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- We can determine distribution of Y given the function g and the distribution of X

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- We have seen many specific examples of this.

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- Then Y is also discrete and $Y \in \{g(x_1), g(x_2), \cdots\}$.
- We can find the pmf of Y as

$$\begin{aligned} f_Y(y) &= p[Y = y] = P[g(X) = y] \\ &= P[X \in \{x_i \ : \ g(x_i) = y\}] \\ &= \sum_{\substack{i: \\ g(x_i)=y}} f_X(x_i) \end{aligned}$$

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$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \ a \leq y \leq b$$

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  where $a = \min(g(\infty), \; g(-\infty))$ and $b = \max(g(\infty), \; g(-\infty))$
- This theorem is useful in some cases to find the densities of functions of continuous random variables

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- Now, expectation does not exist only when $EX^+ = EX^- = \infty$

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- Similar comments apply for a continuous random variable.
- To get a feel for the more formal definition, we look at a couple of examples.

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- But by the formal definition it exists. (Note that here $X^+ = X$ and $X^- = 0$).

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- Hence EX does not exist.

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\int_{-\infty}^{\infty} g(x) \, dx \quad &\triangleq \quad \lim_{c \to \infty, d \to \infty} \int_{-c}^{d} g(x) \, dx \\
&= \quad \lim_{c \to \infty} \int_{-c}^{0} g(x) \, dx + \lim_{d \to \infty} \int_{0}^{d} g(x) \, dx
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- This depends on the definition of infinite integrals

$$\begin{aligned} \int_{-\infty}^{\infty} g(x) \, dx & \triangleq \lim_{c \to \infty, d \to \infty} \int_{-c}^{d} g(x) \, dx \\ & = \lim_{c \to \infty} \int_{-c}^{0} g(x) \, dx + \lim_{d \to \infty} \int_{0}^{d} g(x) \, dx \end{aligned}$$

This is not same as $\lim_{a \to \infty} \int_{-a}^{a} g(x) \, dx,$

which is known as Cauchy principal value of the integral

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- However, $\lim_{a\to\infty} \int_{-a}^{a} x \, \frac{1}{\pi} \, \frac{1}{1+x^2} \, dx = 0$.

Expectation of a random variable

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- Note that expectation is defined for all random variables
- Let us calculate expectations of some of the standard distributions.

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- Thus, for example, $EI_A = P(A)$.

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(Left as an exercise for you!)

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&= \frac{1}{b-a} \, \frac{b^2 - a^2}{2} \\
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&= \mu
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- This theorem is true for all rv's. But we will prove it in only some special cases.

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  That completes the proof.
- The proof goes through even when X (and Y) take countably infinitely many values (because we assume the expectation sum is absolutely convergent).

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- We can similarly show this for the case where $g'(x) < 0, \ \forall x$

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- However, this theorem is true for all random variables.
- Now, for any function, g, we can write

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \quad \text{or} \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$$

Some Properties of Expectation

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \quad \text{or} \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$$

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$$2c^* = 2E[X] \ \Rightarrow \ c^* = E[X]$$

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- Thus $E[(X-c)^2] \geq E[(X-EX)^2], \ \forall c$
- So, $E[(X-c)^2]$ is minimized when $c = EX$ and the minimum value is $E[(X-EX)^2]$

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- This also implies: $E[X^2] \geq (EX)^2$

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$$\begin{aligned}\text{Var}(X) &= EX^2 - (EX)^2 \\ &= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4}\end{aligned}$$

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$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 \\ &= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} \\ &= \frac{4(b^2+ab+a^2) - 3(b^2+2ab+a^2)}{12} \end{aligned}$$

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&= x^2\, \lambda\, \frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty - \int_0^\infty \lambda\, \frac{e^{-\lambda x}}{-\lambda}\, 2x\, dx
\end{aligned}
$$

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E[X^2] &= \int_0^\infty x^2\, \lambda\, e^{-\lambda x}\, dx \\
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- Hence the variance is now given by

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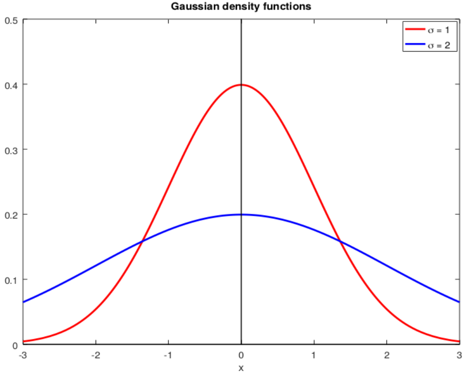

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(Left as an exercise)

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- Not all moments may exist for a given random variable. (For example, m_1 does not exist for Cauchy rv)

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&\leq P[|X|^s < 1] + \int_{|x|\geq 1} |x|^k \, f_X(x)
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&\leq P[|X|^s < 1] + \int_{|x|\geq 1} |x|^k \, f_X(x)
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since for $|x| \geq 1$, $|x|^s < |x|^k$ when $s < k$

- **Theorem**: If $E\left[|X|^k\right] < \infty$ then $E\left[|X|^s\right] < \infty$ for $0 < s < k$.
- For example, if third order moment exists then so do first and second order moments
- **Proof**: We prove it when X is continuous rv. Proof for discrete case is similar.

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- In general

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(Exercise: Differentiate it twice to find EX^2 and hence show that variance is λ).

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- We would consider ϕ_X later in the course

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Hence, we get

$$f_X(0) = P_X(0);\ f_X(1) = \frac{P'_X(0)}{1!};\ f_X(2) = \frac{P''_X(0)}{2!}$$

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- For (positive integer valued) discrete random variables, it is more convenient to deal with generating functions than mgf.

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- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

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- Note that for a given p there can be multiple values for x to satisfy the above.