HW 8

Problem 1:

As per the Approximation Theorem, we want f_K, where $K := \text{Span}(1)$ – ie. the "closest" constant is f projected onto the constants.

What is a basis for "constant functions"? $1! \Rightarrow S = \{1\}$.

$$g = f_S = \sum_{i \in \{1\}} \frac{\langle f, i \rangle}{\|i\|^2} i = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1$$

$$= \frac{\int_1^3 \frac{1}{t} \cdot 1 \, dt}{\int_1^3 1 \cdot 1 \, dt} = \frac{\ln(3) - \ln(1)}{2} = \boxed{\frac{1}{2} \ln(3)}$$

Now we compute $\|g - f\|^2$

$$\|g - f\|^2 = \langle g - f, g - f \rangle = \langle \tfrac{1}{2}\ln(3) - \tfrac{1}{x}, \tfrac{1}{2}\ln(3) - \tfrac{1}{x} \rangle$$

$$= \langle \tfrac{1}{2}\ln(3), \tfrac{1}{2}\ln(3) \rangle - 2 \langle \tfrac{1}{2}\ln(3)\tfrac{1}{x}, \tfrac{1}{2}\ln(3)\tfrac{1}{x} \rangle + \langle \tfrac{1}{x}, \tfrac{1}{x} \rangle$$

$$= \left(\tfrac{1}{2}\ln(3)\right)^2 \langle 1, 1 \rangle - 2\left(\tfrac{1}{2}\ln(3)\right)^2 \langle \tfrac{1}{x}, \tfrac{1}{x} \rangle + \langle \tfrac{1}{x}, \tfrac{1}{x} \rangle.$$

$$= \frac{1}{4}\ln(3)^2 2 - \left(2 \frac{1}{4}\ln(3)^2 \overset{-}{\cancel{+}} 1\right) \langle \tfrac{1}{x}, \tfrac{1}{x} \rangle$$

<Detour:> $\int_1^3 \frac{1}{t^2} dt = \left(-t^{-1}\right)\Big|_1^3 = -\frac{1}{3} - -\frac{1}{1} = -\frac{1}{3} + 1 = \frac{2}{3}$ </Detour>

Resume: $\frac{1}{2}\ln(3)^2 - \frac{1}{2}\ln(3)^2 \cdot \frac{2}{3} - \frac{2}{3} = \cancel{\boxed{\frac{1}{3}\left(\frac{1}{2}\ln(3)^2\right) - \frac{2}{3}}}$

$$\boxed{-\frac{1}{2}(\ln(3))^2 + \frac{2}{3}}$$