Electrostatic Fields: Coulomb's Law & the Electric Field Intensity

EE 141 Lecture Notes
Topic 1

Professor K. E. Oughstun
School of Engineering
College of Engineering & Mathematical Sciences
University of Vermont

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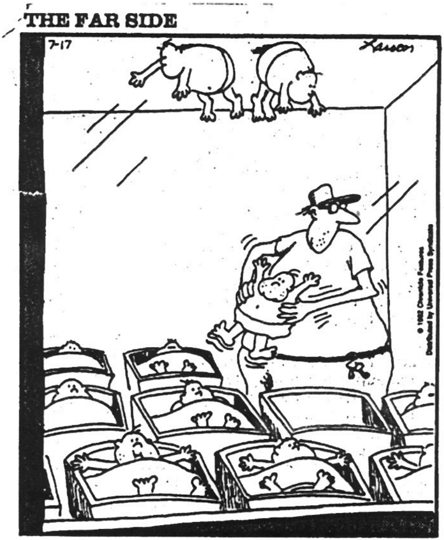
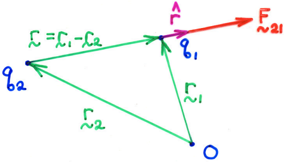
Motivation



Late at night, and without permission, Reuben would often enter the nursery and conduct experiments in static electricity.

Coulomb's Law

Let q_1 be a stationary point charge with position vector $\mathbf{r}_1$ relative to a fixed origin O, and let q_2 be a separate, distinct point charge with position vector $\mathbf{r}_2 \neq \mathbf{r}_1$ relative to the same origin O.



Coulomb's Law then states that the force $\mathbf{F}_{21}$ exerted on q_1 by q_2 is given by

$$\boxed{\mathbf{F}_{21} = \mathcal{K}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}} \tag{1}$$

where $r \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ is the separation distance between q_1 & q_2 and where $\hat{\mathbf{r}} \equiv (\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2| = (\mathbf{r}_1 - \mathbf{r}_2)/r$ is the unit vector in the direction from q_2 to q_1. The force is repulsive if q_1 & q_2 are of the same sign and attractive if they are of the opposite sign.

Coulomb's Law

Reciprocity requires that an equal but oppositely directed force $\mathbf{F}_{12}$ is exerted on q_2 by q_1; that is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{2}$$

In the rationalized MKSA (meter, kilogram, second, ampere) system, the unit of force is the newton (N), the unit of charge is the coulomb (C), and the constant appearing in Coulomb's law is given by

$$\mathcal{K} = \frac{1}{4\pi\epsilon_0} \simeq 8.988 \times 10^9 N \cdot m^2/C^2.$$

Permittivity of free space:

$$\epsilon_0 \simeq 8.8542 \times 10^{-12} F/m,$$

so that

$$\mathcal{K} \simeq 8.988 \times 10^9 m/F$$

where $Farad \equiv Coulomb/volt$ is the unit of capacitance.

Coulomb's Law

Coulomb's Law in MKSA units then becomes

$$\boxed{\mathbf{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}} \quad (N) \tag{3}$$

Coulomb's law directly applies to any pair of point charges that are situated in vacuum and are stationary with respect to each other (Special Theory of Relativity). It also applies in material media if $\mathbf{F}_{21}$ is taken as the direct microscopic force between the two charges q_1 & q_2, irrespective of the other forces arising from all of the other charges in the material medium.

Coulomb's Law

Figure: Charles Augustin de Coulomb (1736–1806)

Coulomb's Law

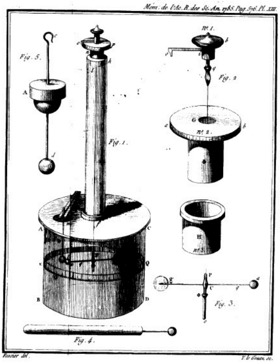


Figure: Coulomb's apparatus (1785)

In 1936, Plimpton & Lawton at Worcester Polytechnic Institute showed that the distance dependency in Coulomb's law deviated from the inverse square law by less than 2 parts in 1 billion; that is, they determined that the force varies as $r^{-(2+\Delta)}$ with $|\Delta| < 2 \times 10^{-9}$.

Coulomb's Law - Principle of Superposition

The Coulombic force satisfies the Principle of Superposition:
The electrostatic force exerted on a stationary point charge q_1 at $\mathbf{r}_1$ by a system of stationary point charges q_k at $\mathbf{r}_k$, $k \neq 1$, is given by the vector sum or linear superposition of all the Coulombic forces exerted on q_1:

$$\mathbf{F}(\mathbf{r}_1) = \sum_{k \neq 1} \mathbf{F}_{k1} = \frac{q_1}{4\pi\epsilon_0} \sum_{k \neq 1} \frac{q_k}{r_{1k}^2} \hat{\mathbf{r}}_{1k} \tag{4}$$

where

$$\mathbf{r}_{1k} = \mathbf{r}_1 - \mathbf{r}_k,$$

$$\hat{\mathbf{r}}_{1k} = \frac{\mathbf{r}_{1k}}{r_{1k}},$$

with $r_{1k} = |\mathbf{r}_{1k}|$.

Electric Field Intensity $\mathbf{E}(\mathbf{r})$

The static Electric Field Intensity (or Electrostatic Field Intensity) $\mathbf{E}(\mathbf{r}) = \mathbf{E}(x, y, z)$, at any fixed point $\mathbf{r} = \hat{\mathbf{1}}_x x + \hat{\mathbf{1}}_y y + \hat{\mathbf{1}}_z z$ in space is defined as the limiting force per unit charge exerted on a test charge q at that point as the magnitude of the test charge goes to zero:

$$\boxed{\mathbf{E}(\mathbf{r}) \equiv \lim_{q \to 0} \frac{\mathbf{F}(\mathbf{r})}{q}} \quad (N/C = V/m) \tag{5}$$

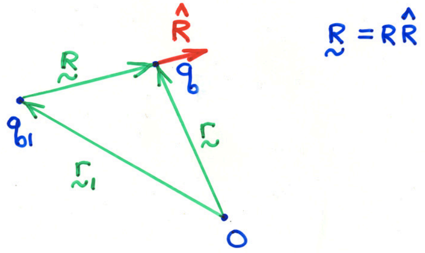
The limit $q \to 0$ is introduced in order that the test charge does not influence the charge sources that produce the electrostatic field. The electric field is then defined in such a way that it is independent of the presence of the test charge.

Electric Field Intensity $\mathbf{E}(\mathbf{r})$

From Eqs. (3) & (5), the electric field intensity at a fixed point $\mathbf{r}$ due to a single point charge q_1 situated at $\mathbf{r}_1$ is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{R^2}\hat{\mathbf{R}} \tag{6}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_1$ denotes the vector from the source point at $\mathbf{r}_1$ to the field point at $\mathbf{r}$, and where $\hat{\mathbf{R}}$ is the unit vector along that direction.



Electric Field Intensity $\mathbf{E}(\mathbf{r})$

As a consequence of the principle of superposition, the electric field intensity at a fixed point $\mathbf{r}$ due to a system of fixed, discrete point charges q_j located at the points $\mathbf{r}_j$, $j = 1, 2, \ldots, n$, is given by the vector sum

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n} \frac{q_j}{R_j^2} \hat{\mathbf{R}}_j} \tag{7}$$

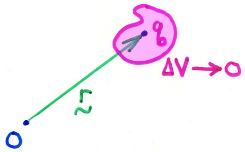
where $\mathbf{R}_j = \mathbf{r} - \mathbf{r}_j$ denotes the vector from the source point at $\mathbf{r}_j$ to the field point at $\mathbf{r}$ with magnitude R_j, and where $\hat{\mathbf{R}}_j$ is the unit vector along that direction.

Charge Density $\rho(\mathbf{r})$

The charge density (or net volume charge density) $\rho(\mathbf{r})$ is a scalar field whose value at any point $\mathbf{r}$ in space, given by the signed net charge per unit volume at that point, is defined by the limiting ratio

$$\boxed{\rho(\mathbf{r}) \equiv \lim_{\Delta\mathcal{V}\to 0} \frac{q}{\Delta\mathcal{V}}} \quad (C/m^3) \tag{8}$$

where q is the net charge in the volume element $\Delta\mathcal{V}$.



From a microscopic perspective, the charge density $\rho(\mathbf{r})$ is zero everywhere except in those regions occupied by fundamental charged particles (electrons & protons).

Charge Density $\varrho(\mathbf{r})$

From a macroscopic perspective, the abrupt spatial variations in the microscopic charge density $\rho(\mathbf{r})$, which are on the scale of interparticle distances, are removed through an appropriate spatial averaging procedure over spatial regions that are small on a macroscopic scale but whose linear dimensions are large in comparison with the particle spacing. The result is the macroscopic charge density

$$\varrho(\mathbf{r}) = \langle\langle \rho(\mathbf{r}) \rangle\rangle \tag{9}$$

The electric field that is determined from such a macroscopic charge density is correspondingly a spatially-averaged field and, as such, is just what would be obtained through an appropriate laboratory measurement.

Electric Field Intensity $\mathbf{E}(\mathbf{r})$

With the introduction of the charge density in Eqs. (8)–(9), the vector summation appearing in Eq. (7) may then be replaced (in the appropriate limit) by a volume integration over the entire region of space containing the source charge distribution. Because

$$\Delta q(\mathbf{r}) = \varrho(\mathbf{r})\Delta\mathcal{V}$$

is the elemental charge contained in the volume element $\Delta\mathcal{V}$ at the point $\mathbf{r}$, then Eq. (7) may be written as

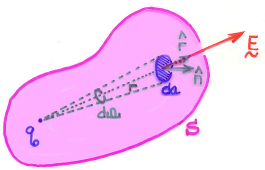
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum \frac{\varrho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\Delta\mathcal{V} \rightarrow \frac{1}{4\pi\epsilon_0}\iiint \frac{\varrho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}d^3r' \quad \textit{as}\ \Delta\mathcal{V} \rightarrow 0 \tag{10}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ is the vector directed from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$ with magnitude R, where $\hat{\mathbf{R}}$ is a unit vector along that direction, and where $d^3r' = dx'dy'dz'$ denotes the volume element at the source point $\mathbf{r}' = (x', y', z')$.

Notice that the integral in (10) is convergent for $\mathbf{r}' = \mathbf{r}$.

Gauss' Law

Consider a point charge q at a fixed point in space together with a simple closed surface $\mathcal{S}$. Let r denote the distance from the point charge to a point on the surface $\mathcal{S}$ with unit vector $\hat{\mathbf{r}}$ directed along the line from the point charge to the surface point, let $\hat{\mathbf{n}}$ be the outwardly directed unit normal vector to the surface $\mathcal{S}$ at that point, and let da be the differential element of surface area at that point.



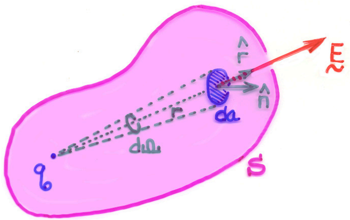
The flux of $\mathbf{E}$ passing through the directed element of area $d\vec{a} = \hat{\mathbf{n}}da$ of $\mathcal{S}$ is then given by

$$\mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{1}{4\pi\epsilon_0} q \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}}{r^2} da \tag{11}$$

Gauss' Law

The differential element of solid angle $d\Omega$ subtended by da at the position of the point charge is given by

$$d\Omega = \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}}{r^2} da \tag{12}$$



With this identification, the flux of $\mathbf{E}$ passing through the directed element of area $d\vec{a} = \hat{\mathbf{n}} da$ of $\mathcal{S}$ becomes

$$\mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{1}{4\pi\epsilon_0} q d\Omega \tag{13}$$

Gauss' Law

The total flux of $\mathbf{E}$ passing through the closed surface $\mathcal{S}$ in the outward direction is then given by integrating Eq. (13) over the entire surface $\mathcal{S}$ as

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{1}{4\pi\epsilon_0} q \oint_{\mathcal{S}} d\Omega \tag{14}$$

where

$$\oint_{\mathcal{S}} d\Omega = \begin{cases} 4\pi, & \text{if } q \in \mathcal{S} \\ 0, & \text{if } q \notin \mathcal{S} \end{cases} \tag{15}$$

Gauss' Law

Gauss Law for a single point charge

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{1}{\epsilon_0} \begin{cases} q, & \text{if } q \in \mathcal{S} \\ 0, & \text{if } q \notin \mathcal{S} \end{cases} \tag{16}$$

For a system of discrete point charges the principle of superposition applies and Gauss' Law becomes

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{1}{\epsilon_0} \sum_j q_j \tag{17}$$

where the summation extends over only those charges that are inside the region enclosed by the surface $\mathcal{S}$.

Gauss' Law

If the charge system is described by the charge density $\varrho(\mathbf{r})$, one finally obtains the Integral Form of Gauss' Law

$$\boxed{\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} d^2 r = \frac{1}{\epsilon_0} \iiint_{\mathcal{V}} \varrho(\mathbf{r}) d^3 r} \tag{18}$$

where $da \to d^2 r$ and $\mathcal{V}$ is the volume enclosed by the surface $\mathcal{S}$.

Notice that the derivation of Gauss' law depends only upon the following three properties:

1. The inverse square law for the force between point charges, as embodied in Coulomb's law.
2. The central nature of the force, also embodied in Coulomb's law.
3. The principle of linear superposition.

Gauss' Law

For a system of discrete point charges q_j located at $\mathbf{r} = \mathbf{r}_j$, the charge density is given by

$$\varrho = \sum_j q_j \delta(\mathbf{r} - \mathbf{r}_j),$$

which recaptures the microscopic description, where

$$\delta(\mathbf{r} - \mathbf{r}_j) \equiv \delta(x - x_j)\delta(y - y_j)\delta(z - z_j)$$

is the three-dimensional Dirac delta function.
With this substitution in Eq. (18), the integral form of Gauss' Law becomes

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} d^2 r = \frac{1}{\epsilon_0} \sum_j q_j \iiint_{\mathcal{V}} \delta(\mathbf{r} - \mathbf{r}_j) d^3 r = \frac{1}{\epsilon_0} \sum_j q_j$$

which is just Gauss' law (17) for a system of discrete point charges with position vectors $\mathbf{r}_j \in \mathcal{V}$.

Gauss' Law

Figure: Carl Friedrich Gauss (1777–1855)

Gauss' Law - Differential Form

With application of the Divergence Theorem

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}} d^2 r = \iiint_{\mathcal{V}} (\nabla \cdot \mathbf{E}) d^3 r \tag{19}$$

the integral form of Gauss' law (18) becomes

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{E} - \varrho/\epsilon_0) d^3 r = 0.$$

Because this expression holds for any region $\mathcal{V}$, the integrand itself must then vanish throughout all of space, so that

$$\boxed{\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\varrho(\mathbf{r})}{\epsilon_0}} \tag{20}$$

which is the differential form of Gauss' law.

This single vector differential relation is not sufficient to completely determine the electric field vector $\mathbf{E}(\mathbf{r})$ for a given charge density $\varrho(\mathbf{r})$.

Coulomb's Law - Differential Form

Helmholtz' Theorem states that a vector field can be specified almost completely (up to the gradient of an arbitrary scalar field) if both its divergence and curl are specified everywhere.
The required curl relation for the electrostatic field follows from the integral form (10) of Coulomb's law, expressed here as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \varrho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r', \tag{21}$$

where the integration extends over all space. Because (Problem 1)

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{22}$$

where $\nabla \equiv \hat{\mathbf{1}}_x \frac{\partial}{\partial x} + \hat{\mathbf{1}}_y \frac{\partial}{\partial y} + \hat{\mathbf{1}}_z \frac{\partial}{\partial z}$ operates only on the unprimed coordinates, then

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \nabla \iiint \frac{\varrho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'. \tag{23}$$

Coulomb's Law - Differential Form & The Scalar Potential

Because the curl of the gradient of any well-behaved scalar function identically vanishes, then Eq. (23) shows that

$$\boxed{\nabla \times \mathbf{E}(\mathbf{r}) = \mathbf{0}} \tag{24}$$

Notice that, in general, this expression for the curl of **E** holds only for an electrostatic field.

From the form of Eq. (23), define a scalar potential for the electric field as

$$\boxed{\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})} \tag{25}$$

where the minus sign is introduced by convention, and

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\varrho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \quad (V)} \tag{26}$$

where the integration extends over all space.

Electrostatic Scalar Potential

The scalar potential at a fixed point P due to a point charge q a distance R away is given by

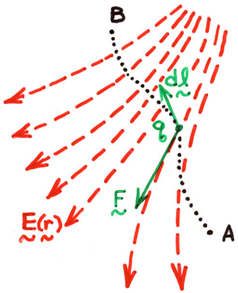
$$V(P) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

By superposition, the scalar potential at a fixed point P due to a system of point charges $q_1, q_2, \cdots, q_N$ at distances $R_1, R_2, \cdots, R_N$ from P, respectively, is then given by

$$V(P) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N}\frac{q_j}{R_j}.$$

Electric Potential & Work

Consider the work done in transporting a test charge q from point A to point B through an externally produced electrostatic field $\mathbf{E}(\mathbf{r})$.



The electric force acting on the test charge q at any point in the field is given by Coulomb's law as

$$\mathbf{F}(\mathbf{r}) = q\mathbf{E}(\mathbf{r}). \tag{27}$$

Electric Potential & Work

The work done in moving the test charge q slowly from A to B (negligible accelerations result in negligible energy loss due to electromagnetic radiation) is given by the path integral

$$W = -\int_A^B \mathbf{F} \cdot d\vec{\ell} = -q \int_A^B \mathbf{E} \cdot d\vec{\ell}, \tag{28}$$

where the minus sign indicates that this is the work done on the test charge against the action of the field. With Eq. (25) this expression becomes

$$W = q \int_A^B \nabla V \cdot d\vec{\ell} = q \int_A^B dV = q(V_B - V_A). \tag{29}$$

This result then shows that the quantity $V(\mathbf{r})$ can be interpreted as the potential energy of the charge q in the electrostatic field. The negative sign in Eq. (25) is then seen to indicate that $\mathbf{E}$ points in the direction of decreasing potential, and hence, decreasing potential energy.

Electric Potential & Work

Eqs. (28) & (29) show that the path integral of the electrostatic field vector $\mathbf{E}(\mathbf{r})$ between any two points is independent of the path and is the negative of the potential difference between the two points, viz.

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\vec{\ell} \tag{30}$$

If the path is closed,

$$\boxed{\oint \mathbf{E} \cdot d\vec{\ell} = 0} \tag{31}$$

Application of Stokes' theorem to this result then yields

$$\oint_{\mathcal{C}} \mathbf{E} \cdot d\vec{\ell} = \iint_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} d^2 r = 0 \implies \nabla \times \mathbf{E} = \mathbf{0}$$

which is just Eq. (24). The electrostatic field $\mathbf{E}(\mathbf{r})$ is then seen to be an irrotational vector field and is therefore conservative.