Chapter 5
Gravitation

In this Chapter we will review the properties of the gravitational force. The gravitational force has been discussed in great detail in your introductory physics courses, and we will primarily focus on specifying the properties of the force and its associated force field, using the vector notions we have introduced in the previous Chapters.

The Gravitational Force

The gravitational force between two point-like particles is proportional to the product of their masses, and inversely proportional to the square of the distance between them. The force is **always** attractive, and directed along the line connection the two particles.

$$\bar{F} = -G\frac{mM}{r^2}\hat{r}$$

The constant G is the gravitational constant, whose value is 6.673 x 10^{-11} Nm^2/kg^2. This relation is known as Newton's law of universal gravitation. The principle of superposition allows us to calculate the force on mass m due to multiple other point-like masses M_1, M_2, ……:

$$\bar{F} = -Gm\sum_{i=1}^{n}\frac{M_i}{r_i^2}\hat{r}_i$$

where r_i is the distance between mass M_i and mass m.

The previous relation is correct only if both masses are point-like objects. If one of the masses is a continuous, we must replace the sum with an integral:

$$\bar{F} = -Gm\int_V \frac{\rho(\bar{r}\,')\hat{r}\,'}{r'^2}dv'$$

The Gravitational Field

The gravitational field generated by the gravitational force is defined in much the same way as we defined the electrostatic field generated by the electrostatic force:

$$\bar{g} = \frac{\bar{F}}{m} = -G\int_V \frac{\rho(\bar{r}\,')\hat{r}\,'}{r'^2}dv'$$

The units of the gravitational field are N/kg = m/s^2.

The Gravitational Potential

The gravitational potential is defined as the scalar function Φ whose gradient is equal to the opposite of the gravitational field:

$$\bar{g} = -\bar{\nabla}\Phi$$

One way to determine if we can find such a scalar function is to calculate the curl of the gravitational field:

$$\bar{\nabla} \times \bar{g} = -\bar{\nabla} \times \bar{\nabla}\Phi = -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \Phi}{\partial x} & \dfrac{\partial \Phi}{\partial y} & \dfrac{\partial \Phi}{\partial z} \end{vmatrix} = 0$$

In the case of the gravitational field we find

$$\bar{\nabla} \times \bar{g} = -GM\left(\bar{\nabla} \times \frac{\hat{r}}{r^2}\right) = -GM\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\right)\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2}\right)\hat{\phi}\right\} = 0$$

and we conclude that there is a scalar function that can generate the gravitational field. Since the gravitational field depends just on r, we expect that the gravitational potential is also just a function of r. The following scalar function can generate the gravitational field:

$$\Phi = -G\frac{M}{r}$$

This is the gravitational potential due to a point mass M. If we have a continuous mass distribution, the gravitational potential will be equal to

$$\Phi = -G\int_V \frac{\rho(\bar{r}')}{r'}dv'$$

In this equation we have assumed that the constant of integration is equal to 0 (or that the gravitational potential is 0 at infinity).

One of the reasons that the gravitational potential is introduced is that even for extended mass distributions, it is in general easier to calculate the gravitational potential (which is a scalar) instead of the gravitational potential energy. Once we have determined the gravitational potential, we can determine the gravitational force by calculating the gradient of the gravitational potential.

The Gravitational Potential Energy

The gravitational potential can be used to determine the gravitational potential energy U of a particle of mass m. Recall from Physics 121 or Physics 141 that the change in the potential energy of an object, when it moves from one position to another position, is the opposite of the work done by the forces acting on the object. Consider an object that moves in the gravitational field of a point mass M located at the origin of a coordinate system. The work done on the object of unit mass by the gravitational force is equal to

$$dW = -\bar{g} \bullet d\bar{r} = \left(\bar{\nabla}\Phi\right) \bullet d\bar{r} = \sum_i \frac{\partial \Phi}{\partial x_i} dx_i = d\Phi$$

If we move the object of unit mass from infinity to a specific position the work done is equal to

$$W(\bar{r}) = \int_{\infty}^{\bar{r}} d\Phi = \Phi(\bar{r})$$

The work done to move an object of mass m to this position is thus equal to

$$W(\bar{r}) = m\Phi(\bar{r})$$

The potential energy of the object at this position is thus equal to

$$U(\bar{r}) = m\Phi(\bar{r})$$

Information about the force on the object can be obtained by taking the gradient of the potential energy:

$$F(\bar{r}) = -\bar{\nabla}U(\bar{r}) = -\frac{\partial}{\partial r}\left(-G\frac{mM}{r}\right)\hat{r} = -G\frac{mM}{r^2}\hat{r}$$

which is of course equal to the gravitational force we used as our starting point.

Visualization of the Gravitational Potential

We can visualize the gravitational potential in a number of different ways. The most common ways are the contour plot, showing equipotential surfaces, and 3D plots showing the gravitational potential as function of the two-dimensional position (x, y). A simple program that can be used to generate such plots is *gravitationalPotential* which can be found in the Mathematica folder under Computing Tools on our website:

```
(* Make a 3D plot of the gravitational potential due to two
point masses of mass 1 and mass 4, located at (-2, -2) and (2,
2), respectively. *)

Plot3D[(1/Sqrt[(x + 2)^2 + (y + 2)^2]) + (4/Sqrt[(x - 2)^2 + (y
- 2)^2]),
  {x, -5, 5}, {y, -5, 5},
  PlotRange -> {0, 7.5}, PlotPoints -> 50,
  Ticks -> {Automatic, Automatic, Automatic}]

(* Make a 2D contour plot of the gravitational potential due to
two point masses of mass 1 and mass 4, located at (-2, -2) and
(2, 2), respectively. *)

ContourPlot[(1/Sqrt[(x + 2)^2 + (y + 2)^2]) + (4/
          Sqrt[(x - 2)^2 + (y - 2)^2]),
    {x, -5, 5}, {y, -5, 5}];
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The plots that are generated using this program are shown in Figure 1.

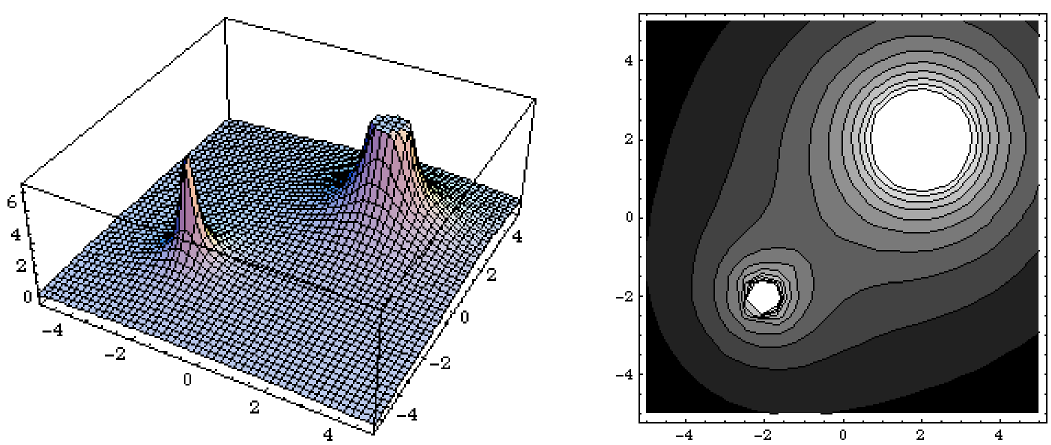

Figure 1. Different ways to visualize the gravitational potential due to two point masses of mass 1 and mass 3, located at (-2,-2) and (2,2), respectively.

The Shell Theorem

When we calculate the gravitational force or the gravitational potential generated by a mass distribution, we can always use the most general expression for these quantities in terms of the volume integral over the mass distribution. However, if the mass distribution has spherical symmetry, we can use the shell theorem to calculate the gravitation force and the gravitational potential. The shell theorem states that:

> **The gravitational potential at any point outside a spherically symmetric mass distribution is independent of the size of the distribution, and we can consider all of its mass to be concentrated at the center of the mass distribution. The gravitational potential is zero at any point inside a spherically symmetric mass distribution.**

Using the shell theorem it is easy to calculate the gravitational potential and the gravitational force due to a spherical shell, which is shown in Figure 2.

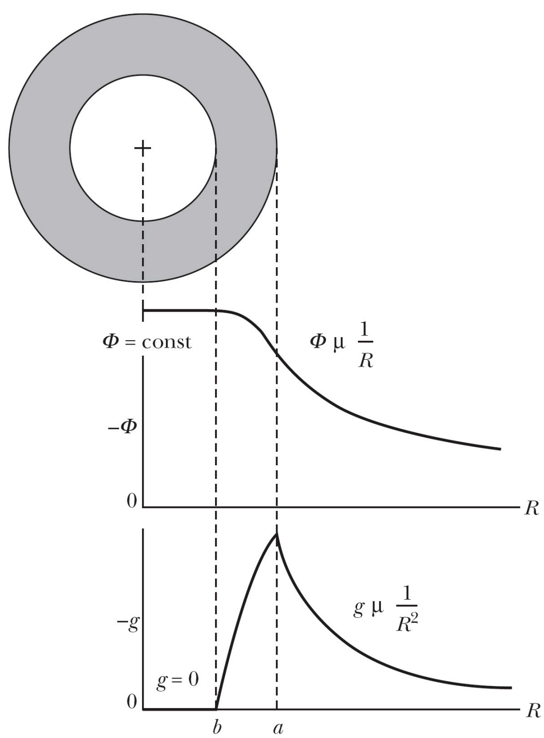

Figure 2. The gravitational potential and the gravitational force due to a spherical shell.

The shell theorem can be used to make important predictions about orbital motion of planets around the central star and solar systems around the center of a galaxy. The observation of the rotational motion of the planets around the sun let to the first determination of its mass. Since the sun is much more massive than any of the planets in the solar system, the motion of the planets could be described in terms of the gravitational force due to just the mass of the sun (see Figure 3.

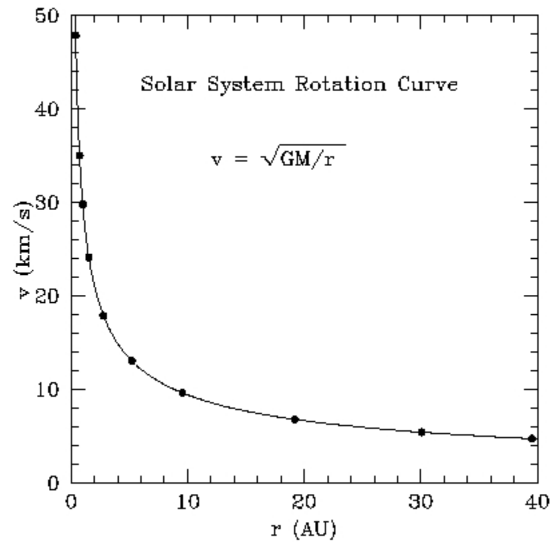

Figure 3. The orbital velocity of the planets in our solar system as function of their distance from the sun. The theoretical dependence, which depends on the mass of the sun, is shown by the solid curve, and does an excellent job describing the trend in the data.

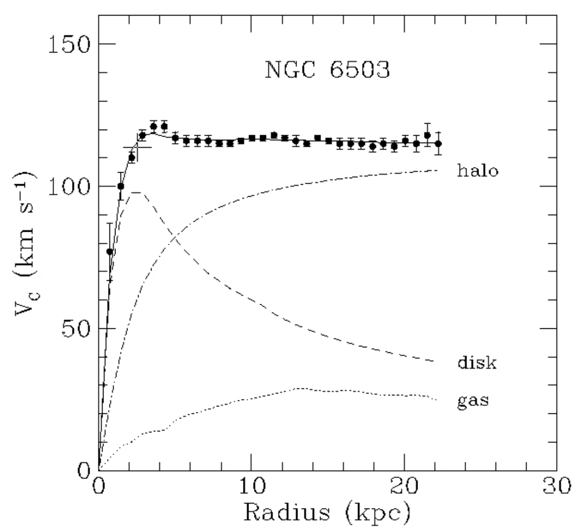

Figure 4. Measured orbital velocity of stars as function of distance from the center of the galaxy. The rotational curve can only be explained if we assume that there is halo of "dark matter" in universe, distributed throughout the galaxy.

The solar systems in most galaxies carry out an orbit around the center of the galaxy. Since it is assumed that a massive black hole is located in the center of most galaxies, we expect to see a trend in the orbital velocity versus distance, similar to the trend seen in our solar system (see

Figure 3). In reality, we see a distribution that decreases at a much smaller rate as function of the distance from the center of the galaxy (see Figure 4). This implies that there is more mass in the galaxy than was assumed, but also that this extra mass is **NOT** located in the center of the galaxy, but throughout the galaxy. This extra matter, that we can not see directly, is called dark matter and your teacher is looking for it in a mine in England.

The Poisson Equation

In Electricity and Magnetism we greatly benefited from the Gauss' law, which related the electric flux through a closed surface with the total charge enclosed by that surface. Given the fact that the nature of the gravitational force (its $1/r^2$ dependence) is similar to the nature of the electric force, we expect that similar "laws" apply to the gravitational force.

Based on the approach we took in Electricity and Magnetism, we define the gravitational flux due a point mass *m* in the following manner:

$$\Phi_{grav} = \int_S \left(\hat{n} \bullet \bar{g} \right) da$$

where *S* is an arbitrary surface surrounding the mass *m* (see Figure 5). Using the definition of the gravitational field, we can rewrite this equation as

$$\Phi_{grav} = -Gm \int_S \frac{\cos\theta}{r^2} da = -Gm \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\frac{\cos\theta}{r^2} \right) r^2 \sin\theta d\theta d\phi = -2\pi Gm \int_{\theta=0}^{\pi} \cos\theta \sin\theta d\theta = -4\pi Gm$$

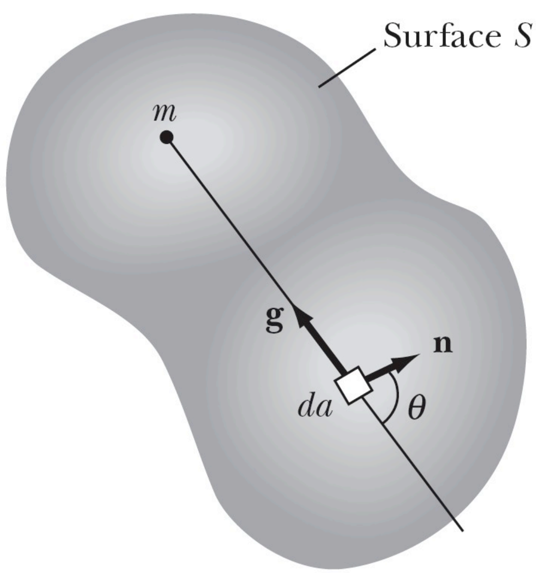

Figure 5. Surface used to calculate gravitational flux associated with a point mass *m*.

When we have a mass distribution inside the surface S we need to replace m by a volume integral over the mass distribution:

$$\Phi_{grav} = -4\pi G \int_V \rho dv$$

The left-hand side of this equation can be rewritten using Gauss's divergence theorem:

$$\Phi_{grav} = \int_S (\hat{n} \bullet \bar{g}) da = \int_V \left(\bar{\nabla} \bullet \bar{g} \right) dv$$

We thus conclude that

$$\int_V \left(\bar{\nabla} \bullet \bar{g} \right) dv + 4\pi G \int_V \rho dv = \int_V \left\{ \left(\bar{\nabla} \bullet \bar{g} \right) + 4\pi G \rho \right\} dv = 0$$

for any volume V and thus

$$\left(\bar{\nabla} \bullet \bar{g} \right) = -4\pi G \rho$$

This equation can also be expressed in terms of the gravitational potential and becomes

$$\bar{\nabla} \bullet \left(-\bar{\nabla} \Phi \right) = -\bar{\nabla}^2 \Phi = -4\pi G \rho$$

or

$$\bar{\nabla}^2 \Phi = 4\pi G \rho$$

This equation is known as **Poisson's equation**.

The Tides

We all know that the tides are caused by the motion of the moon around the earth, but most of us have a more difficult time to explain why we have two high tides a day. Our simple picture would suggest that you have high tide on that side of the earth closest to the moon, but this would only explain one high tide a day.

The calculation of the effect of the moon on the water on earth is complicated by the fact that the earth is not a good inertial system, so we have to assume that we can find a good inertial frame in which we can describe the forces acting on the water (see Figure 6). Consider the forces on a volume of water of mass m, located on the surface of the earth (see Figure 6), due to the moon and the earth:

$$\bar{F}_m = m\ddot{\bar{r}}_m = -\frac{GmM_E}{r^2}\hat{r} - \frac{GmM_m}{R^2}\hat{R}$$

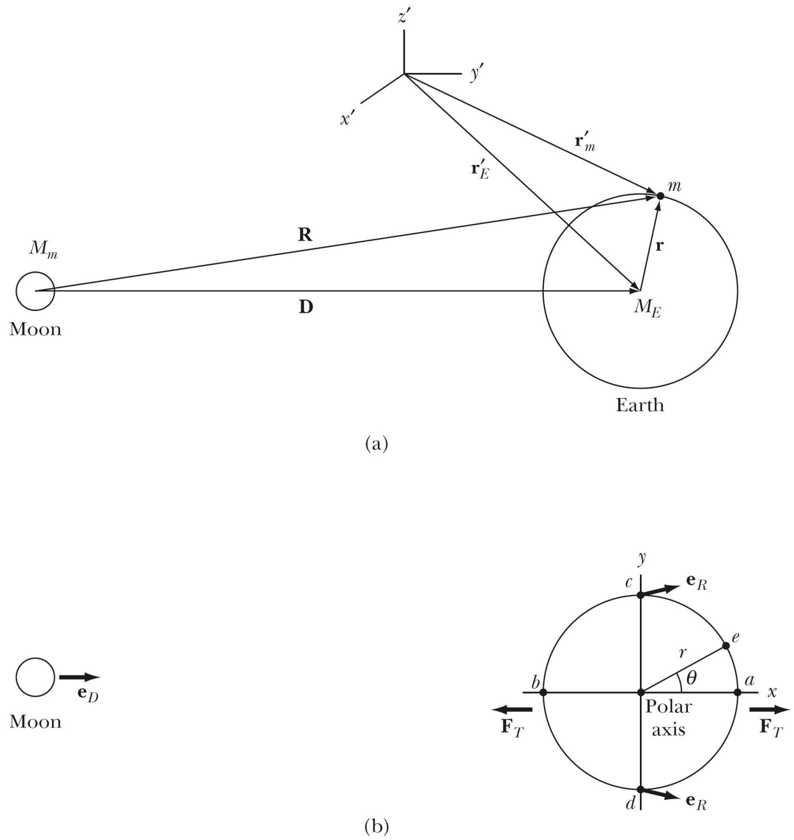

Figure 6. Geometry used to determine the forces on a volume of water of mass *m*, located on the surface of the earth.

The force exerted by the moon on the center of the earth is equal to

$$\bar{F}_E = M_E\ddot{\bar{r}}_E = -\frac{GM_mM_E}{D^2}\hat{D}$$

When we view the motion of the water on earth, we view its motion with respect to the earth. We have used a correct inertial frame to determine the acceleration of the individual components of our system, and we can now transform to a reference frame in which the earth is at rest and centered around the origin (note: this is a non-inertial reference frame). In this reference frame, the acceleration of mass *m* is equal to

$$\ddot{\bar{r}} = \ddot{\bar{r}}_m - \ddot{\bar{r}}_E = -\frac{GM_E}{r^2}\hat{r} - \frac{GM_m}{R^2}\hat{R} + \frac{GM_m}{D^2}\hat{D} = -\frac{GM_E}{r^2}\hat{r} - GM_m\left(\frac{\hat{R}}{R^2} - \frac{\hat{D}}{D^2}\right)$$

The first term on the right-hand side is just the gravitational force on mass m due to the earth. It will be the same anywhere on the surface of the earth and is **not** responsible for the tides. The second part of the right-hand side is equal to the acceleration associated with the tidal force. It is related to the difference between the gravitational pull of the moon on the center of the earth and on the surface of the earth. Let us consider what this equation tells us about the magnitude and the direction of the tidal force at 4 different positions on the surface of the earth (see Figure 6).

- Point a: At point a, the unit vectors associated with R and D are pointing in the same direction. Since $R > D$, the $(1/R^2)$ terms will be smaller than the $(1/D^2)$ term, and the tidal acceleration will be directed towards the right:

$$\ddot{\vec{r}}_T = GM_m\left(\frac{1}{D^2} - \frac{1}{R^2}\right)\hat{D} = GM_m\left(\frac{1}{D^2} - \frac{1}{(D+r)^2}\right)\hat{D} = \frac{GM_m}{D^2}\left(1 - \frac{1}{\left(1+\frac{r}{D}\right)^2}\right)\hat{D} = 2GM_m\frac{r}{D^3}\hat{D}$$

- Point b: At point b, the unit vector associated with R and D are pointing in the same direction. But, since $R < D$, the $(1/R^2)$ terms will be larger than the $(1/D^2)$ term, and the tidal acceleration will be directed towards the left but with the same magnitude as the acceleration at point a:

$$\ddot{\vec{r}}_T = GM_m\left(\frac{1}{D^2} - \frac{1}{R^2}\right)\hat{D} = GM_m\left(\frac{1}{D^2} - \frac{1}{(D-r)^2}\right)\hat{D} = \frac{GM_m}{D^2}\left(1 - \frac{1}{\left(1-\frac{r}{D}\right)^2}\right)\hat{D} = -2GM_m\frac{r}{D^3}\hat{D}$$

- Point c: The x components of the vectors associated with R and D are similar, and the x components of the acceleration will cancel. The vector associated with D will have no y component, and the acceleration will be due to the y component associated with R (which points towards the center of the earth):

$$\ddot{\vec{r}}_T = -GM_m\left(\frac{\hat{R}}{R^2} - \frac{\hat{D}}{D^2}\right) = -GM_m\frac{\hat{R}_y}{R^2} = -GM_m\frac{1}{D^2}\frac{r}{D}\hat{y} = -GM_m\frac{r}{D^3}\hat{y}$$

- Point d: The calculation for the force on point c is similar to the calculation of the force on point a. The force will be directed towards the center of the earth and will be equal to

$$\ddot{\vec{r}}_T = GM_m\frac{r}{D^3}\hat{y}$$

We thus conclude that the acceleration of the water on the surface of the earth is directed away from the surface at two different locations. It will thus be high tide at these two locations and since the moon rotates around the earth in one day, each location will see a high tide twice a day (one time when the moon that location is facing the moon, and one time when the moon is located on the opposite side of the earth from that location). A summary of the tidal forces on the surface of the earth is shown in Figure 7.

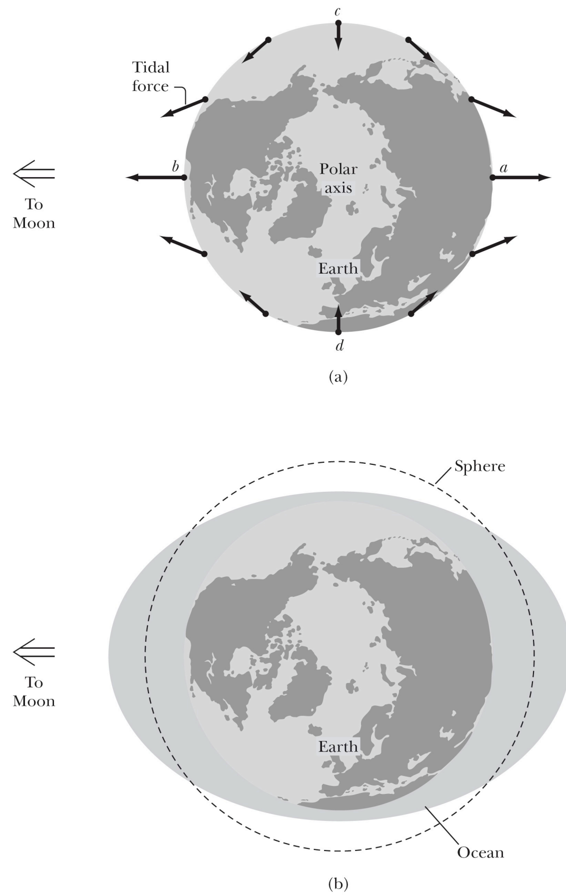

Figure 7. Tidal forces at various places on the surface of the earth.