Recap: Stationary Distribution of markov chain

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- A stationary distribution always exists for a finite chain

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- The limiting fraction of time spent by the chain in transient and null recurrent states is zero.
- **Theorem:** Let x be positive recurrent and let x lead to y. Then y is positive recurrent.

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- We next show that a finite chain cannot have any null recurrent states.

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where we could take the limit inside the sum because C is finite.

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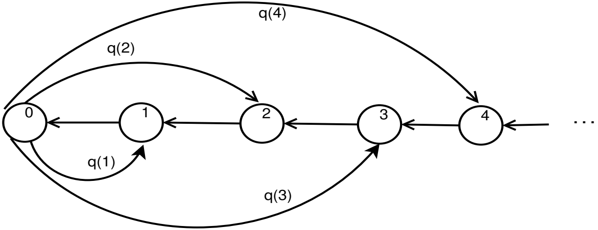
- Hence, a finite chain cannot have a null recurrent state.

Example of null recurrent chain

- Consider the chain with state space $\{0, 1, \cdots\}$ given by



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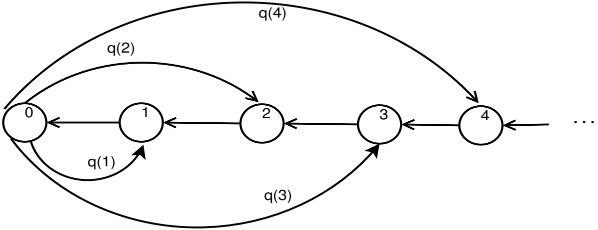

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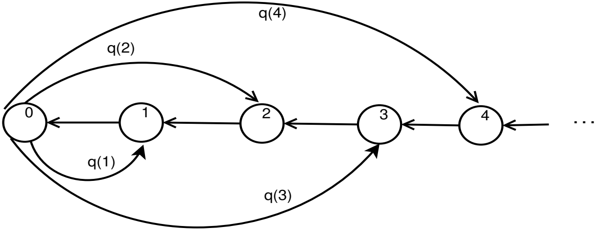

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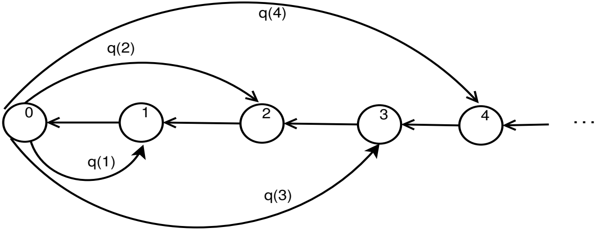

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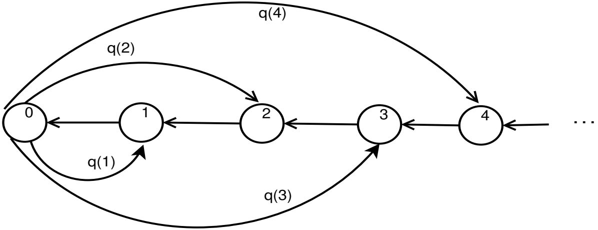

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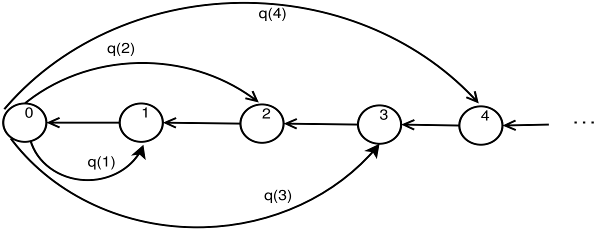

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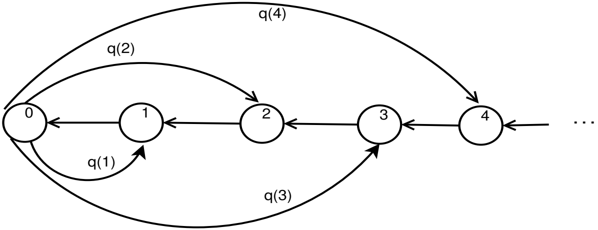

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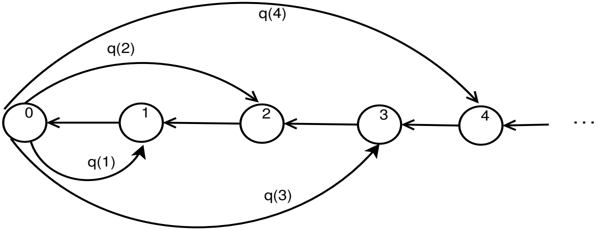

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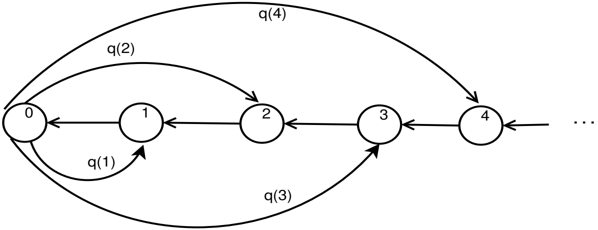

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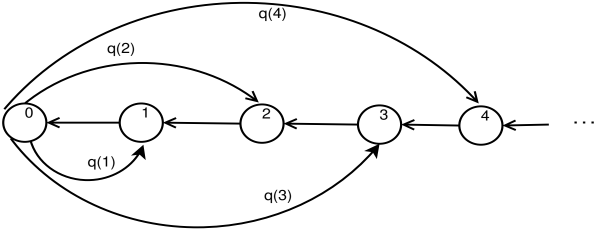

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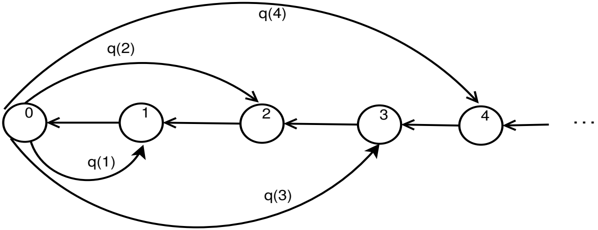

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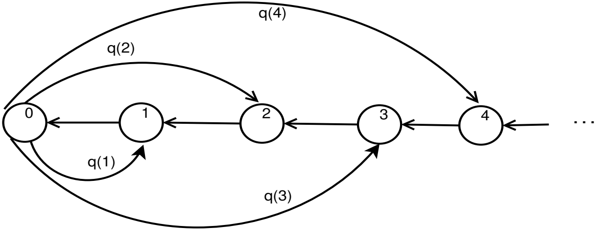

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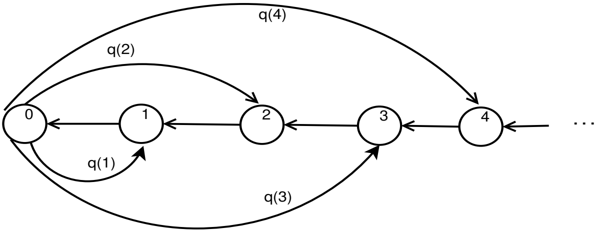

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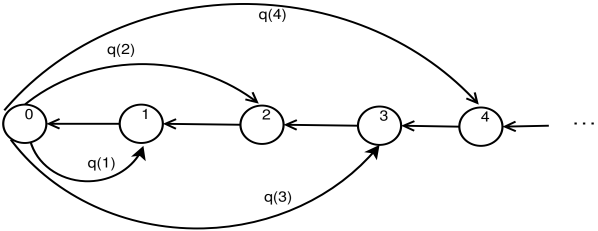
- 

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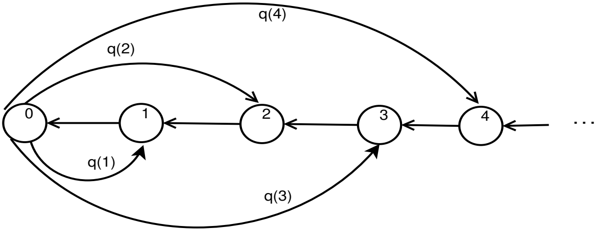
- 

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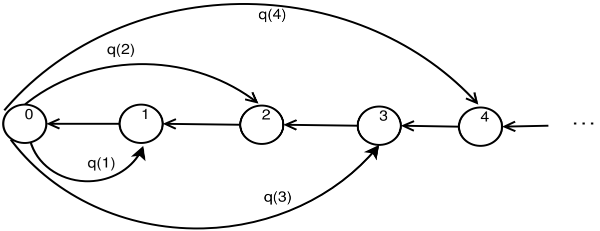
- 

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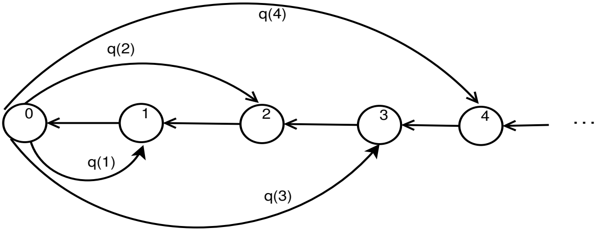
- 

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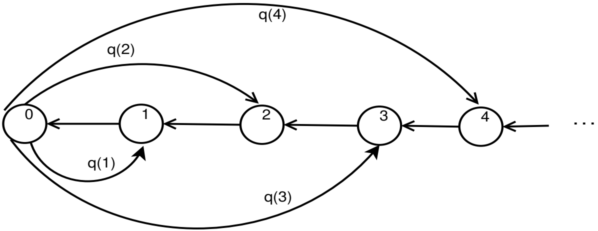
- 

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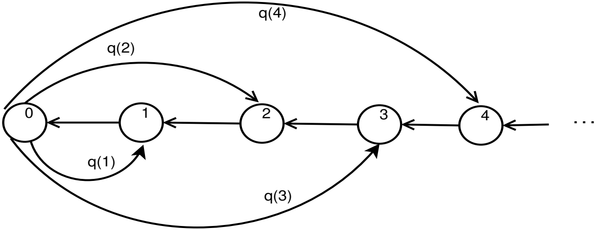
- 

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- The proof is complete if we can take the limit inside the sum

Bounded Convergence Theorem

- **Bounded Convergence Theorem**:
  1. Suppose $a(x) \geq 0, \ \forall x \in S$ and $\sum_x a(x) < \infty$.
  2. Let $b_n(x), \ x \in S$ be such that $|b_n(x)| \leq K, \ \forall x, n$ and
  $$\lim_{n \to \infty} b_n(x) = b(x), \forall x \in S.$$

  Then
  $$\lim_{n \to \infty} \sum_{x \in S} a(x) \ b_n(x) = \sum_{x \in S} a(x) \ \lim_{n \to \infty} b_n(x) = \sum_{x \in S} a(x) \ b(x)$$

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- Hence, if y is transient or null recurrent, then

$$\pi(y) = \sum_x \pi(x) \; \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m(x,y) = 0$$

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- The null recurrent chain we considered earlier is an example of a Markov chain that does not have a stationary distribution.

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- The theorem shows that an irreducible positive recurrent chain has a unique stationary distribution
- Corollary: An irreducible finite chain has a unique stationary distribution
- Corollary: An irreducible chain has a stationary distribution if and only if it is positive recurrent

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- This answers all questions about existence and uniqueness of stationary distributions

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  For example, $a_n = (-1)^n$

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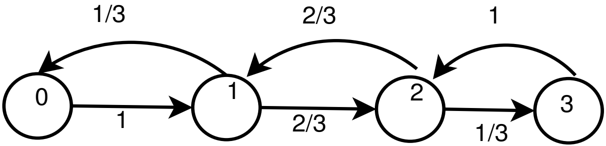
- However, P^n goes to different limits based on whether n is even or odd

- The chain is the folowing



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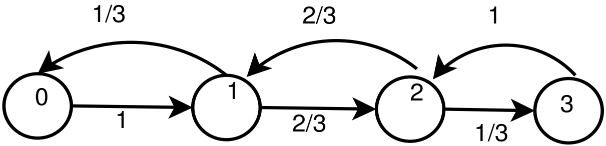

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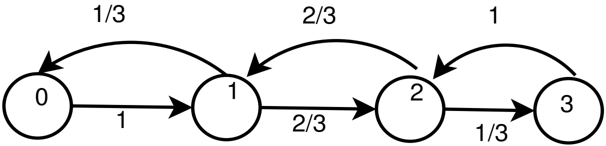

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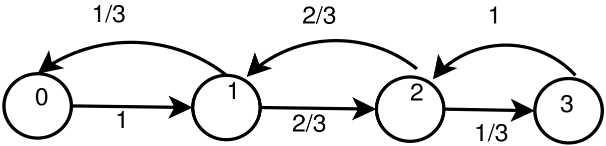

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- We can return to a state only after even number of time steps
- That is why P^n does not go to a limit
- Such a chain is called a periodic chain

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- If the period is 1 then chain is called aperiodic

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- An aperiodic, irreducible, positive recurrent chain is called an ergodic chain

Example

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- Consider the umbrella problem

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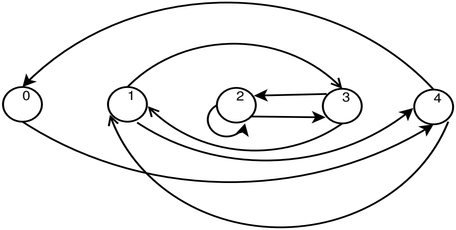

Example

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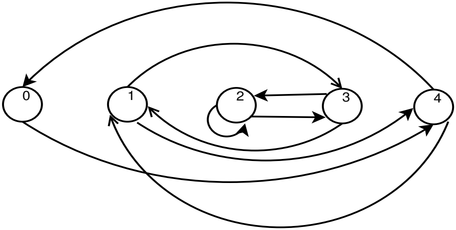
$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1-p & p \\ 2 & 0 & 0 & 1-p & p & 0 \\ 3 & 0 & 1-p & p & 0 & 0 \\ 4 & 1-p & p & 0 & 0 & 0 \end{array}$$



- This is an irreducible, aperiodic positive recurrent chain

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- We are using the fact that this chain converges to the stationary distribution starting with any initial probabilities.

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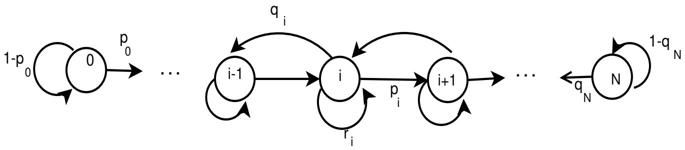
This gives $4\pi(1) + (1-p)\pi(1) = 1$ and hence

$$\pi(i) = \frac{1}{5-p} \; i = 1, 2, 3, 4 \quad \text{and} \quad \pi(0) = \frac{1-p}{5-p}$$

Birth-Death chains

- The following is a finite birth-death chain



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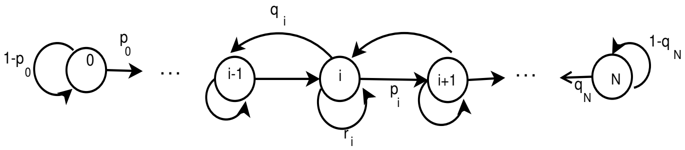

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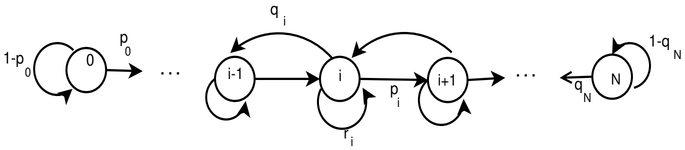

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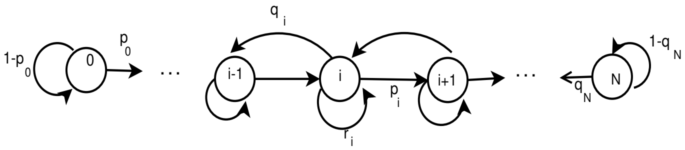

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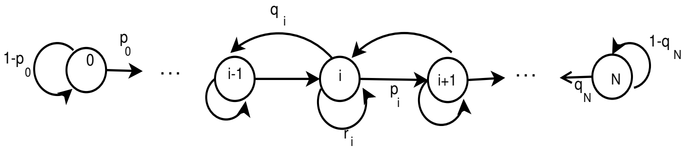

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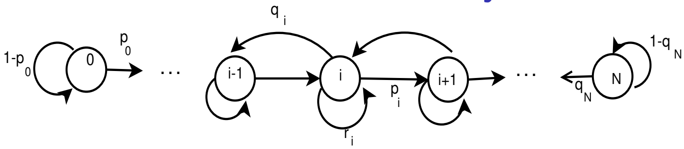
- Then the chain is irreducible, positive recurrent
- It is also aperiodic
- We can derive a general form for its stationary probabilities

birth-death chains – stationary distribution



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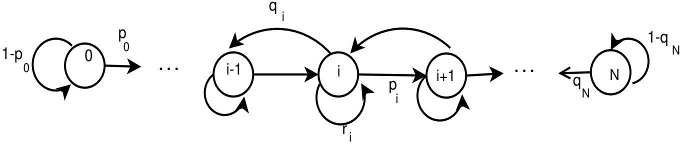

$$\pi(y) = \sum_x \pi(x) P(x, y)$$

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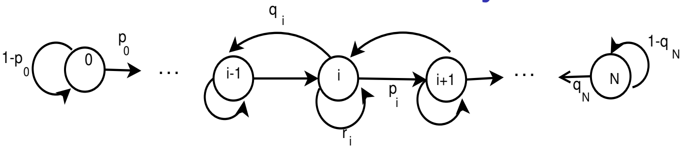

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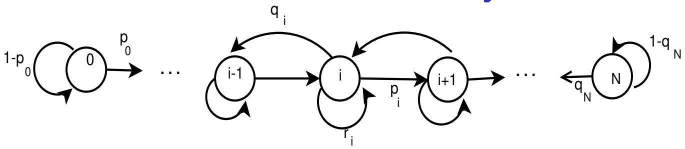

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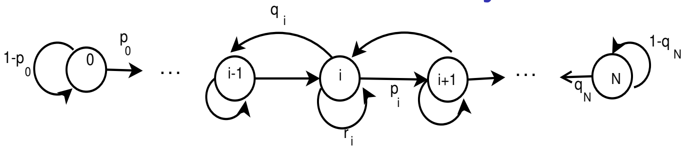

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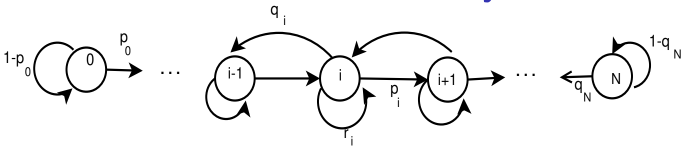

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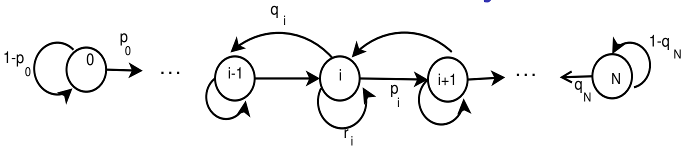

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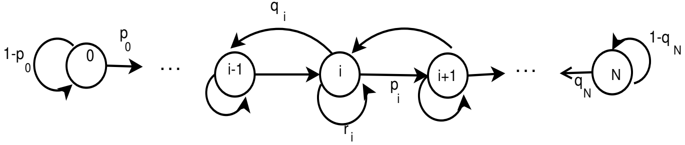

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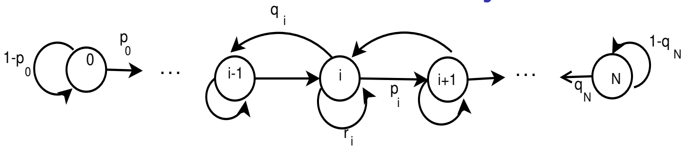

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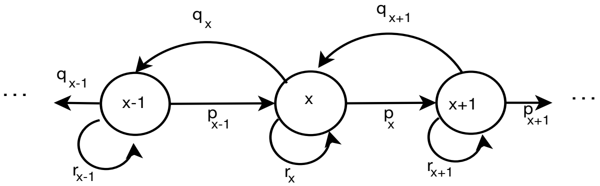
- For any x, the relevant part of the chain is



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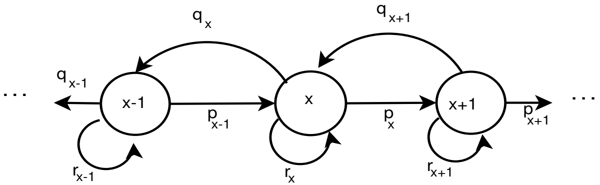

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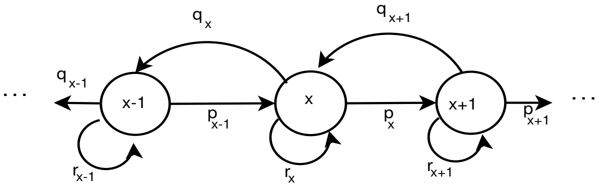

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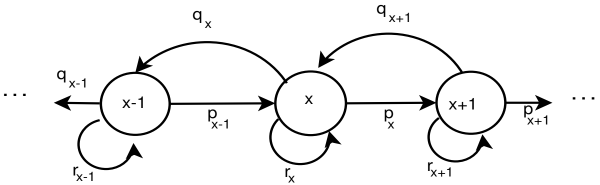

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- Iterating like this, we get

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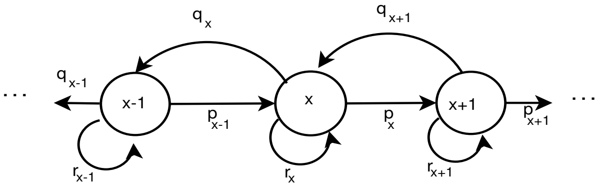
- Note that this process is applicable even for infinite chains with state space $\{0, 1, 2, \cdots\}$ (but there may not be a solution)

- Consider a birth-death chain



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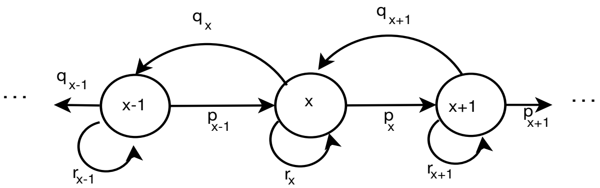

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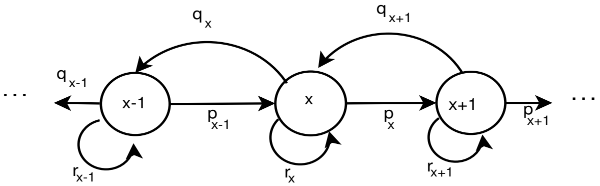

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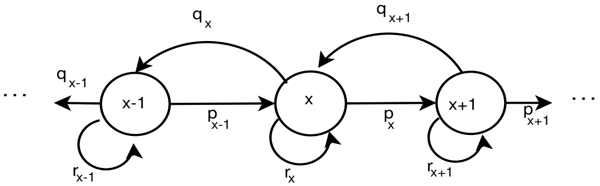
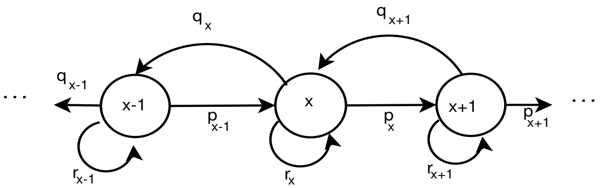


- The chain may be infinite or finite
- Let $a, b \in S$ with $a < b$. Assume $p_x, q_x > 0$, $a < x < b$.
- Define

$$U(x) = P_x[T_a < T_b], \; a < x < b, \quad U(a) = 1, \; U(b) = 0$$

- Consider a birth-death chain



- The chain may be infinite or finite
- Let $a, b \in S$ with $a < b$. Assume $p_x, q_x > 0$, $a < x < b$.
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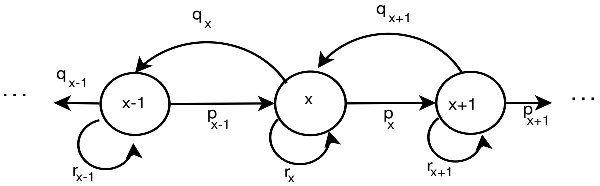

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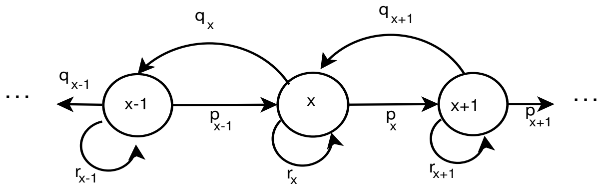
- We want to derive a formula for $U(x)$
- This can be useful, e.g., in the gambler's ruin chain


q x
q x+1
q x-1
. . .
x-1
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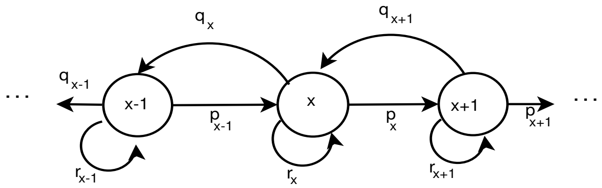
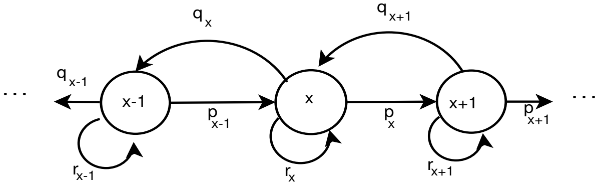
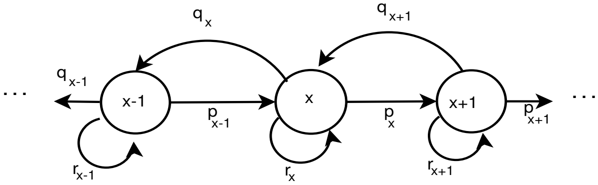
p x-1
p x
p x+1
r x-1
r x
r x+1



$$U(x) = P_x[T_a < T_b] = Pr[T_a < T_b | X_0 = x]$$



$$\begin{aligned} U(x) &= P_x[T_a < T_b] = Pr[T_a < T_b | X_0 = x] \\ &= \sum_{y=x-1}^{x+1} Pr[T_a < T_b | X_1 = y] \; Pr[X_1 = y | X_0 = x] \end{aligned}$$



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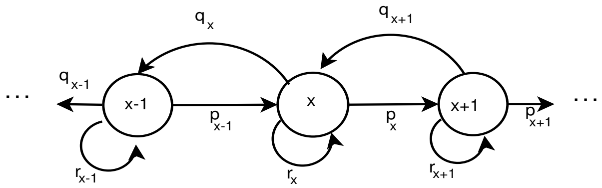
\end{aligned}
$$



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&= U(x-1)q_x + U(x)(1 - p_x - q_x) + U(x+1)p_x
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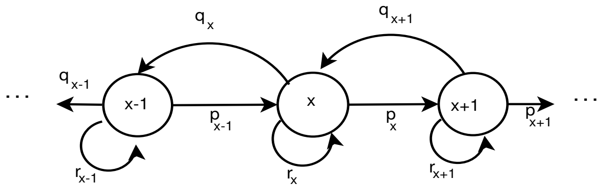

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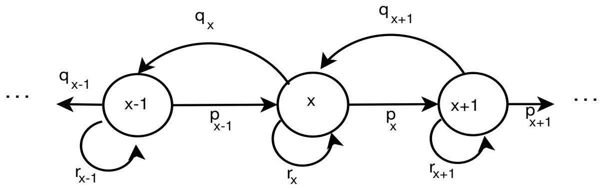
\Rightarrow \qquad q_x[U(x) - U(x-1)] = p_x[U(x+1) - U(x)]
$$



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$$
\begin{aligned}
\Rightarrow \quad & q_x[U(x) - U(x-1)] = p_x[U(x+1) - U(x)] \\
\Rightarrow \quad & U(x+1) - U(x) = \frac{q_x}{p_x} \left[U(x) - U(x-1)\right]
\end{aligned}
$$

$$U(x+1) - U(x) = \frac{q_x}{p_x}\left[U(x) - U(x-1)\right]$$

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U(x+1) - U(x) &= \frac{q_x}{p_x} \left[U(x) - U(x-1) \right] \\
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Let $\gamma_y = \dfrac{q_y q_{y-1}\cdots q_{a+1}}{p_y p_{y-1}\cdots p_{a+1}},\ a < y < b,\ \ \gamma_a = 1$

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Now we get

$$
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$$U(b) - U(b-1) \;=\; \frac{\gamma_{b-1}}{\gamma_a}\,[U(a+1) - U(a)]$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}\left[U(a+1) - U(a)\right]$$

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$$U(b) - U(a)$$

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- Adding all these we get

$$U(b) - U(a) = \frac{1}{\gamma_a}\left[U(a+1) - U(a)\right] \sum_{x=a}^{b-1} \gamma_x$$

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$$0 - 1 = U(b) - U(a) = \frac{1}{\gamma_a}\left[U(a+1) - U(a)\right] \sum_{x=a}^{b-1} \gamma_x$$

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$$\Rightarrow \quad U(a) - U(a+1) = \frac{\gamma_a}{\sum_{x=a}^{b-1}\gamma_x}$$

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$$
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$$
U(b-1) - U(b) = \frac{\gamma_{b-1}}{\sum_{x=a}^{b-1} \gamma_x}
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$$
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- Using these, we get

$$U(x) - U(x+1) = \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)]$$
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- Putting $x = b-1,\ b-2, \cdots, y$ in the above

$$U(b-1) - U(b) = \frac{\gamma_{b-1}}{\sum_{x=a}^{b-1} \gamma_x}$$
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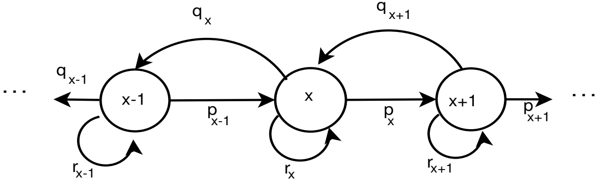
- Adding these we get

$$U(y) - U(b) = U(y) = \frac{\sum_{x=y}^{b-1} \gamma_x}{\sum_{x=a}^{b-1} \gamma_x},\ a < y < b$$

- We are considering birth-death chains



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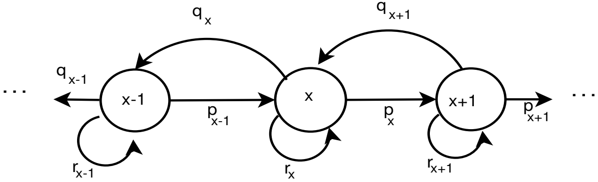

- We have derived, for $a < y < b$,

$$U(y) = P_y[T_a < T_b] = \frac{\sum_{x=y}^{b-1} \gamma_x}{\sum_{x=a}^{b-1} \gamma_x}, \quad \gamma_x = \frac{q_x q_{x-1} \cdots q_{a+1}}{p_x p_{x-1} \cdots p_{a+1}}$$

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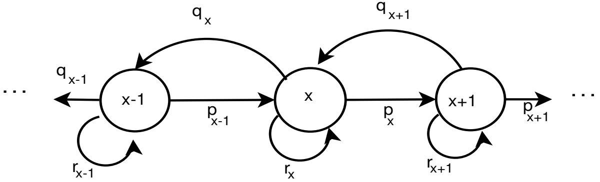

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$$U(y) = P_y[T_a < T_b] = \frac{\sum_{x=y}^{b-1} \gamma_x}{\sum_{x=a}^{b-1} \gamma_x}, \quad \gamma_x = \frac{q_x q_{x-1} \cdots q_{a+1}}{p_x p_{x-1} \cdots p_{a+1}}$$

- Hence we also get

$$P_y[T_b < T_a] = \frac{\sum_{x=a}^{y-1} \gamma_x}{\sum_{x=a}^{b-1} \gamma_x}$$

- Suppose this is a Gambler's ruin chain:
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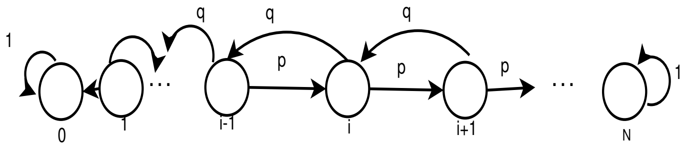

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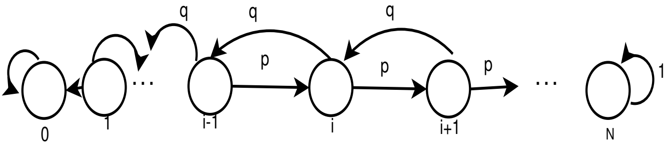

- Then, $\gamma_x = \left(\frac{q}{p}\right)^x$

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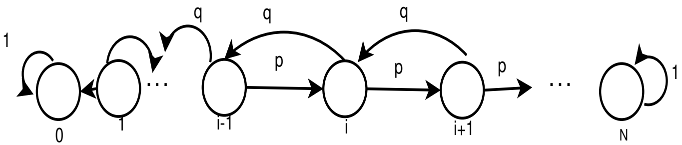

- Then, $\gamma_x = \left(\frac{q}{p}\right)^x$
- Hence, for a Gambler's ruin chain we get, e.g.,

$$P_i[T_N < T_0] = \frac{\sum_{x=0}^{i-1} \gamma_x}{\sum_{x=0}^{N-1} \gamma_x} = \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1}$$

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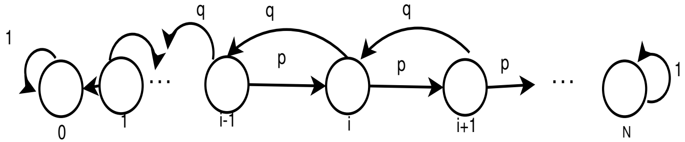

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$$P_i[T_N < T_0] = \frac{\sum_{x=0}^{i-1} \gamma_x}{\sum_{x=0}^{N-1} \gamma_x} = \frac{\left(\frac{q}{p}\right)^i - 1}{\left(\frac{q}{p}\right)^N - 1}$$

- This is the probability of gambler being successful