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- We can view the process as $X : \Omega \times T \to \Re$
  A collection of time functions.

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- The distributions can be specified through joint mass or density functions too.

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- One example is the Markovian dependence
- Other examples: process with independent increments, Gaussian processes

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- This stringent condition is often referred to as strict-sense stationarity

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- For a wide-sense stationary process the autocorrelation is a symmetric functions and its Fourier transform is the power spectral density

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- One sufficient condition is that covariance between $X(t)$ and $X(t+\tau)$ decreases fast with increasing τ.

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- When the above holds we say the process is asymptotically uncorrelated.

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- Thus the sum can be rewritten as

$$\text{Var}(S_n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} C_X(i-j) = \sum_{k=-(n-1)}^{n-1} (n - |k|) C_X(k)$$

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- This shows that $\frac{S_n}{n}$ converges in probability to $\eta = E[X(n)]$.

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- The index set is the interval $[0, \infty)$ and all random variables are discrete and take non-negative integer values.

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- In particular, for all $s > t > 0$, $N(s) - N(t)$ is independent of $N(t) - N(0)$
- The process is said to have stationary increments if $N(t_2) - N(t_1)$ has the same distribution as $N(t_2 + \tau) - N(t_1 + \tau), \forall \tau, \forall t_2 > t_1$

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- Since the process has stationary increments and $N(0) = 0$, $(N(t+s) - N(s))$ would be Poisson with parameter λt for all $s, t > 0$.

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- We will show that both definitions are equivalent

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- For this we need to calculate distribution of $N(t)$

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$$\begin{aligned} P_0(t+h) &= Pr[N(t+h) = 0] \\ &= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \end{aligned}$$

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$$\begin{aligned} P_0(t+h) &= Pr[N(t+h) = 0] \\ &= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \\ &= Pr[N(t) = 0]\, Pr[N(t+h) - N(t) = 0] \end{aligned}$$

(because of independent increments)

- We first show Definition 2 $\Rightarrow$ Definition 1
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$$
\begin{aligned}
P_0(t+h) &= Pr[N(t+h) = 0] \\
&= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \\
&= Pr[N(t) = 0]\ Pr[N(t+h) - N(t) = 0] \\
&\qquad\qquad \text{(because of independent increments)} \\
&= Pr[N(t) = 0]\ Pr[N(h) = 0] \quad \text{(stationary increments)}
\end{aligned}
$$

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&= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \\
&= Pr[N(t) = 0]\ Pr[N(t+h) - N(t) = 0] \\
&\qquad \text{(because of independent increments)} \\
&= Pr[N(t) = 0]\ Pr[N(h) = 0] \quad \text{(stationary increments)} \\
&= P_0(t)(1 - \lambda h + o(h))
\end{aligned}
$$

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&= Pr[N(t) = 0]\ Pr[N(h) = 0] \quad \text{(stationary increments)} \\
&= P_0(t)(1 - \lambda h + o(h))
\end{aligned}
$$

$$
\Rightarrow \quad \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h}
$$

- We first show Definition 2 $\Rightarrow$ Definition 1
- For this we need to calculate distribution of $N(t)$
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$$\begin{aligned} P_0(t+h) &= Pr[N(t+h) = 0] \\ &= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \\ &= Pr[N(t) = 0]\ Pr[N(t+h) - N(t) = 0] \\ &\qquad\qquad \text{(because of independent increments)} \\ &= Pr[N(t) = 0]\ Pr[N(h) = 0] \quad \text{(stationary increments)} \\ &= P_0(t)(1 - \lambda h + o(h)) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \frac{P_0(t+h) - P_0(t)}{h} &= -\lambda P_0(t) + \frac{o(h)}{h} \\ \Rightarrow \quad \frac{d}{dt} P_0(t) &= -\lambda P_0(t) \end{aligned}$$

- Now we can solve this differential equation to get $P_0(t)$

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$$\begin{aligned} \frac{d}{dt}P_0(t) &= -\lambda P_0(t) \\ \Rightarrow \quad \frac{1}{P_0(t)}\frac{d}{dt}P_0(t) &= -\lambda \end{aligned}$$

- Now we can solve this differential equation to get $P_0(t)$

$$\begin{aligned} \frac{d}{dt} P_0(t) &= -\lambda P_0(t) \\ \Rightarrow \quad \frac{1}{P_0(t)} \frac{d}{dt} P_0(t) &= -\lambda \\ \Rightarrow \quad \ln(P_0(t)) &= -\lambda t + c \end{aligned}$$

- Now we can solve this differential equation to get $P_0(t)$

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\frac{d}{dt}P_0(t) &= -\lambda P_0(t) \\
\Rightarrow \quad \frac{1}{P_0(t)}\frac{d}{dt}P_0(t) &= -\lambda \\
\Rightarrow \quad \ln(P_0(t)) &= -\lambda t + c \\
\Rightarrow \quad P_0(t) &= Ke^{-\lambda t}
\end{aligned}$$

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$$
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\frac{d}{dt} P_0(t) &= -\lambda P_0(t) \\
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\Rightarrow \quad \ln(P_0(t)) &= -\lambda t + c \\
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\end{aligned}
$$

- Since $P_0(0) = Pr[N(0) = 0] = 1$, we get $K = 1$ and hence

$$P_0(t) = Pr[N(t) = 0] = e^{-\lambda t}$$

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- Since $P_0(0) = Pr[N(0) = 0] = 1$, we get $K = 1$ and hence
$$P_0(t) = Pr[N(t) = 0] = e^{-\lambda t}$$
- Next we consider $P_n(t)$ for $n > 0$

$$P_n(t+h) \;=\; Pr[N(t+h) = n]$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h) = n] \\
&= Pr[N(t) = n,\ N(t+h) - N(t) = 0] +
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0]+ \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1]+
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

We have

$$
\begin{aligned}
&Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(t+h)-N(t)=k]
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

We have

$$
\begin{aligned}
&Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(h)=k]
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

We have

$$
\begin{aligned}
&Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(h)=k] = o(h),\ \forall k \geq 2
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

We have

$$
\begin{aligned}
&Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(h)=k] = o(h),\ \forall k \geq 2
\end{aligned}
$$

$$
Pr[N(t)=n,\ N(t+h)-N(t)=0] = P_n(t)P_0(h)
$$

$$\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}$$

We have

$$\begin{aligned}
&Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(h)=k] = o(h),\ \forall k \geq 2
\end{aligned}$$

$$Pr[N(t)=n,\ N(t+h)-N(t)=0] = P_n(t)P_0(h)$$

$$Pr[N(t)=n-1,\ N(t+h)-N(t)=1] = P_{n-1}(t)P_1(h)$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h) = n] \\
&= Pr[N(t) = n,\ N(t+h) - N(t) = 0] + \\
&\qquad Pr[N(t) = n-1,\ N(t+h) - N(t) = 1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t) = n-k,\ N(t+h) - N(t) = k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h)
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0]\ + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1]\ + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1-\lambda h + o(h))
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h) = n] \\
&= Pr[N(t) = n,\ N(t+h) - N(t) = 0] + \\
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&\qquad \sum_{k=2}^{n} Pr[N(t) = n-k,\ N(t+h) - N(t) = k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1 - \lambda h + o(h)) + P_{n-1}(t)(\lambda h + o(h)) + o(h)
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1-\lambda h + o(h)) + P_{n-1}(t)(\lambda h + o(h)) + o(h) \\
&= P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h)
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h) = n] \\
&= Pr[N(t) = n,\; N(t+h) - N(t) = 0] + \\
&\qquad Pr[N(t) = n-1,\; N(t+h) - N(t) = 1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t) = n-k,\; N(t+h) - N(t) = k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1 - \lambda h + o(h)) + P_{n-1}(t)(\lambda h + o(h)) + o(h) \\
&= P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1-\lambda h + o(h)) + P_{n-1}(t)(\lambda h + o(h)) + o(h) \\
&= P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}
$$

$$
\Rightarrow \quad \frac{d}{dt}P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)
$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$.

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) = e^{\lambda t}\,\lambda P_{n-1}(t)$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\, \lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\, e^{\lambda t}\right) &= \lambda\, e^{\lambda t}\, P_{n-1}(t) \end{aligned}$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\, \lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\, e^{\lambda t}\right) &= \lambda\, e^{\lambda t}\, P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n.

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\,\lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\,e^{\lambda t}\right) &= \lambda\,e^{\lambda t}\,P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\,\lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\,e^{\lambda t}\right) &= \lambda\,e^{\lambda t}\,P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\,e^{\lambda t}\right) = \lambda\,e^{\lambda t}\,P_0(t)$$

$$\frac{d}{dt} P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt} P_n(t)$

$$\begin{aligned} e^{\lambda t} \left(P_n'(t) + \lambda P_n(t) \right) &= e^{\lambda t} \, \lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt} \left(P_n(t) \, e^{\lambda t} \right) &= \lambda \, e^{\lambda t} \, P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt} \left(P_1(t) \, e^{\lambda t} \right) = \lambda \, e^{\lambda t} \, P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\,\lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\,e^{\lambda t}\right) &= \lambda\,e^{\lambda t}\,P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\,e^{\lambda t}\right) = \lambda\,e^{\lambda t}\,P_0(t) = \lambda e^{\lambda t}e^{-\lambda t} = \lambda$$

$$\Rightarrow \quad e^{\lambda t}P_1(t) = \lambda t + c$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\, \lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\, e^{\lambda t}\right) &= \lambda\, e^{\lambda t}\, P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\, e^{\lambda t}\right) = \lambda\, e^{\lambda t}\, P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\Rightarrow \quad e^{\lambda t} P_1(t) = \lambda t + c \quad \Rightarrow \quad P_1(t) = e^{-\lambda t}(\lambda t + c)$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\,\lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\,e^{\lambda t}\right) &= \lambda\,e^{\lambda t}\,P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\,e^{\lambda t}\right) = \lambda\,e^{\lambda t}\,P_0(t) = \lambda e^{\lambda t}e^{-\lambda t} = \lambda$$

$$\Rightarrow \quad e^{\lambda t}P_1(t) = \lambda t + c \quad \Rightarrow \quad P_1(t) = e^{-\lambda t}(\lambda t + c)$$

- $P_1(0) = Pr[N(0) = 1] = 0 \quad \Rightarrow \quad c = 0$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\,\lambda P_{n-1}(t) \\ \Rightarrow\quad \frac{d}{dt}\left(P_n(t)\,e^{\lambda t}\right) &= \lambda\, e^{\lambda t}\, P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\,e^{\lambda t}\right) = \lambda\, e^{\lambda t}\, P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\Rightarrow\quad e^{\lambda t}P_1(t) = \lambda t + c \quad\Rightarrow\quad P_1(t) = e^{-\lambda t}(\lambda t + c)$$

- $P_1(0) = Pr[N(0) = 1] = 0 \quad\Rightarrow\quad c = 0$
  Hence $P_1(t) = \lambda t\, e^{-\lambda t}$

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These two definitions are equivalent

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- We could also generate Poisson process by generating independent exponential random variables

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- Note that the S_i have to satisfy $S_1 < S_2 < \cdots < S_n$.
- We can show that the conditional joint density of $S_1, \cdots, S_n$ conditioned on $N(t) = n$, would be same as the order statistics of n iid random variables uniform over $[0,\ t]$.

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