- We next consider Markov chains that operate in continuous time – Continuous Time Markov Chains
- This will also be a continuous-time discrete-state process
- As we shall see, Poisson process is a special case of continuous time Markov chain

Continuous-Time Markov Chains

- Let $\{X(t),\ t \geq 0\}$ be a continuous-time discrete-state process
- Let $X(t)$ take non-negative integer values
- It is called a continuous-time markov chain if

$$Pr[X(t+s) = j \mid X(s) = i,\ X(u) \in A_u,\ 0 \leq u < s] \\ = Pr[X(t+s) = j \mid X(s) = i]$$

- Only most recent past matters
- It is called homogeneous chain if

$$Pr[X(t{+}s) = j \mid X(s) = i] = Pr[X(t) = j \mid X(0) = i],\ \forall s$$

- Define

$$P_{ij}(t) = Pr[X(t) = j \mid X(0) = i] = Pr[X(t+s) = j \mid X(s) = i]$$

It is the probability of going from i to j in time t

- Analogous to transition probabilities in the discrete case
- Like in the discrete case, we can show that the Markov condition implies

$$Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(s) = i, X(s'),\ 0 \leq s' < s < t]$$
$$= Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(s) = i]$$

- We can intuitively think of a continuous time Markov chain as follows.
- Here there is no 'regular clock' for state transitions.
- Once the chain comes into a state i, it spends some random time in that state and then transits into some other state j according to some transition probabilities.
- Note that there is no concept of a transition from i to i.
- Let us look at the distribution of time spent in a state before leaving it

- By the Markov property and homogeneity we have

$$
\begin{aligned}
Pr\left[X(s) = i,\ s \in [t,\ t+\tau] \mid X(s') = i,\ 0 \le s' \le t\right] \\
&= Pr[X(s) = i,\ s \in [t,\ t+\tau] \mid X(t) = i] \\
&= Pr[X(s) = i,\ s \in [0,\ \tau] \mid X(0) = i]
\end{aligned}
$$

- let $X(0) = i$ and let T_i be time spent in i before leaving it for the first time

$$
\begin{aligned}
Pr[X(s) = i,\ s \in [t,\ t+\tau] \mid X(s') = i,\ 0 \le s' \le t] \\
= Pr[T_i > t + \tau \mid T_i > t]
\end{aligned}
$$

$$
Pr[X(s) = i,\ s \in [0,\ \tau] \mid X(0) = i] \ = \ Pr[T_i > \tau]
$$

$$
\Rightarrow \quad Pr[T_i > t + \tau \mid T_i > t] \ = \ Pr[T_i > \tau]
$$

$\Rightarrow$ T_i is memoryless and hence exponential

- Once you transit into a state, the time spent in it is exponentially distributed.
- So, the chain can be viewed as follows
- Once you transit to a state, it spends time $T_i \sim$ exponential(ν_i) in it.
- Then, when it leaves i, it transits to state j with probability, say, z_{ij}
- We would have $z_{ij} \geq 0$, $\sum_j z_{ij} = 1$. Also, $z_{ii} = 0$
- Note that $P_{ij}(t)$ is different from these z_{ij}

Example: Birth-Death process

- Consider a chain with state space as non-negative integers.
- From i the process can only go to $i+1$ or $i-1$
- This is generalization of birth-death chains we saw earlier to continuous time
- When in i, a 'birth event' takes it to $i+1$ and a 'death event' takes it to $i-1$

- An example of a Birth-Death process is a queuing sytem or a queuing chain.
- The state represents the number of people in the system. This includes people waiting for service and those being served.
- A birth event would be a new person joining the queue.
- A death event would be a person leaving after finishing service
- Suppose the chain is in state i.
- If a new person joins before anyone leaves, the chain goes to $i + 1$
- If a person leaves before anyone joins the queue, it goes to $i - 1$.

- Suppose, in state n, time till next arrival or birth event is exponential(λ_n).
- Let time till next departure or death event be exponential(μ_n)
  We assume that these two are independent
- Now, these λ_n and μ_n completely determine ν_n and z_{ij} and hence completely specify the chain
- Before we calculate these, we need to recall some properties of exponential random variables

- Let $X \sim \exp(\lambda_1)$ and $Y \sim \exp(\lambda_2)$ and let X, Y be independent.

$$Pr[X < Y] = \int_0^\infty \int_0^y \lambda_1 e^{-\lambda_1 x} \, \lambda_2 e^{-\lambda_2 y} \, dx \, dy$$
$$= \int_0^\infty (1 - e^{-\lambda_1 y}) \, \lambda_2 e^{-\lambda_2 y} \, dy = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

We can also get this using

$$Pr[X < Y] = \int_0^\infty Pr[X < Y | Y = y] \, f_Y(y) \, dy = \int_0^\infty Pr[X < y] \, f_Y(y)$$

- Let $Z = \min(X, Y)$. Then

$$Pr[Z > z] = Pr[X > z, Y > z] = e^{-\lambda_1 z} e^{-\lambda_2 z} = e^{-(\lambda_1 + \lambda_2) z}$$

Thus, $\min(X, Y) \sim \exp(\lambda_1 + \lambda_2)$.

- Consider the birth-death process with birth rate λ_n and death rate μ_n in state n
- Now, we want to calculate ν_n and z_{ij} for the chain.
- $z_{i,i+1}$ is the probability that when the system changes state (while in i) it goes to $i+1$
- Hence it is the probability that a birth event occurs before a death event (when in state i).
- Let $W_1 \sim$ exponential(λ_i) and $W_2 \sim$ exponential(μ_i) be independent. Then

$$z_{i,i+1} = Pr[W_1 < W_2] = \frac{\lambda_i}{\lambda_i + \mu_i}; \quad \Rightarrow \quad z_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

- The time spent in state i, T_i, is exponential(ν_i)
- The chain would be in state i till either a birth event or a death event occurs
- Hence, $T_i = \min(W_1, W_2)$
- Hence, $T_i \sim$ exponential$(\lambda_i + \mu_i)$.
- Thus, $\nu_i = \lambda_i + \mu_i$
- We have taken state space to be non-negative integers.
- Hence, $\mu_0 = 0$ and $\nu_0 = \lambda_0$ and $z_{01} = 1$

- Suppose $\lambda_n = \lambda, \ \forall n$ and $\mu_n = 0, \ \forall n$
- It is called pure birth process
- The process spend time $T_i \sim$ exponential(λ) in state i and then moves to state $i + 1$
- This is the Poisson process

- Consider a queuing system
- Suppose people joining the queue is a Poisson process with rate λ
- Suppose the time to service each customer is independent and exponential with parmeter μ.
- We assume that the arrival and service processes are independent.
- Then this is a birth death process with

$$\lambda_n = \lambda, \; n \geq 0 \quad \text{and} \quad \mu_n = \mu, \; n \geq 1$$

- This is known as an $M/M/1$ queue
- A variation: $M/M/K$ queue

$$\lambda_n = \lambda, \; n \geq 0 \quad \text{and} \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq K \\ K\mu & n > K \end{cases}$$

- Consider an example of some calculations with continuous Markov chains
- Consider a Birth-Death process. Let Y_i be the time that a chain currently in i takes to reach state $i+1$ for the first time.
- We want to calculate $E[Y_i]$. Note that $E[Y_0] = 1/\lambda_0$
- The chain may directly go to $i+1$ or it may go to $i-1$ and then to i and then to $i+1$ or ...
- Define

$$I_i = \begin{cases} 1 & \text{if first transition out of } i \text{ is to } i+1 \\ 0 & \text{if first transition out of } i \text{ is to } i-1 \end{cases}$$

- We can find $E[Y_i]$ by conditioning on I_i.

- Time spent in i is exponential with rate $\lambda_i + \mu_i$.
- Hence, expected time till transition out of i is $1/(\lambda_i + \mu_i)$
- If this transition is to $i+1$ then that is the expected time to reach $i+1$

$$E\left[Y_i \mid I_i = 1\right] = \frac{1}{\lambda_i + \mu_i}$$

- Suppose this transition is to $i-1$.
- Then the expected time to reach $i+1$ is this time plus expected time to reach i from $i-1$ plus expected time to reach $i+1$ from i

$$E\left[Y_i \mid I_i = 0\right] = \frac{1}{\lambda_i + \mu_i} + E[Y_{i-1}] + E[Y_i]$$

- We also have

$$Pr[I_1 = 1] = z_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}; \quad Pr[I_i = 0] = \frac{\mu_i}{\lambda_i + \mu_i}$$

- Now we can calculate $E[Y_i]$ as

$$\begin{aligned} E[Y_i] &= E\left[Y_i \mid I_i = 1\right] \, Pr[I_i = 1] \; + \; E\left[Y_i \mid I_i = 0\right] \, Pr[I_i = 0] \\ &= \frac{\lambda_i}{\lambda_i + \mu_i} \, \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} \, \left(\frac{1}{\lambda_i + \mu_i} + E[Y_{i-1}] + E[Y_i] \right) \\ &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} \, \left(E[Y_{i-1}] + E[Y_i] \right) \end{aligned}$$

$$\begin{aligned} E[Y_i] \left(1 - \frac{\mu_i}{\lambda_i + \mu_i} \right) &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} \, \left(E[Y_{i-1}] \right) \\ E[Y_i] &= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} \, E[Y_{i-1}] \end{aligned}$$

- Thus, we get

$$E[Y_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} \, E[Y_{i-1}], \; i \geq 1$$

- Since we know, $E[Y_0] = 1/\lambda_0$, we have a formula for $E[Y_i]$
- For example,

$$E[Y_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \; \lambda_0} \quad E[Y_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} \left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \; \lambda_0} \right)$$

- We can also calculate expected time to go from i to j for $i < j$ as

$$E[Y_i] + E[Y_{i+1}] + \cdots + E[Y_{j-1}]$$

- Note that all these are only for birth-death processes

- We next consider the transition probability function.
- Recall
$$P_{ij}(t) = Pr[X(t) = j | X(0) = i]$$
- This is the probability that a chain currently in state i would be in state j at time t from now.
- We need a technical assumption.
- We assume that the transition probability function is such that the probability of infinite number of transitions in a finite interval of time is zero. (Called the regularity condition).

- The transition probabilities, $P_{ij}(t)$, satisfy Chapmann-Kolmogorov equations

$$\begin{aligned}
P_{ij}(t+s) &= Pr[X(t+s) = j \mid X(0) = i] \\
&= \sum_k Pr[X(t+s) = j \mid X(s) = k, X(0) = i] \; Pr[X(s) = k \mid X(0) = i] \\
&= \sum_k Pr[X(t+s) = j \mid X(s) = k] \; Pr[X(s) = k \mid X(0) = i] \\
&= \sum_k Pr[X(t) = j \mid X(0) = k] \; Pr[X(s) = k \mid X(0) = i] \\
&= \sum_k P_{kj}(t) P_{ik}(s) = \sum_k P_{ik}(s) P_{kj}(t)
\end{aligned}$$

- For a finite chain we can represent P as a matrix and hence
  $P(t+s) = P(s)P(t) = P(t)P(s)$

- The chapmann-Kolmogorov equations give

$$P_{ij}(t+s) = \sum_k P_{ik}(s) P_{kj}(t)$$

- Hence we have

$$\begin{aligned} P_{ij}(t+h) - P_{ij}(t) &= \sum_k P_{ik}(h) P_{kj}(t) - P_{ij}(t) \\ &= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) - (1 - P_{ii}(h)) P_{ij}(t) \end{aligned}$$

$$\Rightarrow \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \neq i} \lim_{h \to 0} \frac{P_{ik}(h)}{h} P_{kj}(t) - \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} P_{ij}(t)$$

(For an infinite chain there is the issue of whether we can take the limit inside the summation)

- Define

$$q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h}, \quad i \neq j$$

- Then, we have

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \lim_{h \to 0} \frac{\sum_{j \neq i} P_{ij}(h)}{h} = \sum_{j \neq i} q_{ij}$$

- Hence we can write the earlier equation as

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \left(\sum_{k \neq i} q_{ik} \right) P_{ij}(t) = \sum_{k} q_{ik} P_{kj}(t)$$

by taking $q_{ii} = -\sum_{k \neq i} q_{ik}$

- We showed that the transition probability function satisfies

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \left(\sum_{k \neq i} q_{ik} \right) P_{ij}(t), \ \forall i, j$$

- This is known as Kolmogorov Backward equation.
- If we know all the q_{ij}, then we can solve these differential equations and obtain the transition probability function.
- Thus, $P_{ij}(t)$ for all i, j and all t are completely determined by q_{ij}.
- Hence we specify the transition functions of a continuous time Markov chain through the q_{ij}.
- These are referred to as the infinitesimal generator of the chain.

- By definition, $1 - P_{ii}(h)$ is the probability that the chain that started in i is not in i at h.
- Probability of two or more transitions in h is $o(h)$. Transitions out of i occur at the rate of ν_i.
- Hence
$$1 - P_{ii}(h) = \nu_i h + o(h)$$
- We also have
$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \sum_{j \neq i} q_{ij}$$
- Thus $\sum_{j \neq i} q_{ij} = \nu_i$. It is rate of transition out of i

- By definition, $P_{ij}(h) = q_{ij}h + o(h), \ i \neq j$
- Hence q_{ij} is the rate at which transitions out of i into j are occurring.
- Transitions out of i occur with rate ν_i and z_{ij} fraction of these are into j
- Hence, $q_{ij} = \nu_i \ z_{ij}, \ i \neq j$
- Thus, we got

$$\nu_i = \sum_{j \neq i} q_{ij}, \quad z_{ij} = \frac{q_{ij}}{\sum_{j \neq i} q_{ij}}$$

- As mentioned earlier, $\{q_{ij}\}$ are called the infinitesimal generator of the process.
- A continuous time Markov Chain is specified by these q_{ij}

- Consider a Birth-Death process.
- We got earlier

$$\nu_i = \lambda_i + \mu_i, \quad z_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad z_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

- Now we can calculate q_{ij}

$$q_{i,i+1} = (\lambda_i + \mu_i) \, \frac{\lambda_i}{\lambda_i + \mu_i} = \lambda_i, \quad q_{i,i-1} = (\lambda_i + \mu_i) \, \frac{\mu_i}{\lambda_i + \mu_i} = \mu_i$$

- This is intuitively obvious
- We specify a birth-death chain by
  birth rate (i.e., rate of transition from i to $i+1$), λ_i and
  death rate (i.e., rate of transition from i to $i-1$), μ_i.

- Consider a finite chain. The transition probabilities can be represented as a matrix, $P(t)$
- The Chapmann-Kolmogorov equation gives

$$P_{ij}(t+s) = \sum_k P_{ik}(t) P_{kj}(s) \quad \Rightarrow \quad P(t+s) = P(t)\ P(s)$$

- Differentiating the above with respect to t

$$P'(t+s) = P'(t) P(s)$$

- Putting $t = 0$ in the above we get

$$P'(s) = P'(0)\ P(s) = Q\ P(s), \ \text{ where } \ Q = P'(0)$$

- Note that $P_{ii}(0) = 1$ and $P_{ij}(0) = 0, i \neq j$.
- The solution for the above ODE is

$$P(t) = e^{tQ}, \quad \text{because} \quad P(0) = I$$

- Hence we can calculate $P_{ij}(t)$ for any t and i, j

- Recall $P_{ij}(0) = 0, \ i \neq j$ and $P_{ii}(0) = 1$
- We have defined $Q = P'(0)$.

$$P'_{ij}(0) = \lim_{h \to 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} = \lim_{h \to 0} \frac{P_{ij}(h)}{h}, \ i \neq j$$

  This is same as q_{ij} defined earlier.

$$P'_{ii}(0) = \lim_{h \to 0} \frac{P_{ii}(h) - P_{ii}(0)}{h} = \lim_{h \to 0} \frac{P_{ii}(h) - 1}{h} = -\sum_{j \neq i} q_{ij} = q_{ii}$$

- Thus the Q matrix is what we defined earlier as the infinitesimal generator of the chain.
- We have $P'(t) = QP(t)$
- Note that rows of Q sum to zero

- The Chapmann-Kolmogorov equations give us

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq i} P_{ik}(h)\ P_{kj}(t) - (1 - P_{ii}(h)) P_{ij}(t)$$

- Using this we derived

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik}\ P_{kj}(t) - q_{ii}\ P_{ij}(t)$$

  Called Kolmogorov Backward equation
- For a birth-death chain the equation becomes

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

Poisson process as a special case

- Consider the case: $\lambda_i = \lambda$ and $\mu_i = 0.\ \forall i$.
- This would be a Poisson process with rate λ.
- Taking $i = 0$, the differential equation becomes

$$P'_{0j}(t) = \lambda P_{1j}(t) - \lambda P_{0j}(t)$$

- $P_{0j}(t)$ is the probability of j events in an interval of length t which is same as what we had called $P_j(t)$.
- Similarly, $P_{1j}(t)$ is same as what we called $P_{j-1}(t)$ there
- Now one can see that the above ODE is what we got for Poisson process.

- Consider a two-state Birth-Death chain.
- We would have $\mu_0 = \lambda_1 = 0$. Let $\lambda_0 = \lambda$ and $\mu_1 = \mu$
- The two states can be a machine working or failed.
- λ is rate of failure. Time till next failure is exponential(λ)
- μ is rate of repair. Time for repair is exponential(μ)
- We may want to calculate $P_{00}(T)$, the probability that the machine would be working at a time T units later given it is in working condition now
- We can calculate it by solving the ODE's

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

- For the two state chain, these equations are

$$\begin{aligned}
P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\
P'_{01}(t) &= \lambda_0 P_{11}(t) - \lambda_0 P_{01}(t) \\
P'_{10}(t) &= \mu_1 P_{00}(t) - \mu_1 P_{10}(t) \\
P'_{11}(t) &= \mu_1 P_{01}(t) - \mu_1 P_{11}(t)
\end{aligned}$$

- As is easy to see, we get a system of equations like this for any finite chain.
- Solving these we can show

$$P_{00}(t) = \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda + \mu}$$

- For the backward equation, we started with

$$P_{ij}(t+h) = \sum_k P_{ik}(h)\ P_{kj}(t)$$

- The Chapmann-Kolmogorov equation also gives us

$$P_{ij}(t+h) = \sum_k P_{ik}(t)\ P_{kj}(h)$$

- Similar algebra as earlier gives us

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t)\ q_{kj} - q_{jj}\ P_{ij}(t)$$

  (under some assumptions for interchanging limit and summation)
- This is known as Kolmogorov forward equation
- For finite chains, both forward and backward equations are same
- For infinite chains there are some differences

- We can define transient and recurrent states as in the discrete case.
- However, we need to be careful about defining hitting times or first passage times
- We define

$$T_i = \min\{t > 0 \ : \ X(t) \neq i\} \qquad f_i = \min\{t \ : \ t > T_i, \ X(t) = i\}$$

- For a chain started in i we take f_i as first return time to i
- A state i is said to be
  - Transient if $Pr[f_i < \infty \mid X(0) = i] < 1$
  - Recurrent if $Pr[f_i < \infty \mid X(0) = i] = 1$

- Most of the other definitions are also similar to the case of discrete chains
- The chain is said to be irreducible if for all i, j there is a positive probability of going from i to j in some finite time: $P_{ij}(t) > 0$ for some t
- A recurrent state is positive recurrent if mean time to return is finite: $E[f_i \mid X(0) = i] < \infty$
  Otherwise it is null recurrent
- An irreducible positive recurrent chain would have a unique stationary distribution
- There is no concept of periodicity in the continuous time case
- An irreducible positive recurrent chain would be called an ergodic chain

- Define

$$\pi_j(t) = Pr[X(t) = j] = \sum_i \pi_i(0) P_{ij}(t)$$

  This also analogous to the discrete case
- The above equation is true for general infinite chains.
- In the finite case, we can get a more compact expression
- For a finite chain, taking π as a row vector,

$$\pi(t) = \pi(0)\; P(t) = \pi(0)\; e^{Qt}$$

- We say π is a stationary distribution if

$$\pi(0) = \pi \quad \Rightarrow \quad \pi(t) = \pi, \ \forall t$$

- When chain is in the stationary distribution, $\pi'(t) = 0$
- We get from the earlier equation

$$\pi_j(t) = \sum_i \pi_i(0) P_{ij}(t) \quad \text{and hence} \quad \pi_j'(t) = \sum_i \pi_i(0) P_{ij}'(t)$$

- Using the forward equation for $P_{ij}'(t)$, & $\pi'(t) = 0$,

$$\begin{aligned}
\sum_i \pi_i(0) \left(\sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t) \right) &= 0 \\
\Rightarrow \quad \sum_{k \neq j} q_{kj} \sum_i \pi_i(0) P_{ik}(t) - \nu_j \sum_i \pi_i(0) P_{ij}(t) &= 0 \\
\Rightarrow \quad \sum_{k \neq j} q_{kj} \pi_k(t) - \pi_j(t) \sum_{k \neq j} q_{jk} &= 0
\end{aligned}$$

when $\pi_0 = \pi$, a stationary distribution, $\pi(t) = \pi$

- What we showed is that any stationary distribution π has to satisfy

$$\sum_{k \neq j} q_{kj} \pi_k = \pi_j \sum_{k \neq j} q_{jk}$$

- We can interpret this (as we did in discrete case)
- q_{kj} is the rate of transition from k to j and π_k is the fraction present in k.
- Hence $\sum_{k \neq j} q_{kj} \pi_k$ is the net flow into j
- $\pi_j \sum_{k \neq j} q_{jk}$ is the net flow out of j
- At steady state the flows have to be balanced
- The above equation is known as a balance equation

- Any stationary distribution π has to satisfy the balance equation

$$\sum_{k \neq j} q_{kj} \pi_k = \pi_j \sum_{k \neq j} q_{jk}$$

- Suppose there is a π that satisfies

$$q_{kj} \pi_k = \pi_j q_{jk}, \ \forall j, k \ (j \neq k)$$

  Then that π satisfies the balance equation and hence is a stationary distribution.
- This is called detailed balance.