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  - Ω is the sample space
  - $\mathcal{F} \subset 2^\Omega$ set of events; each event is a subset of Ω
  - $P : \mathcal{F} \to [0, 1]$ is a probability (measure) that satisfies the three axioms:

    A1 $P(A) \geq 0$, $\forall A \in \mathcal{F}$

    A2 $P(\Omega) = 1$

    A3 If $A_i \cap A_j = \phi, \forall i \neq j$ then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

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- When Ω is finite with n elements, a special case is $q_i = \frac{1}{n}, \forall i$. (All outcomes equally likely)
- The idea of 'equally likely' assignment can be extended when Ω is a subset of $\Re^n$

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- This holds for multiple event, e.g., $P(ABC) = P(A|BC)P(B|C)P(C)$
- Given a partition, $\Omega = B_1 + B_2 + \cdots + B_m$, for any event, A,

$$P(A) = \sum_{i=1}^{m} P(A|B_i)P(B_i) \quad \text{(Total Probability rule)}$$

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- In general we have

$$P(D_i|T) = \frac{P(T|D_i)P(D_i)}{\sum_j^n P(T|D_j)P(D_j)}$$

where $D_1, \cdots, D_n$ form a partition of Ω

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- If A, B are independent then so are $A \& B^c$, $A^c \& B$ and $A^c \& B^c$.

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$$P(A|BC) = \frac{P(BC|A)P(A)}{P(BC|A)P(A) + P(BC|A^c)P(A^c)}$$

- The above gets simplified if we assume
$P(BC|A) = P(B|A)P(C|A)$,
$P(BC|A^c) = P(B|A^c)P(C|A^c)$

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- For this, we need to first define limit of a sequence of sets.

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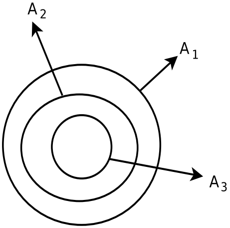

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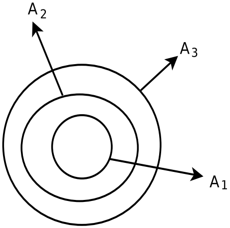

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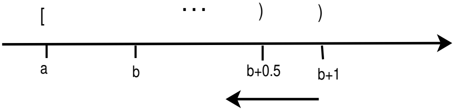

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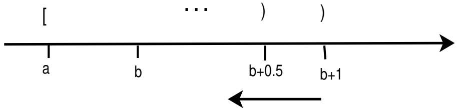

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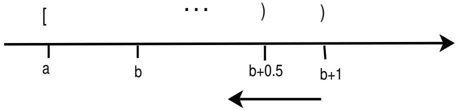

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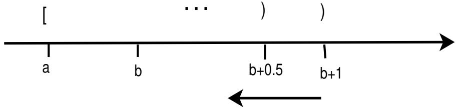

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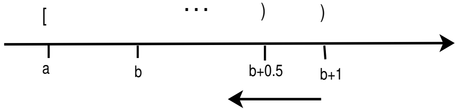

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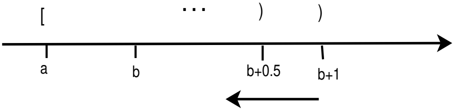

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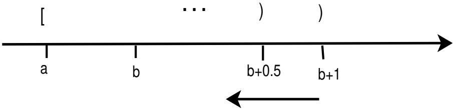

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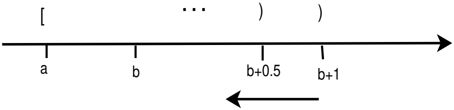

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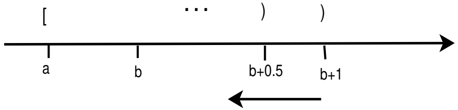

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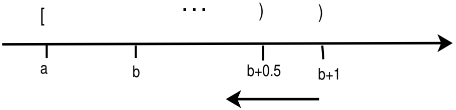

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    For example, $b + 0.01 \notin A_{101} = [a,\ b + \frac{1}{101})$.

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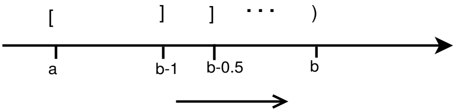

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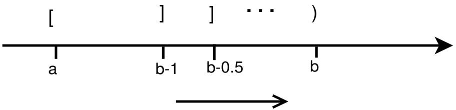

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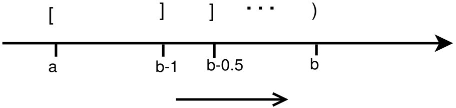

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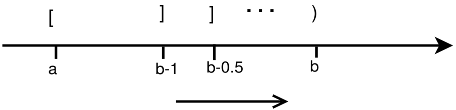

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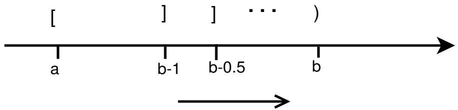

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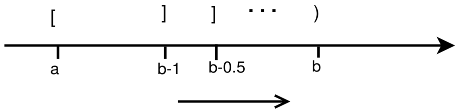

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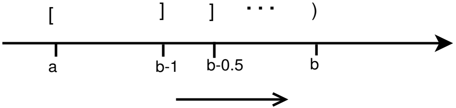

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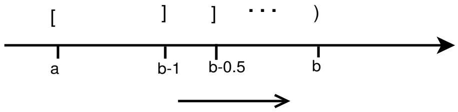

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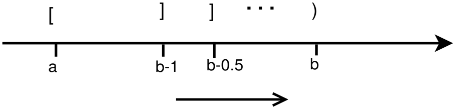

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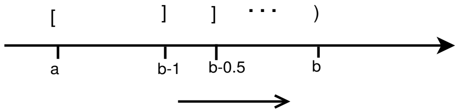

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- These examples also show how using countable unions or intersections we can "convert" one end of an interval from *open* to *closed* or vice versa.

- To summarize, limits of monotone sequences of events are defined as follows

$$A_n \downarrow \quad \lim_{n\to\infty} A_n = \cap_{k=1}^{\infty} A_k$$

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- Having defined the limits, we now ask the question

$$P\left(\lim_{n\to\infty} A_n\right) \stackrel{?}{=} \lim_{n\to\infty} P(A_n)$$

where we assume the sequence is monotone.

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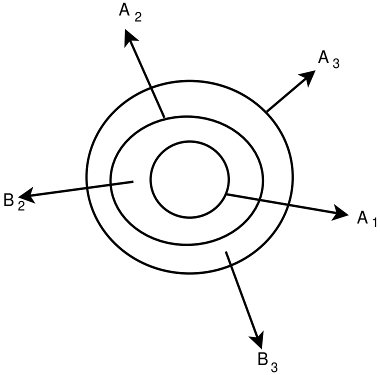

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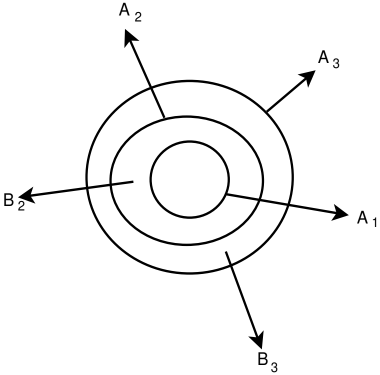

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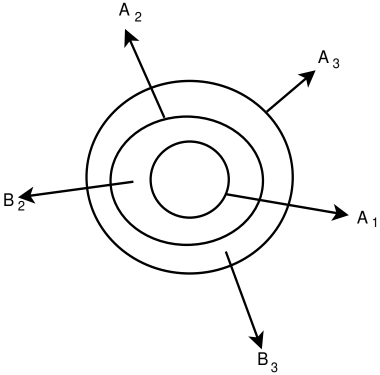

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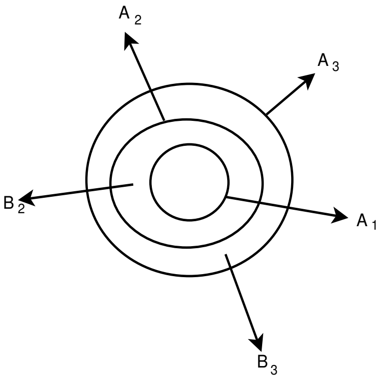

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- If we toss the coin for any fixed N times then we know the sample space can be $\{0,1\}^N$.
- But for our problem, we can not put any fixed limit on the number of tosses and hence our sample space should be for infinite tosses of a coin.

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- Thus "no head in the first n tosses" would be an event.

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- Now we can complete problem

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$$P(\cap_{k=1}^{\infty} A_k) = P(\lim_n A_n) = \lim_n P(A_n)$$

- For $n = 1, 2, \cdots$, define

$$A_n = \{(\omega_1, \omega_2, ...) \ : \ \omega_i = 0, \ i = 1, \cdots, n\}$$

- A_n is the event of no head in the first n tosses and we know $P(A_n) = (0.5)^n$.
- Note that $\cap_{k=1}^{\infty} A_k$ is the event we want.
- Note that $A_n \downarrow$ because $A_{n+1} \subset A_n$.
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- So, uncountable Ω arise naturally if we want to consider infinite repetitions of a random experiment

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- As we already saw, the probability of this event is $(0.5)^2$ which is the length of this interval

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- This is called Borel's normal number theorem and is an interesting result about real numbers.

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  - The tosses are independent
  - Probability of 0 or 3 heads is $1/8$ while that of 1 or 2 heads is $3/8$

- Now consider a P_2 (different from P_1) on the same Ω

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- If we think tosses are independent then we choose P_1. But if we need to model some dependence among tosses, we choose a model like P_2.

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- It is also a useful model.

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This entire course can be considered as studying different random variables.

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- It essentially means we can treat all outcomes as real numbers.
- We can effectively work with $\Re$ as sample space in all probability models

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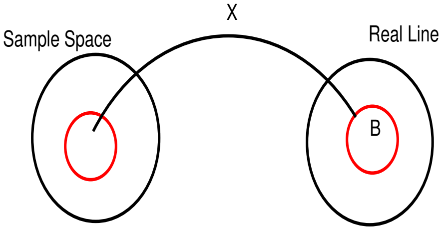

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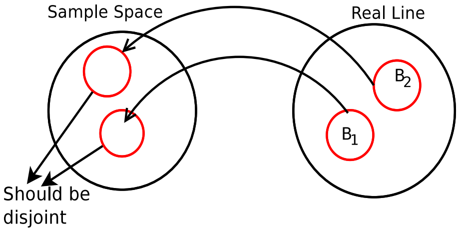

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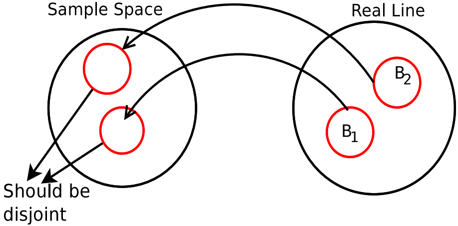

$$P[X \in B_1 \cup B_2] = P[X \in B_1] + P[X \in B_2] = P_X(B_1) + P_X(B_2)$$