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- It essentially results in an induced probability space

$$(\Omega, \mathcal{F}, P) \stackrel{X}{\rightarrow} (\Re, \mathcal{B}, P_X)$$

where $\mathcal{B}$ is the Borel σ-algebra and

$$P_X(B) = P[X \in B] = P\left(\{\omega \in \Omega \ : \ X(\omega) \in B\}\right)$$

Recap: Distribution function of a random variable

- Let X be a random variable. It distribution function, $F_X : \Re \to \Re$, is defined by

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- The distribution function, F_X, completely specifies the probability measure, P_X.

Recap: Properties of distribution function

- The distribution function satisfies
  1. $0 \leq F_X(x) \leq 1, \; \forall x$
  2. $F_X(-\infty) = 0; \; F_X(\infty) = 1$
  3. F_X is non-decreasing: $x_1 \leq x_2 \; \Rightarrow \; F_X(x_1) \leq F_X(x_2)$
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- Let $X \in \{x_1, x_2, \cdots\}$
- Its distribution function, F_X is a stair-case function with jump discontinuities at each x_i and the magnitude of the jump at x_i is equal to $P[X = x_i]$

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- We can calculate the probability of any event as

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Recap: continuous random variable

- X is said to be a continuous random variable if there exists a function $f_X : \Re \rightarrow \Re$ satisfying

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- A continuous rv takes uncountably many distinct values. However, not every rv that takes uncountably many values is a continuous rv

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$$P[X \in B] = \int_B f_X(t)\, dt, \quad \forall B \in \mathcal{B}$$
- In particular,
$$P[a \leq X \leq b] = \int_a^b f_X(t)\, dt$$

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- We can determine distribution of Y given the function g and the distribution of X

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- Then Y is also discrete and $Y \in \{g(x_1), g(x_2), \cdots\}$.
- We can find the pmf of Y as

$$\begin{aligned} f_Y(y) &= p[Y = y] = P[g(X) = y] \\ &= P[X \in \{x_i \ : \ g(x_i) = y\}] \\ &= \sum_{\substack{i: \\ g(x_i)=y}} f_X(x_i) \end{aligned}$$

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- Then Y is a continuous rv with pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \; a \leq y \leq b$$

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- This theorem is useful in some cases to find the densities of functions of continuous random variables

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- Note that expectation is defined for all random variables

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- $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$
- $E\left[(X - c)^2\right] \geq E\left[(X - EX)^2\right], \; \forall c$

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  - $\text{Var}(cX) = c^2\text{Var}(X)$

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- If moment of order k is finite then so is moment of order s for $s < k$.

Recap: Moment Generating function

- The moment generating function – $M_X : \Re \to \Re$

$$M_X(t) = Ee^{tX} = \sum_i e^{tx_i} f_X(x_i) \;\text{ or }\; \int e^{tx} f_X(x)\, dx, \quad t \in \Re$$

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- We say the mgf exists if $E[e^{tX}] < \infty$ for t in some interval around zero
- If $M_X(t)$ exists (for $t \in [-a, \ a]$ for some $a > 0$) then all its derivatives also exist and

$$\left. \frac{d^k M_X(t)}{dt^k} \right|_{t=0} = E[X^k]$$

Generating function

- For $X \in \{0, 1, 2, \cdots\}$ the (probability) generating function of X is defined by

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- We get the pmf from it as

$$f_X(0) = P_X(0); \; f_X(1) = \frac{P'_X(0)}{1!}; \; f_X(2) = \frac{P''_X(0)}{2!}$$

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- We can also get the moments:

$$P_X'(1) = EX, \quad P_X''(1) = E[X(X-1)]$$

quantiles of a distribution

- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

$$P[X \leq x] \geq p \quad \text{and} \quad p[X \geq x] \geq 1 - p$$

is called the quantile of order p or the $100p^{th}$ percentile of rv X.

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- For a given p there can be multiple values for x to satisfy the above.

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- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

$$P[X \leq x] \geq p \quad \text{and} \quad p[X \geq x] \geq 1 - p$$

  is called the quantile of order p or the $100p^{th}$ percentile of rv X.
- If x is quantile of order p, it satisfies

$$p \leq F_X(x) \leq p + P[X = x]$$

- For a given p there can be multiple values for x to satisfy the above.
- For $p = 0.5$, it is called the median.

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$$P[|X - EX| > c] \leq \frac{\mathsf{Var}(X)}{c^2}$$

- With $EX = \mu$ and $\mathsf{Var}(X) = \sigma^2$, we get

$$P[|X - \mu| > k\sigma] \leq \frac{1}{k^2}$$

Recap: A pair of random variables

- Let X, Y be random variables on the probability space $(\Omega, \mathcal{F}, P)$

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\begin{aligned}
P_{XY}(B) &= P[(X,Y) \in B], \ \forall B \in \mathcal{B}^2 \\
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- Given the joint distribution function, probability of any event involving the pair of random variables can be (in principle) calculated.

Recap: Properties of Joint Distribution Function

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  4. For all $x_1 < x_2$ and $y_1 < y_2$

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- Any $F : \Re^2 \to \Re$ satisfying the above would be a joint distribution function.

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- Let $X \in \{x_1, x_2, \cdots\}$ and $Y \in \{y_1, y_2, \cdots\}$ be discrete random variables.

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- Given the joint pmf, we can get the joint df as

$$F_{XY}(x, y) = \sum_{\substack{i: \\ x_i \leq x}} \sum_{\substack{j: \\ y_j \leq y}} f_{XY}(x_i, y_j)$$

Recap: Joint pmf

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- This is because, if we define

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- These are very similar to the properties of the density of a single rv

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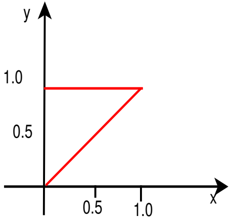
- We can say this density is uniform over the region



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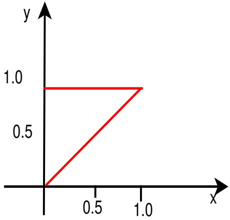

The figure is not a plot of the density function!!

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- Given f_{XY} satisfying the above, define

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- The only property left is the special property of F_{XY} we mentioned earlier.

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&= \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}\, dy\, dx \geq 0
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- This actually shows

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- In general

$$P[(X, Y) \in B] = \int_B f_{XY}(x, y) \, dx \, dy, \ \ \forall B \in \mathcal{B}^2$$

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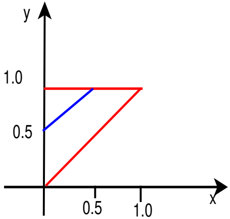
$$= 2\left.\frac{y^2}{2}\right|_{0.5}^{1} - y|_{0.5}^{1} = 1 - 0.25 - 1 + 0.5 = 0.25$$

- We can look at it geometrically



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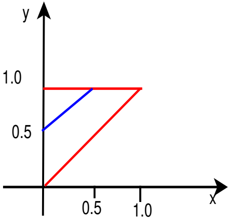

- The probability of the event we want is the area of the small triangle divided by that of the big triangle.

Marginal Distributions

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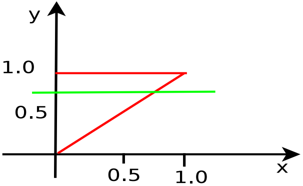

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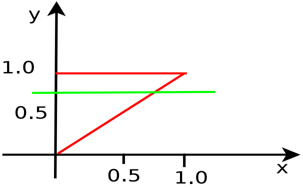

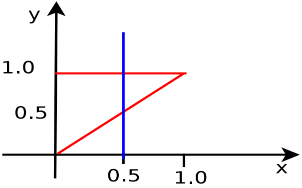

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- We can specify the marginal density of one and the conditional density of the other given the first.
- This may actually be the model of how the the rv's are generated.

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- We can verify it to be a density

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- While often that gives the right result, one needs to be very careful

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