Recap: Monotone Sequences of Sets

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Recap: Monotone Sequential Continuity

- We showed that

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

when $A_n \downarrow$ or $A_n \uparrow$

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- The random variables provide a convenient language to describe different probability models.

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- For now we will assume that any set of $\Re$ that we want would be in $\mathcal{B}$ and hence is an event.

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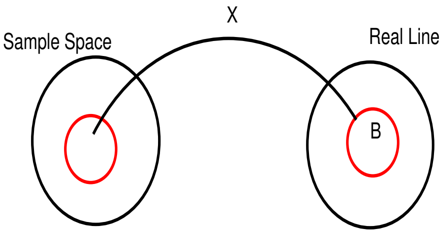

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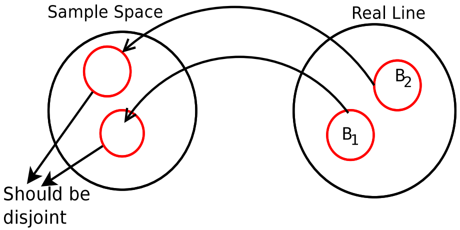

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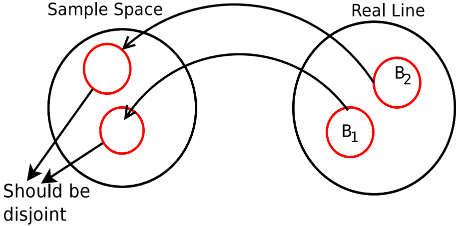

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- Hence

$$P_X((0, 1]) = \frac{3}{8};\;\; P_X((-1.2, 2.78)) = \frac{7}{8}$$

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- We briefly consider this and then move on to studying random variables.

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- Suppose $\Omega = \Re$. If we want every subset of real line to be an event, we cannot construct a probability measure (to satisfy the axioms).

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  (Recall that for countable Ω we can take $\mathcal{F} = 2^{\Omega}$).
- Now the question is what is the best $\mathcal{B}$ we can have?

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- It contains all intervals, all complements, countable unions and intersections of intervals and all sets that can be obtained through complements, countable unions and/or intersections of such sets and so on.

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  6. $[x,\ y), (x,\ y) \in \mathcal{B}$, similarly
- Thus, $\sigma(G)$ is also the smallest σ-algebra containing all intervals.

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- YES!! Infinitely many non-Borel sets exist!

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- We always assume this.

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- How does one represent this probability measure

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- The distribution function, F_X completely specifies the probability measure, P_X.
  (The proof of this important theorem is beyond the scope of this course)

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- Let us look at a simple example.

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Please note that x is a 'dummy variable'

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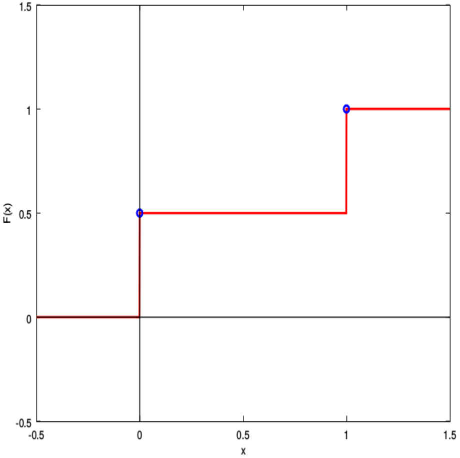
- A plot of this distribution function:



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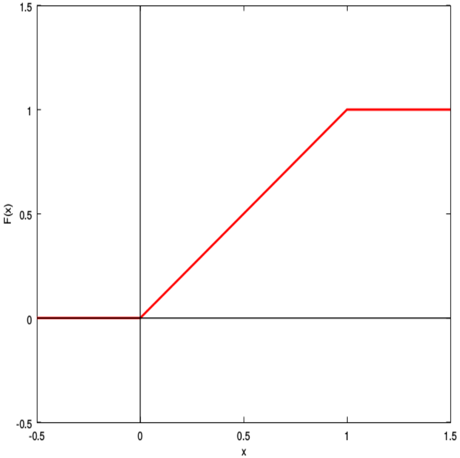
- The plot of this distribution function:



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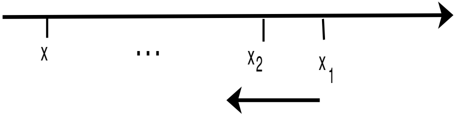

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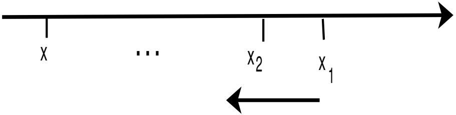

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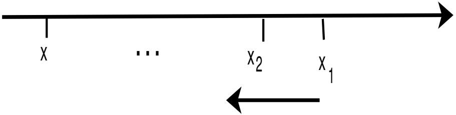

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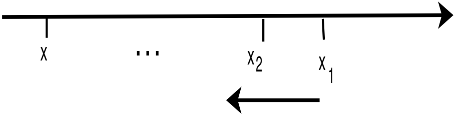

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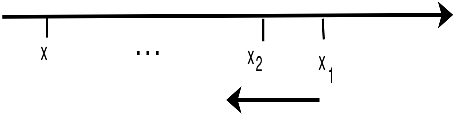

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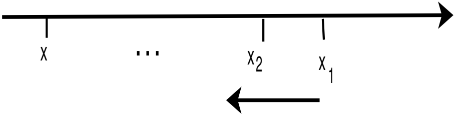

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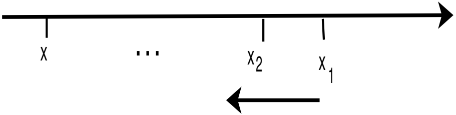

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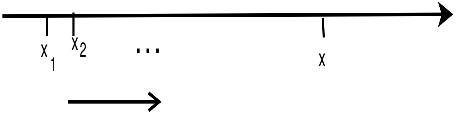

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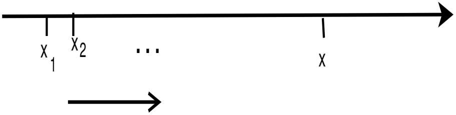

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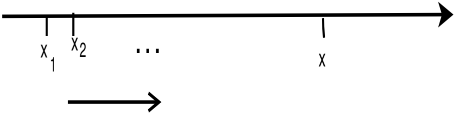

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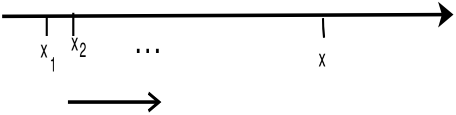

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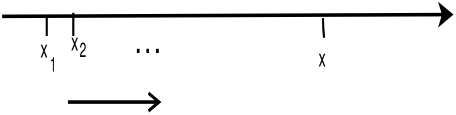

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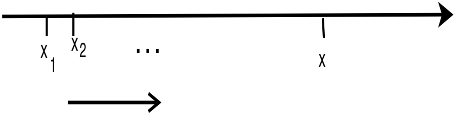

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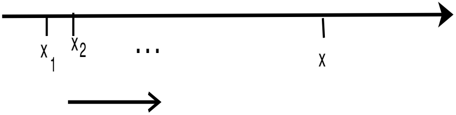

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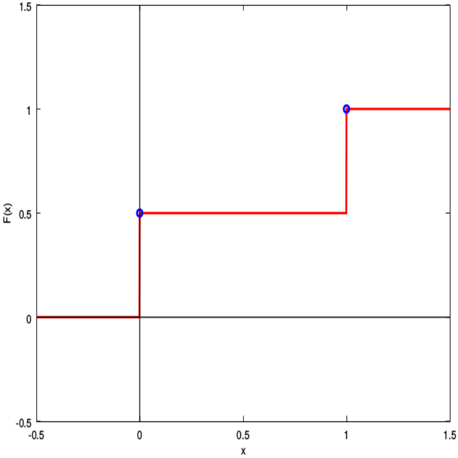
1.5
1
0.5
0
-0.5
F(x)
-0.5
0
0.5
1
1.5
x

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- Any real-valued function of a real variable satisfying the above four properties would be a distribution function of some random variable.

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- Note that the distribution function is defined for **all** random variables.