Definite Integrals

PART A : A Let f(x) be a continuous function defined on [a, b],

$\int f(x)\,dx = F(x) + c$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$ is called definite integral. This formula is known as Newton-Leibnitz formula.

Note : 1. The indefinite integral $\int f(x)\,dx$ is a function of x, where as definite integral $\int_a^b f(x)\,dx$ is a number.

2. Given $\int f(x)\,dx$ we can find $\int_a^b f(x)\,dx$, but given $\int_a^b f(x)\,dx$ we cannot find $\int f(x)\,dx$

Illustration. 1 Evaluate $\int_1^2 \frac{dx}{(x+1)(x+2)}$

Sol. $\because$ $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ (by partial fractions)

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = \left[\log_e(x+1) - \log_e(x+2)\right]_1^2$$

$$= \log_e 3 - \log_e 4 - \log_e 2 + \log_e 3 = \log_e\left(\frac{9}{8}\right)$$

Self Practice Problems

Evaluate the following

1. $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$ **Ans.** $5 - \frac{5}{2}\left(9\log_e\frac{5}{4} - \log_e\frac{3}{2}\right)$

2. $\int_0^{\pi/2} \left(2\sec^2 x + x^3 + 2\right)dx$ **Ans.** $\frac{\pi^4}{1024} + \frac{\pi}{2} + 2$

3. $\int_0^{\pi/3} \frac{x}{1+\sec x}\,dx$ **Ans.** $\frac{\pi^2}{18} - \frac{\pi}{3\sqrt{3}} + 2\log_e\left(\frac{2}{\sqrt{3}}\right)$

PART B : Properties of definite integral

P – 1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. definite integral is independent of variable of integration.

P – 2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P – 3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b].

Illustration 2 If $f(x) = \begin{cases} x+3 & : x < 3 \\ 3x^2+1 & : x \ge 3 \end{cases}$, then find $\int_2^5 f(x)\,dx$

Sol. $\int_2^5 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx$

$$= \int_2^3 (x+3)\,dx + \int_3^5 (3x^2+1)\,dx$$

$$= \frac{9-4}{2} + 3\,(3-2) + 5^3 - 3^3 + 5 - 3 \quad = \frac{211}{2}$$

Illustration 3 Evaluate $\int_2^8 |x-5|\,dx$

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Sol. $\int_2^8 |x-5|\ dx = \int_2^5 (-x+5)\ dx + \int_5^8 (x+5)\ dx = 9$

Illustration 4 Show that $\int_0^2 (2x+1)\ dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

Sol. L.H.S. = $x^2 + x\,]_0^2$ = 4 + 2 = 6
R.H.S. = 25 + 5 – 0 + (4 + 2) – (25 + 5) = 6 $\quad\therefore\quad$ L.H.S. R.H.S

Self Practice Problems Evaluate the following

1. $\int_0^2 |x^2+2x-3|\ dx$ **Ans.** 4

2. $\int_0^3 [x]\ dx$, where [x] is integral part of x. **Ans.** 3

3. $\int_0^9 \left[\sqrt{t}\right] dt$ **Ans.** 13

PART C : P – 4 $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\ dx$

$= 2\int_0^a f(x)\ dx$ if f(–x) = f(x) i.e. f(x) is even

= 0 if f(–x) = –f (x) i.e. f(x) is odd

Illustration 5 Evaluate $\int_{-1}^{1} \frac{e^x+e^{-x}}{1+e^x}\ dx$

Sol. $\int_{-1}^{1} \frac{e^x+e^{-x}}{1+e^x}\ dx = \int_0^1 \left(\frac{e^x+e^{-x}}{1+e^x} + \frac{e^{-x}+e^x}{1+e^{-x}}\right) dx$

$= \int_0^1 \left(\frac{e^x+e^{-x}}{1+e^x} + \frac{e^x(e^{-x}+e^x)}{e^x+1}\right) dx = \int_0^1 (e^x+e^{-x})\ dx = e - 1 + \frac{(e^{-1}-1)}{-1} = \frac{e^2-1}{e}$

Illustration 6 Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$

Sol. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\ dx = 2$ $\quad$ ($\because$ cos x is even function)

Illustration 7 Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx$

Sol. Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right)$

$\Rightarrow$ $f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. f(x) is odd function

$\therefore$ $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

Self Practice Problems : Evaluate the following

1. $\int_{-1}^{1} |x|\ dx$ **Ans.** 1

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\ dx$ **Ans.** 0

3. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\, dx$ **Ans.** 1

PART D : P – 5 $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

Further $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

Illustration 8 Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)}\, dx = \frac{\pi}{4}$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\, dx$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)} = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\, dx$$

on adding, we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)} + \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right) dx = \int_0^{\frac{\pi}{2}} dx \quad \Rightarrow I = \frac{\pi}{4}$$

Note : 1. The above illustration can be remembered as a formula

2. Other similar formulae are

$$\int_0^{\frac{\pi}{2}} \frac{g(\tan x)}{g(\tan x)+g(\cot x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{g(\cot x)}{g(\tan x)+g(\cot x)}\, dx = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \frac{g(\operatorname{cosec} x)}{g(\operatorname{cosec} x)+g(\sec x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{g(\sec x)}{g(\operatorname{cosec} x)+g(\sec x)}\, dx = \frac{\pi}{4}$$

$$\int_0^{a} \frac{g(x)}{g(x)+g(a-x)}\, dx = \frac{a}{2}$$

Self Practice Problems : Evaluate the following

1. $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$ **Ans.** π

2. $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}\, dx$ **Ans.** $\frac{\pi}{2\sqrt{2}} \log_e\left(1+\sqrt{2}\right)$

3. $\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x+\cos^4 x}\, dx$ **Ans.** $\frac{\pi^2}{16}$

4. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$ **Ans.** $\frac{\pi}{12}$

PART E : P – 6 $\int_0^{2a} f(x)\, dx = \int_0^{a} (f(x)+f(2a-x))\, dx$

$= 2\int_0^a f(x)\, dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

Illustration 9 Evaluate $\int_0^{\pi} \sin^3 x \cos^3 x \, dx$

Sol. Let $f(x) = \sin^3 x \cos^3 x \Rightarrow f(\pi - x) = -f(x) \quad \therefore \quad \int_0^{\pi} \sin^3 x \cos^3 x \, dx = 0$

Illustration 10 Evaluate $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} dx$

Sol. Let $f(x) = \frac{1}{1+2\sin^2 x} \Rightarrow f(\pi - x) = f(x)$

$$\Rightarrow \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \, dx}{1+\tan^2 x + 2\tan^2 x}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \, dx}{1+3\tan^2 x} = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\frac{\pi}{2}}$$

$\because$ $\tan\frac{\pi}{2}$ is undefined, we take limit

$$= \frac{2}{\sqrt{3}} \left[\underset{x\to\frac{\pi}{2}^-}{\text{Lt}} \tan^{-1}\left(\sqrt{3}\tan x\right) - \tan^{-1}\left(\sqrt{3}\tan 0\right)\right]$$

$$= \frac{2}{\sqrt{3}} \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

Note : We can evaluate the integral without using this property

Alternatively : $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\text{cosec}^2 x}{\text{cosec}^2 x + 2} dx = \int_0^{\pi} \frac{\text{cosec}^2 x \, dx}{\cot^2 x + 3}$

Observe that we are not converting in terms of tan x as it is not continuous in $(0, \pi)$

$$= -\frac{1}{\sqrt{3}} \left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi} = -\frac{1}{\sqrt{3}} \left[\underset{x\to\pi^-}{\text{Lt}} \tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) - \underset{x\to 0^+}{\text{Lt}} \tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]$$

$$= -\frac{1}{\sqrt{3}} \left[-\frac{\pi}{2} - \frac{\pi}{2}\right] = \frac{\pi}{\sqrt{3}}$$

Note : If we convert in terms of tan x, then we have to break integral using property P – 3.

Illustration 11 Prove that $\int_0^{\frac{\pi}{2}} \log_e \sin x \, dx = \int_0^{\frac{\pi}{2}} \log_e \cos x \, dx = \int_0^{\frac{\pi}{2}} \log_e(\sin 2x) \, dx = -\frac{\pi}{2} \log_e 2$.

Sol. Let $I = \int_0^{\frac{\pi}{2}} \log_e \sin x \, dx$ ....(i)

$\Rightarrow$ $I = \int_0^{\frac{\pi}{2}} \log_e\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$ (by property P – 5)

$I = \int_0^{\frac{\pi}{2}} \log_e(\cos x) \, dx$ ....(ii)

Adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}} \log_e(\sin x \, . \cos x) \, dx = \int_0^{\frac{\pi}{2}} \log_e\left(\frac{\sin 2x}{2}\right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log_e(\sin 2x) \, dx - \int_0^{\frac{\pi}{2}} \log_e 2 \, dx$$

$$2I = I_1 - \frac{\pi}{2}\log^2_e \qquad ....(iii)$$

where $I_1 = \int_0^{\frac{\pi}{2}} \log_e(\sin 2x)\ dx$

put $2x = t \Rightarrow dx = \frac{1}{2}\ dt$

L . L : $x = 0 \Rightarrow t = 0$

U . L : $x = \frac{\pi}{2} \Rightarrow t = \pi \Rightarrow I_1 = \int_0^{\pi} \log_e(\sin t)\ \frac{1}{2}\ dt$

$$= \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log_e(\sin t)\ dt \qquad \text{(by using property P – 6)}$$

$\Rightarrow I_1 = I \quad \therefore$ (iii) gives $I = -\frac{\pi}{2}\log^2_e$

Self Practice Problems

Evaluate the following

1. $\int_0^{\infty} \left(\frac{\log_e\left(x+\frac{1}{x}\right)}{1+x^2}\right) dx$: **Ans :** $\pi \log_e{}^2$

2. $\int_0^{1} \frac{\sin^{-1}x}{x}\ dx$: **Ans :** $\frac{\pi}{2}\log_e{}^2$

3. $\int_0^{\pi} x\log_e \sin x\ dx$ **Ans :** $-\frac{\pi^2}{2}\log_e{}^2$

PART F : P – 7 If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\ dx = n\int_0^{T} f(x)\,dx,\ n \in z$ (ii) $\int_a^{a+nT} f(x)\ dx = n\int_0^{T} f(x)\,dx,\ n \in z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\ dx = (n-m)\int_0^{T} f(x)\,dx,\ m, n \in z$ (iv) $\int_{nT}^{a+nT} f(x)\ dx = \int_0^{a} f(x)\,dx,\ n \in z,\ a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\ dx = \int_a^{a} f(x)\,dx,\ n \in z,\ a, b \in R$

Illustration 12 Evaluate $\int_{-1}^{2} e^{\{x\}}\ dx$

Sol. $\int_{-1}^{2} e^{\{x\}}\ dx = \int_{-1}^{-1+3} e^{\{x\}}\ dx = 3\int_0^{1} e^{\{x\}}\ dx = 3\int_0^{1} e^{\{x\}}\ dx = 3(e-1)$

Illustration 13 Evaluate $\int_0^{n\pi+v} |\cos x|\ dx,\ \frac{\pi}{2} < v < \pi$ and $n \in z$

Sol. $\int_0^{n\pi+v} |\cos x|\,dx = \int_0^{v} |\cos x|\ dx + \int_v^{n\pi+v} |\cos x|\ dx$

$$= \int_0^{\frac{\pi}{2}} \cos x - \int_{\pi}^{v} \cos x\ dx + n\int_0^{\pi} |\cos x|\ dx$$

$$= (1-0) - (\sin v - 1) + 2n\int_0^{\frac{\pi}{2}} \cos x\ dx \qquad = 2 - \sin v + 2n\,(1-0) = 2n + 2 - \sin v$$

Self Practice Problem : Evaluate the following

1. $\int_{-1}^{2} e^{\{3x\}}\, dx$ **Ans.** $3(e-1)$

2. $\int_{0}^{2000\pi} \frac{dx}{1+e^{\sin x}}\, dx$ **Ans.** $1000\,\pi$

3. $\int_{\pi}^{\frac{5\pi}{4}} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\, dx$ **Ans.** $\frac{\pi}{4}$

PART G : P – 8 If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \psi(x)\, dx \le \int_a^b f(x)\, dx \le \int_a^b \phi(x)\, dx$$

P – 9 If $m \le f(x) \le M$ for $a \le x \le b$, then $m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$

Further if f(x) is monotonically decreasing in (a, b) then $f(b)\,(b-a) < \int_a^b f(x)\,dx < f(a)\,(b-a)$ and if f(x) is monotonically increasing in (a, b) then $f(a)\,(b-a) < \int_a^b f(x)\, dx < f(b)\,(b-a)$

P – 10 $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

P – 11 If $f(x) \ge 0$ on [a, b] then $\int_a^b f(x)\,dx \ge 0$

Illustration 14 For $x \in (0, 1)$ arrange $f_1(x) = \frac{1}{\sqrt{4-x^2}}$, $f_2(x) = \frac{1}{\sqrt{4-2x^2}}$ and $f_3(x) = \frac{1}{\sqrt{4-x^2-x^3}}$ in ascending order and hence prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$

Sol. $\because$ $0 < x^3 < x^2 \Rightarrow x^2 < x^2 + x^3 < 2x^2 \Rightarrow -2x^2 < -x^2 - x^3 < -x^2$

$\Rightarrow 4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$

$\Rightarrow \sqrt{4-2x^2} < \sqrt{4-x^2-x^3} < \sqrt{4-x^2} \Rightarrow f_1(x) < f_3(x) < f_2(x)$ for $x \in (0, 1)$

$$\Rightarrow \int_0^1 f_1(x)\,dx < \int_0^1 f_3(x)\,dx < \int_0^1 f_2(x)\,dx$$

$$\sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1$$

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$$

Illustration 15 Estimate the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx$

Sol. Let $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{(\cos x)(x - \tan x)}{x^2} < 0$$

$\Rightarrow$ f(x) is monotonically decreasing function.
f(0) is not defined, so we evaluate

$\underset{x\to 0^+}{\text{Lt}}\ f(x) = \underset{x\to 0^+}{\text{Lt}}\ \frac{\sin x}{x} = 1$. Take $f(0) = \underset{x\to 0^+}{\text{Lt}}\ f(x) = 1$

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\frac{2}{\pi}\cdot\left(\frac{\pi}{2}-0\right) < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < 1\cdot\left(\frac{\pi}{2}-0\right)$$

$$1 < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx < \frac{\pi}{2}$$

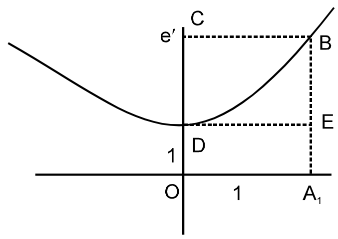
Note : Here by making the use of graph we can make more appropriate approximation as in next illustration.

Illustration 16 Estimate the value of $\int_0^1 e^{x^2}dx$ using (i) rectangle, (ii) triangle

Sol. (i) By using rectangle

Area OAED < $\int_0^1 e^{x^2}dx$ < Area OABC

$$1 < \int_0^1 e^{x^2}dx < 1\,.\,e$$



$$1 < \int_0^1 e^{x^2}dx < e$$

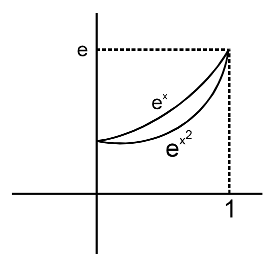
(ii) By using triangle

Area OAED < $\int_0^1 e^{x^2}dx$ < Area OAED + Area of triangle DEB

$$1 < \int_0^1 e^{x^2}dx < 1 + \frac{1}{2}\,.\,1.\,(e-1) \qquad 1 < \int_0^1 e^{x^2}dx < \frac{e+1}{2}$$

Illustration 17 Estimate the value of $\int_0^1 e^{x^2}dx$ by using $\int_0^1 e^{x}dx$

Sol. For $x \in (0, 1)$, $e^{x^2} < e^x$



$$\Rightarrow \qquad 1\times 1 < \int_0^1 e^{x^2}dx < \int_0^1 e^{x}dx$$

$$1 < \int_0^1 e^{x^2}dx < e-1$$

Exercise : Prove the following :

1. $\int_0^1 e^{-x}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x\,dx$ **2.** $0 < \int_0^{\frac{\pi}{2}} \sin^{n+1}x\,dx < \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

3. $e^{-\frac{1}{4}} < \int_0^1 e^{x^2-x}\,dx < 1$ **4.** $-\frac{1}{2} \le \int_0^1 \frac{x^3\cos x}{2+x^2}\,dx < \frac{1}{2}$

5. $1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \sqrt{\frac{\pi}{2}}$ **6.** $0 < \int_0^2 \frac{x\,dx}{16+x^3} < \frac{1}{6}$

PART - H: **Leibnitz Theorem :** If $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then

$$\frac{dF(x)}{dx} = h'(x)\, f(h(x)) - g'(x)\, f(g(x))$$

Proof : Let $P(t) = \int f(t)\, dt$

$\Rightarrow$ $F(x) = \int_{g(x)}^{h(x)} f(t)\, dt = P(h(x)) - P(g(x))$ $\Rightarrow$ $\frac{dF(x)}{dx} = P'(h(x))\, h'(x) - P'(g(x))\, g'(x)$

$= f(h(x))\, h'(x) - f(g(x))\, g'(x)$

Illustration 18 If $F(x) = \int_{x}^{x^2} \sqrt{\sin t}\, dt$, then find $F'(x)$

Sol. $F'(x) = 2x \cdot \sqrt{\sin x^2} - 1 \cdot \sqrt{\sin x}$

Illustration 19 If $F(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\log_e^t}\, dt$, then find first and second derivative of $F(x)$ with respect to $\log_e^x$ at $x = \log_e^2$

Sol. $\frac{dF(x)}{d\left(\log_e^x\right)} = \frac{dF(x)}{dx}\ \frac{dx}{d\,(\log x)} = \left[3 \cdot e^{3x} \cdot \frac{e^{3x}}{\log_e^{e^{3x}}} - 2 \cdot e^{2x}\, \frac{e^{2x}}{\log_e^{e^{2x}}}\right] x = e^{6x} - e^{4x}.$

$\frac{d^2F(x)}{d\left(\log_e^x\right)^2} = \frac{d}{d\left(\log_e^x\right)}\ (e^{6x} - e^{4x}) = \frac{d}{dx}\ (e^{6x} - e^{4x}) \times \frac{1}{\frac{d\log_e^x}{dx}} = (6\, e^{6x} - 4\, e^{4x})\, x$

First derivative of $F(x)$ at $x = \log^2_e$ (i.e. $e^x = 2$) is $2^6 - 2^4 = 48$
Second derivative of $F(x)$ at $x = \log^2_e$ (i.e. $e^x = 2$) is $(6 \cdot 2^6 - 4 \cdot 2^4) \cdot \log_e{}^2 = 5 \cdot 2^6 \cdot \log^2{}_e$.

Illustration 20 Evaluate $\underset{x\to\infty}{\text{Lt}} \frac{\left(\int_0^x e^{t^2}\, dt\right)^2}{\int_0^x e^{2t^2}\, dt}$

Sol. $\underset{x\to\infty}{\text{Lt}} \frac{\left(\int_0^x e^{t^2}\, dt\right)^2}{\int_0^x e^{2t^2}\, dt}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

Applying L′ Hospital rule

$= \underset{x\to\infty}{\text{Lt}} \frac{2 \cdot \int_0^x e^{t^2} dt \cdot e^{x^2}}{1 \cdot e^{2x^2}} = \underset{x\to\infty}{\text{Lt}} \frac{2 \cdot \int_0^x e^{t^2} dt}{e^{x^2}} = \underset{x\to\infty}{\text{Lt}} \frac{2 \cdot e^{x^2}}{2x \cdot e^{x^2}} = 0$

Modified Leibnitz Theorem :

If $F(x) = \int_{g(x)}^{h(x)} f(x, t)\ dt$, then

$$F'(x) = \int_{g(x)}^{h(x)} \frac{\partial f(x, t)}{\partial x}\ dt + f(x, h(x))\, h'(x) - f(x, g(x)) \cdot g'(x)$$

Illustration 21 If $f(x) = \int_{\log_e^x}^{x} \frac{dt}{x + t}$, then find $f'(x)$

Sol. $f'(x) = \int_{\log_e^x}^{x} \frac{-1}{(x+t)^2}\, dt + 1 \cdot \frac{1}{2x} - \frac{1}{x}\ \frac{1}{\left(x + \log_e^x\right)} = \left.\frac{1}{(x+t)}\right]_{\log_e^x}^{x} + \frac{1}{2x} - \frac{1}{x\left(x + \log_e^x\right)}$

$= \frac{1}{2x} - \frac{1}{x + \log_e^x} + \frac{1}{2x} - \frac{1}{x\left(x + \log_e^x\right)} = \frac{1}{x} - \frac{x+1}{x\left(x + \log_e^x\right)} = \frac{\log_e^x - 1}{x\left(x + \log_e^x\right)}$

Alternatively : $f(x) = \int_{\log_e^x}^{x} \frac{dt}{x+t} = \log_e(x+t)\Big]_{\log_e^x}^{x}$ (treating 't' as constant)

$f(x) = \log_e^{2x} - \log_e(x + \log_e^x)$

$f'(x) = \frac{1}{x} - \frac{1}{\left(x+\log_e^x\right)}\left(1+\frac{1}{x}\right) = \frac{\log_e^x - 1}{x\left(x+\log_e^x\right)}$

Definite Integrals dependent on parameters :

Illustration 23 Evaluate $\int_0^1 \frac{x^b - 1}{\log_e^x}$, 'b' being parameter

Sol. Let $I(b) = \int_0^1 \frac{x^b - 1}{\log_e^x}\, dx$

$\frac{dI(b)}{db} = \int_0^1 \frac{x^b \log_e^x}{\log_e^x}\, dx + 0 - 0$

(using modified Leibnitz Theorem)

$= \int_0^1 x^b\, dx = \frac{x^{b+1}}{b+1}\Big]_0^1 = \frac{1}{b+1}$

$I(b) = \log_e(b+1) + c$

$b = 0 \Rightarrow I(0) = 0$

$\therefore$ $c = 0$ $\therefore$ $I(b) = \log_e(b+1)$

Illustration 24 Evaluate $\int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\, dx$, 'a' being parameter

Sol. Let $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\, dx$

$\frac{dI(a)}{da} = \int_0^1 \frac{x}{(1+a^2x^2)} \frac{1}{x\sqrt{1-x^2}}\, dx = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$

Put $x = \sin t \Rightarrow dx = \cos t\, dt$

L.L. : $x = 0 \Rightarrow t = 0$

U.L. : $x = 1 \Rightarrow t = \frac{\pi}{2}$

$\frac{dI(a)}{da} = \int_0^{\frac{\pi}{2}} \frac{1}{1+a^2\sin^2 t} \frac{1}{\cos t} \cos t\, dt = \int_0^{\frac{\pi}{2}} \frac{dt}{1+a^2\sin^2 t}$

$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 t\, dt}{1+(1+a^2)\tan^2 t} = \frac{1}{\sqrt{1+a^2}} \tan^{-1}\left(\sqrt{1+a^2}\,\tan t\right)\Big]_0^{\frac{\pi}{2}}$

$= \frac{1}{\sqrt{1+a^2}} \cdot \frac{\pi}{2}$

$\Rightarrow$ $I(a) = \frac{\pi}{2}\log_e\left(a+\sqrt{1+a^2}\right) + c$

But $I(0) = 0 \Rightarrow c = 0 \Rightarrow I(a) = \frac{\pi}{2}\log_e\left(a+\sqrt{1+a^2}\right)$

Self Practice Problems :

1. If $f(x) = \int_0^{x^3} \sqrt{\cos t}\, dt$, find $f'(x)$. **Ans.** $3x^2\sqrt{\cos x^3}$

2. If $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4}\, dt$ then find the value of $f'(2)$. **Ans.** $\frac{2}{17}$

3. If $x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}$ and $\frac{d^2y}{dx^2} = Ry$ then find R **Ans.** 4

4. If $f(x) = \int_x^{x^2} x^2 \sin t \, dt$ then find $f'(x)$. **Ans.** $x^2 (2x \sin x^2 - \sin x) + (\cos x - \cos x^2)\, x$

5. If $\phi(x) = \cos x - \int_0^x (x - t)\, \phi(t)\, dt$, then find the value of $\phi''(x) + \phi(x)$. **Ans.** $-\cos x$

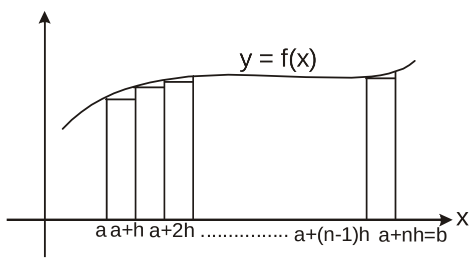
6. Find the value of the function $f(x) = 1 + x + \int_1^x \left((\log_e^t)^2 + 2\log_e^t\right) dt$ where $f'(x)$ vanishes. **Ans.** $1 + \frac{2}{e}$

7. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{\int_0^{x^2} \cos t^2 \, dt}{x \sin x}$. **Ans.** 1

8. Evaluate $\int_0^{\pi} \log_e (1 + b\cos x)\, dx$, 'b' being parameter. **Ans.** $\pi \log_e \left(\frac{1+\sqrt{1-b^2}}{2}\right)$

PART - I: Definite Integral as a Limit of Sum.

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are a, a + h, a + 2h ....a + (n – 1)h, a + nh, where $\frac{b-a}{n} = h$.

Let S_n denotes the area of these n rectangles.
Then, $S_n = hf(a) + hf(a + h) + hf(a + 2h) + \dots + hf(a + (n-1)h)$
Clearly, S_n is area very close to the area of the region bounded by curve y = f(x), x–axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\, dx = \underset{n\to\infty}{\text{Lt}} S_n$

$$\int_a^b f(x)\, dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h\, f(a + rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)\, r}{n}\right)$$

Note : 1. We can also write

$S_n = hf(a + h) + hf(a + 2h) + \dots + hf(a + nh)$ and $\int_a^b f(x)\, dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right) r\right)$

2. If a = 0, b = 1, $\int_0^1 f(x)\, dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

Steps to express the limit of sum as definte integral

Step 1. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\underset{n\to\infty}{\text{Lt}} \Sigma$ by $\int$

Step 2. Evaluate $\underset{n\to\infty}{\text{Lt}} \left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For example $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^p f(x)\, dx$ $\left(\because \underset{n\to\infty}{\text{Lt}} \left(\frac{r}{n}\right)\Big|_{r=1} = 0,\ \underset{n\to\infty}{\text{Lt}} \left(\frac{r}{n}\right)\Big|_{r=np} = p\right)$

Illustration 25 : Evaluate

$$\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{2n}\right]$$

Sol. $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{2n}\right] = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{1}{r+n}$

$$= \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{\left(\frac{r}{n}\right)+1} = \int_0^1 \frac{dx}{x+1} = [\log_e(x+1)]_0^1 = \log_e 2.$$

Illustration 26 : Evaluate $\underset{n\to\infty}{\text{Lt}} \left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+........+\frac{3}{5n}\right]$

Sol. $$\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{2n} \frac{n+r}{n^2+r^2} = \underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{2n} \frac{1}{n} \frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^2}$$

$\because$ $\underset{n\to\infty}{\text{Lt}} \left(\frac{r}{n}\right)$ = 0, when r = 1, lower limit = 0

and $\underset{n\to\infty}{\text{Lt}} \left(\frac{r}{n}\right) = \underset{n\to\infty}{\text{Lt}} \left(\frac{2n}{n}\right)$ = 2, when r = 2n, upper limit = 2

$$\int_0^2 \frac{1+x}{1+x^2}\,dx = \int_0^2 \frac{1}{1+x^2}\,dx + \frac{1}{2}\int_0^2 \frac{2x}{1+x^2}\,dx$$

$$= \tan^{-1}x]_0^2 + \frac{1}{2}\log_e(1+x^2)\Big]_0^2 = \tan^{-1}2 + \frac{1}{2}\log_e 5$$

Illustration 27 : Evaluate

$$\underset{n\to\infty}{\text{Lt}} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$$

Sol. Let $y = \underset{n\to\infty}{\text{Lt}} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

$$\log_e y = \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\log_e\left(\frac{n!}{n^n}\right)$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\log_e\left(\frac{1.2.3........n}{n^n}\right)$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left[\log_e\left(\frac{1}{n}\right)+\log_e\left(\frac{2}{n}\right)+\log_e\left(\frac{3}{n}\right)+.....+\log_e\left(\frac{n}{n}\right)\right]$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\sum_{r=1}^{n}\log_e\left(\frac{r}{n}\right)$$

$$= \int_0^1 \log_e x\,dx = x\log_e x - x\Big]_0^1$$

$$= (0-1) - \underset{x\to 0^+}{\text{Lt}} x\log_e x + 0$$

$$= -1 - 0 = -1 \Rightarrow y = \frac{1}{e}$$

Self Practice Problems :

Evaluate the following limits

1. $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+....+\frac{1}{\sqrt{n^2+n^2}}\right]$ **Ans.** $2\left(\sqrt{2}-1\right)$

2. $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+.......+\frac{1}{5n}\right]$ **Ans.** $\log_e 5$

3. $\underset{n\to\infty}{\text{Lt}} \frac{1}{n^2}\left[\sin^3\frac{\pi}{4n}+2\sin^3\frac{2\pi}{4n}+3\sin^3\frac{3\pi}{4n}+........+n\sin^3\frac{n\pi}{4n}\right]$ **Ans.** $\frac{\sqrt{2}}{9\pi^2}(52-15\pi)$

4. $\underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$ **Ans.** $\frac{\pi}{2}$

5. $\lim\limits_{n\to\infty} \frac{3}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\sqrt{\frac{n}{n+9}}+......+\sqrt{\frac{n}{n+3(n-1)}}\right]$ **Ans.** 2

PART – J: Reduction Formulae in Definite Integrals

1. If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\ dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Proof : $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\ dx$

$$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \,.\, \cos^2 x\ dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \,.\, (1-\sin^2 x)\ dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x\ dx$$

$$I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

Note : 1. $\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \int_0^{\frac{\pi}{2}} \cos^n x\ dx$

2. $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) ..... I_0$ or I_1

according as n is even or odd. $I_0 = \frac{\pi}{2}$, $I_1 = 1$

$$\text{Hence } I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{1}{2}\right).\frac{\pi}{2} & \text{if} \quad n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{2}{3}\right).1 & \text{if} \quad n \text{ is odd} \end{cases}$$

2. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\ dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Sol. $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \,.\, \tan^2 x\ dx$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} (\sec^2 x - 1)\ dx \qquad = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \sec^2 x\ dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\ dx$$

$$= \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2} \qquad \therefore \qquad I_n + I_{n-2} = \frac{1}{n-1}$$

3. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \,.\, \cos^n x\ dx$, then show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2}$, n

Sol. $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x\,(\sin x \cos^n x)\ dx$

$= \left[-\dfrac{\sin^{m-1} x \,.\, \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \dfrac{\cos^{n+1} x}{n+1}\ (m-1) \sin^{m-2} x \cos x\ dx$

$= \left(\dfrac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x \,.\, \cos^n x \,.\, \cos^2 x\ dx$

$= \left(\dfrac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x \,.\, \cos^n x - \sin^m x \,.\, \cos^n x\right) dx$

$= \left(\dfrac{m-1}{n+1}\right) I_{m-2,n} - \left(\dfrac{m-1}{n+1}\right) I_{m,n}$

$\Rightarrow \left(1 + \dfrac{m-1}{n+1}\right) I_{m,n} = \left(\dfrac{m-1}{n+1}\right) I_{m-2,n}$

$I_{m,n} = \left(\dfrac{m-1}{m+n}\right) I_{m-2,n}$

Note : 1. $I_{m,n} = \left(\dfrac{m-1}{m+n}\right)\left(\dfrac{m-3}{m+n-2}\right)\left(\dfrac{m-5}{m+n-4}\right)$ .... $I_{0,n}$ or $I_{1,n}$ according as m is even or odd.

$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x\ dx$ and $I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \,.\, \cos^n x\ dx = \dfrac{1}{n+1}$

2. **<u>Walli's Formula</u>**

$$I_{m,n} = \begin{cases} \dfrac{(n-1)(n-3)(m-5)\,.........(n-1)(n-3)(n-5).......}{(m+n)(m+n-2)(m+n-4)........}\ \dfrac{\pi}{2} & \text{when both m, n are even} \\[2ex] \dfrac{(m-1)(m-3)(m-5)\,.........(n-1)(n-3)(n-5).......}{(m+n)(m+n-2)(m+n-4)........} & \text{otherwise} \end{cases}$$

Illustration 28 : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x(\sin x + \cos x)\ dx$

Sol. Given integral $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \cos^2 x\ dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x\ dx$

$= 0 + 2\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\ dx$ ($\because \sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even)

$= 2.\ \dfrac{1\,.\,2}{5\,.\,3\,.\,1} = \dfrac{4}{15}$

Illustration 29 : Evaluate $\int_0^{\pi} x \sin^5 x \cos^6 x\ dx$

Sol. Let $I = \int_0^{\pi} x \sin^5 x \cos^6 x\ dx$

$I = \int_0^{\pi} (\pi - x) \sin^5(\pi - x) \cos^6(\pi - x)\ dx$

$= \pi \int_0^{\pi} \sin^5 \,.\, \cos^6 x\ dx - \int_0^{\pi} x \sin^5 x \,.\, \cos^6 x\ dx$

$$\Rightarrow \quad 2I = \pi \, . \, 2 \int_0^{\frac{\pi}{2}} \sin^5 x \, . \cos^6 x \, dx$$

$$I = \pi \, \frac{4 \, . \, 2 \, . \, 5 \, . \, 3 \, . \, 1}{11 \, . \, 9 \, . \, 7 \, . \, 5 \, . \, 3 \, . \, 1}$$

$$I = \frac{8\pi}{693}$$

Illustration 30 : Evaluate $\int_0^1 x^3 (1-x)^5 \, dx$

Sol. Put $x = \sin^2\theta \quad \Rightarrow \quad dx = 2 \sin\theta \cos\theta \, d\theta$

L . L : $x = 0 \quad \Rightarrow \quad \theta = 0$

U.L. : $x = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$

$$\therefore \quad \int_0^1 x^3 (1-x)^5 dx = \int_0^{\frac{\pi}{2}} \sin^6\theta (\cos^2\theta)^5 \, 2 \, . \sin\theta \, . \cos\theta \, d\theta$$

$$= 2 \, . \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^{11}\theta \, d\theta$$

$$= 2 \, . \, \frac{6 \, . \, 4 \, . \, 2 \, . \, 10 \, . \, 8 \, . \, 6 \, . \, 4 \, . \, 2}{18 \, . \, 16 \, . \, 14 \, . \, 12 \, . \, 10 \, . \, 8 \, . \, 6 \, . \, 4 \, . \, 2} \quad = \frac{1}{504}$$

Self Practice Problems:

Evaluate the following

1. $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ **Ans.** $\frac{18}{15}$

2. $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x \, dx$ **Ans.** $\frac{8}{315}$

3. $\int_0^1 x^6 \sin^{-1} x \, dx$ **Ans.** $\frac{\pi}{14} - \frac{16}{245}$

4. $\int_0^a x \left(a^2 - x^2\right)^{\frac{7}{2}} dx$ **Ans.** $\frac{a^9}{9}$

5. $\int_0^2 x^{3/2} \sqrt{2-x} \, dx$ **Ans.** $\frac{\pi}{2}$