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- By the definition of limit, the above means

$$\forall \delta > 0, \ \exists N < \infty, \ s.t. \ P\left[|X_n - X_0| > \epsilon\right] < \delta, \ \forall n > N$$

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- In general, almost sure convergence does not imply convergence in r^{th} mean and vice versa

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- We use characteristic functions for proving CLT

Characteristic Function

- Given rv X, its characteristic function, ϕ_X, is defined by

$$\phi_X(u) = E\left[e^{iuX}\right] = \int e^{iux}\, dF_X(x) \quad (i = \sqrt{-1})$$

Characteristic Function

- Given rv X, its characteristic function, ϕ_X, is defined by

$$\phi_X(u) = E\left[e^{iuX}\right] = \int e^{iux}\, dF_X(x) \quad (i = \sqrt{-1})$$

- Since $|e^{iux}| \leq 1$, ϕ_X exists for all random variables

Properties of characteristic function

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- If ν_r is finite, then

$$\phi_X(u) = \sum_{s=0}^{r-1} \mu_s \, \frac{(iu)^s}{s!} + \rho(u) \; \mu_r \, \frac{(iu)^r}{r!}$$

where $|\rho(u)| \leq 1$ and $\rho(u) \to 1$ as $u \to 0$

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  - If $F_n \rightarrow F$ then $\phi_{F_n} \rightarrow \phi_F$
  - If $\phi_{F_n} \rightarrow \psi$ and ψ is continuous at zero, then ψ would be characteristic function of some df, say, F, and $F_n \rightarrow F$

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$$\lim_{n\to\infty} P\left[\tilde{S}_n \leq x\right] = \lim_{n\to\infty} P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt, \quad \forall x$$

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- Without loss of generality let us assume $\mu = 0$.

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Proof:

- Without loss of generality let us assume $\mu = 0$.
- We use characteristic function of $\tilde{S}_n$ for the proof.

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- Recall that we can expand ϕ in a Taylor series

$$\phi(t) = \sum_{s=0}^{r-1} \mu_s \, \frac{(it)^s}{s!} + \rho(t) \; \mu_r \, \frac{(it)^r}{r!}, \quad \rho(t) \to 1, \text{ as } t \to 0$$

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- Here we assume: $EX_i = 0$ and $EX_i^2 = \sigma^2$

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- By Continuity theorem, distribution function of $\tilde{S}_n$ converges to that of standard Normal rv

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- Thus, S_n is well approximated by a normal rv with mean $n\mu$ and variance $n\sigma^2$, if n is large

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&= P\left[\frac{-3}{\sqrt{\frac{5}{3}}} \leq \frac{Z - EZ}{\sqrt{\text{Var}(Z)}} \leq \frac{3}{\sqrt{\frac{5}{3}}}\right] \\
&\approx \Phi\left(\frac{3}{\sqrt{\frac{5}{3}}}\right) - \Phi\left(\frac{-3}{\sqrt{\frac{5}{3}}}\right) \\
&\approx \Phi(2.3) - \Phi(-2.3) \\
&= 0.9893 - 0.0107 \approx 0.98
\end{aligned}
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- Hence probability that the sum differs from true sum by more than 3 is 0.02

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- The approximation is quite good in practice

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- To get a good approximation, to calculate $P[S_n \leq m]$ one uses $P[S_n \leq m + 0.5]$ in the above approximation formula

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(because $\Phi(-x) = (1 - \Phi(x))$)

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- Here, the 0.34 would be the sample mean. The other two numbers can be fixed using CLT

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- There are other ways of handling it

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- If we change ϵ to 0.05, then at variance equal to 0.5 the probability becomes about 0.02 while the Chebyshev bound would be about 0.2