4. (i) For a bivariate gaussian:-

$$C = E\left((Y-\mu)(Y-\mu)^T\right)$$ where $(Y-\mu)$ is a 2×1 vector

$$= E\begin{pmatrix} (Y_1-\mu_1)^2 & (Y_1-\mu_1)(Y_2-\mu_2) \\ (Y_2-\mu_2)(Y_1-\mu_1) & (Y_2-\mu_2)^2 \end{pmatrix}$$

$$= \begin{pmatrix} E(Y_1-\mu_1)^2 & E[(Y_1-\mu_1)(Y_2-\mu_2)] \\ E[(Y_2-\mu_2)(Y_1-\mu_1)] & E(Y_2-\mu_2)^2 \end{pmatrix}$$

Diagonal elements give the variance ~~where~~ and the non-diagonal elements are the covariance.

Since non-diagonal elements are zero $Cov(Y_1, Y_2) = 0$

thus Y_1 & Y_2 are uncorrelated.

Hence False.

PTO

(ii). Since z_1, z_2 are uncorrelated, $Cov(z_1, z_2) = 0$.

The non-diagonal elements of C correspond to the covariance.

Thus, non-diagonal elements are zero ∴

C is a diagonal matrix.

Let $C = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix}$

$$p(z_1, z_2) = \frac{1}{2\pi\sqrt{|C|}} e^{-0.5\left((z-\mu)^T C^{-1} (z-\mu)\right)}$$

where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$= \frac{1}{2\pi\sqrt{|C|}} e^{-0.5\left(\begin{bmatrix} z_1-\mu_1 & z_2-\mu_2 \end{bmatrix} \begin{bmatrix} \frac{1}{c_{11}} & \\ & \frac{1}{c_{22}} \end{bmatrix} \begin{bmatrix} z_1-\mu_1 \\ z_2-\mu_2 \end{bmatrix}\right)}$$

$$= \frac{1}{2\pi\sqrt{|C|}} e^{-0.5\left(\left[\left(\frac{1}{c_{11}}\right)(z_1-\mu_1) \quad \left(\frac{1}{c_2}\right)(z_2-\mu_2)\right] \begin{bmatrix} z_1-\mu_1 \\ z_2-\mu_2 \end{bmatrix}\right)}$$

$$= \frac{1}{2\pi\sqrt{|C|}} e^{-0.5\left(\frac{(z_1-\mu_1)^2}{c_{11}} + \frac{(z_2-\mu_2)^2}{c_{22}}\right)}$$

$$= \frac{e^{-\frac{(z_1-\mu_1)^2}{2c_{11}}}}{\sqrt{c_{11} 2\pi}} \cdot \frac{e^{-\frac{(z_2-\mu_2)^2}{2c_{22}}}}{\sqrt{c_{22} \cdot 2\pi}}$$

$$= \frac{e^{-\frac{(z_1-\mu_1)^2}{2\sigma_1^2}}}{\sigma_1\sqrt{2\pi}} \cdot \frac{e^{-\frac{(z_2-\mu_2)^2}{2\sigma_2^2}}}{\sigma_2\sqrt{2\pi}} \qquad \because c_{ii} = \sigma_i^2$$

$$= p(z_1) \cdot p(z_2)$$

Thus, the random variables z_1, z_2 are independent.