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- We saw many examples.

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- Then the event $[N \leq n-1]$ and hence its complement $[N \geq n]$ is independent of X_n and hence $A2$ holds.

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  If $E[X_i]$ is same for all i, $ES_N = EX_1 \, EN$.

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- IF X, Y are *iid* then $P[X < Y] = 0.5$

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- So, we get: $P[X=k] = \frac{1}{n+1},\ k=0,1,\cdots,n$

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- The Gaussian or normal density is given by

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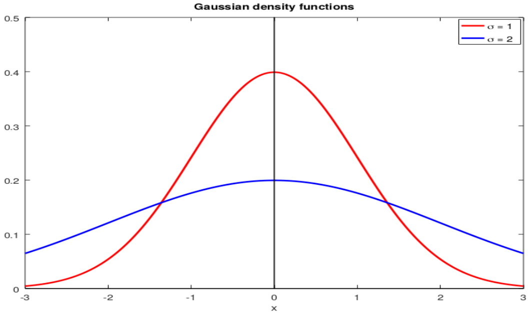

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- We will now show that this is a joint density function.

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- Define $\mathbf{Z} = (Z_1, \cdots, Z_n)^T = L^T \mathbf{Y}$. Then $\mathbf{Y} = L\mathbf{Z}$.
- Recall $|L| = 1$, $|M^{-1}| = (m_1 \cdots m_n)^{-1}$
- Then density of $\mathbf{Z}$ is

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) &= \frac{1}{(2\pi)^{\frac{n}{2}} \, |M^{-1}|^{\frac{1}{2}}} \, e^{-\frac{1}{2}\mathbf{z}^T L^T M L \mathbf{z}} = \frac{1}{(2\pi)^{\frac{n}{2}} \left(\frac{1}{m_1 \cdots m_n}\right)^{\frac{1}{2}}} e^{-\frac{1}{2}\sum_i m_i z_i^2} \\ &= \prod_{i=1}^{n} \sqrt{\frac{1}{2\pi}} \, \frac{1}{\sqrt{\frac{1}{m_i}}} \, e^{-\frac{1}{2} m_i z_i^2} \end{aligned}$$

- Consider $\mathbf{Y}$ with joint density

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This shows that $Z_i \sim \mathcal{N}(0, \frac{1}{m_i})$ and Z_i are independent.

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- Thus, if $\mathbf{Y}$ has density

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Multi-dimensional Gaussian density

- $\mathbf{X} = (X_1, \cdots, X_n)^T$ are said to be jointly Gaussian if

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- This implies X_i are independent.
- If $X_1, \cdots, X_n$ are jointly Gaussian then uncorrelatedness implies independence.

- Let $\mathbf{X} = (X_1, \cdots, X_n)^T$ be jointly Gaussian:

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- Then we saw that $Z_i \sim \mathcal{N}(0, \frac{1}{m_i})$ and Z_i are independent.
- If $X_1, \cdots, X_n$ are jointly Gaussian then there is a 'linear' transform that transforms them into independent random variables.

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- Let $\mathbf{X} = (X_1, \cdots, X_n)^T$ be jointly Gaussian
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- This is the moment generating function of multi-dimensional Normal density

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- Exercise for you – show all this starting with the joint density we have

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- Important to note that 'individually Gaussian' does not mean 'jointly Gaussian'

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- We will prove this using moment generating functions