Recap: Multi-dimensional Gaussian density

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- When X, Y are jointly Gaussian, the joint densityis given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \, e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)}$$

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- If $X_1, \cdots, X_n$ are jointly Gaussian then they are independent if they are uncorrelated.
- If $X_1, \cdots, X_n$ be jointly Gaussian (with zero means) then there is an orthogonal transform $\mathbf{Y} = A\mathbf{X}$ such that $Y_1, \cdots, Y_n$ are jointly Gaussian and independent.
- $X_1, \cdots, X_n$ are jointly Gaussian if and only if $\mathbf{t}^T\mathbf{X}$ is Gaussian for for all non-zero $\mathbf{t} \in \Re^n$.

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- Shows density of X_i is Gaussian for each i. For example, if we take $\mathbf{t} = (1, 0, 0, \cdots, 0)^T$ then $\mathbf{t}^T\mathbf{X}$ would be X_1.

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- This is a defining property of multidimensional Gaussian density

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- Why did we assume A has rank k?

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- For example, if you take A to be

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

then $\mathbf{Y} = (X_1, X_2)^T$

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- X_i are independent iff they are uncorrelated
- $\mathbf{t}^T\mathbf{X}$ being Gaussian for every non-zero $\mathbf{t}$ is a defining property of Gaussian vectors.

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- We want to find out the distribution of S_N.

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- This method is useful for finding distribution of S_N when we can recognize the distribution from its mgf

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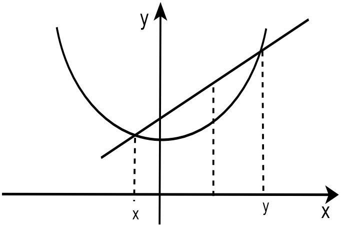

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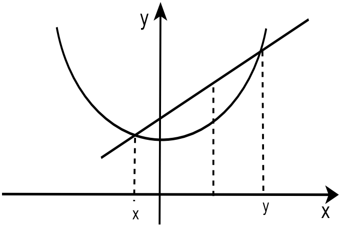

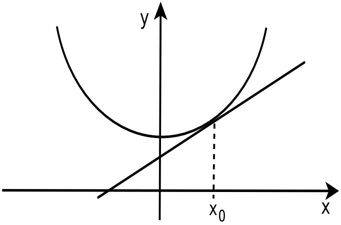

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- This completes the proof

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- A generalization of Cauchy-Schwartz inequality is Holder inequality

Holder Inequality

- For all p, q with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$E[|XY|] \leq (E|X|^p)^{\frac{1}{p}} \ (E|Y|^q)^{\frac{1}{q}}$$

(We assume all the expectations are finite)

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$$\begin{aligned} \left|\text{Cov}(X, Y)\right| &= \left|E[(X - EX)(Y - EY)]\right| \\ &\leq E\left[\left|(X - EX)(Y - EY)\right|\right] \\ &\leq \sqrt{E[(X - EX)^2] \ E[(Y - EY)^2]} \end{aligned}$$

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- This is same as Cauchy-Schwartz inequality. This implies $|\rho_{XY}| \leq 1$.

$$\begin{aligned}
\left|\mathrm{Cov}(X,Y)\right| &= \left|E[(X - EX)(Y - EY)]\right| \\
&\leq E\left[\left|(X - EX)(Y - EY)\right|\right] \\
&\leq \sqrt{E[(X - EX)^2] \ E[(Y - EY)^2]} \\
&= \sqrt{\mathrm{Var}(X) \ \mathrm{Var}(Y)}
\end{aligned}$$

Proof

- First we will show, for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

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- This is true of any event.
- Consider repeatedly performing a random experiment
- X_i be the indicator of event A on i^{th} repetition
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- The fraction of times an event occurs 'converges' to its probability as you repeat the experiment infinite times

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- There are different notions of convergence of a sequence of functions to a function.

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- Thus there would be multiple ways to define convergence of sequence of random variables.

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- A sequence of random variables, X_n, is said to **converge in probability** to a random variable X_0 is

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- We only need marginal distributions of individual X_n to decide whether a sequence converges to a constant in probability

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- Weak law of large numbers says that sample mean converges in probability to the expectation