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- The index need not necessarily represent time. It can represent, for example, space coordinates.

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- The Markov chain we considered is a discrete-time discrete-state process

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- We will denote the random variables as X_t or $X(t)$

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- As we saw, for a Markov chain, π_0 and P together specify all such joint distributions

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  We are using the same symbol, F_X for all.

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- The second order distributions:

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- The n^{th} order distributions:

$$F_X(x_1, \cdots, x_n; t_1, \cdots t_n) = Pr[X_{t_i} \leq x_i,\ i = 1, \cdots, n]$$

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- If all X_t are continuous random variables and if all distributions have density functions, then we denote joint density of $X_{t_1}, \cdots, X_{t_n}$ by $f_X(x_1, \cdots, x_n; t_1, \cdots t_n)$

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- One example is the Markovian assumption.
- As we saw, in a Markov chain, the transition probabilities and initial state probabilities would determine all the distributions
- Another such useful assumption is what is called a process with independent increments

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- We only need up to second order distributions

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- Another example of how all finite dimensional distributions of a process can be specified.

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- This is a rather stringent condition and is often referred to as strict-sense stationarity

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  Autocorrelation depends only on the time difference

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- The process is wide-sense stationary if the first and second order distributions are invariant to translation of time origin
- This is the definition normally used in mathematics books.

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The autocorrelation function of a wide-sense stationary process is a symmetric function of (single) time variable.

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This gives: $R_X(0) \geq R_X(\tau)$.

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- It plays an important role in analysis of linear systems with random inputs

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- But, thus is not implied by law of large numbers; $X(i)$ need not be uncorrelated (e.g., Markov chain)

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- One sufficient condition could be that covariance between $X(t)$ and $X(t+\tau)$ decreases fast with increasing τ.

- Define

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- Hence it is enough if we show
$$\sigma_\tau^2 \triangleq E\left[(\eta_\tau - \eta)^2\right] \ \to \ 0, \quad \text{as} \quad \tau \to \infty$$

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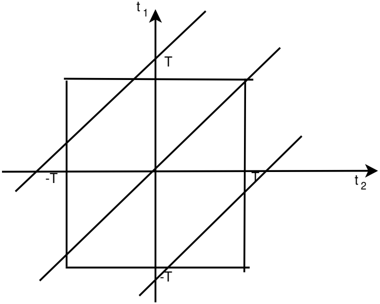

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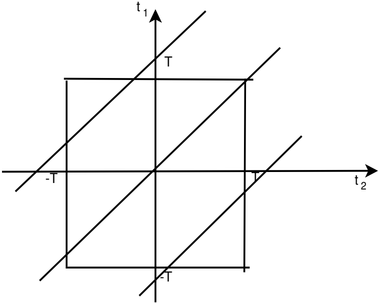

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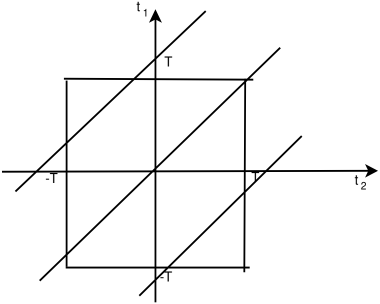

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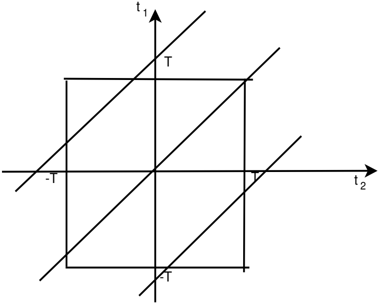

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- This is a sufficient condition for the process being mean-ergodic

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- Similar relation holds for discrete time processes also.