Recap: Random Walk

- Z_i iid with $Pr[Z_i = +s] = Pr[Z_i = -s] = 0.5$
- Define a continuous-time process $X(t)$ by

$$\begin{aligned} X(nT) &= Z_1 + Z_2 + \cdots + Z_n \\ X(t) &= X(nT), \quad \text{for } \; nT \leq t < (n+1)T \end{aligned}$$

- The discrete-time Markov chain, $X(nT), n = 0, 1, \cdots$, is called a (one dimensional) random walk
- The $X(t)$ is defined by piece-wise constant interpolation of $X(nT)$
- For large n, $\frac{X(nT)}{s\sqrt{n}}$ would be Gaussian

$$Pr\left[\frac{X(nT)}{s\sqrt{n}} \leq y\right] \approx \Phi(y)$$

Recap: Limit of Random Walk

- For $t = nT$, $E[X^2(t)] = ns^2 = s^2 \frac{t}{T}$
- If we let $T \to 0$, we need s^2 to go to zero at the same rate as T.
- Define
$$W(t) = \lim_{T \to 0, s^2 = \alpha T} X(t)$$
This is called the Wiener Process or Brownian motion.
- We intuitively expect $W(t)$ to be a process with stationary and independent increments and for each t, $W(t)$ is Gaussian with zero mean and variance proportional to t

- Let $\{X(t),\ t \geq 0\}$ be a continuous-state continuous-time process

  This process is called a Brownian motion if

  1. $X(0) = 0$
  2. The process has stationary and independent increments
  3. For every $t > 0$, $X(t)$ is Gaussian with mean 0 and variance $\sigma^2 t$

- Let $B(t) = \frac{X(t)}{\sigma}$. Then, variance of $B(t)$ is t
- $\{B(t),\ t \geq 0\}$ is called standard Brownian Motion
- Let $Y(t) = X(t) + \mu$. Then $Y(t)$ has non-zero mean
- $\{Y(t),\ t \geq 0\}$ is called Brownian motion with a drift
- The mean can be a function of time

- An important result is that Brownian motion paths are continuous
- Brownian motion is the limit of random walk where both s and T tend to zero
- Intuitively the paths should be continuous.
- The paths are continuous but non-differentiable everywhere
- This is a deep result

- We can use the stationary and independent increments property to calculate the autocorrelation function

$$
\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[X(t_1)\,(X(t_2) - X(t_1) + X(t_1))], \quad (\text{take } \; t_1 < t_2) \\
&= E[X(t_1)(X(t_2) - X(t_1))] + E[X^2(t_1)] \\
&= E[X(t_1)]\; E[X(t_2) - X(t_1)] + E[X^2(t_1)] \\
&= E[X^2(t_1)] \\
&= \sigma^2\, t_1 \quad = \sigma^2 \min(t_1, t_2)
\end{aligned}
$$

- Since $E[X(t)] = 0, \; \forall t$, we have

$$
\text{Cov}(X(t_1), X(t_2)) = E[X(t_1)X(t_2)] = \sigma^2 \; \min(t_1, t_2)
$$

- Let $\{X(t),\ t \geq 0\}$ be a Brownian motion
- The process has stationary increments.
- Hence for $t_2 > t_1$, $X(t_2) - X(t_1)$ has the same distribution as $X(t_2 - t_1)$
- Thus, $X(t_2) - X(t_1)$ is Gaussian with zero mean and variance $\sigma^2(t_2 - t_1)$
- Since increments are also independent, we can show that all n^{th} order distributions are Gaussian

- Suppose we want the joint distribution of $X(t_1), X(t_2), \cdots, X(t_n)$
- Let $t_1 < t_2 < \cdots < t_n$
- Define random variables $Y_1, \cdots, Y_n$ by

$$Y_1 = X(t_1), \;\; Y_2 = X(t_2) - X(t_1), \;\; Y_3 = X(t_3) - X(t_2), \cdots$$

- We know Y_i are independent because the process has independent increments
- This transformation is invertible
- Hence we can get joint density of $X(t_1), \cdots X(t_n)$ in terms of joint density of $Y_1, \cdots, Y_n$
- This is how we can get n^{th} order density for any continuous-state process with independent increments

$$Y_1 = X(t_1), \;\; Y_i = X(t_i) - X(t_{i-1}), \; i = 2, \cdots, n$$

- The transformation is invertible

$$\begin{aligned} X(t_1) &= Y_1 \\ X(t_2) &= Y_1 + Y_2 \\ X(t_3) &= Y_1 + Y_2 + Y_3 \\ &\vdots \\ X(t_n) &= Y_1 + Y_2 + \cdots + Y_n \end{aligned}$$

- $Y_1, \cdots Y_n$ are independent and Gaussian and hence are Jointly Gaussian
- Hence $X(t_1), \cdots, X(t_n)$ are jointly Gaussian
- Thus all n^{th} order distributions are Gaussian

- $X(t_1), X(t_2), \cdots, X(t_n)$ are jointly Gaussian.
- We can write their joint density because we know the means, variances and covariances
- We can also write the density using the transformation considered earlier

$$
\begin{aligned}
X(t_1) &= Y_1 \\
X(t_2) &= Y_1 + Y_2 \\
X(t_3) &= Y_1 + Y_2 + Y_3 \\
&\vdots \\
X(t_n) &= Y_1 + Y_2 + \cdots + Y_n
\end{aligned}
$$

$$
Y_1 = X(t_1), \;\; Y_i = X(t_i) - X(t_{i-1}), \; i = 2, \cdots, n
$$

- Hence we get

$$
f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, \cdots x_n) = f_{Y_1 \cdots Y_n}(x_1, x_2 - x_1, \cdots, x_n - x_{n-1})
$$

- Take $t_1 < t_2 < \cdots < t_n$

$$
f_X(x_1, \cdots, x_n; t_1, \cdots, t_n) \;=\; f_{Y_1}(x_1) f_{Y_2}(x_2 - x_1) \cdots f_{Y_n}(x_n - x_{n-1})
$$

- Since all joint densities are Gaussian and are easy to write, we can also calculate conditional densities

$$
\begin{aligned}
f_{X(s)|X(t)}(x|b) &= \frac{f_{X(s)X(t)}(x,b)}{f_{X(t)}(b)} \quad (s<t) \\
&= \frac{f_{X(s)}(x)\ f_{X(t)-X(s)}(b-x)}{f_{X(t)}(b)} \\
&\propto e^{-\frac{x^2}{2s}}\ e^{-\frac{(b-x)^2}{2(t-s)}} \quad \text{(taking } \sigma^2=1\text{)} \\
&\propto \exp\left(-x^2\left(\frac{1}{2s}+\frac{1}{2(t-s)}\right)+\frac{bx}{t-s}\right) \\
&\propto \exp\left(-\frac{t}{2s(t-s)}\left(x^2-2\frac{sb}{t}x\right)\right) \\
&\propto \exp\left(-\frac{(x-bs/t)^2}{2s(t-s)/t}\right)
\end{aligned}
$$

- Hence the conditional density is Gaussian with mean bs/t and variance $s(t-s)/t$

- We showed, for $s < t$,

$$f_{X(s)|X(t)}(x|b) \propto \exp\left(-\frac{(x - bs/t)^2}{2s(t-s)/t}\right)$$

- Thus, we have

$$E[X(s)|X(t)] = \frac{s}{t}\, X(t)$$

- The conditional variance is $\frac{s}{t}\,(t-s)$

- Let $s < t$. Then

$$E[X(t) \mid X(s)] = E[X(t) - X(s) + X(s) \mid X(s)] = X(s)$$

- Suppose $t_1 < t_2 < t_3$. Then, it is easy to see that

$$E[X(t_3) \mid X(t_2), X(t_1)] = X(t_2)$$

- We can also get the conditional density of $X(t)$ given $X(s)$ for $s < t$

- Let $s < t$. Then

$$
\begin{aligned}
f_{X(t)|X(s)}(x_t|x_s) &= \frac{f_{X(s)X(t)}(x_s, x_t)}{f_{X(s)}(x_s)} \quad (s < t) \\
&= \frac{f_{X(s)}(x_s)\ f_{X(t)-X(s)}(x_t - x_s)}{f_{X(s)}(x_s)} \\
&\propto e^{-\frac{(x_t - x_s)^2}{2(t-s)}}
\end{aligned}
$$

- Thus, conditioned on $X(s)$ the density of $X(t)$ is normal with mean $X(s)$ and variance $(t - s)$

Hitting Times

- Let T_a denote the first time Brownian motion hits a. We take $a > 0$.

$$\begin{aligned} Pr[X(t) \geq a] &= Pr[X(t) \geq a \mid T_a \leq t] \; Pr[T_a \leq t] + \\ &\qquad Pr[X(t) \geq a \mid T_a > t] \; Pr[T_a > t] \end{aligned}$$

- Since Brownian motion paths are continuous, $Pr[X(t) \geq a \mid T_a > t] = 0$
- Brownian motion is a limit of symmetric random walk. Hence if we had already hit a sometime back, then now we are as likely to be above a as below it.

$$\Rightarrow \quad Pr[X(t) \geq a \mid T_a \leq t] = \frac{1}{2}$$

Thus

$$P[X(t) \geq a] = 0.5 Pr[T_a \leq t]$$

- Hence we get

$$
\begin{aligned}
Pr[T_a \leq t] &= 2\ Pr[X(t) \geq a] \\
&= \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}}\ dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-\frac{y^2}{2}}\ dy
\end{aligned}
$$

- Here we have assumed $a > 0$. For $a < 0$ the situation is similar. Hence the above is true even for $a < 0$ except that the lower limit becomes $|a|/\sqrt{t}$
- Another interesting consequence is the following

$$
\begin{aligned}
Pr[\max_{0 \leq s \leq t} X(s) \geq a] &= Pr[T_a \leq t], \text{ by continuity of paths} \\
&= 2Pr[X(t) \geq a]
\end{aligned}
$$

Geometric Brownian Motion

- Let $\{Y(t),\ t \geq 0\}$ is a Brownian motion with drift. Define
$$X(t) = e^{Y(t)}$$
- Then, $\{X(t),\ t \geq 0\}$ is called geometric Brownian motion. It is useful in mathematial finance
- Let $X_0, X_1, \cdots$ be time series of prices of a stock.
- Let $Y_n = X_n / X_{n-1}$ and assume Y_i are iid
$$X_n = Y_n X_{n-1} = Y_n Y_{n-1} X_{n-2} = \cdots = Y_n Y_{n-1} \cdots Y_1 X_0$$
$$\Rightarrow \quad \ln(X_n) = \sum_{i=1}^{n} \ln(Y_i) + \ln(X_0)$$
- Since $\ln(Y_i)$ are iid, with suitable normalization, the interpolated process $\ln(X(t))$ would be Brownian motion and $X(t)$ would be geometric Brownian motion

Gaussian Processes

- A continuous-time continuous-state process $\{X(t),\ t \geq 0\}$ is said to be a Gaussian process if for all n and all $t_1, t_2, \cdots, t_n$, we have that $X(t_1), \cdots, X(t_n)$ are jointly Gaussian.
- The Brownian motion is an example of a Gaussian Process
- The Brwonian motion is a Gaussian process with

$$E[X(t)] = 0, \quad \mathsf{Var}(X(t)) = \sigma^2 t, \quad \mathsf{Cov}(X(s), X(t)) = \sigma^2 \min(s, t)$$

- Recall that the multivariate Gaussian density is specified by the marginal means, variances and the covariances of the random variables
- Hence, a general Gaussian process is specified by the mean function and the variance and covariance functions

- Consider the statistics of the Brownian motion process for $0 < t < 1$ under the condition that $X(1) = 0$
- Consider standard Brownian motion. ($\sigma^2 = 1$)

$$E[X(t)|X(1) = 0] = \frac{t}{1}\, 0 = 0$$

Recall that, for $s < t$, conditional density of $X(s)$ conditioned on $X(t) = b$ is gaussian with mean bs/t and variance $s(t - s)/t$

Now, for $s < t < 1$, since $E[X(s)|X(1) = 0] = 0, \ s < 1$,

$$\begin{aligned}
\text{Cov}(X(s), X(t)|X(1) = 0) \ &\triangleq \ E[X(s)X(t) \mid X(1) = 0] \\
&= E[\ E[X(s)X(t) \mid X(t), X(1) = 0] \mid X(1) = 0] \\
&= \ E[\ X(t)E[X(s) \mid X(t)] \mid X(1) = 0] \\
&= \ E[\ X(t)\frac{s}{t}X(t) \mid X(1) = 0] \\
&= \ \frac{s}{t} \ E[X^2(t) \mid X(1) = 0] \\
&= \ \frac{s}{t} \ t(1-t) \\
&= \ s(1-t)
\end{aligned}$$

Thus, for $0 < t < 1$, conditioned on $X(1) = 0$, this process has mean 0 and covariance function $s(1-t), \ s < t$

- Consider a process $\{Z(t),\ 0 \leq t \leq 1\}$.
- It is called Brownian Bridge process if it is a Gaussian process with mean zero and covariance function $\text{Cov}(Z(s), Z(t)) = s(1-t)$ when $s \leq t$.
- Let $X(t)$ be a standard Brownian motion process.
- Then, $Z(t) = X(t) - tX(1)$, $0 \leq t \leq 1$ is a Brownian Bridge
- Easy to see it is a Gaussian process with mean zero. For $s < t$

$$
\begin{aligned}
\text{Cov}(Z(s), Z(t)) &= \text{Cov}(X(s) - sX(1), X(t) - tX(1)) \\
&= \text{Cov}(X(s), X(t)) - t\text{Cov}(X(s), X(1)) - \\
&\quad s\text{Cov}(X(1), X(t)) + st\text{Cov}(X(1), X(1)) \\
&= s - st - st + st = s(1-t)
\end{aligned}
$$

White Noise

- Consider a process $\{V(t),\ t \geq 0\}$ with

$$E[V(t)] = 0; \quad \mathsf{Var}(V(t)) = \sigma^2 \quad \mathsf{Cov}(V(t), V(s)) = 0,\ s \neq t$$

- This is a kind of generalization of sequence of iid random variables to continuous time
- It is an approximation of what is called White Noise.

- Assume $V(t)$ is Gaussian. Let

$$X(t) = \int_0^t V(\tau)\, d\tau$$

- Then we get $E[X(t)] = 0$ and

$$E[X^2(t)] = \int_0^t \int_0^t E[V(t_1)V(t_2)]\, dt_1\, dt_2 = \int_0^t \sigma^2\, dt_1 = \sigma^2 t$$

$$E[X(t_1)(X(t_2) - X(t_1)] = \int_0^{t_1} \int_{t_1}^{t_2} E[V(t)V(t')]\, dt\, dt' = 0$$

- We see that $X(t)$ is a process with mean zero, variance proportional to t and having uncorrelated increments.
- Since its integral is like a Brownian motion, the process $V(t)$ is like a 'derivative' of Browian motion
- The actual concept involved is rather deep

- We have considered three random processes
- (discrete-time) Markov Chain
  – Example of Discrete-time discrete-state process
- Continuous time Markov Chains (e.g., Poisson Process)
  – Example of continuous-time discrete-state process
- Brownian Motion
  – Example of continuous-time continuous-state process
- We need an example of discrete-time continuous-state process!
- Any sequence of continuous random variables would be a discrete-time continuous-state process

- In general, any 'stochastic' algorithm would generate discrete-state continuous time process.
- If an algorithm uses a random step, then the algorithm would be like

$$X(n+1) = X(n) + \eta_n \, G(X(n), \xi(n))$$

  where $\xi(n)$ would be some random variable which may be dependent on $X(n)$.
- Many algorithms can be written in this general form.
- The $X(n), n = 0, 1, \cdots$ would be a discrete-time continuous-state stocahstic process
- We may want to know the conditions under which we can prove the sequence converges.