Recap: Markov Chain

- $X_n, \; n = 0, 1, \cdots,$ (with X_i discrete) is a Markov chain if

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- For a Markov chain, given the current state, the future evolution is independent of the history of how you reached the current state

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- If S is finite then P can be represented as a matrix

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- For a finite chain, the n-step transition probability matrix is n-fold product of the (1-step) transition probability matrix

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- Intuitively, all transient states would be visited only finitely many times while recurrent states are visited infinitely often.

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- Expected number of visits:

$$G(x, y) \triangleq E_x[N(y)] = \sum_{n=1}^{\infty} P^n(x, y)$$

(Notation: $E_x[Z] = E[Z|X_0 = x]$)

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(i). Let y be transient. Then

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$$P_x[N(y) = \infty] = \rho_{xy}, \ \text{ and } \ G(x, y) = \begin{cases} 0 & \text{if } \ \rho_{xy} = 0 \\ \infty & \text{if } \ \rho_{xy} > 0 \end{cases}$$

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  Theorem: If x is recurrent and x leads to y then y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.
- On the set of recurrent states, 'leads to' is an equivalence relation

Recap: closed and irreducible sets

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- A set of states, $C \subset S$, is said to be closed if $x \in C,\ y \notin C$ implies $\rho_{xy} = 0$.
- Once the chain visits a state in a closed set, it cannot leave that set.

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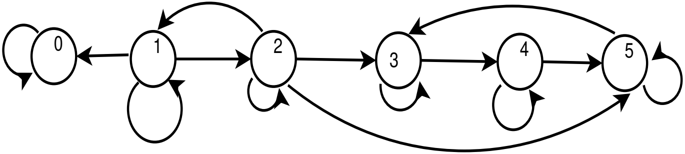
- Eventually the chain spends all its time in one of the C_i

Recap: Example of partition of state space



$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & + & - & - & - & - & - \\
1 & + & + & + & - & - & - \\
2 & - & + & + & + & - & + \\
3 & - & - & - & + & + & - \\
4 & - & - & - & - & + & + \\
5 & - & - & - & + & - & +
\end{array}
$$

- $1, 2$ are transient states.
- We get: $S_T = \{1, 2\}$ and $S_R = \{0\} + \{3, 4, 5\}$

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- By solving this set of linear equations we can get $\rho_c(x), \; x \in S_T$

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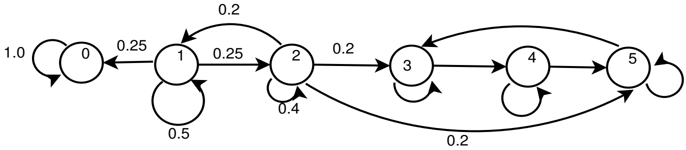

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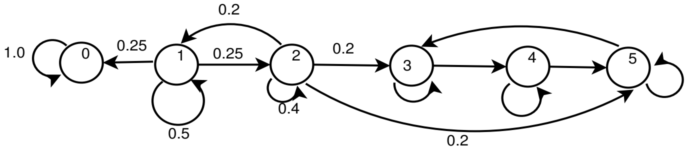

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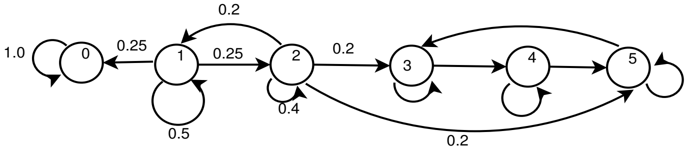

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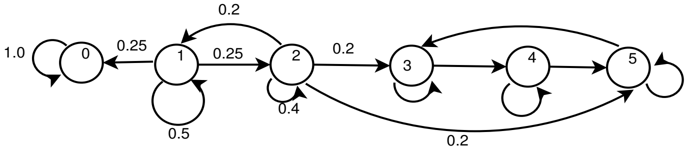

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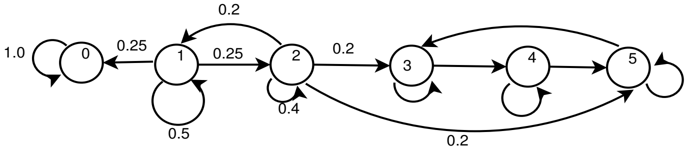

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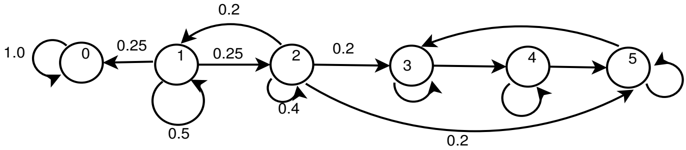

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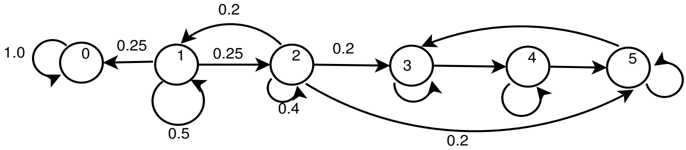

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- Solving these, we get $\rho_{C_1}(1) = 0.6, \quad \rho_{C_1}(2) = 0.2$

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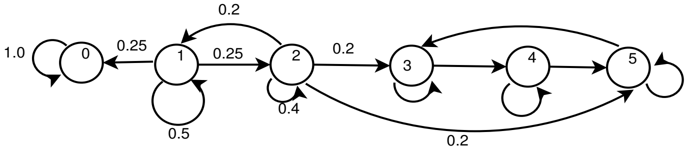

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&= 0.25 + 0.5\rho_{C_1}(1) + 0.25\rho_{C_1}(2) \\
\rho_{C_1}(2) &= 0 + 0.2\rho_{C_1}(1) + 0.4\rho_{C_1}(2)
\end{aligned}$$

- Solving these, we get $\rho_{C_1}(1) = 0.6, \quad \rho_{C_1}(2) = 0.2$
- What would be $\rho_{C_2}(1)$?

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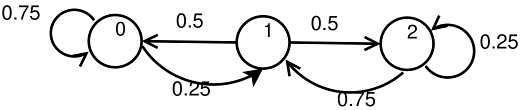
- What about infinite chains?

Example



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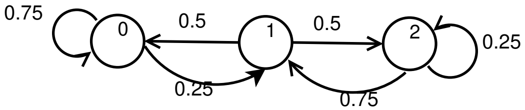

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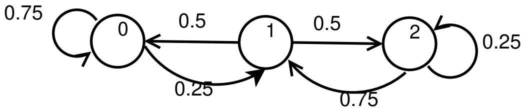

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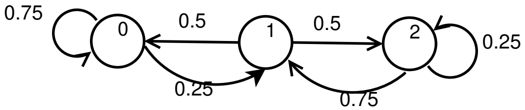

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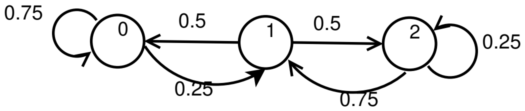

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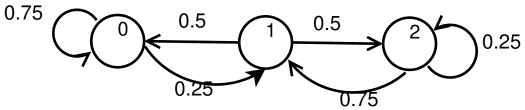

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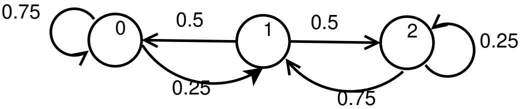
in addition, $\pi(0) + \pi(1) + \pi(2) = 1$



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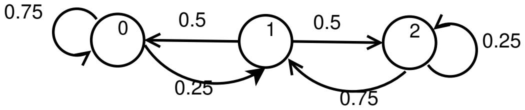
$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.75 & 0.25 \end{bmatrix} = \begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix}$$



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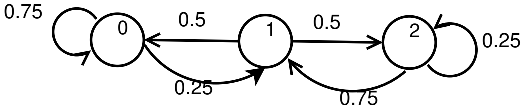

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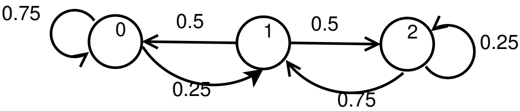
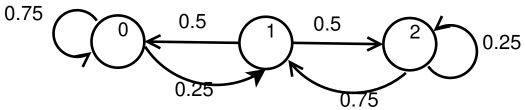
- We have to solve these along with $\pi(0) + \pi(1) + \pi(2) = 1$


0.75
0
0.5
1
0.5
2
0.25
0.25
0.75



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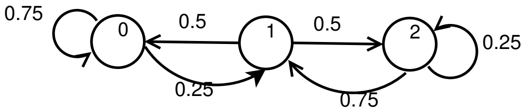

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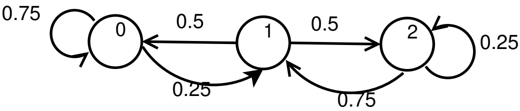

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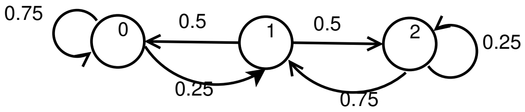

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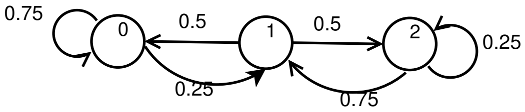

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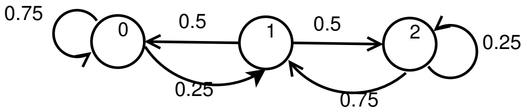
- Now, $\pi(0)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1$ gives $\pi(0) = \frac{6}{11}$



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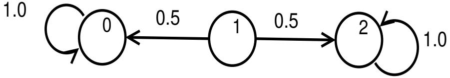
0.75\pi(0) + 0.5\pi(1) &= \pi(0) \quad \Rightarrow \quad \pi(1) = \frac{1}{2}\,\pi(0) \\
0.25\pi(0) + 0.75\pi(2) &= \pi(1) \quad \Rightarrow \quad \pi(2) = \frac{1}{3}\pi(0) \\
0.5\pi(1) + 0.25\pi(2) &= \pi(2) \\
\pi(0) + \pi(1) + \pi(2) &= 1 \quad \Rightarrow \quad \pi(0)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1
\end{aligned}
$$

- Now, $\pi(0)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1$ gives $\pi(0) = \frac{6}{11}$
- We get a unique solution: $\left[\frac{6}{11} \quad \frac{3}{11} \quad \frac{2}{11}\right]$

Example2



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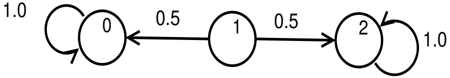

- The stationary distribution has to satisfy

$$\left[\begin{array}{ccc}\pi(0) & \pi(1) & \pi(2)\end{array}\right]\left[\begin{array}{ccc}1.0 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1.0\end{array}\right]=\left[\begin{array}{ccc}\pi(0) & \pi(1) & \pi(2)\end{array}\right]$$

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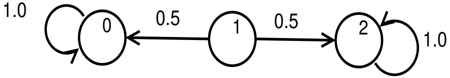

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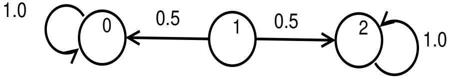

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- We also have to add the equation $\pi(0) + \pi(1) + \pi(2) = 1$
- We now do not have a unique stationary distribution

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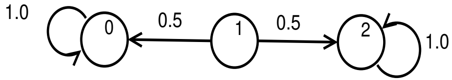

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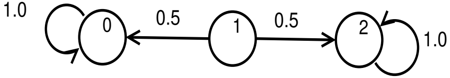

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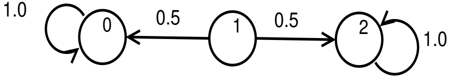

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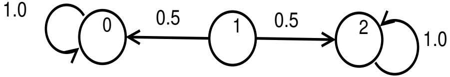

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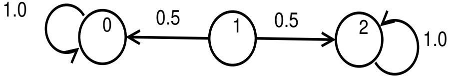

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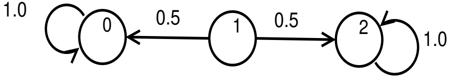

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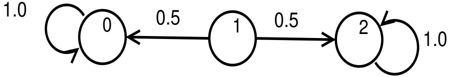

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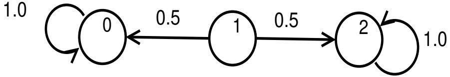

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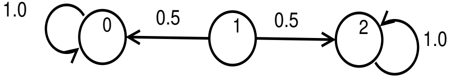

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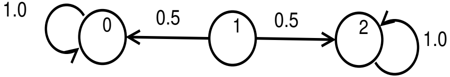

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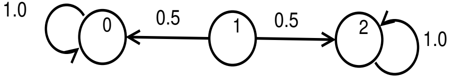

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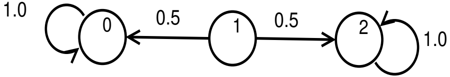

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- Now there are infinitely many solutions.
- Any distribution $[a \ \ 0 \ \ 1-a]$ with $0 \leq a \leq 1$ is a stationary distribution
- This chain is not irreducible; the previous one is irreducible

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- The stationary distribution, when it exists, is related to expected fraction of time spent in different states.
- When the stationary distribution is unique, we also want to know if the chain converges to that distribution starting with any π_0.

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- We will first establish a limit for the above as $n \to \infty$

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- This is intuitively obvious

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- Convergence would be with probability one.

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where $B = [X_{k_1+k_2} = y,\ X_{k_1} = y,\ X_j \neq y, j < k_1 + k_2, j \neq k_1]$

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$$Pr[W_y^3 = k_3 | W_y^2 = k_2, W_y^1 = k_1] \;\; =$$
$$Pr[X_{k_1+k_2+j} \neq y, \; 1 \leq j \leq k_3 - 1, \;\; X_{k_1+k_2+k_3} = y \mid B]$$

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- In general, we get

$$Pr[W_y^r = k_r \mid W_y^{r-1} = k_{r-1}, \cdots, W_y^1 = k_1] = P_y[W_y^1 = k_r]$$

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- Note that this is true even if $m_y = \infty$

- For all n such that $N_n(y) \geq 1$, we have

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- Hence we get

$$\lim_{n\to\infty} \frac{n}{N_n(y)} = m_y, \quad w.p.1$$

or

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1}{m_y}, \quad w.p.1$$

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- Let us characterize y for which $m_y = \infty$

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- Thus the limiting fraction of time spent by the chain in transient and null recurrent states is zero.

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Summing the above for $m = 1, 2, \cdots n$ and dividing by n

$$\frac{1}{n} \sum_{m=1}^{n} P^{n_1+m+n_0}(y,y) \geq P^{n_1}(y,x) \; \frac{1}{n} \sum_{m=1}^{n} P^m(x,x) \;\; P^{n_0}(x,y), \; \forall n$$

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  If we now let $n \to \infty$, the RHS goes to $P^{n_1}(y,x) \ \frac{1}{m_x} \ P^{n_0}(x,y) > 0$.

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- We can write the LHS of above as

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which implies y is positive recurrent

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- Thus, in a closed irreducible set of recurrent states, if one state is positive recurrent then all are positive recurrent.
- Hence, in the partition: $S_R = C_1 + C_2 + \cdots$, each C_i is either wholly positive recurrent or wholly null recurrent.
- We next show that a finite chain cannot have any null recurrent states.