Q4.

4) (a) Likelihood = $1/\theta^n$

For likelihood to be $\max^m$ θ has to $\min^m$.

but $\theta >$ ~~...~~ $x_1, x_2 -- x_n$

$\therefore \hat{\theta}_{ml} = \max\{x_1, x_2, -- x_n\}$

Prior: $P(\theta) \propto \left(\frac{\theta_m}{\theta}\right)^{\alpha}$ $\quad \theta \geq \theta_m$

$= 0$ $\quad$ otherwise.

Posterior $\propto 1/\theta^n \cdot \left(\frac{\theta_m}{\theta}\right)^{\alpha} \propto \frac{\theta_m^{\alpha}}{\theta^{n+\alpha}}$

For this to be $\max^m$ θ has to $\min^m$

but $\theta \geq \{x_1, x_2 .. x_n\}$

$\theta \geq \theta_m$

$\therefore \hat{\theta}_{MAP} = \max\{x_1, x_2 -- x_n, \theta_m\}$

(b) No, because θ_{MAP} ~~depends~~ depends on θ_m but θ_{ml} doesnot

This is not desirable.

Also, θ_{ml} asymptotically converges to the true value, but θ_{map} may or may not.

Q4)

(c) Posterior $\propto \dfrac{\theta_m^{\alpha}}{\theta^{\alpha}}\left(\prod_{i=1}^{n}\dfrac{1}{\theta}\right)$ if $\theta \geq \theta_m$ & $\theta \geq \max_i (x_i)$

$= 0$ otherwise

Posterior $\propto \dfrac{\theta_m^{n+\alpha}}{\theta^{n+\alpha}}$

$\propto \cancel{\max(\theta_m,} \left(\dfrac{C}{\theta}\right)^{n+\alpha}$, where

$C \cancel{= \max(\theta_m, \max(x_i}$

$C = \max(\theta_m, \max_i(x_i))$

So, this is a pareto distribution with parameters C, $n+\alpha$

$\Rightarrow$ Posterior mean $= \dfrac{\int_C^{\infty} \dfrac{C^{n+\alpha}}{\theta^{n+\alpha}} \cdot \theta \, d\theta}{\int_C^{\infty} \dfrac{C^{n+\alpha}}{\theta^{n+\alpha}} \, d\theta}$

$= \left(\dfrac{n+\alpha-1}{n+\alpha-2}\right) \cdot C$

$$\hat{\theta}_{\substack{\text{Posterior}\\ \text{mean}}} = \left(\frac{n+\alpha-1}{n+\alpha-2}\right) \cdot \max\left(\theta_m, \max_i (x_i\right.$$

(d) As is clear, $\lim_{n \to \infty} \hat{\theta}_{\substack{\text{Posterior}\\ \text{mean}}} = \max(\theta_m, \max_i (x_i))$

$$= \hat{\theta}_{map}$$

$$\neq \hat{\theta}_{ML}$$

This is not desirable, because an infinite sample size should mean that likelihood is ~~most~~ much more informative than prior.