E1 222 Stochastic Models and Applications

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Reference Material

- V.K. Rohatgi and A.K.Md.E. Saleh, An Introduction to probability and Statistics, Wiley, 2nd edition, 2018
- S.Ross, 'Introduction to Probability Models', Elsevier, 12th edition, 2019.

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- P G Hoel, S Port and C Stone, Introduction to Stochastic Processes, 1971.
- Scott Sheffield, Probability and Random Variables, Massachusetts Institute of Technology, MIT OpenCourseWare: https://ocw.mit.edu/courses/mathematics/18-600-probability-and-random-variables-fall-2019/
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- Calculus
  - continuity, differentiability, derivatives, functions of several variables, partial derivatives, integration, multiple integrals or integration over $\Re^n$, convergence of sequences and series, Taylor series

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  But we would review the basic probability in the first two classes.

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- It is also useful in many engineering systems, e.g., for analyzing behaviour under noise.

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This is only a 'sample' of possible application scenarios!

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$(\Omega, \mathcal{F}, P)$ is called the **Probability Space**

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$$\text{(i). } \Omega \in \mathcal{F} \quad \text{and} \quad \text{(ii). } A_1, A_2, \cdots \in \mathcal{F} \Rightarrow (\cup_i A_i) \in \mathcal{F}$$

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- As defined, the co-domain of the function P is $\Re$. However, the axioms imply that it takes values in $[0, 1]$.

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$$\begin{aligned} P(U_{i=1}^{n} A_i) &= \sum_i P(A_i) - \sum_i \sum_{j>i} P(A_i \cap A_j) \\ &+ \sum_i \sum_{j>i} \sum_{k>j} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\cap_i A_i) \end{aligned}$$

Known as inclusion-exclusion formula

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- We can easily see this to be true for any event, A.

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- An obvious point worth remembering: specifying P for singleton events fixes it for all other events.

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- This can be done for finite Ω too.

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- Consider a random experiment of tossing a biased coin repeatedly till we get a head. We take the outcome of the experiment to be the number of tails we had before the first head.
- A (reasonable) probability assignment is:

$$P(\{k\}) = (1-p)^k p, k = 0, 1, \cdots$$

where p is the probability of head and $0 < p < 1$.
(We assume you understand the idea of 'independent' tosses here).