Binomial theorem

Properties of Binomial Coefficients

1. ${}^{n}\mathrm{C}_{r} = {}^{n}\mathrm{C}_{r-1}$
2. ${}^{n}\mathrm{C}_{r} = {}^{n}\mathrm{C}_{s} \Rightarrow r = s \quad \text{or} \quad r + s = n$
3. ${}^{n}\mathrm{C}_{r} + {}^{n}\mathrm{C}_{r+1} = {}^{n+1}\mathrm{C}_{r+1}$
4. $\dfrac{{}^{n}\mathrm{C}_{r}}{{}^{n+1}\mathrm{C}_{r+1}} = \dfrac{r+1}{n+1}$
5. $\dfrac{{}^{n}\mathrm{C}_{r}}{{}^{n}\mathrm{C}_{r+1}} = \dfrac{r+1}{n-r}$

Series of Binomial Coefficients

$$(1+x)^{n} = {}^{n}\mathrm{C}_{0} + {}^{n}\mathrm{C}_{1}x + {}^{n}\mathrm{C}_{2}x^{2} + \ldots + {}^{n}\mathrm{C}_{n}x^{n} \qquad \ldots (1)$$

1. Sum of the binomial coefficients in the expansion of $(1 + x)^n = 2^n$

2. $\sum_{r=0}^{n} (-1)^{r}\, {}^{n}\mathrm{C}_{r} = 0$

3. In the expansion $(1 + x)^n$:

Sum of the binomial coefficients at odd position = Sum of the binomial coefficients at even position

$${}^{n}\mathrm{C}_{0} + {}^{n}\mathrm{C}_{2} + {}^{n}\mathrm{C}_{4} + \ldots = {}^{n}\mathrm{C}_{1} + {}^{n}\mathrm{C}_{3} + {}^{n}\mathrm{C}_{5} + \ldots = 2^{n-1}$$

4. ${}^{n}\mathrm{C}_{1} + 2\,{}^{n}\mathrm{C}_{2} + 3\,{}^{n}\mathrm{C}_{3} + \ldots\ldots. + n\,{}^{n}\mathrm{C}_{n} = n\ 2^{n-1}$

5. ${}^{n}\mathrm{C}_{1} - 2\,{}^{n}\mathrm{C}_{2} + 3\,{}^{n}\mathrm{C}_{3} - \ldots + (-1)^{n-1}\, n\,{}^{n}\mathrm{C}_{n} = 0$

6. ${}^{n}\mathrm{C}_{0}\ {}^{n}\mathrm{C}_{r} + {}^{n}\mathrm{C}_{1}\ {}^{n}\mathrm{C}_{r+1} + {}^{n}\mathrm{C}_{2}\ {}^{n}\mathrm{C}_{r+2} + \ldots\ldots + {}^{n}\mathrm{C}_{n-r}\ {}^{n}\mathrm{C}_{n} = {}^{2n}\mathrm{C}_{n+r}$

7. ${}^{n}\mathrm{C}_{0} + 3\,{}^{n}\mathrm{C}_{1} + 5\,{}^{n}\mathrm{C}_{2} + 7\,{}^{n}\mathrm{C}_{3} + \ldots + (2n+1)\,{}^{n}\mathrm{C}_{n} = (n+1)2^{n}$

- The coefficients of the expansions of a binomial are arranged in an array. This array is called Pascal’s triangle. It can be written as

Index	Coefficient(s)
0	${}^{0}\mathrm{C}_{0}$ $(=1)$

1	${}^1C_0$ $(=1)$ ${}^1C_1$ $(=1)$
2	${}^2C_0$ $(=1)$ ${}^2C_1$ $(=2)$ ${}^2C_2$ $(=1)$
3	${}^3C_0$ $(=1)$ ${}^3C_1$ $(=3)$ ${}^3C_2$ $(=3)$ ${}^3C_3$ $(=1)$
4	${}^4C_0$ $(=1)$ ${}^4C_1$ $(=4)$ ${}^4C_2$ $(=6)$ ${}^4C_3$ $(=4)$ ${}^4C_4$ $(=1)$
5	

- **General Term:** The $(r+1)^{th}$ term (denoted by T_{r+1}) is known as the general term of the expansion $(a+b)^n$ and it is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Example 1: In the expansion of $(5x - 7y)^9$, find the general term?

Solution: $T_{r+1} = {}^9C_r (5x)^{9-r} (-7y)^r = (-1)^r\, {}^9C_r (5x)^{9-r} (7y)^r$

- **Middle term in the expansion of $(a + b)^n$:**
  - If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even, $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n}{2}+1\right)^{\text{th}}$ term.
  - If n is odd, then $n + 1$ is even. So, there will be two middle terms in the expansion. They are $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term.
- In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n}{2}+1\right)^{\text{th}}$, i.e., $(n + 1)^{th}$ term [since $2n$ is even].

It is given by ${}^{2n}C_n x^n\left(\frac{1}{x}\right)^n = {}^{2n}C_n$ which is a constant.

This term is called the term independent of x or the constant term.

Note: In the expansion of $(a + b)^n$, r^{th} term from the end = $(n - r + 2)^{th}$ term from the beginning

Example 2: In the expansion of $\left(\frac{x^3}{4}-\frac{12}{x}\right)^4$, find the middle term and find the term which is independent of x.

Solution: As 4 is even, the middle term in the expansion of $\left(\frac{x^3}{4}-\frac{12}{x}\right)^4$ is the $\left(\frac{4}{2}+1\right)^{\text{th}}$ term, i.e., 3^{rd} term, which is given by

$$T_3 = T_{2+1} = {}^4C_2 \left(\frac{x^3}{4}\right)^2 \left(\frac{-12}{x}\right)^2$$

$$= 6 \times \frac{x^6}{16} \times \frac{144}{x^2}$$

$$= 54x^4$$

Now, we will find the term in the expansion which is independent of *x*. Suppose $(r+1)^{\text{th}}$ term is independent of *x*.

The $(r+1)^{\text{th}}$ term in the expansion of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Hence, the $(r+1)^{\text{th}}$ term in the expansion of $\left(\frac{x^3}{4} - \frac{12}{x}\right)^4$ is given by