Recap: Function of a random variable

- If X is a random variable and $g : \Re \rightarrow \Re$ is a function, then $Y = g(X)$ is a random variable.
- More formally, Y is a random variable if g is a Borel measurable function.
- We can determine distribution of Y given the function g and the distribution of X

Recap

- Let X be a rv and let $Y = g(X)$.
- The distribution function of Y is given by

$$\begin{aligned} F_Y(y) &= P[g(X) \leq y] \\ &= P[X \in \{z \,:\, g(z) \leq y\}] \end{aligned}$$

- This probability can be obtained from distribution of X.
- We have seen many specific examples of this.

Recap

- Suppose X is a discrete rv with $X \in \{x_1, x_2, \cdots\}$.
- Suppose $Y = g(X)$.
- Then Y is also discrete and $Y \in \{g(x_1), g(x_2), \cdots\}$.
- We can find the pmf of Y as

$$\begin{aligned} f_Y(y) &= p[Y = y] = P[g(X) = y] \\ &= P[X \in \{x_i \ : \ g(x_i) = y\}] \\ &= \sum_{\substack{i: \\ g(x_i)=y}} f_X(x_i) \end{aligned}$$

Recap

- Let $g : \Re \rightarrow \Re$ be differentiable with $g'(x) > 0, \forall x$ or $g'(x) < 0, \forall x$.
- Let X be a continuous rv and let $Y = g(X)$.
- Then Y is a continuous rv with pdf

$$f_Y(y) = f_X(g^{-1}(y)) \ \left| \frac{d}{dy} g^{-1}(y) \right|, \ a \leq y \leq b$$

  where $a = \min(g(\infty), \ g(-\infty))$ and $b = \max(g(\infty), \ g(-\infty))$
- This theorem is useful in some cases to find the densities of functions of continuous random variables

Recap: Expectation of a random variable

- Let X be a discrete rv with $X \in \{x_1, x_2, \cdots\}$. Then

$$E[X] = \sum_i x_i \, f_X(x_i)$$

- If X is a continuous random variable with pdf, f_X,

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

- Sometimes we use the following notation to denote expectation of both kinds of rv

$$E[X] = \int_{-\infty}^{\infty} x \, dF_X(x)$$

- We take the expectation to exist when the sum or integral above is absolutely convergent
- Note that expectation is defined for all random variables

- The way we have defined existence of expectation, implies that expectation is always finite (when it exists).
- This may be needlessly restrictive in some situations. We redefine it as follows.
- Let X be a non-negative (discrete or continuous) random variable.
- We define its expectation by

$$EX = \sum_i x_i \, f_X(x_i) \quad \text{or} \quad EX = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

  depending on whether it is discrete or continuous
  (In this course we will consider only discrete or continuous rv's)
- Note that the expectation may be infinite.
- But it always exists for non-negative random variables.

- Now let X be a rv that may not be non-negative.
- We define positive and negative parts of X by

$$X^+ = \begin{cases} X & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X^- = \begin{cases} -X & \text{if } X < 0 \\ 0 & \text{otherwise} \end{cases}$$

- Note that both X^+ and X^- are non-negative. Hence their expectations exist.
  (Also, $X(\omega) = X^+(\omega) - X^-(\omega), \ \forall \omega$).
- Now we define expectation of X by

$$EX = EX^+ - EX^-, \quad \text{if at least one of them is finite}$$

  Otherwise EX does not exist.
- Now, expectation does not exist only when $EX^+ = EX^- = \infty$

- This is the formal way of defining expectation of a random variable.
- We first note that if $\sum_i |x_i| \; f_X(x_i) < \infty$ then both EX^+ and EX^- would be finite and we can simply take the expectation as $EX = \sum_i x_i \; f_X(x_i)$.
- Also note that if X takes only finitely many values, the above always holds.
- Similar comments apply for a continuous random variable.
- To get a feel for the more formal definition, we look at a couple of examples.

- Let $X \in \{1, 2, \cdots\}$.
- Suppose $f_X(k) = \frac{C}{k^2}$.
- Since $\sum_k \frac{1}{k^2} < \infty$, we can find C so that $\sum_k f_X(k) = 1$. ($\sum_k \frac{1}{k^2} = \frac{\pi^2}{6}$ and hence $C = \frac{6}{\pi^2}$).
- Hence we get

$$\sum_k |x_k| \ f_X(x_k) = \sum_k x_k \ f_X(x_k) = \sum_k k \ \frac{C}{k^2} = \sum_k \frac{C}{k} = \infty$$

- Here the expectation is infinity.
- But by the formal definition it exists. (Note that here $X^+ = X$ and $X^- = 0$).

- Now suppose X takes values $1, -2, 3, -4, \cdots$ with probabilities $\frac{C}{1^2}$, $\frac{C}{2^2}$, $\frac{C}{3^2}$ and so on.
- Once again $\sum_k |x_k| f_X(x_k) = \infty$.
- But $\sum_k x_k f_X(x_k)$ is an alternating series.
- Here X^+ would take values $2k-1$ with probability $\frac{C}{(2k-1)^2}$, $k = 1, 2, \cdots$
  (and the value 0 with remaining probability).
- Similarly, X^- would take values $2k$ with probability $\frac{C}{(2k)^2}$, $k = 1, 2, \cdots$ (and the value 0 with remaining probability).

$$EX^+ = \sum_k \frac{C}{2k-1} = \infty, \quad \text{and} \quad EX^- = \sum_k \frac{C}{2k} = \infty$$

- Hence EX does not exist.

- Consider a continuous random variable X with pdf

$$f_X(x) = \frac{1}{\pi}\,\frac{1}{1+x^2},\ -\infty < x < \infty$$

- This is called (standard) Cauchy density. We can verify it integrates to 1

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\ \tan^{-1}(x)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 1$$

- What would be EX?

$$EX = \int_{-\infty}^{\infty} x\,\frac{1}{\pi}\,\frac{1}{1+x^2}\,dx \ \stackrel{?}{=} 0 \text{ because } \int_{-a}^{a} \frac{x}{1+x^2} = 0?$$

- The question was

$$EX = \int_{-\infty}^{\infty} x \, \frac{1}{\pi} \, \frac{1}{1+x^2} \, dx \ \stackrel{?}{=} 0$$

- This depends on the definition of infinite integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x) \, dx \quad &\triangleq \quad \lim_{c \to \infty, d \to \infty} \int_{-c}^{d} g(x) \, dx \\
&= \quad \lim_{c \to \infty} \int_{-c}^{0} g(x) \, dx + \lim_{d \to \infty} \int_{0}^{d} g(x) \, dx
\end{aligned}$$

This is not same as $\displaystyle \lim_{a \to \infty} \int_{-a}^{a} g(x) \, dx,$

which is known as Cauchy principal value of the integral

- Here we have

$$\lim_{c\to\infty} \int_{-c}^{0} \frac{x}{1+x^2}\, dx = -\infty; \quad \lim_{d\to\infty} \int_{0}^{d} \frac{x}{1+x^2}\, dx = \infty$$

- Hence $EX = \int_{-\infty}^{\infty} x \, \frac{1}{\pi} \, \frac{1}{1+x^2} \, dx$ does not exist.
- Essentially, both halves of the integral are infinite and hence we get $\infty - \infty$ type expression which is undefined.
- However, $\lim_{a\to\infty} \int_{-a}^{a} x \, \frac{1}{\pi} \, \frac{1}{1+x^2} \, dx = 0$.

Expectation of a random variable

- Let X be a discrete rv with $X \in \{x_1, x_2, \cdots\}$. Then

$$E[X] = \sum_i x_i \, f_X(x_i)$$

- If X is a continuous random variable with pdf, f_X,

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

- Sometimes we use the following notation to denote expectation of both kinds of rv

$$E[X] = \int_{-\infty}^{\infty} x \, dF_X(x)$$

- Note that expectation is defined for all random variables
- Let us calculate expectations of some of the standard distributions.

Binary random variable

- Expectation of a binary rv (e.g., Bernoulli):

$$EX = 0 \times f_X(0) + 1 \times f_X(1) = P[X = 1]$$

- Expectation of a binary random variable is same as the probability of the rv taking value 1.
- Thus, for example, $EI_A = P(A)$.

Expectation of Binomial rv

- Let $f_X(k) = \ {}^nC_k p^k (1-p)^{n-k}, \ k = 0, 1, \cdots, n$.

$$EX = \sum_{k=0}^{n} k \ \frac{n!}{k!(n-k)!} \ p^k \ (1-p)^{n-k} = \sum_{k=1}^{n} k \ \frac{n!}{k!(n-k)!} \ p^k \ (1-p)^{n-k}$$

$$= \ \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \ p^k \ (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!((n-1)-(k-1))!} \ p \ p^{k-1} \ (1-p)^{(n-1)-(k-1)}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \ p^{k-1} \ (1-p)^{(n-1)-(k-1)}$$

$$= \ np \sum_{k'=0}^{n-1} \frac{(n-1)!}{k'!((n-1)-k')!} \ p^{k'} \ (1-p)^{(n-1)-k'} = np$$

Expectation of Poisson rv

- $f_X(k) = \frac{\lambda^k}{k!}\, e^{-\lambda},\ k = 0, 1, \cdots$

$$\begin{aligned} EX &= \sum_{k=0}^{\infty} k\, \frac{\lambda^k}{k!}\, e^{-\lambda} \\ &= \lambda \end{aligned}$$

(Left as an exercise for you!)

Expectation of Geometric rv

- $f_X(k) = (1-p)^{k-1}\, p, \quad k = 1, 2, \cdots$

$$EX = \sum_{k=1}^{\infty} k\, (1-p)^{k-1}\, p$$

- We have

$$\sum_{k=1}^{\infty} (1-p)^k = \frac{1-p}{p} = \frac{1}{p} - 1$$

- Term-wise differentiation of the above gives

$$\sum_{k=1}^{\infty} k\, (1-p)^{k-1} = \frac{1}{p^2}$$

- This gives us $EX = \frac{1}{p}$

Expectation of uniform density

- Let $X \sim U[a,b]$. $f_X(x) = \frac{1}{b-a}, \; a \leq x \leq b$

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} x \, f_X(x) \, dx \\
&= \int_a^b x \, \frac{1}{b-a} \, dx \\
&= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b \\
&= \frac{1}{b-a} \, \frac{b^2 - a^2}{2} \\
&= \frac{b+a}{2}
\end{aligned}
$$

Expectation of exponential density

- $f_X(x) = \lambda \, e^{-\lambda x}, \;\; x > 0.$

$$
\begin{aligned}
EX &= \int_0^\infty x \, \lambda \, e^{-\lambda x} \, dx \\
&= x \, \lambda \, \frac{e^{-\lambda x}}{-\lambda} \bigg|_0^\infty - \int_0^\infty \lambda \, \frac{e^{-\lambda x}}{-\lambda} \, dx \\
&= \int_0^\infty e^{-\lambda x} \, dx \\
&= \frac{e^{-\lambda x}}{-\lambda} \bigg|_0^\infty \\
&= \frac{1}{\lambda}
\end{aligned}
$$

Expectation of Gaussian density

- $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ -\infty < x < \infty$

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} x\, \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&\quad \text{make a change of variable } \ y = \frac{x-\mu}{\sigma} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, (\sigma y + \mu) e^{-\frac{y^2}{2}}\, dy \\
&= \sigma \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, e^{-\frac{y^2}{2}}\, dy + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \\
&= \mu
\end{aligned}
$$

Expectation of a function of a random variable

- Let X be a rv and let $Y = g(X)$.
- **Theorem**: $EY = \int y \, dF_Y(y) = \int g(x) \, dF_X(x)$
- That is, if X is discrete, then

$$EY = \sum_j y_j \, f_Y(y_j) = \sum_i g(x_i) f_X(x_i)$$

- If X and Y are continuous

$$EY = \int y \, f_Y(y) \, dy = \int g(x) \, f_X(x) \, dx$$

- This theorem is true for all rv's. But we will prove it in only some special cases.

- **Theorem**: Let $X \in \{x_1, x_2, \cdots x_n\}$ and let $Y = g(X)$. Then

$$EY = \sum_i g(x_i) \; f_X(x_i)$$

- **Proof**: Let $Y \in \{y_1, y_2, \cdots, y_m\}$.
  Each y_j would be equal to $g(x_i)$ for one or more i.
- Let $B_j = \{x_i \ : \ g(x_i) = y_j\}$. Thus,

$$f_Y(y_j) = P[Y = y_j] = P[X \in B_j] = \sum_{\substack{i: \\ x_i \in B_j}} f_X(x_i)$$

- Note that
  - B_j are disjoint
  - each x_i would be in one (and only one) of the B_j

- Now we have

$$\begin{aligned} EY &= \sum_{j=1}^{m} y_j f_Y(y_j) \\ &= \sum_{j=1}^{m} y_j \sum_{\substack{i: \\ x_i \in B_j}} f_X(x_i) \\ &= \sum_{j=1}^{m} \sum_{\substack{i: \\ x_i \in B_j}} g(x_i) f_X(x_i) \\ &= \sum_{i=1}^{n} g(x_i) f_X(x_i) \end{aligned}$$

  That completes the proof.
- The proof goes through even when X (and Y) take countably infinitely many values (because we assume the expectation sum is absolutely convergent).

- Suppose X is a continuous rv and suppose g is a differentiable function with $g'(x) > 0, \ \forall x$. Let $Y = g(X)$
- Once again we can show $EY = \int g(x) \ f_X(x) \ dx$

$$
\begin{aligned}
EY &= \int_{-\infty}^{\infty} y \ f_Y(y) \ dy \\
&= \int_{g(-\infty)}^{g(\infty)} y \ f_X(g^{-1}(y)) \ \frac{d}{dy} g^{-1}(y) \ dy,
\end{aligned}
$$

change the variable to $\ x = g^{-1}(y) \ \Rightarrow \ dx = \dfrac{d}{dy} g^{-1}(y) \ dy$

$$
= \int_{-\infty}^{\infty} g(x) \ f_X(x) \ dx
$$

- We can similarly show this for the case where $g'(x) < 0, \ \forall x$

- We proved the theorem only for discrete rv's and for some restricted case of continuous rv's.
- However, this theorem is true for all random variables.
- Now, for any function, g, we can write

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \quad \text{or} \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$$

Some Properties of Expectation

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \quad \text{or} \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$$

- If $X \geq 0$ then $EX \geq 0$
- $E[b] = b$ where b is a constant
- $E[ag(X)] = aE[g(X)]$ where a is a constant
- $E[aX + b] = aE[X] + b$ where a, b are constants.
- $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$

- Consider the problem: $\min_c \; E[(X-c)^2]$
- We are asking what is the best constant to approximate a rv with
- We are trying to minimize (weighted) average, over all values X can take, of the square of the error
- We are interested in the best mean-square approximation of X by a constant.

$$E[(X-c)^2] = E[X^2 + c^2 - 2cX] = E[X^2] + c^2 - 2cE[X]$$

- We differentiate this and equate to zero to get the best c

$$2c^* = 2E[X] \;\Rightarrow\; c^* = E[X]$$

- We can derive this in an alternate manner too

$$
\begin{aligned}
E[(X-c)^2] &= E[(X-EX+EX-c)^2] \\
&= E[(X-EX)^2 + (EX-c)^2 + 2(EX-c)(X-EX)] \\
&= E[(X-EX)^2] + (EX-c)^2 + 2(EX-c)E[(X-EX)] \\
&= E[(X-EX)^2] + (EX-c)^2 + 2(EX-c)(EX-EX) \\
&= E[(X-EX)^2] + (EX-c)^2 \\
&\geq E[(X-EX)^2]
\end{aligned}
$$

- Thus $E[(X-c)^2] \geq E[(X-EX)^2], \ \forall c$
- So, $E[(X-c)^2]$ is minimized when $c = EX$ and the minimum value is $E[(X-EX)^2]$

Variance of a Random variable

- We define variance of X as $E[(X - EX)^2]$ and denote it as $\mathsf{Var}(X)$.
- By definition, $\mathsf{Var}(X) \geq 0$.

$$\begin{aligned}
\mathsf{Var}(X) &= E[(X - EX)^2] \\
&= E\left[X^2 + (EX)^2 - 2X(EX)\right] \\
&= E[X^2] + (EX)^2 - 2(EX)E[X] \\
&= E[X^2] - (EX)^2
\end{aligned}$$

- This also implies: $E[X^2] \geq (EX)^2$

Some properties of variance

- $\text{Var}(X + c) = \text{Var}(X)$ where c is a constant

$$\text{Var}(X+c) = E\left[\{(X+c) \ - \ E[X+c]\}^2\right] = E\left[(X - EX)^2\right] = \text{Var}(X)$$

- $\text{Var}(cX) = c^2\text{Var}(X)$ where c is a constant

$$\text{Var}(cX) = E\left[(cX - E[cX])^2\right] = E\left[(cX - cE[X])^2\right] = c^2\text{Var}(X)$$

Variance of uniform rv

- $f_X(x) = \frac{1}{b-a}, \; a \le x \le b$

$$
\begin{aligned}
E[X^2] &= \int_a^b x^2 \, \frac{1}{b-a} \, dx \\
&= \frac{1}{b-a} \; \left. \frac{x^3}{3} \right|_a^b \\
&= \frac{1}{b-a} \, \frac{b^3 - a^3}{3} \\
&= \frac{b^2 + ab + a^2}{3}
\end{aligned}
$$

Variance of uniform rv

- We got $E[X^2] = \frac{b^2+ab+a^2}{3}$. Earlier we showed $EX = \frac{b+a}{2}$
- Now we can calculate $\text{Var}(X)$ as

$$
\begin{aligned}
\text{Var}(X) &= EX^2 - (EX)^2 \\
&= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} \\
&= \frac{4(b^2+ab+a^2) - 3(b^2+2ab+a^2)}{12} \\
&= \frac{(b^2-2ab+a^2)}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

Variance of exponential rv

- $f_X(x) = \lambda \, e^{-\lambda x}, \; x > 0$

$$
\begin{aligned}
E[X^2] &= \int_0^\infty x^2 \, \lambda \, e^{-\lambda x} \, dx \\
&= x^2 \, \lambda \, \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \int_0^\infty \lambda \, \frac{e^{-\lambda x}}{-\lambda} \, 2x \, dx \\
&= \frac{2}{\lambda} \int_0^\infty x \, \lambda \, e^{-\lambda x} \, dx \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

- Hence the variance is now given by

$$
\mathsf{Var}(X) = \frac{2}{\lambda^2} \; - \; \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}
$$

Variance of Gaussian rv

- Let $X \sim \mathcal{N}(0,1)$. That is, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ -\infty < x < \infty$.
- We know $EX = 0$. Hence Var$(X) = EX^2$.

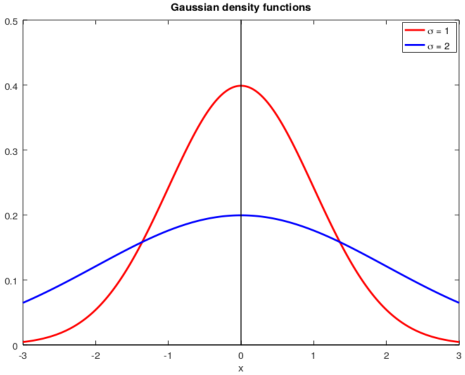
$$
\begin{aligned}
\text{Var}(X) &= EX^2 = \int_{-\infty}^{\infty} x^2 \, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\
&= \int_{-\infty}^{\infty} x \, \left(x \, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) \, dx \\
&= x \, \left. \frac{-1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\
&= 1
\end{aligned}
$$

- Let $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \; -\infty < x < \infty$.
- Let $g(x) = \sigma x + \mu$ and hence $g^{-1}(y) = \frac{y-\mu}{\sigma}$.
- Take $\sigma > 0$ and $Y = g(X)$. By the theorem,

$$f_Y(y) = \left(\frac{d}{dy} g^{-1}(y)\right) \; f_X(g^{-1}(y)) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

- Since $Y = \sigma X + \mu$, we get
  - $EY = \sigma EX + \mu = \mu$
  - $\text{Var}(Y) = \sigma^2 \text{Var}(X) = \sigma^2$
- When $Y \sim \mathcal{N}(\mu, \sigma^2)$, $EY = \mu$ and $\text{Var}(Y) = \sigma^2$.

- Here is a plot of Gaussian densities with different variances



Variance of Binomial rv

- $f_X(k) = \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k},\ k = 0, 1, \cdots, n$
- Here we use the identity, $EX^2 = E[X(X-1)] + EX$

$$\begin{aligned} E[X(X-1)] &= \sum_{k=0}^{n} k(k-1)\, \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k} \\ &= \sum_{k=2}^{n} k(k-1)\, \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k} \end{aligned}$$

$$= \sum_{k=2}^{n} \frac{n(n-1)(n-2)!}{(k-2)!((n-2)-(k-2))!}\, p^2\, p^{k-2}\, (1-p)^{(n-2)-(k-2)}$$

$$= n(n-1)p^2 \sum_{k'=0}^{n-2} \frac{(n-2)!}{k'!((n-2)-k')!}\, p^{k'}\, (1-p)^{(n-2)-k'}$$

$$= n(n-1)p^2$$

- When X is binomial rv, we showed, $E[X(X-1)] = n(n-1)p^2$
- Hence,

$$EX^2 = E[X(X-1)] + EX = n(n-1)p^2 + np = n^2p^2 + np(1-p)$$

- Now we can calculate the variance

$$\text{Var(X)} = EX^2 - (EX)^2 = n^2p^2 + np(1-p) - (np)^2 = np(1-p)$$

Variance of a geometric random variable

- $X \in \{1, 2, \cdots\}$ and $f_X(k) = (1-p)^{k-1}p, \;\; k = 1, 2, \cdots$
- Here also, it is easier to calculate $E[X(X-1)]$

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)(1-p)^{k-1}p = p(1-p)\sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2}$$

- We know

$$\sum_{k=1}^{\infty}(1-p)^k = \frac{1-p}{p} \;\Rightarrow\; \sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2} = \frac{d^2}{dp^2}\left(\frac{1-p}{p}\right)$$

Now you can compute $E[X(X-1)]$ and hence $E[X^2]$ and hence $\mathrm{Var}(X)$ and show it to be equal to $\frac{1-p}{p^2}$.
(Left as an exercise)

moments of a random variable

- We define the k^{th} order moment of a rv, X, by

$$m_k = E[X^k] = \int x^k \, dF_X(x)$$

- $m_1 = EX$ and $m_2 = EX^2$ and so on
- We define the k^{th} central moment of X by

$$s_k = E[(X - EX)^k] = \int (x - EX)^k \, dF_X(x)$$

- $s_1 = 0$ and $s_2 = \text{Var}(X)$.
- Not all moments may exist for a given random variable. (For example, m_1 does not exist for Cauchy rv)

- **Theorem**: If $E\left[|X|^k\right] < \infty$ then $E\left[|X|^s\right] < \infty$ for $0 < s < k$.
- For example, if third order moment exists then so do first and second order moments
- **Proof**: We prove it when X is continuous rv. Proof for discrete case is similar.

$$
\begin{aligned}
E\left[|X|^s\right] &= \int_{-\infty}^{\infty} |x|^s \, f_X(x) \, dx \\
&= \int_{|x|<1} |x|^s \, f_X(x) \, dx + \int_{|x|\geq 1} |x|^s \, f_X(x) \, dx \\
&\leq \int_{|x|<1} f_X(x) \, dx + \int_{|x|\geq 1} |x|^s \, f_X(x) \\
&\leq P[|X|^s < 1] + \int_{|x|\geq 1} |x|^k \, f_X(x) \\
&\qquad \text{since for } |x| \geq 1, \ |x|^s < |x|^k \text{ when } s < k \\
&< \infty \quad \text{because } E\left[|X|^k\right] = \int_{\infty}^{\infty} |x|^k f_X(x) \, dx < \infty
\end{aligned}
$$

Moment generating function

- The moment generating function (mgf) of rv X, $M_X : \Re \to \Re$, is defined by

$$M_X(t) = Ee^{tX} = \sum_i e^{tx_i} f_X(x_i) \text{ or } \int e^{tx} f_X(x)\, dx, \;\; t \in \Re$$

- We say the mgf exists if $E[e^{tX}] < \infty$ for t in some interval around zero
- The mgf may not exist for some random variables.

- The mgf of X is: $M_X(t) = E[e^{tX}]$.
- If $M_X(t)$ exists (for $t \in [-a,\ a]$ for some $a > 0$) then all its derivatives also exist.
- Then we can get the moments of X by successive differentiation of $M_X(t)$.

$$\left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{d}{dt} E\left[e^{tX}\right]\right|_{t=0} = E[Xe^{tX}]\big|_{t=0} = EX$$

- In general

$$\left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0} = E[X^k]$$

- We can easily see this by expanding e^{tX} in Taylor series:

$$\begin{aligned} M_X(t) &= Ee^{tX} = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \cdots\right] \\ &= 1 + \frac{t}{1!}EX + \frac{t^2}{2!}EX^2 + \frac{t^3}{3!}EX^3 + \frac{t^4}{4!}EX^4 + \cdots \end{aligned}$$

- Now we can do term-wise differentiation. For example

$$\frac{d^3M_X(t)}{dt^3} = 0+0+0+\frac{3*2*1*t^0}{3!}EX^3+\frac{4*3*2*t}{4!}EX^4+\cdots$$

- Hence we get

$$\left.\frac{d^3M_X(t)}{dt^3}\right|_{t=0} = E[X^3]$$

Example – Moment generating function for Poisson

- $f_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \; k = 0, 1, \cdots$

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \, \frac{\lambda^k}{k!} \, e^{-\lambda} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda e^t\right)^k \\ &= e^{-\lambda} \, e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

- Now, by differentiating it we can find EX

$$EX = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = e^{\lambda(e^t-1)} \, \lambda e^t \Big|_{t=0} = \lambda$$

(Exercise: Differentiate it twice to find EX^2 and hence show that variance is λ).

mgf of exponential rv

- $f_X(x) = \lambda e^{-\lambda x}, \; x > 0$

$$\begin{aligned}
M_X(t) &= E[e^{tX}] = \int_0^\infty e^{tx}\, \lambda e^{-\lambda x}\, dx \\
&= \int_0^\infty \lambda e^{-x(\lambda - t)}\, dx \\
&\qquad\qquad \text{This is finite if } \; t < \lambda \\
&= \left. \frac{\lambda e^{-x(\lambda-t)}}{-(\lambda - t)} \right|_0^\infty \\
&= \frac{\lambda}{\lambda - t}, \;\; t < \lambda
\end{aligned}$$

- We can use this to compute EX

$$EX = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = \left. \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right) \right|_{t=0} = \left. \frac{\lambda}{(\lambda - t)^2} \right|_{t=0} = \frac{1}{\lambda}$$

- For mgf to exist we need $E[e^{tX}] < \infty$ for $t \in [-a, \ a]$ for some $a > 0$.
- If $M_X(t)$ exists then all moments of X are finite.
- However, all moments may be finite but the mgf may not exist.
- When mgf exists, it uniquely determines the df
- We are not saying moments uniquely determine the distribution; we are saying mgf uniquely determines the distribution

Characteristic Function

- The characteristic function of X is defined by

$$\phi_X(t) = E[e^{itX}] = \int e^{itx}\, dF_X(x) \quad (i = \sqrt{-1})$$

- If X is continuous rv,

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx}\, f_X(x)\, dx$$

- Characteristic function always exists because $|e^{itx}| = 1, \forall t, x$
- For example,

$$\left| \int_{-\infty}^{\infty} e^{itx}\, f_X(x)\, dx \right| \leq \int_{-\infty}^{\infty} \left| e^{itx} \right|\, |f_X(x)|\, dx = \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

- We would consider ϕ_X later in the course

Generating function

- Let $X \in \{0, 1, 2, \cdots\}$
- The (probability) generating function of X is defined by

$$P_X(s) = \sum_{k=0}^{\infty} f_X(k) s^k, \quad s \in \Re$$

- This infinite sum converges (absolutely) for $|s| \leq 1$.
- We have

$$P_X(s) = f_X(0) + f_X(1)s + f_X(2)s^2 + f_X(3)s^3 + \cdots$$

- The pmf can be obtained from the generating function

- $P_X(s) = f_X(0) + f_X(1)s + f_X(2)s^2 + f_X(3)s^3 + \cdots$
- Let $P'_X(s) \triangleq \frac{dP_X(s)}{ds}$ and so on
- We get

$$P'_X(s) = 0 + f_X(1) + f_X(2)\;2s + f_X(3)\;3s^2 + \cdots$$

$$P''_X(s) = 0 + 0 + f_X(2)\;2*1 + f_X(3)\;3*2s^1 + \cdots$$

Hence, we get

$$f_X(0) = P_X(0);\;\; f_X(1) = \frac{P'_X(0)}{1!};\;\; f_X(2) = \frac{P''_X(0)}{2!}$$

- The moments (when they exist) can be obtained from the generating function: $P_X(s) = \sum_{k=0}^{\infty} f_X(k) s^k$

$$P'_X(s) = \sum_{k=0}^{\infty} k \; f_X(k) \; s^{k-1} \;\Rightarrow\; P'_X(1) = EX$$

$$P''_X(s) = \sum_{k=0}^{\infty} k(k-1) \; f_X(k) \, s^{k-2} \;\Rightarrow\; P''_X(1) = E[X(X-1)]$$

- For (positive integer valued) discrete random variables, it is more convenient to deal with generating functions than mgf.

Example – Generating function for binomial rv

- $f_X(k) = \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k},\; k = 0, 1, \cdots, n$

$$
\begin{aligned}
P_X(s) &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k}\, s^k \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, (sp)^k\, (1-p)^{n-k} \\
&= (sp + (1-p))^n = (1 + p(s-1))^n
\end{aligned}
$$

- From the above, we get $P_X'(s) = n(sp + (1-p))^{n-1} p$
- Thus,

$$EX = P_X'(1) = np; \quad f_X(1) = P_X'(0) = n(1-p)^{n-1} p$$

- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

$$P[X \leq x] \geq p \quad \text{and} \quad P[X \geq x] \geq 1-p$$

  is called the quantile of order p or the $100p^{th}$ percentile of rv X.
- Suppose x is a quantile of order p. Then we have
  - $p \leq P[X \leq x] = F_X(x)$
  - $1-p \leq 1 - P[X < x] = 1 - (P[X \leq x] - P[X = x])$
    $\Rightarrow \ \ 1-p \leq 1 - F_X(x) + P[X = x]$
    $\Rightarrow F_X(x) \leq p + P[X = x]$
- Thus, x satisfies (if it is quantile of order p)

$$p \leq F_X(x) \leq p + P[X = x]$$

- Note that for a given p there can be multiple values for x to satisfy the above.