विध्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचितः मानव धर्म प्रणेता
सद्‌गुरु श्री रणछोड़दासजी महाराज

STUDY PACKAGE

Subject : Mathematics

Topic : CONIC SECTION

PARABOLA, ELLIPSE, HYPERBOLA

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Index

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Parabola

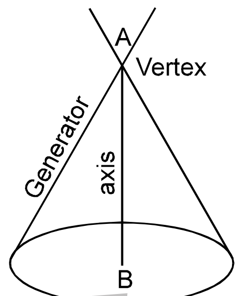
1. Conic Sections:

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.
- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

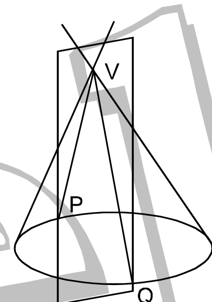
2. Section of right circular cone by different planes

A right circular cone is as shown in the



Circular Base

(i) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the



(ii) Section of a right circular cone by a plane parallel to its base is a circle as shown in the **figure – 3**.

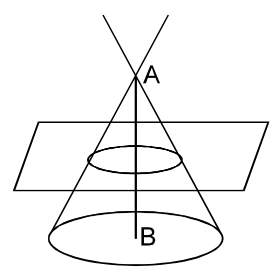
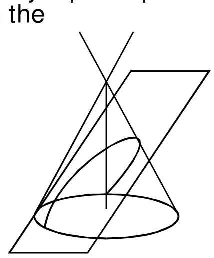


Figure- 3

(iii) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in the



Figure

(iv) Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the **figure – 5 & 6**.

Figure -5

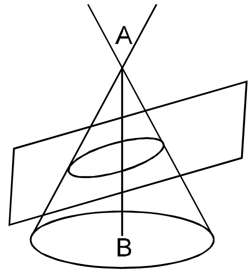
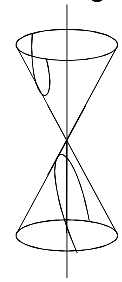
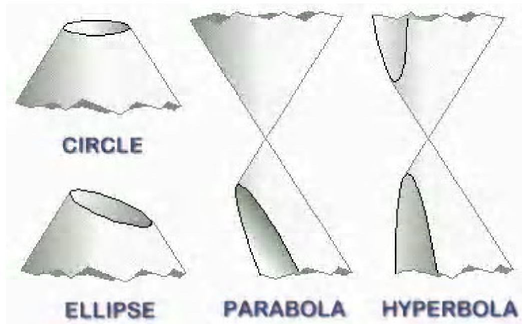


Figure -6



3D View :



3. General equation of a conic: Focal directrix property:

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is:
$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

4. Distinguishing various conics :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) When The Focus Lies On The Directrix.

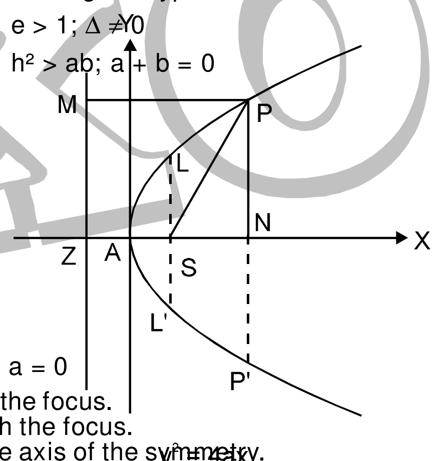
In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if:
$e > 1 \equiv h^2 > ab$ the lines will be real & distinct intersecting at S.
$e = 1 \equiv h^2 \geq ab$ the lines will coincident.
$e < 1 \equiv h^2 < ab$ the lines will be imaginary.

Case (II) When The Focus Does Not Lie On Directrix.

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1;\ \Delta \neq 0,$	$0 < e < 1;\ \Delta \neq 0;$	$e > 1;\ \Delta \neq 0;$	$e > 1;\ \Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab;\ a + b = 0$

PARABOLA

5. Definition and Terminology



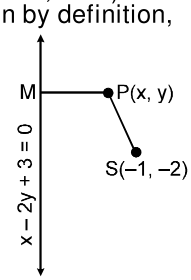
A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix).
Four standard forms of the parabola are
$y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$
For parabola $y^2 = 4ax$:
(i) Vertex is (0, 0) **(ii)** focus is (a, 0)
(iii) Axis is $y = 0$ **(iv)** Directrix is $x + a = 0$

Focal Distance: The distance of a point on the parabola from the focus.
Focal Chord : A chord of the parabola, which passes through the focus.
Double Ordinate: A chord of the parabola perpendicular to the axis of the symmetry.
Latus Rectum: A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus Rectum (L.R.).
For $y^2 = 4ax$. $\Rightarrow$ Length of the latus rectum = 4a.
$\Rightarrow$ ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

NOTE :
(i) Perpendicular distance from focus on directrix = half the latus rectum.
(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.
(iii) Two parabolas are said to be equal if they have the same latus rectum.

Examples :
Find the equation of the parabola whose focus is at (– 1, – 2) and the directrix the line $x - 2y + 3 = 0$.

Solution.
Let P(x, y) be any point on the parabola whose focus is S(– 1, – 2) and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,



SP = PM
$\Rightarrow$ $SP^2 = PM^2$
$\Rightarrow$ $(x + 1)^2 + (y + 2)^2 = \left(\dfrac{x - 2y + 3}{\sqrt{1 + 4}}\right)^2$
$\Rightarrow$ $5\,[(x + 1)^2 + (y + 2)^2] = (x - 2y + 3)^2$
$\Rightarrow$ $5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$

$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

Example :

Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches.

$4y^2 + 12x - 20y + 67 = 0$

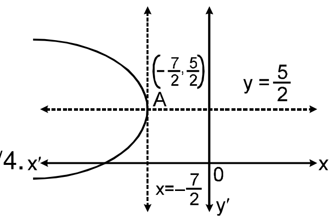
Solution.

The given equation is

$4y^2 + 12x - 20y + 67 = 0 \quad \Rightarrow \quad y^2 + 3x - 5y + \frac{67}{4} = 0$

$\Rightarrow \quad y^2 - 5y = -3x - \frac{67}{4} \quad \Rightarrow \quad y^2 - 5y + \left(\frac{5}{2}\right)^2 = -3x - \frac{67}{4} + \left(\frac{5}{2}\right)^2$

$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4} \quad \Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right)$ ....(i)



Let $\quad x = X - \frac{7}{2},\ y = Y + \frac{5}{2}$ ....(ii)

Using these relations, equation (i) reduces to

$Y^2 = -3X$ ....(iii)

This is of the form $Y^2 = -4aX$. On comparing, we get $4a = 3 \Rightarrow a = 3/4$.

Vertex - The coordinates of the vertex are (X = 0, Y = 0)

So, the coordinates of the vertex are

$\left(-\frac{7}{2}, \frac{5}{2}\right)$ [Putting X = 0, Y = 0 in (ii)]

Axis: The equation of the axis of the parabola is Y = 0.

So, the equation of the axis is

$y = \frac{5}{2}$ [Putting Y = 0 in (ii)]

Focus- The coordinates of the focus are (X = –a, Y = 0)

i.e. (X = – 3/4, Y = 0).

So, the coordinates of the focus are

(–17/4, 5/2) [Putting X = 3/4 in (ii)]

Directrix - The equation of the directrix is X = a i.e. $X = \frac{3}{4}$.

So, the equation of the directrix is

$x = -\frac{11}{4}$ [Putting X = 3/4 in (ii)]

Latusrectum - The length of the latusrectum of the given parabola is 4a = 3.

Self Practice Problems

1. Find the equation of the parabola whose focus is the point (0, 0)and whose directrix is the straight line $3x - 4y + 2 = 0$. **Ans.** $16x^2 + 9y^2 + 24xy - 12x + 16y - 4 = 0$

2. Find the extremities of latus rectum of the parabola $y = x^2 - 2x + 3$.

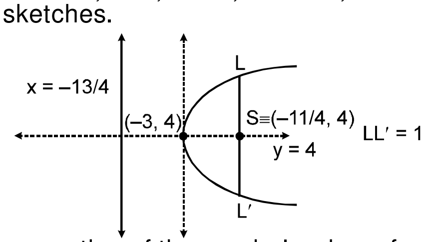
Ans. $\left(\frac{1}{2}, \frac{9}{4}\right)\ \left(\frac{3}{2}, \frac{9}{4}\right)$

3. Find the latus rectum & equation of parabola whose vertex is origin & directrix is x + y = 2.

Ans. $4\sqrt{2}$, $x^2 + y^2 - 2xy + 8x + 8y = 0$

4. Find the vertex, axis, focus, directrix, latusrectum of the parabola $y^2 - 8y - x + 19 = 0$. Also draw their roguht sketches.

Ans.



5. Find the equation of the parabola whose focus is (1, – 1) and whose vertex is (2, 1). Also find its axis and latusrectum.

Ans. $(2x - y - 3)^2 = -20\,(x + 2y - 4)$, Axis $2x - y - 3 = 0$. $LL' = 4\sqrt{5}$.

6. Parametric Representation:

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Example : Find the parametric equation of the parabola $(x - 1)^2 = -12\,(y - 2)$

Solution. $\because \quad 4a = -12 \quad \Rightarrow \quad a = 3, \quad y - 2 = at^2$

$x - 1 = 2\,at \quad \Rightarrow \quad x = 1 - 6t,\ y = 2 - 3t^2$

Self Practice Problems

1. Find the parametric equation of the parabola $x^2 = 4ay$ **Ans.** $x = 2at,\ y = at^2$.

7. Position of a point Relative to a Parabola:

The point $(x_1\ y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Example : Check weather the point (3, 4) lies inside or outside the paabola $y^2 = 4x$.

Solution. $y^2 - 4x = 0$

$\because$ $S_1 \equiv y_1^2 - 4x_1 = 16 - 12 = 4 > 0$

$\therefore$ (3, 4) lies outside the parabola.

Self Practice Problems

1. Find the set of value's of α for which $(\alpha, -2-\alpha)$ lies inside the parabola $y^2 + 4x = 0$.

Ans. $a \in (-4 - 2\sqrt{3}, -4 + 2\sqrt{3})$

8. Line & a Parabola:

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtrless cm \Rightarrow$ condition of tangency is, $c = a/m$.

Length of the chord intercepted by the parabola on the line $y = m\,x + c$ is:

$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)\,(a-mc)}\ .$$

NOTE : 1. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\,y + 2\,at_1t_2 = 0$.

2. If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co–ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

3. Length of the focal chord making an angle α with the x– axis is $4a\operatorname{cosec}^2\alpha$.

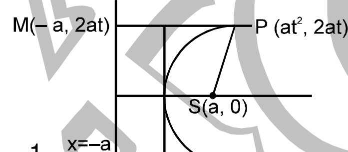
Example : Discuss the position of line $y = x + 1$ with respect to parabolas $y^2 = 4x$.

Solution. Solving we get $(x + 1)^2 = 4x \Rightarrow (x - 1)^2 = 0$

so $y = x + 1$ is tangent to the parabola.

Example :

Prove that focal distance of a point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ $(a > 0)$ is $a(1 + t^2)$.



Solution.

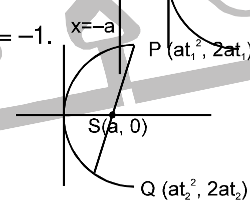
$\because$ $PS = PM$

$= a + at^2$

$PS = a\,(1 + t^2)$.

Example :

If t_1, t_2 are end points of a focal chord then show that $t_1\,t_2 = -1$.



Solution.

Let parabola is $y^2 = 4ax$

since P, S & Q are collinear

$\therefore$ $m_{PQ} = m_{PS}$

$\Rightarrow$ $\dfrac{2}{t_1+t_2} = \dfrac{2t_1}{t_1^2-1}$

$\Rightarrow$ $t_1^2 - 1 = t_1^2 + t_1t_2$

$\Rightarrow$ $t_1t_2 = -1$

Example :

If the endpoint t_1, t_2 of a chord satisfy the relation $t_1\,t_2 = k$ (const.) then prove that the chord always passes through a fixed point. Find the point?

Solution.

Equation of chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$y - 2at_1 = \dfrac{2}{t_1+t_2}\,(x - at_1^2)$

$(t_1 + t_2)\,y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$

$y = \dfrac{2}{t_1+t_2}\,(x + ak)$ $(\because\ t_1t_2 = k)$

$\therefore$ This line passes through a fixed point $(-ak, 0)$.

Self Practice Problems

1. If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct point's then set of value's of 'λ' is

Ans. $(-\infty, 1/3)$

2. Find the midpoint of the chord $x + y = 2$ of the parabola $y^2 = 4x$.

Ans. $(4, -2)$

3. If one end of focal chord of parabola $y^2 = 16x$ is (16, 16) then coordinate of other end is.

Ans. $(1, -4)$

4. If PSQ is focal chord of parabola $y^2 = 4ax$ $(a > 0)$, where S is focus then prove that

$$\frac{1}{PS} + \frac{1}{SQ} = \frac{1}{a}.$$

5. Find the length of focal chord whose one end point is 't'. [Ans. $a\left(t+\frac{1}{t}\right)^2$]

9. Tangents to the Parabola $y^2 = 4ax$:

(i) $yy_1 = 2a(x + x_1)$ at the point (x_1, y_1) ; **(ii)** $y = mx + \frac{a}{m}$ $(m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $ty = x + at^2$ at $(at^2, 2at)$.

NOTE : Point of intersection of the tangents at the point t_1 & t_2 is [at_1t_2, $a(t_1 + t_2)$].

Example : Prove that the straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \frac{a}{m}$

Solution. Equation of tangent of slope 'm' to the parabola $y^2 = 4a(x + a)$ is

$$y = m(x + a) + \frac{a}{m} \quad \Rightarrow \quad y = mx + a\left(m + \frac{1}{m}\right)$$

but the given tangent is $y = mx + c$ $\therefore$ $c = am + \frac{a}{m}$

Example : A tangent to the parabola $y^2 = 8x$ makes an angel of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution. Slope of required tangent's are

$$m = \frac{3 \pm 1}{1 \mp 3}$$

$$m_1 = -2, \qquad m_2 = \frac{1}{2}$$

$\because$ Equation of tangent of slope m to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m}.$$

$\therefore$ tangent's $y = -2x - 1$ at $\left(\frac{1}{2}, -2\right)$

$$y = \frac{1}{2}x + 4 \text{ at } (8, 8)$$

Example :
Find the equation to the tangents to the paabola $y^2 = 9x$ which goes through the point (4, 10).

Solution.
Equation of tangent to parabola $y^2 = 9x$ is

$$y = mx + \frac{9}{4m}$$

Since it passes through (4, 10)

$$\therefore \quad 10 = 4m + \frac{9}{4m} \quad \Rightarrow \quad 16m^2 - 40m + 9 = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$.

Example :
Find the equations to the common tangents of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution.
Equation of tangent to $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \qquad \text{........(i)}$$

Equation of tangent to $x^2 = 4by$ is

$$x = m_1y + \frac{b}{m_1}$$

$$\Rightarrow \quad y = \frac{1}{m_1}x - \frac{b}{(m_1)^2} \qquad \text{........(ii)}$$

for common tangent, (i) & (ii) must represent same line.

$\therefore \quad \dfrac{1}{m_1} = m \quad \& \quad \dfrac{a}{m} = -\dfrac{b}{m_1^2}$

$\Rightarrow \quad \dfrac{a}{m} = -bm^2 \quad \Rightarrow \quad m = \left(-\dfrac{a}{b}\right)^{1/3}$

$\therefore$ equation of common tangent is

$$y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{b}{a}\right)^{1/3}.$$

Self Practice Problems

1. Find equation tangent to parabola $y^2 = 4x$ whose intercept on y–axis is 2.

Ans. $y = \dfrac{x}{2} + 2$

2. Prove that perpendicular drawn from focus upon any tangent of a parabola lies on the tangent at the vertex.

3. Prove that image of focus in any tangent to parabola lies on its directrix.

4. Prove that the area of triangle formed by three tangents to the parabola $y^2 = 4ax$ is half the area of triangle formed by their points of contacts.

10. Normals to the parabola $y^2 = 4ax$:

(i) $y - y_1 = -\dfrac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ;

(ii) $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

NOTE :

(i) Point of intersection of normals at t_1 & t_2 are, $a(t_1^2 + t_2^2 + t_1t_2 + 2);\ -a\,t_1t_2(t_1 + t_2)$.

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \dfrac{2}{t_1}\right)$.

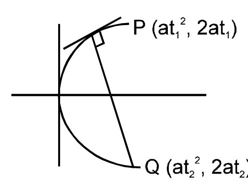
(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

Example :

If the normal at point 't_1' intersects the parabola again at 't_2' then show that $t_2 = -t_1 - \dfrac{2}{t_1}$

Solution.

Slope of normal at P $= -t_1$ and slope of chord PQ $= \dfrac{2}{t_1 + t_2}$



$\therefore \quad -t_1 = \dfrac{2}{t_1 + t_2}$

$t_1 + t_2 = -\dfrac{2}{t_1} \quad \Rightarrow \quad t_2 = -t_1 - \dfrac{2}{t_1}$.

Example :

If the normals at points t_1, t_2 meet at the point t_3 on the parabola then prove that

(i) $t_1t_2 = 2$ (ii) $t_1 + t_2 + t_3 = 0$

Solution.

Since normal at t_1 & t_2 meet the curve at t_3

$\therefore \quad t_3 = -t_1 - \dfrac{2}{t_1}$ .....(i)

$t_3 = -t_2 - \dfrac{2}{t_2}$ .....(ii)

$\Rightarrow \quad (t_1^2 + 2)\,t_2 = t_1(t_2^2 + 2)$

$t_1t_2(t_1 - t_2) + 2(t_2 - t_1) = 0$

$\because \quad t_1 \ne t_2,\ \ t_1t_2 = 2$ ......(iii)

Hence (i) $t_1t_2 = 2$

from equation (i) & (iii), we get

$t_3 = -t_1 - t_2$

Hence (ii) $t_1 + t_2 + t_3 = 0$

Example :

Find the locus of the point N from which 3 normals are drawn to the parabola $y^2 = 4ax$ are such that

(i) Two of them are equally inclined to x-axis

(ii) Two of them are perpendicular to each other

Solution.

Equation of normal to $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Let the normal is passes through N(h, k)

$\therefore \quad k = mh - 2am - am^3 \quad \Rightarrow \quad am^3 + (2a - h)\, m + k = 0$

For given value's of (h, k) it is cubic in 'm'.

Let m_1, m_2 & m_3 are root's

$\therefore \quad m_1 + m_2 + m_3 = 0$ ......(i)

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$ ......(ii)

$m_1m_2m_3 = -\frac{k}{a}$ ......(iii)

(i) If two nromal are equally inclined to x-axis, then $m_1 + m_2 = 0$

$\therefore \quad m_3 = 0 \quad \Rightarrow \quad y = 0$

(ii) If two normal's are perpendicular

$\therefore \quad m_1 m_2 = -1$

from (3) $\quad m_3 = \frac{k}{a}$ .....(iv)

from (2) $\quad -1 + \frac{k}{a}(m_1 + m_2) = \frac{2a-h}{a}$ .....(v)

from (1) $\quad m_1 + m_2 = -\frac{k}{a}$ .....(vi)

from (5) & (6), we get

$$-1 - \frac{k^2}{a} = 2 - \frac{h}{a}$$

$y^2 = a(x - 3a)$

Self Practice Problems

1. Find the points of the parabola $y^2 = 4ax$ at which the normal is inclined at 30° to the axis.

Ans. $\left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right), \left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$

2. If the normal at point P(1, 2) on the parabola $y^2 = 4x$ cuts it again at point Q then Q = ?

Ans. (9, – 6)

3. Find the length of normal chord at point 't' to the parabola $y^2 = 4ax$.

Ans. $\ell = \frac{4a(t^2+1)^{\frac{3}{2}}}{t^2}$

4. If normal chord at a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at the vertex then prove that $t^2 = 2$

5. Prove that the chord of the parabola $y^2 = 4ax$, whose equation is $y - x\sqrt{2} + 4a\sqrt{2} = 0$, is a normal to the curve and that its length is $6\sqrt{3}a$.

6. If the normals at 3 points P, Q & R are concurrent, then show that

(i) The sum of slopes of normals is zero, (ii) Sum of ordinates of points P, Q, R is zero

(iii) The centroid of ΔPQR lies on the axis of parabola.

11. Pair of Tangents:

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by: $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax \quad ; \quad S_1 = y_1^2 - 4ax_1 \quad ; \quad T \equiv y\,y_1 - 2a(x + x_1)$.

Example :

Write the equation of pair of tangents to the parabola $y^2 = 4x$ drawn from a point P(–1, 2)

Solution.

We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore \quad (y^2 - 4x)(4 + 4) = (2y + 2(x - 1))^2$

$\Rightarrow \quad 8y^2 - 32x = 4y^2 + 4x^2 + 4 + 8xy - 8y - 8x$

$\Rightarrow \quad y^2 - x^2 - 2xy - 6x + 2y = 1$

Example :

Find the focus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that

(i) $m_1 + m_2 = m_0$ (const) $\quad$ (ii) $\theta_1 + \theta_2 = \theta_0$ (const)

Sol. Equation of tangent to $y^2 = 4ax$, is

$y = mx + \frac{a}{m}$

Let it passes through P(h, k)

$\therefore \quad m^2h - mk + a = 0$

(i) $\quad m_1 + m_2 = m_0 = \dfrac{k}{h} \quad \Rightarrow \quad y = m_0x$

(ii) $\quad \tan\theta_0 = \dfrac{m_1 + m_2}{1 - m_1 m_2} = \dfrac{k/h}{1 - a/h}$

$\Rightarrow \quad y = (x - a)\tan\theta_0$

Self Practice Problem

1. If two tangents to the parabola $y^2 = 4ax$ from a point P make angles θ_1 and θ_2 with the axis of the parabola, then find the locus of P in each of the following cases.

(i) $\tan^2\theta_1 + \tan^2\theta_2 = \lambda$ (a constant)

(ii) $\cos\theta_1 \cos\theta_2 = \lambda$ (a constant)

Ans. (i) $y^2 - 2ax = \lambda x^2$, (ii) $x^2 = \lambda^2 \{(x - a)^2 + y^2\}$

12. Director Circle:

Locus of the point of intersection of the perpendicular tangents to a curve is called the Director Circle. For parabola $y^2 = 4ax$ it's equation is $x + a = 0$ which is parabola's own directrix.

13. Chord of Contact:

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

NOTE : The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $(y_1^2 - 4ax_1)^{3/2} \div 2a$.

Example :
Find the length of chord of contact of the tangents drawn from point (x_1, y_1) to the parabola $y^2 = 4ax$.

Solution.
Let tangent at $P(t_1)$ & $Q(t_2)$ meet at (x_1, y_1)

$\therefore \quad at_1t_2 = x_1 \quad \& \quad a(t_1 + t_2) = y_1$

$$\because \quad PQ = \sqrt{(at_1^2 - at_2^2)^2 + (2a(t_1 - t_2))^2}$$

$$= a\sqrt{((t_1 + t_2)^2 - 4t_1t_2)((t_1 + t_2)^2 + 4)}$$

$$= \sqrt{\frac{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}{a^2}}$$

Example :
If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Solution.
Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0 \quad$ .....(i)

But given is

$x - y - 1 = 0$

$\therefore \quad \dfrac{4}{1} = \dfrac{-k}{-1} = \dfrac{4h}{-1}$

$\Rightarrow \quad h = -1, k = 4$

$\therefore \quad$ point $\equiv (-1, 4)$

Example :
Find the locus of point whose chord of contact w.r.t to the parabola $y^2 = 4bx$ is the tangents of the parabola $y^2 = 4ax$.

Solution.

Equation of tangent to $y^2 = 4ax$ is $y = mx + \dfrac{a}{m} \quad$ ......(i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is

$yk = 2b(x + h)$

$y = \dfrac{2b}{k}x + \dfrac{2bh}{k} \quad$ .....(ii)

From (i) & (ii)

$m = \dfrac{2b}{k}, \ \dfrac{a}{m} = \dfrac{2bh}{k} \quad \Rightarrow \quad a = \dfrac{4b^2h}{k^2}$

locus of P is

$$y^2 = \frac{4b^2}{a}x.$$

Self Practice Problems

1. Prove that locus of a point whose chord of contact w.r.t. parabola passes through focus is directrix

2. If from a variable point 'P' on the line $x - 2y + 1 = 0$ pair of tangent's are drawn to the parabola $y^2 = 8x$ then prove that chord of contact passes through a fixed point, also find that point.
Ans. (1, 8)

14. Chord with a given middle point:

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is

(x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1) \equiv T = S_1$

Example :
Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given point (p, q).

Solution.
Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,
so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.
Since it passes through (p, q)
$\therefore$ $qk - 2a(p + h) = k^2 - 4ah$
$\therefore$ Required locus is
$y^2 - 2ax - qy + 2ap = 0$.

Example :
Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ whose slope is 'm'.

Solution.
Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,
so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

but slope $= \frac{2a}{k} = m$

$\therefore$ locus is $y = \frac{2a}{m}$

Self Practice Problems

1. Find the equation of chord of parabola $y^2 = 4x$ whose mid point is (4, 2).
Ans. $x - y - 2 = 0$

2. Find the locus of mid - point of chord of parabola $y^2 = 4ax$ which touches the parabola $x^2 = 4by$.
Ans. $y(2ax - y^2) = 4a^2b$

15. Important Highlights:

(i) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of theparabola after reflection.

(ii) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus.

(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(v) If the tangents at P and Q meet in T, then:
$\Rightarrow$ TP and TQ subtend equal angles at the focus S.
$\Rightarrow$ $ST^2 = SP.\ SQ$ & $\Rightarrow$ The triangles SPT and STQ are similar.

(vi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.

(vii) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(viii) If normal are drawn from a point P(h, k) to the parabola $y^2 = 4ax$ then
$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.

$m_1 + m_2 + m_3 = 0$; $\quad m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a - h}{a}$; $m_1 m_2 m_3 = -\dfrac{k}{a}$.

Where m_1, m_2 & m_3 are the slopes of the three concurrent normals. Note that

$\Rightarrow$ algebraic sum of the slopes of the three concurrent normals is zero.

$\Rightarrow$ algebraic sum of the ordinates of the three conormal points on the parabola is zero

$\Rightarrow$ Centroid of the Δ formed by three co–normal points lies on the x–axis.

$\Rightarrow$ Condition for three real and distinct normals to be drawn froma point P (h, k) is

$$h > 2a \ \& \ k^2 < \frac{4}{27a}\,(h - 2a)^3.$$

(ix) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.

(x) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum.

Note : Students must try to proof all the above properties.