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STUDY PACKAGE

Subject : Mathematics

Topic : Area Under Curve (Quadrature)

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7. 14 Yrs. Que. from AIEEE (JEE Main)

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Area Under Curve

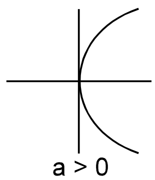
1. Curve Tracing :

To find the approximate shape of a curve, the following procedure is adopted in order:

(a) Symmetry:

(i) Symmetry about x – axis:

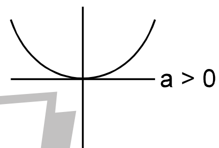
If all the powers of 'y' in the equation are even then the curve is symmetrical about the x – axis.



E.g.: $y^2 = 4ax$.

(ii) Symmetry about y – axis:

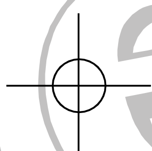
If all the powers of 'x' in the equation are even then the curve is symmetrical about the y – axis.



E.g.: $x^2 = 4ay$.

(iii) Symmetry about both axis;

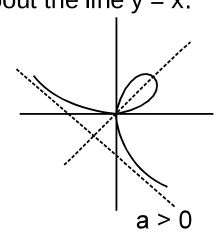
If all the powers of 'x' and 'y' in the equation are even, the curve is symmetrical about the axis of 'x' as well as 'y'.



E.g.: $x^2 + y^2 = a^2$.

(iv) Symmetry about the line y = x:

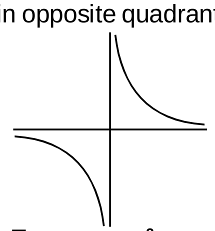
If the equation of the curve remains unchanged on interchanging 'x' and 'y', then the curve is symmetrical about the line y = x.



E.g.: $x^3 + y^3 = 3axy$.

(v) Symmetry in opposite quadrants:

If the equation of the curve remains unaltered when 'x' and 'y' are replaced by – x and – y respectively, then there is symmetry in opposite quadrants.



E.g.: $xy = c^2$.

(b) Find the points where the curve crosses the x–axis and also the y–axis.

(c) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(d) Examine if possible the intervals when f (x) is increasing or decreasing.

(e) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$.

(f) **Asymptotes :**
Asymptoto(s) is (are) line (s) whose distance from the curve tends to zero as point on curve moves towards infinity along branch of curve.

(i) If $\underset{x \to a}{\text{Lt}} f(x) = \infty$ or $\underset{x \to a}{\text{Lt}} f(x) = -\infty$, then x = a is asymptote of y = f(x)

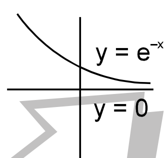
(ii) If $\underset{x \to +\infty}{\text{Lt}} f(x) = k$ or $\underset{x \to -\infty}{\text{Lt}} f(x) = k$, then y = k is asymptote of y = f(x)

(iii) If $\underset{x \to \infty}{\text{Lt}} \frac{f(x)}{x} = m_1$, $\underset{x \to \infty}{\text{Lt}} (f(x) - m_1x) = c$, then $y = m_1x + c_1$ is an asymptote. (inclined to right)

(iv) If $\underset{x \to -\infty}{\text{Lt}} \frac{f(x)}{x} = m_2$, $\underset{x \to -\infty}{\text{Lt}} (f(x) - m_2x) = c_2$, then $y = m_2x + c_2$ is an asymptote (inclined to left)

Example : Find asymptote of $y = e^{-x}$

Solution. $\underset{x \to \infty}{\text{Lim}}\ y = \underset{x \to \infty}{\text{Lim}}\ e^{-x} = 0$



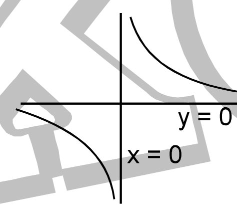
$\therefore$ y = 0 is asymptote.

Example : Find asymptotes of xy = 1 and draw graph.

Solution $y = \frac{1}{x}$

$\underset{x \to 0}{\text{Lim}}\ y = \underset{x \to 0}{\text{Lim}}\ \frac{1}{x} = \infty \Rightarrow x = 0$ is asymptote.

$\underset{x \to \infty}{\text{Lim}}\ y = \underset{x \to \infty}{\text{Lim}}\ \frac{1}{x} = 0 \Rightarrow y = 0$ is asymptote.



Example : Find asymptotes of $y = x + \frac{1}{x}$ and sketch the curve.

Solution $\underset{x \to 0}{\text{Lim}}\ y = \underset{x \to 0}{\text{Lim}} \left(x + \frac{1}{x}\right) = +\infty$ or $-\infty$

$\Rightarrow$ x = 0 is asymptote.

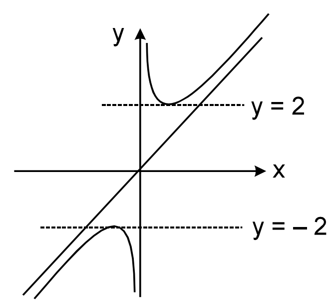
$\underset{x \to 0}{\text{Lim}}\ y = \underset{x \to 0}{\text{Lim}} \left(x + \frac{1}{x}\right) = \infty$

$\Rightarrow$ there is no asymptote of the type y = k.

$\underset{x \to \infty}{\text{Lim}}\ \frac{y}{x} = \underset{x \to \infty}{\text{Lim}} \left(1 + \frac{1}{x^2}\right) = 1$

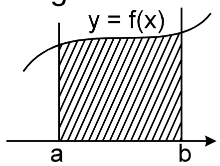
$\underset{x \to \infty}{\text{Lim}}\ (y - x) = \underset{x \to \infty}{\text{Lim}} \left(x + \frac{1}{x} - x\right) = \underset{x \to \infty}{\text{Lim}}\ \frac{1}{x} = 0$

$\therefore$ $y = x + 0 \Rightarrow y = x$ is asymptote.
A rough sketch is as follows



2. Quadrature :

(a) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve y = f(x), x-axis, x-axis, x = a and x = b is $\int_a^b f(x)\,dx$



Example : Find area bounded by the curve $y = \ell n\, x + \tan^{-1}x$ and x-axis between ordinates x = 1 and x = 2.

Solution $y = \ell n\, x + \tan^{-1}x$

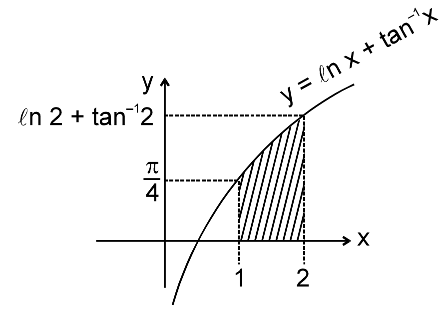
Domain $x > 0 \quad \frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$

It is increasing function

$\underset{x\to\infty}{\text{Lt}}\ y = \underset{x\to\infty}{\text{Lt}}\ (\ell n\, x + \tan^{-1}x) = \infty$

$\underset{x\to 0^+}{\text{Lt}}\ y = \underset{x\to 0^+}{\text{Lt}}\ (\ell n\, x + \tan^{-1}x) = -\infty$

A rought sketch is as follows



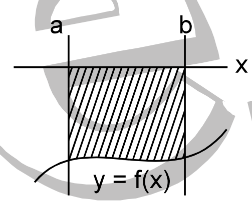
$\therefore$ Required area $= \int_1^2 (\ell n\, x + \tan^{-1}x)\, dx$

$= \left[x\, \ell n\, x - x + x\tan^{-1}x - \frac{1}{2}\ell n\,(1+x^2)\right]_1^2$

$= 2\, \ell n\, 2 - 2 + 2\tan^{-1}2 - \frac{1}{2}\, \ell n\, 5 - 0 + 1 - \tan^{-1}1 + \frac{1}{2}\, \ell n\, 2$

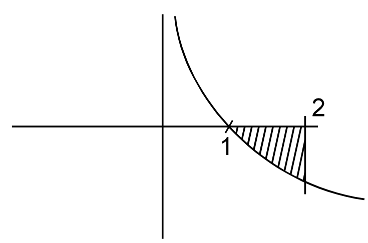
$= \frac{5}{2}\, \ell n\, 2 - \frac{1}{2}\, \ell n\, 5 + 2\tan^{-1}2 - \frac{\pi}{4} - 1$

(b) If $f(x) \le 0$ for $x \in [a, b]$, then area bounded by curve y = f(x), x-axis, x = a and x = b is $-\int_a^b f(x)\, dx$



Example : Find area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between x = 1 and x = 2.

Solution. A rought sketch of $y = \log_{\frac{1}{2}} x$ is as follows



Area $= -\int_1^2 \log_{\frac{1}{2}} x\, dx = -\int_1^2 \log_e x \,.\, \log_{\frac{1}{2}} e\, dx$

$= -\log_{\frac{1}{2}} e \,.\, [x\log_e x - x]_1^2$

$= -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 2 - 0 + 1)$

$= -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 1)$

Note : If y = f(x) does not change sign an [a, b], then area bounded by y = f(x), x-axis between ordinates x = a, x = b is $\left|\int_a^b f(x)\, dx\right|$.

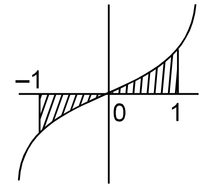
(c) If $f(x) \ge 0$ for $x \in [a,c]$ and $f(x) \le 0$ for $x \in [c,b]$ (a < c < b) then area bounded by curve y = f(x) and x–axis between x = a and x = b is $\int_a^c f(x)dx - \int_c^b f(x)dx$.

Example : Find the area bounded by $y = x^3$ and x–axis between ordinates x = – 1 and x = 1.

Solution Required area $= \int_{-1}^{0} -x^3 dx + \int_{0}^{1} x^3 dx$

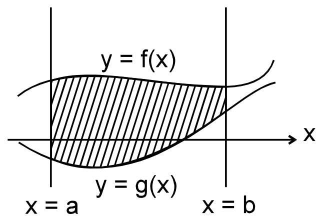
$= -\left.\frac{x^4}{4}\right]_{-1}^{0} + \left.\frac{x^3}{4}\right]_{0}^{1}$

$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$



Note : Area bounded by curve y = f(x) and x–axis between ordinates x = a and x = b is $\int_a^b |f(x)|\,dx$.

(d) If $f(x) \geq g(x)$ for $x \in [a,b]$ then area bounded by curves y = f(x) and y = g(x) between ordinates x = a and x = b is $\int_a^b (f(x) - g(x))dx$.



Example : Find the area enclosed by curve $y = x^2 + x + 1$ and its tangent at (1,3) between ordinates x = – 1 and x = 1.

Solution. $\frac{dy}{dx} = 2x + 1$

$\frac{dy}{dx} = 3$ at x = 1

Equation of tangent is

y – 3 = 3 (x – 1)

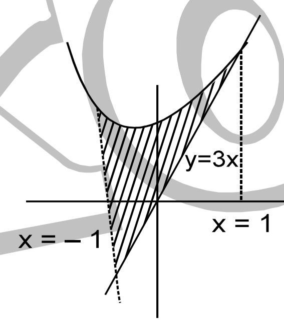
y = 3x

Required area $= \int_{-1}^{1} (x^2 + x + 1 - 3x)\,dx$

$= \int_{-1}^{1} (x^2 - 2x + 1)\,dx = \left.\frac{x^3}{3} - x^2 + x\right]_{-1}^{1}$

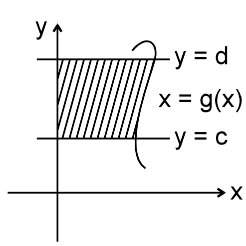
$= \left(\frac{1}{3} - 1 + 1\right) - \left(-\frac{1}{3} - 1 - 1\right)$

$= \frac{2}{3} + 2 = \frac{8}{3}$



Note : Area bounded by curves y = f(x) and y = g (x) between ordinates x = a and x = b is $\int_a^b |f(x) - g(x)|\,dx$.

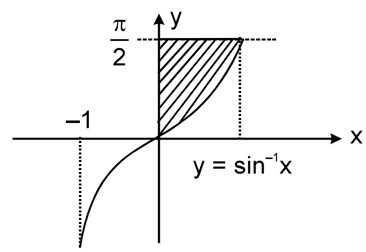
(e) If $g(y) \geq 0$ for $y \in [c,d]$ then area bounded by curve x = g(y) and y–axis between abscissa y = c and y = d is $\int_{y=c}^{d} g(y)dy$



Example : Find area bounded between $y = \sin^{-1}x$ and y–axis between $y = 0$ and $y = \frac{\pi}{2}$.

Solution $y = \sin^{-1} x$

$\Rightarrow$ $x = \sin y$



Required area $= \int_0^{\frac{\pi}{2}} \sin y \, dy$

$= -\cos y \Big]_0^{\frac{\pi}{2}} = -(0-1) = 1$

Note : The area in above example can also evaluated by integration with respect to x.

Area = (area of rectangle formed by $x = 0$, $y = 0$, $x = 1$, $y = \frac{\pi}{2}$) – (area bounded by $y = \sin^{-1}x$, x–axis between $x = 0$ and $x = 1$)

$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1}x \; dx \;\; = \frac{\pi}{2} - (x \sin^{-1}x + \sqrt{1-x^2}\,)^1$

$= \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1\right) = 1$

Some more solved examples

Example : Find the area contained between the two arms of curves $(y - x)^2 = x^3$ between $x = 0$ and $x = 1$.

Solution $(y - x)^2 = x^3 \Rightarrow y = x \pm x^{3/2}$

For arm

$y = x + x^{3/2} \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{2} x^{1/2} > 0 \qquad x \geq 0.$

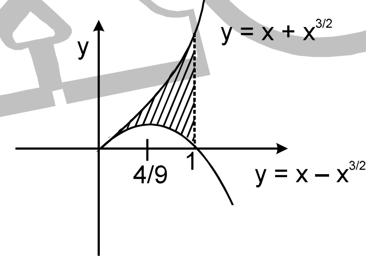
y is increasing function.

For arm

$y = x - x^{3/2} \Rightarrow \frac{dy}{dx} = 1 - \frac{3}{2} x^{1/2}$

$\frac{dy}{dx} = 0 \Rightarrow x = \frac{4}{9}$, $\frac{d^2y}{dx^2} = -\frac{3}{4} x^{-\frac{1}{2}} < 0$ at $x = \frac{4}{9}$

$\therefore$ at $x = \frac{4}{9}$ $y = x - x^{3/2}$ has maxima.



Required are a $= \int_0^1 (x + x^{3/2} - x + x^{3/2})dx$

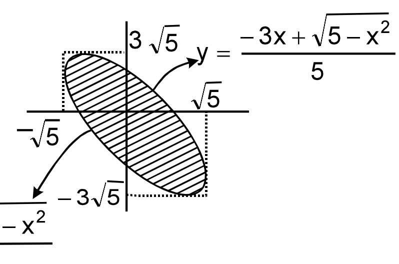
$= 2\int_0^1 x^{3/2}dx = \frac{2\,x^{5/2}}{5/2}\Bigg]_0^1 \qquad = \frac{4}{5}$

Example : Find area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$

Solution. $5y^2 + 6xy + 2x^2 - 1 = 0$

$y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2 - 1)}}{10}$

$y = \frac{-3x \pm \sqrt{5 - x^2}}{5}$



$\because$ y is real $\Rightarrow$ R.H.S. is also real.

$\Rightarrow$ $-\sqrt{5} \leq x \leq \sqrt{5}$

If $x = -\sqrt{5}$, $y = 3\sqrt{5}$

If $x = \sqrt{5}$, $y = -3\sqrt{5}$

If $x = 0$, $y = \pm \frac{1}{\sqrt{5}}$

If $y = 0$, $x = \pm \dfrac{1}{\sqrt{2}}$

Required area $= \displaystyle\int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x+\sqrt{5-x^2}}{5} - \frac{-3x-\sqrt{5-x^2}}{5} \right) dx$

$= \dfrac{2}{5} \displaystyle\int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{5-x^2}\, dx$

$= \dfrac{4}{5} \displaystyle\int_{0}^{\sqrt{5}} \sqrt{5-x^2}\, dx$

Put $x = \sqrt{5} \sin\theta : dx = \sqrt{5} \cos\theta \, d\theta$

L.L : $x = 0 \Rightarrow \theta = 0$

U.L : $x = \sqrt{5} \Rightarrow \theta = \dfrac{\pi}{2}$

$= \dfrac{4}{5} \displaystyle\int_{\theta=0}^{\frac{\pi}{2}} \sqrt{5-5\sin^2\theta}\ \sqrt{5}\cos\theta d\theta$

$= 4 \displaystyle\int_{0}^{\frac{\pi}{2}} \cos^2\theta d\theta = 4\ \frac{1}{2}\ \frac{\pi}{2} = \pi$

Example : Let A (m) be area bounded by parabola $y = x^2 + 2x - 3$ and the line $y = mx + 1$. Find the least area A(m).

Solution. Solving we obtain

$x^2 + (2 - m) x - 4 = 0$

Let α, β be roots $\Rightarrow \alpha + \beta = m - 2$, $\alpha\beta = -4$

$A(m) = \left| \displaystyle\int_{\alpha}^{\beta} (mx + 1 - x^2 - 2x + 3)\, dx \right|$

$= \left| \displaystyle\int_{\alpha}^{\beta} (-x^2 + (m-2)x + 4)\, dx \right|$

$= \left| \left(-\dfrac{x^3}{3} + (m-2)\dfrac{x^2}{2} + 4x \right)_{\alpha}^{\beta} \right|$

$= \left| \dfrac{\alpha^3 - \beta^3}{3} + \dfrac{m-2}{2}(\beta^2 - \alpha^2) + 4(\beta - \alpha) \right|$

$= |\beta - \alpha|. \left| -\dfrac{1}{3}(\beta^2 + \beta\alpha + \alpha^2) + \dfrac{(m-2)}{2}(\beta + \alpha) + 4 \right|$

$= \sqrt{(m-2)^2 + 16} \left| -\dfrac{1}{3}\left((m-2)^2 + 4\right) + \dfrac{(m-2)}{2}(m-2) + 4 \right|$

$= \sqrt{(m-2)^2 + 16} \left| \dfrac{1}{6}(m-2)^2 + \dfrac{8}{3} \right|$

$A(m) = \dfrac{1}{6}\left((m-2)^2 + 16\right)^{3/2}$

Leas $A(m) = \dfrac{1}{6}(16)^{3/2} = \dfrac{32}{3}$.

Self Practice Problems

1. Find the area between curve $y = x^2 - 3x + 2$ and x–axis

(i) bounded between x = 1 and x = 2. **Ans.** $\frac{1}{6}$

(ii) bound between x = 0 and x = 2. **Ans.** 1

2. Find the area included between curves $y = 2x - x^2$ and $y + 3 = 0$.

Ans. $\frac{32}{3}$

3. Find area between curves $y = x^2$ and $y = 3x - 2$ from x = 0 to x = 2.
Ans. 1

4. Curves y = sinx and y = cosx intersect at infinite number of points forming regions of equal area between them calculate area of one such region.

Ans. $2\sqrt{2}$

5. Find the area of the region bounded by the parabola $(y - 2)^2 = (x - 1)$ and the tangent to it at ordinate y = 3 and x–axis.
Ans. 9

6. Find the area included between $y = \tan^{-1}x$, $y = \cot^{-1}x$ and y–axis.
Ans. $\ell n2$

7. Find area common to circle $x^2 + y^2 = 2$ and the parabola $y^2 = x$.

Ans. $\frac{\pi}{3} - \frac{\sqrt{3}}{2} - \frac{2}{3}$

8. Find the area included between curves $y = \frac{4 - x^2}{4 + x^2}$ and $5y = 3|x| - 6$.

Ans. $2\pi - \frac{8}{5}$

9. Find the area bounded by the curve $|y| + \frac{1}{2} = e^{-|x|}$.

Ans. $2\,(1 - \ell n2)$

10. Find the area of loop $y^2 = x\,(x - 1)^2$.

Ans. $\frac{8}{15}$

11. Find the area enclosed by $|x| + |y| \le 3$ and $xy \ge 2$.
Ans. $3 - 4\ell n2$

12. Find are bounded by $x^2 + y^2 \le 2ax$ and $y^2 \ge ax$, $x \ge 0$.

Ans. $\left(\frac{3\pi - 8}{6}\right)a^2$.

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