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- This result is true for all rv though we consider only discrete or continuous.

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- $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$
- $E\left[(X - c)^2\right] \geq E\left[(X - EX)^2\right], \ \forall c$

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- If moment of order k is finite then so is moment of order s for $s < k$.

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- We have

$$f_X(0) = P_X(0);\ f_X(1) = \frac{P'_X(0)}{1!};\ f_X(2) = \frac{P''_X(0)}{2!}\ \cdots$$

$$P'_X(1) = EX;\ P''_X(1) = E[X(X-1)\ \cdots$$

- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

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- Note that for a given p there can be multiple values for x to satisfy the above.

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- Let us see some examples.

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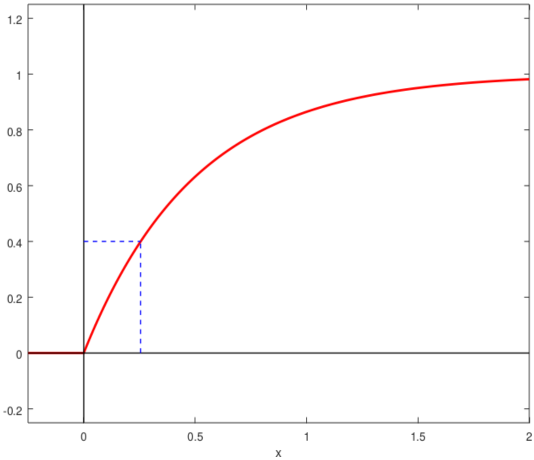

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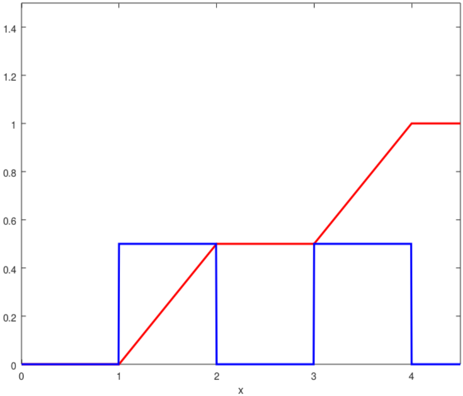

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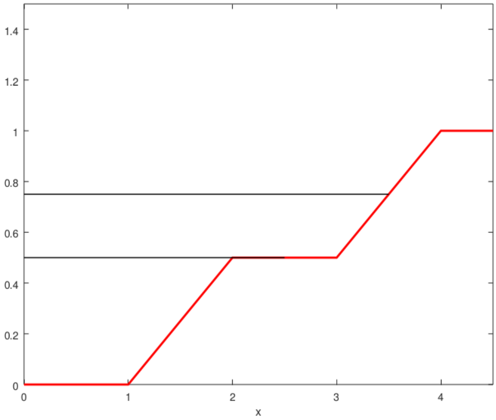

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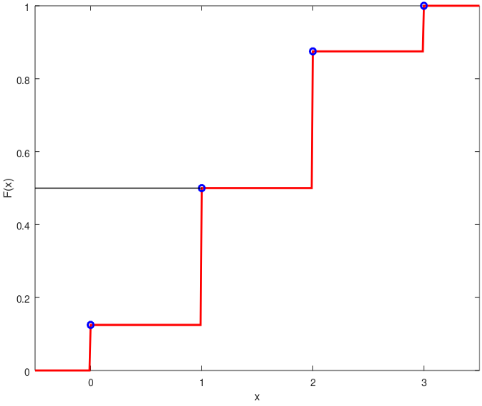
- So, quantile of order p is not unique and all such x qualify.

- This situation is illustrated below



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- Hence we have

$$p < p + \delta_2 = F_X(x_i) < p + \delta_2 + \delta_1 = p + P[X = x_i]$$

- Hence, x_i is quantile of order p.
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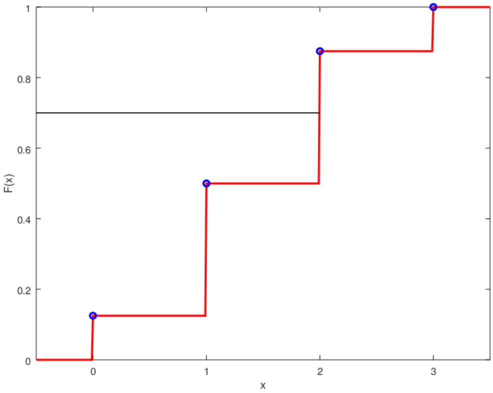
- Thus quantile of order p is unique here.

- This situation is illustrated below



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- One can show that $\int_{-\infty}^{0} f_X(x) \, dx = 0.5$ and hence the median is at the origin.

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  (Exercise: Show this for discrete and continuous rv)

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- For the Gaussian density, the mode, the median and the mean are all same.

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- These help us bound the probabilities of some important events in terms of the moments.

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- Markov inequality is often used in this form.

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- An example of what are called concentration inequalities.

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- This is true for all random variables and the RHS above does not depend on the distribution of X.

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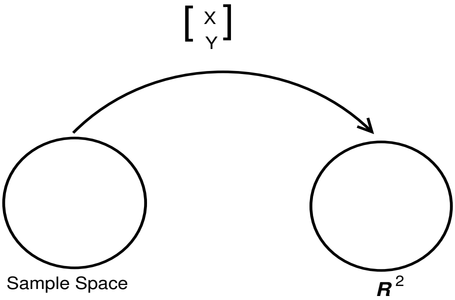

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where $\mathcal{B}$ is the Borel σ-algebra we considered earlier, and $\mathcal{B}^2$ is the set of Borel sets of $\Re^2$.

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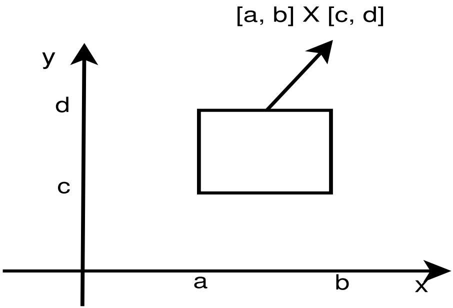

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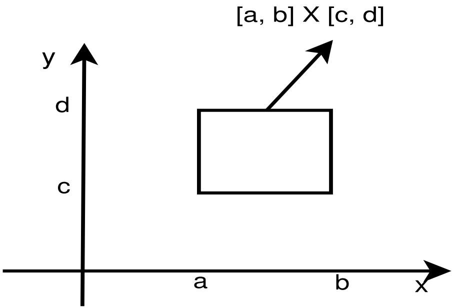

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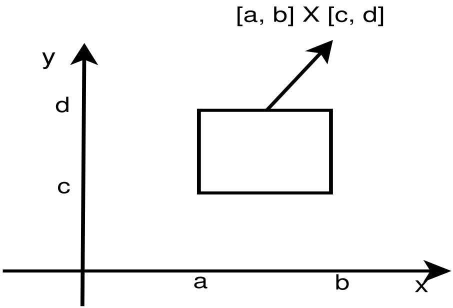

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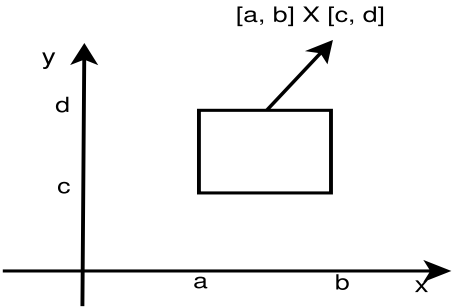

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- The joint distribution function is the probability of the intersection of the events $[X \leq x]$ and $[Y \leq y]$.

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- But there is another crucial property satisfied by F_{XY}.

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- Here, the probability we want is that of the pair of rv's being in a cylindrical set.

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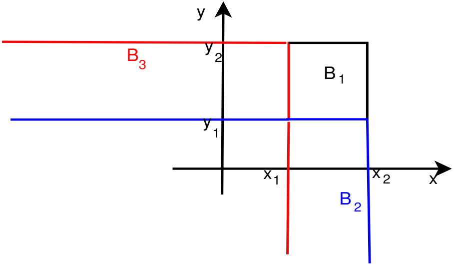

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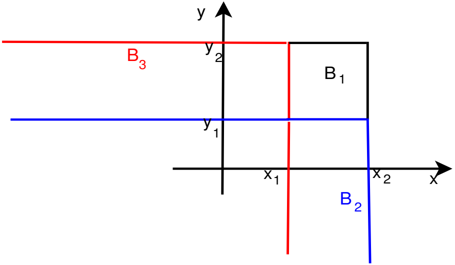

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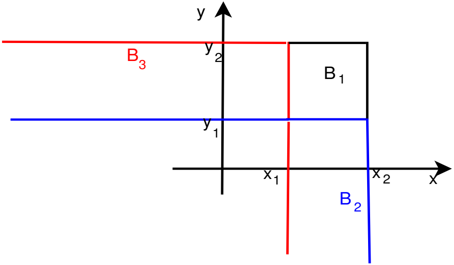

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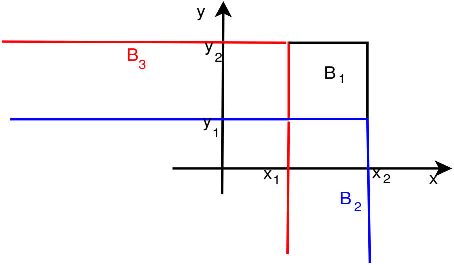

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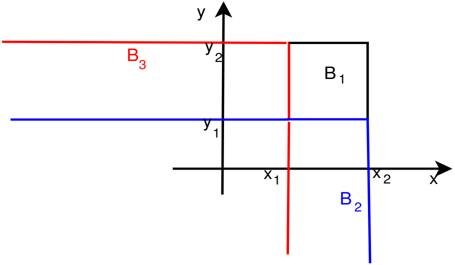

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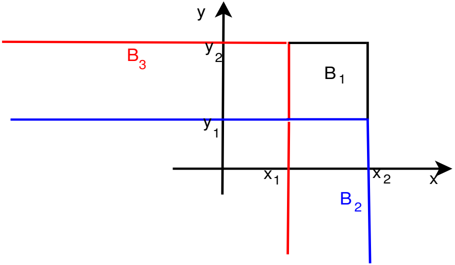

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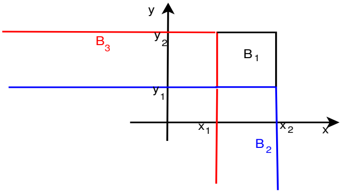
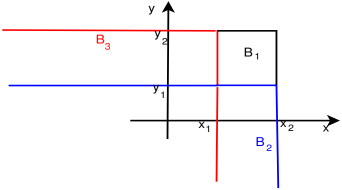

y
y_2
B_3
B_1
y_1
x_1
x_2
x
B_2



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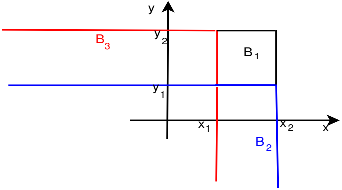

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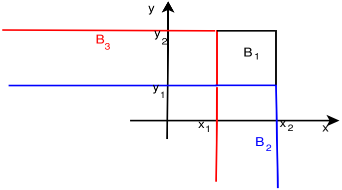

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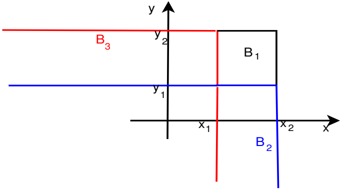

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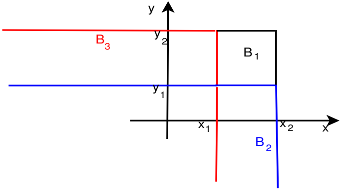

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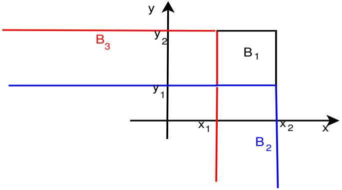

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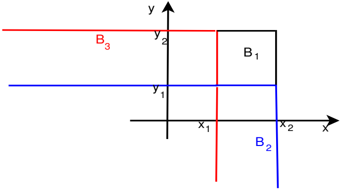

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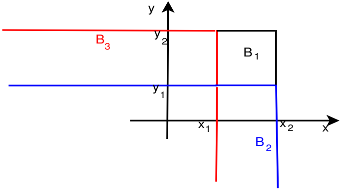

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P[(X, Y) \in B_3] &= P[X \le x_1, Y \le y_2] = F_{XY}(x_1, y_2) \\
P[(X, Y) \in B_2 \cap B_3] &= P[X \le x_1, Y \le y_1] = F_{XY}(x_1, y_1)
\end{aligned}
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\begin{aligned}
P[(X, Y) \in B_1] &= F_{XY}(x_2, y_2) - P[(X, Y) \in (B_2 \cup B_3)] \\
&= F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)
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- This is an additional condition that a function has to satisfy to be the joint distribution function of a pair of random variables

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- Joint distribution function: $F_{XY} : \Re^2 \rightarrow \Re$

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- This is a straight-forward extension of the pmf of a single discrete rv.

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- If $\omega = 0.2576$ then $X(\omega) = 2$ and $Y(\omega) = 5$
- Easy to see that $X, Y \in \{0, 1, \cdots, 9\}$.
- We want to calculate the joint pmf of X and Y

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- Hence the joint pmf of X and Y is

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  $[X = 3, Y = 6] = \{(3, \ 3)\}$,
  $[X = 4, Y = 6] = \{(4, \ 2), \ (2, \ 4)\}$
- So, $P[X = m, Y = n]$ is either $2/36$ or $1/36$ (assuming m, n satisfy other requirements)

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