Recap: Brownian Motion

- $\{X(t),\ t \geq 0\}$ is called a Brownian motion if
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- $Y(t) = X(t) + \mu$ is called Brownian motion with a drift

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- All n^{th} order distributions are Gaussian: $X(t_1), \cdots, X(t_n)$ are jointly Gaussian.

Recap: Gaussian Processes

- A continuous-time continuous-state process $\{X(t),\ t \geq 0\}$ is said to be a Gaussian process if for all n and all $t_1, t_2, \cdots, t_n$, we have that $X(t_1), \cdots, X(t_n)$ are jointly Gaussian.

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$$E[X(t)] = 0, \quad \text{Var}(X(t)) = \sigma^2 t, \quad \text{Cov}(X(s), X(t)) = \sigma^2 \min(s, t)$$

Recap: Conditional Densities

- For $s < t$, $f_{X(s)|X(t)}$ is Gaussian and

$$E[X(s)|X(t)] = \frac{s}{t}\, X(t); \;\; \mathsf{Var}(X(s)|X(t)) = \frac{s}{t}\,(t-s)$$

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- For $s < t$, we also showed $f_{X(t)|X(s)}$ is Gaussian with

$$E[X(t) \mid X(s)] = X(s); \;\; \mathsf{Var}(X(t)|X(s)) = (t-s)$$

Recap: Hitting Times

- Let T_a denote the first time Brownian motion hits a.

$$Pr[T_a \le t] = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-\frac{y^2}{2}} \, dy$$

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- By continuity of sample paths,

$$Pr[\max_{0 \leq s \leq t} X(s) \geq a] = Pr[T_a \leq t] = 2Pr[X(t) \geq a]$$

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$$X(t) = e^{Y(t)}$$
- Then, $\{X(t),\ t \geq 0\}$ is called geometric Brownian motion. It is useful in mathematial finance

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- Any sequence of continuous random variables would be a discrete-time continuous-state process

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- Many algorithms can be written in this general form.
- The $X(n), n = 0, 1, \cdots$ would be a discrete-time continuous-state stocahstic process
- We consider an important class of such processes

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- When X_n is a martingale, we have

$$E[X_{n+1}] = E[X_n],\ \forall n$$

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- Hence, X_n is a martingale.

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- Thus we get

$$E[\, E[h(X)|Y,Z] \mid Y] = E[h(X)|Y]$$

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- Let $Z_n = \sum_{i=1}^{n} W_i$.
- We can show that $Z_n,\ n = 1, 2, \cdots$ is a martingale, assuming that $E[|Z_n|] < \infty$.

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- This shows that Z_n is a martingale

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- If we know the values of $Z_1, \cdots, Z_n$, then we can say whether or not $N = n$.
- The idea is that we can decide to stop the process at N and the decision to stop cannot anticipate the future

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- We call $\{\bar{Z}_n,\ n \geq 1\}$ the stopped process

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- Recall that for a martingale, $E[Z_n] = E[Z_1],\ \forall n$
- The theorem essentially says there is no strategy to have a positive expectation from a fair gambling game.

- $\{X_n,\ n = 0, 1, \cdots\}$ and $E|X_n| < \infty,\ \forall n$ is called a martingale if

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- In the above, the conditioning random variables can be another sequence Y_i if $Y_1, \cdots, Y_n$ determine $X_1, \cdots, X_n$

- A useful result is the martingale convergence theorem.

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- If X_n are bounded, then the condition is always true and the almost sure convergence implies convergence in the mean.
- This is often useful in dealing with many sequences of random variables such as a stochastic algorithm.

Example

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- 2 arms. Response is binary (1 for reward).
- d_i - prob of reward for arm-i. Do not know d_i
- need to play and find which is better arm.
- We choose arm-1 with prob $p(k)$ (and hence arm-2 with prob $(1 - p(k))$ at iteration k and update $p(k)$ based on the outcome.

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- We want to know whether the algorithm converges.

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- So, we can conclude, the algorithm converges almost surely

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- Martingales can be useful in analyzing convergence of many stochastic algorithms
- While we mentioned only discrete-time martingales, one can similarly have continuous-time martingales that satisfy

$$E[X(t)|(X(s'), 0 \leq s' \leq s < t] = X(s)$$