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Recall: Birth-Death chains - stationary distributions

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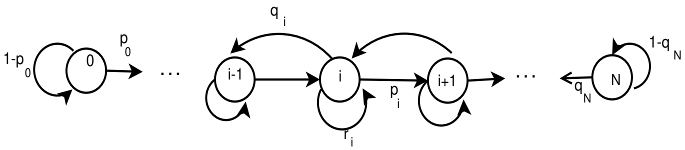

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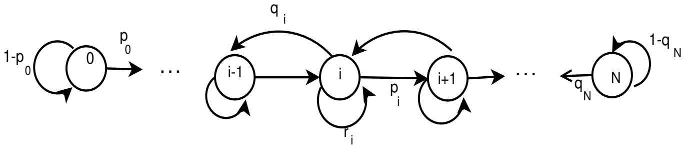

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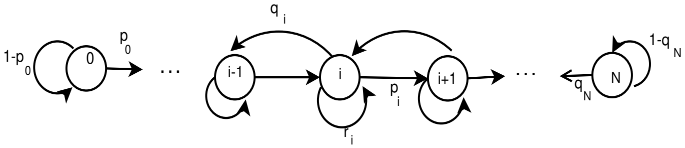

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- This is applicable to an infinite chain also
However, we need $\sum_{j=0}^{N} \eta_j < \infty$ for the stationary distribution to exist.

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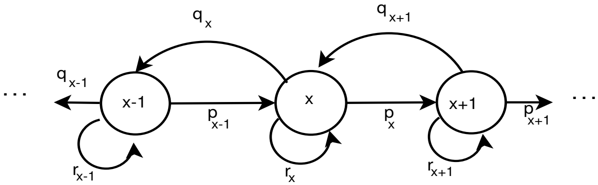
- Consider a finite or infinite birth-death chain



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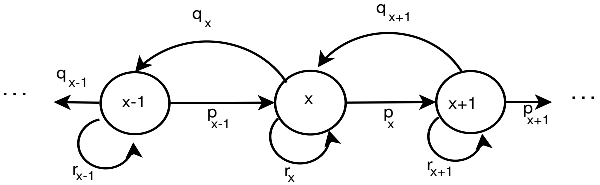

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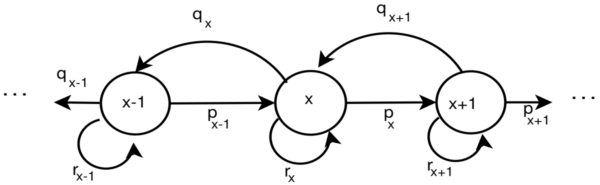

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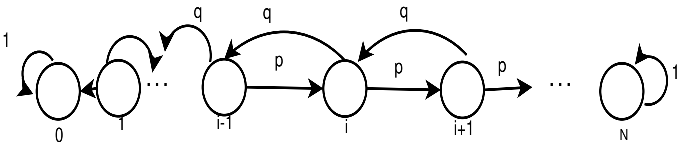

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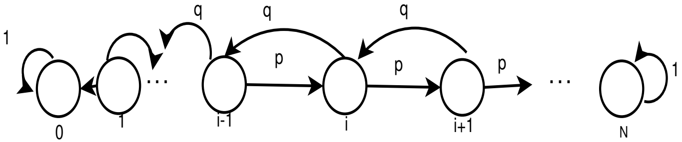

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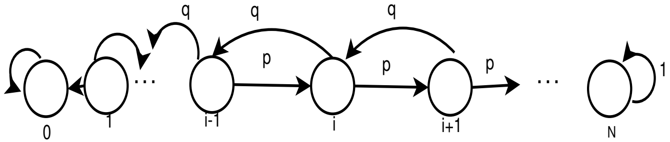

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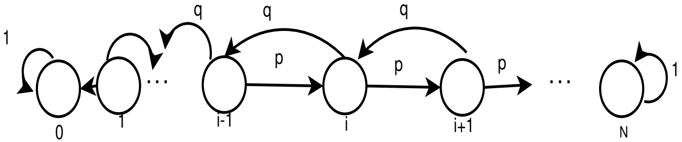

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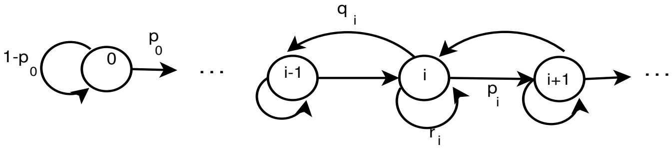

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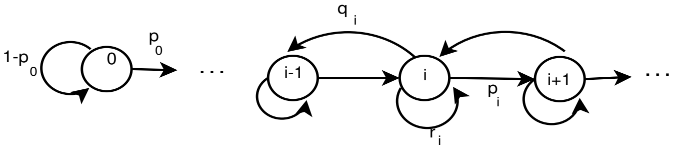

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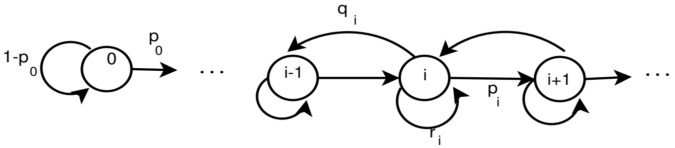

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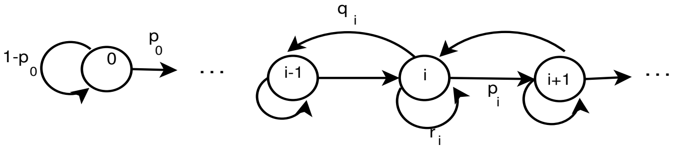

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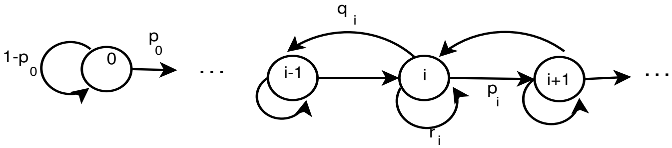

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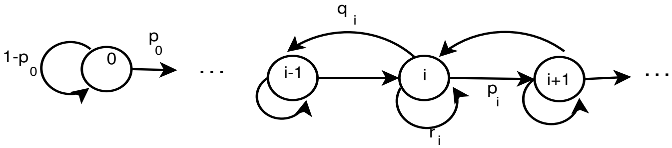

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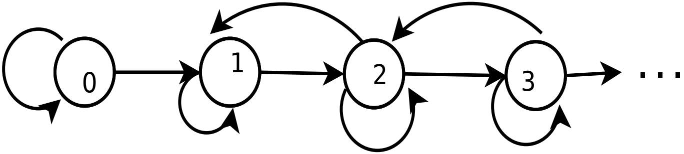

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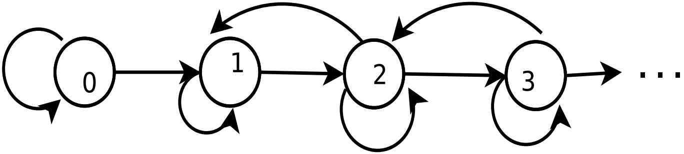

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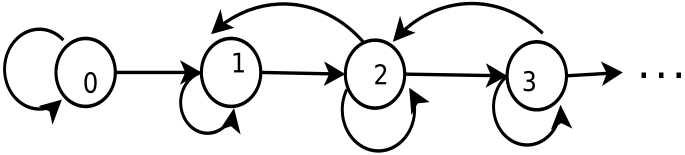

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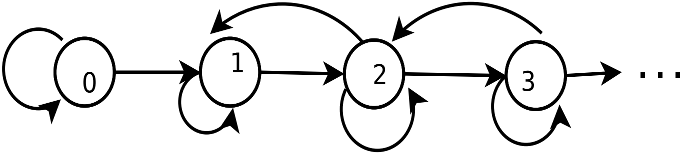

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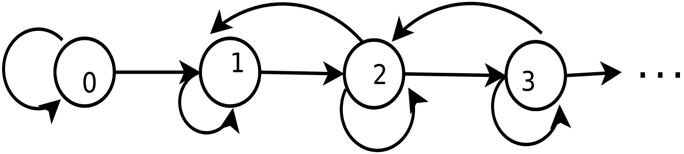

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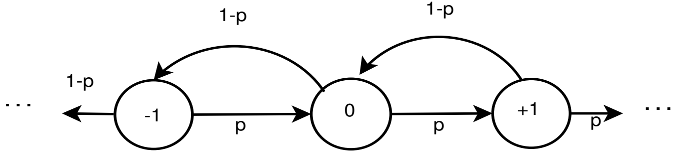

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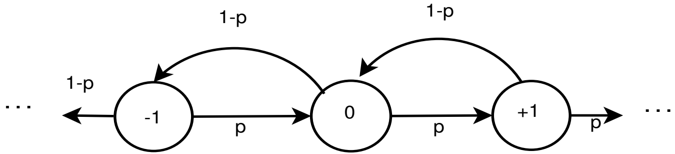

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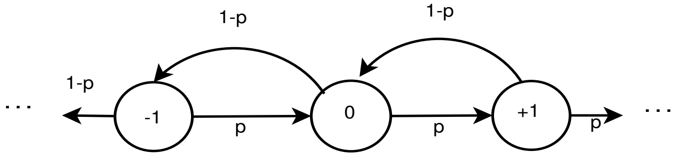

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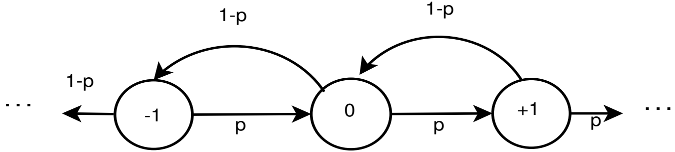

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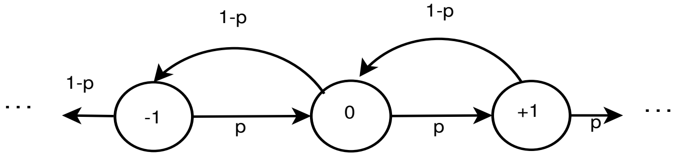

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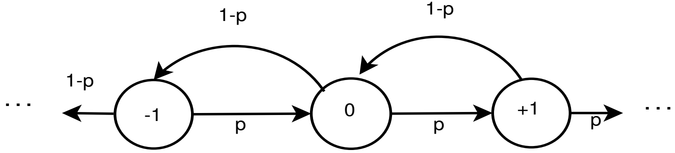

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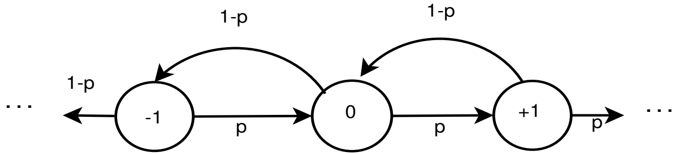

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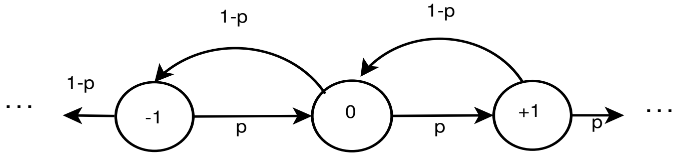

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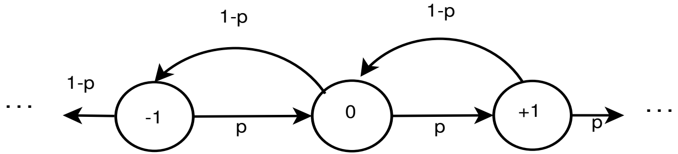

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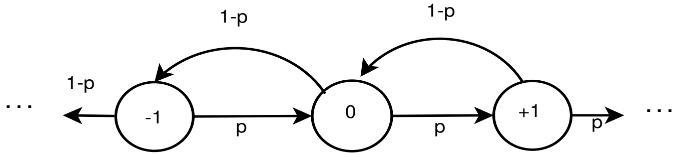

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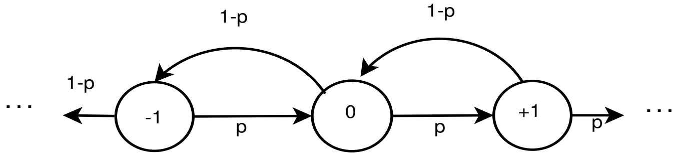

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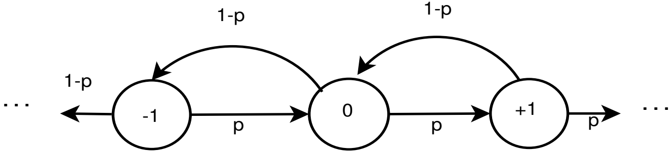

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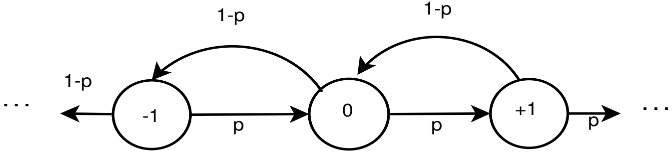

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- This is known as the ergodic theorem for Markov Chains

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- One way to generate samples is to design an ergodic markov chain with stationary distribution π
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- hence, if we can design a Markov chain with a given stationary distribution, we can use that to calculate the expectation.

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- For all these, we need to design a Markov chain with π as stationary distribution

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- Q is called the proposal chain and $\alpha(i,j)$ is called acceptance probabilities

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- We could have chosen Q to be 'uniform over neighbours'

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- We can think of E as the energy function in a Boltzmann distribution

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- Gives rise to interesting optimization technique called simulated annealing

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- This is known as Gibbs sampling