Recap: Continuous-Time Markov Chains

- $\{X(t),\ t \geq 0\}$ is a continuous-time Markov chain if

  $$Pr[X(t+s) = j \mid X(s) = i,\ X(u) \in A_u,\ 0 \leq u < s] \\ = Pr[X(t+s) = j \mid X(s) = i]$$

  implies

  $$Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(s) = i, X(s'),\ 0 \leq s' < s < t] \\ = Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(s) = i]$$

- It is called homogeneous chain if

  $$Pr[X(t+s) = j \mid X(s) = i] = Pr[X(t) = j \mid X(0) = i],\ \forall s$$

Recap: Transition Structure

- Define

$$P_{ij}(t) = Pr[X(t) = j \mid X(0) = i] = Pr[X(t+s) = j \mid X(s) = i]$$

- The Chain can be described as follows.
- By Markov property and homogeneity, time spent in a state i is $T_i \sim$ exponential(ν_i)
- Then, when it leaves i, it transits to state j with probability, say, z_{ij} ($z_{ij} \geq 0$, $\sum_j z_{ij} = 1$, $z_{ii} = 0$)
- Note that $P_{ij}(t)$ is different from these z_{ij}

Recap: Birth-Death process

- From i the process can only go to $i+1$ or $i-1$
- When in i, a 'birth event' takes it to $i+1$ and a 'death event' takes it to $i-1$
- In state i, birth rate is λ_i
  (i.e., time till next birth event is $\exp(\lambda_i)$.
- In state i, death rate is μ_i
  (i.e., time till next death event is $\exp(\mu_i)$.
- For a birth-death process

$$z_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}; \quad z_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}; \quad \nu_i = \lambda_i + \mu_i$$

- Poisson process is a special case with $\lambda_i = \lambda$ and $\mu_i = 0$

Recap: Queuing system

- The state is number of people in the system
- People joining the queue is a Poisson process with rate λ
- The time to service each customer is independent and exponential with parmeter μ.
- We assume that the arrival and service processes are independent.
- Then this is a birth death process with

$$\lambda_n = \lambda,\ n \geq 0 \quad \text{and} \quad \mu_n = \mu,\ n \geq 1$$

- This is known as an $M/M/1$ queue
- A variation: $M/M/K$ queue

$$\lambda_n = \lambda,\ n \geq 0 \quad \text{and} \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq K \\ K\mu & n > K \end{cases}$$

Recap: Kolmogorov backward equation

- The $P_{ij}(t)$ satisfy the chapmann-Kolmogorov equations:

$$P_{ij}(t+s) = \sum_k P_{ik}(s) P_{kj}(t)$$

- For a finite chain, in Matrix notation:
  $P(t+s) = P(s)P(t)$
- Using this, we get Kolmogorov backward equation

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \left(\sum_{k \neq i} q_{ik} \right) P_{ij}(t), \ \forall i, j$$

  where

$$q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h}, \ \ i \neq j$$

- The q_{ij} – the infinitesimal generator of the chain.
- q_{ij} is the rate of transitions out of i into j
- For a birth-death chain, $q_{i,i+1} = \lambda_i$ and $q_{i,i-1} = \mu_i$

Recap: Obtaining $P_{ij}(t)$

- The Kolmogorov backward equation is

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} \; P_{kj}(t) - \left(\sum_{k \neq i} q_{ik} \right) \; P_{ij}(t)$$

- For a birth-death chain the equation becomes

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

- We can solve these to get $P_{ij}(t)$

Recap: 2-state birth-death chain

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

- For the two state chain, these equations are

$$\begin{aligned} P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\ P'_{01}(t) &= \lambda_0 P_{11}(t) - \lambda_0 P_{01}(t) \\ P'_{10}(t) &= \mu_1 P_{00}(t) - \mu_1 P_{10}(t) \\ P'_{11}(t) &= \mu_1 P_{01}(t) - \mu_1 P_{11}(t) \end{aligned}$$

- Solving these we can show

$$P_{00}(t) = \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda + \mu}$$

- We get a system of equations like this for any finite chain.

Recap: Finite chains

- The Chapmann-Kolmogorov equation gives

$$P(t+s) = P(t)\ P(s)$$

- Differentiating the above with respect to t and at $t = 0$,

$$P'(s) = Q\ P(s), \text{ where } Q = P'(0)$$

- The solution for the above ODE is

$$P(t) = e^{tQ}, \quad \text{because} \quad P(0) = I$$

- The Q matrix has elements q_{ij} defined earlier with $q_{ii} = -\sum_{k \neq i} q_{ik}$.

Recap: Kolmogorov Forward equation

- The Chapmann-Kolmogorov equation also gives us

$$P_{ij}(t+h) = \sum_k P_{ik}(t)\ P_{kj}(h)$$

- Similar algebra as earlier gives us

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t)\ q_{kj} - \left(\sum_{k \neq j} q_{jk} \right)\ P_{ij}(t)$$

- This is known as Kolmogorov forward equation

Recap: Transient and recurrent states

- We define

$$T_i = \min\{t > 0 \ : \ X(t) \neq i\} \quad f_i = \min\{t \ : \ t > T_i, \ X(t) = i\}$$

- A state i is said to be
  - Transient if $Pr[f_i < \infty \mid X(0) = i] < 1$
  - Recurrent if $Pr[f_i < \infty \mid X(0) = i] = 1$

Recap: Irreducibility, positive recurrence

- Most of the other definitions are also similar to the case of discrete chains
- The chain is irreducible if for all i, j there is a positive probability of going from i to j in some finite time: $P_{ij}(t) > 0$ for some t
- A recurrent state is positive recurrent if mean time to return is finite: $E[f_i \mid X(0) = i] < \infty$
  Otherwise it is null recurrent
- An irreducible positive recurrent chain would have a unique stationary distribution
- There is no concept of periodicity in the continuous time case
- An irreducible positive recurrent chain would be called an ergodic chain

Recap: Distribution of $X(t)$

- Define

$$\pi_j(t) = Pr[X(t) = j] = \sum_i \pi_i(0) P_{ij}(t)$$

- The above equation is true for general infinite chains.
- For a finite chain, taking π as a row vector,

$$\pi(t) = \pi(0)\ P(t) = \pi(0)\ e^{Qt}$$

Recap: Stationary distribution

- We say π is a stationary distribution if

$$\pi(0) = \pi \quad \Rightarrow \quad \pi(t) = \pi, \ \forall t$$

- Any stationary distribution π has to satisfy

$$\sum_{k \neq j} q_{kj} \pi_k = \pi_j \sum_{k \neq j} q_{jk}$$

- The above equation is known as a balance equation

- All the processes we considered so far are discrete-state processes.
- The random variables are discrete random variables.
- This is so in Markov chains (both discrete and continuous time) and hence in the Poisson process.
- We next consider processes where the random variables are continuous type.
- We consider the Brownian Motion process.
- We first start with a simpler process, namely, the random walk.

Random Walk

- Let Z_i be iid with $Pr[Z_i = +s] = Pr[Z_i = -s] = 0.5$
- Define (for some fixed T) a continuous-time process $X(t)$

$$\begin{aligned} X(nT) &= Z_1 + Z_2 + \cdots + Z_n \\ X(t) &= X(nT), \quad \text{for } \; nT \leq t < (n+1)T \end{aligned}$$

- Viewed as a discrete-time process, $X(nT)$, is a Markov chain.
- Called a (one dimensional) random walk
- It is the position after n random steps
- $X(t)$ is piece-wise constant interpolation of $X(nT)$
- We could have also use piece-wise linear interpolation

- We have $EZ_i = 0$ and $E[Z_i^2] = s^2$
- Hence, $E[X(nT)] = 0$ and $E[X^2(nT)] = ns^2$
- For large n, $\frac{X(nT)}{s\sqrt{n}}$ would be Gaussian

$$Pr\left[\frac{X(nT)}{s\sqrt{n}} \leq y\right] \approx \Phi(y)$$

  where Φ is distribution function of standard Normal
- For any t, $X(t)$ is $X(nT)$ for $n = [t/T]$.
  Large n would mean large t. Hence

$$Pr[X(t) \leq ms] = Pr\left[\frac{X(t)}{s\sqrt{n}} \leq \frac{ms}{s\sqrt{n}}\right] \approx \Phi\left(\frac{m}{\sqrt{n}}\right), \quad \text{for large } t$$

- We are interested in limit of this process as $T \to 0$

- Consider $t = nT$

$$E[X^2(t)] = ns^2 = s^2 \frac{t}{T}$$

- If we let $T \to 0$ then the variance goes to infinity (the process goes to infinity) unless we let s also to go to zero.
- We actuall need s^2 to go to zero at the same rate as T.
- So, we keep $s^2 = \alpha T$ and let T go to zero.
- Define

$$W(t) = \lim_{T \to 0, s^2 = \alpha T} X(t)$$

  This is called the Wiener Process or Brownian motion. This result is known as Donsker's theorem
- Let us intuitively see some properties of $W(t)$

- We have seen that for $n = [t/T]$,

$$Pr[X(t) \leq ms] \approx \Phi\left(\frac{m}{\sqrt{n}}\right)$$

- Let $w = ms$ and $t = nT$. Then

$$\frac{m}{\sqrt{n}} = \frac{w/s}{\sqrt{t/T}} = \frac{w}{\sqrt{t}} \sqrt{\frac{T}{s^2}} = \frac{w}{\sqrt{\alpha t}}$$

- $W(t)$ is limit of $X(t)$ as T goes to zero
- As T goes to zero, any t is 'large n'.
- Hence we can expect

$$Pr[W(t) \leq w] = \Phi\left(\frac{w}{\sqrt{\alpha t}}\right)$$

$$\Rightarrow \quad W(t) \sim \mathcal{N}(0,\ \alpha t)$$

- We had Z_i iid and defined

$$X(nT) = Z_1 + Z_2 + \cdots + Z_n$$

- Hence we get

$$X((m+n)T) - X(nT) = Z_{n+1} + \cdots + Z_{n+m}$$

  Thus, $X(nT)$ is independent of $X((m+n)T) - X(nT)$.
- Hence the $X(nT)$ process has independent increments
- Hence, we can expect $W(t)$ to be a process with independent increments

- We have

$$
\begin{aligned}
X((m+n+k)T) - X((n+k)T) &= Z_{n+k+1} + \cdots + Z_{n+k+m} \\
X((m+n)T) - X(nT) &= Z_{n+1} + \cdots Z_{n+m}
\end{aligned}
$$

  Both are sums of m of the iid Z_i's
- Hence both would have the same distribution
- Thus $X(nT)$ would also have stationary increments.
- Hence we expect $W(t)$ to have stationary increments
- Thus, $W(t)$ should be a process with stationary and independent increments and for each t, $W(t)$ is Gaussian with zero mean and variance proportional to t
- We will now formally define Brownian motion using these properties.

- Let $\{X(t),\ t \geq 0\}$ be a continuous-state continuous-time process

  This process is called a Brownian motion if

  1. $X(0) = 0$
  2. The process has stationary and independent increments
  3. For every $t > 0$, $X(t)$ is Gaussian with mean 0 and variance $\sigma^2 t$
- Let $B(t) = \frac{X(t)}{\sigma}$. Then, variance of $B(t)$ is t
- $\{B(t),\ t \geq 0\}$ is called standard Brownian Motion
- Let $Y(t) = X(t) + \mu$. Then $Y(t)$ has non-zero mean
- $\{Y(t),\ t \geq 0\}$ is called Brownian motion with a drift
- The mean can be a function of time