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- In general, almost sure convergence does not imply convergence in r^{th} mean and vice versa

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- strong law of large numbers: $\frac{S_n}{n} \stackrel{a.s.}{\rightarrow} \mu$

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- **(Lindberg-Levy) Central Limit Theorem**

$$\lim_{n\to\infty} P\left[\tilde{S}_n \leq x\right] = \lim_{n\to\infty} P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt, \quad \forall x$$

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- For example, binomial rv is well approximated by normal for large n

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(because $\Phi(-x) = (1 - \Phi(x))$)

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- This interval is called the $100(1-\delta)\%$ confidence interval.

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- In analyzing any experimental data the confidence intervals or the variance term is important

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- How doe we specify or characterize an infinite collection of random variables?
- We need the joint distribution of every finite subcollection of them.

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- Sum of iid random variables is a Markov chain

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- Number of packets coming into a network switch, number people joining the queue in a bank, can be modeled as Markov chains.
  In such applications, it is called a queing chain.
- Markov chain is a useful model for many dynamic systems or processes

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- It can be more general

Transition Probabilities

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- The joint distribution of $X_{i_1}, \cdots X_{i_k}$ is now calculated as a marginal distribution from the joint distribution of $X_0, \cdots, X_m$

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- This can easily be seen through a graphical notation.

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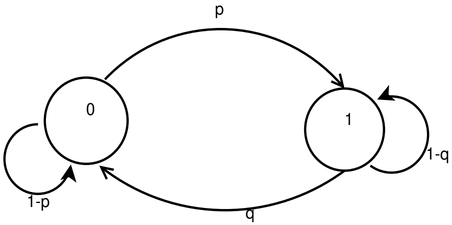

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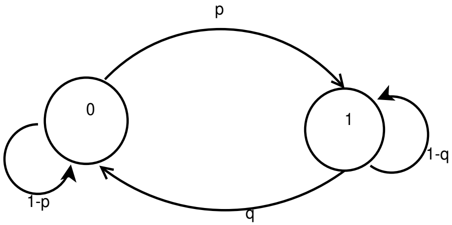

- The nodes represent states. The edges show possible transitions and the probabilities.

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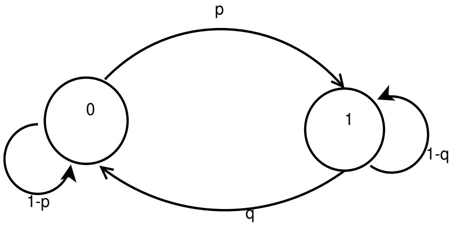

- The nodes represent states. The edges show possible transitions and the probabilities. We can easily see

$$Pr[X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 0] = \pi_0(0)p(1-q)q$$

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- We may want to know about $\lim_{n \to \infty} Pr[X_n = 1]$

Another example

- A man has 4 umbrellas. carries them from home to office and back when needed. Probability of rain in the morning and evening is same, namely, p.

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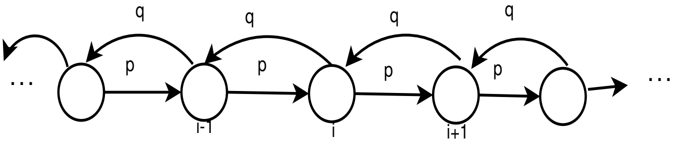
- $S = \{0, 1, \cdots, 5\}$. The transition probabilities are

$$P = \left[\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1-p & p \\ 2 & 0 & 0 & 1-p & p & 0 \\ 3 & 0 & 1-p & p & 0 & 0 \\ 4 & 1-p & p & 0 & 0 & 0 \end{array}\right]$$

Birth-Death chain

- The following Markov chain is known as a birth-death chain



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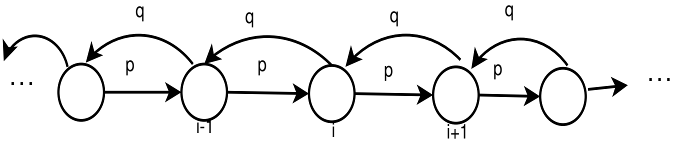

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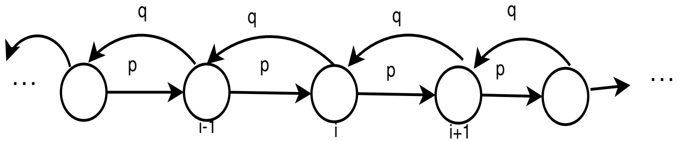

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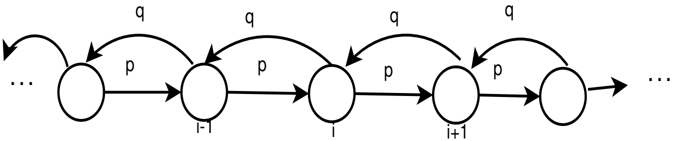

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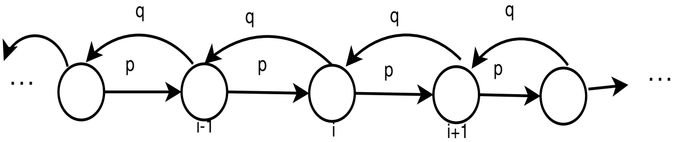

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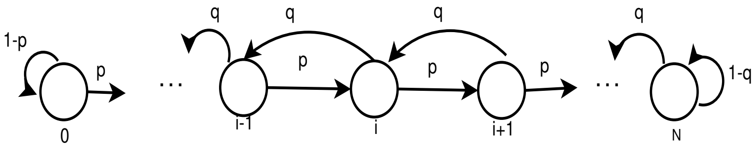
- In general, the transition probabilities can be different for different states.
- Birth-death chains can also have self-loops on states

- We can have birth-death chains with finite state space also



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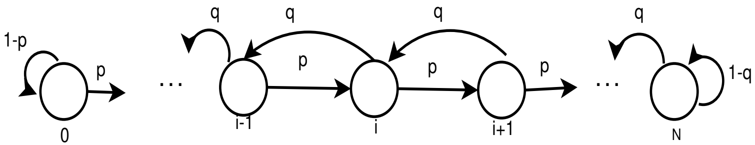

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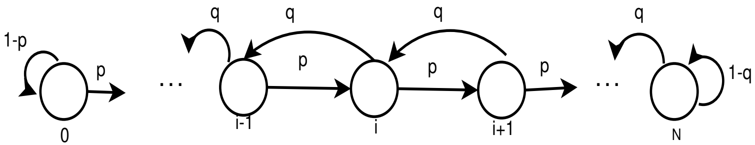

- We can say it has 'reflecting boundaries'
- This chain keeps visiting all the states again and again.

Gambler's Ruin chain

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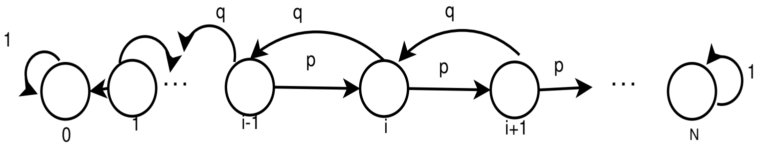

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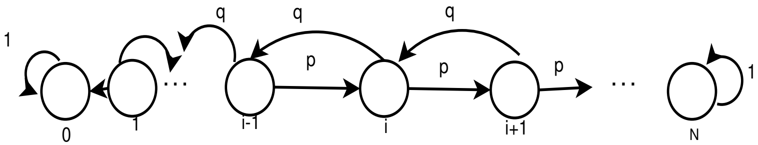

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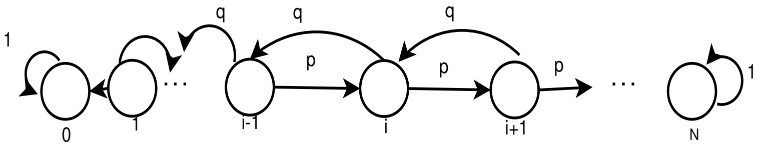

- Here, the chain is ultimately absorbed either in 0 or in N
- Here state can be the current funds that the gambler has

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$$Pr[X_{n+m} = y | X_n = x, X_0] = Pr[X_{n+m} = y | X_n = x]$$

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