CS 215 - Assignment 2

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Q1.

$Z = X + Y$

pdf → $f_Z(z)$

For Z to take z X can take all values x while Y takes value z−x

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx$$

If X & Y are independent, then $f_{XY}(x, z-x) = f_X(x)\, f_Y(z-x)$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx.$$

For $X \le Y$

Y can take all the values y while X takes values smaller than y

$$\therefore P(X \le Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy$$

If X & Y are independent

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

$$P(X \le Y) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{y} f_X(x)\, dx\right) f_Y(y)\, dy$$

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Q2.

2. Given that $X_1, X_2, \ldots X_n$ are independent identically distributed random variables with cdf $F_X(x)$

$$Y_1 = \max(X_1, X_2, \ldots X_n)$$

$$P(Y_1 \le x) = P(\max(X_1, X_2, \ldots X_n) \le x)$$
$$= P((X_1 \le x) \cap (X_2 \le x) \cap \ldots (X_n \le x))$$

Since $X_1, X_2 \ldots X_n$ are independent

$$P((X_1 \le x) \cap (X_2 \le x) \cap \ldots (X_n \le x)) = P(X_1 \le x) \cdot P(X_2 \le x) \ldots P(X_n \le x)$$

Since $X_1, X_2, \ldots X_n$ are identically distributed

$$P(X_1 \le x) = P(X_2 \le x) = \ldots P(X_n \le x) = F_X(x)$$

$$\therefore P(X_1 \le x) \cdot P(X_2 \le x) \ldots P(X_n \le x) = \{F_X(x)\}^n$$

Thus, $P(Y_1 \le x) = \{F_X(x)\}^n$ [cdf]

$n\{F_X(x)\}^{n-1} F_X'(x)$ [pdf]

$$Y_2 = \min(X_1, X_2, --- X_n)$$

$$P(Y_2 \le x) = P(\min(X_1, X_2, --- X_n) \le x)$$

Probability that at least one of $X_1, X_2, --- X_n \le x$ is
$= 1 -$ Probability that all of them are $> x$

$$P(\min(X_1, X_2, --- X_n) \le x) = 1 - P((X_1 > x) \cap (X_2 > x) \cap --- (X_n > x))$$

Since $X_1, X_2, --- X_n$ are independent.

$$1 - P((X_1 > x) \cap (X_2 > x) \cap -- (X_n > x)) = 1 - P(X_1 > x)\, P(X_2 > x) -- P(X_n > x)$$

$$= 1 - (1 - F_X(x))^n$$

$$\therefore \quad P(Y_2 \le x) = 1 - (1 - F_X(x))^n \quad = \text{cdf}$$

$$n(1 - F_X(x))^{n-1} F_X'(x) \quad = \text{pdf}$$

Q3.

For $\tau > 0$

$$X-\mu \geqslant \tau \Rightarrow X-\mu+c \geqslant \tau+c$$

for some $c > 0$

$$\therefore P(X-\mu \geqslant \tau) = P(X-\mu+c \geqslant \tau+c)$$

$\tau + c > 0 \quad \therefore \quad X-\mu+c > 0$

Moreover

$$(X-\mu+c)^2 \geqslant (\tau+c)^2 \Rightarrow X-\mu+c \geqslant \tau+c \quad \text{or}$$

$$X-\mu+c \leqslant -(\tau+c)$$

$$\therefore P(X-\mu+c \geqslant \tau+c) \leq P\left((X-\mu+c)^2 \geqslant (\tau+c)^2\right)$$

By Markov's Inequality

$$P\left((x-\mu+c)^2 \geqslant (\tau+c)^2\right) \leqslant \frac{E\left((X-\mu+c)^2\right)}{(\tau+c)^2}$$

$$E\left((X-\mu+c)^2\right) = E\left((X-\mu)^2\right) + c^2 + \cancel{2E\left((X-\mu)c\right)}^{\,0}$$

$$= \sigma^2 + c^2$$

$$\therefore P(X-\mu \geqslant \tau) \leqslant \frac{\sigma^2+c^2}{(\tau+c)^2}$$

PTO

$f(c) = \dfrac{\sigma^2 + c^2}{(\tau + c)^2}$ has a maximum ~~value~~ at $c = \sigma^2/\tau$

$\therefore$

$$P(X - \mu \geq \tau) \leq \frac{\sigma^2 + \frac{\sigma^4}{\tau^2}}{\left(\tau + \frac{\sigma^2}{\tau}\right)^2} = \frac{\sigma^2(\sigma^2 + \tau^2)}{(\sigma^2 + \tau^2)^2} = \frac{\sigma^2}{\sigma^2 + \tau^2}$$

$\therefore$ for $\tau > 0$ $\boxed{P(X - \mu \geq \tau) \leq \dfrac{\sigma^2}{\sigma^2 + \tau^2}.}$

Let $\tau < 0$ Let $\tau = -k$ where $k > 0$

$$X - \mu \leq \tau \Rightarrow X - \mu \leq -k \Rightarrow \mu - X \geq k$$

$$P(\mu - X \geq k) = P(\mu - X + c \geq k + c), \text{ where } c > 0$$

$$P(\mu - X + c \geq k + c) \leq P((\mu - X + c)^2 \geq (k+c)^2)$$

$$\leq \frac{E(\mu - X + c)^2}{(k+c)^2}$$

$$= \frac{\sigma^2 + c^2}{(k+c)^2}$$

$$\leq \frac{\sigma^2}{\sigma^2 + k^2} \quad [\text{Proved above}]$$

$$= \frac{\sigma^2}{\sigma^2 + \tau^2}$$

$\therefore$

$$P(X - \mu \leq \tau) \leq \frac{\sigma^2}{\sigma^2 + \tau^2}$$

$$\therefore \boxed{P(X - \mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2 + \tau^2}}$$

In the above derivation, the second last step proves a statement about $\boldsymbol{X - \mu \leq \tau}$, the next line should have $\boldsymbol{X - \mu > \tau}$, but, $\boldsymbol{P(X - \mu = \tau) = 0}$, so the last step still holds.

Q4.

~~3.~~
4.

$$e^{-tx}\phi_X(t) = e^{-tx}\int_{-\infty}^{\infty} e^{tx'} f_X(x')\,dx'$$

$$= \int_{-\infty}^{\infty} e^{t(x'-x)} f_X(x')\,dx'$$

$$\geq \int_{x}^{\infty} e^{t(x'-x)} f_X(x')\,dx'$$

$$\geq \int_{x}^{\infty} (1+t(x'-x)) f_X(x')\,dx' \qquad [\because e^a \geq 1+a]$$

$$= \int_{x}^{\infty} f_X(x')\,dx' + \underbrace{\int_{x}^{\infty} t(x'-x)\, f_X(x')\,dx'}$$

this is +ve because $t>0$ & $x'-x>0$ & $f_X(x')>0$

$$\geq \int_{x}^{\infty} f_X(x')\,dx' = P(X \geq x)$$

$$e^{-tx}\phi_X(t) = e^{-tx}\int_{-\infty}^{\infty} e^{+tx'} f_X(x')\,dx'$$

$$= \int_{-\infty}^{\infty} e^{t(x'-x)} f_X(x')\,dx'$$

$$\geq \int_{-\infty}^{\infty} \left(1+t(x'-x)\right) f_X(x')\,dx'$$

$$\geq \int_{-\infty}^{x} \left(1+t(x'-x)\right) f_X(x')\,dx'$$

$$= \int_{-\infty}^{x} f_X(x')\,dx' + \int_{-\infty}^{x} t(x'-x) f_X(x')\,dx'$$

$t<0 \quad (x'-x)<0 \quad f_X(x')>0$

$\therefore t\cdot(x'-x)>0$

$\therefore$ this term is >0

$$\geq \int_{-\infty}^{x} f_X(x')\,dx'$$

$$= P(X \leq x)$$

$$P(X>(1+\delta)\mu) \leq \frac{\phi_X(t)}{e^{+t(1+\delta)\mu}}$$

$$\phi_X(t) = E(e^{Xt}) = E(e^{(X_1+X_2 -- X_n)t})$$
$$= E(e^{X_1t}.e^{X_2t} -- e^{X_nt})$$

Since $X_1, X_2 -- X_n$ are independent
$e^{X_1t}, e^{X_2t} -- e^{X_nt}$ are also independent.

$$\therefore E(e^{X_1t}.e^{X_2t} -- e^{X_nt}) = E(e^{X_1t}).E(e^{X_2t}) -- E(e^{X_nt})$$

$$E(e^{X_it}) = pe^t + (1-p)e^0 = p_ie^t + (1-p_i)$$
$$= 1 + p_i(e^t-1)$$
$$\leq e^{p_i(e^t-1)} \quad [\because 1+x \leq e^x]$$

$$E(e^{X_1t}).E(e^{X_2t}) -- E(e^{X_nt}) \leq e^{\Sigma p_i(e^t-1)} = e^{(e^t-1)\Sigma p_i}$$
$$= e^{\mu(e^t-1)}$$

$$\boxed{\therefore P(X>(1+\delta)\mu) \leq e^{\mu(e^t-1)}/e^{t(1+\delta)\mu}}$$

For the optimal value of t, we can calculate t s.t. RHS is minimum, $\because$ the inequality holds for all t.

$$\Rightarrow \frac{d}{dt}\left(e^{\mu(e^t-1)}/e^{t(1+\delta)\mu}\right) = 0$$

$$\Rightarrow \left(e^{\mu e^t - \mu - \mu t - \delta\mu t}\right)\left(\mu e^t - \mu - \delta\mu\right) = 0$$

$$\Rightarrow e^t = \delta + 1$$

$$t_0 = \ln(1+\delta)$$

It can be seen that this gives the minimal value, by checking the double derivative

Q5.

Derivation-

5. Let $P_{negative}$ be the probability that test on the mixture is negative.

The mixture will be negative if none of them have the disease.

$\therefore P_{negative} = (1-p)^k$

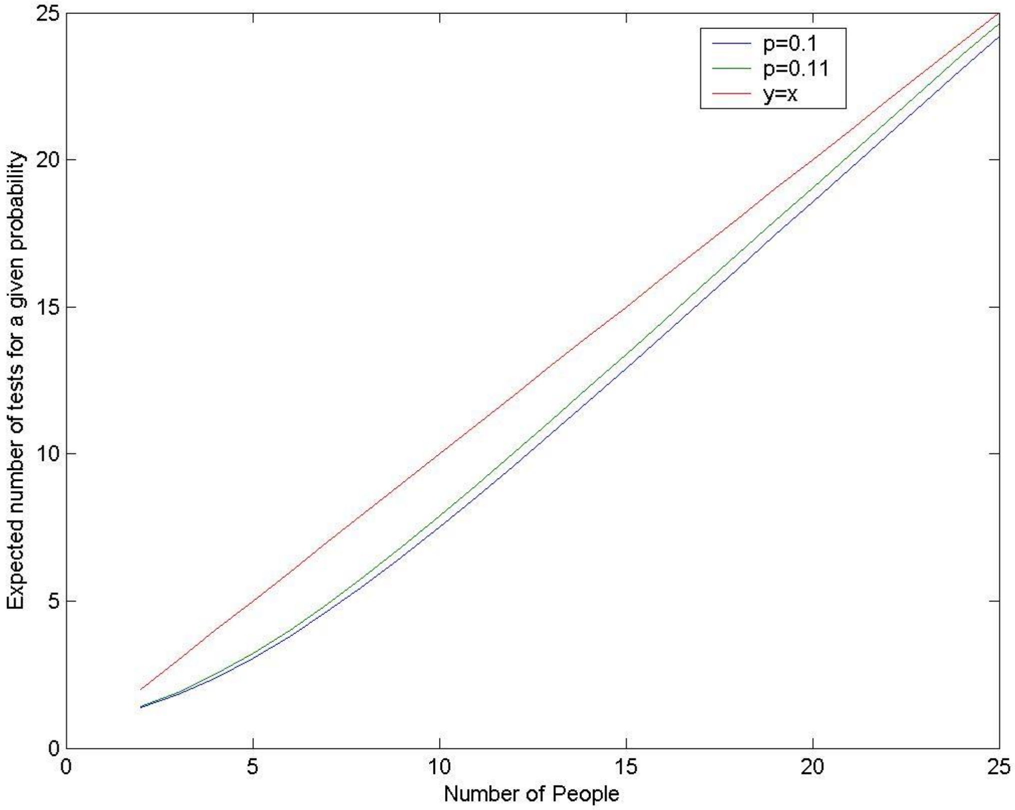
Let N denote number of tests in second case

$$E(N) = P_{negative}(1) + P_{positive}(k+1)$$
$$= (1-p)^k + (k+1)(1-(1-p)^k)$$
$$= (k+1) - k(1-p)^k$$
$$= 1 + k(1-(1-p)^k)$$

Number of tests in the first case $= k$.

$$1 + k(1-(1-p)^k) < k$$
$$k(1-p)^k > 1$$
$$(1-p)^k > 1/k$$
$$(1-p) > (1/k)^{1/k}$$
$$p < 1 - (1/k)^{1/k}$$

Graph of Expected number of tests v/s k



The expected number of tests rises as the probability of occurrence of disease rises. For a smaller number of people, the difference in the number of expected tests is less, while for a greater number of people, it approaches the number k.

Q6

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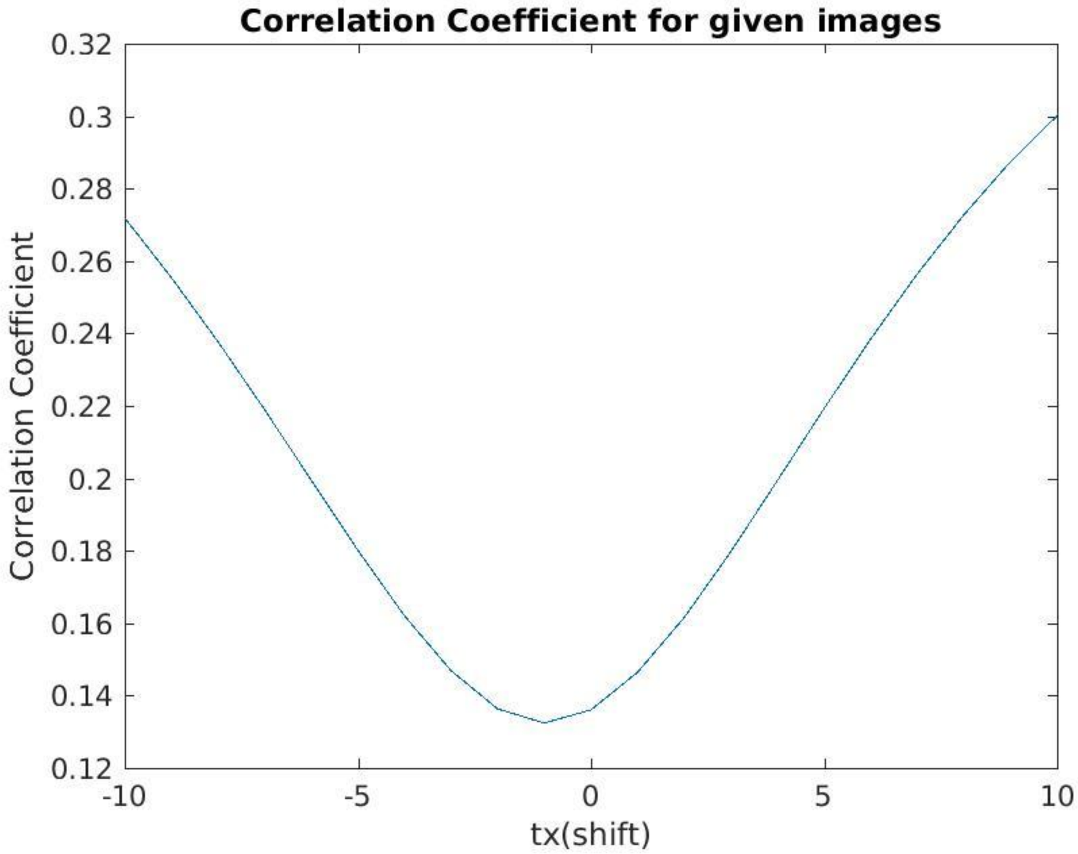
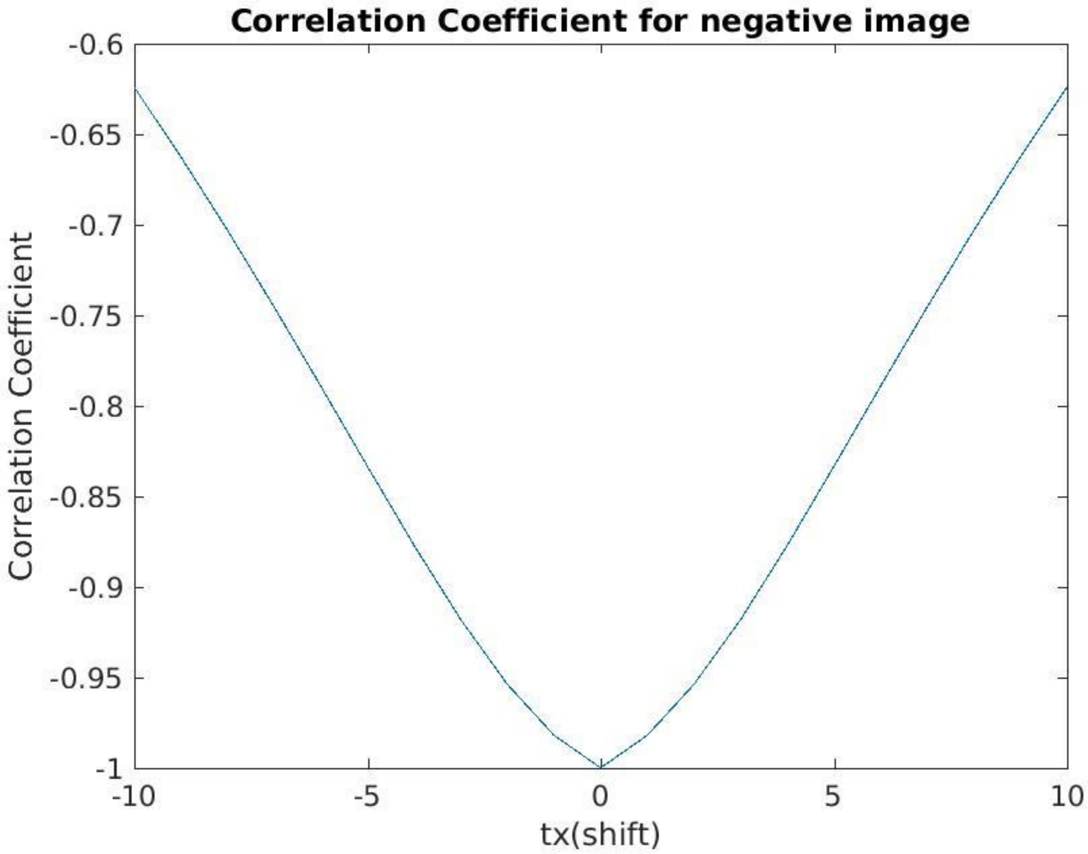
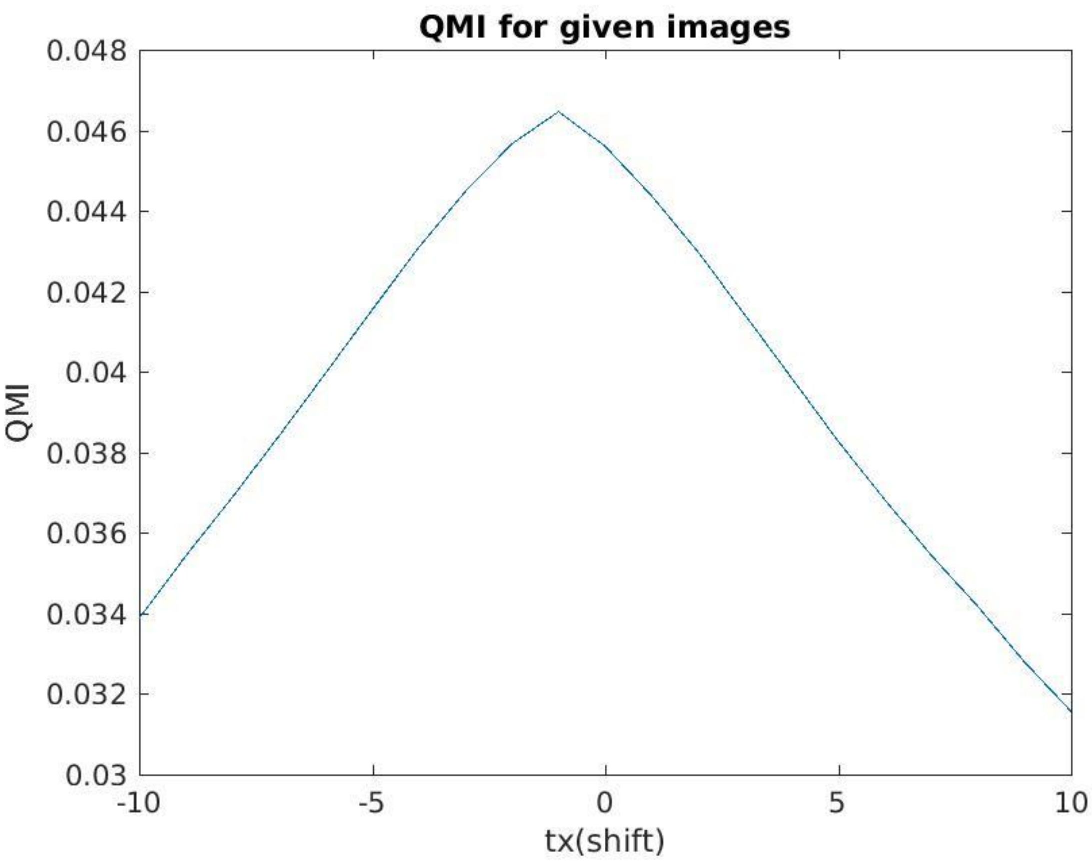
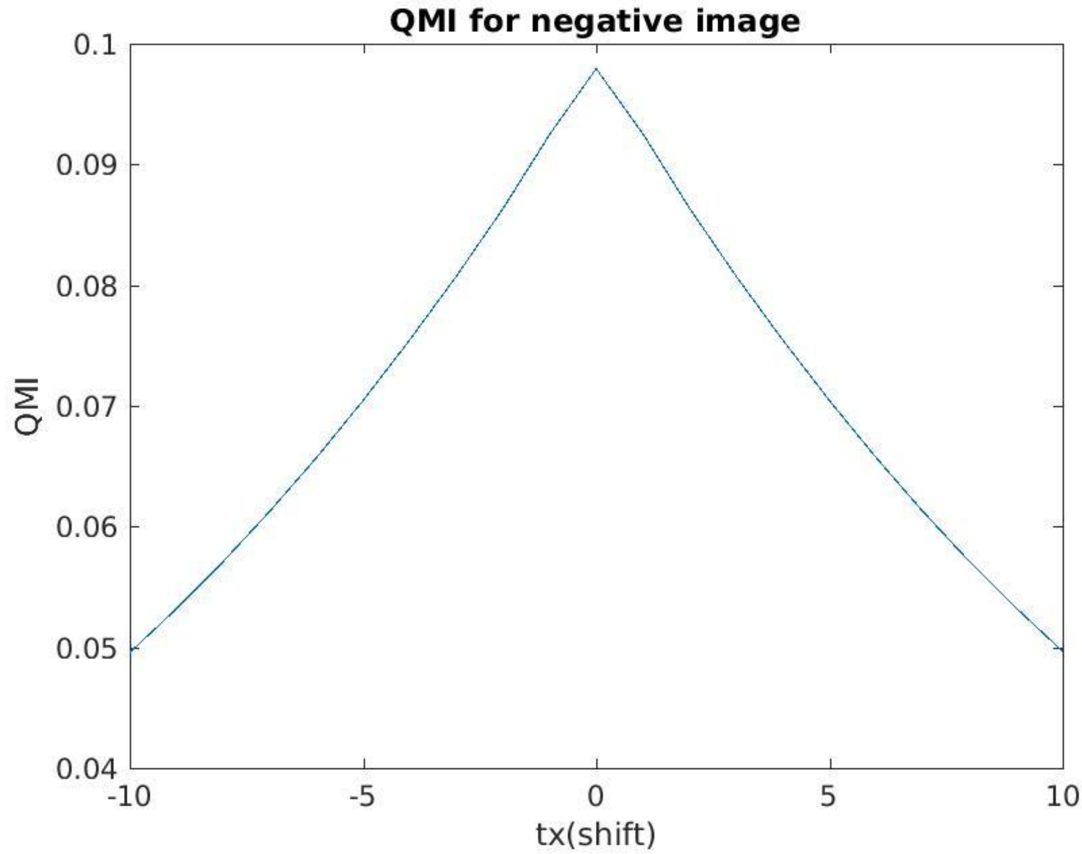

Correlation Coefficient for given images
Correlation Coefficient
tx(shift)
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
-10
-5
0
5
10


Correlation Coefficient for negative image
Correlation Coeficient
tx(shift)
-0.6
-0.65
-0.7
-0.75
-0.8
-0.85
-0.9
-0.95
-1
-10
-5
0
5
10


QMI for given images
QMI
tx(shift)
0.048
0.046
0.044
0.042
0.04
0.038
0.036
0.034
0.032
0.03
-10
-5
0
5
10



QMI for negative of image

Comments – The graphs have an extremum, with the QMI being maximum and correlation coefficient being minimum whenever the two images are aligned perfectly (i.e. at $t_x = 0$). This property can be used to align two images to the same co-ordinate system.

For the correlation coefficient of the negative plot, ρ is -1 at $t_x = 0$, which means that they are perfectly correlated, but one is linearly negatively proportional to the other, as $I_2 = 255 - I_1$.