6u + 9v = 4$ then show that the roots of the equation,
$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2] x + u + v + w = 0$ and
$20x^2 + 10(a-d)^2 x - 9 = 0$ are reciprocals of each other .

Q.9 If the system of equations $x - Ky - z = 0$, $Kx - y - z = 0$ and $x + y - z = 0$ has a non zero solution, then the possible values of K are
(A) $-1, 2$ (B) $1, 2$ (C) $0, 1$ (D) $-1, 1$

Q.10 Prove that for all values of θ, $\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$

Q.11 Find the real values of r for which the following system of linear equations has a non-trivial solution . Also find the non-trivial solutions :
$2rx - 2y + 3z = 0$
$x + ry + 2z = 0$
$2x + rz = 0$

Q.12 Solve for x the equation

$$\begin{vmatrix} a^2 & a & 1 \\ \sin(n+1)x & \sin nx & \sin(n-1)x \\ \cos(n+1)x & \cos nx & \cos(n-1)x \end{vmatrix} = 0$$

Q.13 Test the consistency and solve them when consistent, the following system of equations for all values of λ:

$$x + y + z = 1$$
$$x + 3y - 2z = \lambda$$
$$3x + (\lambda + 2)y - 3z = 2\lambda + 1$$

[REE 2001 (Mains) , 5 out of 100]

Q.14 Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$ represents a straight line.

Q.15 The number of values of k for which the system of equations

$$(k + 1)x + 8y = 4k$$
$$kx + (k + 3)y = 3k - 1$$

has infinitely many solutions is

(A) 0 (B) 1 (C) 2 (D) inifinite

Q.16 The value of λ for which the system of equations $2x - y - z = 12$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ has no solution is

(A) 3 (B) –3 (C) 2 (D) –2

ANSWER KEY [EXERCISE-1]

Q 4. -1 **Q 11.** $(ab' - a'b)(bc' - b'c)(ca' - c'a)$ **Q 14.** $x = -1$ or $x = -2$

Q 15. $x = 0$ or $x = \pm\sqrt{\frac{3}{2}\left(a^2 + b^2 + c^2\right)}$ **Q19.** $\lambda^2\,(a^2 + b^2 + c^2 + \lambda)$

Q 23. If $ab + bc + ca \le 0$, then $x = 0$ is the only real root ; If $ab + bc + ca > 0$, then $x = 0$ or $x = \pm\sqrt{\frac{ab + bc + ca}{3}}$

Q 24. $x = 4$ **Q 28.** Triangle ABC is isosceles.

EXERCISE-2

Q 1. $x = -7$, $y = -4$

Q 2. **(a)** $x = 2$, $y = -1$, $z = 1$; consistent

(b) $x = \frac{13}{3}$, $y = -\frac{7}{6}$, $z = -\frac{35}{6}$; consistent **(c)** inconsistent

Q 3. $x = -(a + b + c)$, $y = ab + bc + ca$, $z = -abc$

Q 4. $K = \frac{33}{2}$, $x : y : z = -\frac{15}{2} : 1 : -3$ **Q7.** 2

Q 9. (a) $\lambda \ne 3$ **(b)** $\lambda = 3, \mu = 10$ **(c)** $\lambda = 3, \mu \ne 10$

Q 10. $x = 1 + 2K$, $y = -3K$, $z = K$, when $p = 1$; $x = 2K$, $y = 1 - 3K$, $z = K$ when $p = 2$; where $K \in R$

Q 11. If $K \ne 2$, $\frac{x}{2(K+6)} = \frac{y}{2K+3} = \frac{z}{6(K-2)} = \frac{1}{2\left(K^2+2K+15\right)}$

If $K = 2$, then $x = \lambda$, $y = \frac{1-2\lambda}{2}$ and $z = 0$ where $\lambda \in R$

Q 12. If $\alpha \ne 1$ or -2, unique solution ;
If $\alpha = -2$ & $m + n + p = 0$, infinite solution ;
If $\alpha = -2$ & $m + n + p \ne 0$, no solution ;
If $\alpha = 1$, infinite solution if $m = n = p$;

If $\alpha = 1$, no solution if $m \neq n$ or $n \neq p$ or $p \neq m$

Q 13. $t = 0$ or 3 ; $x : y : z = 1 : 1 : 1$ **Q 14.** $x = \frac{c-b}{a+b+c}$, $y = \frac{a-c}{a+b+c}$, $z = \frac{b-a}{a+b+c}$

Q 17. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Q18. If $\lambda \neq -5$ then $x = \frac{4}{7}$; $y = -\frac{9}{7}$ and $z = 0$;

If $\lambda = 5$ then $x = \frac{4-5K}{7}$; $y = \frac{13K-9}{7}$ and $z = K$ where $K \in R$

EXERCISE-3

Q 1. **(i)** $p \neq 2$, $q = 3$ **(ii)** $p \neq 2$ & $q \neq 3$ **(iii)** $p = 2$

Q 2. **(i)** d **(ii)** a

Q 3. $\frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$

Q 4. for $c = 0$, $x = -3$, $y = 3$; for $c = -10$, $x = -\frac{1}{2}$, $y = \frac{4}{3}$

Q 5. $k = 1 : (5t+1, -3t, t)$; $k = 2 : (5t-1, 1-3t, t)$ for $t \in R$; no solution

Q 6. B **Q 7.** D **Q 8. (i)** A **Q9.** D

Q 11. $r = 2$; $x = k$; $y = \frac{k}{2}$; $z = -k$ where $k \in R - \{0\}$ **Q 12.** $x = n\pi, n \in I$

Q 13. If $\lambda = 5$, system is consistent with infinite solution given by $z = K$, $y = \frac{1}{2}(3K+4)$ and $x = -\frac{1}{2}(5K+2)$ where $K \in R$

If $\lambda \neq 5$, system is consistent with unique solution given by $x = \frac{1}{3}(1-\lambda)$; $x = \frac{1}{3}(\lambda+2)$ and $y = 0$.

Q15. B **Q.16** D

