$X_1 \sim U(-1,1)$

$X_2 \sim U(-1,1)$

Proof $\rightarrow$ $x_1 = 2u_1 - 1$

$P(x_1) = 1 \cdot \frac{1}{2}$ [over $(-1,1)$]

Probability ($(X_1, X_2)$ is within a circle of radius 1)

$( = P(X_1^2 + X_2^2 \leq 1)$

Let $Y_1 = X_1^2$

for $x \in (0,1]$, $x_1 = \sqrt{y_1}$ $[y_1 \in (0,1)]$

$$P_1(y_1) = P(x_1) \cdot \left| \frac{d}{dy}(x_1) \right|$$

$$= \frac{1}{2} \cdot \frac{1}{2\sqrt{y_1}} = \frac{1}{4\sqrt{y_1}}$$

For $x \in (-1,0)$, $x_1 = -\sqrt{y_1}$

$$P_2(y_1) = \frac{1}{2} \cdot \frac{1}{2\sqrt{y_1}} = \frac{1}{4\sqrt{y_1}}$$

$P(y_1) = P_1 + P_2$ $[y \in (0,1]]$

$= \frac{1}{2\sqrt{y_1}}$ for $y \in (0,1]$

Also, $P(Y_2) = \frac{1}{2\sqrt{y_2}}$

~~Let P=~~

$X = Y_1 + Y_2$

$P(X \leq 1) = P(Y_1 + Y_2 \leq 1)$

$$P(Y_1 + Y_2 \leq 1) = \int_0^1 \int_0^{1-y_2} \frac{1}{2\sqrt{y_2}} \cdot \frac{1}{2\sqrt{y_1}} \, dy_1 \, dy_2$$

$$= \frac{1}{2} \int_0^1 \frac{\sqrt{1-y_2}}{\sqrt{y_2}} \, dy_2$$

$$= \frac{\pi}{4}$$

Hence, randomly sampled points have a probability of $\frac{\pi}{4}$ to be inside the circle.

For a sample of size M, the number of points N s.t. N is inside circle is a binomial variable with $p = \frac{\pi}{4}$.

$$E[N] = \frac{M \cdot \pi}{4}$$

$$\Rightarrow \pi = \frac{4E[N]}{M}.$$

E[N] can be computed empirically.

E[N] is determined by generating uniform random variables and calculating the number of instances lying inside the circle.

Our code handles the case of n>10^8 by not using more than 10^8 numbers at a time, i.e. by using a loop.

The calculated values are-

N=10 2

N=100 3.0800

N=1000 3.2200

N=10000    3.1348

N=100000    3.1408

N=1000000    3.1416

N=10000000    3.1414

N=100000000    3.1415

PTO-

Let $X = \frac{\sum X_i}{n}$ which give the estimate of $\pi/4$. where $X_i$ is the R.V
$= 1$ if $x$ lies in circle
$= 0$ if $x$ lies outside

$$E\left(\frac{\sum X_i}{n}\right) = \frac{n \times \pi/4}{n} = \pi/4$$

$$Var\left(\frac{\sum X_i}{n}\right) = \sum Var\left(\frac{X_i}{n}\right) = \frac{n \times p(1-p)}{n^2}$$

$$= \frac{\pi/4 \times (1-\pi/4)}{n}$$

Assuming N to be large.

$$\frac{\frac{\sum X_i}{n} - \pi/4}{\sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}} \sim N(0,1)$$

by Central limit theorem.

for $\pi$ estimate to lie in $[\pi - 0.01, \pi + 0.01]$

$$\frac{\pi}{4} - \frac{0.01}{4} < \frac{\sum X_i}{n} < \frac{\pi}{4} + \frac{0.01}{4}$$

$$\frac{-0.01}{4 \times \sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}} < \frac{\frac{\sum X_i}{n} - \pi/4}{\sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}} < \frac{0.01}{4 \times \sqrt{\frac{\pi/4 (1-\pi/4)}{n}}}$$

For $P\left(\frac{-0.01}{4 \times \sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}} < \frac{\sum X_i - \pi/4}{\sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}} < \frac{0.01}{4 \times \sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}}\right) = 0.95$

$$\frac{0.01}{4 \times \sqrt{\frac{\pi/4 \times (1-\pi/4)}{n}}} = 1.96$$

$$n \approx 103598.92$$

Since this value of n is large enough for CLT to hold, our initial assumption was correct

The value of pi calculated at this n was 3.1356, which is within 0.01 of actual value.

Script to run- P1_driver.m, it will generate 8 values for pi, then will generate another value for n=103598.