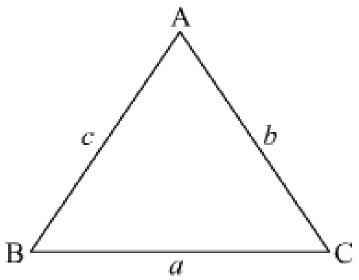
Trigonometry

Properties and Solutions of Triangles

A triangle is a polygon having three sides and three angles. Consider the ΔABC whose lengths of the sides AB, BC and CA are *c*, *a* and *b* respectively.



Some Geometrical Properties Related to ΔABC

1. ∠A + ∠B + ∠C = 180° = π radians

2. Perimeter, $2s = a + b + c$

Semiperimeter, $s = \frac{a+b+c}{2}$

3. Sum of any two sides of a triangle is always greater than the third side.

$a + b > c$, $b + c > a$ and $c + a > b$

4. Difference of any two sides of a triangle is always less than the third side.

$|a-b|<c, |b-c|<a$ and $|c-a|<b$

Sine Rule

This rule states that the sines of the angles of a triangle are proportional to the lengths of their opposite sides.

In any ΔABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Let *R* be the radius of the circumcircle. Thus, we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Cosine Rule

In any ΔABC,

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$
$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$
$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

Projection Formulae

In any ΔABC,

$$a=b\cos C+c\cos B$$
$$b=c\cos A+a\cos C$$
$$c=a\cos B+b\cos A$$

Napier's Analogy

In any ΔABC,

$$\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$$
$$\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$$
$$\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$$

Half-Angle Formulae

In any ΔABC,

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}},\ \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$
$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\ \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\ \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$$
$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\ \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Area of Triangle

Let the area of ΔABC is denoted by Δ. Thus, we have:

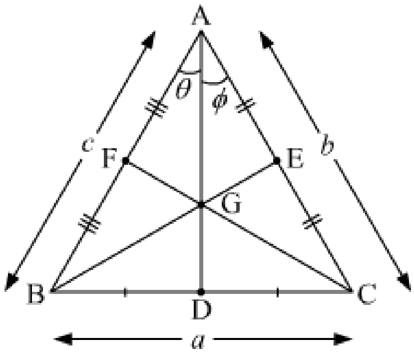
$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}\ \ Heron's Formula$$

$$\Delta=\ \tfrac{1}{2}bc\sin A=\ \tfrac{1}{2}ca\sin B=\ \tfrac{1}{2}ab\sin C\ \Delta\ =\frac{b^2\sin C\,\sin A}{2\sin B}\ =\frac{c^2\sin A\,\sin B}{2\sin C}\ =\frac{a^2\sin B\,\sin C}{2\sin A}\ \ \Delta\ =\frac{abc}{4R}\ =rs$$

Here, r and R are the radii of the incircle and circumcircle of ΔABC respectively.

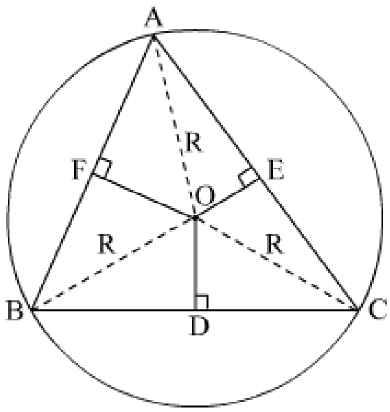
Some Terms Related to Triangle

- **Centroid, G:** The point of intersection of the medians of a triangle is known as the centroid of the triangle.



A centroid divides every median in the ratio 2:1, i.e., AG:GD = BG:GE = CG:GF = 2:1.

- **Circumcentre, O:** The point of intersection of perpendicular bisectors of all sides of a triangle is called its circumcentre. This point is equidistant from the three vertices of the circle passing through them.

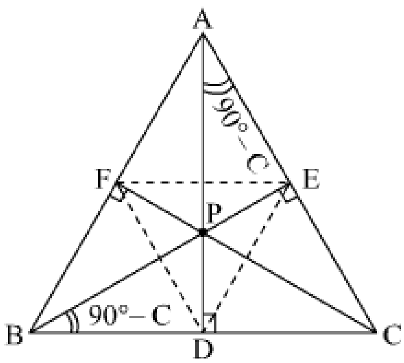


The circle drawn with this point as the centre is called circumcircle.

OA = OB = OC = *R*

Here, *R* is the circumradius.

- **Orthocentre, P:** The point of intersection of the altitudes of a triangle is called the orthocentre of the triangle.



In ΔABC, P is the orthocentre and ΔDEF is the pedal triangle.

The orthocentre lies outside the triangle for an obtuse triangle, inside the triangle for an acute triangle and on the right angle for a right-angled triangle.

1. Orthocentre P, centroid G and circumcentre O are always collinear. Centroid G divides PO in the ratio 2:1.

2. AD = $c \sin B = b \sin C$

BE = $a \sin C = c \sin A$

CF = $a \sin B = b \sin A$

3. BF:FA = $a \sec A : b \sec B$

BD:DC = $c \sec C : b \sec B$

CE:EA = $a \sec A : c \sec C$

4. ∠APB = ∠A + ∠B

∠APC = ∠A + ∠C

∠BPC = ∠B + ∠C

5. PA = $2R \cos A$

PB = $2R \cos B$

PC = $2R \cos C$

6. PD = $2R \cos B \cos C$

PE = $2R \cos C \cos A$

PF = $2R \cos A \cos B$

Properties of Pedal ΔDEF

1. Circumradius of ΔDEF = $\frac{R}{2}$ = Half of the circumradius of ΔABC

2. ∠EDF = 180° − 2∠A

∠DFE = 180° − 2∠C

∠FED = 180° − 2∠B

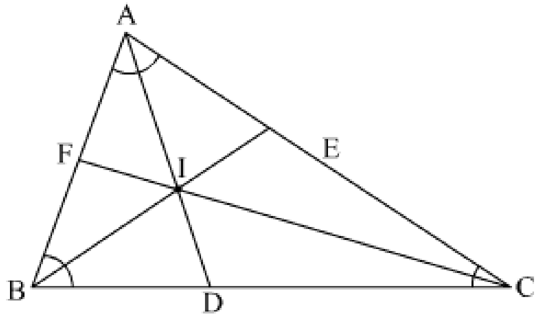
3. EF = $a \cos A$

FE = $b \cos B$

DE = $c \cos C$

4. The incentre of ΔDEF is the orthocentre of ΔABC.

- **Incentre I:** The point of intersection of the interior angle bisectors of a triangle is called the incentre of the triangle. The circle drawn taking this point as the centre and touching all the sides of the triangle is called the incircle of the triangle. The radius of this circle is called inradius; it is denoted by *r*.



In ΔABC, I is the incentre. The incentre always lies inside a triangle.

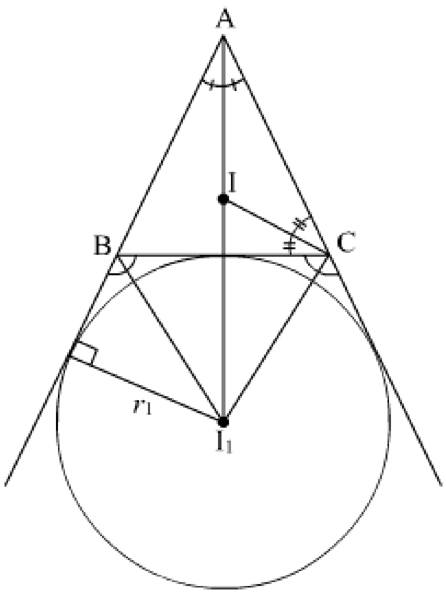
1. $r=\dfrac{\Delta}{s}$

2. $r=(s-a)\tan\dfrac{A}{2}=(s-b)\tan\dfrac{B}{2}=(s-c)\tan\dfrac{C}{2}$

3. $r=\dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=\dfrac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}}=\dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

4. $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

- **Excentres (I_1, I_2, I_3):** In ΔABC, the bisectors of the exterior angles ∠B and ∠C obtained on producing the sides AB and AC, respectively, intersect each other at point I_1. The circle having the centre I_1 and touching the side BC and the extended sides AB and AC is called the excircle of ΔABC.



I_1 is called the excentre opposite to ∠A of the triangle. Three excircles are possible in a triangle. Excentres always lie outside a triangle.

Let r_1, r_2 and r_3 be the radii of excircles opposite to ∠A, ∠B and ∠C, respectively. Thus, we have:

1. $r_1=\dfrac{\Delta}{s-a}=s\tan\dfrac{A}{2}=\dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}=4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

2. $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = \dfrac{b\cos\dfrac{C}{2}\cos\dfrac{A}{2}}{\cos\dfrac{B}{2}} = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$

3. $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = \dfrac{c\cos\dfrac{A}{2}\cos\dfrac{B}{2}}{\cos\dfrac{C}{2}} = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

- **Distance between the circumcentre O and the orthocentre P**

$$\text{OP} = R\sqrt{1-8\cos A\cos B\cos C}$$

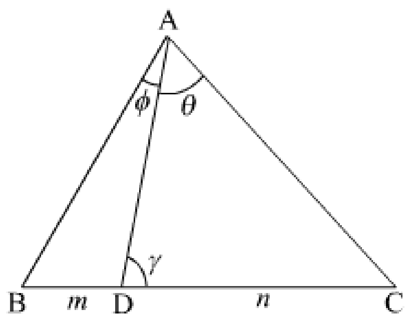
- **Distance between the circumcentre O and the incentre I**

$$\text{OI} = R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2-2rR}$$

- **Ptolemy's Theorem**

In a cyclic quadrilateral PQRS, PR × QS = PQ × RS + QR × PS.

- ***m–n* Theorem:** If D is a point on the side BC of ΔABC such that BD:DC = m:n and ∠BAD = Φ, ∠CAD = θ and ∠ADC = γ, then:



1. $(m+n)\cot\gamma = m\cot\phi - n\cot\theta$

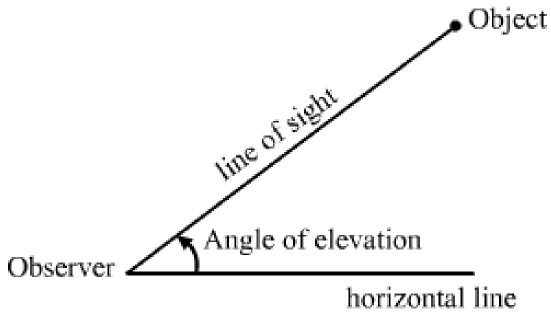
2. $(m+n)\cot\gamma = n\cot B - m\cot C$

Line of Sight

The line of sight is a line drawn from the eye of an observer to the point in the object viewed by the observer.

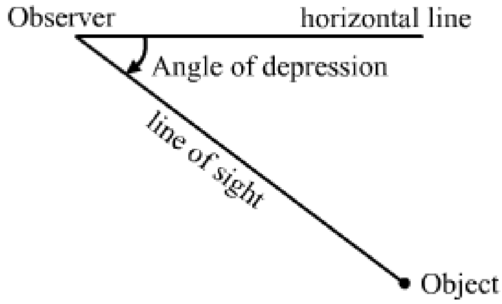
Angle of Elevation

The angle of elevation is the angle formed by the line of sight with a horizontal line along the eye of an observer when an object lies above the horizontal line.



Angle of Depression

The angle of depression is the angle formed by the line of sight with a horizontal line along the eye of an observer when an object lies below the horizontal line.



- If in a circle of radius *r*, an arc of length *l* subtends an angle of θ radians, then $l = r\theta$.
- Radian measure $= \frac{\pi}{180} \times \text{Degree measure}$
- Degree measure $= \frac{180}{\pi} \times \text{Radian measure}$
- A degree is divided into 60 minutes and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as 1', and one sixtieth of a minute is called a second, written as1".
  Thus, $1^\circ = 60'$ and $1' = 60''$
- **Signs of trigonometric functions in different quadrants:**

Trigonometric function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
sin *x*	+ ve *Increases from 0 to 1*	+ ve *Decreases from 1 to 0*	–ve (Decreases from 0 to -1)	–ve (Increases from -1 to 0)
cos *x*	+ ve *Decreases from 1 to 0*	–ve (Decreases from 0 to -1)	–ve (Increases from -1 to 0)	+ ve *Increases from 0 to 1*
tan *x*	+ ve *Increases from 0 to* ∞	–ve (Increases from -∞ to 0)	+ ve *Increases from 0 to* ∞	–ve (Increases from -∞ to 0)
cot *x*	+ ve *Decreases from* ∞ *to 0*	–ve(Decreases from 0 to -∞)	+ ve *Decreases from* ∞ *to 0*	–ve (Decreases from 0 to -∞)
sec *x*	+ ve *Increases from 1 to* ∞	–ve (Increases from -∞ to -1)	–ve (Decreases from -1 to -∞)	+ ve *Decreases from* ∞ *to 1*
cosec *x*	+ ve *Decreases from* ∞ *to 1*	+ ve *Increases from 1 to* ∞	–ve (Increases from -∞ to -1)	–ve (Decreases from -1 to -∞)

Example 1:

If $\sin\theta = -\frac{1}{\sqrt{3}}$, where $\pi < \theta < \frac{3\pi}{2}$, then find the value of $3\tan\theta - \sqrt{3}\sec\theta$.

Solution:

Since θ lies in the third quadrant, therefor tan θ is positive and cos θ (or sec θ) is negative.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\left(-\frac{1}{\sqrt{3}}\right)^2} = \pm\sqrt{1-\frac{1}{3}} = \pm\sqrt{\frac{2}{3}}$$

$$\therefore \cos\theta = -\sqrt{\frac{2}{3}}$$

$$\Rightarrow \sec\theta = -\sqrt{\frac{3}{2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{\sqrt{3}}}{-\sqrt{\frac{2}{3}}} = \frac{1}{\sqrt{2}}$$

$$\therefore 3\tan\theta - \sqrt{3}\sec\theta = 3\times\frac{1}{\sqrt{2}} - \sqrt{3}\times\left(-\sqrt{\frac{3}{2}}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 3\sqrt{2}$$

Example 2: Find the value of cos 390° cos 510° + sin 390° cos (–660°).

Solution:

$$\cos 390° = \cos(2\times180° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\cos 510° = \cos(3\times180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

$$\sin 390° = \sin(2\times180° + 30°) = \sin 30° = \frac{1}{2}$$

$$\cos(-660°) = \cos 660° = \cos(4\times180° - 60°) = \cos 60° = \frac{1}{2}$$

$$\therefore \cos 390° \cos 510° + \sin 390° \cos(-660°)$$

$$= \frac{\sqrt{3}}{2}\times\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right)$$

$$= -\frac{3}{4} + \frac{1}{4}$$

$$= -\frac{2}{4}$$

$$= -\frac{1}{2}$$

- **Domain and Range of trigonometric functions:**

Trigonometric function	Domain	Range
sin x	**R**	$-1, 1$
cos x	**R**	$-1, 1$
tan x	**R** - $\left\{x : x = \frac{(2n+1)\pi}{2}, n \in Z\right\}$	**R**
cot x	**R** - $\{x : x = n\pi, n \in \mathbf{Z}\}$	**R**
sec x	**R** - $\left\{x : x = \frac{(2n+1)\pi}{2}, n \in Z\right\}$	**R** - $-1, 1$
cosec x	**R** - $\{x : x = n\pi, n \in \mathbf{Z}\}$	**R** - $-1, 1$

- **Trigonometric identities and formulas:**
  - $\operatorname{cosec} x = \frac{1}{\sin x}$
  - $\sec x = \frac{1}{\cos x}$
  - $\tan x = \frac{\sin x}{\cos x}$
  - $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
  - $\cos^2 x + \sin^2 x = 1$
  - $1 + \tan^2 x = \sec^2 x$
  - $1 + \cot^2 x = \operatorname{cosec}^2 x$
  - $\cos(2n\pi + x) = \cos x, n \in Z$
  - $\sin(2n\pi + x) = \sin x, n \in Z$
  - $\sin(-x) = -\sin x$
  - $\cos(-x) = \cos x$
  - $\cos(x + y) = \cos x \cos y - \sin x \sin y$
  - $\cos(x - y) = \cos x \cos y + \sin x \sin y$
  - $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
  - $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
  - $\sin(x + y) = \sin x \cos y + \cos x \sin y$
  - $\sin(x - y) = \sin x \cos y - \cos x \sin y$
  - $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
  - $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
  - $\cos(\pi - x) = -\cos x$
  - $\sin(\pi - x) = \sin x$
  - $\cos(\pi + x) = -\cos x$
  - $\sin(\pi + x) = -\sin x$
  - $\cos(2\pi - x) = \cos x$
  - $\sin(2\pi - x) = -\sin x$
  - If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \text{ and } \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

  - If none of the angles x, y and $(x \pm y)$ is a multiple of π, then
    $\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cos x}$, and $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$
- In particular, $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2} - 1 = 1 - 2\sin^2 \frac{x}{2} = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$
- $\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$
- In particular, $\sin x = 2\sin \frac{x}{2} \cos \frac{x}{2} = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
- In particular,

- **General solutions of some trigonometric equations:**
  - $\sin x = 0 \Rightarrow x = n\pi,$, where $n \in \mathbf{Z}$
  - $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$,where $n \in \mathbf{Z}$
  - $\sin x = \sin y \Rightarrow x = n\pi + -1^n y$, where $n \in \mathbf{Z}$
  - $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$, where $n \in \mathbf{Z}$
  - $\tan x = \tan y \Rightarrow x = n\pi + y$, where $n \in \mathbf{Z}$

Example 1: Solve $\cot x \cos^2 x = 2 \cot x$
Solution:

$\cot x \cos^2 x = 2\cot x$

$\Rightarrow \cot x \cos^2 x - 2\cot x = 0$

$\Rightarrow \cot x (\cos^2 x - 2) = 0$

$\Rightarrow \cot x = 0$ or $\cos^2 x = 2$

$\Rightarrow \frac{\cos x}{\sin x} = 0$ or $\cos x = \pm\sqrt{2}$

$\Rightarrow \cos x = 0$ or $\cos x = \pm\sqrt{2}$

Now, $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$

and $\cos x = \pm\sqrt{2}$

But this is not possible as $-1 \le \cos x \le 1$

Thus, the solution of the given trigonometric equation is $x = (2n+1)\frac{\pi}{2}$ where $n \in$ Z.

Example 2: Solve $\sin 2x + \sin 4x + \sin 6x = 0$.
Solution:

$\sin 4x + (\sin 2x + \sin 6x) = 0$

$\Rightarrow \sin 4x + 2\sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right) = 0$

$\Rightarrow \sin 4x + 2\sin 4x \cos 2x = 0$

$\Rightarrow \sin 4x(1 + 2\cos 2x) = 0$

$\Rightarrow \sin 4x = 0$ or $1 + 2\cos 2x = 0$

$\Rightarrow \sin 4x = 0$ or $\cos 2x = -\frac{1}{2}$

$\sin 4x = 0$

$\Rightarrow 4x = n\pi, n \in Z$

$\Rightarrow x = \frac{n\pi}{4}, n \in Z$

$\cos 2x = -\frac{1}{2}$

$\Rightarrow \cos 2x = \cos\frac{2\pi}{3}$

$\Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3}, m \in Z$

$\Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in Z$

Thus, $x = \frac{n\pi}{4}$ or $x = m\pi \pm \frac{\pi}{3}$, where m, $n \in$ **Z**