Recap: Process with stationary and independent increments

- A (continuous-time) random process $\{N(t),\ t \geq 0\}$ has independent increments if for all $t_1 < t_2 \leq t_3 < t_4$, $N(t_2) - N(t_1)$ is independent of $N(t_4) - N(t_3)$

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- In particular, for all $s > t > 0$, $N(s) - N(t)$ is independent of $N(t) - N(0)$
- The process has stationary increments if $N(t_2) - N(t_1)$ has the same distribution as $N(t_2 + \tau) - N(t_1 + \tau)$, $\forall \tau, \forall t_2 > t_1$

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- A random process $\{N(t),\ t \geq 0\}$ is called a counting process if $N(t)$ is non-negative integer-valued and is non-decreasing

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- The two definitions are equivalent.

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$$Pr[N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3]$$
$$= Pr[N(t_1) = n_1] \; Pr[N(t_2) - N(t_1) = n_2 - n_1]$$
$$Pr[N(t_3) - N(t_2) = n_3 - n_2]$$
$$= Pr[N(t_1) = n_1] \; Pr[N(t_2 - t_1) = n_2 - n_1] \; Pr[N(t_3 - t_2) = n_3 - n_2$$

where we assumed $t_1 < t_2 < t_3$

Recap: mean and autocorrelation

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&\Rightarrow \quad \text{not stationary}
\end{aligned}$$

Recap: Inter-arrival or waiting times

- Let T_1 denote the time of first event and let T_n denote the time between n^{th} and $(n-1)st$ events.

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- Since T_i are iid, exponential, S_n is Gamma with parameters n, λ

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- Conditioned on $N(t) = 1$, S_1 is uniform over $[0, t]$

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- Note that the S_i have to satisfy $S_1 < S_2 < \cdots < S_n$.
- We showed that the conditional joint density of $S_1, \cdots, S_n$ conditioned on $N(t) = n$, would be same as the order statistics of n iid random variables uniform over $[0,\ t]$.

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- Thus we have for $0 < t_1 < \cdots < t_n < t$,

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$$f_{S_1 \cdots S_n | N(t)}(t_1, \cdots, t_n | n) = \frac{n!}{t^n}, \ 0 < t_1 < \cdots < t_n < t$$

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We calculate $Pr[S_4 > t | N(1) = 2]$ and use it to find the above expectation

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- Recall that if X is a non-negative continuous rv, then

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- Since S_4 is non-negative, we can use this

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$$Pr[S_4 > t|N(1) = 2] = e^{-\lambda(t-1)} + \lambda(t-1)e^{-\lambda(t-1)}, \; t > 1$$

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- Using this

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- Intuitively reasonable because expected inter-arrival time is $\frac{1}{\lambda}$

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Theorem: $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are Poisson processes with rate λp and $\lambda(1-p)$ respectively, and they are independent

$$Pr[N_1(t) = n, N_2(t) = m]$$

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$$= \sum_k Pr[N_1(t) = n, N_2(t) = m \mid N(t) = k] \; Pr[N(t) = k]$$

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- This shows that $N_1(t)$ and $N_2(t)$ are independent Poisson

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- The answer is 3 because the two processes are independent

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- This is sometimes referred to as thinning of a Poisson process

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where we have used independence of N_1 and N_2

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- Dealing with $\max$ of non-independent random variables is difficult.

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- Hence, by Wald's identity, $E[X] = E[N] \; E[T_i] = E[N]$

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- But we need the expected number of events by that time
- Let N denote the number of events by time X. Then

$$X = \sum_{i=1}^{N} T_i, \quad T_i \textit{ iid} \text{ exponential with mean 1}$$

- Hence, by Wald's identity, $E[X] = E[N]\ E[T_i] = E[N]$
- This completes the solution of the problem.

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The earlier case corresponds to $p_i(s) = p_i, \forall s$.

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  An individual infected at s would show symptoms by t with probability $G(t - s)$

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- Using this we can approximate

$$E[N_2(t)] \approx \hat{\lambda} \int_0^t (1 - G(y))\ dy$$

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- One can show that $N(t+s) - N(t)$ is Poisson with parameter $m(t+s) - m(t)$ where $m(\tau) = \int_0^\tau \lambda(s)\, ds$

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