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Probability axioms

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- A1 $P(A) \geq 0, \; \forall A \in \mathcal{F}$
- A2 $P(\Omega) = 1$
- A3 If $A_i \cap A_j = \phi, \forall i \neq j$ then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

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- An obvious point worth remembering: specifying P for singleton events fixes it for all other events.

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- This can be done for finite Ω too.

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- Consider a random experiment of tossing a biased coin repeatedly till we get a head. We take the outcome of the experiment to be the number of tails we had before the first head.
- A (reasonable) probability assignment is:

$$P(\{k\}) = (1-p)^k p, k = 0, 1, \cdots$$

where p is the probability of head and $0 < p < 1$.
(We assume you understand the idea of 'independent' tosses here).

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  (There are many issues that need more attention here).

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- We can take $\Omega = \{(x, y) \ : \ 0 < x < y < 1\} \subset \Re^2$.
- For the pieces to make a triangle, sum of lengths of any two should be more than the third.

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- So the event of interest is:

$$A = \{(x,y) \;:\; y > 0.5;\; x < 0.5;\; y < x{+}0.5,\; 0 < x, y < 1\}$$

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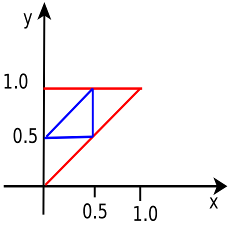

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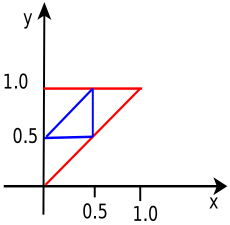

- The required probability is area of A divided by area of Ω which gives the answer as 0.25

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- Once we understand condional probability is a new probability assignment, we go back to the ‘standard notation’

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- It is not as if we talk about conditional probability only for subsets of B. Conditional probability is also with respect to the original probability space. Every element of $\mathcal{F}$ has conditional probability defined.

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- There are probabilistic methods to capture causation (but far beyond the scope of this course!)

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- This is a very useful in many situations. ("arguing by cases")

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An urn contains r red balls and b black balls. We draw a ball at random, note its color, and put back that ball along with c balls of the same color. We keep repeating this process. Let R_n (B_n) denote the event of drawing a red (black) ball at the n^{th} draw. We want to calculate the probabilities of all these events.

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- These different cases are important in understanding the role of false positives rate.

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  - The knowledge we need is $P(T_+|D)$, $P(T_+|D^c)$ which can be determined through experiment or modelling of channel.

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- Example: I have three coins with probability of heads being 0.1, 0.5, 0.8. I choose one at random and toss it twice and see heads both times. What is the probability it is the fair coin?

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- Independence is an important (often confusing!) concept.

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  Similarly we can show for others.

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- Hence whether or not two events are independent is a matter of 'calculation'

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- This also shows that A^c and B are independent and so are A^c and B^c.

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- Using $A = AB + AB^c$, and $AB \subset A$, we have

$$P(AB^c) = P(A-AB) = P(A)-P(AB) = P(A)(1-P(B)) = P(A)P(B^c)$$

- This also shows that A^c and B are independent and so are A^c and B^c.
- For example, in the previous problem, once we saw that F and C are independent, we can conclude M and C are also independent (because in this example we are taking $F^c = M$).

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- Hence, in multiple tosses, assuming all outcomes are equally likely implies outcome of one toss is independent of another.

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- How should we then assign these probabilities?
- If we assume tosses are independent then we can assign probabilities easily.

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  (We will look at it more formally when we consider multiple random variables).

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- This is often used, at an intuitive level, to justify assumption of independence.

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- But, $P(E_1 E_2 E_3) = 0.25 \neq (0.5)^3$

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- Events may be conditionally independent but not independent. (e.g., 'independent' multiple tests for confirming a disease)
- It is also possible that A, B are independent but are not conditionally independent given some other event C.

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- Consider the old example, where now we repeat the test for the disease.
- Take: $A = D$, $B = T^1_+$, $C = T^2_+$.
- Assuming conditional independence we can calculate the new posterior probability using the same information we had about true positive and false positive rate.

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$$P(D \mid T_+^1 T_+^2) \;=\; \frac{P(T_+^1 T_+^2 \mid D)P(D)}{P(T_+^1 T_+^2 \mid D)P(D) + P(T_+^1 T_+^2 \mid D^c)P(D^c)}$$

- Let us consider the example with $P(T_+|D) = 0.99$, $P(T_+|D^c) = 0.05$. $P(D) = 0.1$.
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$$= \frac{P(T_+^1 \mid D)P(T_+^2 \mid D)P(D)}{P(T_+^1 \mid D)P(T_+^2 \mid D)P(D) + P(T_+^1 \mid D^c)P(T_+^2 \mid D^c)P(D^c)}$$

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$$= \frac{0.99 * 0.99 * 0.1}{0.99 * 0.99 * 0.1 + 0.05 * 0.05 * 0.9} = 0.97$$