# Recap: Random Processes

- A random process or a stochastic process is a collection of random variables: $\{X(t),\ t \in T\}$
- The index set $T$ can be countable or uncountable (discrete-time or continuous-time processes)
- The $X(t)$ can be discrete or continuous (discrete-state or continuous-state processes)
- We can view the process as $X : \Omega \times T \to \Re$
  A collection of time functions.

# Recap: Distributions of a random process

- The first order distributions:

$$F_X(x;t) = Pr[X(t) \leq x] = F_{X(t)}(x)$$

- The second order distributions:

$$F_X(x_1, x_2; t_1, t_2) = Pr[X(t_1) \leq x_1,\ X(t_2) \leq x_2]$$

- The $n^{th}$ order distributions:

$$F_X(x_1, \cdots, x_n; t_1, \cdots t_n) = Pr[X(t_i) \leq x_i,\ i = 1, \cdots, n]$$

- The distributions can be specified through joint mass or density functions too.

# Recap

- One often makes some assumptions on the process so that all $n^{th}$ order distributions are easily specified implicitly.
- One example is the Markovian dependence
- Other examples: process with independent increments, Gaussian processes

# Recap: Mean, Autocorrelation, autocovariance

- The mean or mean function is

$$\eta_X(t) = E[X(t)], \ \ t \in T$$

- The autocorrelation of the process is

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

- The autocovariance of the process is

$$\begin{aligned} C_X(t_1, t_2) &= E\left[(X(t_1) - E[X(t_1)])(X(t_2) - E[X(t_2)])\right] \\ &= R_X(t_1, t_2) - \eta_X(t_1)\eta_X(t_2) \end{aligned}$$

# Recap: Stationarity

- A random process $\{X(t),\ t \in T\}$ is said to be stationary if

  *for all* $n$*, for all* $t_1, \cdots, t_n$*, for all* $x_1, \cdots x_n$ *and for all* $\tau$ *we have*

$$F_X(x_1, \cdots, x_n \,;\, t_1, \cdots, t_n) = F_X(x_1, \cdots, x_n \,;\, t_1+\tau, \cdots, t_n+\tau)$$

- For a stationary process, the distributions are unaffected by translation of the time axis.
- This stringent condition is often referred to as strict-sense stationarity

# Recap: Wide-sense stationarity

- $\{X(t),\ t \in T\}$ is said to be be wide-sense stationary if

1. $\eta_X(t) = \eta_X$, a constant
2. $R_X(t_1, t_2)$ depends only on $t_1 - t_2$

- This would be so if the first and second order distributions are invariant to change of time origin.
- For a wide-sense stationary process the autocorrelation is a symmetric functions and its Fourier transform is the power spectral density

# Recap: Ergodicity

- Let $X(t)$ be wide-sense stationary
- Ergodicity is a question of whether time-averages converge to ensemble-averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X(i) \ \stackrel{?}{=}\ E[X(n)] = \eta_X$$

Or, more generally

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(X(i)) \ \stackrel{?}{=}\ E[g(X(n))]$$

For a continuous time process we can write this as

$$\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} X(t)\ dt \ \stackrel{?}{=}\ E[X(t)] = \eta_X$$

- One sufficient condition is that covariance between $X(t)$ and $X(t+\tau)$ decreases fast with increasing $\tau$.

# Recap: Mean Ergodicity

- Let $X(t)$ be wide-sense stationary with $E[X(t)] = \eta$.
- Define
$$\eta_\tau = \frac{1}{2\tau} \int_{-\tau}^{\tau} X(t)\, dt \quad (\tau > 0)$$
- We say the process is mean-ergodic if
$$\eta_\tau \xrightarrow{P} \eta, \quad \text{as } \ \tau \to \infty$$
- We showed this holds if
$$\int_{-\infty}^{\infty} |C_X(z)|\, dz \ < \ \infty$$

- Similar sufficient condition holds in case of discrete time processes also.
- A wide-sense stationary process $\{X(n),\ n = 0, 1, \cdots\}$ is said to be mean ergodic if

$$\frac{1}{n}\sum_{i=0}^{n-1} X(i) \ \stackrel{P}{\to}\ EX(n) = \eta$$

- Note that this is a generalization of (weak) law of large numbers to the case where the random variables may not be uncorrelated.
- The above holds if

$$\sum_{k=0}^{\infty} |C_X(k)| \leq \infty, \quad \text{where} \quad C_X(k) = \text{Cov}(X(n), X(n{+}k))$$

- When the above holds we say the process is asymptotically uncorrelated.

- The proof in the discrete case is similar to that in the continuous case.
- Let $S_n = \sum_{i=0}^{n-1} X(i)$.

$$\mathsf{Var}(S_n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \mathsf{Cov}(X(i), X(j)) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} C_X(i-j)$$

- The above sum can be viewed as summing all the elements in a (Toeplitz) matrix where each (left-right) diagonal has all entries same.
- Thus the sum can be rewritten as

$$\mathsf{Var}(S_n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} C_X(i-j) = \sum_{k=-(n-1)}^{n-1} (n-|k|) C_X(k)$$

- Let $S_n = \sum_{i=0}^{n-1} X(i)$.

$$\text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2} \sum_{k=-(n-1)}^{n-1} (n-|k|) C_X(k) = \frac{1}{n} \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) C_X(k)$$

- Now, using $\sum_k |C_X(k)| < \infty$ we can show that

$$\lim_{n\to\infty} \text{Var}\left(\frac{S_n}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n}\right) C_X(k) = 0$$

- This shows that $\frac{S_n}{n}$ converges in probability to $\eta = E[X(n)]$.

# Poisson Process

- This is the next process we study
- This is a discrete-state continuous-time process
- The index set is the interval $[0, \infty)$ and all random variables are discrete and take non-negative integer values.

- A random process $\{N(t),\ t \geq 0\}$ is called a counting process if
  1. $N(t) \geq 0$ and is integer-valued
  2. If $s < t$ then, $N(s) \leq N(t)$

  $Generally, N(t)$ represents number of 'events' till $t$
- The counting process has independent increments if for all $t_1 < t_2 \leq t_3 < t_4$, $N(t_2) - N(t_1)$ is independent of $N(t_4) - N(t_3)$
- In particular, for all $s > t > 0$, $N(s) - N(t)$ is independent of $N(t) - N(0)$
- The process is said to have stationary increments if $N(t_2) - N(t_1)$ has the same distribution as $N(t_2 + \tau) - N(t_1 + \tau), \forall \tau, \forall t_2 > t_1$

- We start with two definitions of Poisson process
- **Definition 1** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(t) = n] = e^{-\lambda t} \, \frac{(\lambda t)^n}{n!}, \; n = 0, 1, \cdots$
- $N(t)$ is Poisson with parameter $\lambda t$
- $E[N(t)] = \lambda t$ and hence $\lambda$ is called rate
- Since the process has stationary increments and $N(0) = 0$, $(N(t+s) - N(s))$ would be Poisson with parameter $\lambda t$ for all $s, t > 0$.

- **Definition 2** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(h) = 1] = \lambda h + o(h)$ and $Pr[N(h) \geq 2] = o(h)$
- We say $g(h)$ is $o(h)$ if

$$\lim_{h \to 0} \frac{g(h)}{h} = 0$$

- This definition tells us when Poisson process may be a good model
- We will show that both definitions are equivalent

- We first show Definition 2 $\Rightarrow$ Definition 1
- **Definition 1** $\{N(t), t \geq 0\}$ is a counting process with
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(t) = n] = e^{-\lambda t} \, \frac{(\lambda t)^n}{n!}, \; n = 0, 1, \cdots$
- **Definition 2** $\{N(t), t \geq 0\}$ is a counting process with
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(h) = 1] = \lambda h + o(h)$ and
     $Pr[N(h) \geq 2] = o(h)$
- For this we need to calculate distribution of $N(t)$

- We first show Definition 2 $\Rightarrow$ Definition 1
- For this we need to calculate distribution of $N(t)$
- Let $P_n(t) = Pr[N(t) = n]$

$$\begin{aligned}
P_0(t+h) &= Pr[N(t+h) = 0] \\
&= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \\
&= Pr[N(t) = 0]\ Pr[N(t+h) - N(t) = 0] \\
&\qquad \text{(because of independent increments)} \\
&= Pr[N(t) = 0]\ Pr[N(h) = 0] \quad \text{(stationary increments)} \\
&= P_0(t)(1 - \lambda h + o(h))
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \frac{P_0(t+h) - P_0(t)}{h} &= -\lambda P_0(t) + \frac{o(h)}{h} \\
\Rightarrow \quad \frac{d}{dt} P_0(t) &= -\lambda P_0(t)
\end{aligned}$$

- Now we can solve this differential equation to get $P_0(t)$

$$\begin{aligned} \frac{d}{dt}P_0(t) &= -\lambda P_0(t) \\ \Rightarrow \quad \frac{1}{P_0(t)}\frac{d}{dt}P_0(t) &= -\lambda \\ \Rightarrow \quad \ln(P_0(t)) &= -\lambda t + c \\ \Rightarrow \quad P_0(t) &= Ke^{-\lambda t} \end{aligned}$$

- Since $P_0(0) = Pr[N(0) = 0] = 1$, we get $K = 1$ and hence
$$P_0(t) = Pr[N(t) = 0] = e^{-\lambda t}$$
- Next we consider $P_n(t)$ for $n > 0$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k]
\end{aligned}
$$

We have

$$
\begin{aligned}
&Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(t+h)-N(t)=k] \\
&\qquad = Pr[N(t)=n-k]\ P[N(h)=k] = o(h),\ \forall k \geq 2
\end{aligned}
$$

$$
Pr[N(t)=n,\ N(t+h)-N(t)=0] = P_n(t)P_0(h)
$$

$$
Pr[N(t)=n-1,\ N(t+h)-N(t)=1] = P_{n-1}(t)P_1(h)
$$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1-\lambda h + o(h)) + P_{n-1}(t)(\lambda h + o(h)) + o(h) \\
&= P_n(t) - \lambda h P_n(t) + \lambda h P_{n-1}(t) + o(h)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}
$$

$$
\Rightarrow \quad \frac{d}{dt}P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)
$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain $P_n$
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\, \lambda P_{n-1}(t) \\ \Rightarrow \quad \frac{d}{dt}\left(P_n(t)\, e^{\lambda t}\right) &= \lambda\, e^{\lambda t}\, P_{n-1}(t) \end{aligned}$$

- We need $P_{n-1}$ to solve for $P_n$. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\, e^{\lambda t}\right) = \lambda\, e^{\lambda t}\, P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\Rightarrow \quad e^{\lambda t} P_1(t) = \lambda t + c \quad \Rightarrow \quad P_1(t) = e^{-\lambda t}(\lambda t + c)$$

- $P_1(0) = Pr[N(0) = 1] = 0 \quad \Rightarrow \quad c = 0$
  Hence $P_1(t) = \lambda t\, e^{-\lambda t}$

- We showed: $P_0(t) = e^{-\lambda t}$ and $P_1(t) = \lambda t e^{-\lambda t}$
- We need to show: $P_k(t) = e^{-\lambda t} \; \frac{(\lambda t)^k}{k!}$
- Assume it is true till $k = n-1$

$$\frac{d}{dt}\left(P_n(t)\, e^{\lambda t}\right) = \lambda\, e^{\lambda t}\, P_{n-1}(t) = \lambda e^{\lambda t} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda^n \frac{t^{n-1}}{(n-1)!}$$

$$\Rightarrow \;\; e^{\lambda t} P_n(t) = \lambda^n \frac{t^n}{n} \frac{1}{(n-1)!} + c \;\; \Rightarrow \;\; P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

where $c = 0$ because $P_n(0) = 0$.

- This completes the proof that Definition 2 implies Definition 1

- **Definition 1** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(t) = n] = e^{-\lambda t} \, \frac{(\lambda t)^n}{n!}, \ n = 0, 1, \cdots$
- **Definition 2** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(h) = 1] = \lambda h + o(h)$ and
     $Pr[N(h) \geq 2] = o(h)$

- Now we prove Definition 1 implies Definition 2
- We need to only show point(3) of Definition 2 using point (3) of Definition 1

$$\text{Let} \quad Pr[N(t) = k] = e^{-\lambda t} \, \frac{(\lambda t)^k}{k!}$$

$$Pr[N(h) = 1] = \lambda \, h \, e^{-\lambda h} = \lambda \, h + \lambda \, h \left(e^{-\lambda h} - 1\right) \, = \lambda \, h + o(h)$$

because

$$\lim_{h \to 0} \frac{\lambda \, h \left(e^{-\lambda h} - 1\right)}{h} = \lim_{h \to 0} \lambda \left(e^{-\lambda h} - 1\right) \, = 0$$

- We showed $Pr[N(h) = 1] = \lambda h + o(h)$

- Now we need to show $Pr[N(h) \geq 2] = o(h)$

$$\begin{aligned} Pr[N(h) \geq 2] &= 1 - Pr[N(h) = 0] - Pr[N(h) = 1] \\ &= 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} \end{aligned}$$

- This goes to zero as $h \to 0$
- We can use L'Hospital rule

$$\lim_{h \to 0} \frac{1 - e^{-\lambda h} - \lambda h e^{-\lambda h}}{h} = \lim_{h \to 0} \frac{\lambda e^{-\lambda h} - \lambda e^{-\lambda h} + \lambda^2 h e^{-\lambda h}}{1} = 0$$

- This completes the proof that Definition 2 implies Definition 1

# These two definitions are equivalent

- **Definition 1** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(t) = n] = e^{-\lambda t} \, \frac{(\lambda t)^n}{n!}, \; n = 0, 1, \cdots$
- **Definition 2** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(h) = 1] = \lambda h + o(h)$ and
     $Pr[N(h) \geq 2] = o(h)$

- Since the process has stationary increments, for $t_2 > t_1$,

$$
\begin{aligned}
Pr[N(t_2) - N(t_1) = k]] &= Pr[N(t_2 - t_1) - N(0) = k] \\
&= e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^k}{k!}
\end{aligned}
$$

- The first order distribution of the process is: $N(t) \sim \text{Poisson}(\lambda t)$
- This, along with stationary and independent increments property determines all distributions

$$
\begin{aligned}
&Pr[N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3] \\
&\quad = Pr[N(t_1) = n_1]\, Pr[N(t_2) - N(t_1) = n_2 - n_1] \\
&\qquad Pr[N(t_3) - N(t_2) = n_3 - n_2] \\
&= Pr[N(t_1) = n_1]\, Pr[N(t_2 - t_1) = n_2 - n_1]\, Pr[N(t_3 - t_2) = n_3 - n_2
\end{aligned}
$$

where we assumed $t_1 < t_2 < t_3$

- We can easily calculate mean and autocorrelation of the process

$$\eta_N(t) = E[N(t)] = \lambda\, t \quad \Rightarrow \quad \text{not stationary}$$

With $t_2 > t_1$, we have

$$\begin{aligned}
R_N(t_1, t_2) &= E[N(t_2)N(t_1)] \\
&= E[N(t_1)(N(t_2) - N(t_1) + N(t_1))] \\
&= E[N(t_1)(N(t_2) - N(t_1))] + E[N(t_1)^2] \\
&= E[N(t_1)]\; E[N(t_2) - N(t_1)] + E[N(t_1)^2] \\
&= E[N(t_1)]\; E[N(t_2 - t_1)] + E[N(t_1)^2] \\
&= \lambda t_1(\lambda(t_2 - t_1)) + (\lambda t_1 + \lambda^2 t_1^2) \\
&= \lambda t_1 + \lambda^2 t_1 t_2
\end{aligned}$$

$$\Rightarrow \; R_N(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda\; \min(t_1, t_2)$$

# Inter-arrival or waiting times

- Let $T_1$ denote the time of first event and let $T_n$ denote the time between $n^{th}$ and $(n-1)st$ events.
- Let $S_n = \sum_{i=1}^{n} T_i$ – time of $n^{th}$ event

$$Pr[T_1 > t] = Pr[N(t) = 0] = e^{-\lambda t}$$

$$\Rightarrow \quad T_1 \sim \text{exponential}(\lambda)$$

$$\begin{aligned} Pr[T_2 > t | T_1 = s] &= Pr[0 \text{ events in } (s,\ s+t] \mid T_1 = s] \\ &= Pr[0 \text{ events in } (s,\ s+t]\ ] = e^{-\lambda t} \\ \Rightarrow \quad Pr[T_2 > t] &= \int Pr[T_2 > t | T_1 = s]\ f_{T_1}(s)\ ds = e^{-\lambda t} \end{aligned}$$

- $T_n$ are iid exponential with parameter $\lambda$

- The time of $n^{th}$ event is

$$S_n = \sum_{i=1}^{n} T_i$$

Since $T_i$ are iid, exponential, $S_n$ is Gamma with parameters $n, \lambda$

- Let $0 < s < t$.

$$\begin{aligned}
Pr[S_1 \leq s | N(t) = 1] &= \frac{Pr[S_1 \leq s, \ N(t) = 1]}{Pr[N(t) = 1]} \\
&= \frac{Pr[1 \text{ event in } [0, \ s], \ 0 \text{ in } (s, \ t]]}{Pr[N(t) = 1]} \\
&= \frac{\lambda s e^{-\lambda s} \ e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}
\end{aligned}$$

- Conditioned on $N(t) = 1$, $S_1$ is uniform over $[0, t]$

- This can be used, e.g., in simulating Poisson process
- We can cut time axis into small intervals of length $h$.
- In each interval we can decide whether or not there is an event, with prob $\lambda h$.
- If there is an event, we choose its time uniformly in the interval.
- Called Bernoulli approximation of Poisson process
- We could also generate Poisson process by generating independent exponential random variables

- Let $S_1, \cdots, S_n$ be the times of the first $n$ events.
- We calculated conditional density of $S_1$ conditioned on $N(t) = 1$.
- Suppose we want to calculate the conditional joint density of $S_1, \cdots, S_n$ conditioned on $N(t) = n$.
- Note that the $S_i$ have to satisfy $S_1 < S_2 < \cdots < S_n$.
- We can show that the conditional joint density of $S_1, \cdots, S_n$ conditioned on $N(t) = n$, would be same as the order statistics of $n$ iid random variables uniform over $[0,\ t]$.

- Take $t_i, 1 \leq i \leq n$ satisfying $0 < t_1 < t_2 < \cdots < t_n < t$.
- Let $h_i$ be small positive numbers such that $t_i + h_i < t_{i+1}, \forall i$.

$$
\begin{aligned}
& Pr[t_i \leq S_i \leq t_i + h_i, i = 1, \cdots, n \mid N(t) = n] \\
= & \frac{Pr[t_i \leq S_i \leq t_i + h_i, i = 1, \cdots, n, N(t) = n]}{Pr[N(t) = n]} \\
= & \frac{Pr[1 \text{ event in each } [t_i,\ t_i + h_i], 1 \leq i \leq n,\ 0 \text{ in rest of } [0,\ t]]}{Pr[N(t) = n]} \\
= & \frac{\left(\prod_{i=1}^n \lambda h_i e^{-\lambda h_i}\right)\ e^{-\lambda(t - h_1 - \cdots - h_n)}}{((\lambda t)^n / n!) e^{-\lambda t}} \\
= & \frac{n!\ h_1 \cdots h_n}{t^n}
\end{aligned}
$$

- Thus we have for $0 < t_1 < \cdots < t_n < t$,

$$\frac{Pr[t_i \leq S_i \leq t_i + h_i, i = 1, \cdots, n \mid N(t) = n]}{h_1 \cdots h_n} = \frac{n!}{t^n}$$

- If we now take limit as all $h_i$ go to zero, the LHS above would be the conditional joint density of $S_1, \cdots, S_n$ conditioned on $N(t) = n$.
- Thus

$$f_{S_1 \cdots S_n | N(t)}(t_1, \cdots, t_n | n) = \frac{n!}{t^n}, \ 0 < t_1 < \cdots < t_n < t$$

- Let $X_1, \cdots, X_n$ be iid continuous random variables with common density $f_x$.
- Recall that $X_{(k)}$ denotes the $k^{th}$ smallest of them.
- Then the joint density of $X_{(1)}, \cdots, X_{(n)}$ is given by

$$f(x_1, \cdots, x_n) = n! \prod_{i=1}^{n} f_x(x_i), \ x_1 < \cdots < x_n$$

- If $X_i$ are uniform over $[0, \ t]$

$$f(t_1, \cdots, t_n) = \frac{n!}{t^n}, \ 0 < t_1 < \cdots < t_n < t$$

# Examples

- We look at a few simple example problems using Poisson process.

$$\begin{aligned} E[N(4) - N(2) | N(1) = 3] &= E[N(4) - N(2)] \\ &= E[N(2) - 0] = 2\lambda \end{aligned}$$

- Another example;

$$E[S_4] = E\left[\sum_{i=1}^{4} T_i\right] = \frac{4}{\lambda}$$

- The memoryless property of exponential rv can be useful

$$Pr[S_3 > t | N(1) = 2] = \begin{cases} 1 & \text{if } t < 1 \\ e^{-\lambda(t-1)} & \text{if } t \geq 1 \end{cases}$$