11. Linear Combinations:

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, ...$ then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, ...$ for any x, y, z..... $\in$ R. We have the following results:

(a) If $\vec{a}, \vec{b}$ are non zero, non–collinear vectors then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'\, ; y = y'$

(b) **FundamentalTheorem:** Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. Then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$

i.e. There exist some uniquly x, y $\in$ R such that $x\vec{a} + y\vec{b} = \vec{r}$.

(c) If $\vec{a}, \vec{b}, \vec{c}$ are non–zero, non–coplanar vectors then:

$$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x'\ ,\ y = y'\ ,\ z = z'$$

(d) **Fundamental Theorem In Space:** Let $\vec{a}, \vec{b}, \vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$, can be uniquly expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique x,y $\in$ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(e) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are n non zero vectors, & k_1, k_2,.....k_n are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ... k_n\vec{x}_n = 0 \Rightarrow k_1 = 0, k_2 = 0.....k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are **Linearly Independent Vectors.**

(f) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are not **Linearly Independent** then they are said to be **Linearly Dependent** vectors. i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are said to be **Linearly Dependent.**

Note 1: If $k_r \neq 0$; $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 + ... + k_r\vec{x}_r + ... + k_n\vec{x}_n = 0$

$$-k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_{r-1}.\vec{x}_{r-1} + k_{r+1}.\vec{x}_{r+1} + ... + k_n\vec{x}_n$$

$$-k_r\frac{1}{k_r}\vec{x}_r = k_1\frac{1}{k_r}\vec{x}_1 + k_2\frac{1}{k_r}\vec{x}_2 + ... + k_{r-1}.\frac{1}{k_r}\vec{x}_{r-1} + ... + k_n\frac{1}{k_r}\vec{x}_n$$

$$\vec{x}_r = c_1\vec{x}_1 + c_2\vec{x}_2 + ...+ c_{r-1}\vec{x}_{r-1} + c_r\vec{x}_{r-1} + ... + c_n\vec{x}_n$$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors.

$\vec{x}_1, \vec{x}_2, ...\vec{x}_{r-1}, \vec{x}_{r+1}, ...\vec{x}_n$

Hence $\vec{x}_r$ with $\vec{x}_1, \vec{x}_2, ...\vec{x}_{r-1}, \vec{x}_{r+1}....\vec{x}_n$ forms a linearly dependent set of vectors.

Note 2: ☞ If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a **Linear Combination** of vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ Also, $\vec{a}$, $\hat{i}$, $\hat{j}$, $\hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.

☞ $\hat{i}, \hat{j}, \hat{k}$ are **Linearly Independent** set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = K_2 = K_3 = 0$

☞ Two vectors $\vec{a}$ & $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0 \Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq 0$ then $\vec{a}$ & $\vec{b}$ are linearly independent.

☞ If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}, \vec{b}, \vec{c}] = 0$, conversely, if $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then the vectors are linearly independent.

Solved Example: Given A that the points $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$, $-7\vec{b} + 10\vec{c}$, A, B, C have position vector prove that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Solution. Let A, B, C be the given points and O be the point of reference then

$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\overrightarrow{OB} = 2\vec{a} + 3\vec{b} - 4\vec{c}$ and $\overrightarrow{OC} = -7\vec{b} + 10\vec{c}$

Now $\overrightarrow{AB}$ = p.v. of B – p.v. of A

$= \overrightarrow{OB} - \overrightarrow{OA} = (\vec{a} + 5\vec{b} - 7\vec{c}) = -\overrightarrow{AB}$

$\therefore\ \overrightarrow{AC} = \lambda\overrightarrow{AB}$ where $\lambda = -1$. Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent

Solved Example: Prove that the vectors $5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$ are linearly dependent $\vec{a}, \vec{b}, \vec{c}$ being linearly independent vectors.

Solution. We know that if these vectors are linearly dependent, then we can express one of them as a linear combination of the other two.

Now let us assume that the given vector are coplanar, then we can write

$5\vec{a} + 6\vec{b} + 7\vec{c} = \ell(7\vec{a} - 8\vec{b} + 9\vec{c}) + m(3\vec{a} + 20\vec{b} + 5\vec{c})$
where ℓ, m are scalars

Comparing the coefficients of $\vec{a}, \vec{b}$ and $\vec{c}$ on both sides of the equation
$5 = 7\ell + 3$ ....(i) $6 = -8\ell + 20m$ ....(ii)
$7 = 9\ell + 5m$ ....(iii)
From (i) and (iii) we get

$4 = 8\ell \Rightarrow \ell = \frac{1}{2} = m$ which evidently satisfies (ii) equation too.

Hence the given vectors are linearly dependent .

Self Practice Problems :

1. Does there exist scalars u, v, w such that $u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 = \vec{i}$ where $\vec{e}_1 = \vec{k}$, $\vec{e}_2 = \vec{j} + \vec{k}$, $\vec{e}_3 = -\vec{j} + 2\vec{k}$?
Ans. No

2. Consider a base $\vec{a}, \vec{b}, \vec{c}$ and a vector $-2\vec{a} + 3\vec{b} - \vec{c}$. Compute the co-ordinates of this vector relatively to the base p, q, r where $\vec{p} = 2\vec{a} - 3\vec{b}$, $\vec{q} = \vec{a} - 2\vec{b} + \vec{c}$, $\vec{r} = -3\vec{a} + \vec{b} + 2\vec{c}$. **Ans.** (0, –7/5, 1/5)

3. If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A} = (x + 4y)\vec{a} + (2x + y + 1)\vec{b}$ and $\vec{B} = (y - 2x + 2)\vec{a} + (2x - 3y - 1)\vec{b}$, find x and y such that $3\vec{A} = 2\vec{B}$. **Ans.** x = 2, y = –1

4. If vectors $\vec{a}, \vec{b}, \vec{c}$ be linearly independent, then show that :(i) $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - 4\vec{c}$, $-\vec{b} + 2\vec{c}$ are linearly dependent (ii) $\vec{a} - 3\vec{b} + 2\vec{c}$, $-2\vec{a} - 4\vec{b} - \vec{c}$, $3a + 2\vec{b} - \vec{c}$ are linearly independent.

5. Given that $\hat{i} - \hat{j}$, $\hat{i} - 2\hat{j}$ are two vectors. Find a unit vector coplanar with these vectors and perpendicular to the first vector $\hat{i} - \hat{j}$. Find also the unit vector which is perpendicular to the plane of the two given vectors. Do you thus obtain an orthonormal triad? **Ans.** $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$; k; Yes

6. If with reference to a right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$ $\vec{\alpha} = 3\vec{i} - \vec{j}$, $\vec{\beta} = 2\vec{i} + \vec{j} - 3\vec{k}$ express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$ where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ & $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.
Ans. $\vec{\beta}_1 = \frac{3}{2}\vec{i} - \frac{1}{2}\vec{j}$, $\vec{\beta}_2 = \frac{1}{2}\vec{i} + \frac{3}{2}\vec{j} - 3\vec{k}$

7. Prove that a vector $\vec{r}$ in space can be expressed linearly in terms of three non-coplanar, non-null vectors $\vec{a}, \vec{b}, \vec{c}$ in the form $\vec{r} = \frac{[\vec{r}\,\vec{b}\,\vec{c}]\,\vec{a} + [\vec{r}\,\vec{c}\,\vec{a}]\,\vec{b} + [\vec{r}\,\vec{a}\,\vec{b}]\,\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Note: **Test Of Collinearity:** Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where x + y + z = 0.

Note: Test Of Coplanarity: Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where, x + y + z + w = 0.

Solved Example Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors.
Solution. Let, the given vectors be coplanar.
Then one of the given vectors is expressible in terms of the other two.
Let $2\vec{a} - \vec{b} + 3\vec{c} = x(\vec{a} + \vec{b} - 2\vec{c}) + y(\vec{a} + \vec{b} - 3\vec{c})$, for some scalars x and y.
$\Rightarrow$ $2\vec{a} - \vec{b} + 3\vec{c} = (x + y)\vec{a}\ (x + y)\vec{b} + (-2x - 3y)\vec{c}$
$\Rightarrow$ 2 = x + y, –1 = x + y and 3 = 2x – 3y.
Solving, first and third of these equations, we get x = 9 and y = –7.
Clearly, these values do not satisfy the thrid equation.
Hence, the given vectors are not coplanar.

Solved Example: Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Solution. Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So, let

$$\overrightarrow{PQ} = x\,\overrightarrow{PR} + y\,\overrightarrow{PS}$$

$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = x(\vec{a} + \vec{b} - \vec{c}) + y(-\vec{a} - 9\vec{b} - 7\vec{c}) \Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\vec{a} + (x - 9y)\vec{b} + (-x + 7y)\vec{c}$

$\Rightarrow x - y = -1,\ x - 9y = -5,\ -x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first of these three equations, we get $x = -\frac{1}{2}$, $y = \frac{1}{2}$.

These values also satisfy the third equation. Hence, the given four points are coplanar.

Self Practice Problems :

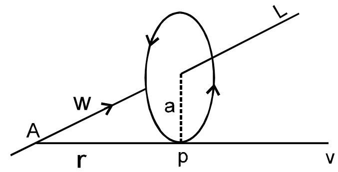
1. If, $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that the terminal A, B, C, D of these vectors are coplanar. Find the point at which AC and BD meet. Find the ratio in which P divides AC and BD.

2. Show that the vector $\vec{a} - \vec{b} + \vec{c}$, $\vec{b} - \vec{c} - \vec{a}$ and $2\vec{a} - 3\vec{b} - 4\vec{c}$ are non-coplanar, where $\vec{a}, \vec{b}, \vec{c}$, are any non-coplanar vectors.

3. Find the value of λ for which the four points with position vectors $-\hat{j} - \hat{k}$, $4\hat{i} + 5\hat{j} + \lambda\hat{k}$. $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar. **Ans.** $\lambda = 1$

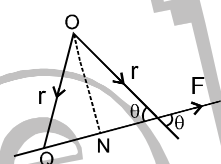
12. Application Of Vectors:

(a) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $\vec{W} = \vec{F}.\vec{s}$ **(b)** The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V} = \vec{w} \times \vec{r}$ where $\vec{r}$ is the pv of the point P.



(c) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the pv of P wrt 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}, \vec{F}$ & $\vec{M}$ form a right handed system.

(d) Moment of the couple $= (\vec{r}_1 - \vec{r}_2) \times \vec{F}$ where $\vec{r}_1 \& \vec{r}_2$ are pv's of the point of the application of the forces $\vec{F} \& -\vec{F}$.



Solved Example: Forces of magnitudes 5 and 3 units acting in the directions $6\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} + 6\hat{k}$ respectively act on a particle which is displaced from the point (2, 2, –1) to (4, 3, 1). Find the work done by the forces.

Solution. Let $\vec{F}$ be the resultant force and $\vec{d}$ be the displacement vector. Then,

$$\vec{F} = 5\frac{(6\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{36+4+9}} + 3\frac{(3\hat{i} + 2\hat{j} + 6\hat{k})}{\sqrt{9+4+36}} = \frac{1}{7}(39\hat{i} + 4\hat{j} + 33\hat{k})$$

and, $\vec{d} = (4\hat{i} + 3\hat{j} + \hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$

$\therefore$ Total work done $= \vec{F} \cdot \vec{d} = \frac{1}{7}(39\hat{i} + 4\hat{j} + 33\hat{k}) \cdot (2\hat{i} + \hat{j} + 2\hat{k})$

$$= \frac{1}{7}(78 + 4 + 66) = \frac{148}{7} \text{ units.}$$

Self Practice Problems :1. A point describes a circle uniformly in the $\hat{i}$, $\hat{j}$ plane taking 12 seconds to complete one revolution. If its initial position vector relative to the centre is $\hat{i}$, and the rotation is from $\hat{i}$ to $\hat{j}$, find the position vector at the end of 7 seconds. Also find the velocity vector. **Ans.** 1 / 2 $(\hat{j} - \sqrt{3}\,\hat{i})$, p/12 $(\hat{i} - \sqrt{3}\,\hat{j})$

2. The force represented by $3\hat{i} + 2\hat{k}$ is acting through the point $5\hat{i} + 4\hat{j} - 3\hat{k}$. Find its moment about the point $\hat{i} + 3\hat{j} + \hat{k}$. **Ans.** $2\hat{i} - 20\hat{j} - 3\hat{k}$

3. Find the moment of the comple formed by the forces $5\hat{i} + \hat{k}$ and $-5\hat{i} - \hat{k}$ acting at the points (9, –1, 2) and (3, –2, 1) respectively **Ans.** $\hat{i} - \hat{j} - 5\hat{k}$

Miscellaneous Solved Examples

Solved Example: Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

Solution. We have, $\overrightarrow{AB}$ = Position vector of B – Position vector of A

$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$= (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$

and, $\overrightarrow{CA}$ = Position vector of A – Position vector of C

$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$

Since $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = (-\hat{i} - 2\hat{j} - 6\hat{k}) + (2\hat{i} - \hat{j} + \hat{k}) + (-\hat{i} + 3\hat{j} + 5\hat{k}) = \vec{0}$

So, A, B and C are the vertices of a triangle.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k}) = -2 - 3 + 5 = 0$

$\Rightarrow \quad \overrightarrow{BC} \perp \overrightarrow{CA} \quad \Rightarrow \quad \angle BCA = \dfrac{\pi}{2}$ Hence, ABC is a right angled triangle.

Since a is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Therefore

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k})}{\sqrt{(-1)^2 + (-2)^2 + (-6)^2}\sqrt{1^2 + (-3)^2 + (-5)^2}}$$

$$= \frac{-1 + 6 + 30}{\sqrt{1 + 4 + 36}\sqrt{1 + 9 + 25}} = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}} \qquad A = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\cos B = \frac{\overrightarrow{BA}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(\hat{i} + 2\hat{j} + 6\hat{k})\,(2\hat{i} - \hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + 6^2}\sqrt{2^2 + (-1)^2 + (1)^2}} \Rightarrow \cos B = \frac{2 - 2 + 6}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}} \Rightarrow B = \cos^{-1}\sqrt{\frac{6}{41}}$$

Solved Example: If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Solution.: Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, therefore $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$ ....(i)

Now, $\left|\vec{a} + \vec{b} + \vec{c}\right|^2$

$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a}$

$= |\vec{a}|^2 \,|+\, |\vec{b}|^2 + |\vec{c}|^2$ [Using (i)]

$= 3\lambda^2$ $[\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$

$\therefore \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\lambda$ ....(ii)

Suppose $\vec{a} + \vec{b} + \vec{c}$ makes angles θ_1, θ_2, θ_3 with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a} \cdot (\vec{a} + \vec{b} + \vec{c})}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \frac{\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \frac{|\vec{a}|}{|\vec{a} + \vec{b} + \vec{c}|} = \frac{\lambda}{\sqrt{3}\lambda} = \frac{1}{\sqrt{3}} \qquad \text{[Using (ii)]}$$

$\therefore \qquad \theta_1 \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Similarly, $\theta_2 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ $\quad \therefore \quad \theta_1 = \theta_2 = \theta_3$.

Hence, $\vec{a} + \vec{b} + \vec{c}$ is equally inclineded with $\vec{a}, \vec{b}$ and $\vec{c}$

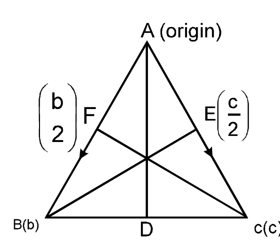
Solved Example: Prove using vectors : If two medians of a triangle are equal, then it is isosceles.

Solution. : Let ABC be a triangle and let BE and CF be two equal medians. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

P.V. of E = $\dfrac{1}{2}\vec{c}$ and, P.V. of F = $\dfrac{1}{2}\vec{b}$ $\quad \therefore \quad \overrightarrow{BE} = \dfrac{1}{2}(\vec{c} - 2\vec{b})$

$\overrightarrow{CF} = \dfrac{1}{2}(\vec{b} - 2\vec{c})$

Now, BE = CF $\Rightarrow |\overrightarrow{BE}| = |\overrightarrow{CF}|$



$\Rightarrow \quad |\overrightarrow{BE}|^2 = |\overrightarrow{CF}|^2 \Rightarrow \left|\dfrac{1}{2}(\vec{c} - 2\vec{b})\right|^2 = \left|\dfrac{1}{2}(\vec{b} - 2\vec{c})\right|^2$

$\Rightarrow \quad \dfrac{1}{4}|\vec{c} - 2\vec{b}|^2 = \dfrac{1}{4}|\vec{b} - 2\vec{c}|^2 \Rightarrow |\vec{c} - 2\vec{b}|^2 = |\vec{b} - 2\vec{c}|^2$

$\Rightarrow \quad (\vec{c} - 2\vec{b}) \cdot (\vec{c} - 2\vec{b}) = (\vec{b} - 2\vec{c}) \cdot (\vec{b} - 2\vec{c})$

$\Rightarrow \quad \vec{c}\,.\,\vec{c} - 4\vec{b}\,.\,\vec{c} + 4\vec{b}\,.\,\vec{b} = \vec{b}\,.\,\vec{b} - 4\vec{b}\,.\,\vec{c} + 4\vec{c}\,.\,\vec{c}$

$\Rightarrow \quad |\vec{c}|^2 - 4\vec{b}\,.\,\vec{c} + 4\,|\vec{b}|^2 = |\vec{b}|^2 - 4\vec{b}\,.\,\vec{c} + 4\,|\vec{c}|^2$

$\Rightarrow \quad 3\,|b|^2 = 3\,|\vec{c}|^2 \qquad \Rightarrow \qquad |b|^2 = |\vec{c}|^2$

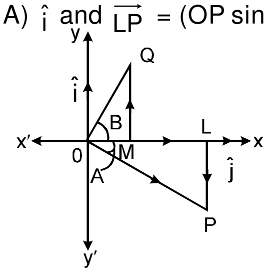
$\Rightarrow \quad AB = AC$ Hence, triangle ABC is an isosceles triangle.

Solved Example: Using vectors : Prove that cos (A + B) = cos A cos B – sin A sin B

Solution. Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively. Let $\angle XOP = A$ and $\angle XOQ = B$. Drawn $PL \perp OX$ and $QM \perp OX$.

Clearly angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is A + B

In ΔOLP, OL = OP cos A and LP = OP sin A. Therefore $\overrightarrow{OL}$ = (OP cos A) $\hat{i}$ and $\overrightarrow{LP}$ = (OP sin A) $(-\hat{j})$



Now. $\overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OP}$

$\Rightarrow \quad \overrightarrow{OP} = OP\,[(\cos A)\,\hat{i} - (\sin A)\,\hat{j}]$

In ΔOMQ, OM = OQ cos B and MQ = OQ sin B.
Therefore,

$\overrightarrow{OM} = (OQ\cos B)\,\hat{i}$, $\overrightarrow{MQ} = (OQ\sin B)\,\hat{j}$

Now, $\overrightarrow{OM} + \overrightarrow{MQ} = \overrightarrow{OQ}$

From (i) and (ii), we get

$\overrightarrow{OP}\cdot\overrightarrow{OQ} = OP\,[(\cos A)\,\hat{i} - (\sin A)\,\hat{j}]\,.\,OQ\,[(\cos B)\,\hat{i} + (\sin B)\,\hat{j}]$

$= OP\,.\,OQ\,[\cos A\cos B - \sin A\sin B]$

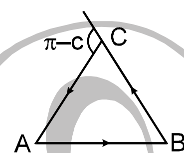
But, $\overrightarrow{OP}\cdot\overrightarrow{OQ} = |\overrightarrow{OP}|\,|\overrightarrow{OQ}|\cos(A+B) = OP\,.\,OQ\cos(A+B)$

$\therefore \quad OP\,.\,OQ\cos(A+B) = OP\,.\,OQ\,[\cos A\cos B - \sin A\sin B]$

$\Rightarrow \quad \cos(A+B) = \cos A\cos B - \sin A\sin B$

Solved Example: Prove that in any triangle ABC

(i) $c^2 = a^2 + b^2 - 2ab\cos C$ (ii) $c = b\cos A + a\cos B$.



Solution. (i) In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

or, $\overrightarrow{BC} + \overrightarrow{CA} = -\overrightarrow{AB}$ ......(i)

Squaring both sides

$(\overrightarrow{BC})^2 + (\overrightarrow{CA})^2 + (\overrightarrow{BC})\cdot\overrightarrow{CA} + (\overrightarrow{AB})^2$

$\Rightarrow \quad a^2 + b^2 + 2\,(\overrightarrow{BC}\,.\,\overrightarrow{CA}) = c^2 \qquad \Rightarrow \qquad c^2 = a^2 + b^2 = 2\,ab\cos(\pi - C)$

$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos C$

(ii) $(\overrightarrow{BC} + \overrightarrow{CA})\cdot\overrightarrow{AB} = -\overrightarrow{AB}\cdot\overrightarrow{AB}$

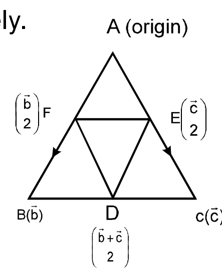
$\overrightarrow{BC}\cdot\overrightarrow{AB} + \overrightarrow{CA}\cdot\overrightarrow{AB} = -c^2$

$-ac\cos B - bc\cos A = -c^2$

$a\cos B + b\cos A = c$.

Solved Ex.: If D, E, F are the mid-points of the sides of a triangle ABC, prove by vector method that area of $\Delta DEF = \frac{1}{4}$ (area of ΔABC)

Solution. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of D, E and F are $\frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.



Now, $\overrightarrow{DE} = \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b}+\vec{c}) = \frac{-\vec{b}}{2}$

and $\overrightarrow{DF} = \frac{1}{2}\vec{b} - \frac{1}{2}((\vec{b}+\vec{c}) = \frac{-\vec{c}}{2}$

$\therefore$ Vector area of ΔDEF $\frac{1}{2}(\overrightarrow{DE}\times\overrightarrow{DF}) = \frac{1}{2}\left(\frac{-\vec{b}}{2}\times\frac{-\vec{c}}{2}\right)$

$= \frac{1}{8}(\vec{b}\times\vec{c}) = \frac{1}{4}\left\{\frac{1}{2}(\overrightarrow{AB}\times\overrightarrow{AC})\right\}$

$= \frac{1}{4}$(vector area of ΔABC) Hence, area of $\Delta DEF = \frac{1}{4}$ area of ΔABC.

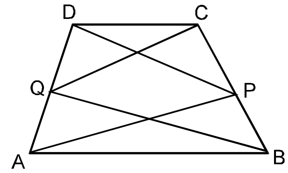
Solved Example: P, Q are the mid-points of the non-parallel sides BC and AD of a trapezium ABCD. Show that $\Delta APD = \Delta CQB$.

Solution. Let $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AD} = \vec{d}$

Now DC is parallel to AB $\Rightarrow$ there exists a acalar t sush that $\overrightarrow{DC} = t\,\overrightarrow{DB} = t\,\vec{b}$

$\therefore \quad \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{d} + t\,\vec{b}$

The position vectors of P and Q are $\frac{1}{2}$ $(\vec{b}+\vec{d}+t\,\vec{b})$ and $\frac{1}{2}$ $\vec{d}$ respectively.



Now $2\Delta\ \overrightarrow{APD} = \overrightarrow{AP} \times \overrightarrow{AD}$

$= \frac{1}{2}\ (\vec{b}+\vec{d}+t\,\vec{b}) \times \vec{d} = \frac{1}{2}\ (1+t)\ (\vec{b}\times\vec{d})$

Also $2\Delta\ \overrightarrow{CQB} = \overrightarrow{CQ} \times \overrightarrow{CB} = \left[\frac{1}{2}\vec{d}-(\vec{d}+t\vec{b})\right] \times [\vec{b}-(\vec{d}+t\,\vec{b})]$

$= \left[-\frac{1}{2}d-t\,\vec{b}\right]\times[-d+(1-t)\,b] \quad = -\frac{1}{2}(1-t)\,\vec{d}\times\vec{b}+t\,\vec{b}\times\vec{d}$

$= \frac{1}{2}(1-t+2t)\,\vec{b}\times\vec{d} \quad = \frac{1}{2}(1+t)\,\vec{b}\times\vec{d} \quad = \quad 2\Delta\overrightarrow{APD}$ Hence the result.

Solved Example: Let $\vec{u}$ and $\vec{v}$ are unit vectors and $\vec{w}$ is a vector such that $\vec{u}\times\vec{v}+\vec{u} = \vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$ then find the value of $[\vec{u}\ \ \vec{v}\ \ \vec{w}]$.

Solution. Given $\vec{u}\times\vec{v}+\vec{u} = \vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$

$\Rightarrow$ $(\vec{u}\times\vec{v}+\vec{u}) \times \vec{u}=\vec{w}\times\vec{u}$ $\Rightarrow$ $(\vec{u}\times\vec{v}) \times \vec{u}+\vec{u}\times\vec{u} = \vec{v}$ (as, $\vec{w}\times\vec{u}=\vec{v}$)

$\Rightarrow$ $(\vec{u}\,.\,\vec{u})\ \vec{v}-(v\,.\,\vec{u})\ \vec{u}+\vec{u}\times\vec{u}=\vec{v}$ (using $\vec{u}\ .\ \vec{u} = 1$ and $\vec{u}\times\vec{u} = 0$, since unit vector)

$\Rightarrow$ $\vec{v}-(\vec{v}\,.\,\vec{u})\,\vec{u}=\vec{v}$ $\Rightarrow$ $(\vec{u}\,.\,\vec{v})\,\vec{u}=\vec{0}$

$\Rightarrow$ $\vec{u}\,.\,\vec{v} = 0$ (as; $\vec{u} \neq 0$) ....(i)

$\Rightarrow$ $\vec{u}\,.\,(\vec{v}\times\vec{w})$

$= \vec{u}\,.\,(\vec{v}\times(\vec{u}\times\vec{v}+\vec{u}))$ (given $\vec{w}=\vec{u}\times\vec{v}+u$)

$= \vec{u}\,.\,(\vec{v}\times(\vec{u}\times\vec{v})+\vec{v}\times\vec{u})$ $= \vec{u}\,.\,((\vec{v}\,.\,\vec{v})\,\vec{u}-(\vec{v}\,.\,\vec{u})\,\vec{v}+\vec{v}\times\vec{u})$

$= \vec{u}\,.\,(|\vec{v}|^2\,u-0+\vec{v}\times\vec{u})$ (as; $\vec{u}\,.\,\vec{v}=0$ from (i))

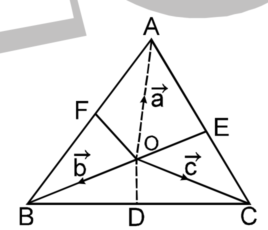
$= |\vec{v}|^2\ (\vec{u}\,.\,\vec{u}) - \vec{u}\ .\ (\vec{v}\times\vec{u})$ $= |\vec{v}|^2\ |\vec{u}|^2 - 0$ (as, $[\vec{u}\ \ \vec{v}\ \ \vec{u}] = 0$)

$= 1$ (as; $|\vec{u}| = |\vec{v}| = 1$) $\therefore$ $[\vec{u}\ \ \vec{v}\ \ \vec{w}] = 1$

Sol. Ex.: In any triangle, show that the perpendicular bisectors of the sides are concurrent.

Solution. Let ABC be the triangle and D, E and F are respectively middle points of sides BC, CA and AB. Let the perpendicular of D and E meet at O join OF. We are required to prove that OF is $\perp$ to AB. Let the position vectors of A, B, C with O as origin of reference be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore\ \overrightarrow{OD} = \frac{1}{2}\ (\vec{b}+\vec{c})$, $\overrightarrow{OE} = \frac{1}{2}\ (\vec{c}+\vec{a})$ and $\overrightarrow{OF} = \frac{1}{2}\ (\vec{a}+\vec{b})$



Also $\overrightarrow{BC} = \vec{c} - \vec{b}$, $\overrightarrow{CA} = \vec{a} - \vec{c}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$

Since OD $\perp$ BC, $\frac{1}{2}\ (\vec{b}+\vec{c})\,.\,(\vec{c}-\vec{b}) = 0$

$\Rightarrow b^2 = c^2$ ....(i)

Similarly $\frac{1}{2}\ (\vec{c}+\vec{a})\,.\,(\vec{a}+\vec{c}) = 0$

$\Rightarrow a^2 = c^2$ ....(ii)

from (i) and (ii) we have $a^2 - b^2 = 0$

$\Rightarrow (\vec{a}+\vec{b})\,.\,(\vec{b}+\vec{a}) = 0$ $\Rightarrow \frac{1}{2}(\vec{b}+\vec{a})\,.\,(\vec{b}-\vec{a}) = 0$

Solved Example: A, B, C, D are four points in space. using vector methods, prove that $AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$ what is the implication of the sign of equaility.

Solution.: Let the position vector of A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively then

$AC^2 + BD^2 + AD^2 + BC^2 = (\vec{c}-\vec{a})\ .\ (\vec{c}-\vec{a}) + (\vec{d}-\vec{b})\ .\ (\vec{d}-\vec{b}) + (\vec{d}-\vec{a})\ .\ (\vec{d}-\vec{a}) + (\vec{c}-\vec{b})\ .\ (\vec{c}-\vec{b})$

$= |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a}\,.\,\vec{c} + |\vec{d}|^2 + |\vec{b}|^2 - 2\vec{d}\,.\,\vec{b} + |\vec{d}|^2 + |\vec{a}|^2 - 2\vec{a}\,.\,\vec{d} + |\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b}\,.\,\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\,.\,\vec{b} + |\vec{c}|^2 + |\vec{d}|^2 - 2\vec{c}\,.\,\vec{d} + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2$

$+ 2\vec{a}\,.\,\vec{b} + 2\vec{c}\,.\,\vec{d} - 2\vec{a}\,.\,\vec{c} - 2\vec{b}\,.\,\vec{d} - 2\vec{a}\,.\,\vec{d} - 2\vec{b}\,.\,\vec{c}$

$= (\vec{a}-\vec{b})\ .\ (\vec{a}-\vec{b}) + (\vec{c}-\vec{d})\ .\ (\vec{c}-\vec{d}) + (\vec{a}-\vec{b}-\vec{c}-\vec{d}) \geq AB^2 + CD^2$

$= AB^2 + CD^2 + (\vec{a}+\vec{b}-\vec{c}-\vec{d})\ .\ (\vec{a}+\vec{b}-\vec{c}-\vec{d}) \leq AB^2 + CD^2$

$\therefore$ $AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$

for the sign of equality to hold, $\vec{a}+\vec{b}-\vec{c}-\vec{d} = 0$

$\vec{a}-\vec{c}=\vec{d}-\vec{b}$

$\Rightarrow$ $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are collinear the four points A, B, C, D are collinear