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- The $X(t)$ is defined by piece-wise constant interpolation of $X(nT)$
- For large n, $\frac{X(nT)}{s\sqrt{n}}$ would be Gaussian

$$Pr\left[\frac{X(nT)}{s\sqrt{n}} \leq y\right] \approx \Phi(y)$$

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- We intuitively expect $W(t)$ to be a process with stationary and independent increments and for each t, $W(t)$ is Gaussian with zero mean and variance proportional to t

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- The mean can be a function of time

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- Brownian motion is the limit of random walk where both s and T tend to zero
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- The paths are continuous but non-differentiable everywhere
- This is a deep result

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R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
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- Since $E[X(t)] = 0,\ \forall t$, we have

$$
\text{Cov}(X(t_1), X(t_2)) = E[X(t_1)X(t_2)] = \sigma^2\ \min(t_1, t_2)
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- Thus, $X(t_2) - X(t_1)$ is Gaussian with zero mean and variance $\sigma^2(t_2 - t_1)$
- Since increments are also independent, we can show that all n^{th} order distributions are Gaussian

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- This is how we can get n^{th} order density for any continuous-state process with independent increments

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- We can also write the density using the transformation considered earlier

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- Hence we get

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$$f_X(x_1, \cdots, x_n; t_1, \cdots, t_n) = f_{Y_1}(x_1) f_{Y_2}(x_2 - x_1) \cdots f_{Y_n}(x_n - x_{n-1})$$

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&\propto \exp\left(-x^2\left(\frac{1}{2s}+\frac{1}{2(t-s)}\right)+\frac{bx}{t-s}\right) \\
&\propto \exp\left(-\frac{t}{2s(t-s)}\left(x^2-2\frac{sb}{t}x\right)\right)
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&\propto e^{-\frac{x^2}{2s}}\ e^{-\frac{(b-x)^2}{2(t-s)}} \quad (\text{taking } \sigma^2=1) \\
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\end{aligned}
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- Since all joint densities are Gaussian and are easy to write, we can also calculate conditional densities

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- Hence the conditional density is Gaussian with mean bs/t and variance $s(t-s)/t$

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- Thus, conditioned on $X(s)$ the density of $X(t)$ is normal with mean $X(s)$ and variance $(t - s)$

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Thus

$$P[X(t) \geq a] = 0.5 Pr[T_a \leq t]$$

- Hence we get

$$Pr[T_a \leq t] \;=\; 2\, Pr[X(t) \geq a]$$

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$$P[T_a \le t] = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-\frac{1}{2}y^2}\, dy$$

$$\lim_{t \to \infty} P[T_a \le t] = 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}y^2}\, dy = 1$$

For every a

$$P[T_a < \infty] = 1$$

Ex1: $E[X(t_1)X(t_2)]$ $t_1 < t_2$

$s < t$
$X(t)$ given $X(s)$ mean $X(s)$ var $(t-s)$

$$E[E[X(t_1)X(t_2) \mid X(t_1)]] = E[X(t_1)\cdot E[X(t_2) \mid X(t_1)]]$$
$$= E[X(t_1)\cdot X(t_1)]$$
$$= E[X^2(t_1)]$$

Ex2: $X(t) \sim$ BM with $\sigma^2 = 1$

$$E[X(t_1)X(t_2)X(t_3)] = E[E[X(t_1)X(t_2)X(t_3) \mid X(t_1)X(t_2)]]$$
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$$= E[X(t_1)\,E[X^2(t_2) \mid X(t_1)]]$$
$$= E[X(t_1)(X^2(t_1) + (t_2 - t_1))]$$
$$= E[X(t_1)(t_2-t_1)] + E[X^3(t_1)]$$
$$= 0$$

Ex: Density $X(t) + X(8)$ $\quad t > 8, \ \sigma^2 = 1$

$$X(t) + X(8) = \underbrace{2X(8)}_{} + \underbrace{X(t) - X(8)}_{}$$

$$2X(8) \sim \mathcal{N}(0, 48)$$

$$X(t) + X(8) \sim \mathcal{N}\left(0, \ 48 + (t-8)\right) = \mathcal{N}(0, 3\cdot 8 + t)$$

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- Since $\ln(Y_i)$ are iid, with suitable normalization, the interpolated process $\ln(X(t))$ would be Brownian motion and $X(t)$ would be geometric Brownian motion

Gaussian Processes

- A continuous-time continuous-state process $\{X(t),\ t \geq 0\}$ is said to be a Gaussian process if for all n and all $t_1, t_2, \cdots, t_n$, we have that $X(t_1), \cdots, X(t_n)$ are jointly Gaussian.

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- Recall that the multivariate Gaussian density is specified by the marginal means, variances and the covariances of the random variables
- Hence, a general Gaussian process is specified by the mean function and the variance and covariance functions

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Recall that, for $s < t$, conditional density of $X(s)$ conditioned on $X(t) = b$ is gaussian with mean bs/t and variance $s(t-s)/t$

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\end{aligned}
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Thus, for $0 < t < 1$, conditioned on $X(1) = 0$, this process has mean 0 and covariance function $s(1-t), \ s < t$

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- The actual concept involved is rather deep

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- We may want to know the conditions under which we can prove the sequence converges.