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- Any $F : \Re^2 \to \Re$ satisfying the above would be a joint distribution function.

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$$P[(X, Y) \in B] = \sum_{\substack{i,j: \\ (x_i, y_j) \in B}} f_{XY}(x_i, y_j)$$

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- Two cont rv X, Y have a joint density f_{XY} if

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- We also have

$$P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}\, dy\, dx$$

  and, $P[(X,Y) \in B] = \int_B f_{XY}(x,y)\, dx\, dy, \ \ \forall B \in \mathcal{B}^2$

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- If X, Y have joint pdf f_{XY} then the marginal pdf are

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- For each y, $F_{X|Y}(x|y)$ is a df in x.
- If X, Y have a joint density or if X is continuous and Y is discrete, $F_{X|Y}$ would have a density.

Recap: Contional density (or mass) fn

- Let X be a discrete random variable. Conditional mass fn of X conditioned on Y is

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This exists if X, Y have a joint density or when Y is discrete.

Recap

- When X, Y are both discrete or they have a joint density

$$f_{XY}(x, y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

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- The above relation gives rise to the total probability rules and Bayes rule for rv's

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- If X is also continuous, the $f_X, f_{X|Y}$ are densities; If X is discrete, they are mass functions (Where needed we assume the conditional density exists)

Recap: Bayes rule

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- We need to interpret f_X or $f_{X|Y}$ as mass functions when X is discrete and as densities when X is a continuous and so on

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- If they are continuous, they have a joint density if

$$F_{XYZ}(x, y, z) = \int_{-\infty}^{z} \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XYZ}(x', y', z')\, dx'\, dy'\, dz'$$

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  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \; f_{XYZ}(x, y, z) \; dx \; dy \; dz = 1$

Recap

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- For example when Z is continuous

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- Independence implies that the marginals would determine the joint distribution.
- If $X_1, X_2, \cdots, X_n$ are independent and if each X_i has the same distribution, they are said to be **independent and identically distributed** or **iid** random variables.

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- For example, if X, Y are discrete, then

$$f_Z(z) = P[Z = z] = P[g(X, Y) = z] = \sum_{\substack{x_i, y_j: \\ g(x_i, y_j) = z}} f_{XY}(x_i, y_j)$$

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- Is this correct for all values of z, w?

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- $X_{(1)} = \min(X_1, \cdots, X_n), \quad X_{(n)} = \max(X_1, \cdots, X_n)$
- The joint distribution of $X_{(1)}, \cdots X_{(n)}$ is called the order statistics.

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- Let $X_1, \cdots, X_n$ be iid with density f.
- Let $X_{(k)}$ denote the k^{th} smallest of these.
- That is, $X_{(k)} = g_k(X_1, \cdots, X_n)$ where $g_k : \Re^n \to \Re$ and the value of $g_k(x_1, \cdots, x_n)$ is the k^{th} smallest of the numbers $x_1, \cdots, x_n$.
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- Earlier, we calculated the order statistics for the case $n = 2$.
- It can be shown that

$$f_{X_{(1)} \cdots X_{(n)}}(x_1, \cdots x_n) = n! \prod_{i=1}^{n} f(x_i), \quad x_1 < x_2 < \cdots < x_n$$

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- Hence we get

$$F_{X_{(k)}}(y) = \sum_{j=k}^{n} {}^nC_j(F(y))^j(1 - F(y))^{n-j}$$

We can get the density by differentiating this.

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- This gives us the density of Z

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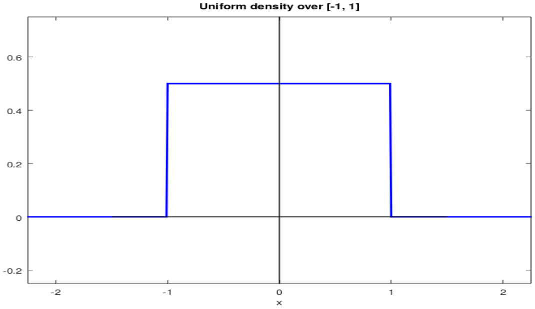

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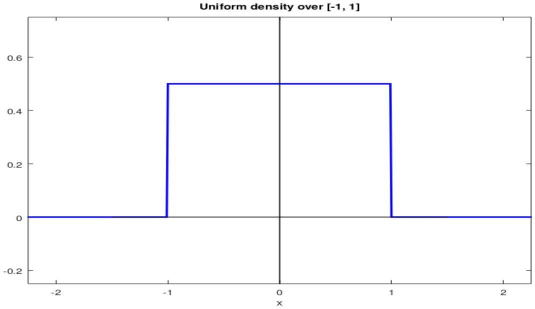

- f_Z is convolution of this density with itself.

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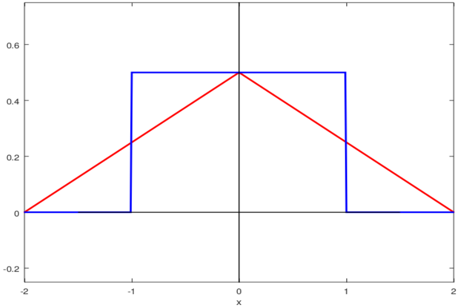

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- If $\mathbf{X}, \mathbf{Y}$ are vector random variables (or random vectors), independence implies $[\mathbf{X} \in B_1]$ is independent of $[\mathbf{Y} \in B_2]$ for all borel sets B_1, B_2 (in appropriate spaces).
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- Then $g(\mathbf{X})$ would be independent of $h(\mathbf{Y})$.
- That is, suppose $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are independent.
- Then, $g(X_1, \cdots, X_m)$ is independent of $h(Y_1, \cdots, Y_n)$.

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- Exercise for you: Find density of $X_1 + X_2 + X_3$ where X_1, X_2, X_3 are iid uniform over $(0,\ 1)$.

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- Thus, sum of independent exponential random variables has gamma distribution:

$$f_Z(z) = \lambda z\, \lambda e^{-\lambda z}, \quad z > 0$$

Sum of independent gamma rv

- Gamma density with parameters $\alpha > 0$ and $\lambda > 0$ is given by
$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha \, x^{\alpha-1} \, e^{-\lambda x}, \quad x > 0$$

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Because

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- There is a calculation trick that is often useful with Gaussian density

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because

$$
\frac{1}{\sqrt{2\pi K}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-b)^2}{2K}\right)\ dx = 1
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- This result is only for continuous random variables.

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- Let h be the inverse of g. That is

$$X_1 = h_1(Y_1, \cdots, Y_n) \quad \cdots \quad X_n = h_n(Y_1, \cdots, Y_n)$$

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- Each of g_i, h_i are $\Re^n \to \Re$ functions and we can write them as

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  We denote the partial derivatives of these functions by $\frac{\partial x_i}{\partial y_j}$ etc.

- The jacobian of the inverse transformation is

$$J = \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

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- **Theorem**: Under the above conditions, we have

$$f_{Y_1 \cdots Y_n}(y_1, \cdots, y_n) = |J| f_{X_1 \cdots X_n}\left(h_1(y_1, \cdots, y_n), \cdots, h_n(y_1, \cdots, y_n)\right)$$

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Or, more compactly, $f_{\mathbf{Y}}(\mathbf{y}) = |J| f_{\mathbf{X}}(h(\mathbf{y}))$

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- Called multidimensional change of variable formula

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- Now we get density of Z as

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)\, dw$$