

Probability

• Complementary events

For an event E such that $0 \leq P(E) \leq 1$ of an experiment, the event \bar{E} represents 'not E ', which is called the complement of the event E . We say, E and \bar{E} are **complementary** events.

$$P(E) + P(\bar{E}) = 1$$

$$\Rightarrow P(\bar{E}) = 1 - P(E)$$

Example:

A pair of dice is thrown once. Find the probability of getting a different number on each die.

Solution:

When a pair of dice is thrown, the possible outcomes of the experiment can be listed as:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The number of all possible outcomes = $6 \times 6 = 36$

Let E be the event of getting the same number on each die.

Then, \bar{E} is the event of getting different numbers on each die.

Now, the number of outcomes favourable to E is 6.

$$\therefore P(\bar{E}) = 1 - P(E) = 1 - \frac{6}{36} = \frac{5}{6}$$

Thus, the required probability is $\frac{5}{6}$.

• Algebra of events

- **Complementary event:** For every event A , there corresponds another event A' called the complementary event to A . It is also called the event 'not A '.

$$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A.$$

- **The event 'A or B':** When sets A and B are two events associated with a sample space, then the set $A \cup B$ is the event 'either A or B or both'.

That is, event 'A or B' = $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$

- **The event 'A and B':** When sets A and B are two events associated with a sample space, then the set $A \cap B$ is the event 'A and B'.

That is, event ' A and B ' $= A \cap B = \{\omega: \omega \in A \text{ and } \omega \in B\}$

- **The event ' A but not B ':** When sets A and B are two events associated with a sample space, then the set $A - B$ is the event ' A but not B '.

That is, event ' A but not B ' $= A - B = A \cap B' = \{\omega: \omega \in A \text{ and } \omega \notin B\}$

Example: Consider the experiment of tossing 2 coins. Let A be the event 'getting at least one head' and B be the event 'getting exactly two heads'. Find the sets representing the events

(i) complement of ' A or B '

(ii) A and B

(iii) A but not B

Solution:

Here, $S = \{HH, HT, TH, TT\}$

$A = \{HH, HT, TH\}$, $B = \{HH\}$

(i) A or $B = A \cup B = \{HH, HT, TH\}$

Hence, complement of A or $B = (A \text{ or } B)' = (A \cup B)' = U - (A \cup B) = \{TT\}$

(ii) A and $B = A \cap B = \{HH\}$

(iii) A but not $B = A - B = \{HT, TH\}$

• Mutually Exclusive Events

Two events, A and B , are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event i.e., if they cannot occur simultaneously.

In this case, sets A and B are disjoint i.e., $A \cap B = \emptyset$

If E_1, E_2, \dots, E_n are n events of a sample space S , and if

$\bigcup_{i=1}^n E_i = S$, $E_1 \cup E_2 \cup \dots \cup E_n = \bigcup_{i=1}^n E_i = S$ then

E_1, E_2, \dots, E_n are called mutually exclusive and exhaustive events.

In other words, at least one of E_1, E_2, \dots, E_n necessarily occurs whenever the experiment is performed.

The events E_1, E_2, \dots, E_n , i.e., n events of a sample space (S) are called mutually exclusive and exhaustive events if

$E_i \cap E_j = \emptyset$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint, and

$\bigcup_{i=1}^n E_i = S$

Example: Consider the experiment of tossing a coin twice. Let A and B be the event of "getting at least one head" and "getting exactly two tails" respectively. Are the events A and B mutually exclusive and exhaustive?

Solution:

Here, $S = \{HH, HT, TH, TT\}$

$$A = \{HH, HT, TH\}$$

$$B = \{TT\}$$

Now, $A \cap B = \emptyset$ and $A \cup B = \{HH, HT, TH, TT\} = S$

Thus, A and B are mutually exclusive and exhaustive events.

- The number $P(\omega_j)$ i.e., the probability of the outcome ω_j , is such that
 - $0 \leq P(\omega_j) \leq 1$
 - $\sum P(\omega_j) = 1$ for all $\omega_j \in S$
 - For any event A , $P(A) = \sum P(\omega_j)$ for all $\omega_j \in A$
- For a finite sample space S , with equally likely outcomes, the probability of an event A is denoted as $P(A)$ and it is given by

$$P(A) = \frac{n(A)}{n(S)},$$

- Where, $n(A)$ = Number of elements in set A and $n(S)$ = Number of elements in set S
 - If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

- If A is any event, then

$$P(A') = 1 - P(A)$$

Example: Consider the experiment of tossing a die. Let A be the event “getting an even number greater than 2” and B be the event “getting the number 4”. Find the probability of

(i) getting an even number greater than 2 or the number 4

(ii) getting a number, which is not the number 4, on the top face of the die

Solution: Here, $S = \{1, 2, 3, 4, 5, 6\}$

$$A = \{4, 6\}, B = \{4\}$$

$$A \cap B = \{4\}$$

$$p(A) = \frac{2}{6}, p(B) = \frac{1}{6}, p(A \cap B) = \frac{1}{6}$$

$$\begin{aligned} \text{(i) Required probability} &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= \frac{2}{6} + \frac{1}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

$$\text{(ii) } P(B) = \frac{1}{6}$$

$$\therefore P(\text{not } B) = 1 - P(B) = 1 - \frac{1}{6} = \frac{5}{6}$$

Hence, the required probability of not getting number 4 on the top face of the die is $\frac{5}{6}$.

Example: 20 cards are selected at random from a deck of 52 cards. Find the probability of getting at least 12 diamonds.

Solution: 20 cards can be selected at random from a deck of 52 cards in ${}^{52}C_{20}$ ways. Hence,

$$\text{Total possible outcomes} = {}^{52}C_{20}$$

$$\begin{aligned} P(\text{at least 12 diamonds}) &= P(12 \text{ diamonds or } 13 \text{ diamonds}) \\ &= P(12 \text{ diamonds}) + P(13 \text{ diamonds}) \end{aligned}$$

$$\begin{aligned}
&= \frac{{}^{13}C_{12} \times {}^{39}C_8}{{}^{52}C_{20}} + \frac{{}^{13}C_{13} \times {}^{39}C_7}{{}^{52}C_{20}} \\
&= \frac{13 \times {}^{39}C_8}{{}^{52}C_{20}} + \frac{{}^{39}C_7}{{}^{52}C_{20}} \\
&= \frac{13 \times {}^{39}C_8 + {}^{39}C_7}{{}^{52}C_{20}} \\
&= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39!}{32! \times 7!}}{{}^{52}C_{20}} \\
&= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{39! \times 8}{32 \times 31! \times 7! \times 8}}{{}^{52}C_{20}} \\
&= \frac{13 \times \frac{39!}{31! \times 8!} + \frac{8}{32} \times \frac{39!}{31! \times 8!}}{{}^{52}C_{20}} \\
&= \frac{\frac{53}{4} \times \frac{39!}{31! \times 8!}}{{}^{52}C_{20}} \\
&= \frac{53}{4} \times \frac{{}^{39}C_8}{{}^{52}C_{20}}
\end{aligned}$$