On the Utility of Chirp Modulation for Digital Signaling

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Abstract—The issue of signal selection in binary data transmission is presented. The question of the relative utility of linear frequency sweeping (LFS or chirp), compared to PSK and FSK, in terms of error probability and spectrum usage, is discussed. The transmission media considered are the coherent, partially coherent, Rayleigh, and Rician channel models. Theoretically, LFS has unconditionally superior characteristics in the partially coherent and fading cases, for certain ranges of channel conditions. This is due to the more negative values of cross-coherence parameters possible with the LFS signal set over the FSK signal set. For the fading channel, theoretical supremacy of LFS over FSK depends upon the specular-to-Rayleigh signal power ratio and the adjustability of in-phase cross coherence, with a constraint upon quadrature phase cross coherence.

From a practical standpoint, coherent reception of the LFS signal set has severe limitations. These are manifested primarily in two aspects: the need for phase synchronization of a chirp signal set, and the fact that the optimum value of cross coherence is highly sensitive to synchronization channel signal-to-noise ratio (SNR), and/or spectral-to-Rayleigh signal power ratio. The latter would require that modulation characteristics track the channel conditions in order to achieve the supremacy in performance theoretically predicted by optimization of the cross-coherence parameter in LFS.

INTRODUCTION

This paper deals with the issue of signal selection in binary data transmission and is motivated by a recurrent interest [1]–[3] in the use of linear-frequency-swept signal sets for transmission of digital data. This paper takes up some of the questions raised in the literature. In particular, a comparison of linear frequency sweeping (LFS), PSK, and FSK is made on the basis of error probability, data rate, and spectrum usage. The channel models are coherent, partially coherent, and nonselective fading.

SIGNAL REPRESENTATION

This paper considers the utility of linear frequency modulation (LFS), sometimes referred to as a chirp signal set, for use in digital data transmission. The problem of optimal signaling has been treated by Kotelnikov [4], Middleton [5], and Van Trees [6]. Work in [4]–[6] has shown that the bit error probability depends upon certain parameters that can be defined on a general signal set. The physical parameter of primary concern in this paper is the cross-coherence factor ρ. For equal energy signals, ρ is defined as

$$\rho \triangleq \frac{1}{E}\int_0^T s_1(t)s_2(t)\,dt \qquad (1)$$

where E is the bit pulse energy

$$E \triangleq \int_0^T s_i^2(t)\,dt, \qquad i = 1, 2.$$

Paper approved by the Communication Theory Committee of the IEEE Communications Society for publication without oral presentation. Manuscript received March 10, 1972; revised November 29, 1972. This research was supported in part by the Department of Electrical Engineering and the Bureau of Engineering Research, College of Engineering, University of Texas, Austin, under Grant TX 78712.

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Application of Schwarz's inequality to (1) reveals that the cross coherence is bounded as $-1 \leqslant \rho \leqslant +1$.

The three modulation schemes considered are PSK, FSK, and LFS or chirp.

The PSK signal set is

$$s_1(t) = A\cos[\omega_c t]$$
$$s_2(t) = A\cos[\omega_c t + \pi], \qquad 0 \leqslant t \leqslant T. \qquad (2)$$

Since $s_1(t) = -s_2(t)$, the signal energy is the same for both signals:

$$E = \int_0^T A^2\cos^2[\omega_c t]\,dt = A^2T/2$$

and also $\rho = -1$. This assumes that the period of the carrier frequency is small with respect to the bit interval. A bandwidth of $2/T$ Hz centered at the carrier frequency contains 90 percent of the signal energy. This is usually taken as the nominal bandwidth for PSK.

The FSK signal set is given by

$$s_1(t) = A\cos[\omega_1 t]$$
$$s_2(t) = A\cos[\omega_2 t], \qquad 0 \leqslant t \leqslant T. \qquad (3)$$

In an FSK system the carrier frequency is shifted in accordance with the information bit being conveyed. If, as just stated, the carrier frequency is large compared to the reciprocal of the baud length, the signal energy is again $A^2T/2$.

The cross-coherence coefficient (1) is more involved for FSK. Neglecting the integral over high frequencies yields

$$\rho = \frac{\sin[\Delta\omega T]}{\Delta\omega T} \qquad (4)$$

where

$$\Delta\omega = \omega_2 - \omega_1.$$

The value of ρ varies as $\sin x/x$, where x is the product of frequency separation and baud length for the signal set. The bounds on ρ for FSK are

$$-0.218 \leqslant \rho \leqslant 1 \text{ FSK}. \qquad (5)$$

Note that this definition of FSK requires phase coherency between the two signal states; this requirement is also made for the LFS signal set.

The spectral occupancy of the FSK signal follows from (3):

$$S(j\omega) = S_1(j\omega) + S_2(j\omega)$$
$$s_i(j\omega) = \tfrac{1}{2}AT\frac{\sin[(\omega_i+\omega)\frac{1}{2}T]}{[(\omega_i+\omega)\frac{1}{2}T]}\exp\{-j[(\omega_i+\omega)\tfrac{1}{2}T]\} + \tfrac{1}{2}AT\frac{\sin[(\omega_i-\omega)\frac{1}{2}T]}{[\omega_i-\omega)\frac{1}{2}T]}\exp\{+j[(\omega_i-\omega)\tfrac{1}{2}T]\}. \qquad (6)$$

From the amplitude spectra of (6), it is conventionally assumed that band limiting is achieved at the first zero points. The bandwidth of each signal in the FSK set is equal to that of PSK.

$$W = \frac{4\pi}{T}\text{ rad/s}.$$

The general LFS signal set is

$$s_1(t) = A \cos [\omega_1 t + \tfrac{1}{2} mt^2]$$
$$s_2(t) = A \cos [\omega_2 t - \tfrac{1}{2} mt^2]. \qquad (7)$$

Taking the overall frequency change mT to be much less than the carrier frequencies, the signal energy is also $A^2 T/2$.

The cross-coherence coefficient for LFS cannot be expressed in simple form. Employing (7) in (1) yields a Fresnel integral,

$$\rho = \frac{1}{T} \int_0^T \cos [(\omega_1 - \omega_2) t + mt^2] \, dt \qquad (8)$$

where the integral over the high-frequency terms is not significant. The integral can be rewritten into a standard Fresnel form [7]:

$$\rho = \frac{1}{g} \{\cos [\tfrac{1}{2} \pi p^2] [C_f(p) - C_f(p - g)] + \sin [\tfrac{1}{2} \pi p^2] [S_f(p) - S_f(p - g)]\} \qquad (9)$$

where

$$p = \frac{\omega_2 - \omega_1}{(2\pi m)^{1/2}}; \; g = \left(\frac{2m}{\pi}\right)^{1/2} T$$

$$C_f(x) = \int_0^x \cos \tfrac{1}{2} \pi t^2 \, dt; \; S_f(x) = \int_0^x \sin \tfrac{1}{2} \pi t^2 \, dt. \qquad (10)$$

Numerical evaluation indicates that ρ is bounded by

$$-0.629 \leqslant \rho \leqslant +1 \text{ LFS}. \qquad (11)$$

Fig. 1 illustrates the variation of ρ dependent upon T, the bit duration, and hence the data rate; the values of ρ will be likewise dependent upon T. However, the optimum value of ρ for any data rate can be obtained by the variation of the sweep rate m and the frequency range $\Delta\omega$.

The overall frequency sweep in a baud length of T, mT is, to a good approximation, the radian frequency bandwidth W.

Manipulation of (10) yields the following interdependence of bandwidth, initial frequency spacing, and data rate:

$$\Delta\omega = pg\pi/T$$
$$mT = \pi g^2/2T \qquad (12)$$

where W is the radian half-power bandwidth of the signal, and $\Delta\omega$ is the initial frequency spacing $\omega_1 - \omega_2$.

This establishes the relative values of spectrum occupancy, energy, and cross-coherence coefficient for the PSK, FSK, and LFS signal sets. The parameters are employed in terms of the effects upon bit error probability for various channels.

Coherent Reception

This channel situation is analytically specified with the received waveform given by the original transmitted signal and additive zero-mean Gaussian noise,

$$r(t) = A \cos [\omega_i t + m_i(t)] + n(t), \qquad i = 1, 2.$$

The additive noise is taken to be of flat spectral density over the frequency range of interest.

Since the preceding model is simple, it best serves to introduce the signal selection issue.

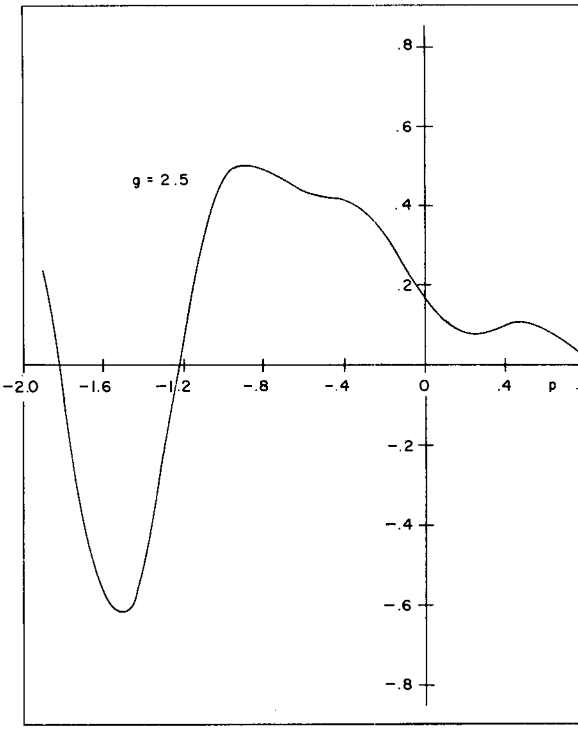

Fig. 1. Real part of cross-coherence coefficient for LFS signals.

The signal set parameters are

$$m_1(t) = 0, m_2(t) = \pi; \omega_1 = \omega_2 \text{ PSK}$$
$$m_1(t) = 0, m_2(t) = 0; \omega_1 \neq \omega_2 \text{ FSK}$$
$$m_1(t) = \tfrac{1}{2} mt^2, m_2(t) = \tfrac{1}{2} mt^2 \text{ LFS}.$$

In general, the $\{m_i(t)\}$ can be any waveforms that are low pass with respect to the carrier frequencies.

For equal *a priori* signal probabilities, flat noise spectral density with two-sided height $N_0/2$, assuming equal energy signals, the error probabilities of the first and second kind [6] are

$$\alpha = \text{erfc}\left(\frac{\delta}{2}\right) = \tfrac{1}{2} - \text{erf}\left(\frac{\delta}{2}\right)$$

$$\beta = \tfrac{1}{2} - \text{erf}\left(\frac{\delta}{2}\right)$$

where

$$\delta^2 \triangleq \frac{4}{N_0} E(1 - \rho).$$

The overall probability of error is then

$$P(e) = P(H_1)\alpha + P(H_2)\beta = \tfrac{1}{2} - \text{erf}\left(\frac{\delta}{2}\right) \qquad (13)$$

and decreases with an increase in δ; in the limit $\text{erf}_{\delta \to \infty}(\delta/2) = \frac{1}{2} = P(e)_{\delta \to \infty} = 0$.

The parameter δ is specified by design considerations. The signal energy E is a quantity fixed by transmitter power and transmission characteristics, $N_0/2$ is the noise power spectral density level set by the receiver noise figure, leaving the cross-coherence parameter ρ to be optimized. It can be seen from (13) that δ increases monotonically as ρ decreases for fixed energy to noise ratio E/N_0. The most negative value of ρ possible is -1, indicating that an antipodal signal set is optimum. Referring to (2), it is seen that the PSK signal set is optimal for this channel. Direct numerical evaluation shows that FSK is capable of achieving a cross-coherence coefficient of -0.218, while LFS can be as negative as -0.629. From Fig. 1 this value of the cross-coherence coefficient is produced for the LFS signal set when $p = 1.53$, $g = 2.50$. For these values, (12) yields

$$W = 9.85\left(\frac{1}{T}\right); \Delta\omega = 12\left(\frac{1}{T}\right) \text{ LFS.}$$

Since the signals are sweeping into one another, the overall radian frequency bandwidth of the signal set for this case is determined by the initial frequency separation

$$B = \Delta\omega = 12\left(\frac{1}{T}\right).$$

An FSK signal set with cross-coherence coefficient of -0.218 yields from (4)

$$\Delta\omega = \frac{4.4}{T}, \; W = 4\pi\left(\frac{1}{T}\right) = 12.5\left(\frac{1}{T}\right).$$

The overall system bandwidth B is then

$$B = \Delta\omega + W = 17\left(\frac{1}{T}\right).$$

Table I gives a comparison of FSK, PSK, and LFS for equal probability of error in terms of energy and bandwidth requirements. There is a 2.12-dB improvement from FSK to PSK, and a 1.3-dB improvement from FSK to LFS. A comparison of the three modulation techniques shown in Table I indicates that there is no motivation for the use of LFS over PSK. However, LFS is superior to FSK in terms of required signal energy and bandwidth for a given probability of error.

Partially Coherent Reception

In partially coherent reception, precisely synchronous detection is not achieved. This is analytically represented as the exact knowledge of the RF phase of the received signal component not being available to the receiver. The phase is then treated as a random variable with some appropriate probability distribution. The received waveform is

$$r(t) = A\cos\left[\omega_i t + m_i(t) + \theta\right] + n(t), \qquad i = 1, 2.$$

The statements pertaining to $m(t)$, ω, etc., for the various modulation schemes are the same as in the coherent case. The phase variable θ is taken to have a probability density function $p(\theta)$ modeled [9] by

$$p(\theta) \triangleq \frac{\exp\left[\Gamma\cos\theta\right]}{2\pi I_0(\Gamma)}; \; \Gamma = \frac{A\tau}{N_0/2}, \qquad 0 \leqslant \theta \leqslant 2\pi$$
$$= 0, \qquad \text{elsewhere}$$

where $I_0(\Gamma)$ is the zeroth-order Bessel function of the first kind, and τ is a measure of the phase instability/drift.

As $\Gamma \to \infty$, $p(\theta) \to \delta(\theta)$ and the receiver LO becomes coherent; when $\Gamma = 0$, $p(\theta)$ is uniform, corresponding to the noncoherent situation. In practice, the situation is more nearly one of partial coherence, such as when θ is acquired through the use of a pilot carrier or phase lock loop, etc.

The probability of error is evaluated assuming that the test

TABLE I

Modulation	Bandwidth B (rad/s)	Relative Signal Energy E
PSK	6.283 $(1/T)$	E_0
FSK	17 $(1/T)$	$1.64\,E_0$
LFS	12 $(1/T)$	$1.22\,E_0$

statistic is conditionally Gaussian given the phase variable θ, by averaging over θ

$$P(e) = \int_{-\pi}^{\pi} p(e)P(e/\theta)\,d\theta. \tag{14}$$

It has been shown by Viterbi [9] that the conditional error probability is

$$P(e/\theta) = Q\left(\frac{k_1}{\sqrt{2}}, \frac{k_2}{\sqrt{2}}\right) - \tfrac{1}{2}\exp\left\{-\frac{k_1^2 + k_2^2}{4}\right\} I_0\left(\frac{k_1 k_2}{2}\right) \tag{15}$$

where

$$Q(x, y) \triangleq \int_y^{\infty} \exp\left\{-(t^2 + x^2/2\right\} I_0(xt)\,dt$$

is the Marcum Q function [10].

To obtain the average probability of error $P(e)$, (15) is substituted into (14) and the expression is integrated numerically. The analytical results are plotted versus Γ for fixed signal-to-noise ratio (SNR) d and quantized values of coherence coefficients ρ_1 and ρ_2, where the parameters are

$$d^2 = 2E/N_0$$
$$k_1^2 = (E[y_1])^2 + (E[y_2])^2 \quad k_2^2 = (E[y_3])^2 + (E[y_4])^2$$

$$E\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \frac{1}{\nu d[2(2+\nu)]^{1/2}} \begin{bmatrix} \Gamma(1 + \nu - \rho_1) + d^2(1+\nu)\cos\theta \\ \Gamma\rho_2 + d^2\nu(1+\nu)\sin\theta \\ \Gamma(1+\nu-\rho_1) + d^2\nu(\rho_1\cos\theta - \rho_2\sin\theta) \\ -\Gamma\rho_2 + d^2\nu(\rho_1\sin\theta + \rho_2\cos\theta) \end{bmatrix}$$

$$\nu = (1 - \rho_1^2 - \rho_2^2)^{1/2}$$
$$\rho_1 = \frac{A^2}{E}\int \cos\left[\omega_1 t + m_1(t)\right]\cos\left[\omega_2 t + m_2(t)\right]dt$$
$$\rho_2 = \frac{A^2}{E}\int \sin\left[\omega_1 t + m_1(t)\right]\cos\left[\omega_2 t + m_2(t)\right]dt.$$

The results indicate that P_e is an increasing function of the magnitude of the quadrature cross-coherence coefficient $|\rho_2|$ and has a minimum at $\rho_2 = 0$ for any quantization of ρ_1. One such case, Fig. 2, is for $\rho_2 = 0$ and values of ρ_1 corresponding to FSK, PSK, and LFS. It can be seen that for values of Γ between 2 and 8, partially coherent reception, a signal set with an intermediate value of cross coherence excels over PSK. Fig. 3 shows $|\rho_2|$ for LFS and indicates that values of ρ_1 near the minimum are obtainable with $|\rho_2| = 0$. The use of FM modulation to achieve the desired cross coherence has an advantage over an AM technique in view of the peak-to-average-power limitations of high power transmission. However, the required phase recovery system for LFS, presumably a phase locked pilot tone, and sensitivity of the optimal modulation

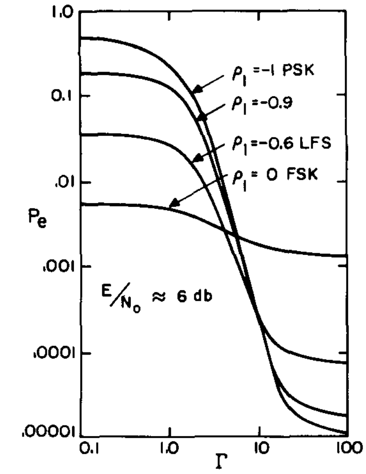

Fig. 2. Bit error probability for partially coherent reception, zero quadrature cross coherence $\rho_2 = 0$.

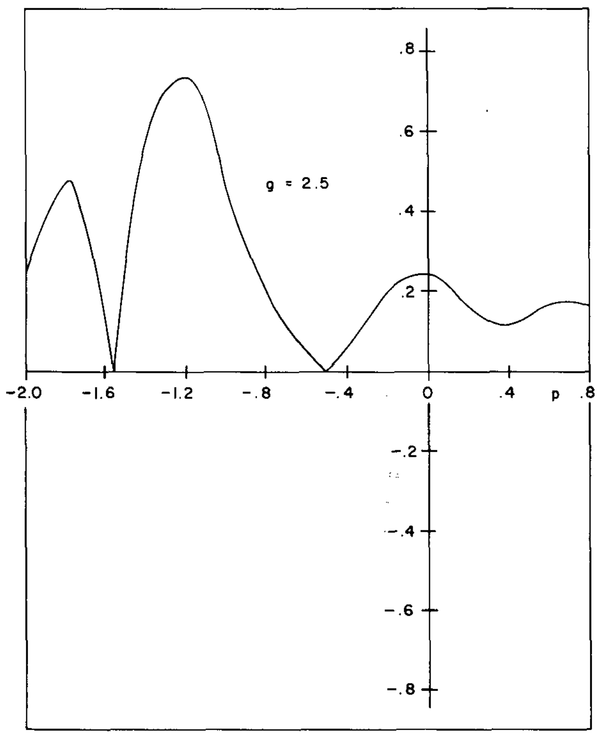

Fig. 3. Magnitude of imaginary part of the cross-coherence coefficient for LFS signals.

parameters to channel conditions (Fig. 2) make LFS unappealing for most applications.

Fading Channel

For transmission over a Rician channel, the received waveform is modeled by a deterministic component and a scatter component whose envelope is specified by a Rayleigh probability distribution and a uniformly distributed phase. The fading is assumed not frequency selective and slow compared to T.

The received waveform is

$$r(t) = vA \cos [\omega_i t + m_i(t) + \theta] + A \cos [\omega_i t + m_i(t)] + n(t), \qquad T_1 \leqslant t \leqslant T_2, i = 1, 2.$$

The random variables v and θ are Rayleigh and uniformly distributed, respectively.

Turin [11] has shown that the generalized orthogonal signal set $\rho_1 = 0$; $\rho_2 = 0$ is desirable for the range of spectral-to-Rayleigh ratios encountered in most applications. Thus the PSK signal set is excluded while the FSK and LFS signal sets are candidates. However, the relatively simpler implementation of FSK rules out the use of LFS for orthogonal signaling in the nonselective slow fading channel.

Discussion

For ideal (coherent) reception over stable channels, the antipodal signal set PSK excels over FSK and LFS, in terms of both error probability and bandwidth. For partially coherent reception, a signal set intermediate to the antipodal (PSK) and orthogonal (FSK) signal sets would be optimum; the extent of cross coherence desirable would be dependent upon the degree of phase coherency.

For the Rician channel, for large values of spectral-to-Rayleigh, a situation similar to that for partially coherent reception prevails. The fading case parallels partially coherent reception with the added complexity of a fluctuating signal amplitude. Thus, in both instances, the error rate is an increasing function of $|\rho_2|$; the inquadrature cross coherence should be zero and the optimum value of in-phase cross coherence ρ_1 is dependent upon the spectral-to-Rayleigh ratio.

Theoretical analyses based upon the relationship of error probability to the signal parameters ρ_1, ρ_2 overlook certain practical constraints. Since the modulation cannot be removed in a manner such as squaring in PSK, a pilot tone is required in addition to the data transmission. Comparisons between modulation schemes must be made at equal total power levels and data rate; an analysis based upon power division between pilot signal and data is then required as well.

Finally, an obvious limitation is manifested in the implications of Fig. 2; the optimal value of the cross-coherence coefficient is highly sensitive to the SNR in the phase synchronization channel, for a wide range of SNR. Since communication systems almost always have a requirement to operate over a range of SNR's, and since the modulation characteristics would be exceedingly difficult to adaptively change as a function of SNR, the benefits of optimization would be quite limited in practice.

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