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- For a Markov chain, given the current state, the future evolution is independent of the history of how you reached the current state

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- The P and π_0 together determine all joint distributions:

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  Using this we can find joint distribution of any finite number of X_i's

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- This relationship is intuitively clear

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- That is why we use P^n for n-step transition probabilities

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- For any state y define

$$I_y(X_n) = \begin{cases} 1 & \text{if } X_n = y \\ 0 & \text{otherwise} \end{cases}$$

- Now, the total number of visits to y is given by

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&= \rho_{yy}^{m-1}\, \rho_{xy} - \rho_{yy}^{m}\, \rho_{xy} = \rho_{xy}\, \rho_{yy}^{m-1}(1 - \rho_{yy}) \\
P_x(N_y = 0) &= 1 - P_x(N_y \geq 1) = 1 - \rho_{xy}
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- $G(x,y)$ is the expected number of visits to y for a chain that is started in x.

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  Theorem: If x is recurrent and x leads to y then y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.

  Proof:
- Take $x \neq y$, wlog. Since $\rho_{xy} > 0$, $\exists n$ s.t. $P^n(x,y) > 0$
- Take least such n. Then we have states $y_1, \cdots, y_{n-1}$, none of which is x (or y) such that

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- Hence, if x is recurrent and x leads to y then $\rho_{yx} = 1$

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- This completes proof of the theorem

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- Easy to check it is an equivalence relation.

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- In our example, each equivalence class corresponds to a rational number.
- Here, C_i contains all fractions that are equal to that rational number

- The state space of any Markov chain can be partitioned into the transient and recurrent states: $S = S_T + S_R$:

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- Hence we get a partition: $S_R = C_1 + C_2 + \cdots$ where C_i are equivalence classes.

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- All states in any C_i lead to each other or communicate with each other
- If $i \neq j$ and $x \in C_i$ and $y \in C_j$, then, $\rho_{xy} = \rho_{yx} = 0$. x and y do not communicate with each other.

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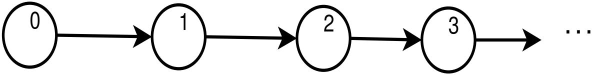
- For example, in the umbrellas problem, the chain is irreducible and hence all states are recurrent.
- An infinite irreducible chain may be wholly transient
- Here is a trivial example of non-irreducible transient chain:



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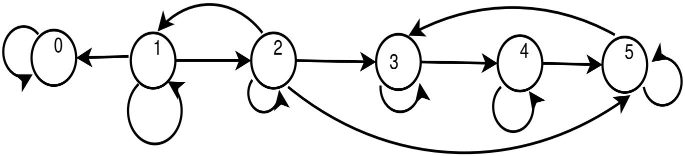
- By looking at the structure of transition probability matrix we can get this partition

Example



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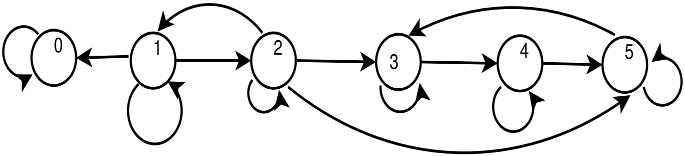

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\hline
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{cccccc}
+ & - & - & - & - & - \\
+ & + & + & - & - & - \\
- & + & + & + & - & + \\
- & - & - & + & + & - \\
- & - & - & - & + & + \\
- & - & - & + & - & +
\end{array}\right]
\end{array}
$$

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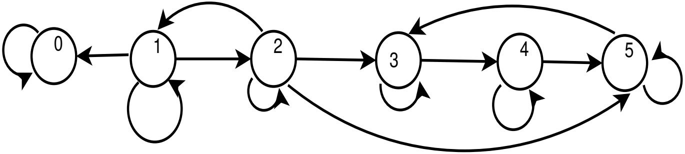

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & + & - & - & - & - & - \\
1 & + & + & + & - & - & - \\
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3 & - & - & - & + & + & - \\
4 & - & - & - & - & + & + \\
5 & - & - & - & + & - & +
\end{array}
$$

- State 0 is called an absorbing state. $\{0\}$ is a closed irreducible set.

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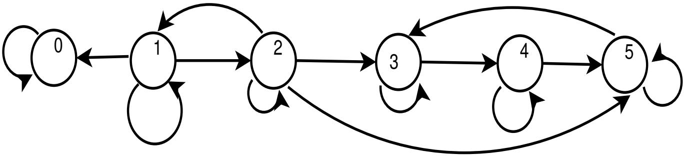

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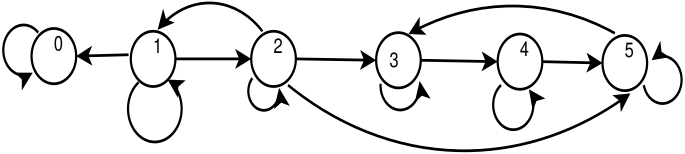
\begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{cccccc}
+ & - & - & - & - & - \\
+ & + & + & - & - & - \\
- & + & + & + & - & + \\
- & - & - & + & + & - \\
- & - & - & - & + & + \\
- & - & - & + & - & +
\end{array}\right]
\end{array}
$$

- State 0 is called an absorbing state. $\{0\}$ is a closed irreducible set.
- $1, 2$ are transient states.

Example



$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & + & - & - & - & - & - \\ 1 & + & + & + & - & - & - \\ 2 & - & + & + & + & - & + \\ 3 & - & - & - & + & + & - \\ 4 & - & - & - & - & + & + \\ 5 & - & - & - & + & - & + \end{array}$$

- State 0 is called an absorbing state. $\{0\}$ is a closed irreducible set.
- $1, 2$ are transient states.
- We get: $S_T = \{1, 2\}$ and $S_R = \{0\} + \{3, 4, 5\}$