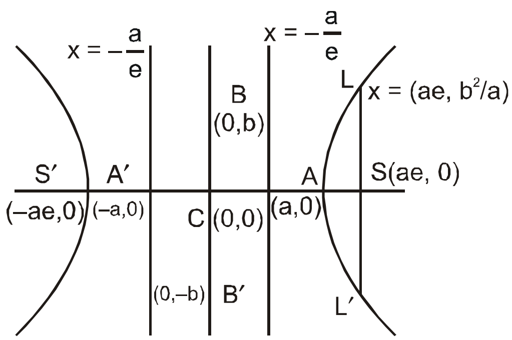
Hyperbola

The **Hyperbola** is a conic whose eccentricity is greater than unity. ($e > 1$).

1. Standard Equation & Definition(s)



Standard equation of the hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

where $b^2 = a^2 (e^2 - 1)$.

Eccentricity (e) : $e^2 = 1 + \dfrac{b^2}{a^2} = 1 + \left(\dfrac{C.A}{T.A}\right)^2$

Foci : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

Equations Of Directrices :

$x = \dfrac{a}{e}$ & $x = -\dfrac{a}{e}$.

Transverse Axis : The line segment A′A of length 2a in which the foci S′ & S both lie is called the transverse axis of the hyperbola.

Conjugate Axis : The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola.

Principal Axes : The transverse & conjugate axis together are called Principal Axes of the hyperbola.

Vertices : $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the transverse axis is called a double ordinate.

Latus Rectum (ℓ) :

The focal chord perpendicular to the transverse axis is called the latus rectum.

$$\ell = \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a\,(e^2 - 1).$$

Note : ℓ (L.R.) = 2 e (distance from focus to directrix)

Centre : The point which bisects every chord of the conic drawn through it is called the centre of the conic. $C \equiv (0 \cdot 0)$ the origin is the centre of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

General Note :

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

Example : Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus (1, 2) and eccentricity $\sqrt{3}$.

Solution. Let P 9x,y) be any point on the hyperbola.

Draw PM perpendicular from P on the directrix.

Then by definition $SP = e\,PM$

$\Rightarrow$ $(SP)^2 = e^2 (PM)^2$

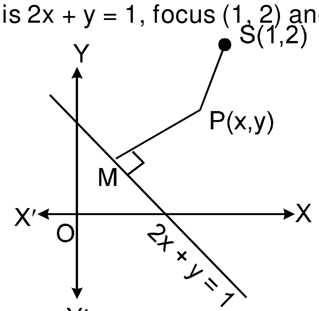
$\Rightarrow$ $(x-1)^2 + (y-2)^2 = 3\left\{\dfrac{2x + y - 1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow$ $5\,(x^2 + y^2 - 2x - 4y + 5\}$

$= 3\,(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow$ $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.



Example : Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Solution. Let the equation of hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Then transverse axis = 2a and latus–rectum = $\frac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow \quad 2b^2 = a^2 \qquad (\because b^2 = a^2(e^2 - 1))$

$\Rightarrow \quad 2a^2(e^2 - 1) = a^2 \quad \Rightarrow \quad 2e^2 - 2 = 1$

$\Rightarrow \quad e^2 = \frac{3}{2} \quad \therefore \quad e = \sqrt{\frac{3}{2}}$

Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

2. Conjugate Hyperbola :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called Conjugate Hyperbolas of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

Note : (a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similiar if they have the same eccentricity.

(d) Two similar hyperbolas are said to be equal if they have same latus rectum.

(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

Example : Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, lengths of the latus-rectum and equations of the directrices of the following hyperbolas $16x^2 - 9y^2 = -144$.

Solution. The equation $16x^2 - 9y^2 = -144$ can be written as

$$\frac{x^2}{9} - \frac{y^2}{16} = -1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore \quad a^2 = 9,\ b^2 = 16 \quad \Rightarrow \quad a = 3,\ b = 4$

Length of transverse axis : The length of transverse axis = 2b = 8

Length of conjugate axis : The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The coªordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum : The length of latus–rectum = $\frac{2a^2}{b}$

$$= \frac{2(3)^2}{4}\ \frac{9}{2}$$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e}$$

$$y = \pm \frac{4}{(5/4)} \qquad y = \pm \frac{16}{5}$$

Self Practice Problems :

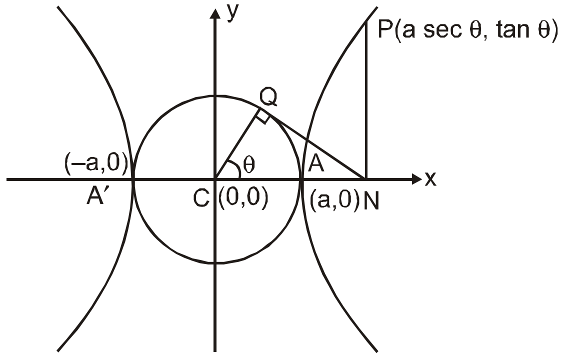
1. Find the equation of the hyperbola whose focis are (6, 4) and (– 4, 4) and eccentricity is 2.
Ans. $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

2. Obtain the equation of a hyperbola with coordinates axes as principal axes given that the distances of one of its vertices from the foci are 9 and 1 units. **Ans.** $\frac{x^2}{16} - \frac{y^2}{9} = 1,\ \frac{y^2}{16} - \frac{x^2}{9} = 1$

3. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of the hyperbola if its eccentricity is 2. **Ans.** $3x^2 - y^2 - 12 = 0$.

3. Auxiliary Circle :

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.
Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle.



4. Parametric Representation : The equations $x = a \sec\theta$ & $y = b\tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations; $x = a\cosh\phi$, $y = b\sinh\phi$ also represents the same hyperbola.

Note that if $P(\theta) \equiv (a\sec\theta, b\tan\theta)$ is on the hyperbola then;

$Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

The equation to the chord of the hyperbola joining two points with eccentric angles α & β is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}.$$

5. Position Of A Point 'P' w.r.t. A Hyperbola :

The quantity $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative according as the point (x_1, y_1) lies inside, on or outside the curve.

Example : Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution. Since $9(5)^2 - (-4)^2 = 1 = 225 - 16 - 1 = 208 > 0$,

So the point (5,–4) inside the hyperbola $9x^2 - y^2 = 1$.

6. Line And A Hyperbola : The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 >$ or $=$ or $< a^2m^2 - b^2$, respectively.

7. Tangents :

(i) Slope Form : $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope 'm'.

(ii) Point Form : Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(iii) Parametric Form : Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point.

(a sec θ, b tan θ) is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Note : **(i)** Point of intersection of the tangents at θ_1 & θ_2 is $x = a\,\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$, $y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangetns at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touches the hyperbola at the extremities of a diameter.

Example :

Prove that the straight line $\ell x + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\ell^2 - b^2m^2 = n^2$.

Solution.

The given line is $\quad \ell x + my + n = 0$

or $\quad y = -\frac{\ell}{m}x - \frac{n}{m}$

Comparing this line with $y = Mx + c$

$\therefore \quad M = -\frac{\ell}{m}$ and $c = -\frac{n}{m}$ ....(1)

This line (1) will touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ if } c^2 = a^2M^2 - b^2$$

$$\Rightarrow \quad \frac{n^2}{m^2} = \frac{a^2 l^2}{m^2} - b^2$$

or $\quad a^2\ell^2 - b^2m^2 = n^2$

Example :

Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution.

Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore \quad m \times 1 = -1 \quad \Rightarrow \quad m = -1$

Since $\quad x^2 - 4y^2 = 36 \quad$ or $\quad \frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \therefore \quad a^2 = 36$ and $b^2 = 9$

So the equation of tangents are

$$y = (-1)\,x \pm \sqrt{36 \times (-1)^2 - 9}$$

$$y = -x \pm \sqrt{27}$$

$$\Rightarrow \quad x + y \pm 3\sqrt{3} = 0$$

Example : Find the equation and the length of the common tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Solution. Tangent at $(a \sec \phi\ b \tan \phi)$ on the 1st hyperbola is

$$\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1 \quad .....(1)$$

Similarly tangent at any point $(b \tan \theta, a \sec \theta)$ on 2nd hyperbolas is

$$\frac{y}{a} \sec \theta - \frac{x}{b} \tan \theta = 1 \quad .....(2)$$

If (1) and (2) are common tangents then they should be identical. Comparing the co–effecients of x and y

$$\Rightarrow \quad \frac{\sec\theta}{a} = -\frac{\tan\phi}{b} \quad .....(3)$$

and $\quad -\frac{\tan\theta}{b} = \frac{\sec \phi}{a}$

or $\quad \sec \theta = -\frac{a}{b} \tan\phi \quad ......(4)$

$\because \quad \sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \quad \frac{a^2}{b^2} \tan^2\phi - \frac{b^2}{a^2} \sec^2\phi = 1 \quad$ {from (3) and (4)}

or $\quad \frac{a^2}{b^2} \tan^2\phi - \frac{b^2}{a^2} (1 + \tan^2\phi) = 1$

or $\quad \left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right) \tan^2\phi = 1 + \frac{b^2}{a^2}$

$$\tan^2\phi = \frac{b^2}{a^2 - b^2}$$

and $\quad \sec^2\phi = 1 + \tan^2\phi = \frac{a^2}{a^2 - b^2}$

Hence the point of contanct are

$$\left\{\pm\frac{a^2}{\sqrt{(a^2 - b^2)}}, \pm\frac{b^2}{\sqrt{(a^2 - b^2)}}\right\} \text{ and } \left\{\pm\frac{b^2}{\sqrt{(a^2 - b^2)}}, \pm\frac{a^2}{\sqrt{(a^2 - b^2)}}\right\} \text{ \{from (3) and (4)\}}$$

Length of common tangent i.e., the distance between the above points is $\sqrt{2}\frac{(a^2 + b^2)}{\sqrt{(a^2 - b^2)}}$ and equation of common tangent on putting the values of secϕ and tanϕ in (1) is

$$\pm \frac{x}{\sqrt{(a^2-b^2)}} \mp \frac{y}{\sqrt{(a^2-b^2)}} = 1 \qquad \text{or} \qquad x \mp y = \pm \sqrt{(a^2-b^2)}$$

Self Practice Problems :

1. Show that the line x cos α + y sin α = p touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2 \cos^2\alpha - b^2 \sin^2\alpha = p^2$. **Ans.** $p^2 = a^2 \cos^2\alpha - b^2 \sin^2\alpha$

2. For what value of λ does the line y = 2x + λ touches the hyperbola $16x^2 - 9y^2 = 144$?
Ans. $\lambda = \pm 2\sqrt{5}$

3. Find the equation of the tangent to the hyperbola $x^2 - y^2 = 1$ which is parallel to the line 4y = 5x + 7.
Ans. 4y = 5x ± 3

8. NORMALS:

(a) The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P (x_1, y_1) on it is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2.$$

(b) The equation of the normal at the point P (a sec θ, b tan θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

(c) Equation of normals in terms of its slope 'm' are $y = mx \pm \frac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}$.

Example : A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola
$a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

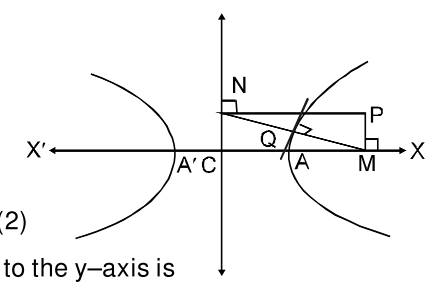
Solution. The equation of normal at the point Q (a sec ϕ, b tan ϕ) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax \cos\phi + by \cot\phi = a^2 + b^2 \qquad ....(1)$$

The normal (1) meets the x–axis in

$$M\left(\frac{a^2+b^2}{a}\sec\phi,\ 0\right) \text{ and y–axis in}$$

$$N\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$$



$\therefore$ Equation of MP, the line through M and perpendicular to x–axis, is

$$x = \left(\frac{a^2+b^2}{a}\right)\sec\phi \text{ or } \sec\phi = \frac{ax}{(a^2+b^2)} \qquad ....(2)$$

and the equation of NP, the line through N and perpendicular to the y–axis is

$$y = \left(\frac{a^2+b^2}{b}\right)\tan\phi \text{ or } \tan\phi = \frac{by}{(a^2+b^2)} \qquad ....(3)$$

The locus of the point of intersection of MP and NP will be obtained by eliminating ϕ from (2) and (3), we have

$$\sec^2\phi - \tan^2\phi = 1$$

$$\Rightarrow \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = \mathbf{1}$$

or $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

is the required locus of P.

Self Practice Problems :

1. Prove that the line lx + my – n = 0 will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

Ans. $\frac{a^2}{\ell^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.

2. Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Ans. $(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = x^2y^2 (a^2 + b^2)$

9. Pair of Tangents:

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by: $SS_1 = T^2$ where :

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \quad ; \quad S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \; ; \quad T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$$

Example : How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Find the equation these tangents & angle between them.

Solution.

Given point $P \equiv (4, 3)$

Hyperbola $S \equiv \frac{x^2}{16} - \frac{y^2}{9} - 1 = 0$

$\because \quad S_1 \equiv \frac{16}{16} - \frac{9}{9} - 1 = -1 < 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the hyperbola.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is

$SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right) . (-1) = \left(\frac{4x}{16} - \frac{3y}{9} - 1\right)^2$$

$$\Rightarrow -\frac{x^2}{16} + \frac{y^2}{9} + 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 - \frac{xy}{6} - \frac{x}{2} + \frac{2y}{3}$$

$\Rightarrow \quad 3x^2 - 4xy - 12x + 16y = 0$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

Example : Find the locus of point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution. Let P(h, k) be the point of intersection of two perpendicular tangents

equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} - \frac{ky}{b^2} - 1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(-\frac{k^2}{b^2} - 1\right) - \frac{y^2}{b^2}\left(\frac{h^2}{a^2} - 1\right) + \ldots\ldots = 0 \qquad \ldots\ldots\text{(i)}$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(-\frac{k^2}{b^2} - 1\right) - \frac{1}{b^2}\left(\frac{h^2}{a^2} - 1\right) = 0$$

$\Rightarrow \quad -k^2 - b^2 - h^2 + a^2 = 0 \qquad \Rightarrow \qquad$ locus is $x^2 + y^2 = a^2 - b^2$ **Ans.**

10. Director Circle :

The locus of the intersection point of tangents which are at right angles is known as the Director Circle of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real.

If $b^2 = a^2$ (rectangular hyperbola) the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no pair of tangents at right angle can be drawn to the curve.

11. Chord of Contact:

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$T = 0$, where $T = \dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} - 1$

Example :

If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Solution.

Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\dfrac{xh}{a^2} - \dfrac{yk}{b^2} = 1$ ....(i)

which touches the parabola

equation of tangent to parabola $y^2 = 4ax$

$y = mx - \dfrac{a}{m} \Rightarrow mx - y = -\dfrac{a}{m}$ ....(ii)

equation (i) & (ii) as must be same

$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(-\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1} \quad \Rightarrow \quad m = \frac{h}{k}\,\frac{b^2}{a^2} \;\&\; m = -\frac{ak}{b^2}$$

$$\therefore \quad \frac{hb^2}{ka^2} = -\frac{ak}{b^2} \quad \Rightarrow \quad \text{locus of P is } y^2 = -\frac{b^4}{a^3}\,.\,x \qquad \textbf{Ans.}$$

12. Chord with a given middle point:

Equation of the chord of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is $T = S_1$,

where $S_1 = \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} - 1$; $T \equiv \dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} - 1$.

Example : Find the locus of the mid - point of focal chords of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

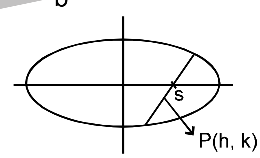
Solution. Let $P \equiv (h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given $\dfrac{xh}{a^2} - \dfrac{yk}{b^2} - 1 = \dfrac{h^2}{a^2} - \dfrac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either $(ae, 0)$ or $(-ae, 0)$

If it passes trhrough $(ae, 0)$



$\therefore$ locus is $\dfrac{ex}{a} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

If it passes through $(-ae, 0)$

$\therefore$ locus is $-\dfrac{ex}{a} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ **Ans.**

Example : Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\dfrac{x^2}{2a^2} - \dfrac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line $x + y = b$.

Solution. Let the line $x + y = b$ bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b)

$$\therefore \quad \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

$$\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0$$

$$\alpha = 0, \quad \alpha = \frac{1}{\frac{1}{a} + \frac{1}{b}} \qquad \therefore \quad a \neq \pm b$$

Example : Find the locus of the mid point of the chords of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Solution. let (h,k) be the mid–point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{b^2} - \frac{k^2}{b^2} - 1 \qquad \text{or} \qquad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{........(i)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chored (1) is obtained by making homogeneous hyperbola with the help of (1)

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2 = \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \quad \text{.......(2)}$$

The lines represented by (2) will be at right angle if coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{k^2}{b^4}$$

hence, the locus of (h,k) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$$

Self Practice Problem

1. Find the equation of the chord $\frac{x^2}{36} - \frac{y^2}{9} = 1$ which is bisected at (2, 1). **Ans.** $x = 2y$

2. Find the point 'P' from which pair of tangents PA & PB are drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ in such a way that (5, 2) bisect AB **Ans.** $\left(\frac{375}{4}, 12\right)$

3. From the points on the circle $x^2 + y^2 = a^2$, tangent are drawn to the hyperbola $x^2 - y^2 = a^2$, prove that the locus of the middle points of the chords of contact is the curve $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$.
Ans. $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$.

13. Diameter :

The locus of the middle points of a system of parallel chords with slope 'm' of an hyperbola is called its diameter. It is a straight line passing through the centre of the hyperbola and has the equation $y = -\frac{b^2}{a^2m}x$. **NOTE :** All diameters of the hyperbola passes through its centre.

14. Asymptotes :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the hyperbola.

Equations of Asymptote : $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

Pair of asymptotes : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

NOTE : (i) A hyperbola and its conjugate have the same asymptote.

(ii) The equation of the pair of asymptotes differs from the equation of hyperbola (or conjugate hyperbola) by the constant term only.

(iii) The asymptotes pass through the centre of the hyperbola & are equally inclined to the transverse axis of the hyperbola. Hence the bisectors of the angles between the asymptotes are the principle axes of the hyperbola.

(iv) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(v) Asymptotes are the tangent to the hyperbola from the centre.

(vi) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let f (x, y) = 0 represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

Example : Find the asymptotes $xy - 3y - 2x = 0$.

Solution. Since equation of a hyperbola and its asymptotes differ in constant terms only,

$\therefore$ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$

where λ is any constant such that it represents two straight lines.

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow \quad 0 + 2 \times -\frac{3}{2} \times -1 \times \frac{1}{2} - 0 - 0 - \lambda \left(\frac{1}{2}\right)^2 = 0$$

$\therefore$ $\lambda = 6$

From (1), the asymptotes of given hyperbola are given by

$xy - 3y - 2x + 6 = 0$ or $(y - 2)(x - 3) = 0$

$\therefore$ Asymptotes are $x - 3 = 0$ and $y - 2 = 0$

Example : The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3), show that its equation is

$(2x + 3y - 8)(3x + 2y + 7) = 154$

Solution. Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since asymptotes passes through (1,2), then $\lambda = -8$ and $\mu = -7$

Thus the equation of asympotes are

$2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$

Let the equation of hyperbola be

$(2x + 3y - 8)(3x + 2y - 7) + v = 0$ ......(1)

It passes through (5,3), then

$(10 + 9 - 8)(15 + 6 - 7) + v = 0$

$\Rightarrow$ $11 \times 14 + v = 0$

$\therefore$ $v = -154$

putting the value of v in (1) we obtain

$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$

which is the equation of required hyperbola.

Self Practice Problems :

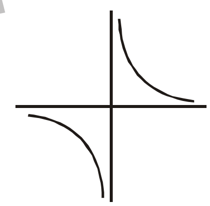
1. Show that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and that the portion of it intercepted between the asymptotes is bisected at the point of contact.

15. Rectangular Or Equilateral Hyperbola :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

Rectangular Hyperbola ($xy = c^2$) :

It is referred to its asymptotes as axes of co–ordinates.



Vertices : (c, c) & (– c, – c);

Foci : $\left(\sqrt{2}\,c, \sqrt{2}\,c\right)$ & $\left(-\sqrt{2}\,c, -\sqrt{2}\,c\right)$,

Directrices : $x + y = \pm\sqrt{2}\,c$

Latus Rectum (l) :

$\ell = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of a chord joining the points P (t_1) & Q(t_2) is $x + t_1 t_2 y = c(t_1 + t_2)$.

Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P (t) is $\frac{x}{t} + t\,y = 2c$.

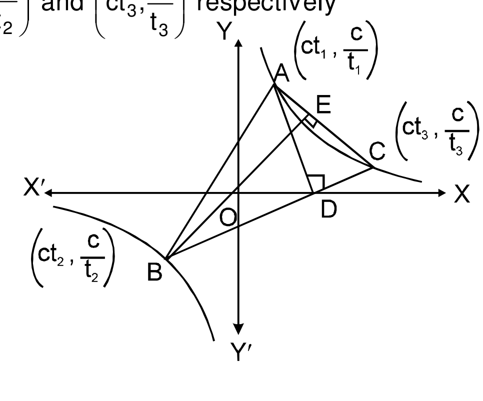
Equation of the normal at P (t) is $x\,t^3 - y\,t = c\,(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Example : A triangle has its vertices on a rectangle hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution. Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ Co–ordinates of A,B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively



Now lope of BC is $\frac{t_3 - t_2}{ct_3 - ct_2} = -\frac{1}{t_2 t_3}$

$\therefore$ Slope of AD is $t_2 t_3$

Equation of Altitude AD is

$$y - \frac{c}{t_1} = t_2 t_3 (x - ct_1)$$

or $t_1y - c = x\, t_1t_2t_3 - ct_1^2t_2t_3$ .....(1)

Similarly equation of altitude BE is

$t_2y - c = x\, t_1t_2t_3 - ct_1t_2^2t_3$ ......(2)

Solving (1) and (2), we get the orthocentre $\left(-\dfrac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Which lies on $xy = c^2$.

Example : A, B, C are three points on the rectangular hyperbola $xy = c^2$, find

(i) The area of the triangle ABC

(ii) The area of the triangle formed by the tangents A, B and C.

Sol. Let co–ordinates of A,B and C on the hyperbola $xy = c^2$ are $\left(ct_1, \dfrac{c}{t_1}\right)$, $\left(ct_2, \dfrac{c}{t_2}\right)$ and $\left(ct_3, \dfrac{c}{t_3}\right)$ respectively.

(i) $\therefore$ Area of triangle ABC $= \dfrac{1}{2}\left|\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2}\end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3}\end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1}\end{vmatrix}\right|$

$$= \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\left|t_1^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\,|\,(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\,|$$

(ii) Equations of tangents at A,B,C are

$x + t_1^2 - 2ct_1 = 0$

$x + yt_2^2 - 2ct_2 = 0$

and $x + yt_3^2 - 2ct_3 = 0$

$\therefore$ Required Area $= \dfrac{1}{2\,|\,C_1C_2C_3\,|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3\end{vmatrix}^2$ ....(1)

where $C_1 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2\end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2\end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2\end{vmatrix}$

$\therefore$ $C_1 = t_3^2 - t_2^2$, $C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From (1) $= \dfrac{1}{2\left|(t_3^2 - t_2^2)\,(t_1^2 - t_3^2)(t_2^2 - t_1^2)\right|}\; 4c^2.(t_1 - t_2)^2\,(t_2 - t_3)^2\,(t_3 - t_1)^2$

$$= 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)\,(t_2 + t_3)\,(t_3 + t_1)}\right|$$

$\therefore$ Required area is, $2c^2\left|\dfrac{(t_1 - t_2)\,(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)\,(t_2 + t_3)\,(t_3 + t_1)}\right|$

Example : Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

Solution. Let rectangular hyperbola is $x^2 - y^2 = a^2$

Let equations of PQ and DE are

$y = mx + c$ ......(1)

and $y = m_1x + c_1$ ....(2)

respectively.

Be any two focal chords of any rectangular hyperbola $x^2 - y^2 = a^2$ through its focus. We have to prove PQ = DE. Since PQ $\perp$ DE.

$\therefore$ $mm_1 = -1$ ......(3)

Also PQ passes through S $(a\sqrt{2}, 0)$ then from (1),

$0 = ma\sqrt{2} + c$

or $c^2 = 2a^2m^2$ ......(4)

Let (x_1,y_1) and (x_2,y_2) be the co–ordinates of P and Q then

$(PQ)^2 = (x_1 - x_2^2) + (y_1 - y_2)^2$ ......(5)

Since (x_1,y_1) and (x_2,y_2) lie on (1)

$\therefore$ $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

$\therefore$ $(y_1 - y_2) = m\,(x_1 - x_2)$ ....(6)

From (5) and (6)

$(PQ)^2 = (x_1 - x_2)^2\,(1 + m^2)$ ....(7)

Now solving y = mx + c and $x^2 - y^2 = a^2$ then

$x^2 - (mx + c)^2 = a^2$

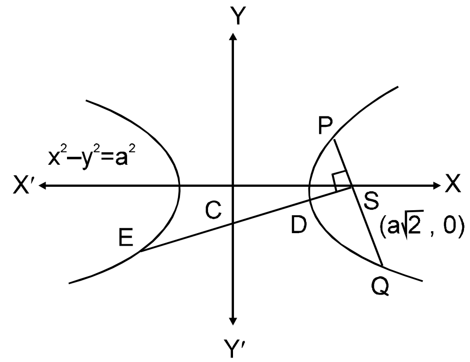
or $(m^2 - 1)x^2 + 2mcx + (a^2 + c^2) = 0$

$\therefore \quad x_1 + x_2 = \dfrac{2mc}{m^2-1}$ and $x_1x_2 = \dfrac{a^2+c^2}{m^2-1}$

$\Rightarrow \quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$$= \frac{4m^2c^2}{(m^2-1)^2} - \frac{4(a^2+c^2)}{(m^2-1)}$$

$$= \frac{4\{a^2+c^2-a^2m^2\}}{(m^2-1)^2}$$



$$= \frac{4a^2(m^2+1)}{(m^2-1)^2} \qquad \{\because c^2 = 2a^2m^2\}$$

From (7), $\quad (PQ)^2 = 4a^2\left(\dfrac{m^2+1}{m^2-1}\right)$

Similarly, $\quad (DE)^2 = 4a^2\left(\dfrac{m_1^2+1}{m_1^2-1}\right)^2$

$$= 4a^2\left(\frac{\left(1-\frac{1}{m}\right)^2+1}{\left(-\frac{1}{m}\right)^2-1}\right)^2 \qquad (\because mm_1 = -1)$$

$$= 4a^2\left(\frac{m^2+1}{m^2-1}\right)$$

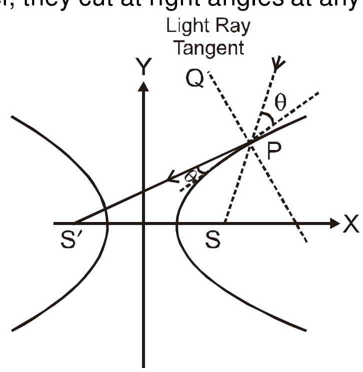
$$= (PQ)^2$$

Thus $(PQ)^2 = (DE)^2 \Rightarrow PQ = DE$.

Hence perpendicular focal chords of a rectangular hyperbola are equal.

15. Important Results :

- Locus of the feet of the perpendicular drawn from focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.
- The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.
- The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ & the hyperbola $\dfrac{x^2}{a^2-k^2} - \dfrac{y^2}{k^2-b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

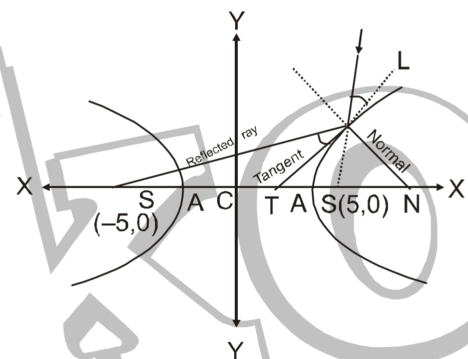
- The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.
- If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.
- Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

- The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2-b^2y^2)=(a^2+b^2)^2$.
- If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.
- A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(c\,t_i, \frac{c}{t_i}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1 t_2 t_3}, -c t_1 t_2 t_3\right)$.
- If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then
  (a) $t_1 t_2 t_3 t_4 = 1$
  (b) the centre of the mean position of the four points bisects the distance between the centres of the two curves.
  (c) the centre of the circle through the points t_1, t_2 & t_3 is :
  $$\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1+t_2+t_3\right)\right\}$$

Example :

A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lies in first quadrant.

Solution.



Given hyperbola is
$9x^2 - 16y^2 = 144$. This equation can be rewritten as $\frac{x^2}{16}-\frac{y^2}{9}=1$ ......(1)

Since x co–ordinate of P is 8. Let y co–ordinate of P ia α.

$\because$ $(8,\alpha)$ lies on (1)

$\therefore$ $\frac{64}{16}-\frac{\alpha^2}{9}=1$

$\Rightarrow$ $\alpha^2 = 27$

$\Rightarrow$ $a = 3\sqrt{3}$ ($\because$ P lies in first quadrant)

Hence coªordinate of point P is $(8,3\sqrt{3})$.

$\because$ Equation of reflected ray passing through P $(8,3\sqrt{3})$ and S′(–5,0)

$\therefore$ Its equation is $y-3\sqrt{3}=\frac{0-3\sqrt{3}}{-5-8}(x-8)$

or $13y - 39\sqrt{3} = 3\sqrt{3}\,x - 24\sqrt{3}$

or $3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$.