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- Called multidimensional change of variable formula

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This shows: $f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{x}\right| f_{XY}\left(x, \frac{z}{x}\right)\ dx$

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- Exchangeable random variables are identically distributed but they may not be independent.

Recap: Expectation of function of multiple rv's

- Let $Z = g(X_1, \cdots X_n) = g(\mathbf{X})$. Then

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- Specifically, $E[X+Y] = E[X] + E[Y]$

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- Uncorrelated random variables need not necessarily be independent

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- $|\rho_{XY}| = 1$ iff $X = aY$

Recap: mean square estimation

- The best mean-square approximation of Y as a 'linear' function of X is

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- If $\rho_{XY} = 0$ the final error is $\text{Var}(Y)$

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- $\text{Var}(\mathbf{a}^T\mathbf{X}) = \mathbf{a}^T\Sigma_X\,\mathbf{a}$
- Σ_X is a real symmetric and positive semidefinite matrix.

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- We define the characteristic function of $\mathbf{X}$ by

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- The last property above follows directly from the definition.

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- Similarly

$$E[\ E[Y|X]\ ] = E\left[\frac{1+X}{2}\right] = \frac{2}{3} = E[Y]$$

Example

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$$EX = 1; \quad \text{Var}(X) = 1; \quad EY = 2; \quad \text{Var}(Y) = 2$$

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Thus, $E[Y|X] = 1 + X$

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&= 1\, p + (1+EX)(1-p)
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If we can guess value of $E[R_n]$ then we can prove it using mathematical induction

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- Hence a good guess is $E[R_n] = n$.
- We verify it using mathematical induction. We know $E[R_1] = 1$

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- So, final number of comparisons depends on the 'number of rounds'

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- Question: 'on the average' how many comparisons?

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- Define: $X = j$ if pivot is j^{th} smallest
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$$\begin{aligned} E[M_n] &= E[\ E[M_n|X]\ ] = \sum_{j=1}^{n} E[M_n|X=j]\ P[X=j] \\ &= \sum_{j=1}^{n} E[(n-1) + M_{j-1} + M_{n-j}]\ \frac{1}{n} \\ &= (n-1) + \frac{2}{n}\sum_{k=1}^{n-1} E[M_k], \quad \text{(taking } M_0 = 0) \end{aligned}$$

- This is a recurrence relation. (A little complicated to solve)

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- Let us prove this.

- We want to show that for all g

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&= \big(g(X) - E[Y \mid X]\big)^2 + \big(E[Y \mid X] - Y\big)^2 \\
&\quad + 2\big(g(X) - E[Y \mid X]\big)\big(E[Y \mid X] - Y\big)
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- Now we can take expectation on both sides.
- We first show that expectation of last term on RHS above is zero.

First consider the last term

$$E\Big[(g(X) - E[Y \mid X])(E[Y \mid X] - Y)\Big]$$

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\begin{aligned}
& E\Big[(g(X) - E[Y \mid X])(E[Y \mid X] - Y)\Big] \\
= \;& E\Big[\;\; E\Big\{(g(X) - E[Y \mid X])(E[Y \mid X] - Y) \mid X\Big\} \quad\;\; \Big]
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$$\text{because} \quad E[Z] = E[\, E[Z|X]\,]$$

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= \; & E\Big[\; (g(X) - E[Y \mid X]) \; E\Big\{(E[Y \mid X] - Y) \mid X\Big\} \; \Big] \\
& \qquad \text{because} \quad E[h_1(X)h_2(Z)|X] = h_1(X) \; E[h_2(Z)|X]
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= \;& E\Big[\;\; (g(X) - E[Y \mid X]) \;\; E\Big\{(E[Y \mid X] - Y) \mid X\Big\} \;\;\Big] \\
= \;& E\Big[\;\; (g(X) - E[Y \mid X]) \;\; \Big(E\Big\{(E[Y \mid X])|X\Big\} \; - \; E\{Y \mid X\}\Big) \;\;\Big]
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= \; & E\left[\; (g(X)-E[Y \mid X]) \;\; \left(E\left\{(E[Y \mid X]) | X\right\} \; - \; E\{Y \mid X\}\right) \;\right] \\
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= \;& E\Big[\; (g(X) - E[Y \mid X]) \;\; (E[Y \mid X] \; - \; E[Y \mid X)) \;\Big] \\
= \;& 0
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- Since the above is true for all functions g, we get

$$g^*(X) = E\,[Y \mid X]$$