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- We also have
$$P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}\ dy\ dx$$
  and, in general,
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$$f_{X|Y}(x_i|y_j) = \frac{f_{Y|X}(y_j|x_i) f_X(x_i)}{\sum_i f_{Y|X}(y_j|x_i) f_X(x_i)}$$

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Called a mixture density model

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- This gives us further versions of total probability rule and Bayes rule.

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- Earlier we derived the same formula when X, Y have a joint density.

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- In general, one refers to these always as densities since the actual meaning would be clear from context.

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- We want to use the Bayes rule to calculate this

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- In practice we only want to know whether $f_{Y|X}(1|x) > f_{Y|X}(0|x)$

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- $f_{X|Y}(x|1)$ is Gaussian with mean 5 and variance σ^2 and $f_{X|Y}(x|0)$ is Gaussian with mean zero and variance σ^2

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- So, if $X > 2.5$ we will conclude bit 1 is sent. Intuitively obvious!

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  Define a function $f : \Re^2 \to \Re$ by
  $$f(x, y) = f_1(x) f_2(y) \left[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)\right]$$
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- For this, we note the following

$$\int_{-\infty}^{\infty} f_1(x)\ F_1(x)\ dx = \left.\frac{(F_1(x))^2}{2}\right|_{-\infty}^{\infty} = \frac{1}{2}$$

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$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy &= \int_{-\infty}^{\infty} f_1(x)\, dx \int_{-\infty}^{\infty} f_2(y)\, dy \\ &+\alpha \int_{-\infty}^{\infty} (2f_1(x)F_1(x) - f_1(x))\, dx \int_{-\infty}^{\infty} (2f_2(y)F_2(y) - f_2(y))\, dy\end{aligned}$$

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because $2\int_{-\infty}^{\infty} f_1(x)\;F_1(x)\;dx = 1$.

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- An important special case where this is possible is that of independent random variables

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- **Theorem**: X, Y are independent if and only if $F_{XY}(x, y) = F_X(x) F_Y(y)$.

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- So, X, Y are independent if and only if $f_{XY}(x,y) = f_X(x)f_Y(y)$

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- Let X, Y be independent continuous rv

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- So, X, Y are independent if and only if $f_{XY}(x,y) = f_X(x)f_Y(y)$

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- This is true for all the four possibilities of X, Y being continuous/discrete.

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- If they are continuous , they have a joint density if

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- Generalizing the special property of the df (relating to probability of cylindrical sets) is a little more complicated. (An exercise for you!)
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- We obtain the marginal mass functions for a subset of the rv's also similarly where we sum over the remaining variables.

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- With these we can generally calculate most quantities of interest.

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- For example when Z is continuous

$$F_{XY|Z}(x, y|z) = \lim_{\delta \to 0} P[X \leq x, Y \leq y|Z \in [z, z + \delta]]$$

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- For example, the first one above follows from

$$P[X = x, Y = y|Z = z] = \frac{P[X = x, Y = y, Z = z]}{P[Z = z]}$$

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- We use similar notation for marginal and conditional distributions

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- For independent random variables, the joint mass function (or density function) is product of individual mass functions (or density functions)

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First let us determine K.

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$$\begin{aligned} f_{XZ}(x, z) &= \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dy \\ &= \int_{z}^{x} 6\, dy, \quad 0 < z < x < 1 \\ &= 6(x - z), \quad 0 < z < x < 1 \end{aligned}$$

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&= \int_0^1 \left(6x^2 - 6\,\frac{x^2}{2}\right)\,dx \\
&= 3\left.\frac{x^3}{3}\right|_0^1 = 1
\end{aligned}$$

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- Hence,

$$f_{Y|XZ}(y|x, z) = \frac{f_{XYZ}(x, y, z)}{f_{XZ}(x, z)} = \frac{1}{x - z}, \quad 0 < z < y < x < 1$$

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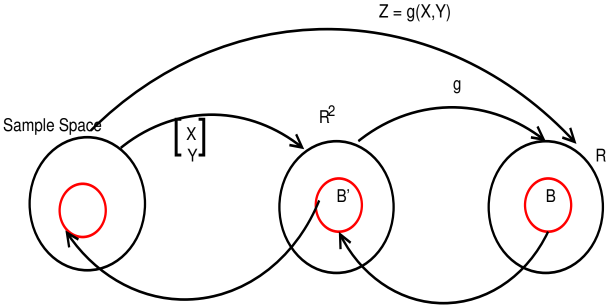
- This is analogous to functions of a single rv



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- For example, if X, Y are discrete, then

$$f_Z(z) = P[Z = z] = P[g(X, Y) = z] = \sum_{\substack{x_i, y_j: \\ g(x_i, y_j) = z}} f_{XY}(x_i, y_j)$$

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&= \sum_{y>z} P[X = z, Y = y] + \sum_{x>z} P[X = x, Y = z] \\
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- Now suppose X, Y are independent and both of them have geometric distribution with the same parameter, p.
- Such random variables are called **independent and identically distributed** or **iid** random variables.

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f_Z(z) &= P[X = z, Y > z] + P[Y = z, X > z] + P[X = Y = z] \\
&= P[X = z]P[Y > z] + P[Y = z]P[X > z] + P[X = z]P[Y = z]
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&= P[X = z]P[Y > z] + P[Y = z]P[X > z] + P[X = z]P[Y = z] \\
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&= p(1-p)^{z-1}(1-p)^z * 2 + \left(p(1-p)^{z-1}\right)^2 \\
&= 2p(1-p)^{z-1}(1-p)^z + \left(p(1-p)^{z-1}\right)^2 \\
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&= F_{X_1 \cdots X_n}(z, \cdots, z)
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&= P[X_1 \le z, X_2 \le z, \cdots, X_n \le z] \\
&= F_{X_1 \cdots X_n}(z, \cdots, z) \\
&= F_{X_1}(z) \cdots F_{X_n}(z), \quad \text{if they are independent} \\
&= (F_X(z))^n, \quad \text{if they are iid} \\
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- For example if all X_i are uniform over $(0, 1)$ and ind, then $F_Z(z) = z^n,\ 0 < z < 1$

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- We can once again find density of Z if X, Y are continuous

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- Notice that $P[X > z] = (1 - z)$.
- We get the density of Z as

$$f_Z(z) = 2(1 - z), \ \ 0 < z < 1$$

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- Hence, when X_i are iid, the df of Z is

$$F_Z(z) = 1 - (1 - F_X(z))^n$$

where F_X is the common df