Q2.

2) $y = -\frac{1}{\lambda} \log(x)$

$x = e^{-\lambda y} = g^{-1}(y)$

By transformation of RVs.

$$P(y) = P(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right|$$

$$= 1 \cdot \lambda e^{-\lambda y}$$

$$= \lambda e^{-\lambda y}$$

Thus, distribution of $y \Rightarrow P(Y=y) = \lambda e^{-\lambda y}$

Likelihood $= \prod_i \lambda e^{-\lambda y_i} = \lambda^n e^{-\lambda \sum_i y_i}$

Prior $= \text{Gamma}(\lambda, \alpha, \beta) \propto \lambda^{\alpha-1} e^{-\beta\lambda}$.

Posterior $\propto \lambda^{\alpha+n-1} e^{-\lambda(\sum y_i + \beta)}$

$\propto \text{Gamma}(\lambda, \alpha+n, \beta+\sum y_i)$

mean of posterior $=$ mean of Gamma$(\lambda, \alpha+n, \beta+\sum y_i)$

$$= \frac{n+\alpha}{\beta + \sum y_i}$$

Contd.

For maximum likelihood estimate

$$\text{Likelihood} = \lambda^n e^{-\lambda \Sigma y_i}$$

$$\log(\text{likeli.}) = n\log\lambda - \lambda \Sigma y_i$$

$$\frac{\partial ll}{\partial \lambda} = n/\lambda - \Sigma y_i$$

for log likelihood to be max^m

$$\frac{\partial ll}{\partial \lambda} = 0$$

$$n/\lambda = \Sigma y_i$$

$$\lambda = n/\Sigma y_i$$

(c)

As can be seen from the graph, as N increases, the error for both the estimates decreases, along with their spread(variance) around the error value.

However, for smaller values of N, the posterior mean is a better estimator as it has the least error. Hence it is preferred.

Both the graphs have been plotted seperately, since plotting them on the same graph decreased visibility.