Recap: Continuous-Time Markov Chains

- $\{X(t),\ t \geq 0\}$ is a continuous-time Markov chain if

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- It is called homogeneous chain if

$$Pr[X(t+s) = j \mid X(s) = i] = Pr[X(t) = j \mid X(0) = i],\ \forall s$$

Recap: Transition Structure

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- Note that $P_{ij}(t)$ is different from these z_{ij}

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- Poisson process is a special case with $\lambda_i = \lambda$ and $\mu_i = 0$

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$$\lambda_n = \lambda, \ n \geq 0 \quad \text{and} \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq K \\ K\mu & n > K \end{cases}$$

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$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \left(\sum_{k \neq i} q_{ik} \right) P_{ij}(t), \ \forall i, j$$

where

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- The q_{ij} – the infinitesimal generator of the chain.
- q_{ij} is the rate of transitions out of i into j
- For a birth-death chain, $q_{i,i+1} = \lambda_i$ and $q_{i,i-1} = \mu_i$

Recap: Obtaining $P_{ij}(t)$

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Recap: 2-state birth-death chain

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P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\
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- We get a system of equations like this for any finite chain.

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- The Q matrix has elements q_{ij} defined earlier with $q_{ii} = -\sum_{k \neq i} q_{ik}$.

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- This is known as Kolmogorov forward equation

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- There is no concept of periodicity in the continuous time case
- An irreducible positive recurrent chain would be called an ergodic chain

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- The above equation is true for general infinite chains.
- For a finite chain, taking π as a row vector,

$$\pi(t) = \pi(0)\ P(t) = \pi(0)\ e^{Qt}$$

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- The above equation is known as a balance equation

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- We first start with a simpler process, namely, the random walk.

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- Called a (one dimensional) random walk

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- Viewed as a discrete-time process, $X(nT)$, is a Markov chain.
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- We could have also use piece-wise linear interpolation

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- We are interested in limit of this process as $T \to 0$

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- Let us intuitively see some properties of $W(t)$

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$$\Rightarrow \quad W(t) \sim \mathcal{N}(0,\ \alpha t)$$

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- We will now formally define Brownian motion using these properties.

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- The mean can be a function of time