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- As we shall see, Poisson process is a special case of continuous time Markov chain

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$$Pr[X(t{+}s) = j \mid X(s) = i] = Pr[X(t) = j \mid X(0) = i],\ \forall s$$

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- Analogous to transition probabilities in the discrete case
- Like in the discrete case, we can show that the Markov condition implies

$$\begin{aligned} &Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(s) = i, X(s'),\ 0 \leq s' < s < t] \\ &\quad = Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(s) = i] \end{aligned}$$

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- Let us look at the distribution of time spent in a state before leaving it

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$\Rightarrow$ T_i is memoryless and hence exponential

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- Note that $P_{ij}(t)$ is different from these z_{ij}

Example: Birth-Death process

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- Now, these λ_n and μ_n completely determine ν_n and z_{ij} and hence completely specify the chain
- Before we calculate these, we need to recall some properties of exponential random variables

- Let $X \sim \exp(\lambda_1)$ and $Y \sim \exp(\lambda_2)$ and let X, Y be independent.

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We can also get this using

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Thus, $\min(X, Y) \sim \exp(\lambda_1 + \lambda_2)$.

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- Hence, $\mu_0 = 0$ and $\nu_0 = \lambda_0$ and $z_{01} = 1$

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$$\lambda_n = \lambda, \; n \geq 0 \quad \text{and} \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq K \\ K\mu & n > K \end{cases}$$

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$$E[Y_i] + E[Y_{i+1}] + \cdots + E[Y_{j-1}]$$

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- Note that all these are only for birth-death processes

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- We need a technical assumption.
- We assume that the transition probability function is such that the probability of infinite number of transitions in a finite interval of time is zero. (Called the regularity condition).

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(For an infinite chain there is the issue of whether we can take the limit inside the summation)

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by taking $q_{ii} = -\sum_{k \neq i} q_{ik}$

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- Thus, $P_{ij}(t)$ for all i, j and all t are completely determined by q_{ij}.

- We showed that the transition probability function satisfies

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- If we know all the q_{ij}, then we can solve these differential equations and obtain the transition probability function.
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- These are referred to as the infinitesimal generator of the chain.

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- Hence we can calculate $P_{ij}(t)$ for any t and i, j

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- Now one can see that the above ODE is what we got for Poisson process.

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- We may want to calculate $P_{00}(T)$, the probability that the machine would be working at a time T units later given it is in working condition now
- We can calculate it by solving the ODE's

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

- For the two state chain, these equations are

$$\begin{aligned}
P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\
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- As is easy to see, we get a system of equations like this for any finite chain.

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

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$$\begin{aligned} P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\ P'_{01}(t) &= \lambda_0 P_{11}(t) - \lambda_0 P_{01}(t) \\ P'_{10}(t) &= \mu_1 P_{00}(t) - \mu_1 P_{10}(t) \\ P'_{11}(t) &= \mu_1 P_{01}(t) - \mu_1 P_{11}(t) \end{aligned}$$

- As is easy to see, we get a system of equations like this for any finite chain.
- Solving these we can show

$$P_{00}(t) = \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda + \mu}$$

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- This is called detailed balance.