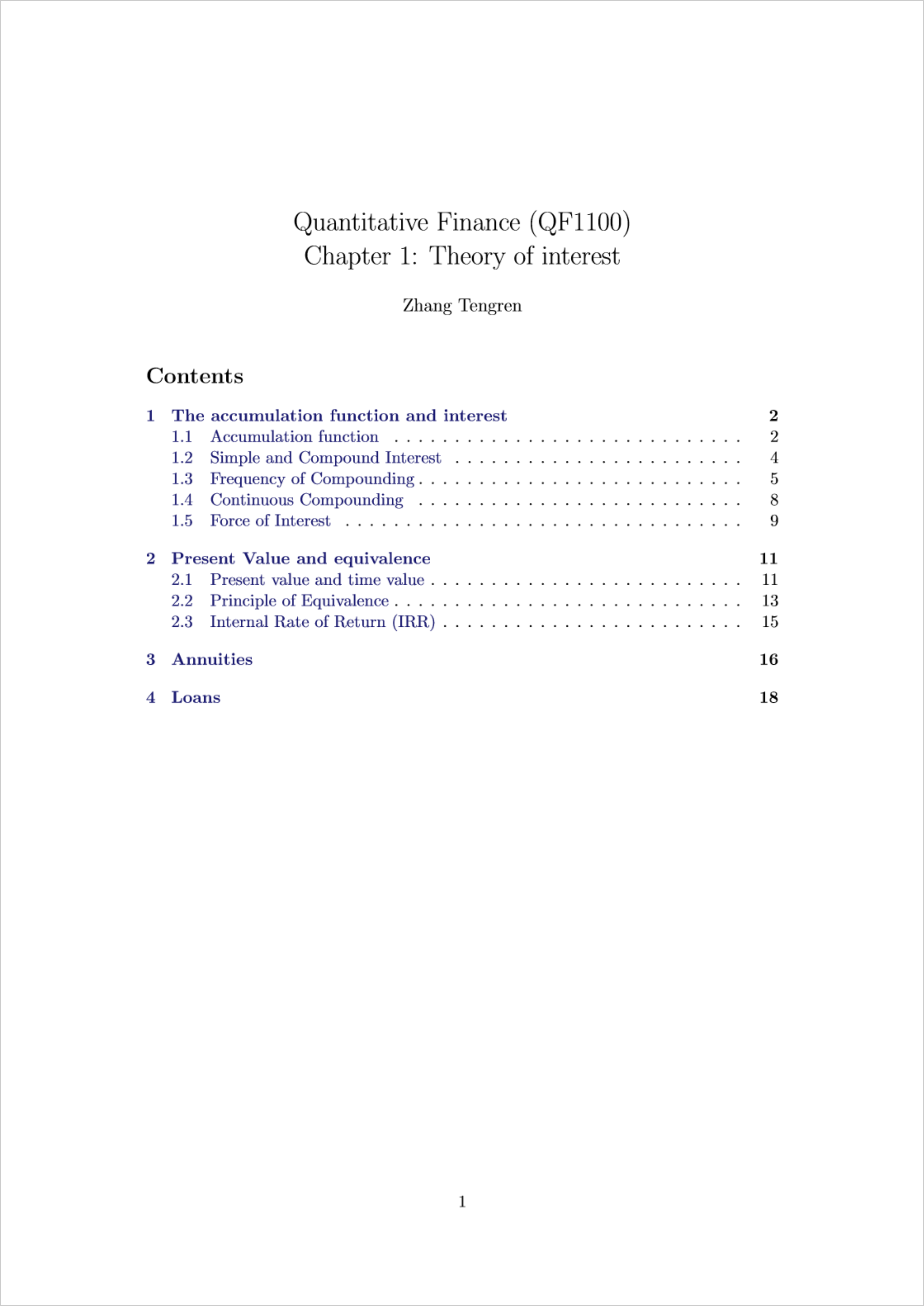
ch1

Monday, August 14, 2023 12:04 PM

ch1



Quantitative Finance (QF1100)
Chapter 1: Theory of interest

Zhang Tengren

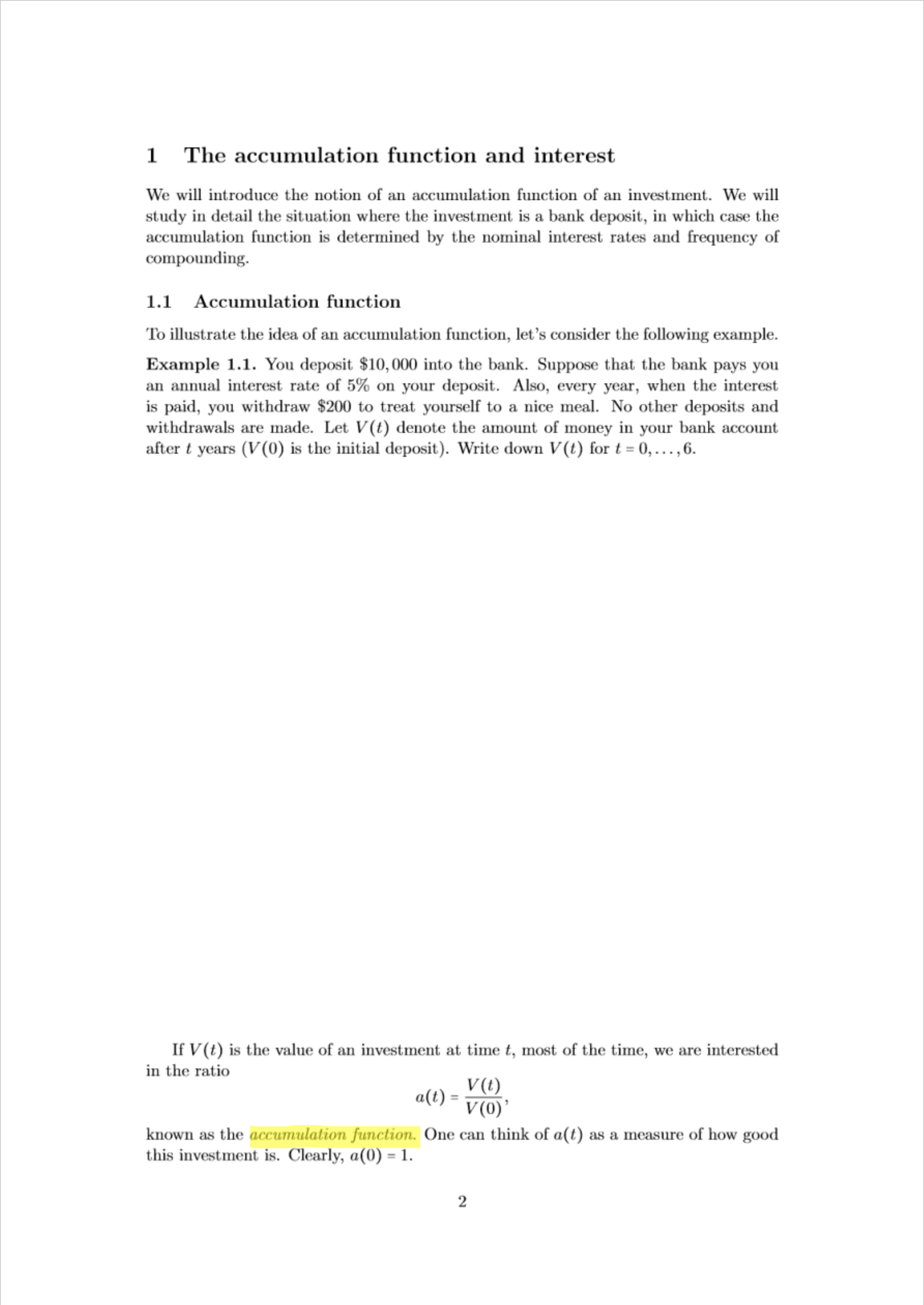
Contents

1 The accumulation function and interest **2**
- 1.1 Accumulation function 2
- 1.2 Simple and Compound Interest 4
- 1.3 Frequency of Compounding 5
- 1.4 Continuous Compounding 8
- 1.5 Force of Interest 9

2 Present Value and equivalence **11**
- 2.1 Present value and time value 11
- 2.2 Principle of Equivalence 13
- 2.3 Internal Rate of Return (IRR) 15

3 Annuities **16**

4 Loans **18**



1 The accumulation function and interest

We will introduce the notion of an accumulation function of an investment. We will study in detail the situation where the investment is a bank deposit, in which case the accumulation function is determined by the nominal interest rates and frequency of compounding.

1.1 Accumulation function

To illustrate the idea of an accumulation function, let's consider the following example.

Example 1.1. You deposit \$10,000 into the bank. Suppose that the bank pays you an annual interest rate of 5% on your deposit. Also, every year, when the interest is paid, you withdraw \$200 to treat yourself to a nice meal. No other deposits and withdrawals are made. Let $V(t)$ denote the amount of money in your bank account after t years ($V(0)$ is the initial deposit). Write down $V(t)$ for $t = 0, \ldots, 6$.

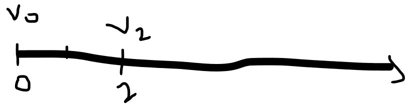
If $V(t)$ is the value of an investment at time t, most of the time, we are interested in the ratio

$$a(t) = \frac{V(t)}{V(0)},$$

known as the *accumulation function.* One can think of $a(t)$ as a measure of how good this investment is. Clearly, $a(0) = 1$.

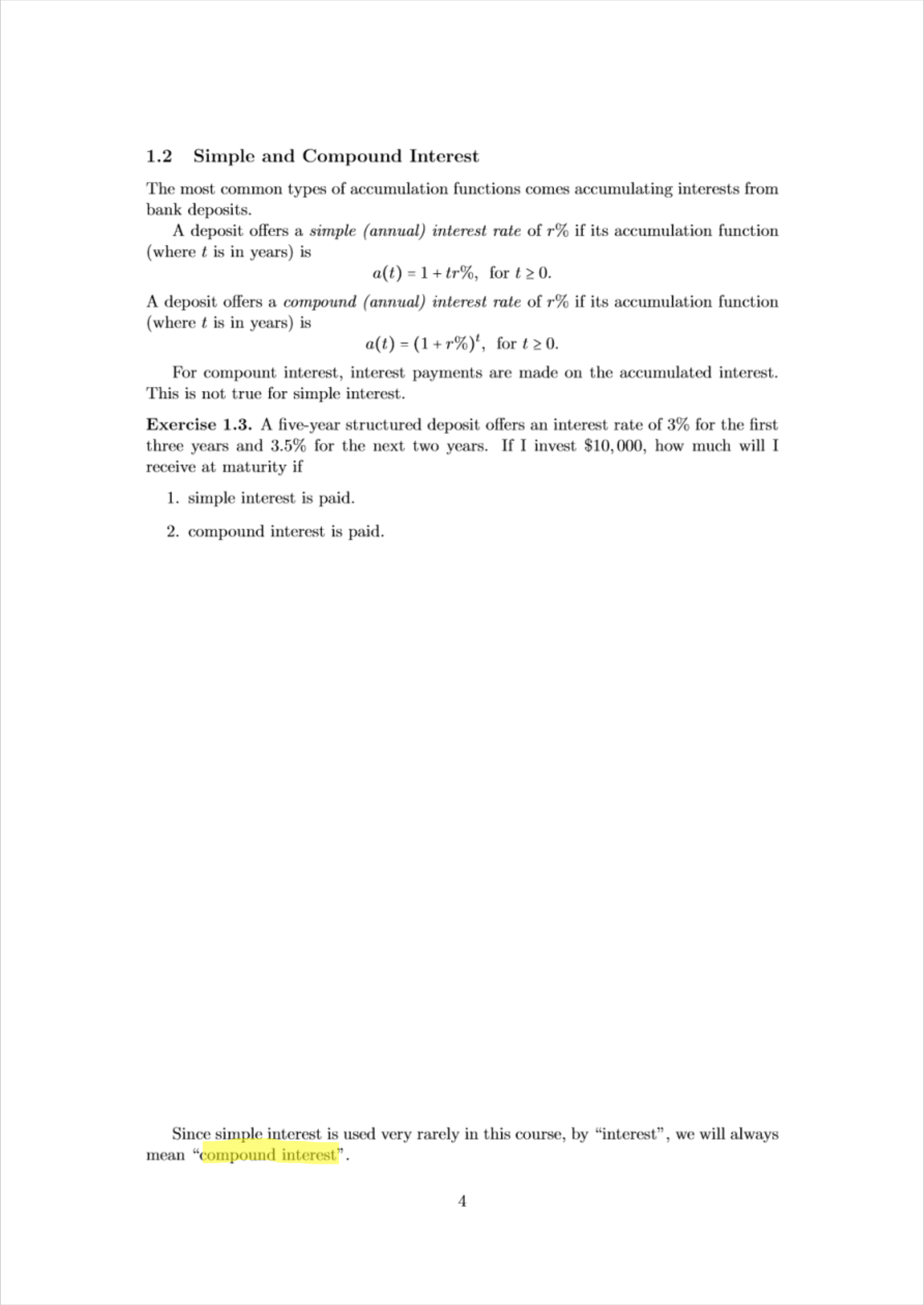
$V(t) = 1.05(v(t-1)) -200 =$
$=10000 * 1.05^t - 200(1.05)^t - 1 /$
(0.05)
$V(0) = 10000$
$v(1) =$
Geometric series
$S = 1 + r + r^2 \ldots$
$S = (r^{(k+1)}-1)/(r-1)$

We use a diagram called **cash flow diagram** to visualize $a(t)$.



The lower half of a cash flow diagram denotes time, and the upper half denotes payments. In a cash flow diagram, the unit for the time parameter t is usually in years, but sometimes also in months or days.

Example 1.2. Draw the cash flow diagram for the investment in Exercise 1.1.



1.2 Simple and Compound Interest

The most common types of accumulation functions comes accumulating interests from bank deposits.

A deposit offers a *simple (annual) interest rate* of $r\%$ if its accumulation function (where t is in years) is

$$a(t) = 1 + tr\%, \quad \text{for } t \geq 0.$$

A deposit offers a *compound (annual) interest rate* of $r\%$ if its accumulation function (where t is in years) is

$$a(t) = (1 + r\%)^t, \quad \text{for } t \geq 0.$$

For compount interest, interest payments are made on the accumulated interest. This is not true for simple interest.

Exercise 1.3. A five-year structured deposit offers an interest rate of 3% for the first three years and 3.5% for the next two years. If I invest \$10,000, how much will I receive at maturity if

1. simple interest is paid.
2. compound interest is paid.

Since simple interest is used very rarely in this course, by "interest", we will always mean "compound interest".

.

1. Y0 = 10000
2. Y1 = 10000(1.03)
3. Y2 = 10000(1.03)^2
4. Y3 = 10000(1.03)^2(1.035)
5. Y4 = 10000(1.03)^3(1.035)
6. Y5 = 10000(1.03)^3(1.035)^2 = 11705.56
7. 10000(1 + 3%)^3(1+3.5%)^2 = 11705.6

Simple 1.10000 (1+ 3% + 3% + 3% + 3.5%+3.5%) = 11600

1.3 Frequency of Compounding

A *nominal* (annual) interest rate of $r\%$ is *compounded p times annually* (or equivalently, convertible p-thly) if the year is divided into p equal time periods, and the interest to be paid over each period, is $\frac{r\%}{p}$. We call p the *frequency of compounding.*

Remark 1.4.

1. Sometimes, we write the superscript (p) for the nominal interest rate $r = r^{(p)}$ to indicate the frequency of compounding p.
2. $p = 2, 4$ and 12 correspond to semi-annual, quarterly and monthly compounding respectively.

Example 1.5. Suppose that a savings account offers an interest rate of $r\%$ compounded monthly. If you deposit \$100, calculate the amount of of money you have in the savings account after one month, two months, and one year (assuming that no other deposits and withdrawals are made).

v(t) = amoutn of $ in bank
v(0) = 100
v(1/12) = 100(1+r%/12)
v(2/12) = 100(1+r%/12)^2
v(1) = 100(1+r%/12)^12

If the nominal interest rate of $r\%$ is compounded p times annually, then the quantity

$$r_e\% := \left(1 + \frac{r\%}{p}\right)^p - 1$$

is called the *effective* (annual) interest rate. The accumulation function is then

$$a(t) = (1 + r_e\%)^t = \left(1 + \frac{r\%}{p}\right)^{pt}.$$

Exercise 1.6. At what nominal rate convertible monthly will money be doubled in 5 years?

(1+x)^5 = 2
X = 0.1486 <- effective interest rate
X = (1+r%/12)^12-1
R = 13.94%

Two nominal interest rates are said to be **equivalent** if and only if they yield the same effective interest rate, i.e.

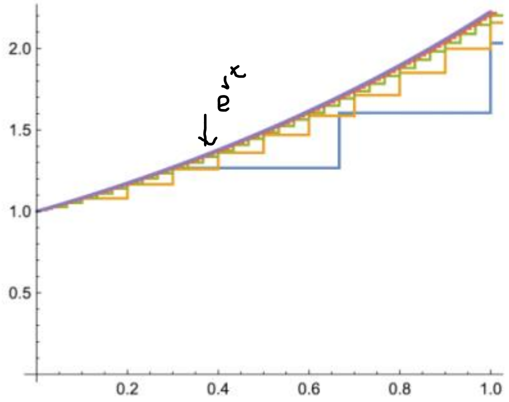
$$\left(1+\frac{r^{(p)}}{p}\right)^p = \left(1+\frac{r^{(q)}}{q}\right)^q.$$

Exercise 1.7. Find the nominal rate convertible monthly that is equivalent to a 4% nominal rate convertible quarterly.

(1+0.04/4)^4 = 1.0406
(1 + r/12)^12
R = 3.987%

1.4 Continuous Compounding

Fix a nominal interest rate r. One can show that the more we increase the frequency of compounding, the larger the effective interest rate.



We say that interest is *compounded continuously* when the frequency of compounding tends to infinity. In that case, the effective interest rate satisfies

$$1 + r_e = \lim_{p\to\infty}\left(1+\frac{r}{p}\right)^p = e^r,$$

and so the corresponding accumulation function is

$$a(t) = (1+r_e)^t = e^{rt}.$$

Exercise 1.8. An investor makes a deposit today to earn an effective interest rate of 4% over the first 3 years, and a nominal rate of 3% compounded monthly for the next 2 years. What is the nominal continuously compounded interest rate over the subsequent five years in order for the original investment to be tripled by the end of ten years?

Y3 = x(1+0.04)^3
Y5 = Y3(1+0.03/12)^24
Y10 = 3 = (1+0.04)^2(1+0.03/12^24(e^5r)
2.61233 = 2.61233 e^5r

R = 0.192

1.5 Force of Interest

Suppose that you buy an investment product where the accumulation function is $a(t)$. The **force of interest** at time t, denoted by $\delta(t)$, is

$$\delta(t) = \frac{a'(t)}{a(t)} = [\ln(a(t))]'.$$

The force of interest at time t is a measure of how good this investment product is at the time t. If the investment product accumulates like continuous compounded interests, i.e. $a(t) = e^{\delta t}$, then notice that $\delta(t) = \delta$. In other words, if $\delta(t)$ is the force of interest at time t of an investment product, then at time t, one is indifferent between this investment product and a deposit with nominal interest rate of $\delta(t)$ compounded continuously.

It follows from the definition of the force of interest that

$$a(t) = \exp\left(\int_0^t \delta(u)\,du\right).$$

Furthermore, if $0 < s < t$, then

$$a(s,t) := \frac{a(t)}{a(s)} = \exp\left(\int_s^t \delta(u)\,du\right)$$

is the value of the investment at time t when \$1 is invested at time s. This implies the *principle of consistency*: For $t_0 < t_1 < t_2 < \ldots < t_n$,

$$a(t_0,t_n) = a(t_0,t_1)a(t_1,t_2)\cdots a(t_{n-1},t_n).$$

Exercise 1.9. If a fund accumulates at force of interest $\delta(t) = 0.02t$ and a deposit of \$1000 is placed in the account at time $t = 2$, find the accumulated value after another 3 years ($t = 5$).

2 Present Value and equivalence

Investment products usually have payouts that are paid at different times in the future. How do we compare investment products with different payout schedules? The key idea is to compare the investment product with a bank deposit.

2.1 Present value and time value

Let $a(t)$ be the accumulation function of a bank deposit. Let c be an amount that you are guaranteed to receive T time periods later. Then the *present value* of c is the quantity

$$\frac{c}{a(T)}.$$

Intuitively, the present value of c is the amount of money one has to deposit into the bank so that after time T, the value of the deposit is c. In other words, one is indifferent between receiving $\frac{c}{a(T)}$ now, or c at time T.

More generally, for a cash flow

$$\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \dots, (c_n, t_n)\}$$

consisting of a series of payments, with c_i received at time t_i, the *present value* of this cash flow, denoted by $PV(\mathbf{C})$, is defined by

$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}.$$

In other words, one is indifferent between receiving $PV(\mathbf{C})$ now and receiving payments according to the cash flow $\mathbf{C}$.

Similarly, the **time value** of the cash flow $\mathbf{C}$ at time t, denoted by $TV_t(\mathbf{C})$, is given by

$$TV_t(\mathbf{C}) = PV(\mathbf{C}) \times a(t).$$

Notice that the time value of $\mathbf{C}$ at $t = 0$ is exactly the present value of $\mathbf{C}$. One is indifferent between receiving $TV_t(\mathbf{C})$ at time t and receiving payments according to the cash flow $\mathbf{C}$.

In the case when the effective annual interest rate is constant at $r\%$ and t is measured in years, then $a(t) = (1 + r\%)^t$, and so

$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i}} \quad \text{and} \quad TV_t(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i - t}}.$$

Also, for the special case when $t_i = i - 1$, the cash flow

$$\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \dots, (c_n, t_n)\} = \{(c_1, 0), (c_2, 1), \dots, (c_n, n-1)\}$$

can be written as $(c_1, c_2, \dots, c_n)$.

Exercise 2.1. To start saving for retirement on her 20th birthday, Miss Saver will invest \$2000 a year in a 20-year structured deposit that earns a nominal 3% interest compounded monthly. The first payment will be made at the end of the first month. At age 40 when the structured deposit is redeemed fully, she will re-invest the accumulated amount in a 20-year endowment fund that pays 5% annual interest compounded continuously. How much will she receive at age 60?

2.2 Principle of Equivalence

Two cash flows are **equivalent** if and only if they have the same present value (equivalently, they have the same time value at any t). In other words, one is indifferent between the two cash flows.

We have seen a special case of this: The cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ is equivalent to a single payment of $PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}$ at time $t = 0$.

Exercise 2.2. At 2% effective annual rate of interest, the cash flow

$$\mathbf{C}_1 = (1, 0, 3, 0, 5)$$

is equivalent to the cash flow

$$\mathbf{C}_2 = (\underbrace{0, 0, \ldots, 0,}_{k \text{ zeros}} 4.74, 4.74)$$

Find

1. the present value of the cash flow stream $\mathbf{C}_1$.
2. the integer k.

Exercise 2.3. Find the time value at $t = 7$ of the cash flow $\mathbf{C} = (-30, 20, -40, 50)$ at an effective annual rate of 3%. Hence, find

1. the time value of $\mathbf{C}$ at $t = 8$,
2. the present value of $\mathbf{C}$,

2.3 Internal Rate of Return (IRR)

Given a cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$, the equation

$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i}} = 0,$$

(with variable r) is known as the *equation of value*. Any non-negative solution to the equation of value is called the *yield*, or *internal rate of return (IRR)*, of the cash flow.

One should think of the IRR of $\mathbf{C}$ as the amount that the prevailing effective interest rate needs to be in order for one to be indifferent between the cash flow $\mathbf{C}$ and \$0.

Exercise 2.4. Find all the IRRs of an investment project with cash flow stream $(-100, 0, 50, 0, 150)$.

3 Annuities

An **annuity** is a series of payments made at regular intervals. A **perpetuity** is an annuity with an infinite number of payments.

Example 3.1 (Annuities with varying payments)**.** The first payment of \$100 of a 10-year annuity is made at the beginning of the first year. Every subsequent payment increases by 10%. Assuming an annual interest rate of 5%, calculate the present value of this annuity.

Exercise 3.2 (Perpetuity). A perpetual annuity pays

$$\begin{array}{ll} \$2 & \text{at } t = 2, 4, 6, 8, \ldots, 30. \\ \$1 & \text{at } t = 31, 41, 51, 61, \ldots \end{array}$$

where t is in years. Find, to 4 significant figures, the present value of this annuity given that interest is 3% compounded continuously.

4 Loans

Loans are normally repaid by a series of installment payments made at periodic intervals. The size of each installment can be determined using a present-value analysis. Specifically, if we let L be the amount of loan taken at time $t = 0$ and let $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ be the series of repayments, then

$$L = \text{ the present value of } C.$$

Example 4.1. If you borrow \$1000 for a term of 4 years at an annual interest rate of 5% and wish to completely pay off the loan by equal installments, with the first payment made at the end of the first year, how much should you pay per year?

We can also compute the balance of the loan at any point in time. The *loan balance* immediately after the m-th installment has been paid is the time value at $t = m$ of the remaining $(n - m)$ installment payments.

Exercise 4.2. A loan is being repaid with 20 annual payments of $1000, with payment made at the end of each year. Immediately after the 5th payment has been made, the borrower wishes to pay an additional $2000 and then repay the balance over 12 years by annual installment of $\$x$. If the effective annual interest rate is 9%, find x.

We now turn to problems in which the number of payments n is to be determined given the loan amount L, the stream of repayments, assumed to be an annual repayment of the same amount, A and the interest rates. Quite often, the value of n is not an integer. In this case, the loan will be repaid with $[n]$ ($[n]$ is the greatest integer less than or equal to n) full payments of A plus a final payment, B made at some time $T > [n]$, where B is determined from the equation

$$L = \text{ present value of } \overbrace{(A, A, \ldots, A)}^{n \text{ payments}} + \text{ present value of } \{(B, T)\}$$

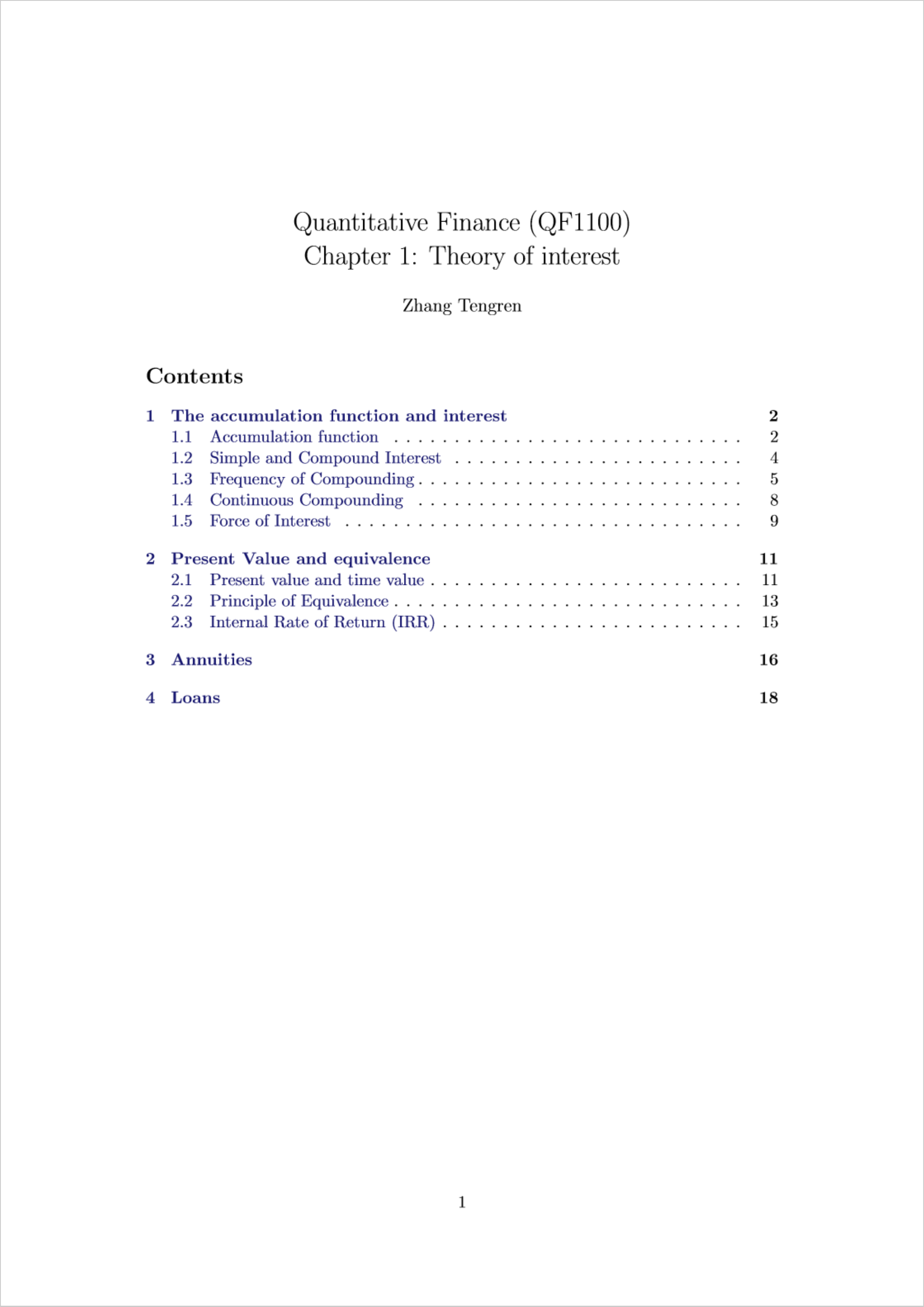
Exercise 4.3. A loan of \$10,000 is to be repaid by annual installments of \$1000. The effective interest rate is 4.5% per annum. Determine the total number of payments to be made and the amount of the last payment, given that the last payment will be made together with the last installment payment of \$1000.

n

ch1 (2)

Friday, August 18, 2023 1:05 PM

ch1 (2)



Quantitative Finance (QF1100)
Chapter 1: Theory of interest

Zhang Tengren

Contents

1 The accumulation function and interest **2**
- 1.1 Accumulation function 2
- 1.2 Simple and Compound Interest 4
- 1.3 Frequency of Compounding 5
- 1.4 Continuous Compounding 8
- 1.5 Force of Interest 9

2 Present Value and equivalence **11**
- 2.1 Present value and time value 11
- 2.2 Principle of Equivalence 13
- 2.3 Internal Rate of Return (IRR) 15

3 Annuities **16**

4 Loans **18**

1 The accumulation function and interest

We will introduce the notion of an accumulation function of an investment. We will study in detail the situation where the investment is a bank deposit, in which case the accumulation function is determined by the nominal interest rates and frequency of compounding.

1.1 Accumulation function

To illustrate the idea of an accumulation function, let's consider the following example.

Example 1.1. You deposit \$10,000 into the bank. Suppose that the bank pays you an annual interest rate of 5% on your deposit. Also, every year, when the interest is paid, you withdraw \$200 to treat yourself to a nice meal. No other deposits and withdrawals are made. Let $V(t)$ denote the amount of money in your bank account after t years ($V(0)$ is the initial deposit). Write down $V(t)$ for $t = 0, \ldots, 6$.

$V(0) = \underline{\$10,000}$

$V(1) = 10,000 \times (1.05) - 200 = \underline{10,300}$

$V(2) = (10,000 \times (1.05) - 200) \times (1.05) - 200$

$= 10,000 \times 1.05^2 - 200 \times 1.05 - 200 = \underline{10,615}$

$V(3) = 10,000 \times 1.05^3 \underbrace{- 200 \times 1.05^2 - 200 \times 1.05 - 200}_{-200(1.05^2 + 1.05 + 1)} = \underline{10,945.75}$

$\vdots$

$V(t) = 10,000 \times 1.05^t - 200 \sum_{i=0}^{t-1} 1.05^i$

$= 10,000 \times 1.05^t - 200 \frac{(1.05)^t - 1}{0.05}$

$s = \overbrace{1 + r + r^2 + r^3 + \ldots + r^k}^{k+1}$

$rs = \cancel{r} + \cancel{r^2} + r^3 + \ldots + r^{k+1}$

$rs - s = r^{k+1} - 1$

$\Rightarrow s = \frac{r^{k+1} - 1}{r - 1}$

$V(4) = \underline{11,293.09}$

$V(5) = \ldots$

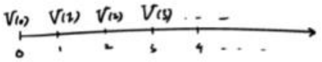
$V(6) = \ldots$

If $V(t)$ is the value of an investment at time t, most of the time, we are interested in the ratio

$$a(t) = \frac{V(t)}{\underline{V(0)}},$$

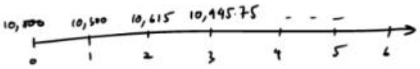
known as the *accumulation function*. One can think of $a(t)$ as a measure of how good this investment is. Clearly, $a(0) = 1$.

We use a diagram called **cash flow diagram** to visualize $a(t)$.



The lower half of a cash flow diagram denotes time, and the upper half denotes payments. In a cash flow diagram, the unit for the time parameter t is usually in years, but sometimes also in months or days.

Example 1.2. Draw the cash flow diagram for the investment in Exercise 1.1.



1.2 Simple and Compound Interest

The most common types of accumulation functions comes accumulating interests from bank deposits.

A deposit offers a *simple (annual) interest rate* of $r\%$ if its accumulation function (where t is in years) is

$$a(t) = 1 + tr\%, \quad \text{for } t \geq 0.$$

A deposit offers a *compound (annual) interest rate* of $r\%$ if its accumulation function (where t is in years) is

$$a(t) = (1 + r\%)^t, \quad \text{for } t \geq 0.$$

For compount interest, interest payments are made on the accumulated interest. This is not true for simple interest.

Exercise 1.3. A five-year structured deposit offers an interest rate of 3% for the first three years and 3.5% for the next two years. If I invest \$10,000, how much will I receive at maturity if

1. simple interest is paid.
2. compound interest is paid.

1) $10{,}000 \cdot (1 + 3\% + 3\% + 3\% + 3.5\% + 3.5\%) = \$11{,}600$.

2) $10{,}000\,(1+3\%)^3(1+3.5\%)^2 = \$11{,}705.60$

Since simple interest is used very rarely in this course, by "interest", we will always mean "compound interest".

1.3 Frequency of Compounding

A *nominal* (annual) interest rate of $r\%$ is *compounded p times annually* (or equivalently, convertible p-thly) if the year is divided into p equal time periods, and the interest to be paid over each period, is $\frac{r\%}{p}$. We call p the *frequency of compounding.*

Remark 1.4. 1. Sometimes, we write the superscript (p) for the nominal interest rate $r = r^{(p)}$ to indicate the frequency of compounding p.

2. $p = 2, 4$ and 12 correspond to semi-annual, quarterly and monthly compounding respectively.

Example 1.5. Suppose that a savings account offers an interest rate of $r\%$ compounded monthly. If you deposit \$100, calculate the amount of of money you have in the savings account after one month, two months, and one year (assuming that no other deposits and withdrawals are made).

$V(t)$ = amount of money in the bank after t years.

$V(0) = \$100$.

$V(\frac{1}{12}) = 100 \times (1 + \frac{r\%}{12})$

$V(\frac{2}{12}) = 100 \times (1 + \frac{r\%}{12})^2$

$\vdots$

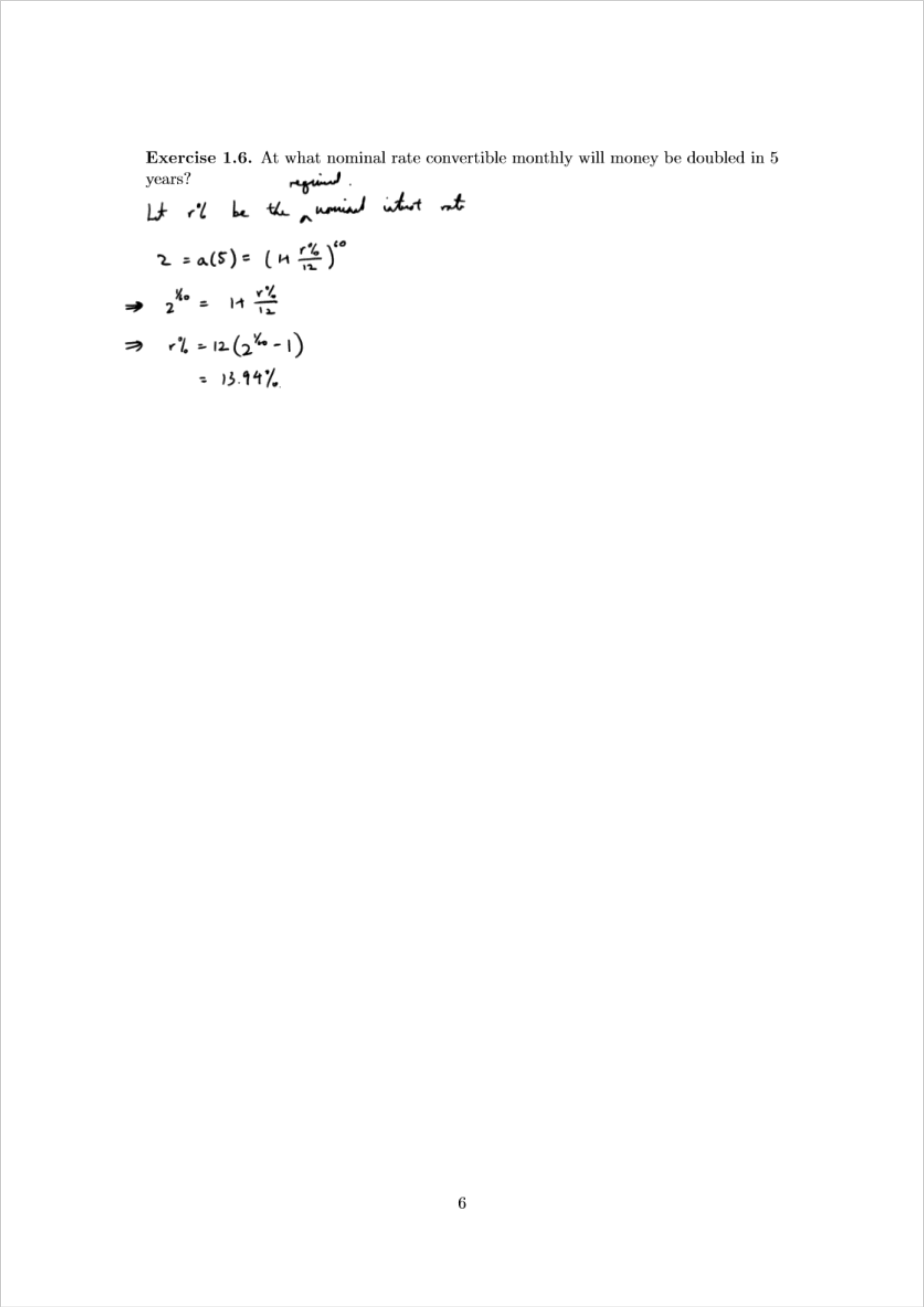
$V(1) = 100 \times (1 + \frac{r\%}{12})^{12} \neq 100 \times (1 + r\%)$

If the nominal interest rate of $r\%$ is compounded p times annually, then the quantity

$$r_e\% := \left(1 + \frac{r\%}{p}\right)^p - 1$$

is called the *effective* (annual) interest rate. The accumulation function is then

$$a(t) = (1 + r_e\%)^t = \left(1 + \frac{r\%}{p}\right)^{pt}.$$



Exercise 1.6. At what nominal rate convertible monthly will money be doubled in 5 years?

Let r% be the required nominal interest rate.

$$2 = a(5) = \left(1 + \frac{r\%}{12}\right)^{60}$$

$$\Rightarrow 2^{1/60} = 1 + \frac{r\%}{12}$$

$$\Rightarrow r\% = 12\left(2^{1/60} - 1\right)$$

$$= 13.94\%$$

Two nominal interest rates are said to be **equivalent** if and only if they yield the same effective interest rate, i.e.

$$\left(1+\frac{r^{(p)}}{p}\right)^p = \left(1+\frac{r^{(q)}}{q}\right)^q.$$

Exercise 1.7. Find the nominal rate convertible monthly that is equivalent to a 4% nominal rate convertible quarterly.

Let r% be the required nominal rate.

$$\left(1+\frac{4\%}{4}\right)^4 = \left(1+\frac{r\%}{12}\right)^{12}$$

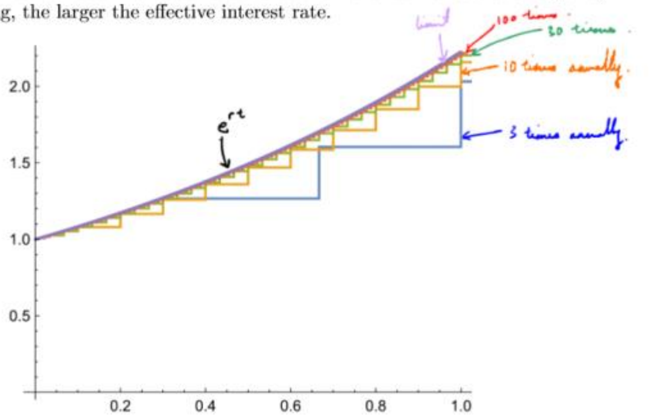
$$\Rightarrow (1.01)^{\frac{1}{3}} = 1+\frac{r\%}{12}$$

$$\Rightarrow r\% = 12\times\left(1.01^{1/3}-1\right)$$

$$= 3.9867\%.$$

1.4 Continuous Compounding

Fix a nominal interest rate r. One can show that the more we increase the frequency of compounding, the larger the effective interest rate.



We say that interest is *compounded continuously* when the frequency of compounding tends to infinity. In that case, the effective interest rate satisfies

$$1 + r_e = \lim_{p\to\infty}\left(1+\frac{r}{p}\right)^p = e^r,$$

and so the corresponding accumulation function is

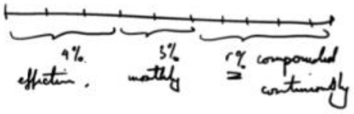
$$a(t) = (1 + r_e)^t = e^{rt}.$$

$$a(p) = \left(1+\frac{r}{p}\right)^p \xrightarrow[p\to\infty]{} ?$$

$$\lim_{p\to\infty}\ln(a(p)) = \lim_{p\to\infty} p\ln\left(1+\frac{r}{p}\right) = \lim_{p\to\infty}\frac{\ln\left(1+\frac{r}{p}\right)}{\frac{1}{p}} = \lim_{p\to\infty}\left(\frac{-rp^{-2}}{1+rp^{-1}}\right)\Big/ -p^{-2} = \lim_{p\to\infty}\frac{r}{1+rp^{-1}} = r.$$

$$\therefore \lim_{p\to\infty} a(p) = e^r$$

Exercise 1.8. An investor makes a deposit today to earn an effective interest rate of 4% over the first 3 years, and a nominal rate of 3% compounded monthly for the next 2 years. What is the nominal continuously compounded interest rate over the subsequent five years in order for the original investment to be tripled by the end of ten years?



$$3 = a(10)$$

$$= (1+4\%)^3\left(1+\frac{3\%}{12}\right)^{24} e^{\frac{r}{100}\times 5}$$

$$\Rightarrow e^{\frac{r}{20}} = \frac{3}{1.04^3\times\left(1+\frac{1}{400}\right)^{24}}$$

$$\Rightarrow r = \ln\left(\frac{3}{1.04^3+\left(1+\frac{1}{400}\right)^{24}}\right)\times 20$$

$$= 18.42\%.$$

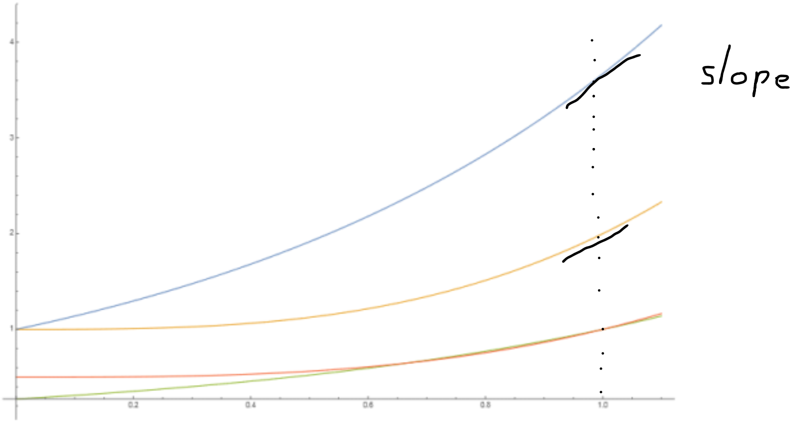
1.5 Force of Interest

Suppose that you buy an investment product where the accumulation function is $a(t)$. The **force of interest** at time t, denoted by $\delta(t)$, is

$$\delta(t) = \frac{a'(t)}{a(t)} = [\ln(a(t))]'.$$

How well the investment is doiing per dollar.

In words, the force of interest in the infinitesimal rate of change of the per dollar value of an investment with respect to time.



A1(t) =
A2(t)

The force of interest at time t is a measure of how good this investment product is at the time t. If the investment product accumulates like continuous compounded interests, i.e. $a(t) = e^{\delta t}$, then notice that $\delta(t) = \delta$. In other words, if $\delta(t)$ is the force of interest at time t of an investment product, then at time t, one is indifferent between this investment product and a deposit with nominal interest rate of $\delta(t)$ compounded continuously.

It follows from the definition of the force of interest that

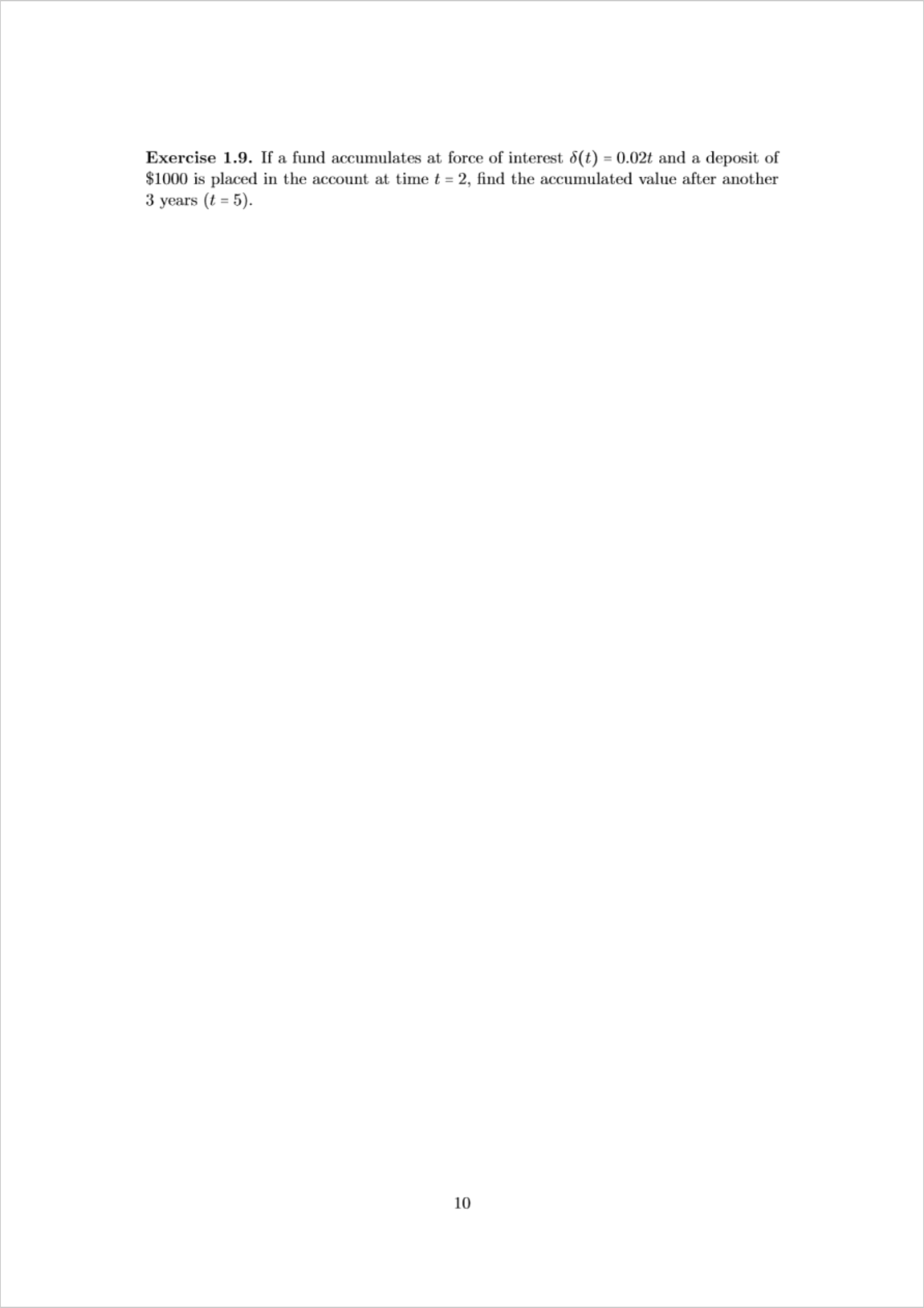
$$a(t) = \exp\left(\int_0^t \delta(u)\,du\right).$$

Furthermore, if $0 < s < t$, then

$$a(s,t) := \frac{a(t)}{a(s)} = \exp\left(\int_s^t \delta(u)\,du\right)$$

is the value of the investment at time t when \$1 is invested at time s. This implies the *principle of consistency*: For $t_0 < t_1 < t_2 < \ldots < t_n$,

$$a(t_0, t_n) = a(t_0, t_1)a(t_1, t_2)\cdots a(t_{n-1}, t_n).$$



Exercise 1.9. If a fund accumulates at force of interest $\delta(t) = 0.02t$ and a deposit of \$1000 is placed in the account at time $t = 2$, find the accumulated value after another 3 years ($t = 5$).

t=2 1000
v(2,5) = a(5)/a(2) * 1000
=
exp($\int_2^5 0.02t dt)$ $* 1000$
= 1233.68

2 Present Value and equivalence

Investment products usually have payouts that are paid at different times in the future. How do we compare investment products with different payout schedules? The key idea is to compare the investment product with a bank deposit.

2.1 Present value and time value

How much C in future is worth today

Let $a(t)$ be the accumulation function of a bank deposit. Let c be an amount that you are guaranteed to receive T time periods later. Then the *present value* of C is the quantity

$$\frac{C}{a(T)}.$$

Intuitively, the present value of C is the amount of money one has to deposit into the bank so that after time T, the value of the deposit is C. In other words, one is indifferent between receiving $\frac{C}{a(T)}$ now, or C at time T.

More generally, for a cash flow

$$\mathbf{C} = \{(C_1, t_1), (C_2, t_2), \ldots, (C_n, t_n)\}$$

consisting of a series of payments, with C_i received at time t_i, the *present value* of this cash flow, denoted by $PV(\mathbf{C})$, is defined by

$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{C_i}{a(t_i)}.$$

In other words, one is indifferent between receiving $PV(\mathbf{C})$ now and receiving payments according to the cash flow $\mathbf{C}$.

Similarly, the **time value** of the cash flow $\mathbf{C}$ at time t, denoted by $TV_t(\mathbf{C})$, is given by

$$TV_t(\mathbf{C}) = PV(\mathbf{C}) \times a(t).$$

Notice that the time value of $\mathbf{C}$ at $t = 0$ is exactly the present value of $\mathbf{C}$. One is indifferent between receiving $TV_t(\mathbf{C})$ at time t and receiving payments according to the cash flow $\mathbf{C}$.

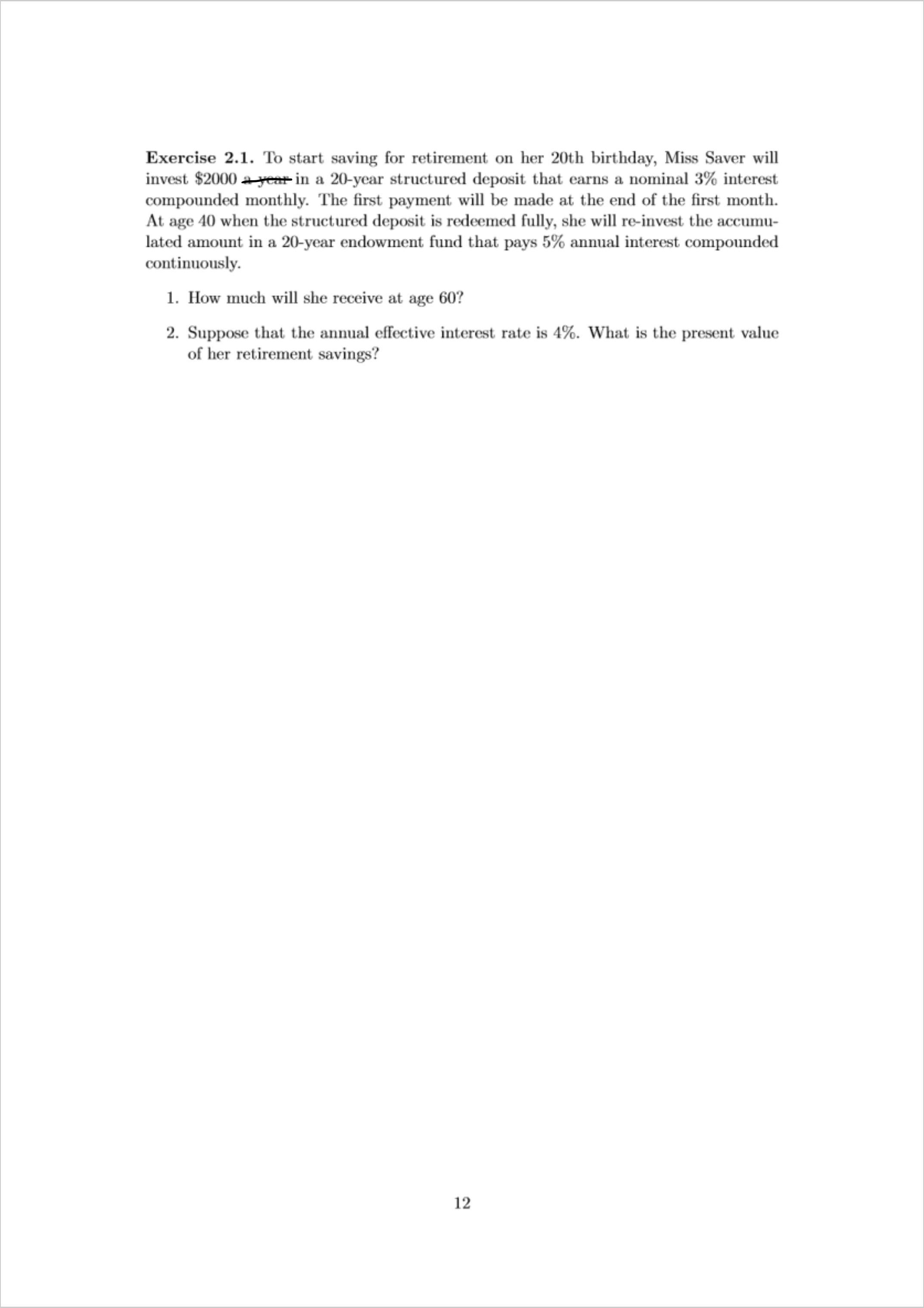
In the case when the effective annual interest rate is constant at $r\%$ and t is measured in years, then $a(t) = (1 + r\%)^t$, and so

$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{C_i}{(1+r)^{t_i}} \quad \text{and} \quad TV_t(\mathbf{C}) = \sum_{i=1}^{n} \frac{C_i}{(1+r)^{t_i - t}}.$$

Also, for the special case when $t_i = i - 1$, the cash flow

$$\mathbf{C} = \{(C_1, t_1), (C_2, t_2), \ldots, (C_n, t_n)\} = \{(C_1, 0), (C_2, 1), \ldots, (C_n, n-1)\}$$

can be written as $(C_1, C_2, \ldots, C_n)$.



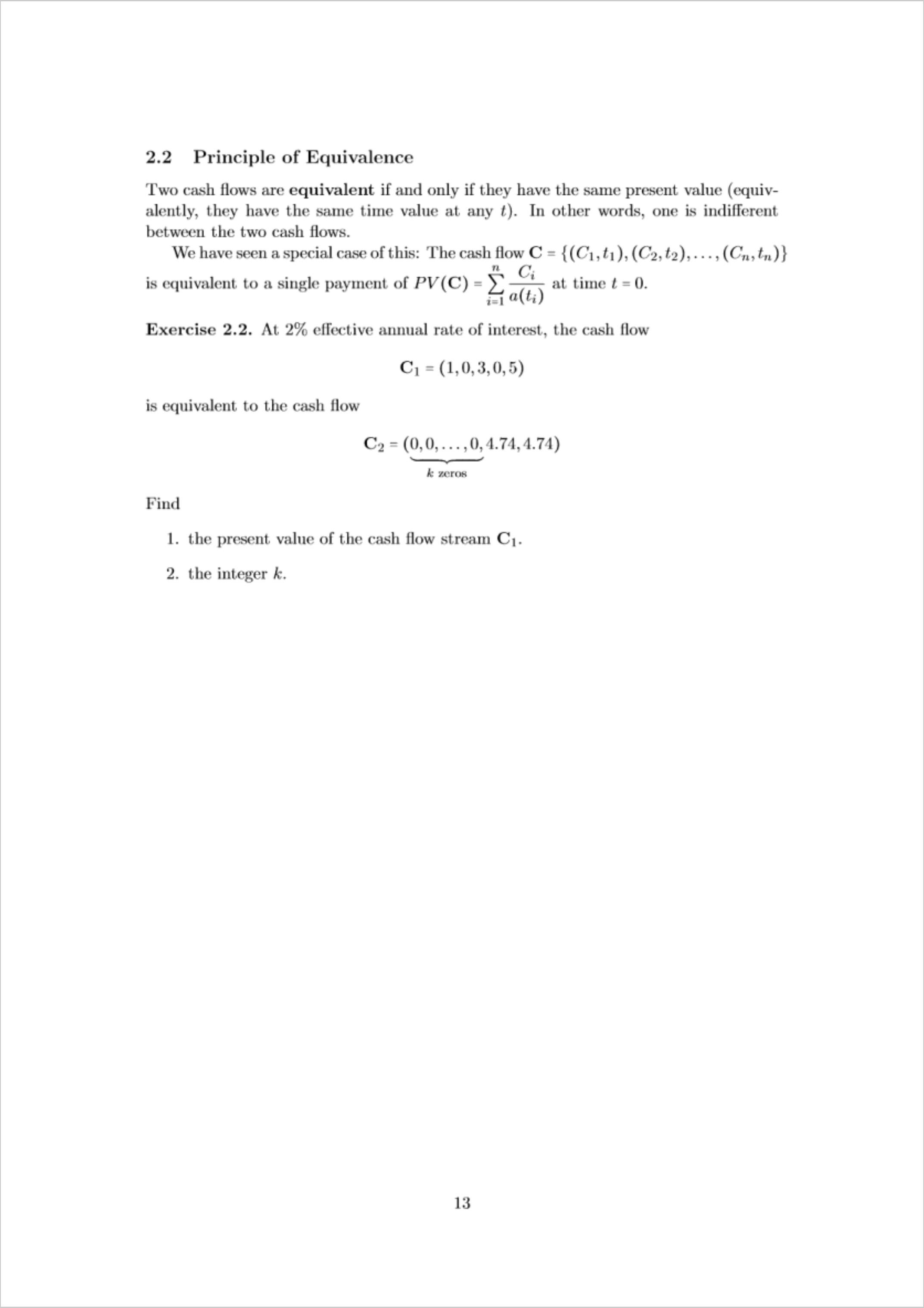
Exercise 2.1. To start saving for retirement on her 20th birthday, Miss Saver will invest \$2000 ~~a year~~ in a 20-year structured deposit that earns a nominal 3% interest compounded monthly. The first payment will be made at the end of the first month. At age 40 when the structured deposit is redeemed fully, she will re-invest the accumulated amount in a 20-year endowment fund that pays 5% annual interest compounded continuously.

1. How much will she receive at age 60?
2. Suppose that the annual effective interest rate is 4%. What is the present value of her retirement savings?

1. 2000(1+0.03/12)^(20*12) = 3641.51

3641.51(e^(0.05*20)) = 9898.65

2. *PV(9898.65) = 9898.65/((1+ 0.04)^40) = 2061.7804*



2.2 Principle of Equivalence

Two cash flows are **equivalent** if and only if they have the same present value (equivalently, they have the same time value at any t). In other words, one is indifferent between the two cash flows.

We have seen a special case of this: The cash flow $\mathbf{C} = \{(C_1, t_1), (C_2, t_2), \dots, (C_n, t_n)\}$ is equivalent to a single payment of $PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{C_i}{a(t_i)}$ at time $t = 0$.

Exercise 2.2. At 2% effective annual rate of interest, the cash flow

$$\mathbf{C}_1 = (1, 0, 3, 0, 5)$$

is equivalent to the cash flow

$$\mathbf{C}_2 = (\underbrace{0, 0, \dots, 0}_{k \text{ zeros}}, 4.74, 4.74)$$

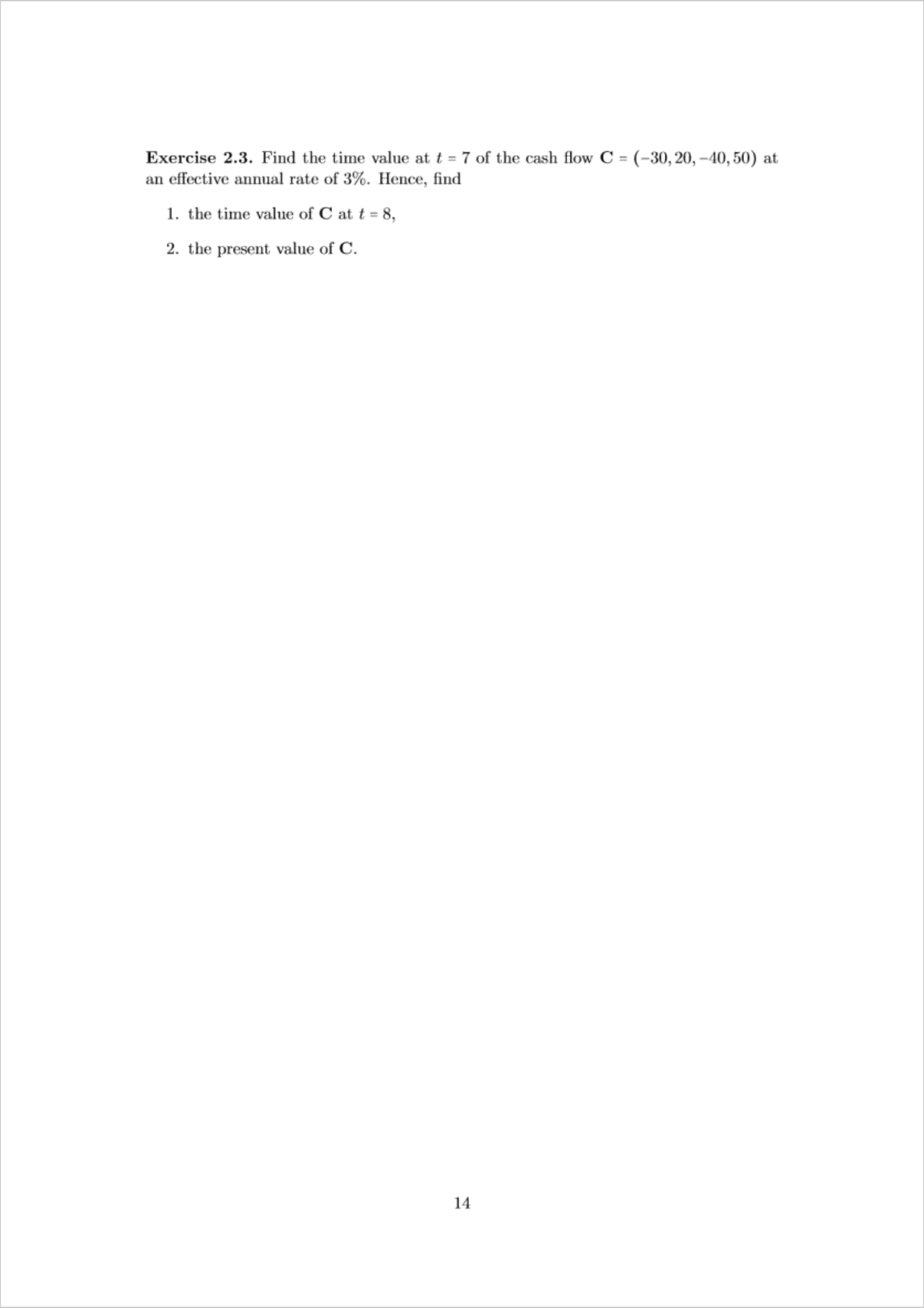
Find

1. the present value of the cash flow stream $\mathbf{C}_1$.
2. the integer k.

1. PV(C1) = 1+0+3/(1+2%)^2+ 5/(1+ 0.02)^4 = 8.5027
2. PV(C2) = 8.5027 = 4.74/(1+ 0.02)^k + 4.74/(1+0.02)^k+1

8.5027/4.74 = 1.7938 = 1/(1.02)^k + 1/(1.02)^k+1 = 1.02^-k + 1.02^-k-1

K = 5



Exercise 2.3. Find the time value at $t = 7$ of the cash flow $\mathbf{C} = (-30, 20, -40, 50)$ at an effective annual rate of 3%. Hence, find

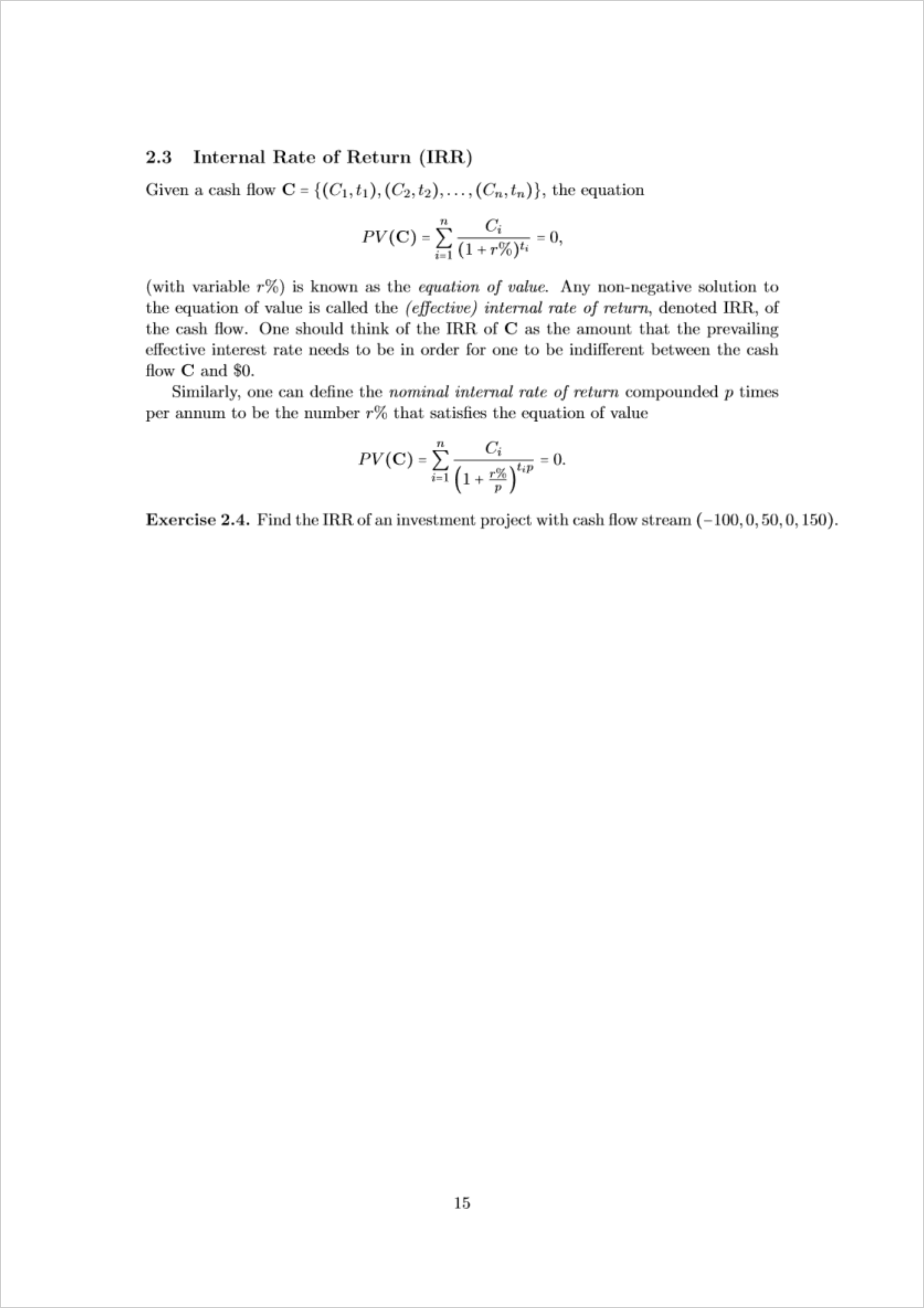
1. the time value of $\mathbf{C}$ at $t = 8$,
2. the present value of $\mathbf{C}$.

14

1.
2. TV(7) = -30(1+3%)^7
   +20(1+3%)^6
   -40(1+3%)^5
   +50(1+3%)^4
   = -3.111

   TV(8) = -3.11(1+0.03)^1 = TV(7)*1.03
   =
   -30(1+3%)^8+20(1+3%)^7
   -40(1+3%)^6
   +50(1+3%)^5
   = -3.204

   PV© = -3.111/(1+3%)^7 = -2.5295



2.3 Internal Rate of Return (IRR)

Given a cash flow $\mathbf{C} = \{(C_1, t_1), (C_2, t_2), \dots, (C_n, t_n)\}$, the equation

$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{C_i}{(1 + r\%)^{t_i}} = 0,$$

(with variable $r\%$) is known as the *equation of value.* Any non-negative solution to the equation of value is called the *(effective) internal rate of return*, denoted IRR, of the cash flow. One should think of the IRR of $\mathbf{C}$ as the amount that the prevailing effective interest rate needs to be in order for one to be indifferent between the cash flow $\mathbf{C}$ and \$0.

Similarly, one can define the *nominal internal rate of return* compounded p times per annum to be the number $r\%$ that satisfies the equation of value

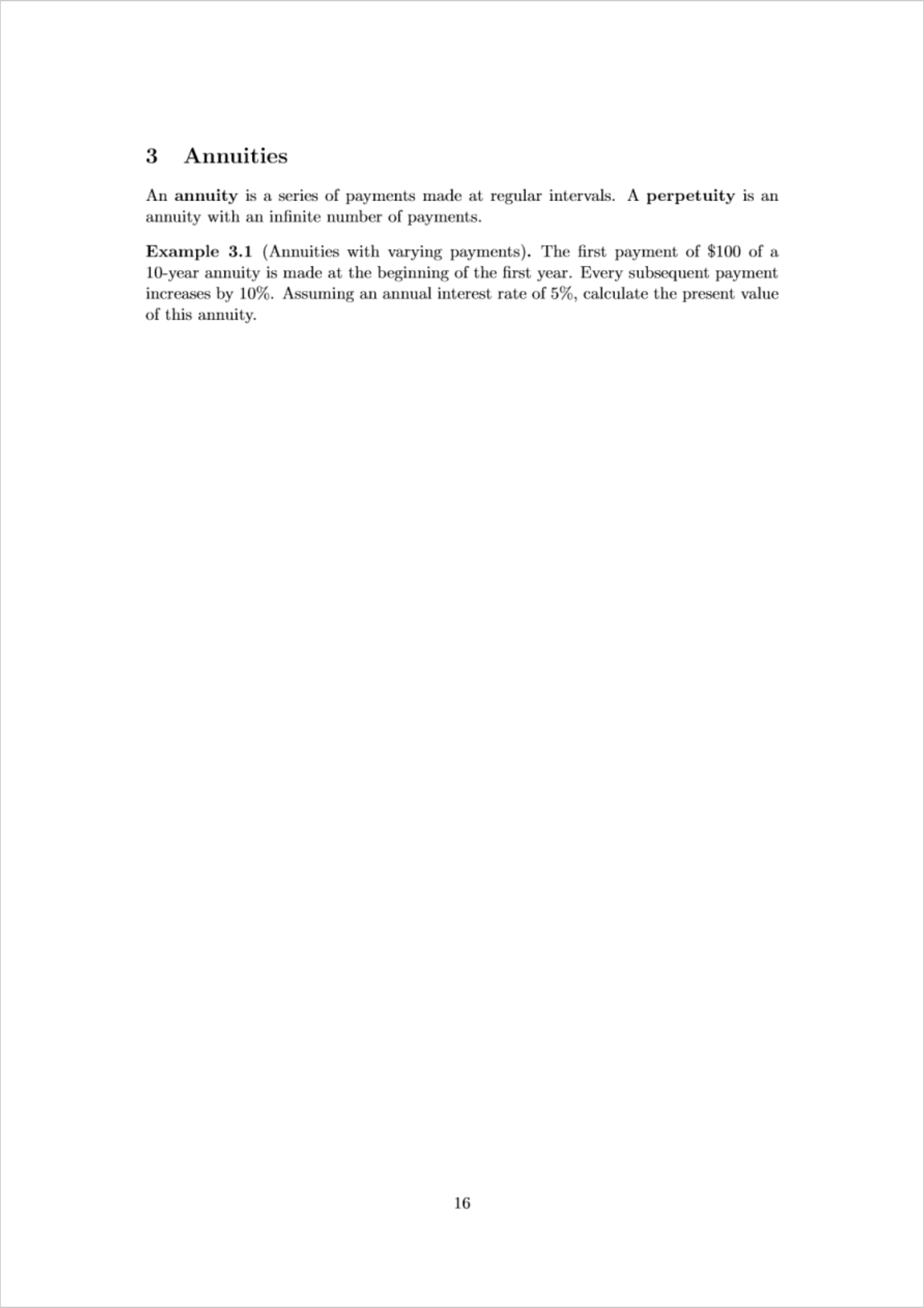
$$PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{C_i}{\left(1 + \frac{r\%}{p}\right)^{t_i p}} = 0.$$

Exercise 2.4. Find the IRR of an investment project with cash flow stream $(-100, 0, 50, 0, 150)$.

PV©= $-100/(1+r)^4 + 50/(1+r)^2 + 150/(1+r)^0$

Find r% s.t. $0 = -100 + 50/(1+r\%)^2 + 150/(1+r\%)^4$
$0 = 3 + (1+r\%)^2 - 2(1+r\%)^4$
$= (1 + (1+r\%)^2)\,(3-2(1+r\%)^2)$
$3=2(1+r\%)^2$
$r\% = 22.47\%$

$0 = 3 + x^2 - 2x^4$ if $x= (1+r\%)$
$= (1+x^2)(3-2x^2)$



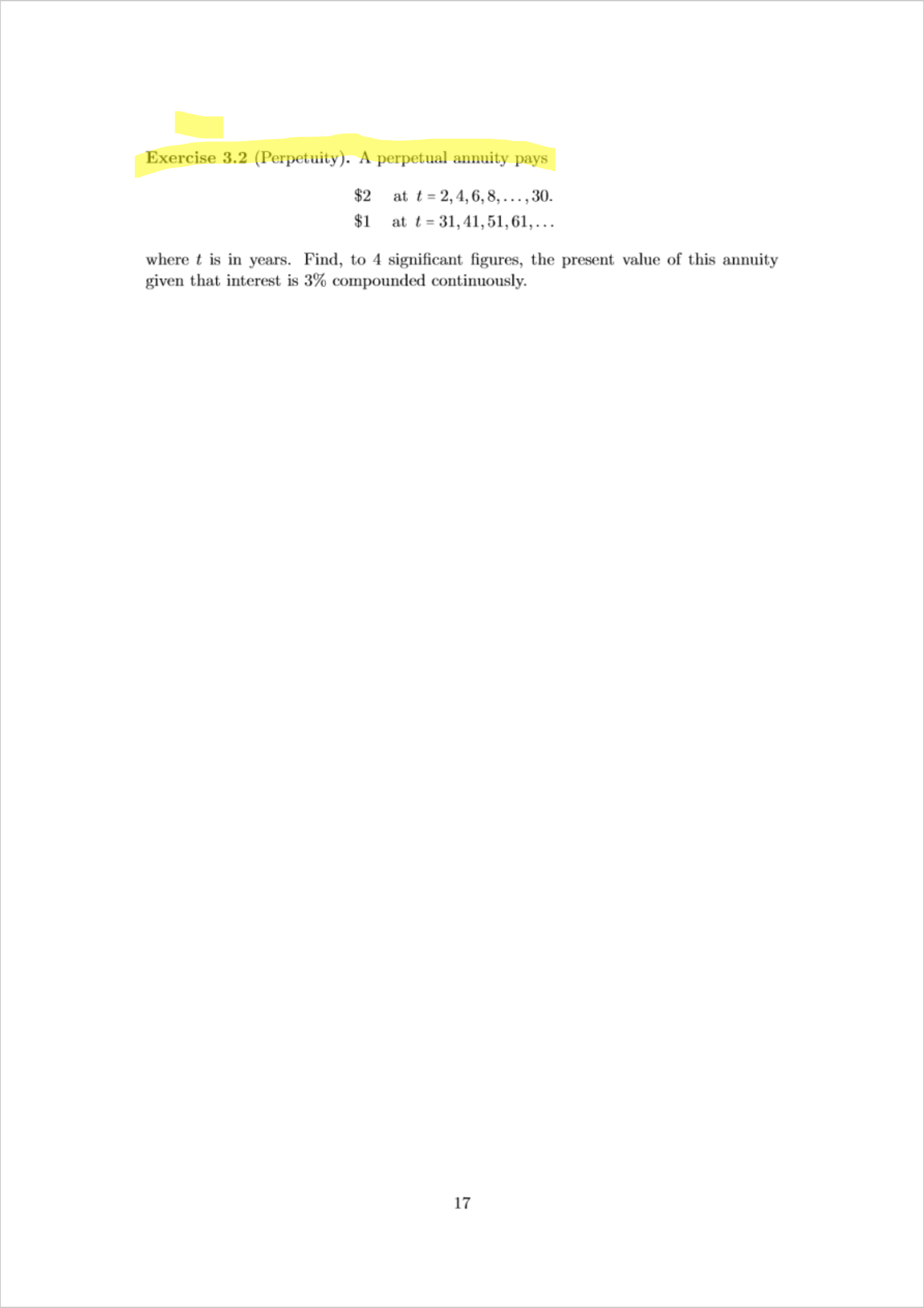
3 Annuities

An **annuity** is a series of payments made at regular intervals. A **perpetuity** is an annuity with an infinite number of payments.

Example 3.1 (Annuities with varying payments)**.** The first payment of $100 of a 10-year annuity is made at the beginning of the first year. Every subsequent payment increases by 10%. Assuming an annual interest rate of 5%, calculate the present value of this annuity.

$T_0 = 100$
$T_1 = 1.1 * 100$
$T_2 = 1.1^2 * 100$
…
$T_9 = 1.1^9 * 100$

$PV = 100 + 1.1*100/1.05 + 1.1^2*100/1.05^2 + \ldots + 1.1^9*100/1.05^9$
$= 100(1 + 1.1/1.05 + \ldots + 1.1^9/1.05^9)$
$= 100((1-(1.1/1.05)^{10})/(1-(1.1/1.05)) = 1243.899$



Exercise 3.2 (Perpetuity). A perpetual annuity pays

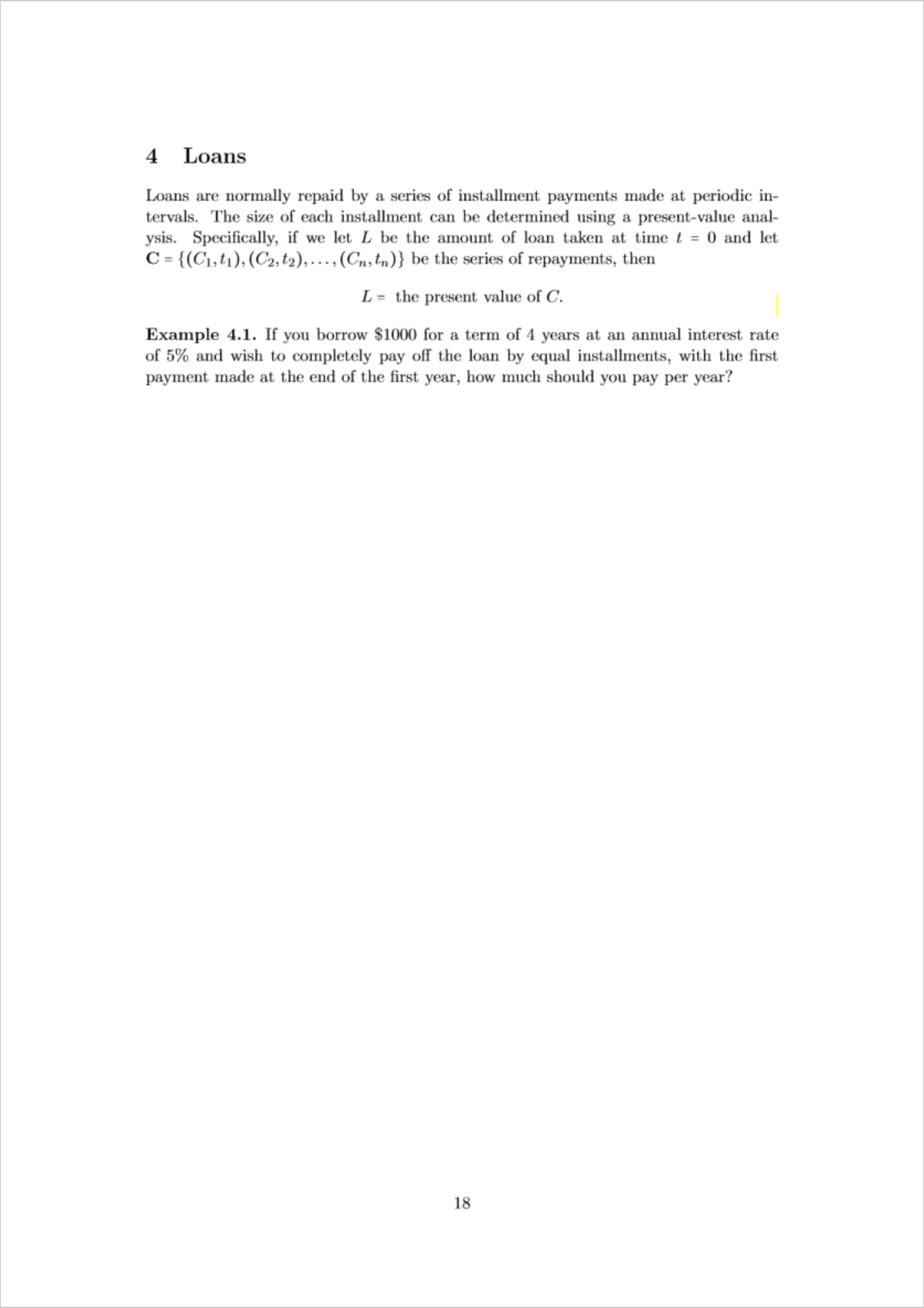
$$\begin{array}{ll} \$2 & \text{at } t = 2, 4, 6, 8, \ldots, 30. \\ \$1 & \text{at } t = 31, 41, 51, 61, \ldots \end{array}$$

where t is in years. Find, to 4 significant figures, the present value of this annuity given that interest is 3% compounded continuously.

PV = 2/e^(3%*2) + 2/e^(3%*4) + ... + 2/(e^(3%*30) + 1/e^(3%*31) + ...
= 2(1 - (1/e)^30)/(1 - 1/e))

1 + r + ... r^n = (1-r^n+1)/(1-r)
1+ r +.... = 1/(1-r)

PV = 2*(1-r^2n+2)/(1-r) + 1/(1-1/e)^10
= 2/e^3%*2 (1 + 1/e^35*2 + ... + 1/(e^3%^2)^14) + 1/(e^3%*31)(1+1/e^3%*10 +...)
= 20.716



4 Loans

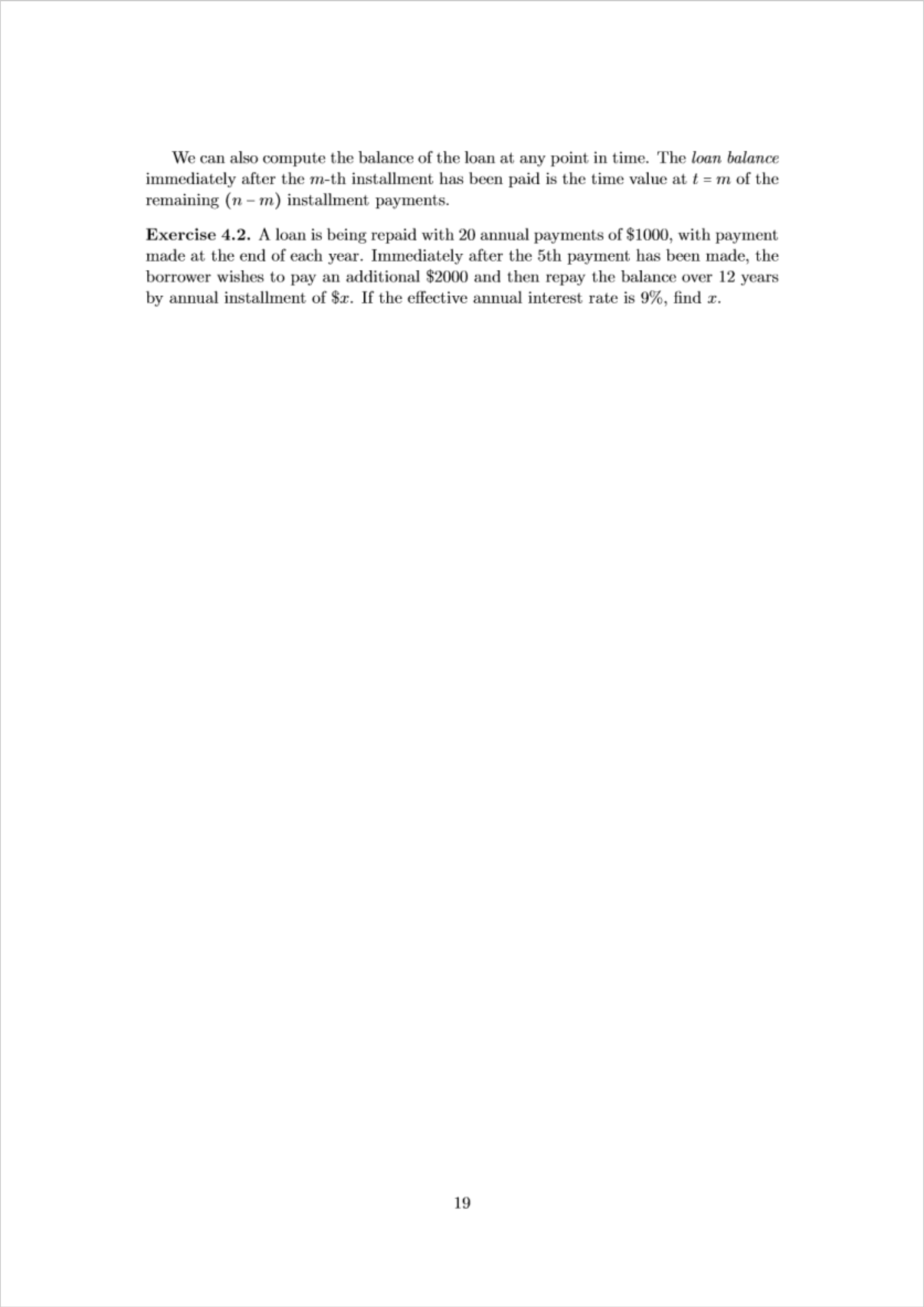
Loans are normally repaid by a series of installment payments made at periodic intervals. The size of each installment can be determined using a present-value analysis. Specifically, if we let L be the amount of loan taken at time $t = 0$ and let $\mathbf{C} = \{(C_1, t_1), (C_2, t_2), \ldots, (C_n, t_n)\}$ be the series of repayments, then

$$L = \text{ the present value of } C.$$

Example 4.1. If you borrow \$1000 for a term of 4 years at an annual interest rate of 5% and wish to completely pay off the loan by equal installments, with the first payment made at the end of the first year, how much should you pay per year?

Cash flow
T0 = 1000
T1 = -x
T2= -x
T3 = -x
T4 = -x

PV = 0 = 1000 - x/(1+0.05) - x/(1+0.05)^2 - x/(1+0.05)^3 - x/(1+1.05)^4
= 1000 - x*1/1.05(1+1/1.05 + 1/1.05^2 +1/1.05^3)
X = 1000 * 1.05 / ((1-1/1.05)^3/(1 - 1/1.05)) = 282.01



We can also compute the balance of the loan at any point in time. The *loan balance* immediately after the m-th installment has been paid is the time value at $t = m$ of the remaining $(n - m)$ installment payments.

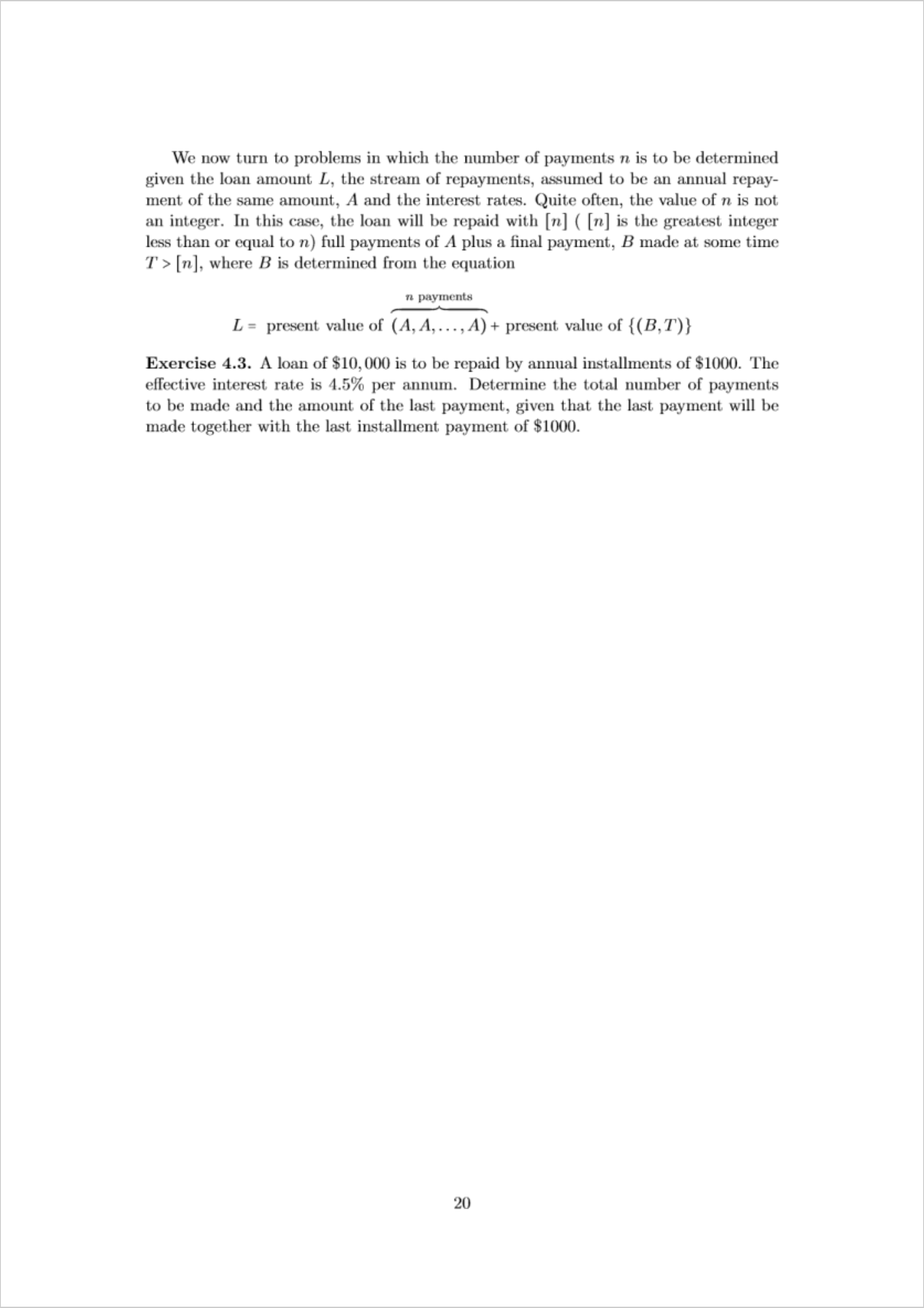
Exercise 4.2. A loan is being repaid with 20 annual payments of \$1000, with payment made at the end of each year. Immediately after the 5th payment has been made, the borrower wishes to pay an additional \$2000 and then repay the balance over 12 years by annual installment of $\$x$. If the effective annual interest rate is 9%, find x.

$TV(1) = 1000/(1.09)^{19}$
$PV(OG) = L - 1000/1.09 - 1000/1.09^2 \ldots 1000/1.09^{20} = 0$
$L = 1000/1.09(1 + \ldots + 1000/1.09^{19})$
$PV(new) = L - 1000/1.09 - \ldots -1000/1.09^5 - 2000/1.09^6 - x/1.09^7 - \ldots x/1.09^{17}$

$1000/1.09(1 + 1/1.09^2 + \ldots 1/1.09^4) + 2000/1.09^6 +x/1.09(1/1.09^6 + 1/1.09^7 \ldots. + 1/1.09^{16}) = L$
$L - 5082.185 = x/1.09(1/1.09^6 + 1/1.09^7 \ldots. + 1/1.09^{16})$
=

Cash flow
T1 = -1000
T2 = -1000
T3 = -1000
T4 = -1000
T5 = -3000
T6 = -x
…
T17 = -x

PV = 0 =



We now turn to problems in which the number of payments n is to be determined given the loan amount L, the stream of repayments, assumed to be an annual repayment of the same amount, A and the interest rates. Quite often, the value of n is not an integer. In this case, the loan will be repaid with $[n]$ ($[n]$ is the greatest integer less than or equal to n) full payments of A plus a final payment, B made at some time $T > [n]$, where B is determined from the equation

$$L = \text{present value of } \overbrace{(A, A, \ldots, A)}^{n \text{ payments}} + \text{present value of } \{(B, T)\}$$

Exercise 4.3. A loan of \$10,000 is to be repaid by annual installments of \$1000. The effective interest rate is 4.5% per annum. Determine the total number of payments to be made and the amount of the last payment, given that the last payment will be made together with the last installment payment of \$1000.

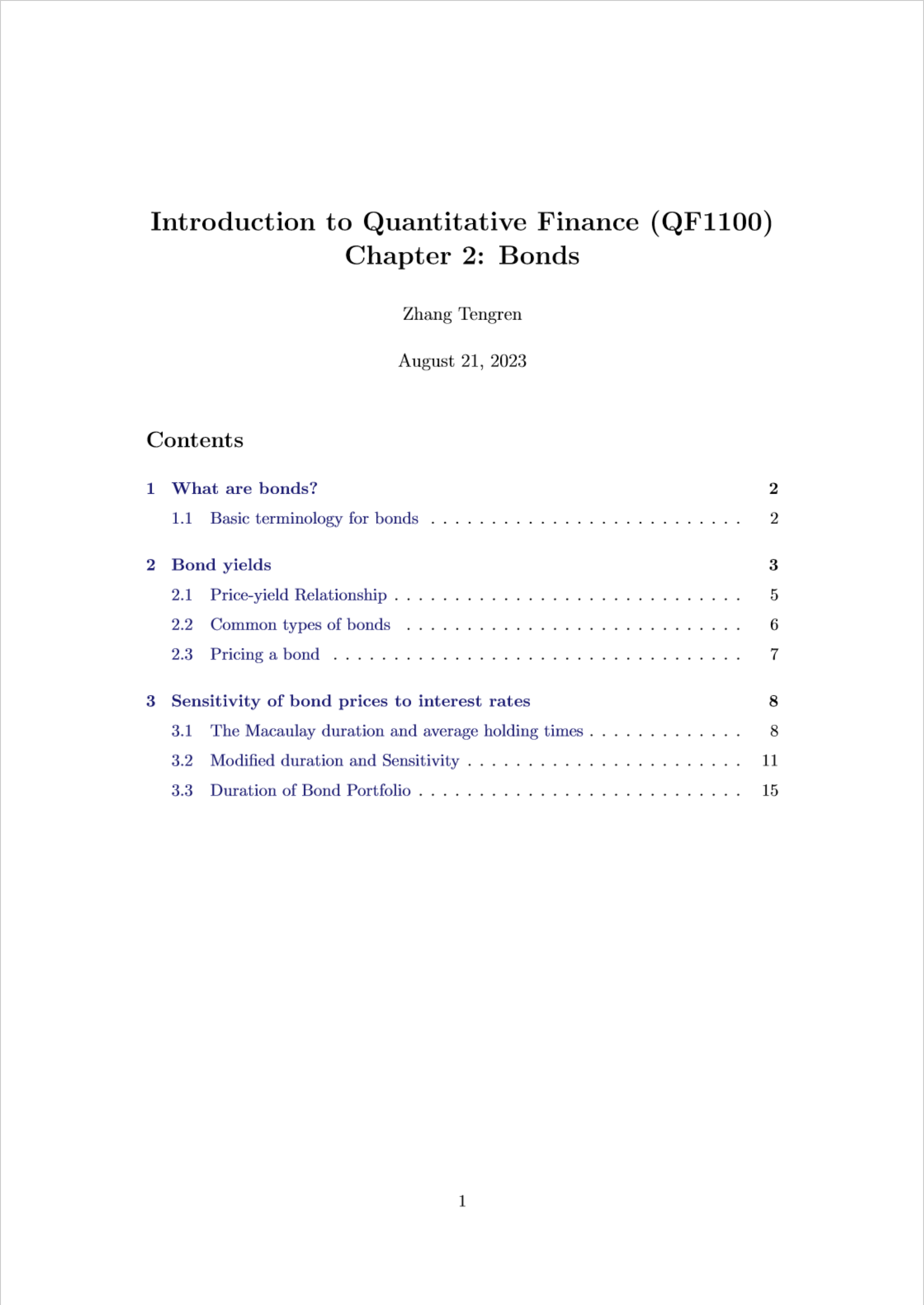
Cash flow
T0 = 10000
T1 = 1000
T2 = 1000
…
TN = 1000 + B (B < 1000)

N payment < 10000
N+1 payment > 10000

Ch2- bond

Monday, August 28, 2023 6:24 PM

ch2



Introduction to Quantitative Finance (QF1100)
Chapter 2: Bonds

Zhang Tengren

August 21, 2023

Contents

1 What are bonds? 2
- 1.1 Basic terminology for bonds 2

2 Bond yields 3
- 2.1 Price-yield Relationship 5
- 2.2 Common types of bonds 6
- 2.3 Pricing a bond 7

3 Sensitivity of bond prices to interest rates 8
- 3.1 The Macaulay duration and average holding times 8
- 3.2 Modified duration and Sensitivity 11
- 3.3 Duration of Bond Portfolio 15

1 What are bonds?

Governments or corporations can borrow money by issuing bonds to investors. A bond is a written contract between the issuer (borrower) and the investors (lenders/bond holders). There are various types of bonds: coupon-paying bonds, zero-coupon bonds, callable bonds, convertible bonds e.t.c.

Bonds, like stocks, are tradable in the financial markets. Fluctuations in bond prices due to changes in interest rates and failure of bond issuers to pay coupons (known as bond default) constitute some of the risks bond investors have to face. Other risks include liquidity risk (the risk of being unable to find a buyer) and reinvestment risk (the inability to invest coupons received at a required rate of return).

1.1 Basic terminology for bonds

Typically, when a bond is issued, the following information is specified:

1. *Face value* (also called par value) of the bond - this is the amount based on which periodic interest payments are computed. This is usually denoted F.
2. *Redemption/maturity value* of the bond - the amount to be repaid at the end of the loan. This is usually denoted R. Typically, the redemption value is the same as the face value.
3. *Coupon rate* (for coupon-paying bonds) - the bond's interest payments, as a percentage of the par value, to be made to investors at regular intervals during the term of the loan. This is usually denoted $c\%$.
4. *Maturity date* (also called redemption date) of the bond - the date on which the loan will be fully repaid. Along side this, the number of coupon payments per year, usually denoted m, and the total number of coupon payments, usually denoted n, is specified.

Thus, the cash flow of the bond takes the following form:

$$\Big((-F,0),(\frac{c\%F}{m},\frac{1}{m}),(\frac{c\%F}{m},\frac{2}{m}),\ldots,(\frac{c\%F}{m}+R,\frac{n}{m})\Big).$$

In other words, the cash flow of the bond is made up of:

1. coupon payments of $\frac{c\%F}{m}$ at time $t=\frac{1}{m},\frac{2}{m},\ldots,\frac{n}{m}$ (a total of n payments)
2. redemption/maturity value R at $t=\frac{n}{m}$.

For the rest of this course, unless we explicitly say so, we will always assume that the face value and the redemption value of the bond are the same.

2 Bond yields

For any time t in the lifetime of the bond, the *nominal yield* of the bond is the nominal internal rate of return compounded m times per annum of holding the bond from time t to maturity. In other words, if $P(t)$ denotes the price of the bond at time t, then the nominal yield of the bond $\lambda(t)\%$ at time t satisfies

$$P(t) = \frac{R}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{i}}. \tag{2.1}$$

Hence, if $R = F$, then

$$\begin{aligned} P(t) &= F\left[\frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \frac{c}{\lambda(t)}\left(1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right)\right] \\ &= F + F\left(\frac{c - \lambda(t)}{\lambda(t)}\right)\left[1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right]. \end{aligned}$$

A bond is said to be priced at time t

1. at a *premium* if $P(t) > F$
2. at *par* if $P(t) = F$
3. at a *discount* if $P(t) < F$.

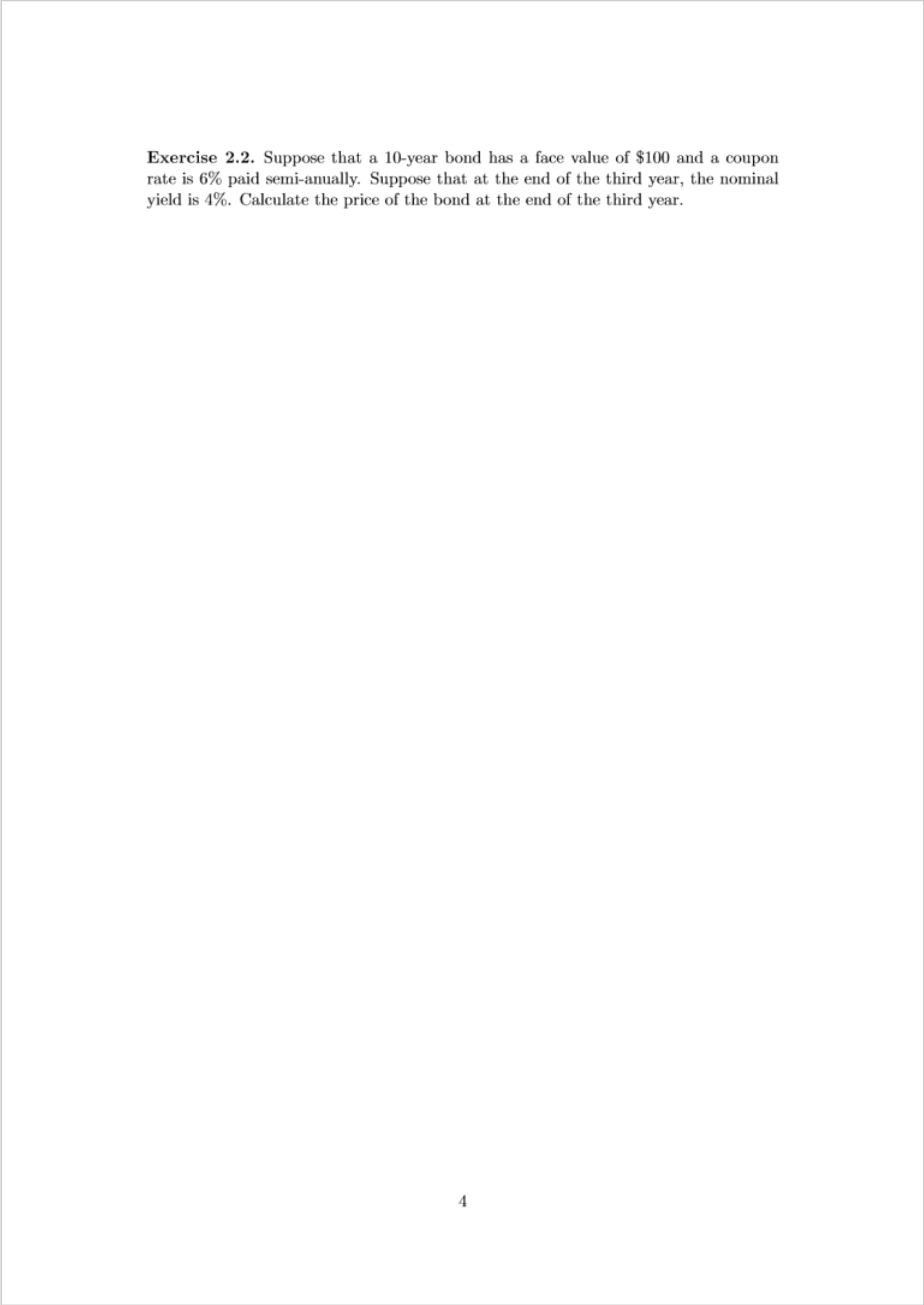
From the preceding bond pricing formula, it is easy to see that

1. $P(t) > F$ if and only if $c > \lambda(t)$,
2. $P(t) = F$ if and only if $c = \lambda(t)$,
3. $P(t) < F$ if and only if $c < \lambda(t)$.

Similarly, one can define the *effective yield* of the bond to be the quantity $\lambda_e(t)\%$ that satisfies

$$P(t) = \frac{R}{(1 + \lambda_e(t)\%)^{\frac{n}{m} - t}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{(1 + \lambda_e(t)\%)^{\frac{i}{m}}}.$$

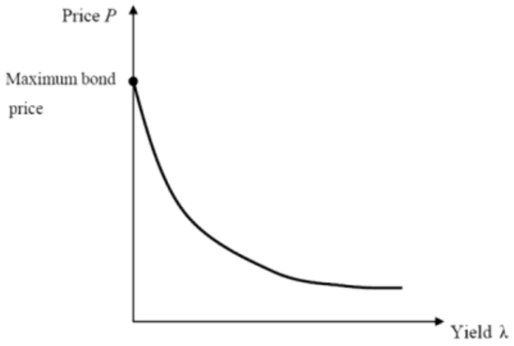
Notice that $P(\frac{n}{m}) = R$.



Exercise 2.2. Suppose that a 10-year bond has a face value of \$100 and a coupon rate is 6% paid semi-anually. Suppose that at the end of the third year, the nominal yield is 4%. Calculate the price of the bond at the end of the third year.

2.1 Price-yield Relationship

A fundamental property of the equation of value for bonds is that the price $P(t)$ of the bond at time t is a decreasing function of the nominal yield $\lambda(t)$ (and also of the effective yield $\lambda_e(t)$). Furthermore, it can be shown that the graph of the price-yield curve is convex.



The diagram below shows a typical price-yield curve. Note that the maximum possible bond price is attained when the yield is zero, in which case stream of cash flow is *undiscounted*, that is, the bond price is simply the sum of all coupon payments and the redemption value.

2.2 Common types of bonds

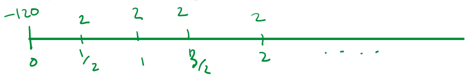
A *zero coupon bond* is a bond that pays no coupons. Hence, at any time t in the lifetime of the bond, the price $P(t)$ and effective yield $\lambda_e\%$ of a N-year zero-coupon bond with maturity value R satisfies

$$P(t) = \frac{R}{(1+\lambda_e(t)\%)^{N-t}}.$$

A *perpetual bond* or *consol* is a bond that never matures. At any time $t > 0$, the price $P(t)$ and nominal yield $\lambda(t)\%$ of a perpetual bond with coupon c paid m times annually satisfies

$$P(t) = \frac{cF}{\lambda(t)}.$$

Exercise 2.3. A perpetual bond of face value \$100 pays coupons semi-annually, with nominal coupon rate 4% per year. Assuming that the price of the bond at $t = 0$ is \$120, find the nominal yield of this bond at $t = 0$.



let $\lambda(0)$ be nominal yield at $t=0$

$120 = P(0)$

how to derive

$$= \frac{2}{1+\frac{\lambda(0)\%}{2}} + \frac{2}{\left(1+\frac{\lambda(0)\%}{2}\right)^2} + \cdots$$

$$= \frac{2}{1+\frac{\lambda(0)\%}{2}}\left(1 + \frac{1}{1+\frac{\lambda(0)\%}{2}} + \frac{1}{\left(1+\frac{\lambda(0)\%}{2}\right)^2} + \cdots\right)$$

$$= \frac{2}{1+\frac{\lambda(0)\%}{2}}\left(\frac{1}{1-\frac{1}{1+\frac{\lambda(0)\%}{2}}}\right)$$

$$= \frac{2}{1+\frac{\lambda(0)\%}{2}-1}$$

same formula

$$= \frac{4}{\lambda(0)\%}$$

$$\lambda(0) = \frac{4}{120} = 3.33\%$$

2.3 Pricing a bond

In order to price the bonds, we make the following simplifying assumptions:

- Interest rates are constant over the lifetime of the bond. This is reasonable if economic conditions are stable.
- The yield at any point in time has to equal the interest rates. This is reasonable if the bond does not have significant default or liquidity risks (for example, US treasury bonds). Then the price of the bond is equal to its present value.

default risk

bank can't pay coupon payments

liquidity risk

whether you can freely sell / buy this bond

Henceforth, we always make these assumptions. These then allow us to determine the price of the bond at every point in time once we know the current interest rates. We simply replace $\lambda(t)$ in Equation (2.1) with the current interest rates.

Exercise 2.4. A 10-year bond with face value \$100 pays coupons annually at 4%. Given that the price of the bond at $t = 0$ is \$90, find the price of the bond at $t = 4$. Give your answer to 3 significant figures.

Interest rate risk

at $t=0$ $P=90$ at $t=4$ $P=$

$P(4) = \frac{cF}{\lambda(4)\%}$

$P(0) = \frac{c100}{(1+\lambda(t))^{10-t}} = 90$

$P(0) = \frac{c \cdot F}{(1+\lambda(t))^{t-\gamma}}$

1. calc yield

$P(0) = 90 = \frac{4}{1+\lambda\%} + \frac{4}{(1+\lambda\%)^2} + \cdots \frac{4+100}{(1+\lambda\%)^{10}}$

$90 = 4\left(\frac{1}{1+\lambda\%} + \frac{1}{(1+\lambda\%)^2} + \cdots \frac{1}{(1+\lambda\%)^{10}}\right) + \frac{100}{(1+\lambda\%)^{10}}$

$90 = 4\left(\frac{1-\left(\frac{1}{1+\lambda\%}\right)^9}{1-\left(\frac{1}{1+\lambda}\right)}\right) + \frac{100}{(1+\lambda\%)^{10}}$

~~$\lambda = 0.0518215$~~ $5.3149\% = \lambda$

$P(4) = \frac{4}{1+\lambda\%} + \frac{4}{(1+\lambda\%)^2} + \cdots \frac{4}{(1+\lambda\%)^6} + \frac{100}{(1+\lambda\%)^6}$

$= \frac{4}{1+\lambda\%}\left(\frac{1-\left(\frac{1}{1+\lambda\%}\right)^6}{1-\left(\frac{1}{1+\lambda\%}\right)}\right) + \frac{100}{(1+\lambda\%)^6}$

$P(4) = \$93.40$

3 Sensitivity of bond prices to interest rates

We say earlier that the price of a bond and its yield are inversely related. We would now like to quantify how sensitive this relation is.

3.1 The Macaulay duration and average holding times

D

The *Macaulay duration* of any cash flow $\mathbf{C} = ((C_1, t_1), (C_2, t_2), \dots, (C_n, t_n))$ is the quantity

weighted avg

$$D = \frac{\sum_{i=1}^{n} t_i \cdot PV(C_i)}{PV(\mathbf{C})} = \sum_{i=1}^{n} w_i t_i, \tag{3.1}$$

where $PV(C_i)$ is the present value of C_i and $w_i = \dfrac{PV(C_i)}{PV(\mathbf{C})}$. In words, the Macaulay duration is the average time each dollar in $PV(\mathbf{C})$ needs to be held before it can be redeemed by the investor.

Observe that

1. if $C_i \geq 0$ for all i, then $t_1 \leq D \leq t_n$ (this is the case for any bond)
2. if $C_i = 0$ for all $i < n$, then $D = t_n$ (this is the case for any zero-coupon bond).

We can extend the definition of D to any infinite cash flow $\mathbf{C} = ((C_1, t_1), (C_2, t_2), \dots)$:

$$D = \frac{\sum_{i=1}^{\infty} t_i \cdot PV(C_i)}{PV(\mathbf{C})}. \tag{3.2}$$

Now, consider a bond that is redeemable at par and pays a total of n coupons at a frequency of m payments a year. Suppose that the nominal bond yield and nominal coupon rate are constant at $\lambda\%$ and $c\%$ respectively. The cash flow in this case is

$$\mathbf{C} = \left(\left(\frac{c\%F}{m}, t_1\right), \dots, \left(\frac{c\%F}{m}, t_{n-1}\right), \left(\frac{c\%F}{m} + F, t_n\right)\right),$$

where $t_i = \frac{i}{m}$, so that

$$D = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{\frac{c\%F}{m}}{(1+\frac{\lambda\%}{m})^i}\frac{i}{m} + \frac{F}{(1+\frac{\lambda\%}{m})^n}\frac{n}{m}\right], \tag{3.3}$$

where

$$P = \sum_{i=1}^{n} \frac{\frac{c\%F}{m}}{(1+\frac{\lambda\%}{m})^i} + \frac{F}{(1+\frac{\lambda\%}{m})^n}. \tag{3.4}$$

It can be shown that

$$\boxed{D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%} - \frac{1+\frac{\lambda\%}{m} + n(\frac{c\%}{m} - \frac{\lambda\%}{m})}{c\%\left[\left(1+\frac{\lambda\%}{m}\right)^n - 1\right] + \lambda\%}.}$$

As the time to maturity tends to infinity (that is, $n \to \infty$), i.e. for a **perpetual bond**,

$$D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%}.$$

When a bond is priced at par, i.e. when $\lambda = c$, then

$$D = \frac{1+\frac{c\%}{m}}{c\%}\left(1 - \frac{1}{\left(1+\frac{c\%}{m}\right)^n}\right).$$

Exercise 3.5. A bond with 10% coupon payable quarterly matures in 5 years and gives a 8% nominal yield. Find Macaulay duration of the bond.

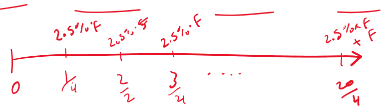
← does not depend on price of bond
how long a dollar has to be held to get it back

$$D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%} - \frac{1+\frac{\lambda\%}{m}+n\left(\frac{c\%}{m}-\frac{\lambda\%}{m}\right)}{c\%\left[\left(1+\frac{\lambda\%}{m}\right)^n-1\right]+\lambda\%}$$

$$D = \frac{1+\frac{8\%}{4}}{8\%} - \frac{1+\frac{8\%}{4}+20\left(\frac{10\%}{4}-\frac{8\%}{4}\right)}{10\%\left[\left(1+\frac{8\%}{4}\right)^{20}-1\right]+\lambda\%} = 4.04 \text{ yrs}$$

← avg time a dollar must be held b4 you get it back

F is face val



← numerator

$$\sum_{i=1}^{20} \frac{2.5\% \times F}{\left(1+\frac{8\%}{4}\right)^i} \cdot \frac{i}{4} + \frac{F}{(1+2\%)^{20}} \cdot \frac{20}{4}$$

$$\sum_{i=1}^{n} i r^{i-1} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$$

$$\Rightarrow \frac{2.5\% \cdot F}{4(1+2\%)} \sum_{i=1}^{20} \frac{1}{1.02^{i-1}} + \frac{5F}{1.02^{20}}$$

numerator

$$= \frac{2.5\% \cdot F}{4.08}\left(\frac{1-21\left(\frac{1}{1.02}\right)^{20}+20\left(\frac{1}{1.02}\right)^{21}}{\left(1-\frac{1}{1.02}\right)^2}\right) + \frac{5F}{1.02^{20}}$$

$$D = \frac{\quad}{\quad}$$

denominator

$$= \sum_{i=1}^{20} \frac{2.5\% \times F}{(1+2\%)^i} + \frac{F}{(1+2\%)^{20}}$$

$$= \frac{2.5\% \times F}{1.02}\left(\frac{1-\left(\frac{1}{1.02}\right)^{20}}{1-\frac{1}{1.02}}\right) + \frac{F}{1.02^{20}}$$

3.2 Modified duration and Sensitivity

so must ÷ by P

The modified duration measures the sensitivity per unit dollar of the present value P of a cashflow (this is the bond price in our case) to interest rates λ. More precisely, the *modified duration* D_M is given by

$$D_M := -\frac{\left(\frac{dP}{d\lambda}\right)}{P}.$$

We can write the modified duration in terms of the Macaulay duration as follows. Since Equation (3.4) gives a formula for the price of the bond, by differentiating Equation (3.4) with respect to λ, we get

$$\begin{aligned}\frac{dP}{d\lambda} &= -\sum_{i=1}^{n} \frac{\frac{i}{m}\frac{cF}{m}}{(1+\frac{\lambda}{m})^{i+1}} - \frac{\frac{n}{m}F}{(1+\frac{\lambda}{m})^{n+1}} \\ &= -\frac{1}{1+\frac{\lambda}{m}}\left[\sum_{i=1}^{n} \frac{\frac{i}{m}\frac{cF}{m}}{(1+\frac{\lambda}{m})^{i}} + \frac{\frac{n}{m}F}{(1+\frac{\lambda}{m})^{n}}\right] \\ &= -\frac{1}{1+\frac{\lambda}{m}}DP,\end{aligned}$$

$$\frac{dP}{d\lambda} = -\frac{1}{1+\frac{\lambda}{m}}DP$$

where the last equality follows from the formula for the Macaulay duration given by Equation (3.3). Thus, we have derived the *price sensitivity formula*

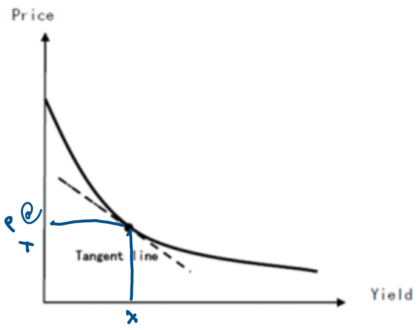
$$\underset{\text{modified}}{D_M} = \frac{1}{1+\frac{\lambda}{m}}\overset{\text{macaulay}}{D}. \tag{3.6}$$

$$D_M = \frac{1}{1+\frac{\lambda}{m}}D$$

Remark 3.7. In general, Equation (3.6) holds for any cashflow (not just for bonds).

On the price-yield curve (the graph of P against λ, $-D_M P$ is the slope of the tangent line drawn at the point where the yield is λ.

$$\frac{1}{1+\frac{\lambda}{m}}D$$



If we let $\Delta\lambda$ denote a small change in λ, then by linear approximation,

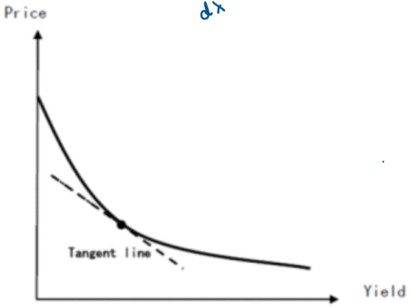
$$P(\lambda+\Delta\lambda) \approx P(\lambda) - (D_M P)\cdot\Delta\lambda. \tag{3.8}$$

This allows us to estimate the bond price when the annual yield rate λ changes by a small amount $\Delta\lambda$. The corresponding change in P, $\Delta P = P(\lambda+\Delta\lambda) - P(\lambda)$ is

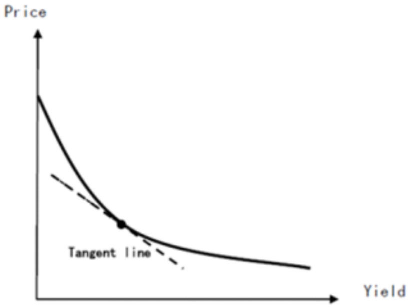
$$\Delta P \approx -\underbrace{(D_M P)}_{\frac{dP}{d\lambda}} \cdot \Delta\lambda. \tag{3.9}$$

$$D = -\left(1+\frac{\lambda}{m}\right)\frac{\frac{dP}{d\lambda}}{P}$$

if you dont know duration of it



Remark 3.10. By the convexity of the price-yield curve, we see that linear approximation tends to overestimate (underestimate) the change in bond price when the yield increases (decreases). The approximation also gets worse as the change in yield becomes larger.



Example 3.11. A 25-year bond pays coupons semiannually at 6% and has a 9% nominal yield.

1. Calculate the Macaulay duration of this bond.
2. Suppose the face value of the bond is \$100. Estimate the change in bond price when it is first issued if the yield increases by 1 basis point (100 basis points = 1%).

①

$\lambda = 9\%$

$c = 6\%$

$m = 2$

$$D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%} - \frac{1+\frac{\lambda\%}{m} + n\left(\frac{c\%}{m} - \frac{\lambda\%}{m}\right)}{c\%\left[\left(1+\frac{\lambda\%}{m}\right)^n - 1\right] + \lambda\%}$$

$$= \frac{1+\frac{0.09}{2}}{0.09} - \frac{1+\frac{0.09}{2} + 50\left(\frac{0.06}{2} - \frac{0.09}{2}\right)}{0.06\left[\left(1+\frac{0.09}{2}\right)^{50} - 1\right] + 0.09} \approx 11.095$$

②

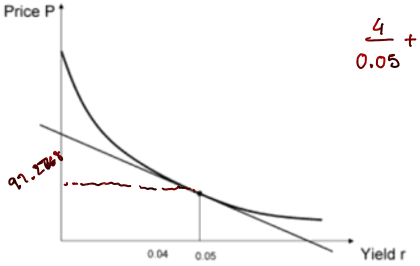
$F = 100$ $\quad \Delta P = -D_m P \cdot \Delta\lambda$ $\quad D_M = \frac{1}{1+\frac{0.09}{2}} D = 10.61755$

$$P(0) = \cancel{0} + \frac{3}{1+0.045} + \frac{3}{(1.045)^2} + \cdots + \frac{3}{(1.045)^{50}} + \frac{100}{1.045^{50}}$$

$$\Rightarrow \frac{3}{1.045}\left(\frac{1-\left(\frac{1}{1.045}\right)^{50}}{1-\left(\frac{1}{1.045}\right)}\right) + \frac{100}{1.045^{50}} = 70.3570$$

ans. $\Delta P = -D_M P \cdot \left(\frac{0.01}{100}\right) = -0.0747$

Exercise 3.12. A bond with face value \$100 pays annual coupons at 4%. The diagram below shows the price-yield curve of this bond. When the effective yield is 5%, the bond price is 97.2768, and the tangent line to the price-yield curve has slope −267.222.



$$\frac{4}{0.05} + \frac{4}{0.05^2} + \cdots \frac{4}{0.05^n} + \frac{100}{0.05^n} = 97.2768$$

$$\frac{4}{0.05}\left[\frac{1-\left(\frac{1}{0.05}\right)^n}{1-\left(\frac{1}{0.05}\right)}\right] + \frac{100}{0.05^n} = 97.2768$$

$$\frac{dP}{dx} = -\frac{1}{1+\frac{x}{m}} DP$$

1. Calculate, to 3 decimal places, the Macaulay duration of the bond when the yield is 5%.
2. Find, to 2 decimal places, the approximate percentage change in bond price when the bond yield decreases by 1 basis point from 5%.

①

$$-267.222 = -\frac{1}{1+0.05} \cdot D \cdot 97.2768$$

$$\boxed{D_m = 2.884}$$

$$D_M = \frac{1}{1+\frac{x}{m}} D = 2.747$$

2.)

~~−100~~

$$\Delta P = -D_m P \cdot \Delta x \qquad D_M = \frac{1}{1+\frac{0.09}{2}} D = 10.61755$$

$$= -D_m \cdot \cancel{P} \cdot \left(\frac{-0.01}{100}\right) \Delta x$$

0.0002747

$$\frac{\Delta P}{P} = \frac{-D_m P \cdot \Delta \cdot x}{P}$$

$$= -D_m \cdot \Delta x = \frac{-D}{1+\frac{x}{m}} \cdot \Delta x$$

3.3 Duration of Bond Portfolio

Since the Macaulay duration makes sense for any cash flow, we can talk about the Macaulay duration of a bond portfolio.

We assume that all the bonds in the portfolio have constant (effective) yield λ_e, all of which are equal to the current effective interest rate. This again, is reasonable only if all the bonds in the portfolio do not have significant default or liquidity risks, and the economic conditions are stable.

Consider a portfolio Π consisting of α_i units of a bond b_i, where $i = 1, 2, \dots, n$. The *portfolio weight* w_i of the bond b_i in the portfolio Π is the quantity

$$w_i := \frac{\alpha_i P_i}{\sum\limits_{i=1}^{n} \alpha_i P_i},$$

where P_i is the current price of the bond b_i. One can then show from the definition of the Macaulay duration, that if D_i is the duration of the bond b_i and D_Π is the duration of the portfolio Π, then

$$D_\Pi = \sum_{i=1}^{n} w_i D_i. \tag{3.13}$$

Exercise 3.14. A company is required to make a huge payment in seven years, the present value of which is 1 million. The company's investment manager plans to fund this obligation using two types of securities: three-year zero-coupon bonds and perpetual bonds paying annual coupons. Assume that the effective interest rates over the next seven years are constant at 10%. How much should the manager invest in each security so that the duration of this investment matches that of the liability (i.e. 7 years)?

↑ macauly

$D = \frac{1+\frac{x}{m}}{x}$ perpetual

π α is weight of 3-yr zero coupon bonds
$1-\alpha$ is weight of perpetual bond

$7 = D_\pi$

$7 = \omega_1 D_1 + \omega_2 D_2 = \alpha D_1 + (1-\alpha) D_2$

$3\alpha + 11 - 11\alpha$

$= 11 - 8\alpha$

$\boxed{\alpha = 0.5}$

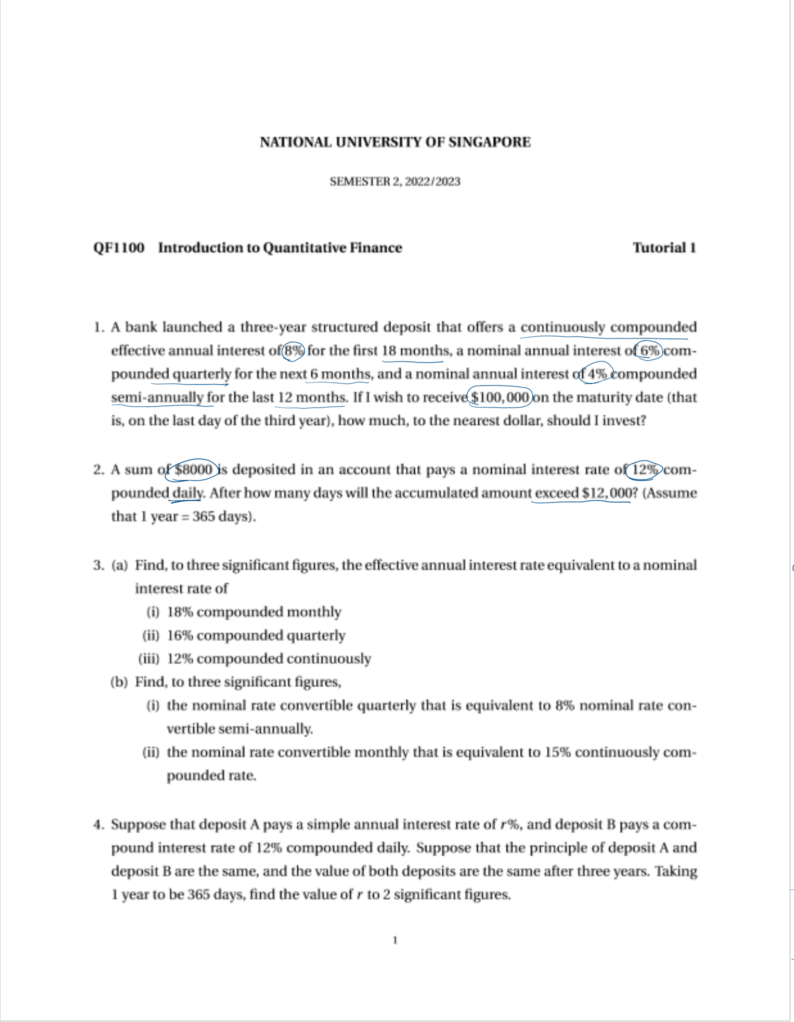
$D_2 = \frac{1+0.1}{0.1} = 11$

$D_1 = 3$

QF1100-Tutorial1

Wednesday, August 30, 2023 11:41 PM

QF1100-Tutorial1



NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2, 2022/2023

QF1100 Introduction to Quantitative Finance **Tutorial 1**

1. A bank launched a three-year structured deposit that offers a continuously compounded effective annual interest of 8% for the first 18 months, a nominal annual interest of 6% compounded quarterly for the next 6 months, and a nominal annual interest of 4% compounded semi-annually for the last 12 months. If I wish to receive $100,000 on the maturity date (that is, on the last day of the third year), how much, to the nearest dollar, should I invest?

2. A sum of $8000 is deposited in an account that pays a nominal interest rate of 12% compounded daily. After how many days will the accumulated amount exceed $12,000? (Assume that 1 year = 365 days).

3. (a) Find, to three significant figures, the effective annual interest rate equivalent to a nominal interest rate of
   (i) 18% compounded monthly
   (ii) 16% compounded quarterly
   (iii) 12% compounded continuously

   (b) Find, to three significant figures,
   (i) the nominal rate convertible quarterly that is equivalent to 8% nominal rate convertible semi-annually.
   (ii) the nominal rate convertible monthly that is equivalent to 15% continuously compounded rate.

4. Suppose that deposit A pays a simple annual interest rate of r%, and deposit B pays a compound interest rate of 12% compounded daily. Suppose that the principle of deposit A and deposit B are the same, and the value of both deposits are the same after three years. Taking 1 year to be 365 days, find the value of r to 2 significant figures.

1

x = amount invested

$$x\left(e^{0.08\cdot1.5}\right)\left(1+\frac{0.06}{4}\right)^2\left(1+\frac{0.04}{2}\right)^2 = 100000$$

Effective means the compounding doesn't matter
If it says nominal, use the e^rt

x = 82747

(1+8%)^3/2(1+^%/4)^2(1+4%/2)^2

$$8000\left(1+\frac{0.12}{365}\right)^{365\cdot x} = 12000$$

x = 3.38 of years

a.) i. $\left(1+\frac{0.18}{12}\right)^{12}-1 = r_{\text{effective}}$

$= 0.196$

ii. $\left(1+\frac{0.16}{4}\right)^4 - 1 = 0.190$

iii $\left(e^{0.12}\right)-1 = 0.127$

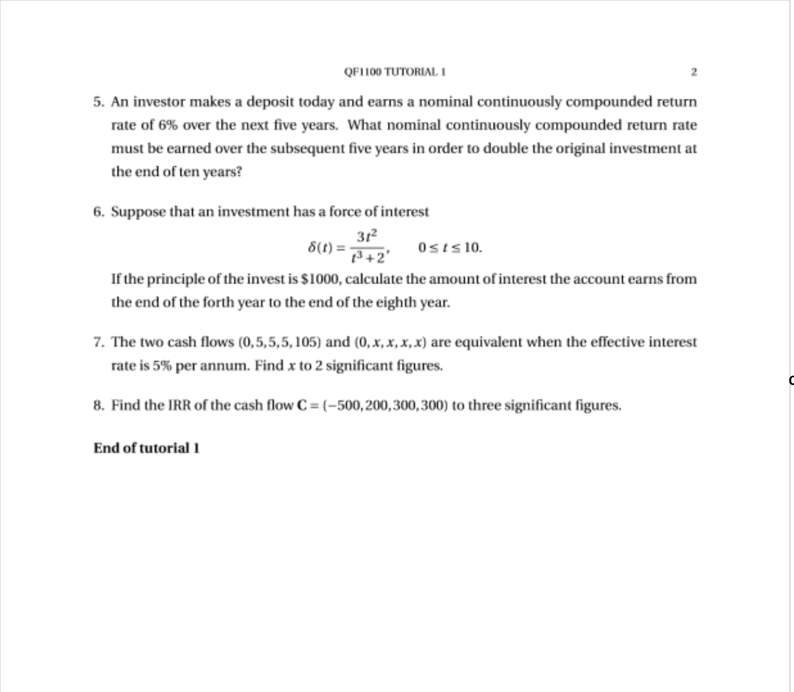
b.) i $\left(1+\frac{0.08}{2}\right)^2 = \left(1+\frac{x}{4}\right)^4$

$\boxed{x = 0.792}$

ii. $\left(e^{0.15}\right) = \left(1+\frac{x}{12}\right)^{12}$ $\boxed{x = 0.151}$

I

$I(1+3r) = \cancel{I}\left(1+\frac{0.12}{365}\right)^{365\cdot3}$ $r = 0.14$



QF1100 TUTORIAL 1 2

5. An investor makes a deposit today and earns a nominal continuously compounded return rate of 6% over the next five years. What nominal continuously compounded return rate must be earned over the subsequent five years in order to double the original investment at the end of ten years?

6. Suppose that an investment has a force of interest
$$\delta(t) = \frac{3t^2}{t^3+2}, \qquad 0 \le t \le 10.$$
If the principle of the invest is $1000, calculate the amount of interest the account earns from the end of the forth year to the end of the eighth year.

7. The two cash flows $(0, 5, 5, 5, 105)$ and $(0, x, x, x, x)$ are equivalent when the effective interest rate is 5% per annum. Find x to 2 significant figures.

8. Find the IRR of the cash flow $\mathbf{C} = (-500, 200, 300, 300)$ to three significant figures.

End of tutorial 1

D = initial

$D(e^{0.06\cdot5}) \leftarrow$ @5yr

$\cancel{D}(e^{0.06\cdot5})\cdot(e^{r\cdot5}) = 2\cancel{D}$

$r = 0.0786$

$$a(t) = e^{\int_0^t \frac{3t^2}{t^3+2}dt}$$

$$1000\cdot\int_4^8 \frac{3t^2}{t^3+2}dt = \ln|t^3+2|$$

$= 2065.89$

1000(a(8)) -1000(a(4))
Exp($\int_0^8 (2t^2)/(t^3+1)dt$)
⬚
⬚
Exp($\int_0^4$ ⬚
1000*Exp[ln(514)-ln(2)} - 1000*Exp[ln(66)-ln(2)]
= 1000(a(8) - a(4)) = 224000

Q7

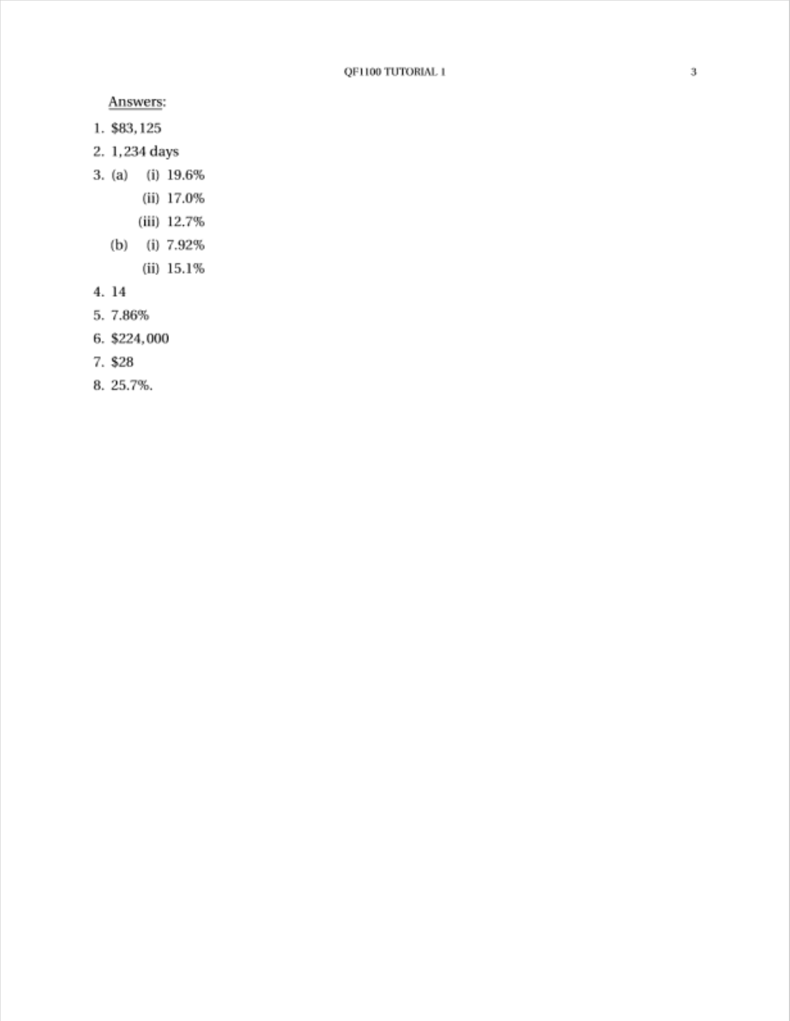
$\frac{5}{(1+0.05)^1} + \frac{5}{(1+0.05)^2} + \frac{5}{(1+0.05)^3}$ + 105/(1+0.05)^4 = x/(1+0.05) + x/(1+0.05)^2 + x/(1.05)^3 + x/(1.05)^4

100 = (x/1.05) (1 + 1/1.05 + 1/1.05^2 + 1/1.05^3)
X = 28

Q8 :

0 = -500 + 200/(1+r) + 300(1+r)^2 + 200(1+r)^3
5 = 2/(1+r) + 3/(1+r)^2 + 2/(1+r)^3
Use excel - 25.7%

5 = 2/(1+r) + 3/(1+r)^2 + 2/(1+r)^3
Use excel - 25.7%



Answers:

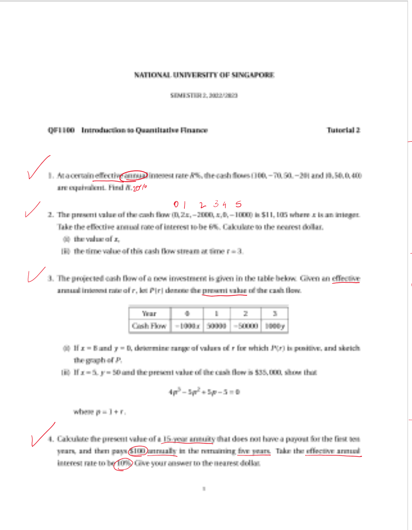
1. \$83,125
2. 1,234 days
3. (a) (i) 19.6%
    - (ii) 17.0%
    - (iii) 12.7%

    (b) (i) 7.92%
    - (ii) 15.1%
4. 14
5. 7.86%
6. \$224,000
7. \$28
8. 25.7%.

QF1100-Tutorial2 (1)

Wednesday, September 6, 2023 11:22 AM

QF1100-Tutorial2 (1)



NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2, 2022/2023

QF1100 Introduction to Quantitative Finance **Tutorial 2**

✓ 1. At a certain effective annual interest rate $R\%$, the cash flows $(100, -70, 50, -20)$ and $(0, 50, 0, 40)$ are equivalent. Find R. 20%

0 1 2 3 4 5

✓ 2. The present value of the cash flow $(0, 2x, -2000, x, 0, -1000)$ is \$11, 105 where x is an integer. Take the effective annual rate of interest to be 6%. Calculate to the nearest dollar.
   (i) the value of x,
   (ii) the time value of this cash flow stream at time $t = 3$.

✓ 3. The projected cash flow of a new investment is given in the table below. Given an effective annual interest rate of r, let $P(r)$ denote the present value of the cash flow.

Year	0	1	2	3
Cash Flow	$-1000x$	50000	-50000	$1000y$

   (i) If $x = 8$ and $y = 0$, determine range of values of r for which $P(r)$ is positive, and sketch the graph of P.
   (ii) If $x = 5$, $y = 50$ and the present value of the cash flow is \$35,000, show that

$$4p^3 - 5p^2 + 5p - 5 = 0$$

   where $p = 1 + r$.

✓ 4. Calculate the present value of a 15-year annuity that does not have a payout for the first ten years, and then pays \$100 annually in the remaining five years. Take the effective annual interest rate to be 10%. Give your answer to the nearest dollar.

1

$$100 - \frac{70}{(1+R)} + \frac{50}{(1+R)^2} - \frac{20}{(1+R)^3} = \frac{50}{(1+R)} + \frac{40}{(1+R)^3}$$

$$10 = \frac{12}{(1+R)} - \frac{5}{(1+R)^2} + \frac{6}{(1+R)^3}$$

$$\boxed{R = 20\%}$$

$$11105 = \frac{2x}{(1.06)} - \frac{2000}{1.06^2} + \frac{x}{1.06^3} - \frac{1000}{1.06^5}$$

$$13632.25105 = x\left(\frac{2}{1.06} + \frac{1}{1.06^3}\right)$$

i. $\boxed{x = 5000}$

ii. $0 + 2x(1.06)^2 - 2000(1.06) + x + 0 - \frac{1000}{1.06^2}$

$= \boxed{13226}$

$C = (-8000, 50000, -50000, 0)$

$$PV = -8000 + \frac{50000}{1+r} - \frac{50000}{(1+r)^2} > 0$$

i. $\frac{1}{1+r} - \frac{1}{(1+r)^2} > \frac{8}{50}$ $\boxed{0.25 < r < 4}$

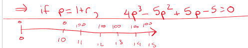
$r = 0.25 \to r = 4$

$$PV = 35\cancel{0}\cancel{0}\cancel{0} = -5\cancel{0}\cancel{0}\cancel{0} + \frac{5\cancel{0}\cancel{0}\cancel{0}\cancel{0}}{1+r} - \frac{5\cancel{0}\cancel{0}\cancel{0}\cancel{0}}{(1+r)^2} + \frac{5\cancel{0}\cancel{0}\cancel{0}\cancel{0}}{(1+r)^3}$$

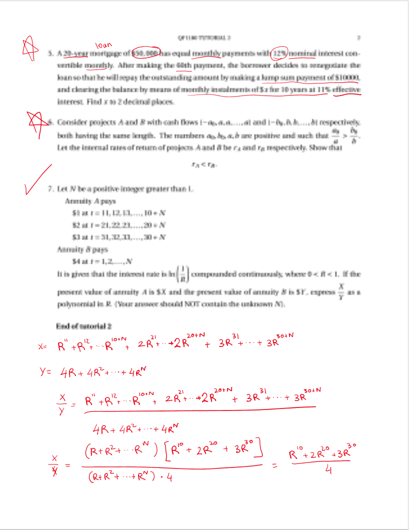
$$4\cancel{0} = \frac{5\cancel{0}}{1+r} - \frac{5\cancel{0}}{(1+r)^2} + \frac{5\cancel{0}}{(1+r)^3}$$

$$4(1+r)^3 = 5(1+r)^2 - 5(1+r) + 5$$

$\Rightarrow$ if $p = 1+r$, $4p^3 - 5p^2 + 5p - 5 = 0$



$$PV = \frac{100}{1.1^{10}} + \frac{100}{1.1^{11}} + \frac{100}{1.1^{12}} + \frac{100}{1.1^{13}} + \frac{100}{1.1^{14}} = 160.766$$



QF1100 TUTORIAL 2 2

☆ 5. A 20-year mortgage of \$50,000 (loan) has equal monthly payments with 12% nominal interest convertible monthly. After making the 60th payment, the borrower decides to renegotiate the loan so that he will repay the outstanding amount by making a lump sum payment of \$10000, and clearing the balance by means of monthly instalments of $\$x$ for 10 years at 11% effective interest. Find x to 2 decimal places.

☆ 6. Consider projects A and B with cash flows $(-a_0, a, a, \ldots, a)$ and $(-b_0, b, b, \ldots, b)$ respectively, both having the same length. The numbers a_0, b_0, a, b are positive and such that $\frac{a_0}{a} > \frac{b_0}{b}$. Let the internal rates of return of projects A and B be r_A and r_B respectively. Show that

$$r_A < r_B.$$

✓ 7. Let N be a positive integer greater than 1.

Annuity A pays
\$1 at $t = 11, 12, 13, \ldots, 10+N$
\$2 at $t = 21, 22, 23, \ldots, 20+N$
\$3 at $t = 31, 32, 33, \ldots, 30+N$

Annuity B pays
\$4 at $t = 1, 2, \ldots, N$

It is given that the interest rate is $\ln\left(\frac{1}{R}\right)$ compounded continuously, where $0 < R < 1$. If the present value of annuity A is $\$X$ and the present value of annuity B is $\$Y$, express $\frac{X}{Y}$ as a polynomial in R. (Your answer should NOT contain the unknown N).

End of tutorial 2

$$X = R^{11} + R^{12} + \cdots R^{10+N} + 2R^{21} + \cdots + 2R^{20+N} + 3R^{31} + \cdots + 3R^{30+N}$$

$$Y = 4R + 4R^2 + \cdots + 4R^N$$

$$\frac{X}{Y} = \frac{R^{11} + R^{12} + \cdots R^{10+N} + 2R^{21} + \cdots + 2R^{20+N} + 3R^{31} + \cdots + 3R^{30+N}}{4R + 4R^2 + \cdots + 4R^N}$$

$$\frac{X}{Y} = \frac{(R + R^2 + \cdots R^N)\left[R^{10} + 2R^{20} + 3R^{30}\right]}{(R + R^2 + \cdots + R^N) \cdot 4} = \frac{R^{10} + 2R^{20} + 3R^{30}}{4}$$

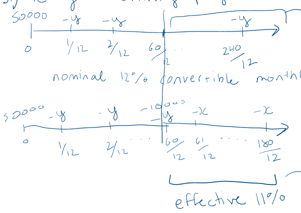
$$50000 = \frac{y}{\left(1+\frac{0.12}{12}\right)^1} + \frac{y}{\left(1+\frac{0.12}{12}\right)^2} + \cdots + \frac{y}{\left(1+\frac{0.12}{12}\right)^{240}} = y\left[\frac{1-\left(\frac{1}{1.01}\right)^{241}}{1-\left(\frac{1}{1.01}\right)}\right]$$

$y = 544.547$

$$\frac{y}{\left(1+\frac{0.12}{12}\right)^1} + \frac{y}{\left(1+\frac{0.12}{12}\right)^2} + \cdots + \frac{y}{\left(1+\frac{0.12}{12}\right)^{60}} + 10000 + \frac{x}{1.11^{61}} + \cdots \frac{x}{1.11^{180}} = 50000$$

$$\Rightarrow y\left[\frac{1-\left(\frac{1}{1.01}\right)^{61}}{1-\left(\frac{1}{1.01}\right)}\right] + \frac{x}{1.11^{60}}\left[\frac{1-\left(\frac{1}{1.11}\right)^{120}}{1-\left(\frac{1}{1.11}\right)}\right] = 40000$$

5.) let y be original monthly payments



$$50000 = \frac{y}{1+\frac{0.12}{12}} + \frac{y}{1.01^2} + \frac{y}{1.01^3} + \cdots + \frac{y}{1.01^{240}}$$

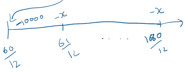
$$= \frac{y}{1.01}\left(\frac{1-\left(\frac{1}{1.01}\right)^{240}}{1-\frac{1}{1.01}}\right)$$

$$y = \frac{50000 \cdot 0.01}{1-\left(\frac{1}{1.01}\right)^{240}}$$

Take PV at $t = \frac{60}{12}$ of remaining payment

$$\frac{y}{1.01} + \frac{y}{1.01^2} + \cdots + \frac{y}{1.01^{180}}$$

$$\Rightarrow \frac{y}{1.01}\left[\frac{1-\left(\frac{1}{1.01}\right)^{180}}{1-\left(\frac{1}{1.01}\right)}\right] = 50000 \cdot$$



$$\frac{50000 \cdot \left(1-\frac{1}{1.01^{180}}\right)}{1-\frac{1}{1.01^{240}}} = 10000 + \frac{x}{(1+11\%}$$

$x =$

6.) $0 = -a_0 + \frac{a}{1+r_A} + \frac{a}{(1+r_A)^2} + \cdots \frac{a}{(1+r_A)^n}$

$$\frac{a_0}{a} = \frac{1}{1+r_A} + \cdots \frac{1}{(1+r_A)^n}$$

$$\frac{b_0}{b} = \frac{1}{1+r_B} + \cdots + \frac{1}{(1+r_B)^n}$$

Suppose for contradiction that $r_A \geq r_B$

$$1 + r_A \geq 1 + r_B$$

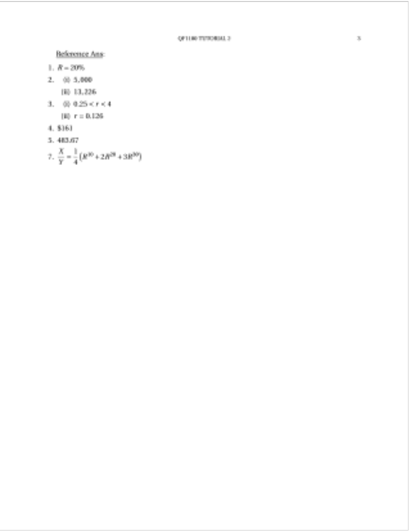
$$\frac{1}{1+r_A} \leq \frac{1}{1+r_B}$$

$\Rightarrow \frac{1}{(1+r_A)^k} \leq \frac{1}{(1+r_B)^k}$ for all +ive ints k

$$\underbrace{\sum_{i=1}^{k} \frac{1}{(1+r_A)^k}}_{= \frac{a_0}{a}} \leq \underbrace{\sum_{i=1}^{k} \frac{1}{(1+r_B)^k}}_{= \frac{b_0}{b}}$$

$\Rightarrow \frac{a_0}{a} < \frac{b_0}{b}$

$\therefore r_A < r_B$



QF1100 TUTORIAL 2 3

Reference Ans:

1. $R = 20\%$
2. (i) 5,000
   (ii) 13,226
3. (i) $0.25 < r < 4$
   (ii) $r = 0.126$
4. \$161
5. 483.67
7. $\frac{X}{Y} = \frac{1}{4}(R^{10} + 2R^{20} + 3R^{30})$

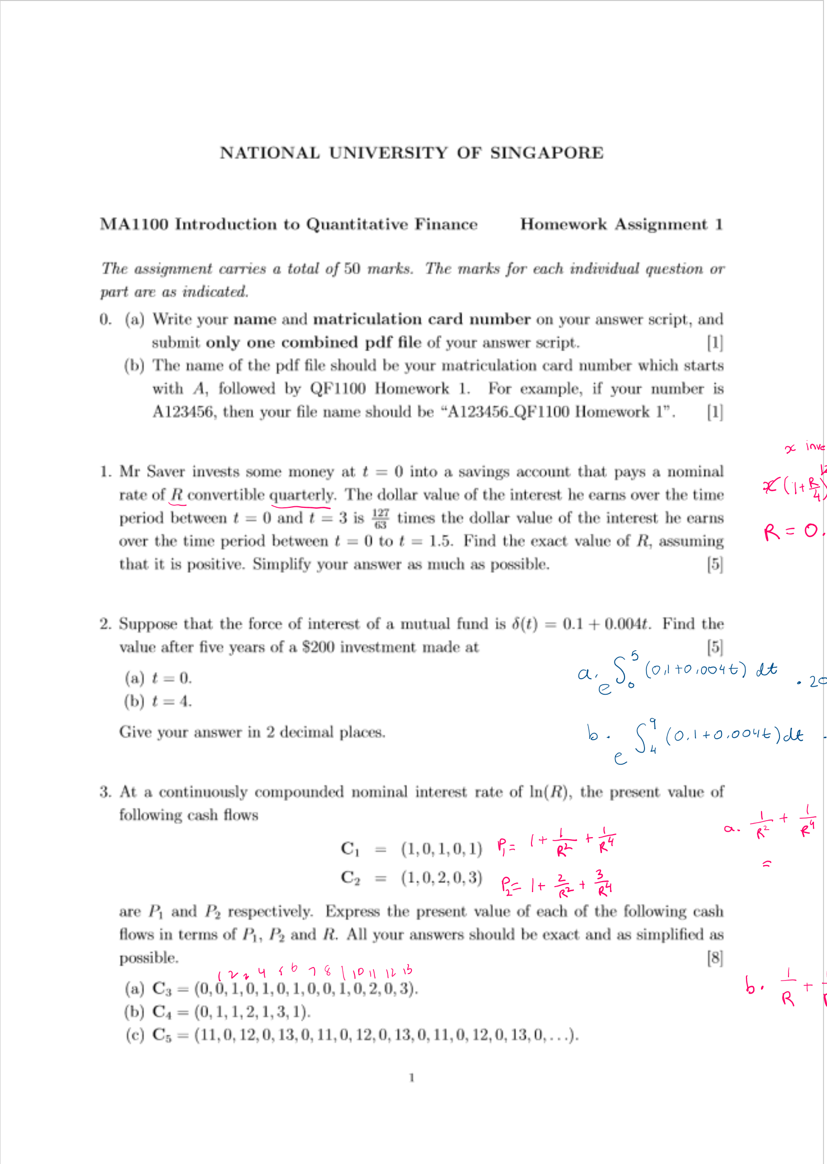
$$\frac{\left(1-\frac{1}{1.01^{180}}\right)}{1-\frac{1}{1.01^{240}}}$$

$$)^{1/12}+\frac{x}{(1+11\%)^{2/12}}+\cdots+\frac{x}{1.01^{120/12}}$$

QF1100-Assignment 1

Friday, September 8, 2023 5:31 PM

QF1100-Assignment 1



NATIONAL UNIVERSITY OF SINGAPORE

MA1100 Introduction to Quantitative Finance **Homework Assignment 1**

The assignment carries a total of 50 marks. The marks for each individual question or part are as indicated.

0. (a) Write your **name** and **matriculation card number** on your answer script, and submit **only one combined pdf file** of your answer script. [1]
   (b) The name of the pdf file should be your matriculation card number which starts with A, followed by QF1100 Homework 1. For example, if your number is A123456, then your file name should be "A123456_QF1100 Homework 1". [1]

1. Mr Saver invests some money at $t = 0$ into a savings account that pays a nominal rate of R convertible quarterly. The dollar value of the interest he earns over the time period between $t = 0$ and $t = 3$ is $\frac{127}{63}$ times the dollar value of the interest he earns over the time period between $t = 0$ to $t = 1.5$. Find the exact value of R, assuming that it is positive. Simplify your answer as much as possible. [5]

x invested

$$\cancel{x}\left(1+\frac{R}{4}\right)^{\cancel{12}\,6} = \cancel{x}\cancel{\left(1+\frac{R}{4}\right)^{6}}\cdot\frac{127}{63}$$

$R = 0.49576$

2. Suppose that the force of interest of a mutual fund is $\delta(t) = 0.1 + 0.004t$. Find the value after five years of a \$200 investment made at [5]
   (a) $t = 0$.
   (b) $t = 4$.

   Give your answer in 2 decimal places.

a. $e^{\int_0^5 (0.1+0.004t)\,dt} \cdot 200 = 346.65$

b. $e^{\int_4^9 (0.1+0.004t)\,dt} \cdot 200 = 375.52$

3. At a continuously compounded nominal interest rate of $\ln(R)$, the present value of following cash flows

$$\mathbf{C}_1 = (1, 0, 1, 0, 1) \qquad P_1 = 1 + \frac{1}{R^2} + \frac{1}{R^4}$$
$$\mathbf{C}_2 = (1, 0, 2, 0, 3) \qquad P_2 = 1 + \frac{2}{R^2} + \frac{3}{R^4}$$

are P_1 and P_2 respectively. Express the present value of each of the following cash flows in terms of P_1, P_2 and R. All your answers should be exact and as simplified as possible. [8]

1 2 3 4 5 6 7 8 1 10 11 12 13

(a) $\mathbf{C}_3 = (0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 2, 0, 3)$.
(b) $\mathbf{C}_4 = (0, 1, 1, 2, 1, 3, 1)$.
(c) $\mathbf{C}_5 = (11, 0, 12, 0, 13, 0, 11, 0, 12, 0, 13, 0, 11, 0, 12, 0, 13, 0, \ldots)$.

a. $\frac{1}{R^2} + \frac{1}{R^4} + \frac{1}{R^6} + \frac{1}{R^9} + \quad + \frac{2}{R^{11}} + \frac{3}{R^{13}}$

$=$

b. $\frac{1}{R} + \frac{1}{R^2} + \frac{2}{R^3} + \frac{1}{R^4} + \frac{3}{R^5} + \frac{1}{R^6}$

1

4. The present value of the cash flow $(\overset{0}{-x-100}, \overset{1}{2000}, \overset{2}{x}, \overset{3}{0}, \overset{4}{3x})$ is \$2,000. Suppose that the effective annual interest rate is 3%. [5]

   (a) Find the value of x, and calculate to the nearest cent.

   (b) Up to 3 decimal places, find the future value of this cash flow at time $t = 4$.

5. Consider the cash flow $\mathbf{C} = (-50, -a, -a, 0, 120 + 3a)$, where $a > 0$. [5]

   (a) Write down the equation of value for this cash flow. Your answer should be in terms of a.

   (b) Show that every internal rate of return of $\mathbf{C}$ is also an internal rate of return of the cash flow

$$(-50, -a, -a, 0, 120 + 3a, -50, -a, -a, 0, 120 + 3a).$$

6. Suppose that the continuously compounded nominal interest rate is 10%. Calculate the time value, at the end of the tenth year, of a 10-year annuity that pays, starting from the end of the first month, \$200 every month for the first five years and \$300 every month for the remaining five years. Give your answer to the nearest dollar. [5]

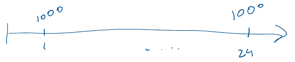
✓ 7. Starting from her 30th birthday ($t = 0$), Miss Saver deposited some money **each year** in a savings account for the next 35 years. The first deposit of \$1000 was made on her 30th birthday, she made 35 deposits in total, and every subsequent deposit increased by 4%. After she retired on her 65th birthday, she withdrew an equal amount each year from her savings account. The first withdrawal was made on her 65th birthday. After the 30th withdrawal, her bank savings account became empty. Assuming the effective annual interest rate is 3%, compute the value of her annual from her bank saving account after she retired. Give your answer to the nearest integer. [7]

8. Lee made a nonrefundable deposit of his first month's rent to secure a two-year lease of Apartment A, whose monthly rent is \$1000. Later, on the same day, Lee finds a Apartment B, which he likes just as well, but its monthly rent for a two-year lease is \$$M$. Assume a nominal rate interest rate of 4% convertible monthly. The rent will be paid monthly and at the beginning of each month. [8]

   (a) Write down the cash flow and compute the present value (to 2 decimal numbers) if Lee rents Apartment A for two years.

   (b) Find the value of M (to 2 decimal places) so that Lee is indifferent between staying with his two-year lease of Apartment A and switching to a two-year lease of Apartment B.

a.



$$PV_A = 1000 + \frac{1000}{\left(1+\frac{0.04}{12}\right)^1} + \cdots + \frac{1000}{\left(1+\frac{0.04}{12}\right)^{24}} = 1000 + \frac{1000}{\left(1+\frac{0.04}{12}\right)}\left[\frac{1-\left(\frac{0.04}{12}\right)^{24}}{1-\left(\frac{0.04}{12}\right)}\right]$$

$$= 2000.0\overline{111}$$

b. $$PV_B = \frac{M}{1+\frac{0.04}{12}} + \frac{M}{\left(1+\frac{0.04}{12}\right)^2} + \cdots \frac{M}{\left(1+\frac{0.04}{12}\right)^{24}} = PV_A$$

$$2000 \overset{?}{=} PV = -x - 100 + \frac{2000}{(1.03)} + \frac{x}{1.03^2} + \frac{3x}{1.03^4}$$

$$158.25 = -x + \frac{x}{1.03^3} + \frac{3x}{1.03^4}$$

$$\frac{(-1.03^4 + 1.03 + 3)x}{1.03^4}$$

$$x = 61.3238$$

$$FV = (-x-100)(1.03)^4 + 2000(1.03)^3 + x(1.03)^2 + 3x$$

$$= 2252.912526$$

a $$PV = -50 - \frac{a}{(1+r)} - \frac{a}{(1+r)^2} + \frac{120+3a}{(1+r)^4}$$

b.

120 mnths 60

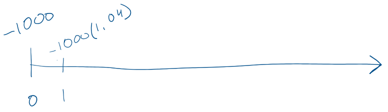
6.)

$$TV(10) = 200(1.1)^{119} + 200(1.1)^{118} + \cdots + 200(1.1)^{60} + 300(1.1)^{59} + \cdots + 300(1.1)^1 + 300$$

$$= 200 \cdot 1.1^{60}\left(\frac{1-(1.1)^{60}}{1-1.1}\right) + 300(1.1)\left(\frac{1-1.1^{60}}{1-1.1}\right)$$

$$= 1858106633.8$$

7.)



$$1000 + 1000(1.04) + \cdots + 1000(1.04)^{34} =$$

$$TV(65) =$$

ch3 (1)

Monday, September 11, 2023 2:30 PM

ch3 (1)

Introduction to Quantitative Finance (QF1100)
Chapter 3: Mean-Variance Analysis

Zhang Tengren

September 6, 2023

Contents

1 Review of Probability Theory **2**
- 1.1 Probability spaces and random variables 2
- 1.2 Expected Value 4
- 1.3 Variance 5
- 1.4 Covariance 6

2 Return and Risk of an Asset **9**
- 2.1 Rate of Return of an Asset 9
- 2.2 Risks and Correlations of Assets 9

3 Return and Risk of a Portfolio **11**
- 3.1 Portfolio Weights and Short Selling 11
- 3.2 Portfolio mean and variance 12
- 3.3 Diversification 14

4 Portfolios of Two Assets **16**
- 4.1 Global Minimum-variance Portfolio 16
- 4.2 Portfolio Graph and feasible sets 18

5 Logarithmic Return **23**

1 Review of Probability Theory

We will first review some basics of probability theory.

1.1 Probability spaces and random variables

A *finite probability space* is a pair (Ω, P), where

- Ω is a finite, non-empty set, called the *sample space*,
- $P : \Omega \to [0,1]$ is a function, called the *probability function*,

such that

$$\sum_{a\in\Omega} P(a) = 1.$$

We refer to any point in Ω as an *sample* and every subset in Ω as an *event*. Given any event $A \subset \Omega$, we define the probability of A to be

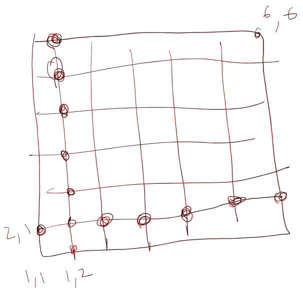
$$P(A) = \sum_{a\in A} P(a).$$

Remark 1.1. One can also define the notion of a probability space without requiring the sample space to be finite (for example, one can take $\Omega = \mathbb{R}$). However, the definition is much more complicated, and we will not be using this notion much in this course. On the other hand, the expected value, variance, and covariance can defined below can be extended to general probability spaces, and Propositions 1.5, 1.6, and 1.9 also hold for more general probability spaces.

Example 1.2. Suppose that we are tossing a pair of fair, six-sided die. Then Ω will be the set of pairs (a, b), where a and b both take values from 1 to 6, and P assigns to every sample in Ω the number $\frac{1}{36}$. An example of an event is the set of points (a, b) in Ω where either a or $b = 2$, i.e. the number 2 shows up in the toss.

$$P(A) = \sum_{a\in A} P(a)$$

$$= \frac{11}{36}$$



Given a finite probability space (Ω, P), a *random variable* X on Ω is a function

$$X : \Omega \to \mathbb{R}.$$

Example 1.3. Let Ω and P be as in Example 1.2. The map X that assigns to every sample in Ω the sum of its entries (i.e. $X(a, b) = a + b$) is a random variable.

1.2 Expected Value

If X is a random variable on a finite probability space (Ω, P), then the *expected value* (or *expectation* or *mean*) of X, denoted by μ_X or $\mathbf{E}(X)$, is

$$\mu_X = \mathbf{E}(X) = \sum_{x \in X(\Omega)} x \cdot P(X = x),$$

where $P(X = x)$ denotes the probability of the event

$$\{a \in \Omega : X(a) = x\}$$

occurring. In other words, the expected value of X is a weighted average of the values of X, where the weights are given by the probability function.

Exercise 1.4. Find the expected value of the random variable X given in Example 1.3.

$X(\Omega) = \{2, \cdots, 12\}$

$P(\Omega_2) = \frac{1}{36}, P(\Omega_3) = \frac{2}{36} \cdots P(\Omega_7) = \frac{6}{36}, P(\Omega_8) = \frac{5}{36} \cdots P(\Omega_{12}) = \frac{1}{36}$

$E(x) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + \cdots + 12 \cdot \frac{1}{36}$

$= 7$

The map $X(\Omega) \to [0, 1]$ given by $x \mapsto P(X = x)$ is called the *density function* of X. Note that to calculate the expected value of X, we only need to know its density function. In fact, if we are interested only in X, the probability space (Ω, P) is not that relevant; we only need to know the density function of X. For this reason, it is common to omit mentioning (Ω, P) when we discuss random variables.

The following proposition states some easily verified properties of the expected value.

Proposition 1.5. *Let a, b, c be constants and X, Y be random variables on a finite probability space. Then*

$$\mathbf{E}(c) = c, \qquad \mathbf{E}(cX) = c\mathbf{E}(X),$$
$$\mathbf{E}(X + c) = \mathbf{E}(X) + c, \qquad \mathbf{E}(aX + bY) = a\mathbf{E}(X) + b\mathbf{E}(Y).$$

1.3 Variance

If X is a random variable on a finite probability space with mean μ_X, then the *variance* of X, denoted by σ_X^2 or $\mathbf{Var}(X)$, is defined by

$$\sigma_X^2 = \mathbf{Var}(X) = \mathbf{E}[(X - \overset{\text{mean}}{\mu_x})^2] = \mathbf{E}(X^2) - 2\mathbf{E}(X)\mu_X + \mu_X^2 = \boxed{\mathbf{E}(X^2) - [\mathbf{E}(X)]^2}.$$

The *standard deviation* of X, denoted by σ_X or $\mathbf{SD}(X)$ is defined by

$$\sigma_X = \mathbf{SD}(X) = \sqrt{\mathbf{Var}(X)}.$$

Notice that like the expected value, to calculate the variance of X, we only need to know density function of X (instead of the probability function P).

The following proposition states some easily verified properties of the variance.

Proposition 1.6. *Let c be a constant and X be a random variable on a finite probability space. Then*

$$\mathbf{Var}(c) = 0, \quad \mathbf{Var}(cX) = c^2\mathbf{Var}(X), \quad \mathbf{Var}(X + c) = \mathbf{Var}(X)$$

Exercise 1.7. Find the variance of the random variable X given in Example 1.3.

$X^2 : \Omega \to \mathbb{R}$

$(a, b) \longmapsto (a+b)^2$

$X^2(\Omega) = \{4, 9 \cdots, 144\}$

$= \{r^2 : r \in X(\Omega)\}$

$P(x^2 = 4) = P(x = 2)$

$\vdots$

$P(x^2 = r^2) = P(x = r)$

$E(x^2) = \sum_{x \in X^2(\Omega)} x \cdot P(x^2 = x)$

$= \sum_{r^2 \in x^2(\Omega)} r^2 \cdot P(x^2 = r^2)$

$= \sum_{r \in x(\Omega)} r^2 P(x^2 = r^2)$

$= 4 \cdot \frac{1}{36} + 9 \cdot \frac{2}{36} + 16 \cdot \frac{3}{36}$

$+ 25 \cdot \frac{4}{36} + 36 \cdot \frac{5}{36}$

$+ 49 \cdot \frac{6}{36} + 64 \cdot \frac{5}{36}$

$+ 81 \cdot \frac{4}{36} + 100 \cdot \frac{3}{36}$

$+ 121 \cdot \frac{2}{36} + 144 \cdot \frac{1}{36}$

$=$ ans

$\text{var} = E(x^2) - E(x)^2$

$= \frac{35}{6}$

1.4 Covariance

Just as the expected value and the variance of a single random variable give us information about this random variable, the covariance between two random variables give us information about the relationship between the random variables.

Let X and Y be two random variables on a finite probability space (Ω, P). The *covariance* between X and Y, denoted by $\sigma_{X,Y}$ or $\mathbf{Cov}(X,Y)$, is defined by

$$\sigma_{X,Y} = \mathbf{Cov}(X,Y) := \mathbf{E}\left[(X - \mathbf{E}(X))(Y - \mathbf{E}(Y))\right] = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y).$$

Explicitly,

$$\mathbf{E}(XY) = \sum_{(x,y) \in X(\Omega) \times Y(\Omega)} xy \cdot P(X = x, Y = y),$$

where $P(X = x, Y = y)$ denotes the probability of the event

$$\{(\underset{\omega}{a,b}) \in \Omega : (X(\underset{\omega}{a,b}), Y(\underset{\omega}{a,b})) = (x,y)\}$$

occurring.

The map $X(\Omega) \times Y(\Omega) \to [0,1]$ given by $(x,y) \mapsto P(X = x, Y = y)$ is called the *joint density function* of X and Y. It follows that to calculate the covariance of X and Y, we only need to know density functions of X and Y as well as the joint density function of X and Y (instead of the probability function P).

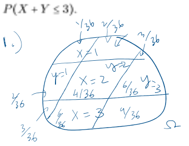
The *correlation coefficient* of X and Y is

$$\rho_{X,Y} = \frac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X)\,\mathbf{Var}(Y)}}.$$

Exercise 1.8. Suppose X, Y are random variables that takes values in $\{1,2,3\}$. Suppose that the joint density function $f(x,y) = \frac{1}{36}xy$. Find

1. the probability density functions of X and Y respectively;
2. $\rho_{X,Y}$; correlation coefficient
3. $P(X + Y \le 3)$.

1.)



$P(x=1) = 6/36$

$P(x=2) = 12/36$

$P(x=3) = 18/36$

$$P(X=x) = P(x,1) + P(x,2) + P(x,3)$$
$$= \frac{1}{36}x + \frac{2}{36}x + \frac{3}{36}x$$
$$= \frac{1}{6}x$$

$$P(Y=y) = P(1,y) + P(2,y) + P(3,y)$$
$$= \frac{1}{36}y + \frac{2}{36}y + \frac{3}{36}y = \frac{1}{6}y$$

2.)

$$\rho_{xy} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6} = \frac{7}{3}$$

$$E(Y) = \frac{7}{3}$$

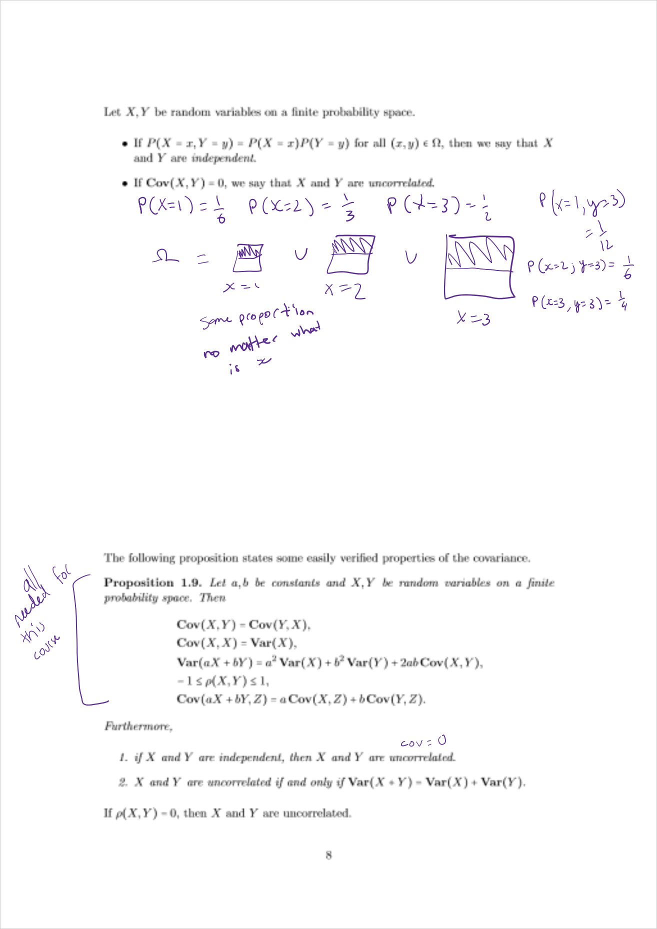
$$E(XY) = \sum_{(x,y)\,x y \in \{1,2,3\}} xy \cdot \frac{1}{36} xy$$
$$= \frac{1}{36} \sum x^2 y^2 = \frac{1}{36}\left[1+4+4+16+9+9+36+36+81\right]$$
$$= \frac{196}{36} = \frac{49}{9}$$

$$\mathrm{Cov}(X,Y) = \frac{49}{9} - \frac{7}{3} \times \frac{7}{3} = E(XY) - E(X) - E(Y)$$
$$= 0$$

$$\therefore \rho_{xy} = 0$$

3.) $P(X + Y \le 3)$

$$= P(x=1, y=1) + P(x=2, y=1) + P(x=1, y=2)$$
$$= \frac{1}{36} \cdot 1 \cdot 1 + \frac{1}{36} \cdot 2 \cdot 1 + \frac{1}{36} \cdot 1 \cdot 2 = \boxed{\frac{5}{36}}$$



Let X, Y be random variables on a finite probability space.

- If $P(X = x, Y = y) = P(X = x)P(Y = y)$ for all $(x, y) \in \Omega$, then we say that X and Y are *independent*.
- If $\mathbf{Cov}(X, Y) = 0$, we say that X and Y are *uncorrelated*.

$P(X=1) = \frac{1}{6}$ $P(X=2) = \frac{1}{3}$ $P(X=3) = \frac{1}{2}$

$P(x=1, y=3) = \frac{1}{12}$

$\Omega =$ $\cup$ $\cup$

$x = 1$ $x = 2$ $x = 3$

$P(x=2, y=3) = \frac{1}{6}$

$P(x=3, y=3) = \frac{1}{4}$

same proportion no matter what is x

The following proposition states some easily verified properties of the covariance.

all needed for this course

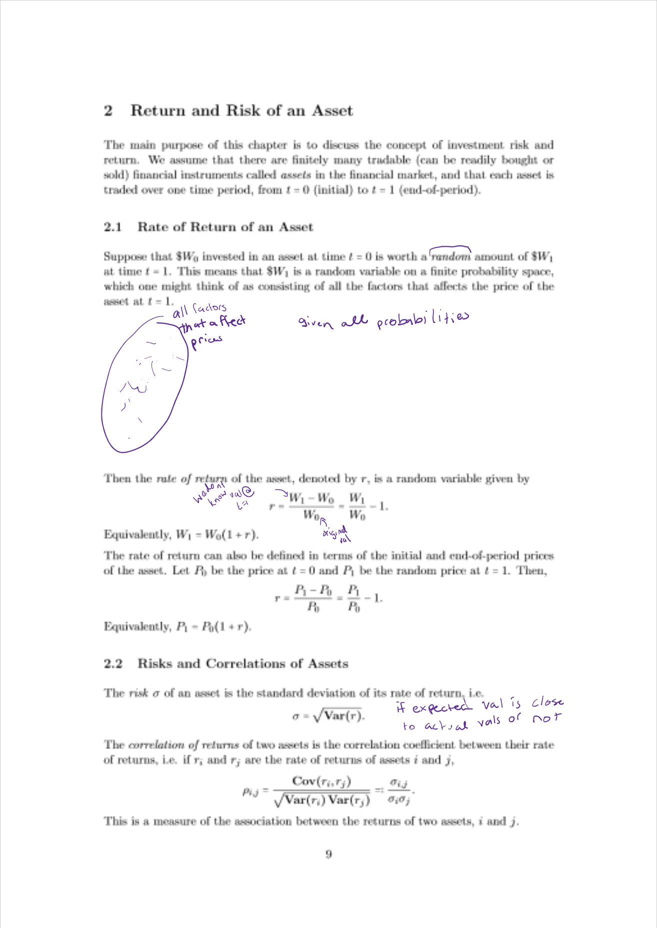
Proposition 1.9. *Let a, b be constants and X, Y be random variables on a finite probability space. Then*

$$\begin{aligned}
&\mathbf{Cov}(X,Y) = \mathbf{Cov}(Y,X),\\
&\mathbf{Cov}(X,X) = \mathbf{Var}(X),\\
&\mathbf{Var}(aX+bY) = a^2\,\mathbf{Var}(X) + b^2\,\mathbf{Var}(Y) + 2ab\,\mathbf{Cov}(X,Y),\\
&-1 \le \rho(X,Y) \le 1,\\
&\mathbf{Cov}(aX+bY,Z) = a\,\mathbf{Cov}(X,Z) + b\,\mathbf{Cov}(Y,Z).
\end{aligned}$$

Furthermore,

1. *if X and Y are independent, then X and Y are* $\overset{\text{cov} = 0}{\textit{uncorrelated}}$.
2. *X and Y are uncorrelated if and only if* $\mathbf{Var}(X+Y) = \mathbf{Var}(X) + \mathbf{Var}(Y)$.

If $\rho(X,Y) = 0$, then X and Y are uncorrelated.



2 Return and Risk of an Asset

The main purpose of this chapter is to discuss the concept of investment risk and return. We assume that there are finitely many tradable (can be readily bought or sold) financial instruments called *assets* in the financial market, and that each asset is traded over one time period, from $t = 0$ (initial) to $t = 1$ (end-of-period).

2.1 Rate of Return of an Asset

Suppose that $\$W_0$ invested in an asset at time $t = 0$ is worth a *random* amount of $\$W_1$ at time $t = 1$. This means that $\$W_1$ is a random variable on a finite probability space, which one might think of as consisting of all the factors that affects the price of the asset at $t = 1$.

all factors that affect prices

given all probabilities

Then the *rate of return* of the asset, denoted by r, is a random variable given by

we don't know val @ t=1

$$r = \frac{W_1 - W_0}{W_0} = \frac{W_1}{W_0} - 1.$$

original val

Equivalently, $W_1 = W_0(1 + r)$.

The rate of return can also be defined in terms of the initial and end-of-period prices of the asset. Let P_0 be the price at $t = 0$ and P_1 be the random price at $t = 1$. Then,

$$r = \frac{P_1 - P_0}{P_0} = \frac{P_1}{P_0} - 1.$$

Equivalently, $P_1 = P_0(1 + r)$.

2.2 Risks and Correlations of Assets

The *risk* σ of an asset is the standard deviation of its rate of return, i.e.

$$\sigma = \sqrt{\mathbf{Var}(r)}.$$

if expected val is close to actual vals or not

The *correlation of returns* of two assets is the correlation coefficient between their rate of returns, i.e. if r_i and r_j are the rate of returns of assets i and j,

$$\rho_{i,j} = \frac{\mathbf{Cov}(r_i, r_j)}{\sqrt{\mathbf{Var}(r_i)\,\mathbf{Var}(r_j)}} =: \frac{\sigma_{i,j}}{\sigma_i \sigma_j}.$$

This is a measure of the association between the returns of two assets, i and j.

Exercise 2.1. A hypothetical end-of-period prices of shares of company ABC are given below.

possible prices at $t=1$

Share Price	3.50	4.20	5.00	5.50	6.00	P_1
Probability	0.15	0.10	0.30	0.20	0.25	$P(P_1 = \rightarrow)$

The initial share price is \$5.00. A man buys 200 shares of ABC. Calculate the expected profit/loss, mean rate of return and standard deviation of the rate of return.

$t_0 = 5$

$E(1) = 3.5(0.15) + 4.2(0.1) + 5(0.3) + 5.5(0.2) + 6(0.25) = 5.045$

$E(\text{Profit}) = 9$

$\text{Profit} = 200 \times P_1 - 200 \cdot P_0$

$= 200(P_1 - P_0)$

$$E\left(\frac{P_1 - P_0}{P_0}\right) = \frac{1}{5} E(P_1) - 1 = 0.9\%$$

for var

$$E\left(\left(\frac{P_1 - P_0}{P_0}\right)^2\right) = \frac{1}{25} E(P_1^2 - 2P_1P_0 + P_0^2)$$

$$= \frac{1}{25}\left(E(P_1^2) - 10E(P_1) + 25\right)$$

$$= \frac{1}{25}\left(E(P_1^2) - 10 \cdot 5.045 + 25\right)$$

$$E(P_1^2) = 3.5^2(0.15) + 4.2^2(0.1) + 5^2(0.3) + 5.5^2(0.2) + 6^2(0.25) = 5.045$$

$\Rightarrow$

$$\text{Var}\left(\frac{P_1 - P_0}{P_0}\right) = E\left(\left(\frac{P_1 - P_0}{P_0}\right)^2\right) - E\left(\left(\frac{P_1 - P_0}{P_0}\right)\right)^2$$

3 Return and Risk of a Portfolio

We will now generalize the discussion in the previous section to a *portfolio* of assets, which is a simultaneous investment into different assets.

3.1 Portfolio Weights and Short Selling

If an individual invests in a portfolio p with n assets, then for each $i \in \{1, \ldots, n\}$, let W_i be the amount of money invested into asset i.

It is possible that $W_i < 0$ for some i, in which case we say that the individual *short sells asset i*. This means that the individual borrows a certain number of units of the asset from another individual (the lender) at $t = 0$ and sells them immediately to receive an amount $\$W_i$. At some pre-agreed date $t = 1$, the short-seller will buy the same number of units of the asset for an amount $\$W_i'$ and return the asset to the lender. Assuming there are no transaction costs, the borrower makes a profit of $\$(W_i - W_i')$ which is positive if and only if the value of the asset falls.

Note that the loss from short-selling an asset can be potentially unlimited (theoretically, the asset value W_i' can be arbitrarily large) while the gain is bounded above by W_i.

Notice that the net amount invested into p at time $t = 0$ is

$$W := \sum_{i=1}^{n} W_i.$$

We will always assume that $W > 0$. (In reality, this is usually forced by trading regulations.) Then we may define the *portfolio weight of asset i* by

$$w_i := \frac{W_i}{W}.$$

In other words, w_i is the proportion of the net value of p that is invested in asset i at time $t = 0$. We call the vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^T$ the *portfolio weight vector* of p. These portfolio weights are useful in analyzing the per-dollar performance of p because they factor out the value of the initial investment of p.

3.2 Portfolio mean and variance

Given a portfolio p with portfolio weight vector $(w_1, \dots, w_n)^T$, the rate of return, r_p of the portfolio is a random variable that is related to the rate of return of individual assets, r_i by

$$r_p = \sum_{i=1}^{n} w_i r_i$$

It follows that the expected rate of return of p, or simply, the *portfolio mean* of p, is

$$\mu_p = \mathbf{E}(r_p) = \sum_{i=1}^{n} w_i \mu_i$$

We may use

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \dots \mu_n)^T$$

to denote the vector of expected rates of return of the assets in p. We shall call this vector the *mean vector* for simplicity.

As for the variance of the rate of return of p, or simply the *portfolio variance* of p is

$$\begin{aligned} \sigma_p^2 &= \mathbf{Var}(r_p) \\ &= \mathbf{Var}\left(\sum_{i=1}^{n} w_i r_i\right) \\ &= \mathbf{Cov}\left(\sum_{i=1}^{n} w_i r_i, \sum_{j=1}^{n} w_j r_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}. \end{aligned}$$

Since $\sigma_{i,i} = \sigma_i^2 = \mathbf{Var}(r_i)$ and $\sigma_{i,j} = \sigma_{j,i}$, we can also write

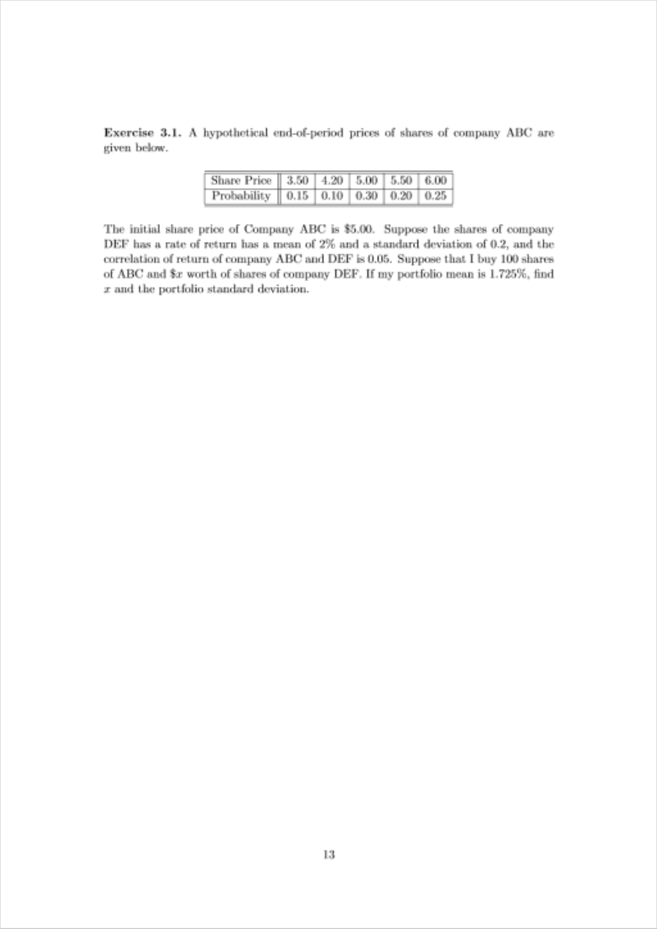
risk of ~~asset~~ portfolio

$$\begin{aligned} \sigma_p^2 &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j \neq i} w_i w_j \sigma_{i,j} \\ &= \underbrace{\sum_{i=1}^{n} w_i^2 \sigma_i^2}_{\text{sum of risk from indiv assets}} + 2\sum_{i=1}^{n}\sum_{j<i} w_i w_j \sigma_{i,j}. \end{aligned}$$

risk how related assets are

Just as it was in the case of assets, the standard deviation σ_p of the portfolio p is a measure of its risk.

$$\sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j<i} w_i w_j \sigma_{i,j}$$



Exercise 3.1. A hypothetical end-of-period prices of shares of company ABC are given below.

Share Price	3.50	4.20	5.00	5.50	6.00
Probability	0.15	0.10	0.30	0.20	0.25

The initial share price of Company ABC is \$5.00. Suppose the shares of company DEF has a rate of return has a mean of 2% and a standard deviation of 0.2, and the correlation of return of company ABC and DEF is 0.05. Suppose that I buy 100 shares of ABC and \$$x$ worth of shares of company DEF. If my portfolio mean is 1.725%, find x and the portfolio standard deviation.

3.3 Diversification

Suppose that the portfolio weights of a portfolio p are all the same. For the special case when the returns of assets are mutually uncorrelated, i.e. $\sigma_{i,j} = 0$ for $i \neq j$, the portfolio risk as measured by the standard deviation of its rate of return is

$$\sigma_p = \sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}.$$

as $n \to \infty$ $\sigma_p \to 0$

So more terms/asset, risk $\to 0$

*asset uncorrelated

Under certain conditions (for example the risk of all assets have a uniform upper bound), $\sigma_p^2 \to 0$ as the number of assets tends to infinity. This demonstrates the power of diversification: adding securities whose returns are uncorrelated with one another help reduce the portfolio risk.

In general, it is unlikely that asset returns are mutually uncorrelated. Let $\bar{\sigma}^2$ and $\bar{\phi}$ be the average variance and average covariance of an n assets, that is

$$\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 \text{ and } \bar{\phi} = \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sigma_{i,j}.$$

One should think of $\bar{\sigma}^2$ as the asset-specific risks of the portfolio and $\bar{\phi}$ as the market-wide risk of the portfolio.

Suppose that $\bar{\sigma}^2 \to \sigma^2$ and $\bar{\phi} \to \phi$ as $n \to \infty$ (this happens if the risk of all assets have a uniform upper bound). Then, for an equally weighted portfolio, it can be shown that

$$\sigma_p^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 + \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sigma_{i,j} \to 0 + \phi = \phi.$$

This shows the limitation of diversification as a tool to reduce portfolio risk: while the asset-specific risk $\bar{\sigma}^2$ can be driven to zero, the market-wide risk $\bar{\phi}$ cannot be eliminated no matter how one increases the number of assets in the portfolio.

Exercise 3.2. In a financial market, there are n securities with rates of return r_i $(i = 1, 2, \ldots, n)$. It is given that

$$\sigma_{i,j} = \begin{cases} 4\sigma^2 & \text{if } i = j, \\ \sigma^2 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find an expression for σ_p for the equally-weighted portfolio, and determine $\lim_{n\to\infty}\sigma_p$.

$$\sigma_p^{\cancel{2}} = \sqrt{\frac{1}{n^2}\sum_{i=0}^{n}\sigma_i^2 + \frac{1}{n^2}\sum_{i,j=1}^{n}\sigma_{i,j}}$$

$$\lim_{n\to\infty}\sigma_p = \sqrt{\frac{1}{n^2}\sum_{i,j=1}^{n}\sigma_{i,j}} = \sqrt{\frac{1}{n^2}\left(n4\sigma^2+n\sigma^2\right)} = \sqrt{\frac{1}{n}\left(5\sigma^2\right)}$$

$$n\cdot 4\sigma^2 + n\sigma^2$$

$\boxed{\text{ANS}}$

$$\sigma_p^2 = \frac{1}{n^2}\sum_{i=0}^{n}\sigma_i^2 + \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sigma_{i,j} = \frac{1}{n^2}\left(4\sigma^2\right) + \frac{1}{n^2}\left[\sigma_{1,2} + \sigma_{2,3} + \sigma_{3,2} + \sigma_{3,3} + \cdots\right]$$

$$\sigma_p^2 = \frac{4\sigma^2 n}{n^2} + \frac{2(n-1)\sigma^2}{n^2}$$

$$\lim_{n\to\infty}\sigma_p = \sqrt{\lim_{n\to\infty}\left(\underset{\substack{\Downarrow\\0}}{\frac{4\sigma^2}{n}} + \underset{\substack{\Downarrow\\0}}{\frac{2(n-1)\sigma^2}{n^2}}\right)} = 0$$

4 Portfolios of Two Assets

Let Asset 1 have mean μ_1 and variance σ_1^2, let Asset 2 have mean μ_2 and variance σ_2^2, and let the correlation of returns of Assets 1 and 2 be $\rho_{1,2}$. Consider a portfolio p with these two assets, and whose portfolio weights for Assets 1 and 2 are α and $1-\alpha$ respectively. Then the portfolio mean μ_p and variance σ_p^2 of p are given by

$$\mu_p = \alpha\mu_1 + (1-\alpha)\mu_2 \tag{4.1}$$

correlation coef.

and

$$\sigma_p^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_{1,2} = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2. \tag{4.2}$$

$$\sigma_p^2 = \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2 + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2$$

4.1 Global Minimum-variance Portfolio

minimize risk/σ_p

A risk averse individual seeks a portfolio with the smallest risk. He will thus seek the value of α that minimizes σ_p^2.

It can be shown (assuming $\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2 \neq 0$) that the minimum portfolio variance σ_p^2 occurs when

$$\alpha = \alpha^* = \frac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}.$$

derivative

$$\frac{d(\sigma_p^2)}{d\alpha}$$

$$\alpha = \frac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}$$

Then the minimum portfolio variance is

$$(\sigma_p^2)^* = \frac{\sigma_1^2\sigma_2^2(1-\rho_{1,2}^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2},$$

and the corresponding portfolio mean can then be determined from

$$(\mu_p)^* = \alpha^*\mu_1 + (1-\alpha^*)\mu_2$$

We call the portfolio with minimum variance the *global minimum-variance portfolio (GMVP)*.

$$\sigma_p^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho_{1,2}^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}$$

$$\mu_p = \alpha\mu_1 + (1-\alpha)\mu_2$$

Exercise 4.3. The standard deviation of the rate of return of assets A and B are 0.27 and 0.16 respectively. The correlation of these return rates is ρ. An investor is considering investing half of his initial capital in asset A and the other half in either cash (with zero risk) or in asset B. His objective is to minimise investment risk

1. If $\rho = -0.5$, would he choose cash or asset B?
2. Find the range of ρ for which he would prefer cash to B.

$\sigma_A = 0.27^2$

$\sigma_B = 0.16^2$

$\rho_{A,B} = -0.5$

$$\sigma_p^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2$$

$$B \Rightarrow \quad = \left(\tfrac{1}{2}\right)^2(0.27)^2 + \left(\tfrac{1}{2}\right)^2 0.16^2 + 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)(-0.5)(0.27)(0.16) = 0.013825$$

$$\text{cash} \Rightarrow \quad \left(\tfrac{1}{2}\right)^2(0.27^2) + \left(\tfrac{1}{2}\right)^2 \cdot 0 + 0 = 0.018225$$

choose B

* * *

For general ρ

$$\sigma_{p_1}^2 = \frac{0.27^2}{4} + \frac{0.16^2}{4} + \frac{0.27 \cdot 0.16 \cdot \rho}{2}$$

Find ρ s.t. $\sigma_{p_1}^2 < \sigma_{p_2}^2$

$$\Rightarrow 0.018225 < \frac{0.27^2}{4} + \frac{0.16^2}{4} + \frac{0.27 \cdot 0.16 \cdot \rho}{2}$$

$$\rho > -\frac{8}{27}$$ prefer cash

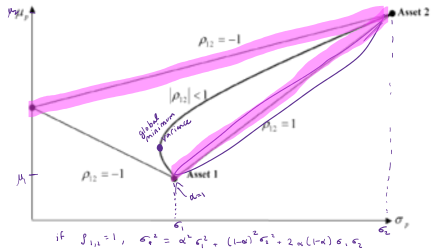
4.2 Portfolio Graph and feasible sets

Suppose that $\mu_1 \neq \mu_2$ and $\sigma_1 \neq \sigma_2$. Eliminating α from (4.1) and (4.2), we obtain an equation of the form

$$\sigma_p^2 = A\mu_p^2 + B\mu_p + C \tag{4.4}$$

for some constants A, B and C, with $A > 0$. Here, A, B, and C depend on μ_1, μ_2, σ_1, σ_2, and $\rho_{1,2}$.

The graph of μ_p against σ_p for all allowable portfolio weights w is also known as the *feasible set* for two given assets.



if $\rho_{1,2} = 1$, $\sigma_p^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2$

$$= (\alpha\sigma_1 + (1-\alpha)\sigma_2)^2$$

$$\sigma_p = \alpha(\sigma_1 - \sigma_2) + \sigma_2$$

$$= \frac{\mu_p - \mu_2}{\mu_1 - \mu_2}(\sigma_1 - \sigma_2) + \sigma_2$$

if $\rho_{1,2} = -1$, $\sigma_p^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 - 2\alpha(1-\alpha)\sigma_1\sigma_2$

$$= (\alpha\sigma_1 - (1-\alpha)\sigma_2)^2$$

$$= \left(\frac{\mu_p-\mu_2}{\mu_1-\mu_2}\sigma_1 - \frac{\mu_1-\mu_p}{\mu_1-\mu_2}\sigma_2\right)^2$$

$$\sigma_p = \begin{cases} \dfrac{\mu_p\sigma_1 - \mu_2\sigma_1 - \mu_1\sigma_2 + \mu_p\sigma_2}{\mu_1-\mu_2} & \text{if } > 0 \\[2ex] \dfrac{\mu_1\sigma_2 - \mu_p\sigma_2 - \mu_p\sigma_1 + \mu_2\sigma_1}{\mu_1-\mu_2} & \text{if } < 0 \end{cases}$$

For example, when $0 \le \alpha \le 1$ (that is short-selling is **not allowed**), a typical feasible set for the various cases outlined above is depicted in the diagram babove. If short-selling is possible, the feasible set can be obtained by simply extending the corresponding graph below beyond the end points.

Example 4.5. A portfolio is to be constructed from two assets whose mean and variance are summarised in the table below. The correlation of these assets is ρ.

Asset	Mean	Variance
1	0	0.2
2	0.2	0.8

If $\rho = \frac{5}{8}$,

1. find the portfolio weights that minimize risk and the corresponding mean and variance.
2. sketch the (σ_p, μ_p)-graph, that is the graph of μ_p against σ_p as the portfolio weights vary. Indicate the point representing the GMVP.

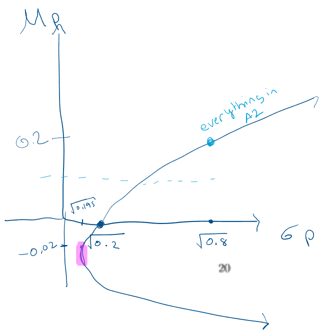
1.)

$$\sigma_p^2 = \frac{\sigma_1^2 \sigma_2^2 (1 - \rho_{1,2}^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2} = \frac{(0.2)^2 (0.8)^2 \left(1 - \left(\frac{5}{8}\right)^2\right)}{(0.2)^2 + (0.8)^2 - 2\left(\frac{5}{8}\right)(0.2)(0.8)}$$

$$=$$

$$\mu_p = \alpha \mu_1 + (1-\alpha)\mu_2$$

$$\alpha = \frac{\sigma_2 (\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}$$



minimizing risk
$\Rightarrow$ negative return

$\boxed{\text{ANS}}$

$$\sigma_p^2 = \alpha^2 (0.2) + (1+\alpha)^2 (0.8) + 2\alpha(1-\alpha)\sqrt{0.2 \cdot 0.8} \cdot \frac{5}{8}$$

$$\frac{d(\sigma_p')}{d\alpha} = 0.4\alpha - 1.6(1-\alpha) + 2(1-2x)\frac{5 \cdot \sqrt{0.16}}{8}$$

$$= 0.4\alpha - 1.6 + 1.6\alpha - \frac{1}{2} - \alpha$$

$$= \alpha - 1.1 = 0$$

$$\alpha = 1.1$$

When $\alpha = 1.1$, $\sigma_p^2 = (1.1)^2 \cdot 0.2 + (0.1)^2 \cdot 0.8 + 2(-0.1)(1.1)\sqrt{0.2 \cdot 0.8} \cdot \frac{5}{8}$

$$= \boxed{0.195}$$

$$\mu_p = 1.1 \cdot \mu_1 - 0.1\mu_2 =$$

Exercise 4.6. A portfolio is to be constructed from two assets whose mean and variance are summarised in the table below. The correlation of these assets is ρ.

Asset	Mean	Variance
1	0	0.2
2	0.2	0.8

If $\rho = \frac{5}{8}$,

1. find the weight, mean and standard deviation of the portfolio that has the smallest variance under the condition that short-selling is **not allowed**.
2. find the weight, mean and standard deviation of the portfolio that has the smallest variance under the condition that portfolio mean is at least 0.1. ← limit graph

Eq 4.1, 4.2

required mean $\mu_p = 0.1$

$$0.1 = \mu_p = \alpha \cdot 0 + (1-\alpha) \cdot 0.2$$

$$1-\alpha = \frac{0.1}{0.2} = \frac{1}{2}$$

required weight $\alpha = \frac{1}{2}$

$$\sigma_p^2 = \left(\tfrac{1}{2}\right)^2(0.2) + \left(\tfrac{1}{2}\right)^2(0.8) + 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\sqrt{0.2 \cdot 0.8}\,\overset{\rho_{12}}{\left(\tfrac{5}{8}\right)}$$

$$= 3/8$$

required stDev $= \sqrt{3/8}$

---

5 Logarithmic Return

If \$$W_0$ invested in an asset at time $t = 0$ is worth a random amount of \$$W_1$ at time $t = 1$, then the *logarithmic return* (or *log return* for short) of the asset, denoted by R, is the random variable given by

$$R = \ln\left(\frac{W_1}{W_0}\right).$$

Equivalently, $W_1 = W_0 \exp(R)$.

If r is the rate of return, and R is the log return, then

$$\exp(R) = 1 + r$$

or equivalently,

$$R = \ln(1 + r).$$

The rate of return can also be defined in terms of the initial and end-of-period prices of the asset. Let P_0 be the price at $t = 0$ and P_1 be the random price at $t = 1$. Then,

$$R = \ln\left(\frac{P_1}{P_0}\right).$$

Equivalently, $P_1 = P_0 \exp(R)$.

Exercise 5.1 (Log return version of Example 3.1). A hypothetical end-of-period prices of shares of company ABC are given below.

log retur $\ln\left(\frac{3.5}{5}\right)$ $\ln\left(\frac{4.2}{5}\right)$ '' '' ''

Share Price	3.50	4.20	5.00	5.50	6.00
Probability	0.15	0.10	0.30	0.20	0.25

The initial share price is \$5.00. Calculate the mean log return and standard deviation of the log return.

rate of return

mean price

$$0.15(3.5)+0.1(4.2)+0.3(5)+0.2(5.5)+(0.25)(6) = 5.045$$

mean log return = $5.045 - 5 = 0.045$

$e^R = 1+r$

$e^R = 1.045$

$R = 0.04402$

ANS

$$0.15\left(\ln\frac{3.5}{5}\right)+0.1\left(\ln\frac{4.2}{5}\right)+0.2\left(\ln\frac{5.5}{5}\right)+0.25\left(\ln\frac{6}{5}\right)$$

$= -0.00629$ ← mean log return

For variance

$$\underbrace{E\left((\text{log return})^2\right)}_{0.15\left(\left(\ln\frac{3.5}{5}\right)^2\right)+0.1\left(\left(\ln\frac{4.2}{5}\right)^2\right)+0.2\left(\left(\ln\frac{5.5}{5}\right)^2\right)+0.25\left(\left(\ln\frac{6}{5}\right)^2\right)} - (-0.00629)^2 = V(\text{log return})$$

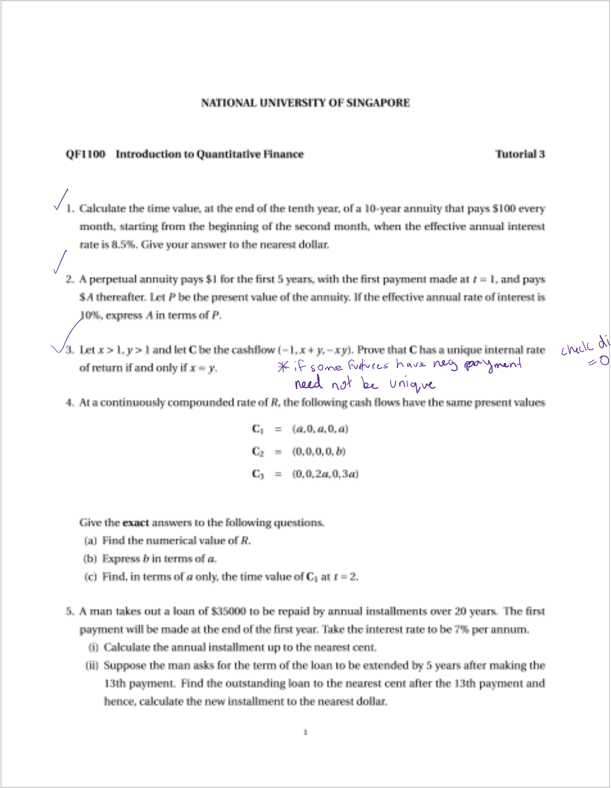
$= 0.03221$

$\sigma = 0.1795$

Tut3

Wednesday, September 13, 2023 12:08 AM

QF1100-Tutorial3



NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 3**

✓ 1. Calculate the time value, at the end of the tenth year, of a 10-year annuity that pays $100 every month, starting from the beginning of the second month, when the effective annual interest rate is 8.5%. Give your answer to the nearest dollar.

✓ 2. A perpetual annuity pays $1 for the first 5 years, with the first payment made at $t = 1$, and pays $A thereafter. Let P be the present value of the annuity. If the effective annual rate of interest is 10%, express A in terms of P.

✓ 3. Let $x > 1$, $y > 1$ and let **C** be the cashflow $(-1, x + y, -xy)$. Prove that **C** has a unique internal rate of return if and only if $x = y$.

* if some futures have neg payment need not be unique

check discriminant $=0$

4. At a continuously compounded rate of R, the following cash flows have the same present values

$$\begin{aligned} \mathbf{C}_1 &= (a, 0, a, 0, a) \\ \mathbf{C}_2 &= (0, 0, 0, 0, b) \\ \mathbf{C}_3 &= (0, 0, 2a, 0, 3a) \end{aligned}$$

Give the **exact** answers to the following questions.

(a) Find the numerical value of R.

(b) Express b in terms of a.

(c) Find, in terms of a only, the time value of $\mathbf{C}_1$ at $t = 2$.

5. A man takes out a loan of $35000 to be repaid by annual installments over 20 years. The first payment will be made at the end of the first year. Take the interest rate to be 7% per annum.

(i) Calculate the annual installment up to the nearest cent.

(ii) Suppose the man asks for the term of the loan to be extended by 5 years after making the 13th payment. Find the outstanding loan to the nearest cent after the 13th payment and hence, calculate the new installment to the nearest dollar.

1

120 -

1.)

$$TV(10) = 100(1+0.085)^{\frac{119}{12}} + 100(1.085)^{\frac{118}{12}} + \cdots + 100$$

$$= 100\left[\frac{1-\left(1.085^{1/12}\right)^{120}}{1-\left(1.085^{1/12}\right)}\right] = \boxed{18485}$$

2.)

$$\overset{}{0}\ \overset{1}{1}\ \overset{1}{2}\ \overset{1}{3}\ \overset{1}{4}\ \overset{1}{5}\ \overset{A}{6}\ \cdots\cdots$$

$$P = \frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.1^3} + \frac{1}{1.1^4} + \frac{1}{1.1^5} + \frac{A}{1.1^6} + \frac{A}{1.1^7} + \cdots$$

$$\frac{1}{1-\frac{1}{1.1}} \Rightarrow \frac{1}{\frac{0.1}{1.1}}$$

$$= 3.7907 + \frac{A}{1.1^6}\left[1 + \frac{1}{1.1} + \frac{1}{1.1^2} + \cdots\right]$$

$$P = \frac{1.1\left(1-\left(\frac{1}{1.1}\right)^6\right)}{0.1} + \frac{A}{1.1^6} \cdot \overset{\Downarrow}{\frac{1.1}{0.1}}$$

$$\frac{P}{10} = 1.1 - \frac{1}{1.1^5} + \frac{A}{1.1^5}$$

$$\boxed{A = \frac{P(1.1)^5}{10} - 1.1^5 + 1}$$

3.)

$$-1 + \frac{x+y}{1+r\%} - \frac{xy}{(1+r\%)^2} = 0$$

$$\frac{-xy}{(1+r\%)^2} + \frac{x+y}{1+r\%} - 1 = 0$$

$$\left(-1 + \frac{x}{1+r\%}\right)\left(1 - \frac{y}{1+r\%}\right) = 0$$

$$-1 + \frac{x}{1+r\%} = 0 \quad \text{or} \quad 1 - \frac{y}{1+r\%} = 0$$

$$\frac{x}{1+r\%} = 1 \qquad 1 = \frac{y}{1+r\%}$$

$$x = 1+r\% \qquad y = 1+r\%$$

4.) a.

$$a + \frac{a}{e^{2R}} + \frac{a}{e^{4R}} = \frac{2a}{e^{2R}} + \frac{3a}{e^{4R}}$$

$$1 = \frac{1}{e^{2R}} + \frac{2}{e^{4R}} = \frac{1}{e^{2R}} + 2\left[\frac{1}{e^{2R}}\right]^2$$

$$2\left[\frac{1}{e^{2R}}\right]^2 + \frac{1}{e^{2R}} - 1 = 0$$

$$2x^2 + x - 1 = 0$$

i.

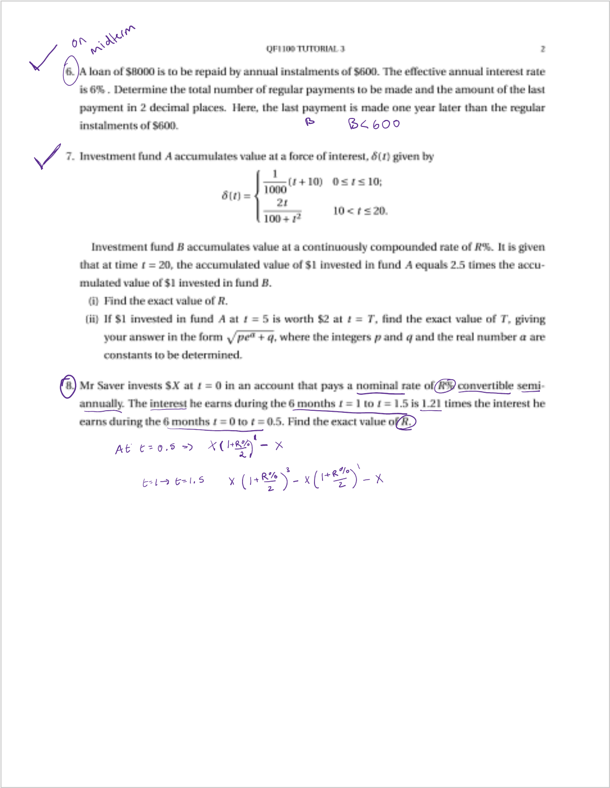
$$35000 = \frac{x}{(1.07)} + \frac{x}{1.07^2} + \cdots + \frac{x}{1.07^{20}}$$

$$= \frac{x}{1.07}\left[\frac{1-\left(\frac{1}{1.07}\right)^{20}}{1-\left(\frac{1}{1.07}\right)}\right]$$

$$x = 3303.75$$

12 more yrs

ii.



✓ on midterm

QF1100 TUTORIAL 3 2

6. A loan of $8000 is to be repaid by annual instalments of $600. The effective annual interest rate is 6% . Determine the total number of regular payments to be made and the amount of the last payment in 2 decimal places. Here, the last payment is made one year later than the regular instalments of $600.

B $B < 600$

✓ 7. Investment fund A accumulates value at a force of interest, $\delta(t)$ given by

$$\delta(t) = \begin{cases} \dfrac{1}{1000}(t+10) & 0 \le t \le 10; \\ \dfrac{2t}{100+t^2} & 10 < t \le 20. \end{cases}$$

Investment fund B accumulates value at a continuously compounded rate of R%. It is given that at time $t = 20$, the accumulated value of $1 invested in fund A equals 2.5 times the accumulated value of $1 invested in fund B.

(i) Find the exact value of R.

(ii) If $1 invested in fund A at $t = 5$ is worth $2 at $t = T$, find the exact value of T, giving your answer in the form $\sqrt{pe^a + q}$, where the integers p and q and the real number a are constants to be determined.

8. Mr Saver invests $X at $t = 0$ in an account that pays a nominal rate of R% convertible semi-annually. The interest he earns during the 6 months $t = 1$ to $t = 1.5$ is 1.21 times the interest he earns during the 6 months $t = 0$ to $t = 0.5$. Find the exact value of R.

$$\text{At } t = 0.5 \Rightarrow X\left(1+\frac{R\%}{2}\right)^1 - X$$

$$t=1 \rightarrow t=1.5 \qquad X\left(1+\frac{R\%}{2}\right)^3 - X\left(1+\frac{R\%}{2}\right)^1 - X$$

6.) actual n payments

$$8000 > \frac{600}{1.06} + \frac{600}{1.06^2} + \cdots + \frac{600}{1.06^n}$$

$$8000 > \frac{600}{1.06}\left[\frac{1-\left(\frac{1}{1.06}\right)^n}{1-\left(\frac{1}{1.06}\right)}\right]$$

⋮

$$n < 27.6$$

so $\boxed{n = 27}$ yrs of regular payments

$$8000 = \frac{600}{1.06}\left[\frac{1-\left(\frac{1}{1.06}\right)^{27}}{1-\left(\frac{1}{1.06}\right)}\right] + \frac{B}{1.06^{28}}$$

$$\boxed{B = 376.63}$$

7.) i. Let a_A and a_B be accumulation func.

$$a_A(20) = 2.5\, a_B(20)$$

$$e^{r\% t}$$

$$= e^{\int_0^{20} \delta(t)\,dt}$$

$$\int_0^{20} \delta(t)\,dt = \frac{1}{1000}\left[\frac{1}{2}t^2 + 10t\right]_0^{10} + \ln|100+t^2|\Big]_{10}^{20}$$

$$= \frac{3}{20} + \ln\left|\frac{5}{2}\right|$$

$$\Rightarrow e^{\frac{3}{20} + \ln\left|\frac{3}{2}\right|}$$

$$\frac{5}{2}e^{3/20} = 2.5 \times e^{\frac{R \cdot 20}{100}}$$

$$\boxed{R = \frac{3}{4}}$$

ii. 1 dollar invested in A at $t = 5 \Rightarrow$ 2 dollars @ $t = T$

$$\frac{a_A(T)}{a_A(5)} = 2 = \frac{e^{\int_0^T \delta(t)\,dt}}{e^{\int_0^5 \delta(t)\,dt}} = e^{\int_0^T \delta(t)\,dt - \int_0^5 \delta(t)\,dt}$$

$$2 = e^{\int_5^T \delta(t)\,dt}$$

$$\ln 2 = \int_5^T \delta(t)\,dt$$

check @ 10

$$\int_5^{10} \frac{t+10}{1000}\,dt = \frac{1}{1000}\left[\frac{1}{2}t^2 + 10t\right]_5^{10} = \frac{87.5}{1000} < \ln|2|$$

$$\ln 2 = \int_5^{10} \delta(t)\,dt + \int_{10}^{T} \delta(t)\,dt$$

$$= \frac{87.5}{1000} + \left[\ln(100+T^2)\right]_0^T$$

$$=$$

$$\boxed{T = \sqrt{400e^{-\frac{87.5}{1000}} - 100}}$$

Reference Ans:

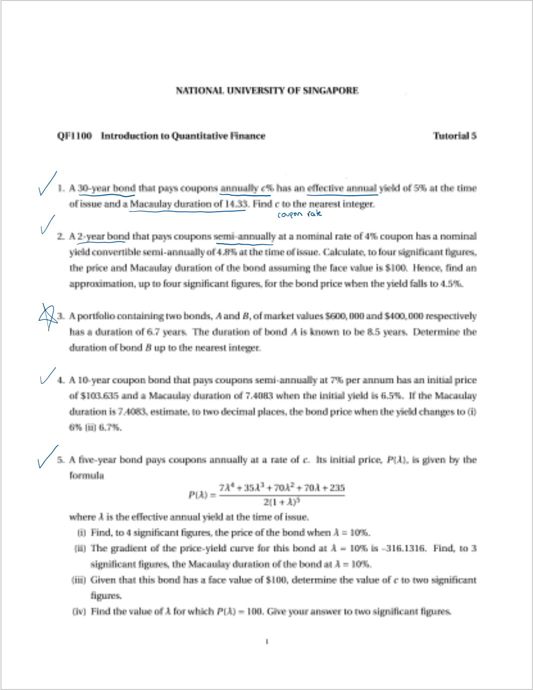
1. \$18,485
2. $A = \left(\frac{P}{10}\right) 1.1^5 - 1.1^5 + 1$
4. (1) $R = \frac{\ln 2}{2}$
   (2) $b = 7a$
   (3) $7a/2$

5. (i) \$3,303.75
   (ii) \$17,804.88 and \$2,242
6. Make regular payment of \$600 for 27 years. The last payment is \$376.63.
7. (i) $R = 0.75$
   (ii) $T = \sqrt{400 \exp(-7/80) - 100}$
8. $R = 20$

QF1100-Tutorial5

Thursday, October 5, 2023 12:02 AM

QF1100-Tutorial5



NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 5**

✓ 1. A 30-year bond that pays coupons annually $c\%$ has an effective annual yield of 5% at the time of issue and a Macaulay duration of 14.33. Find c to the nearest integer.

coupon rate

✓ 2. A 2-year bond that pays coupons semi-annually at a nominal rate of 4% coupon has a nominal yield convertible semi-annually of 4.8% at the time of issue. Calculate, to four significant figures, the price and Macaulay duration of the bond assuming the face value is \$100. Hence, find an approximation, up to four significant figures, for the bond price when the yield falls to 4.5%.

3. A portfolio containing two bonds, A and B, of market values \$600,000 and \$400,000 respectively has a duration of 6.7 years. The duration of bond A is known to be 8.5 years. Determine the duration of bond B up to the nearest integer.

✓ 4. A 10-year coupon bond that pays coupons semi-annually at 7% per annum has an initial price of \$103.635 and a Macaulay duration of 7.4083 when the initial yield is 6.5%. If the Macaulay duration is 7.4083, estimate, to two decimal places, the bond price when the yield changes to (i) 6% (ii) 6.7%.

✓ 5. A five-year bond pays coupons annually at a rate of c. Its initial price, $P(\lambda)$, is given by the formula

$$P(\lambda) = \frac{7\lambda^4 + 35\lambda^3 + 70\lambda^2 + 70\lambda + 235}{2(1+\lambda)^5}$$

where λ is the effective annual yield at the time of issue.

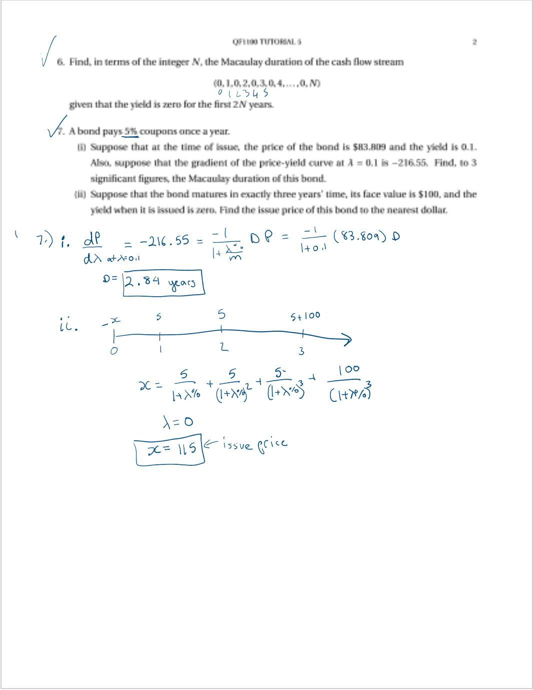
(i) Find, to 4 significant figures, the price of the bond when $\lambda = 10\%$.

(ii) The gradient of the price-yield curve for this bond at $\lambda = 10\%$ is −316.1316. Find, to 3 significant figures, the Macaulay duration of the bond at $\lambda = 10\%$.

(iii) Given that this bond has a face value of \$100, determine the value of c to two significant figures.

(iv) Find the value of λ for which $P(\lambda) = 100$. Give your answer to two significant figures.

1



QF1100 TUTORIAL 5 2

✓ 6. Find, in terms of the integer N, the Macaulay duration of the cash flow stream

$$(0, 1, 0, 2, 0, 3, 0, 4, \ldots, 0, N)$$

0 1 2 3 4 5

given that the yield is zero for the first $2N$ years.

✓ 7. A bond pays 5% coupons once a year.

(i) Suppose that at the time of issue, the price of the bond is \$83.809 and the yield is 0.1. Also, suppose that the gradient of the price-yield curve at $\lambda = 0.1$ is −216.55. Find, to 3 significant figures, the Macaulay duration of this bond.

(ii) Suppose that the bond matures in exactly three years' time, its face value is \$100, and the yield when it is issued is zero. Find the issue price of this bond to the nearest dollar.

7.) i. $\left.\frac{dP}{d\lambda}\right|_{at\ \lambda=0.1} = -216.55 = \frac{-1}{1+\frac{\lambda\%}{m}} DP = \frac{-1}{1+0.1}(83.809)D$

$D = \boxed{2.84 \text{ years}}$

ii.

$-x$ 5 5 5+100

0 1 2 3

$x = \frac{5}{1+\lambda\%} + \frac{5}{(1+\lambda\%)^2} + \frac{5}{(1+\lambda\%)^3} + \frac{100}{(1+\lambda\%)^3}$

$\lambda = 0$

$\boxed{x = 115}$ ← issue price

1.) $14.33 = D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%} - \frac{1+\frac{\lambda\%}{m}+n\left(\frac{c\%}{m}-\frac{\lambda\%}{m}\right)}{c\%\left[\left(1+\frac{\lambda\%}{m}\right)^n-1\right]+\lambda\%}$

$= \frac{1+5\%}{5\%} - \frac{1+5\%+30(c\%-5\%)}{c\%\left[(1.05)^{30}-1\right]+0.05}$

$\frac{1.05+30c\%-1.5}{c\%\left[1.05^{30}-1\right]+0.05} = 6.67$

$1.05+30c\%-1.5 = 6.67c\%\left[1.05^{30}-1\right]+0.3335$

$c\%\left[30-6.67(1.05^{30}-1)\right] = 0.7835$

$c = 0.0999 \approx 0.1$

10%

2.) $D = \frac{1+\frac{4.8\%}{2}}{4.8\%} - \frac{1+\frac{4.8\%}{2}+4\left(\frac{4\%}{2}-\frac{4.8\%}{2}\right)}{4\%\left[\left(1+\frac{4.8\%}{2}\right)^4-1\right]+4.8\%} = 1.941$ ← Macauly duration

$P(0) = \frac{2}{1+\frac{0.048}{2}} + \frac{2}{\left(1+\frac{0.048}{2}\right)^2} + \frac{2}{\left(1+\frac{0.048}{2}\right)^3} + \frac{2}{\left(1+\frac{0.048}{2}\right)^4} + \frac{100}{\left(1+\frac{0.048}{2}\right)^4}$

$= 98.49$ ← price of bond

$\Delta P \approx -(D_M P)\Delta\lambda = 0.56$

$D_M = \frac{1}{1+\frac{\lambda}{m}} D = \frac{1}{1+\frac{0.048}{2}} \cdot 1.941 = 1.896$

new price = 99.05

3.) α is weigt of A $PV_A = 600k$ $PV_B = 400k$

4.) i. $\Delta P \approx -(D_M P)\Delta\lambda = 3.718$

$D_M = \frac{1}{1+\frac{\lambda}{m}} D = \frac{1}{1+\frac{6.5\%}{2}} \cdot 7.4083$

new price = 107.35

ii. $\Delta P \approx 102.15$

5.) i. $P(10\%) = \boxed{75.36}$

ii. $\frac{dP}{d\lambda} = -316.1316 = \frac{-1}{1+\frac{\lambda\%}{m}} DP = \frac{-1}{1+0.1} D \cdot (75.36)$

$D = \boxed{4.61 \text{ years}}$

iii. $75.36 = \frac{c\% \cdot 100}{1+0.1} + \frac{c\% \cdot 100}{1.1^2} + \frac{c\% \cdot 100}{1.1^3} + \frac{c\% \cdot 100}{1.1^4} + \frac{c\% \cdot 100}{1.1^5} + \frac{106}{1.1^5}$

$\frac{100c\%}{1.1}\left(\frac{1-\left(\frac{1}{1.1}\right)^5}{1-\frac{1}{1.1}}\right) + \frac{100}{1.1^5}$

$\boxed{c = 3.5\%}$

iv. if $P = F = 100$

then $\lambda = c = \boxed{3.5\%}$

6.) $\frac{1\cdot1+2\cdot3+3\cdot5+4\cdot7+\cdots+N(2N-1)}{(1+2+\cdots N)}$ $\sum_{i=1}^{N} i(2i-1)$

$1+2+3+4+5 = 15$

$N\left(\frac{N}{2}\right)$

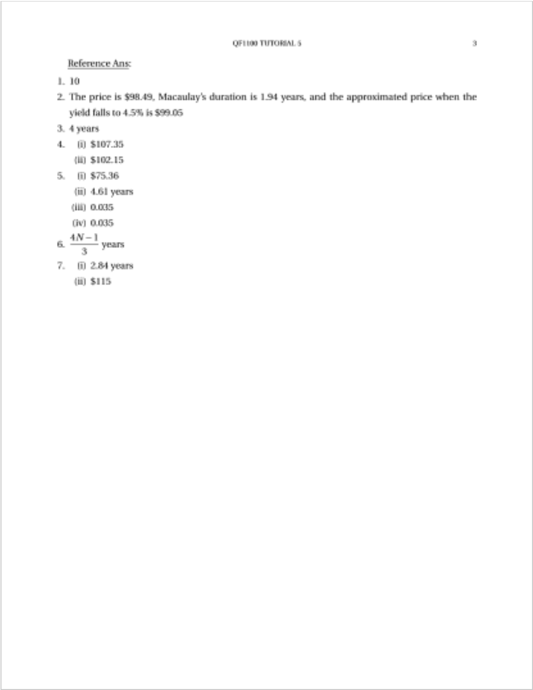
$2\sum_{i=1}^{N} i^2 - \sum_{i=1}^{N} i$

N

$\frac{2}{\cancel{N}(\cancel{N+1})}\left(\frac{\cancel{N}(\cancel{N+1})(2N+1)}{3} - \frac{\cancel{N}(\cancel{N+1})}{2}\right) = 2\cdot\left(\frac{2(2N+1)-3}{\cancel{6}\,3}\right)$

$= \frac{4N+2-3}{3}$

$= \boxed{\frac{4N-1}{3} \text{ years}}$



Reference Ans:

1. 10
2. The price is \$98.49, Macaulay's duration is 1.94 years, and the approximated price when the yield falls to 4.5% is \$99.05
3. 4 years
4. (i) \$107.35
   (ii) \$102.15
5. (i) \$75.36
   (ii) 4.61 years
   (iii) 0.035
   (iv) 0.035
6. $\frac{4N-1}{3}$ years
7. (i) 2.84 years
   (ii) \$115

QF1100-Tutorial4

Wednesday, September 20, 2023 10:10 AM

QF1100-Tutorial4

NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 4**

Result A: For any $y \in (0,1)$,

$$\sum_{i=1}^{\infty} i y^i = \frac{y}{(1-y)^2}$$

(You can try proving the above result by differentiating both sides of the identity

$$\sum_{i=1}^{\infty} y^i = \frac{y}{1-y}$$

w.r.t. y.)

Result B: For any $y \in (0,1)$,

$$\sum_{i=1}^{n} i y^i = \frac{y(1-y^n) - n y^{n+1}(1-y)}{(1-y)^2}$$

(You can try proving the above result by differentiating both sides of the identity

$$\sum_{i=1}^{n} y^i = \frac{1-y^{n+1}}{1-y}$$

w.r.t. y.)

1. A 20-year annuity makes payments at the end of every year. The first payment of \$$x$ is made at the end of the first year, and subsequent payments increase by 4% each year. If the present value of this annuity is \$7,229.50, find the value of x to the nearest integer. Take the effective annual rate of interest to be 7%.

2. A 30-year loan of \$10,000 is to be repaid by annual repayments. The interest on the loan is guaranteed at a rate of 8% for the first 5 years of the loan.
   (i) Calculate the annual repayment \$$A$, assuming that the interest on the loan is 8% throughout the 30-year term. Give your answer in 2 decimal places.
   (ii) Calculate the loan balance immediately after the fifth payment has been made. Give your answer in 2 decimal places.

1 i

$$7229.5 = PV = 7229.5 = \frac{x}{1.07} + \frac{x(1.04)}{(1.07^2)} + \cdots + \frac{x(1.04)^{19}}{1.07^{20}}$$

$$\frac{x}{1.07} + \frac{x}{1.07}\left(\frac{(1.04)}{1.07} + \cdots + \left(\frac{1.04}{1.07}\right)^{19} \right)$$

2.) i

$$10000 = \frac{A}{1.08} + \frac{A}{1.08^2} + \cdots + \frac{A}{1.08^{30}}$$

$$= A\left[\frac{1-\left(\frac{1}{1.08}\right)^{31}}{1-\left(\frac{1}{1.08}\right)}\right]$$

   (iii) Assume that after the fifth payment is made, the interest rate is increased to 9%. Calculate the new annual repayment if the outstanding loan is to be fully paid at the end of year 30. Give your answer in 2 decimal places.
   (iv) If instead, the same annual repayment (i.e. your answer to (i)) is made throughout the term of the loan, how many more years does it take to fully repay the loan? Here, the last payment is made one year later than the regular instalments of \$$A$. Give your answer to the nearest integer.

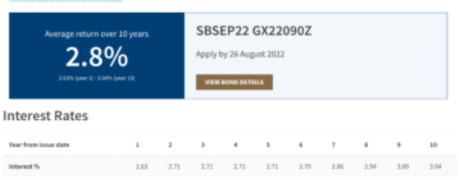
3. At a continuously compounded rate of $\ln r$ $(r > 1)$, a loan of \$$P$ can be fully repaid by either of the following methods:
   Method 1: An annual payment of \$$4A$ at $t = 1,2,\ldots,10$ and an annual payment of \$$2A$ at $t = 21,22,\ldots,30$.
   Method 2: A payment of \$$B$ at $t = 11,12,\ldots,20$.
   Show that
   $$B \geq 4\sqrt{2}A.$$

4. A perpetual annuity pays \$$t$ at $t = 1,2,\ldots,20$. Starting from $t = 21$, the annuity pays \$40 constantly every year. It is given that money accumulates zero interest (i.e. interest rate is 0%) for the first 20 years ($t = 0$ to $t = 20$). From $t > 20$, the interest rate is $\ln(1.2)$ compounded **continuously**. Find the present value of this annuity.

5. A 10-year bond with face value \$100 pays semi-annual coupons at a nominal rate of 8.4%. Given that the bond yields 10% convertible semi-annually, find the price of the bond when it is issued.

6. Singapore Government Securities (SGS) bonds are tradable debt securities that pay a semi-annual coupon, are redeemable at face value, and are priced at par when they are issued.

F=price



This Month's Bond

Average return over 10 years

2.8%

2.63% (year 1) - 3.04% (year 10)

SBSEP22 GX22090Z

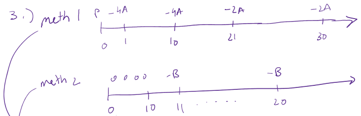
Apply by 26 August 2022

VIEW BOND DETAILS

Interest Rates

Year from issue date	1	2	3	4	5	6	7	8	9	10
Interest %	2.63	2.71	2.71	2.71	2.71	2.79	2.86	2.94	3.00	3.04

3.) meth 1



$$P = \frac{4A}{r} + \frac{4A}{r^2} + \cdots + \frac{4A}{r^{10}} + \frac{2A}{r^{21}} + \frac{2A}{r^{22}} + \cdots + \frac{2A}{r^{30}}$$

$$= \frac{4A}{r}\left[\frac{1-\left(\frac{1}{r}\right)^{10}}{1-\left(\frac{1}{r}\right)}\right] + \frac{2A}{r^{21}}\left[\frac{1-\frac{1}{r^{10}}}{1-\frac{1}{r}}\right]$$

$$= \left[\frac{1-\left(\frac{1}{r}\right)^{10}}{1-\left(\frac{1}{r}\right)}\right]\left[\frac{2A}{r}\right]\left[2+\frac{1}{r^{20}}\right]$$

$$P = \frac{B}{r^{11}} + \frac{B}{r^{12}} + \cdots + \frac{B}{r^{20}} = \frac{B}{r^{11}}\left[\frac{1-\frac{1}{r^{10}}}{1-\frac{1}{r}}\right]$$

show $B \geq 4\sqrt{2}A$

$= 2A(2\sqrt{2})$

$$\frac{B}{r^{11}}\left[\cancel{\frac{1-\frac{1}{r^{10}}}{1-\frac{1}{r}}}\right] = \left[\cancel{\frac{1-\frac{1}{r^{10}}}{1-\frac{1}{r}}}\right]\left[\frac{2A}{r}\right]\left(2+\frac{1}{r^{10}}\right)$$

$$B = 2A\left(2r^{10}+\frac{1}{r^{10}}\right)$$

Show minimum over all possible r, of $2r^{10}+\frac{1}{r^{10}}$ is at least $2\sqrt{2}$

$f(x) = 2x^{10} + \frac{1}{x^{10}}$ $f'(x) = 20x^9 - \frac{10}{x^{11}}$ implies min

$f''(x) = 180x^8 + \frac{110}{x^{12}}$

$0 < f'(x) = 20x^9 - \frac{10}{x^{11}}$

$20x^9 > \frac{10}{x^{11}}$

$x^{20} > \frac{1}{2}$

$x^{10} > \frac{1}{\sqrt{2}}$

$$f\left(\left(\frac{1}{\sqrt{2}}\right)^{1/10}\right) = 2\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{1/\sqrt{2}}$$

$= \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2}$

4.) 1 2 3 4

5 · 2

21th

$$PV = 1+2+\cdots+20 + \frac{40}{1.2} + \frac{40}{1.2^2} + \cdots$$

$$(21)(10) + 40\left(\frac{1/1.2}{1-1/1.2}\right) = \boxed{410}$$

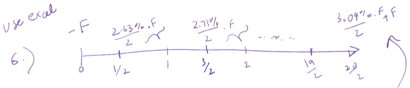
5.) coupon amount = 8.4

$$P(0) = \frac{8.4}{\left(1+\frac{0.1}{2}\right)} + \frac{8.4}{(1+0.05)^2} + \cdots + \frac{8.4}{1.05^{20}} + \frac{100}{1.05^{20}}$$

$$8.4\left(\frac{1-\left(\frac{1}{1.05}\right)^{21}}{1-\frac{1}{1.05}}\right) + \frac{100}{1.05^{20}}$$

use excel

6.)



$$\cancel{F} = \frac{\frac{2.63\% \cdot \cancel{F}}{2}}{\left(1+\frac{\lambda\%}{2}\right)} + \frac{\frac{2.63\% \cdot \cancel{F}}{2}}{\left(1+\frac{\lambda\%}{2}\right)^2} + \cdots + \frac{\frac{3.04\%}{2}+1}{\left(1+\frac{\lambda}{2}\right)^{20}}$$

find $\lambda = 2.8\%$

What is written as "Interest Rates" above are the nominal coupon rates (which in this case vary from year to year). What is the nominal yield of the bond (convertible semi-annually) when it is issued?

7. (a) A perpetual annuity pays $(1 + k) at time $t = k$, $k \in \mathbb{Z}^+$, with the first payment made at $t = 1$. If the interest rate is r compounded continuously, prove that the present value (at t = 0) of this annuity is

$$\frac{2e^r - 1}{(e^r - 1)^2}.$$

   (b) Another perpetual annuity pays $3k at time $t = 2k$, $k \in \mathbb{Z}^+$, with the first payment made at $t = 4$. If the effective annual rate is 10%, calculate the value of the present value (at $t = 0$) of this annuity up to two decimal places.

8. An annuity pays $(1 + 2k) at time $t = k$, $k \in \mathbb{Z}^+$, with the first payment made at $t = 1$, and the last payment made at $t = 30$. If the interest rate is 3% compounded continuously, calculate the present value (at $t = 0$) of this annuity up to two decimal places.

$t=1$

$$PV = \frac{3}{e^{(0.03)}} + \frac{5}{e^{(0.03)\cdot 2}} + \frac{7}{e^{0.03\cdot 3}} + \cdots + \frac{61}{e^{(0.03)\cdot 30}}$$

$$PV = \sum_{k=1}^{30} \frac{1+2k}{e^{3\% k}} = \sum_{k=1}^{30} \frac{1}{e^{0.03k}} + 2\sum_{k=1}^{30} \frac{k}{e^{0.03k}}$$

* use result B?

$$= \frac{1}{e^{0.03}}\left[\frac{1 - \frac{1}{e^{0.03\cdot 30}}}{1 - \frac{1}{e^{0.03}}}\right] + 2\left[\frac{\frac{1}{e^{3\%}}\left(1 - \frac{1}{e^{3\%\cdot 30}}\right) - 30\cdot\frac{1}{e^{3\%\cdot 31}}\left(1 - \frac{1}{e^{3\%}}\right)}{\left(1 - \frac{1}{e^{3\%}}\right)^2}\right]$$

$$= \underline{\text{ans}}$$

7.) a.

$$PV = \frac{2}{e^r} + \frac{3}{e^{2r}} + \frac{4}{e^{3r}} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{k+1}{e^{kr}} = \sum_{k=1}^{\infty} \frac{k}{e^{kr}} + \sum_{k=1}^{\infty} \frac{1}{e^{kr}}$$

$$= \frac{e^{-r}}{1-e^{-r}} + \frac{1}{e^r}\left(\frac{1}{1-e^{-r}}\right)$$

2.5

$$= \frac{e^{-r} + e^{-r}(1-e^{-r})}{(1-e^{-r})} = \frac{e^r + (e^r - 1)}{(e^r-1)^2} = \underline{\underline{\frac{2e^r - 1}{(e^r-1)^2}}}$$

b.

$$PV = \frac{6}{1.1^4} + \frac{9}{1.1^6} + \frac{12}{1.1^8} + \cdots \frac{3k}{1.1^{2k}}$$

$$= \sum_{1}^{\infty} \frac{3k}{(1.1)^{2k}} = 3\left(\sum_{k=1}^{\infty} \frac{k}{1.21^k} - \frac{1}{1.21}\right)$$

$$= 3\left(\frac{\frac{1}{1.21}}{\left(1 - \frac{1}{1.21}\right)^2} - \frac{1}{1.21}\right) = \underline{\underline{79.83}}$$

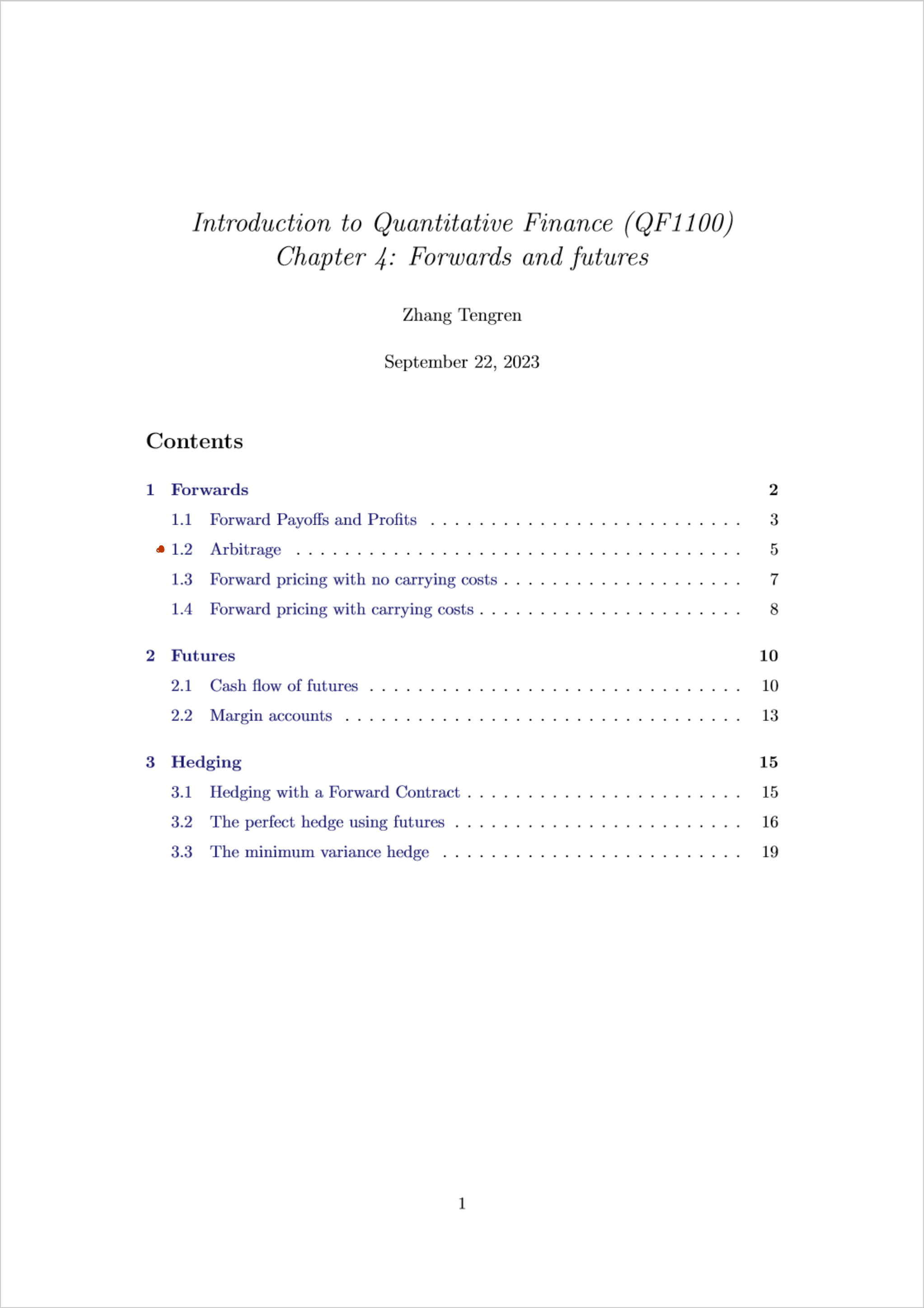
Reference Ans:

1. 500
2. (i) $888.27
   (ii) $9,482.13
   (iii) $965.34
   (iv) 13 more years, and the last instalment is $512.
4. $410
5. $90.03
6. 2.8%
7. (b) $79.83
8. 537.12

ch4

Friday, September 22, 2023 6:46 PM

ch4



Introduction to Quantitative Finance (QF1100)
Chapter 4: Forwards and futures

Zhang Tengren

September 22, 2023

Contents

1 Forwards **2**

1.1 Forward Payoffs and Profits 3

1.2 Arbitrage 5

1.3 Forward pricing with no carrying costs 7

1.4 Forward pricing with carrying costs 8

2 Futures **10**

2.1 Cash flow of futures 10

2.2 Margin accounts 13

3 Hedging **15**

3.1 Hedging with a Forward Contract 15

3.2 The perfect hedge using futures 16

3.3 The minimum variance hedge 19

1 Forwards

A *forward contract* (also sometimes called a forward) is an obligation to buy or sell an (underlying) asset for a certain price F_0 agreed at time $t = 0$ on an agreed future date $t = T \geq 0$.

Example 1.1. Suppose that you want to buy 5,000 bushels of wheat in 3 months. One way to do so is via a forward contract to purchase 5,000 bushels of wheat at 8 USD per ton in exactly 3 months. If the price of wheat is 9 USD per bushel in 3 months, then the contract would save you 1 USD per bushel. On the other hand, if the price of wheat is 7 USD per bushel in 3 months, then the contract would lose you 1 USD per bushel.

Forward contracts are an example of *derivatives*, which are financial instruments whose value are derived from the value of some underlying assets such as stocks and bonds. The forward contract in the example is a derivative because its value is derived from the price of wheat.

As part of the forward contract, the quantity of the asset to be traded and the price at which it is to be traded is specified. We will assume that there are no costs to enter into a forward contract.

taking the
↙ risk

Notice that for any forward contract, there is a buyer (called the *long side* or the holder of a *long position*) and there is a seller (called the *short side* or the holder of a *short* ← betting prices go down *position*). Both parties are obligated to carry out the transaction.

We fix the following notation and terminology for forward contracts:

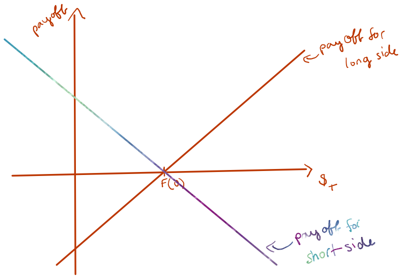
- T is the *expiration/delivery/maturity date* of the forward contract. This is when the underlaying asset is to be delivered.
- F_0 is the *forward price*. This is the price agreed on between the buyer and the seller at time t for the underlying asset for delivery at the delivery date. This price is only paid on the delivery date.
- The open market for immediate delivery of the *underlying asset* is called the *spot market*. The price of the underlying asset, also known as the *spot price*, at time t is denoted $S(t)$ or S_t. (This terminology is used for any derivative.)

1.1 Forward Payoffs and Profits

For a forward contract signed at $t = 0$, the payoff P (or equivalently, the profit) of

- a long forward contract is $S(T) - F(0)$. spot price − forward hoping Spot > future (0)
- a short forward contract is $F(0) - S(T)$. buy asset on spot mkt at price T then sell at F(0) hoping spot @T < F(0)

The payoffs can be illustrated in a payoff diagram, which is a plot of the payoff against the spot price for the contract.



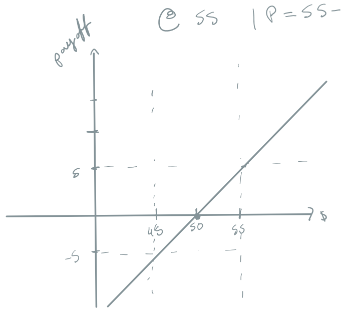
Note that offsetting positions in a forward contract are equivalent to a *zero-sum* game. Any gains in the long position equals the losses of the short position.

Exercise 1.2. Suppose you take the long side of a forward contract on an asset today (at time $t = 0$) for a delivery price of $50. Suppose the contract expires in 6 months ($T = 0.5$ years). Find the payoff if the asset price at maturity is one of the following prices: $45, $50, $55. Illustrate graphically the payoff diagram.

payoff @ 45 | $P = 45 - 50 = -5$

@ 50 | $P = 50 - 50 = 0$

@ 55 | $P = 55 - 50 = 5$



1.2 Arbitrage

Suppose $V(0)$ is the value of a portfolio today and $V(t)$ is the value at some future time t, then an *arbitrage opportunity* is a situation where one can earn a net positive return in future without requiring a net initial cash outlay. For example, an arbitrage opportunity arises if either of the following conditions hold:

1. $V(0) \leq 0$, shorting a lot
2. $\mathbb{P}(V(t) < 0) = 0$ for some $t > 0$
3. If $V(0) = 0$, then $\mathbb{P}(V(t) = 0) < 1$.

Example 1.3. Bank A and Bank B offer to loan money or accept deposits at the same rate of interests, say 10% and 12%. There is an *arbitrage opportunity.*

Borrow \$100 from bank A
" " " " B

initial outlay = 0

Profit = \$2

Example 1.4. Two portfolio strategies have the same initial cost and one has always a better final payoff than the other one with probability one.

$P_1 : r_1$ $r_2 > r_1$

$P_2 : r_2$

long asset w. higher return

sell short asset w. less return

π, short P_1, buy P_2

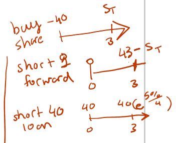
$\overline{W}$ is cost of P_1 and P_2

Initial outlay $= \overline{W} + (-W) = 0$

Return $:= -\overline{W}(1+r_1) + \overline{W}(1+r_2) = W(r_2 - r_1) > 0$

Exercise 1.5. Consider a forward contract to purchase a stock in 3 months. Assume the current stock price is \$40 and the 3-month risk-free interest rate is 5% per annum compound continuously.

1. Suppose the forward price is at $F(0) = \$43$. Prove the following portfolio is an arbitrage opportunity. At $t = 0$,
   - Long 1 share (i.e. hold the share).
   - Short 1 forward contract (to sell one share in 3-months).
   - Short \$40 (i.e. borrow) for 3-months.



initial outlay $= -40 + 40 = 0$

~~initially~~

$$\text{Profit} = \cancel{S_T} + (43 - \cancel{S_T}) + (\quad +40e^{5\%/4})$$

$$= 43 - 40e^{5\%/4} > 0$$

so arbitrage opportunity

2. Suppose the forward price $F(0)$ is unknown instead. Find the range of $F(0)$ such that the above portfolio is an arbitrage opportunity.

initial $= -40 + 40 = 0$

$$\text{Future} = \cancel{S_T} + x - \cancel{S_T} + 40(e^{5\%/4}) > 0$$

$$\left[F(0) = x > 40(e^{5\%/4}) \right.$$

1.3 Forward pricing with no carrying costs

When calculating prices of forwards, we will assume the *no-arbitrage principle*, i.e. arbitrage opportunities do not exist. This is reasonable in the situation where there is perfect information and perfect liquidity in the spot and future markets.

Consider a forward contract to purchase a non-dividend-paying stock in T years. Assume the current stock price is $\$S_0$ at $t = 0$ and the nominal risk-free interest rate is $r\%$ per annum compounded continuously. Suppose the forward price is at $\$F_0$.

Example 1.6. 1. If $F_0 > S_0 e^{r\%T}$, then construct an arbitrage opportunity.

2. If $F_0 < S_0 e^{r\%T}$, then construct an arbitrage opportunity.

1. π: buy 1 share of stock $\quad -S(0) \longrightarrow S(t)$

short 1 forward $\quad 0 \longrightarrow -S(t)+F(0)$

borrow S_0 from bank $\quad S(0) \longrightarrow -S0e^{r\%T}$

$F(0) - S(0)e^{r\%T} > 0$

2. $\Pi_1 =$ short 1 share

buy 1 forward

deposit S_0 to bank $\Big]$ $\quad 0 \longrightarrow -S(T) - F_0 + S(T) + S(0)e^{r\%T} > 0$

$0 \qquad T$

Therefore, by the no-arbitrage principle, the price of a forward contract is

$$F_0 = S_0 e^{r\%T}. \tag{1.7}$$

1.4 Forward pricing with carrying costs

Often, even though there is no cost to get into either the long side or short side of a forward contract, there are sometimes costs (storage, maintainence, etc) for holding onto an asset. These are called *carrying costs.*

Example 1.8. Consider a forward contract to purchase an underlying asset in T years. Suppose there are carrying costs for the long position of the forward contract:

$$\{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$$

where $t_1 < t_2 < \ldots < t_n \leq T$. Assume the current underlying price is $\$S_0$ at $t = 0$ and the risk-free interest rate is $r\%$ per annum compounded continuously. Suppose the forward price is at $\$F_0$.

1. If $F_0 > S_0 e^{r\%T} + c_1 e^{r\%(T-t_1)} + c_2 e^{r\%(T-t_2)} + \ldots + c_n e^{r\%(T-t_n)}$, then construct an arbitrage opportunity.
2. If $F_0 < S_0 e^{r\%T} + c_1 e^{r\%(T-t_1)} + c_2 e^{r\%(T-t_2)} + \ldots + c_n e^{r\%(T-t_n)}$, then construct an arbitrage opportunity.

Therefore, by the no-arbitrage principle, the price of a forward contract is

$$F_0 = S_0 e^{r\%T} + \sum_{i=1}^{n} c_i e^{r\%(T-t_i)}. \tag{1.9}$$

Exercise 1.10. The current price of wheat is 9.4 USD per bushel. The carrying cost of wheat is 0.1 USD per bushel per month, to be paid at the beginning of the month. And the interest (nominal) rate is constant at 9% per annum. Determine the forward price of wheat to be delivered in 5 months.

2 Futures

Forward contracts are private agreements between a financial institution and one of its corporate clients or between two financial institutions. They are not usually liquid, and traders often have to maintain their positions until the contracted settlement date. In addition, since the contracts are private agreements, credit risk is a major concern. Futures contracts attempt to overcome liquidity and credit risk problems.

2.1 Cash flow of futures

A *futures contract* is a standardized legal contract to buy an asset at a predetermined price (usually denoted $F(s)$ if one enters into the futures contract at time s) for delivery at a specified delivery date in the future (usually denoted $T \geq s$). The predetermined price $F(s)$ is called the *futures price* at time s. Future contracts are also a derivatives, as their value depends on the (spot) price of the underlying asset.

Recall that for forward contracts, one enters into the long or short position of the forward contract at time 0, and can only exit that position at the delivery date. This is not the case for futures contract: one can enter into a futures contract at any time s prior to the delivery date. However, the price $F(s)$ at which the underlying asset is traded on the delivery date depends on s.

Also, unlike forward contracts, which are settled at the end of the delivery date, futures contracts are settled daily, and so one can also exit the long or short side of the future contract at the end of any day. This daily settlement is done according to the following the daily cash flows from futures contracts (for the long position).

Daily Cash Flows of a Futures Contract

Date	Futures Price	Spot Price	Cash Flow from Futures
0	$F(0)$	$S(0)$	-
1	$F(1)$	$S(1)$	$F(1) - F(0)$
2	$F(2)$	$S(2)$	$F(2) - F(1)$
	.	.	.
	.	.	.
	.	.	.
$T-1$	$F(T-1)$	$S(T-1)$	$F(T-1) - F(T-2)$
T	$F(T)$	$S(T)$	$F(T) - F(T-1)$

Note that:

- At the maturity date, T, the futures price for immediate delivery must equal the spot price, i.e. $F(T) = S(T)$, otherwise an arbitrage opportunity will exist. Indeed, if the futures price was higher than the spot price, an astute investor would buy the spot, sell the futures and deliver immediately to capture riskless arbitrage profits. Similarly, if the futures price were below the spot price, the astute investor would buy the futures, take immediate delivery and then sell the spot to earn riskless profits.

- The sum of the entries of the cash flow are

  $$[F(T)-F(T-1)]+[F(T-1)-F(T-2)]+\ldots+[F(1)-F(0)]=S(T)-F(0).$$

  This is exactly the profit from a forward contract with delivery date T and forward price $F(0)$.

- Assume that the nominal interest rate is r compounded continuously. Let $R=\exp(r/365)$. We have the following table of *accrued* profits at time $t=T$.

Accrued Profit on a Long Position in a Futures Contract

Day	Futures Price	Cash-Flow	Accrued Profit (to date T)
0	$F(0)$	-	-
1	$F(1)$	$F(1)-F(0)$	$[F(1)-F(0)]R^{T-1}$
2	$F(2)$	$F(2)-F(1)$	$[F(2)-F(1)]R^{T-2}$
3	$F(3)$	$F(3)-F(2)$	$[F(3)-F(2)]R^{T-3}$
.	.	.	.
.	.	.	.
$T-1$	$F(T-1)$	$F(T-1)-F(T-2)$	$[F(T-1)-F(T-2)]R$
T	$F(T)$	$F(T)-F(T-1)$	$[F(T)-F(T-1)]$

The *accrued* total profit is

$$\begin{aligned}&[F(T)-F(T-1)]+[F(T-1)-F(T-2)]R+\ldots+[F(1)-F(0)]R^{T-1}\\=&\sum_{i=1}^{T}[F(i)-F(i-1)]R^{T-i}.\end{aligned}$$

Thus, unlike forward contracts, the total profit on a futures position will depend on the sequence of price moves over the period. For example, if futures prices gradually increase and then decrease, the long position would first generate a sequence of early profits, which can earn interest over a long period, followed by a sequence of losses, which can be financed over shorter periods. Clearly, the long position is better off than if futures prices initially decreased and then returned to the same level.

Exercise 2.1. You took a long position on futures for 500,000 barrels of oil to be delivered in one week, i.e. when $t = 7$. Suppose that the futures price (per barrel of oil delivered on $t = 7$) over the next seven days are as followed:

t	0	1	2	3	4	5	6	7
F(t)	93	94	95	94	93	92	93	94

1. Find the spot price of oil (per barrel) at $t = 7$.
2. Suppose you maintain your long position until the delivery date. Calculate your accrued total profit.
3. Suppose that on $t = 3$, you exit your long position and entered a short position on futures for 500,000 barrels of oil to be delivered at $t = 7$. Calculate your accrued total profit.

2.2 Margin accounts

In practice, in order to enter into a long or short position on a futures contract, one needs to open a *margin account* with the broker. Margin accounts not only serve as accounts to collect or pay out daily profits they also guarantee that contract holders will not default on their obligation.

The broker would require an *initial margin amount*, which consists of a fraction of the total contract value. For instance, 50% of the total contract value by the Federal Reserve Board's Regulation T. If the value of a margin account should drop below a defined *maintenance margin* level (for instance, 25% of the total contract value by the Federal Reserve Board's Regulation T), a *margin call* is issued to the contract holder, demanding additional margin. If the additional margin is not provided, the futures position will be closed out.

The margin accounts are deposits with the broker, and so will typically accumulate at the risk free interest rate. However, for the purpose of clarifying concepts, we will assume that no interest are paid on margin accounts.

Example 2.2. Suppose that an investor takes a long position of ten contracts in corn ($50,000$ bushels) for December delivery at price of \$0.14 (per bushel). Suppose the broker requires margin of 50% of the total contract value with a maintenance margin of 25% of the total contract value.

1. Find the total contract value.
2. Find the initial margin.
3. Find the maintenance margin.
4. Suppose on the 2nd day the price of this contract drops to \$0.12. Find the value of the margin account on the 2nd day.

Exercise 2.3. Suppose that an investor takes a long position of ten contracts in corn (50,000 bushels) for December delivery at price of \$0.14 (per bushel). Suppose the broker requires margin of 50% of the total contract value with a maintenance margin of 25% of the total contract value.

1. Suppose on the 3rd day the price of this contract drops to \$0.09. Find the value of the margin account on the 3rd day. What will happen to Mr Saver's margin account?
2. Suppose on the 3rd day the price of this contract increases to \$0.13. Find the value of the margin account on the 3rd day. What will happen to Mr Saver's margin account?

3 Hedging

We will now discuss how one can use futures and forwards to manage risks via a process called *hedging*.

3.1 Hedging with a Forward Contract

Using forward contracts to do hedging is relatively straightforward: one just enters into an agreement to buy/sell the required commodity at an agreed price to remove the risks of price fluctuations in the spot market.

Example 3.1. A gold-mining firm plans to mine and sell 100,000 ounces of gold precisely one year from now. The firm can either

1. **remain unhedged**, i.e. sell the gold at the spot price of $\$S(1)$ per ounce in one year, or
2. **hedge using a forward contract**, i.e. lock in a price for gold in one year by entering into the short position of a forward contract for delivery of 100,000 ounces of gold in one year.

Suppose that the cost of mining one ounce of gold is \$380, and the forward price is \$420 per ounce. Calculate their profit of the firm when it remains unhedged and when it hedges.

Thus, if the gold mining firm chooses to hedge, then its profits are completely insulated from the gold prices. In particular, the hedging protects the firm from loss should gold prices decline. However, hedging also removes the firm's opportunity to make even greater profit if the gold prices do rise.

3.2 The perfect hedge using futures

One can also use futures contracts to hedge against risk. The simplest hedging strategy is the *perfect hedge*, where the risk associated with a future commitment to deliver or receive an asset is completely eliminated by taking an *equal and opposite* position in the futures market.

Example 3.2. A gold-mining firm plans to mine and sell 100, 000 ounces of gold precisely one year from now (assume that the year has 365 days). Suppose that there are gold futures, each for 20, 000 ounces of gold to be delivered one year from now.

1. What is the perfect hedge to eliminate risks from fluctuations in the price of gold?
2. Suppose that the effective annual interest rate is 10%, the cost of mining the gold is \$380, and the futures price is \$420. Calculate the profit if the firm remains unhedged, and if the firm hedges using the perfect hedge.

Exercise 3.3. On 9 July, a US firm wants to sell equipment to a German customer in 3 months. The price is specified as 500,000 Euro, which will be paid upon delivery. Suppose that in the market, there are Euro futures contracts, each for 125,000 Euro (with futures price in USD) to be delivered on 9 October.

1. What is the perfect hedge to eliminate the exchange rate risks from the equipment sale?
2. Suppose the effective annual interest rate is 4%, and the spot prices of 1 Euro is 1.0182 USD and 0.9735 USD on 9 July and 9 October respectively.
   (a) Calculate the decrease in the value of the sale in USD if the firm remains unhedged.
   (b) What is the decrease in the value of the sale in USD if the firm uses the perfect hedge.

A perfect hedge completely eliminates the risk. However, perfect hedges are rare. To establish a perfect hedge,

1. the trader has to match the delivery date of the asset to the delivery date of the futures,
2. the commodity to be hedged must exactly match the commodity underlying the futures contract,
3. the amount of the asset obligated must be an integral multiple of the contract size, and

If any of these features is missing then a perfect hedge is not possible. In such circumstances risk can still be reduced but not eliminated.

ch4 (1)

Monday, October 2, 2023 6:35 PM

ch4 (1)

Introduction to Quantitative Finance (QF1100)
Chapter 4: Forwards and futures

Zhang Tengren

September 30, 2023

Contents

1 Forwards **2**
- 1.1 Forward Payoffs and Profits 3
- 1.2 Arbitrage 5
- 1.3 Forward pricing with no carrying costs 7
- 1.4 Forward pricing with carrying costs 9

2 Futures **11**
- 2.1 Cash flow of futures 11
- 2.2 Margin accounts 14

3 Hedging **16**
- 3.1 Hedging with a Forward Contract 16
- 3.2 The perfect hedge using futures 17
- 3.3 The minimum variance hedge 20

1 Forwards

A *forward contract* (also sometimes called a forward) is an obligation to buy or sell an (underlying) asset for a certain price F_0 agreed at time $t = 0$ on an agreed future date $t = T \geq 0$.

Example 1.1. Suppose that you want to buy 5,000 bushels of wheat in 3 months. One way to do so is via a forward contract to purchase 5,000 bushels of wheat at 8 USD per ton in exactly 3 months. If the price of wheat is 9 USD per bushel in 3 months, then the contract would save you 1 USD per bushel. On the other hand, if the price of wheat is 7 USD per bushel in 3 months, then the contract would lose you 1 USD per bushel.

Forward contracts are an example of *derivatives*, which are financial instruments whose value are derived from the value of some underlying assets such as stocks and bonds. The forward contract in the example is a derivative because its value is derived from the price of wheat.

As part of the forward contract, the quantity of the asset to be traded and the price at which it is to be traded is specified. We will assume that there are no costs to enter into a forward contract.

Notice that for any forward contract, there is a buyer (called the *long side* or the holder of a *long position*) and there is a seller (called the *short side* or the holder of a *short position*). Both parties are obligated to carry out the transaction.

We fix the following notation and terminology for forward contracts:

- T is the *expiration/delivery/maturity date* of the forward contract. This is when the underlying asset is to be delivered.
- F_0 is the *forward price*. This is the price agreed on between the buyer and the seller at time t for the underlying asset for delivery at the delivery date. This price is only paid on the delivery date.
- The open market for immediate delivery of the *underlying asset* is called the *spot market*. The price of the underlying asset, also known as the *spot price*, at time t is denoted $S(t)$ or S_t. (This terminology is used for any derivative.)

1.1 Forward Payoffs and Profits

For a forward contract signed at $t = 0$, the payoff P (or equivalently, the profit) of

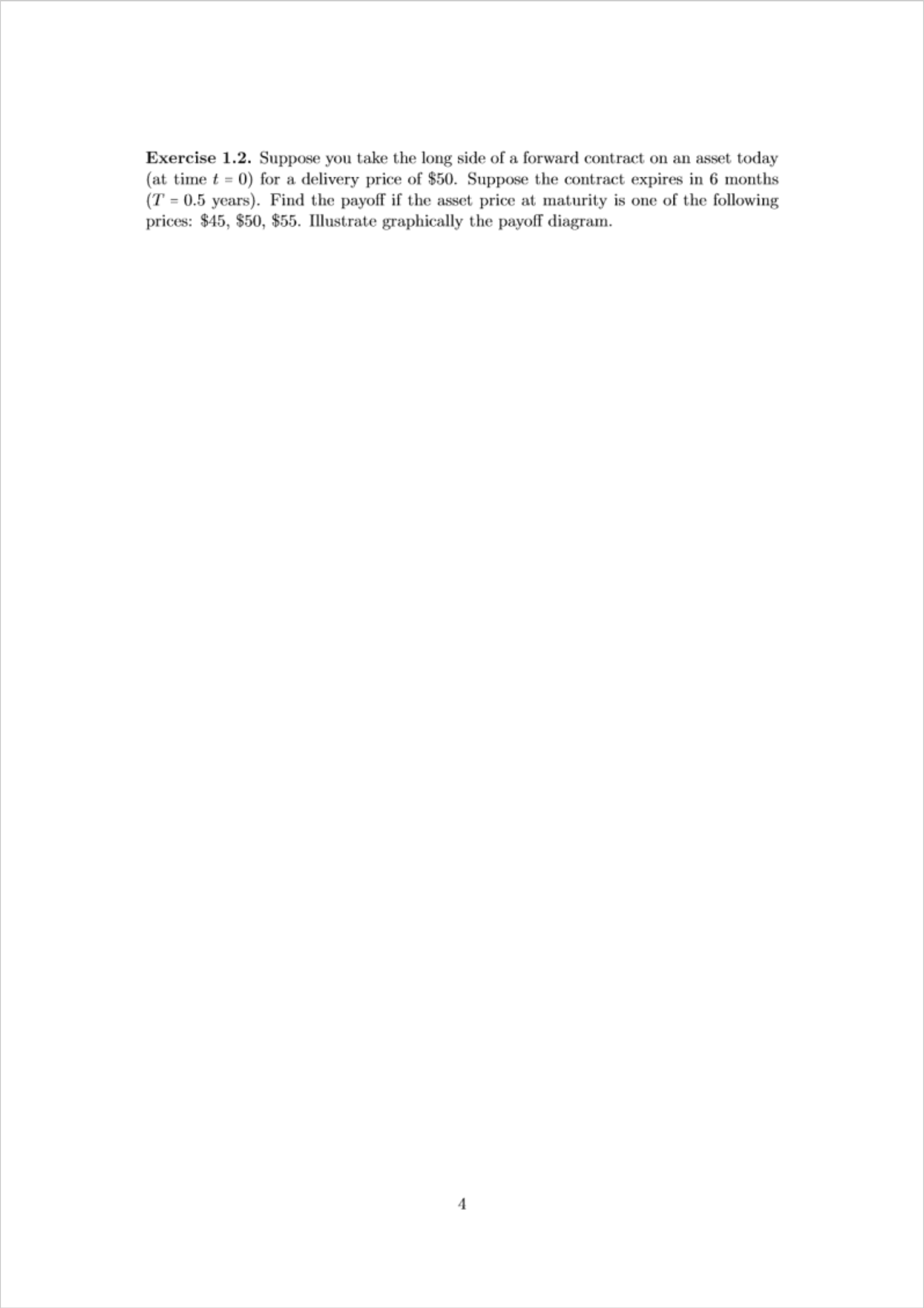
- a long forward contract is $S(T) - F_0$.
- a short forward contract is $F_0 - S(T)$.

The payoffs can be illustrated in a payoff diagram, which is a plot of the payoff against the spot price for the contract.

The cashflow of the forward contract is given by

Note that

- unlike cashflow of bonds, the cashflow of a forward contract is random.
- the offsetting positions in a forward contract are equivalent to a *zero-sum* game, i.e. any gains in the long position equals the losses of the short position.



Exercise 1.2. Suppose you take the long side of a forward contract on an asset today (at time $t = 0$) for a delivery price of \$50. Suppose the contract expires in 6 months ($T = 0.5$ years). Find the payoff if the asset price at maturity is one of the following prices: \$45, \$50, \$55. Illustrate graphically the payoff diagram.

1.2 Arbitrage

Suppose $V(0)$ is the value of a portfolio today and $V(t)$ is the value at some future time t, then an *arbitrage opportunity* is a situation where one can earn a net positive return in future without requiring a net initial cash outlay.

Example 1.3. Bank A and Bank B offers annual effective interest rates of 10% and 12% respectively (for both loans and deposits). There is an *arbitrage opportunity.*

Example 1.4. Suppose that two portfolio strategies have the same initial cost, and one has always a better final payoff than the other one with probability one.

Exercise 1.5. Consider a forward contract to purchase a stock in 3 months. Assume the current stock price is \$40 and the 3-month risk-free interest rate is 5% per annum compound continuously. If the forward price is at $F_0 = \$43$, prove the following portfolio is an arbitrage opportunity. At $t = 0$,

- Long 1 share (i.e. hold the share).
- Short 1 forward contract (to sell one share in 3-months).
- Short \$40 (i.e. borrow) for 3-months.

1.3 Forward pricing with no carrying costs

When calculating prices of forwards, we will assume the *no-arbitrage principle*, i.e. arbitrage opportunities do not exist. This is reasonable in the situation where there is perfect information and perfect liquidity in the spot and future markets.

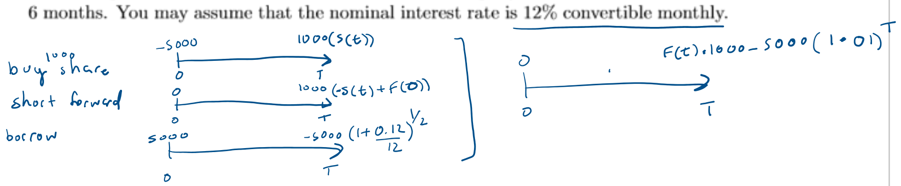
Example 1.6. Consider a forward contract to purchase a non-dividend-paying stock in T years. Assume the current stock price is $\$S_0$ at $t = 0$ and the nominal risk-free interest rate is $r\%$ per annum compounded continuously. Suppose the forward price is at $\$F_0$.

1. If $F_0 > S(0)e^{r\% T}$, then construct an arbitrage opportunity.
2. If $F_0 < S(0)e^{r\% T}$, then construct an arbitrage opportunity.

Therefore, by the no-arbitrage principle, the price of a forward contract is

$$F_0 = S(0)e^{r\% T}. \tag{1.7}$$

Exercise 1.8. Suppose that the current stock price of a company is \$5 per share. Calculate price of a forward contract for 1000 of the company's shares to be delivered in 6 months. You may assume that the nominal interest rate is 12% convertible monthly.



No arbitrage $\Rightarrow$ $1000\,F(0) = 5000(1.01)^6$

$$F(0) = 5(1.01)^6$$

1.4 Forward pricing with carrying costs

Often, even though there is no cost to get into either the long side or short side of a forward contract, there are sometimes costs or payouts for holding onto an asset. These *carrying costs* can be positive (e.g. storage costs, maintenence,...) or negative (e.g. dividends, revenue from rents,...)

Example 1.9. Consider a forward contract to purchase an underlying asset in T years. Suppose there are carrying costs for holding the asset:

$$\{(c_1, t_1), (c_2, t_2), \dots, (c_n, t_n)\}$$

where $t_1 < t_2 < \dots < t_n \le T$. Denote by $S(0)$ the current spot price of the asset, and F_0 the forward price. Assume that the nominal interest rate is $r\%$ compounded continuously.

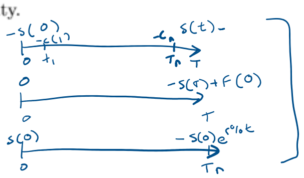
1. If $F_0 > S(0)e^{r\%T} + c_1e^{r\%(T-t_1)} + c_2e^{r\%(T-t_2)} + \dots + c_ne^{r\%(T-t_n)}$, construct an arbitrage opportunity.
2. If $F_0 < S(0)e^{r\%T} + c_1e^{r\%(T-t_1)} + c_2e^{r\%(T-t_2)} + \dots + c_ne^{r\%(T-t_n)}$, construct an arbitrage opportunity.

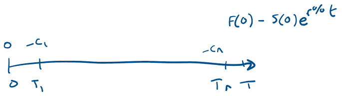
1.)

Buy 1 asset @ t=0

Short 1 forward

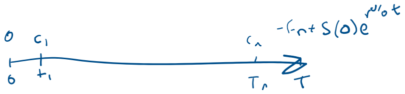
Borrow





$$\left\{ \text{Payout of portfolio} = F(0) - S(0)e^{r\%T} - c_1e^{r\%(T-t_1)} - \dots - c_ne^{r\%(T-T_n)} = 0 \right.$$

2.) short prev. portfolio



$$\left[\text{Payout} = -F(0) + S(0)e^{r\%t} + c_1e^{r\%(T-t_1)} + \dots + c_ne^{r\%(T-t_n)} \ge 0 \right.$$

Therefore, by the no-arbitrage principle, the price of a forward contract is

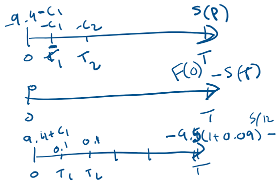
$$F_0 = S(0)e^{r\%T} + \sum_{i=1}^{n} c_ie^{r\%(T-t_i)}. \tag{1.10}$$

Exercise 1.11. The current price of wheat is 9.4 USD per bushel. The carrying cost of wheat is 0.1 USD per bushel per month, to be paid at the beginning of the month. The effective interest rate is constant at 9% per annum. Determine the forward price per bushel of wheat to be delivered in 5 months.

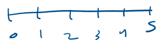
Buy 1 asset

Short 1 forward

borrow



$-9.5(1+0.09)^{5/12} - 0.1(1.09)^{4/12} - \dots$



$$\text{payout} = F(0) - 9.5(1.09)^{5/12} - c_1(1.09)^{4/12} - c_2(1.09)^{3/12} - c_3(1.09)^{2/12} - c_4(1.09)^{1/12} = 0$$

$$+9.5(1.09)^{5/12} + 0.1\sum_{i=1}^{4}(1.09)^{i/12} = F(0)$$

make sure payout = 0

$$F(0) = \$10.25$$

2 Futures

Forward contracts are private agreements between a financial institution and one of its corporate clients or between two financial institutions. They are not usually liquid, and traders often have to maintain their positions until the contracted settlement date. In addition, since the contracts are private agreements, credit risk is a major concern. Futures contracts attempt to overcome liquidity and credit risk problems.

2.1 Cash flow of futures

A *futures contract* is a standardized legal contract to buy an asset at a predetermined price (usually denoted $F(s)$ if one enters into the futures contract at time s) for delivery at a specified delivery date in the future (usually denoted $T \geq s$). The predetermined price $F(s)$ is called the *futures price* at time s. Futures contracts are also derivatives, as their value depends on the (spot) price of the underlying asset.

Don't have
to pay anything
to get into
futures contract

Recall that for forward contracts, one enters into the long or short position of the forward contract at time 0, and can only exit that position at the delivery date. This is not the case for futures contracts: one can enter into a futures contract at any time s prior to the delivery date. However, the price $F(s)$ at which the underlying asset is traded on the delivery date depends on s.

Also, unlike forward contracts, which are settled at the end of the delivery date, futures contracts are settled daily, and so one can also exit the long or short side of the future contract at the end of any day. This daily settlement is done according to the following the daily cash flows from futures contracts (for the long position).

people are
willing to buy &
sell at this
price at delivery
date

Daily Cash Flows of a Futures Contract

Date	Futures Price	Spot Price	Cash Flow from Futures
0	$F(0)$	$S(0)$	-
1	$F(1)$	$S(1)$	$F(1) - F(0)$
2	$F(2)$	$S(2)$	$F(2) - F(1)$
	.	.	.
	.	.	.
	.	.	.
$T-1$	$F(T-1)$	$S(T-1)$	$F(T-1) - F(T-2)$
T	$F(T)$	$S(T)$	$F(T) - F(T-1)$

broker pay
the diff.
if Future
price↑

Note that:

- At the maturity date, T, the futures price for immediate delivery must equal the spot price, i.e. $F(T) = S(T)$, otherwise an arbitrage opportunity will exist. Indeed, if the futures price was higher than the spot price, an astute investor would buy the spot, sell the futures and deliver immediately to capture riskless arbitrage profits. Similarly, if the futures price were below the spot price, the astute investor would buy the futures, take immediate delivery and then sell the spot to earn riskless profits.
- The sum of the entries of the cash flow are

  $$[F(T) - F(T-1)] + [F(T-1) - F(T-2)] + \ldots + [F(1) - F(0)] = S(T) - F(0).$$

  This is exactly the profit from the long side of a forward contract with delivery date T and forward price $F(0)$.
- Assume that the nominal interest rate is r compounded continuously. Let $R = \exp(r/365)$. We have the following table of *accrued* profits at time $t = T$.

Accrued Profit on a Long Position in a Futures Contract

You still need to discount cuz youre getting paid over time

Day	Futures Price	Cash-Flow	Accrued Profit (to date T)
0	$F(0)$	-	-
1	$F(1)$	$F(1) - F(0)$	$[F(1) - F(0)]R^{T-1}$
2	$F(2)$	$F(2) - F(1)$	$[F(2) - F(1)]R^{T-2}$
3	$F(3)$	$F(3) - F(2)$	$[F(3) - F(2)]R^{T-3}$
.	.	.	.
.	.	.	.
$T-1$	$F(T-1)$	$F(T-1) - F(T-2)$	$[F(T-1) - F(T-2)]R$
T	$F(T)$	$F(T) - F(T-1)$	$[F(T) - F(T-1)]$

initial payouts are worth more

The *accrued* total profit is

$$\begin{aligned} & [F(T) - F(T-1)] + [F(T-1) - F(T-2)]R + \ldots + [F(1) - F(0)]R^{T-1} \\ = & \sum_{i=1}^{T} [F(i) - F(i-1)]R^{T-i}. \end{aligned}$$

Thus, unlike forward contracts, the total profit on a futures position will depend on the sequence of price moves over the period. For example, if futures prices gradually increase and then decrease, the long position would first generate a sequence of early profits, which can earn interest over a long period, followed by a sequence of losses, which can be financed over shorter periods. Clearly, the long position is better off than if futures prices initially decreased and then returned to the same level.

Exercise 2.1. Suppose that the nominal interest rate compounded daily is 7.3% per year. You took a long position on futures for 500,000 barrels of oil to be delivered in one week, i.e. when $t = 7$. Suppose that the futures price (per barrel of oil delivered on $t = 7$) over the next seven days are as followed:

t	0	1	2	3	4	5	6	7
F(t)	93	94	95	94	93	92	93	94

1. Find the spot price of oil (per barrel) at $t = 7$. $S(7) = F(7) = 94$
2. Suppose you maintain your long position until the delivery date. Calculate your accrued total profit.
3. Suppose that at $t = 3$, right after the daily settlement, you exit your long position and entered a short position on futures for 500,000 barrels of oil to be delivered at $t = 7$. Calculate your accrued total profit.

1	2	3	4	5	6	7
1	1	-1	1	1	-1	-1

$$\left(1+\frac{0.073}{365}\right)^6 + \left(1+\frac{0.073}{365}\right)^5 - \left(1+\frac{0.073}{366}\right)^4$$

$$+\left(1+\frac{0.073}{365}\right)^3 + \left(1+\frac{0.072}{365}\right)^2 - \left(1+\frac{0.073}{365}\right)$$

$$-\left(1-\frac{0.073}{365}\right)^0 = 1.0022$$

$$\underline{501100.4601}$$

$$1\left(1+\frac{0.073}{365}\right)^6 + \left(1+\frac{0.073}{365}\right)^{\overset{5}{\cancel{1825}}}$$

$$-1\left(1+\frac{0.073}{365}\right)^{\overset{\cancel{1460}}{4}} - \left(1+\frac{0.023}{365}\right)^{\overset{3}{\cancel{1095}}}$$

$$-1\left(1+\frac{0.073}{365}\right)^{\overset{\cancel{730}}{2}} + \left(1+\frac{0.073}{365}\right)^{\overset{\cancel{360}}{1}}$$

$$+ 1 = 1.0006$$

$$1.0006 \cdot 500000 =$$

2.2 Margin accounts

In practice, in order to enter into a long or short position on a futures contract, one needs to open a *margin account* with the broker. Margin accounts not only serve as accounts to collect or pay out daily profits they also guarantee that contract holders will not default on their obligation.

The broker would require an *initial margin amount*, which consists of a fraction of the total contract value. For instance, 50% of the total contract value by the Federal Reserve Board's Regulation T. If the value of a margin account should drop below a defined *maintenance margin* level (for instance, 25% of the total contract value by the Federal Reserve Board's Regulation T), a *margin call* is issued to the contract holder, demanding additional margin. If the additional margin is not provided, the futures position will be closed out.

The margin accounts are deposits with the broker, and so will typically accumulate at the risk free interest rate.

Example 2.2. Suppose that an investor takes a long position of ten contracts in corn (50,000 bushels) for December delivery at price of \$0.14 (per bushel). Suppose the broker requires margin of 50% of the total contract value with a maintenance margin of 25% of the total contract value. The effective risk free-interest rate is 5% per year.

1. Find the total contract value.
2. Find the initial margin.
3. Find the maintenance margin.
4. Suppose on the 2nd day the price of this contract drops to \$0.12. Find the value of the margin account on the 2nd day.

1. Total contract val $= 10 \cdot 50000 \cdot 0.14 = \underline{\overset{\$}{70000}}$

2. initial margin → 50% $\Rightarrow \underline{\$35000}$

3. maintence margen 25% $\Rightarrow \underline{\$17500}$

4. Value of margin acc.

$$35000 \cdot (1.05)^{1/365} + \overset{\text{bushels}}{500{,}000}\ \overset{\text{lose } 2¢/\text{bush}}{(0.12 - 0.14)}$$

get 1 days interest

$$= 35004.68 + (-10000)$$

$$= \boxed{25004.68} > 17500$$

no margin call

Exercise 2.3. Suppose that an investor takes a long position of ten contracts in corn (50,000 bushels) for December delivery at price of \$0.14 (per bushel). Suppose the broker requires margin of 50% of the total contract value with a maintenance margin of 25% of the total contract value. The effective risk free-interest rate is 5% per year.

1. Suppose on the 3rd day the price of this contract drops to \$0.09. Find the value of the margin account on the 3rd day. What will happen to Mr Saver's margin account?
2. Suppose on the 3rd day the price of this contract increases to \$0.13. Find the value of the margin account on the 3rd day. What will happen to Mr Saver's margin account?

1.) $10 \cdot 50000 = 500000$ bushels

$\cdot 0.14 =$

$10 \cdot 50000 \cdot 0.14 = \underline{\$70000}$

initial marsin → 50% => $\underline{\$35000}$

maintence margen 25% => $\underline{\$17500}$

$25004.68(1.05)^{1/365} + 500000(0.09 - 0.12)$

$= 25{,}008.02 - 15000 = 10008.02 <$ maintence margin

margin call

2.)

$25004.68(1.05)^{1/365} + 500000(0.13 - 0.12)$

$= 30008.02 >$ margin

no margin call

3 Hedging

We will now discuss how one can use futures and forwards to manage risks via a process called *hedging*.

3.1 Hedging with a Forward Contract

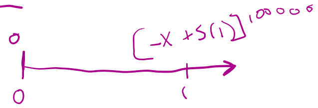
Using forward contracts to do hedging is relatively straightforward: one just enters into an agreement to buy/sell the required commodity at an agreed price to remove the risks of price fluctuations in the spot market.

Example 3.1. A gold-mining firm plans to mine and sell 100,000 ounces of gold precisely one year from now. The firm can either

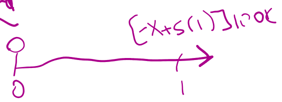
1. **remain unhedged**, i.e. sell the gold at the spot price of $\$S(1)$ per ounce in one year, or
2. **hedge using a forward contract**, i.e. lock in a price for gold in one year by entering into the short position of a forward contract for delivery of 100,000 ounces of gold in one year.

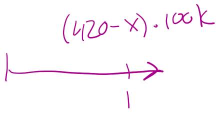
Suppose that the cost of mining one ounce of gold is a random variable $\$X$, and the forward price is \$420 per ounce. Calculate, in terms of X, their profit of the firm when it remains unhedged and when it hedges.

Remain unhedged

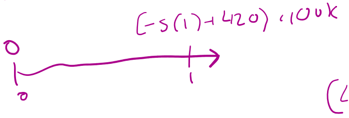


Hedge w. Forward





forward:



$(420-X)\cdot 100K$

Thus, if the gold mining firm chooses to hedge, then its profits are completely insulated from the gold prices. In particular, the hedging protects the firm from loss should gold prices decline. However, hedging also removes the firm's opportunity to make even greater profit if the gold prices do rise.

3.2 The perfect hedge using futures

One can also use futures contracts to hedge against risk. The simplest hedging strategy is the *perfect hedge*, where the risk associated with a future commitment to deliver or receive an asset is completely eliminated by taking an *equal and opposite* position in the futures market.

Example 3.2. A gold-mining firm plans to mine and sell 100,000 ounces of gold precisely one year from now (assume that the year has 365 days). Suppose that there are gold futures, each for 20,000 ounces of gold to be delivered one year from now.

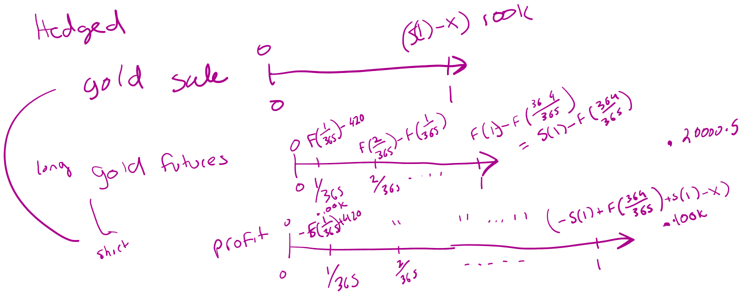
1. What is the perfect hedge to eliminate risks from fluctuations in the price of gold?
2. Suppose that the interest rate is 0%, the cost of mining the gold is a random variable $X, and the futures price is $420. Calculate the profit if the firm remains unhedged, and the (accumulated) profit if the firm hedges using the perfect hedge.

1. short 5 gold futures

2.) unhedged

$$\text{profit} = (S(1) - X)\cdot 100000$$

Hedged



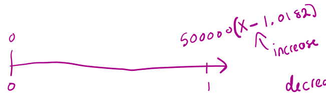
$$\text{hedged profit} = 100000\left(-F\left(\tfrac{1}{365}\right) + 420 - F\left(\tfrac{2}{365}\right) + F\left(\tfrac{1}{365}\right) - F\left(\tfrac{3}{365}\right) + F\left(\tfrac{2}{365}\right) \cdots - F\left(\tfrac{364}{365}\right) + F\left(\tfrac{363}{365}\right) - S(1) + F\left(\tfrac{364}{365}\right) + S(1) - X\right)$$

$$= 100600\,(420 - X)$$

Exercise 3.3. On 9 July, a US firm wants to sell equipment to a German customer in 3 months. The price is specified as 500,000 Euro, which will be paid upon delivery. Suppose that in the market, there are Euro futures contracts, each for 125,000 Euro (with futures price in USD) to be delivered on 9 October.

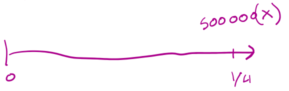
1. What is the perfect hedge to eliminate the exchange rate risks from the equipment sale? long 4 euro futures
2. Suppose the futures price of 1 Euro is 1 USD, and the spot price of 1 Euro is 1.0182 USD on 9 July. Also, let X denote the spot price of 1 Euro on 9 October (this is a random variable).
   - (a) Calculate the decrease in the value of the sale in USD if the firm remains unhedged.
   - (b) What is the decrease in the value of the sale in USD if the firm uses the perfect hedge.

a. unhedged

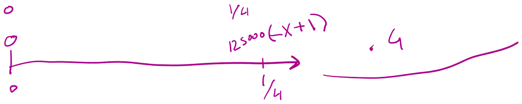


decrease ⇒ 500000(1.0182 − X)

b. hedged − long 4 euro futures



4 · short 1 future



actually, you are getting paid differences but here since 0 interest rates we push to end

increase in value of sale = 500k − 500k(1.0182)
= −9100

decrease ⇒ 9100

A perfect hedge completely eliminates the risk. However, perfect hedges are rare. To establish a perfect hedge,

1. the trader has to match the delivery date of the asset to the delivery date of the futures,
2. the commodity to be hedged must exactly match the commodity underlying the futures contract,
3. the amount of the asset obligated must be an integral multiple of the contract size, and

If any of these features is missing then a perfect hedge is not possible. In such circumstances risk can still be reduced but not eliminated.

3.3 The minimum variance hedge

Even if a perfect hedge for the price of an asset is not available, one can still manage (but not completely eliminate) risks by using futures of some other asset.

Suppose that you have k units of Asset A, and would like to sell it at some time T in the future. You would like to hedge against fluctuations of the price of Asset A, but you only have futures of Asset B available to you to do so (we assume that the delivery date of Asset B is larger than T). You can hedge by shorting k' units of futures of Asset B, where k' is some some number (k' might be negative). Our goal is to find k' to minimize the risk of the cashflow at time T of this portfolio. This is known as the *minimum variance hedge.*

For the sake of simplicity, we will assume that interest rates are 0 when calculating the minimum variance hedge. This will imply that the payoff for taking a long position (resp. short position) of a futures contract has payoff $F_B(T) - F_B(0)$ (resp. $F_B(0) = F_B(T)$).

Let $S_A(t)$ denote the spot price of Asset A at time t, let $F_B(t)$ be the futures price of Asset B at time t. If you remain unhedged, then the cash flow at time T is $kS_A(T)$. On the other hand, if you hedge by shorting k' units of futures of Asset B, then the cash flow at time T, denoted Y_T, is

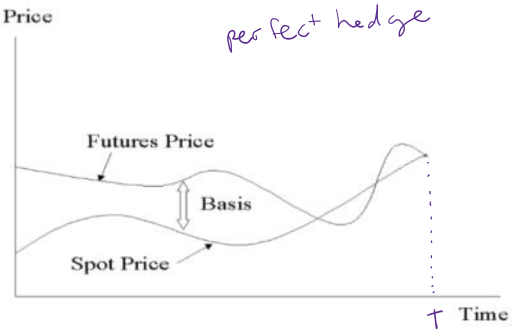
$$Y_T := kS_A(T) - k'(F_B(T) - F_B(0)) = k\left(S_A(T) - hF_B(T)\right) - k'F_B(0),$$

where $h := \frac{k'}{k}$ is called the *hedge ratio.*

The random variable

$$X_T := S_A(T) - hF_B(T)$$

is called the *basis* of the portfolio. Notice that if $A = B$ and T is the delivery date of the futures of Asset A, then the perfect hedge corresponds to choosing $h = 1$, in which case the basis is the constant 0. Thus, one can think of the basis as a measurement of how far the hedge is from the perfect hedge.



In general, if $\sigma_{S_A(T)}$ is the standard deviation of $S_A(T)$, $\sigma_{F_B(T)}$ is the standard deviation of $F_B(T)$, and ρ is the correlation coeffecient between $F_B(T)$ and $S_A(T)$, then variance of basis

note that the variance of X_T is

$$\sigma^2_{X_T} = \sigma^2_{S_A(T)} + h^2\sigma^2_{F_B(T)} - 2h\sigma_{S_A(T)}\sigma_{F_B(T)}\rho,$$

and the variance of the cashflow of Π at time T is

$$\sigma^2_{Y_T} = k^2\sigma^2_{X_T}.$$

Thus, minimizing the risk of the portfolio is equivalent to minimizing the basis risk. In the case of the perfect hedge, the basis risk is 0.

Since $\sigma^2_{X_T}$ is a quadratic function of h with positive leading coefficient, when σ_{X_T} is minimized,

$$\frac{\mathrm{d}\sigma^2_{X_T}}{\mathrm{d}h} = 2h\sigma^2_{F_B(T)} - 2\sigma_{S_A(T)}\sigma_{F_B(T)}\rho = 0.$$

The solution to this equation, also called the *optimal hedge ratio*, is

$$\boxed{h = \frac{\sigma_{S_A(T)}}{\sigma_{F_B(T)}}\rho.}$$

Thus, the risk of the minimum variance hedge is

$$\sigma_{Y_T} = k\sigma_{X_T} = k\sigma_{S_A(T)}\sqrt{1-\rho^2}.$$

if $\rho \uparrow$, $\sigma \downarrow$

Notice that the stronger the correlation between the $F_B(T)$ and $S_A(T)$, the lower the risk of the minimum variance hedge. In the case of the perfect hedge, this risk is driven down to 0.

Example 3.4. A gold-mining firm plans to mine and sell 100,000 ounces of gold precisely one year from now (assume that the year has 365 days). Suppose that there are silver futures, each for 30,000 ounces of ~~gold~~ silver to be delivered two years from now. If the standard deviation of the gold spot price in a year is \$40, the standard deviation of silver futures price in a year is \$20, and the correlation coefficient between the gold spot price and the silvers future price is 0.8, find the risk of the hedge that minimizes risk, assuming that the number of silver futures bought must be an interger.

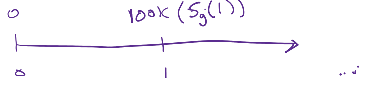
done long way

$\sigma_g(t) = 40$

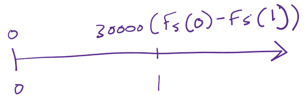
$\sigma_s(t) = 20$

$\rho_{g,s} = 0.8$

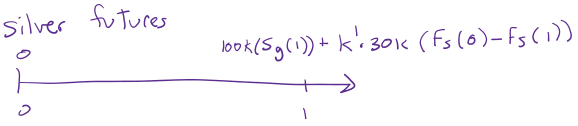
gold sale



short silver future



Short k' silver futures



$$Y_1 = 100000\,S(1) + 30000\,k'\,(F_s(0)-F_s(1))$$

$$\sigma_{y_1}^2 = 100000^2\,\sigma_{S_g(1)}^2 + 30000k'^2\,\sigma_{F_s(1)}^2 - 2(100000)(30000k')\,\sigma_{S_g(1)}\,\sigma_{F_s(1)}\,\rho$$

$$= 10000^2(40)^2 + 10^8 \cdot 9k'^2(20)^2 - 10^9 \cdot \cancel{2}\,k'(40)(20)(0.8)$$

$$= 10^{10}(1600 + 36k'^2 - 384k')$$

$$\frac{d(\sigma_{y_1}^2)}{dk'} = 10^{10}(72k' - 384) = 0$$

round

$$k' = \frac{384}{72} = \frac{16}{3} \approx \boxed{5}$$

$$\sigma_{y_1}^2\Big|_{k'=5} = 10^{10}(1600 + 36 \cdot 25 - 384 \cdot 5)$$

$$\sigma_{y_1}^2\Big|_{k'=5} = \boxed{2408318.92}$$

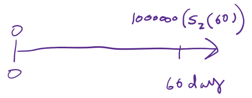
Exercise 3.5. A U.S. corporation has obtained a large order from a small developing country, whose currency is the zee. Payment will be in 60 days in the amount of the 1 million zees. The corporation would like to hedge their exchange rate risks using Japanese yen futures. Assume that:

- Current exchange rates of zee is $Z_0 = 0.164$ dollar/zee,
- Current exchange rates of yen: $Y_0 = 0.0125$ dollar/yen, and
- $\sigma_z = 2.5\% Z_0$, $\sigma_y = 3\% Y_0$, $\rho_{z,y} = 0.8$ (obtained from historical data).

risk of spot p of zee

risk of future price of yen

Find the minimum variance hedge for the exchange rate risks of the coporation.



$$h = \frac{\sigma_{S_z}(T)}{\sigma_{S_y}(T)} \rho = \frac{0.025 \cdot 0.164}{0.03 \cdot 0.0125} \cdot 0.8 = \frac{k'}{1000000}$$

$$k' = 8746666.67$$

Exercise 3.6. Suppose a trader has a holding period of 2 months to sell rice. Assume the standard deviation of spot prices per kilogram of rice over two month periods is σ_S = \$0.18 and the volatility of the futures contracts per kilogram of rice (with delivery in 1 year) over the same period is σ_F = \$0.29. The correlation of the two changes in prices is $\rho = 0.85$.

1. Find the optimal hedge ratio.
2. By what factor, does the minimum-variance hedge reduce the risk per kilogram of rice?

1. $$h = \frac{\sigma_S}{\sigma_{F_r}} \cdot \rho = \frac{0.18}{0.29} \cdot (0.85) = 0.5279$$

2. unhedged risk $= 0.18$

min variance hedged risk
$$= \overset{1}{k}\ \sigma_{S_A}(t)\sqrt{1-\rho^2}$$
$$= 1 \cdot (0.18)\sqrt{1-0.85^2}$$
$$= \boxed{0.0948}$$

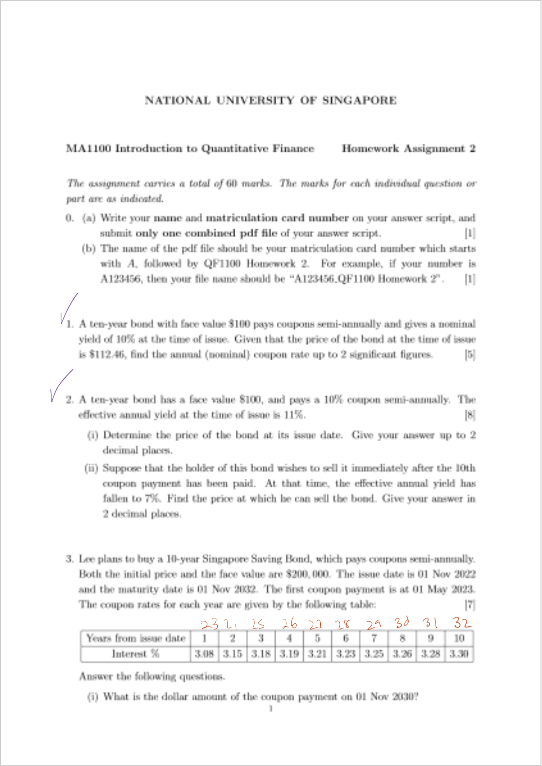
reduction in risk $= \dfrac{0.18 - 0.0948}{0.18}$

$$= 0.473$$

QF1100-Assignment 2

Thursday, October 5, 2023 5:59 PM

QF1100-Assignment 2



NATIONAL UNIVERSITY OF SINGAPORE

MA1100 Introduction to Quantitative Finance **Homework Assignment 2**

The assignment carries a total of 60 marks. The marks for each individual question or part are as indicated.

0. (a) Write your **name** and **matriculation card number** on your answer script, and submit **only one combined pdf file** of your answer script. [1]
   (b) The name of the pdf file should be your matriculation card number which starts with A, followed by QF1100 Homework 2. For example, if your number is A123456, then your file name should be "A123456_QF1100 Homework 2". [1]

1. A ten-year bond with face value \$100 pays coupons semi-annually and gives a nominal yield of 10% at the time of issue. Given that the price of the bond at the time of issue is \$112.46, find the annual (nominal) coupon rate up to 2 significant figures. [5]

2. A ten-year bond has a face value \$100, and pays a 10% coupon semi-annually. The effective annual yield at the time of issue is 11%. [8]
   (i) Determine the price of the bond at its issue date. Give your answer up to 2 decimal places.
   (ii) Suppose that the holder of this bond wishes to sell it immediately after the 10th coupon payment has been paid. At that time, the effective annual yield has fallen to 7%. Find the price at which he can sell the bond. Give your answer in 2 decimal places.

3. Lee plans to buy a 10-year Singapore Saving Bond, which pays coupons semi-annually. Both the initial price and the face value are \$200,000. The issue date is 01 Nov 2022 and the maturity date is 01 Nov 2032. The first coupon payment is at 01 May 2023. The coupon rates for each year are given by the following table: [7]

23 24 25 26 27 28 29 30 31 32

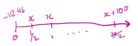
Years from issue date	1	2	3	4	5	6	7	8	9	10
Interest %	3.08	3.15	3.18	3.19	3.21	3.23	3.25	3.26	3.28	3.30

Answer the following questions.

(i) What is the dollar amount of the coupon payment on 01 Nov 2030?

1

1.)



$112.46 = \frac{x}{1+\frac{0.1}{2}} + \frac{x}{(1+0.05)^2} + \cdots + \frac{x}{(1+0.05)^{20}} + \frac{100}{1.05^{20}}$

$112.46 = \frac{x}{1.05}\left(\frac{1-\left(\frac{1}{1.05}\right)^{20}}{1-\frac{1}{1.05}}\right) + \frac{100}{1.05^{20}}$

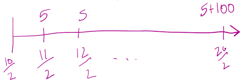
$x \approx 6$

$c\% = 6\%$ annual coupon rate = $\boxed{12\%}$

2.) i. $P = \frac{5}{(1+0.11)^{1/2}} + \frac{5}{1.11^{1}} + \frac{5}{1.11^{3/2}} + \cdots + \frac{5}{1.11^{20/2}} + \frac{100}{1.11^{20/2}}$

$= \frac{5}{1.11^{1/2}}\left(\frac{1-\left(\left(\frac{1}{1.11}\right)^{1/2}\right)^{20}}{1-\left(\frac{1}{1.11}\right)^{1/2}}\right) + \frac{100}{1.11^{10}} = \boxed{\begin{matrix}95.69\\ \cancel{75.989}\end{matrix}}$

ii.



$PV\left(\frac{10}{2}\right) = \frac{5}{(1+0.07)^{1/2}} + \frac{5}{1.07^{2/2}} + \cdots + \frac{5}{1.07^{10/2}} + \frac{100}{1.07^{10/2}} = \frac{5}{(1.07)^{1/2}}\left(\frac{1-\left(\left(\frac{1}{1.07}\right)^{1/2}\right)^{10}}{1-\left(\frac{1}{1.07}\right)^{1/2}}\right) + \frac{100}{1.07^{10/2}}$

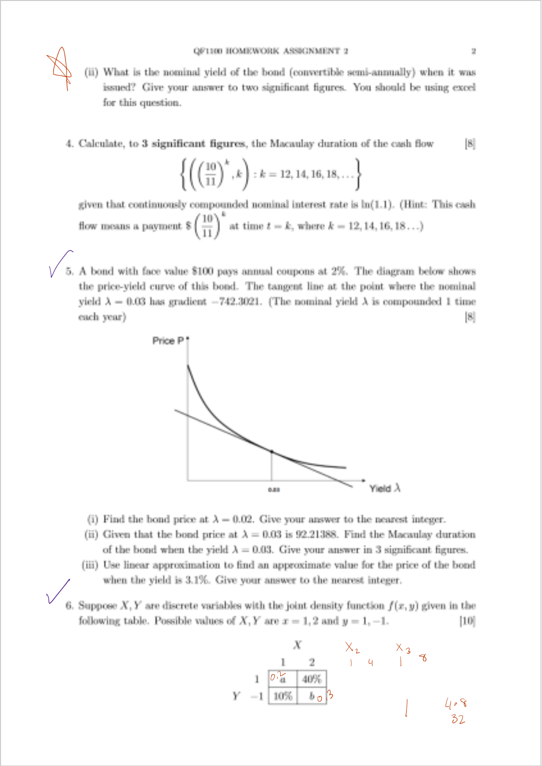
$= \boxed{\begin{matrix}\cancel{107.625}\\ 113.01\end{matrix}}$

3.) $P = F$ at par

i. $(200\,000)(0.0326) = \boxed{6520}$ /2 = 3260

ii.

=6520/2



(ii) What is the nominal yield of the bond (convertible semi-annually) when it was issued? Give your answer to two significant figures. You should be using excel for this question.

4. Calculate, to **3 significant figures**, the Macaulay duration of the cash flow [8]

$$\left\{\left(\left(\frac{10}{11}\right)^k, k\right) : k = 12, 14, 16, 18, \ldots\right\}$$

given that continuously compounded nominal interest rate is $\ln(1.1)$. (Hint: This cash flow means a payment \$ $\left(\frac{10}{11}\right)^k$ at time $t = k$, where $k = 12, 14, 16, 18\ldots$)

5. A bond with face value \$100 pays annual coupons at 2%. The diagram below shows the price-yield curve of this bond. The tangent line at the point where the nominal yield $\lambda = 0.03$ has gradient -742.3021. (The nominal yield λ is compounded 1 time each year) [8]

(i) Find the bond price at $\lambda = 0.02$. Give your answer to the nearest integer.
(ii) Given that the bond price at $\lambda = 0.03$ is 92.21388. Find the Macaulay duration of the bond when the yield $\lambda = 0.03$. Give your answer in 3 significant figures.
(iii) Use linear approximation to find an approximate value for the price of the bond when the yield is 3.1%. Give your answer to the nearest integer.

6. Suppose X, Y are discrete variables with the joint density function $f(x, y)$ given in the following table. Possible values of X, Y are $x = 1, 2$ and $y = 1, -1$. [10]

		X	
		1	2
Y	1	0.2 a	40%
	−1	10%	b 0.3

X_2 1 4 X_3 1 8

1 4.8 32

4.) $\frac{\sum i \cdot PV(C_i)}{PV(C)} = \frac{\frac{12\cdot\left(\frac{10}{11}\right)^{12}}{1.1^{12}} + \frac{14\cdot\left(\frac{10}{11}\right)^{14}}{1.1^{14}} + \cdots}{\frac{\left(\frac{10}{11}\right)^{12}}{1.1^{12}} + \frac{\left(\frac{10}{11}\right)^{14}}{1.1^{14}} + \frac{\left(\frac{10}{11}\right)^{16}}{1.1^{16}} + \cdots} = \frac{\sum_{i=6}^{\infty} \frac{2i\cdot\left(\frac{10}{11}\right)^{2i}}{1.1^{2i}}}{\frac{\left(\frac{10}{11}\right)^{12}}{1.1^{12}}\left(1+\frac{\left(\frac{10}{11}\right)^{2}}{1.1^{2}}+\frac{\left(\frac{10}{11}\right)^{4}}{1.1^{4}}+\cdots\right)} = \frac{2\sum_{i=6}^{\infty} \frac{i\cdot\left(\frac{10^2}{11^2}\right)^{i}}{1.1^{2i}}}{\frac{\left(\frac{10}{11}\right)^{12}}{1.1^{12}}\left(\frac{1}{1-\left(\frac{(10/11)^2}{1.1^2}\right)}\right)}$

5.) a. $c = \lambda$

then price = 100 = Face val 1

b. $\underset{-742.3021}{\frac{dP}{d\lambda}} = -\frac{1}{1+\frac{\overset{0.03}{\lambda}}{\underset{1}{m}}}\, DP \rightarrow 92.21388$

$\boxed{D = 8.291}$

c. $\Delta P = -(D_m P)\cdot\Delta\lambda$

$= -\left(-\frac{dP/d\lambda}{P}\cdot P\right)\cdot(3.1-3.0)$

0.7423021

Price = 91.47

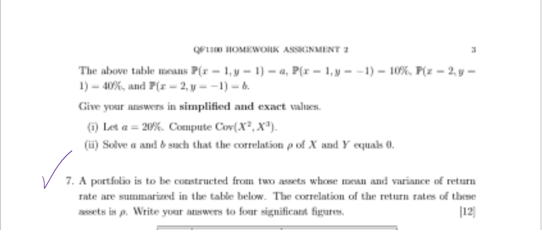
$\boxed{91}$

6.) a. $a = 20\%$
$b = 30\%$

$\mathrm{Cov}(X^2, X^3) = E(X^2\cdot X^3) - E(X^2)E(X^3)$

$= 1(0.3) + 32(0.7) - \left[(1\cdot 0.3 + 4\cdot 0.7)(1\cdot 0.3 + 8\cdot 0.7)\right]$

$= \boxed{4.41}$



The above table means $\mathbb{P}(x = 1, y = 1) = a$, $\mathbb{P}(x = 1, y = -1) = 10\%$, $\mathbb{P}(x = 2, y = 1) = 40\%$, and $\mathbb{P}(x = 2, y = -1) = b$.

Give your answers in **simplified and exact** values.

(i) Let $a = 20\%$. Compute $\mathrm{Cov}(X^2, X^3)$.
(ii) Solve a and b such that the correlation ρ of X and Y equals 0.

7. A portfolio is to be constructed from two assets whose mean and variance of return rate are summarized in the table below. The correlation of the return rates of these assets is ρ. Write your answers to four significant figures. [12]

b. $0 = \rho_{x,y} = \frac{\mathrm{cov}(x,y)}{\sqrt{\mathrm{var}(x)\,\mathrm{var}(y)}}$ $\rightarrow E(XY) - E(X)E(Y) =$

$E(x^2) - E(x)^2$

$1(0.1+a) + 4(0.4+b) - \left(1(0.1+a) + 2(0.4+b)\right)^2$

$E(xy) = 1(a) + -1(0.1) + 2(0.4) - 2(b)$

$E(x) =$

$E(xy) = E(x)E(y)$

$a - 0.1 + 0.8 - 2b = (a + 0.1 + 2(0.4+b))(0.4 + a - 0.1 - b)$

$a + 0.7 - 2b = (a + 0.9 + 2b)(a + 0.3 - b)$

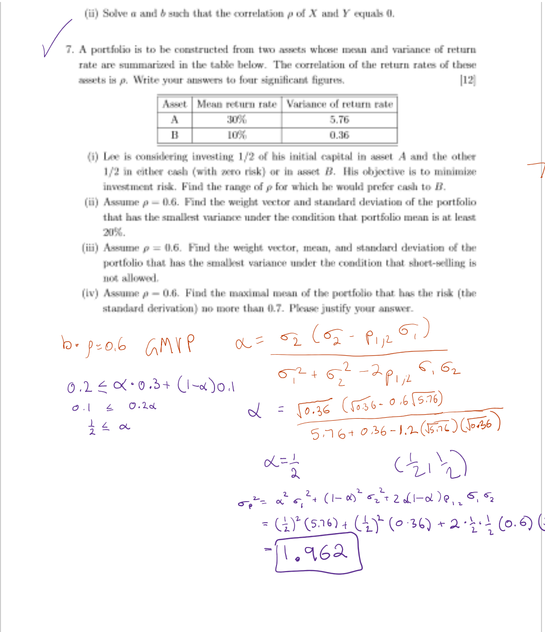
$a + 0.7 - 1 + 2a = (a + 0.9 + 1 - 2a)(a + 0.3 - 0.5 + a)$

$b = 0.5 - a$

$0 = (1.9-a)(2a-0.2) - 2a + 0.3$

$3.8a - 0.38 + 2a^2 + 0.2a - 2a + 0.3$

$= 2a + 2a^2 - 0.08$



(ii) Solve a and b such that the correlation ρ of X and Y equals 0.

✓

7. A portfolio is to be constructed from two assets whose mean and variance of return rate are summarized in the table below. The correlation of the return rates of these assets is ρ. Write your answers to four significant figures. [12]

Asset	Mean return rate	Variance of return rate
A	30%	5.76
B	10%	0.36

(i) Lee is considering investing 1/2 of his initial capital in asset A and the other 1/2 in either cash (with zero risk) or in asset B. His objective is to minimize investment risk. Find the range of ρ for which he would prefer cash to B.

(ii) Assume $\rho = 0.6$. Find the weight vector and standard deviation of the portfolio that has the smallest variance under the condition that portfolio mean is at least 20%.

(iii) Assume $\rho = 0.6$. Find the weight vector, mean, and standard deviation of the portfolio that has the smallest variance under the condition that short-selling is not allowed.

(iv) Assume $\rho = 0.6$. Find the maximal mean of the portfolio that has the risk (the standard derivation) no more than 0.7. Please justify your answer.

$$a-0.1+0.8-2b = (a+0.1+2(0.4+b))(0.4+a-0.1-b)$$
$$a+0.7-2b = (a+0.9+2b)(a+0.3-b)$$
$$a+0.7-1+2a = (a+0.9+1-2a)(a+0.3-0.5+a)$$
$$3a-0.3 = (1.9-a)(2a-0.2)$$
$$\boxed{a = 0.\tfrac{1}{0.4} \quad b = 0.4|0.1}$$

$b = 0.5-a$

$3.8a - 0.38 + 2a^2 + 0.2a - 2a + 0.0$
$= 2a + 2a^2 - 0.08$

7.) a. cash: $\sigma_{p_c}^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2 \overset{\sigma_2^2}{} + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2$

$= \left(\frac{1}{2}\right)^2 \cdot 5.76 + \left(\frac{1}{2}\right)^2 \cdot 0 + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \cdot 0 =$

B $\sigma_{p_B}^2 = \left(\frac{1}{2}\right)^2 \cdot 5.76 + \left(\frac{1}{2}\right)^2 \cdot 0.36 + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \cdot \rho \cdot \sqrt{5.76} \cdot \sqrt{0.36}$

$\cancel{\sigma_{p_B}^2 < \sigma_{p_c}^2}$ $\quad \sigma_{p_B}^2 > \sigma_{p_c}^2$

$\frac{1}{4} \cdot 0.36 + \frac{1}{2}\rho\sqrt{5.76}\sqrt{0.36} > 0$

$\cancel{-1 < \rho < -0.125}$

$\boxed{\rho > -0.125}$

b. $\rho = 0.6$ GMVP $\quad \alpha = \dfrac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}$

$0.2 \leq \alpha \cdot 0.3 + (1-\alpha)0.1$
$0.1 \leq 0.2\alpha$
$\frac{1}{2} \leq \alpha$

$\alpha = \dfrac{\sqrt{0.36}(\sqrt{0.36} - 0.6\sqrt{5.76})}{5.76 + 0.36 - 1.2(\sqrt{5.76})(\sqrt{0.36})}$

$\alpha = \frac{1}{2} \qquad \left(\frac{1}{2}, \frac{1}{2}\right)$

$\sigma_p^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2$
$= \left(\frac{1}{2}\right)^2(5.76) + \left(\frac{1}{2}\right)^2(0.36) + 2 \cdot \frac{1}{2} \cdot \frac{1}{2}(0.6)(\sqrt{0.36})(\sqrt{5.76})$
$= \boxed{1.962}$

c. $\rho = 0.6$
GMVP $\quad \alpha = \dfrac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2} = \dfrac{\sqrt{0.36}(\sqrt{0.36} - 0.6\sqrt{5.76})}{5.76 + 0.36 - 1.2(\sqrt{5.76})(\sqrt{0.36})}$

$= -0.114$

since neg α not allowed
$\Rightarrow \alpha = 0 \quad$ mean = 10% $\quad SD = \sqrt{0.36} = 0.6$
invest 100% into asset B $\quad (0, 1)$

d.) $\rho = 0.6$
$0.7 \geq \alpha_p^2$
$$0.7^2 = \alpha^2(5.76) + (1-\alpha)^2(0.36) + 2\alpha(1-\alpha)0.6\sqrt{5.76}\sqrt{0.36}$$
$\alpha = \frac{1}{6} \quad \left(\frac{1}{6}, \frac{5}{6}\right) \quad 0.092$
~~0.077384~~
$\mu = \frac{1}{6} \cdot 0.3 + \frac{5}{6} \cdot 0.1 = \boxed{\cancel{\tfrac{2}{15}}}$ $\quad$ ~~97~~ $\boxed{0.1184}$

$\alpha \cdot 0.3 + (1-\alpha) \cdot 0.1$

QF1100-Tutorial6

Friday, October 6, 2023 6:03 PM

QF1100-Tutorial6

NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 6**

1. Find the exact value of $\mathbf{E}(X)$ and $\mathbf{Var}(X)$ where X is the outcome when we roll an unbalanced die. The probability of each face is given as follows.

k	1	2	3	4	5	6
$\mathbb{P}(k)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

E(X) = 1/12 + 2/12 + 1/2 + 2/3 + 5/6 + 2 = 4.25
Var(X) = 1/12 + 4/12 + 9/6 + 16/6 + 25/6 + 36/6 = 14.75

2. The joint density function of the random variables X, Y, Z is

$$p(1,2,3) = p(2,1,1) = p(2,2,1) = p(2,3,2) = \frac{1}{4}$$

Find the exact values of the following:
(i) the density functions of X, Y, and Z;
(ii) $\mathbf{E}(X)$, $\mathbf{E}(Y)$, and $\mathbf{E}(Z)$;
(iii) $\mathbf{Var}(X)$, $\mathbf{Var}(Y)$, and $\mathbf{Var}(Z)$;
(iv) $\rho_{X,Y}$, $\rho_{X,Z}$, and $\rho_{Y,Z}$;

✓ 3. The following table shows the one-year return distribution of Startup, Inc.

Probability	40%	20%	20%	10%	10%
Return	−100%	−75%	−50%	−25%	1000%

Calculate the exact value of
(i) The expected return.
(ii) The standard deviation of the return.

$E = -100 \cdot 0.4 + -75 \cdot 0.2 + -50 \cdot 0.2$
$+ -25 \cdot 0.1 + 1000 \cdot 0.1$

i. $= 32.5\%$

ii. $V = E(x^2) - E(x)^2$
$\doteq 323.46\%$

✓ 4. The hypothetical end-of-period price of an asset is given in the table below

Share price	11.40	9.50	11.00	10.90	11.20
Probability	0.1	0.2	0.4	0.1	0.2

The initial price of the asset is $10.00.

E Increase =
10.77

$10 = \frac{10.77}{(1+r\%)}$

(i) Calculate the expected one-period rate of return of the asset to 2 significant figures. ↙ 0.077

7.7%

(ii) A man buys 100 shares of the asset and \$6000 worth of another asset whose mean rate of return is 3.5%. Calculate the one-period mean rate of return of this portfolio to 2 significant figures.

$100 \cdot 10 + 6000 = 7000$

$\frac{1000}{7000} \cdot 7.7 + \frac{6000}{7000} \cdot 3.5 =$

$= 0.041$

5. Consider an equally-weighted portfolio of n securities with return rates r_i $(i = 1, 2, \ldots, n)$. It is given that

$$\sigma_{i,j} = \begin{cases} 4 & \text{if } i = j; \\ 2 & \text{if } i \neq j. \end{cases}$$

Show that $\sigma_p = \sqrt{2\left(1 + \frac{1}{n}\right)}$.

What does this tell you about the diversification benefits of investing in many assets?

$$\sigma_p = \sqrt{\frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 + \frac{1}{n^2}\sum_{i,j=1}^{n} \sigma_{ij}}$$

$$= \sqrt{\frac{1}{n^2}\left[16n\right] + \frac{1}{n^2}\frac{n(n+1)}{\cancel{2}} \cdot \cancel{2}}$$

$$= \sqrt{\frac{1}{n} \cdot 16 + \frac{1}{n} \cdot (n+1)}$$

$$= \sqrt{\frac{1}{n}(16 + n + 1)}$$

6. A financial market consists of n $(n \geq 2)$ assets whose rates of return $r_1, r_2, \ldots, r_n$ are such that

$$\sigma_{i,j} = \begin{cases} 1 & \text{if } |i - j| = n - 1; \\ 2 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Q is a portfolio whose weight vector is of the form $\boldsymbol{w} = \left(1 - (n-1)\alpha, \underbrace{\alpha, \alpha, \ldots \alpha}_{(n-1)\text{ terms}}\right)^T$. That is, Q is equally weighed in assets $2, 3, \ldots, n$.

(i) Find, in terms of α and n, the variance of portfolio Q.

(ii) Find, in terms of n, the value of α that minimises the variance of portfolio Q.

Simplify your answers as far as possible.

7. The variance of the rate of return of two investment trusts, A and B are 0.25 and 0.09 respectively. The correlation coefficient of these return rates is c. An investor is considering investing 60% of his initial capital in asset A and the other 40% in either cash (with **zero** risk) or in asset B. If his objective is to minimise investment risk, find the range of values of c for which he would he (strictly) prefers asset B to cash.

Reference Ans:

1. $\mathbf{E}(X) = 17/4$, $\mathbf{Var}(X) = 43/16$
2. (i) The density function of X is $p_X(1) = 1/4$, $p_X(2) = 3/4$
   The density function of Y is $p_Y(1) = 1/4$, $p_Y(2) = 1/2$, $p_Y(3) = 1/4$
   The density function of Z is $p_Z(1) = 1/2$, $p_Z(2) = 1/4$, $p_Z(3) = 1/4$

(ii) $\mathbf{E}(X) = 7/4$, $\mathbf{E}(Y) = 2$, $\mathbf{E}(Z) = 7/4$

(iii) $\mathbf{Var}(X) = 3/16$, $\mathbf{Var}(Y) = 1/2$, $\mathbf{Var}(Z) = 11/16$

(iv) $\rho_{X,Y} = 0$, $\rho_{X,Z} = -5/\sqrt{33}$, $\rho_{Y,Z} = \sqrt{2/11}$

3. (i) 0.325

   (ii) $\frac{\sqrt{16741}}{40}$

4. (i) 0.077

   (ii) 0.041

6. (i) $2(n-1)^2\alpha^2 - 2(2n-3)\alpha + 2$

   (ii) $\alpha = \dfrac{2n-3}{2(n-1)^2}$

7. $-1 \le c < -1/5$

QF1100-Tutorial7

Wednesday, October 18, 2023 10:51 PM

QF1100-Tutorial7

NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 7**

1. A financial market consists of four risky assets whose rates of return r_1, r_2, r_3 and r_4 have means $1, \frac{1}{2}, 1$, and 2 respectively, and

$$\sigma_{i,j} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i^2 - j^2| = 5 \quad 2,3 \\ 0 & \text{otherwise.} \end{cases}$$

Let α be a real number. A portfolio P has weight vector $\mathbf{w} = (\alpha, 2\alpha, \alpha, 1-4\alpha)^T$, and has the lowest risk among all such portfolios.

(i) Find the value of α.
(ii) Find the mean and variance of the portfolio P.

✓ 2. A portfolio P is to be constructed from two stocks whose mean and variance of return rate are given below

Asset	Mean return	Variance of Return
1	0.2	0.01
2	0.3	0.16

The two assets have perfectly positively correlated returns, that is $\rho = 1$.
(i) Plot the portfolio graph (graph of μ_P against σ_P), and find its equation.
(ii) Find the mean return of the global minimum-risk portfolio.

3. The rates of return of two assets, A and B have mean, standard deviation and correlation coefficient given by

$$\mu_A = 0.1, \quad \mu_B = 0.2, \quad \sigma_A = 0.2, \quad \sigma_B = 0.3, \quad \rho_{A,B} = \rho$$

(i) Find the range of values of ρ for which the global minimum-variance portfolio does not involve short-selling.
(ii) Suppose that $\rho = -\frac{1}{4}$. Find the equation of the graph of the feasible set, and sketch this graph, indicating clearly the points representing each of these assets and the minimum variance point.

✓ 4. You are given two assets, A and B whose rates of returns are given in the table below:

Stock	Mean	Variance
A	0	0.1
B	0.2	0.15

It is also given that the covariance of the return rates is −0.1. Calculate the portfolio weights that will minimize the portfolio risk if
(i) there are no restrictions.
(ii) the desired mean portfolio return is at least 8%.
(iii) the desired mean portfolio return is at least 9%.
Find the equation of the feasible set (that is, the $\mu-\sigma$ curve) and indicate clearly on the graph your answers for (i), (ii) and (iii).

5. A market has two risky assets, A and B, whose rates of return have means $\mu_A = 2$ and $\mu_B = 1$. The equation of the feasible set (that is, the $\mu-\sigma$ curve) is

$$\sigma^2 = 6\mu^2 - 21\mu + k$$

for some constant k. The global minimum-variance portfolio (GMVP) has variance $\frac{5}{8}$.
(i) Find the mean of the GMVP.
(ii) Find the value of k.
(iii) Find the weight vector of the GMVP.
(iv) Find the variance of asset A, the variance of asset B, and the covariance of A and B.

6. A market has two risky assets, Asset 1 and Asset 2, whose rates of return have variances $\sigma_1^2 = 2$ and $\sigma_2^2 = 3$ respectively. The mean of Asset 1 is $\mu_1 = \frac{1}{4}$. The equation of the feasible set (that is, the $\mu-\sigma$ curve) is

$$\sigma^2 = 48\mu^2 + B\mu + C$$

for some constants B and C. The portfolio mean of the global minimum-variance portfolio (GMVP) is $\frac{1}{8}$. The mean of Asset 2 is higher than the mean of GMVP.
(i) Find the value of B.
(ii) Find the value of C.
(iii) Find the variance of the GMVP.
(iv) Find the mean of Asset 2.
(v) Find the weight vector of the GMVP.
(vi) Find the covariance of Asset 1 and Asset 2.
(vii) Find the mean of the portfolio that has the smallest variance under the condition that short selling is not allowed.

(viii) Find the weight vector of the portfolio that has the smallest variance under the condition that portfolio mean is at least 0.5.

1.) i.

$\sigma_\pi^2 = \alpha^2\sigma_1^2 + (2\alpha)^2\sigma_2^2 + \alpha^2\sigma_3^2 + (1-4\alpha)^2\sigma_4^2$
$+ 2(\alpha)(2\alpha)\sigma_{1,2} + \alpha^2\sigma_{1,3} + \alpha(1-4\alpha)\sigma_{1,4}$
$\boxed{+2\alpha^2\sigma_{2,3}} + (2\alpha)(1-4\alpha)\sigma_{2,4} + \alpha(1-4\alpha)\sigma_{3,4}$

$= 2(\alpha^2 + 4\alpha^2 + \alpha^2 + 1 - 8\alpha + 16\alpha^2) - 4\alpha^2$

$= 2(20\alpha^2 - 8\alpha + 1)$

$\frac{d(\sigma_\pi^2)}{d\alpha} = 40\alpha - 8 = 0$ $\boxed{\alpha = \frac{1}{5}}$ ← risk minimized here

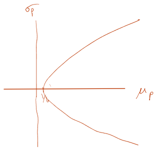
ii.

2.)

$\alpha = \frac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}$

$= \frac{\sqrt{0.16}(\sqrt{0.16} - 1\cdot\sqrt{0.01})}{0.01 + 0.16 - 2\sqrt{0.16}\sqrt{0.01}} = \frac{4}{3}$

$\frac{4}{3}(0.2) + (-\frac{1}{3})(0.3)$
$= \boxed{\frac{1}{6}}$



3.)

$\alpha = \frac{0.2(0.2 - \rho\cdot 0.3)}{0.3^2 + 0.2^2 - 2\rho\cdot 0.3\cdot 0.2} \geq 0$

A = 2
B = 1

$0.2(0.2 - \rho\cdot 0.3) \geq 0$ $\quad 0.3^2 + 0.2^2 - 2\rho\cdot 0.3\cdot 0.2 > 0$

$\frac{2}{3} \geq \rho$ $\quad \frac{13}{12} > \rho$

$0.2 - \rho\cdot 0.3 \leq 0$ $\quad 0.3^2 + 0.2^2 - 2\rho\cdot 0.3\cdot 0.2 < 0$

$\frac{2}{3} \leq \rho$ $\quad \rho > \frac{13}{12}$

4.) i. cov(A,B) = −0.1

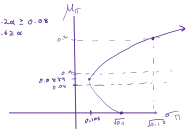
$\rho_{A,B} = \frac{-0.1}{\sqrt{0.1\cdot 0.15}} = -0.8165$

$\alpha = \frac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2} = \frac{\sqrt{0.15}(\sqrt{0.15} - \rho\sqrt{0.1})}{0.1 + 0.15 - 2\rho\sqrt{0.1}\sqrt{0.15}} = \frac{5}{9}$

$\boxed{\langle\frac{5}{9}, \frac{4}{9}\rangle^T}$

ii. $\mu = \alpha\cdot 0 + (1-\alpha)\cdot 0.2 = 0.2 - 0.2\alpha \geq 0.08$
$0.6 \geq \alpha$
$\boxed{\langle\frac{5}{9}, \frac{4}{9}\rangle^T}$

iii. $(1-\alpha)\cdot 0.2 \geq 0.09$
$0.2 - 0.2\alpha \geq 0.09$
$0.55 \geq \alpha$
$\boxed{\langle 0.55, 0.45\rangle^T}$



5.) i.

$\sigma_p^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho_{1,2}^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2} = \frac{5}{8} =$

$1 + \alpha = \frac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2} + 1 = \mu_p$

$\mu_p = \alpha\mu_1 + (1-\alpha)\mu_2$
$= 2\alpha + (1-\alpha)$
$\mu_p = 1 + \alpha$

6.)

3.)

a.) $\sigma_\pi^2 = \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B + 2\alpha(1-\alpha)\sigma_A\sigma_B\rho$
$= 0.04\alpha^2 + 0.09(1-\alpha)^2 + 0.12\rho\alpha(1-\alpha)$

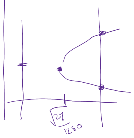
$\frac{d(\sigma_\pi^2)}{d\alpha} = 0$
$0.08\alpha - 0.18(1-\alpha) + 0.12\rho(1-2\alpha) = 0$

$\alpha = \frac{9-6\rho}{13-12\rho}$ weight of GMV

$0 \leq \alpha \leq 1$

$0 \leq \frac{9-6\rho}{13-12\rho} \leq 1$ $\quad 9 - 6\rho \leq 13 - 12\rho$

$6\rho \leq 4 \quad \rho \leq \frac{2}{3}$



No short selling of GMVP
ie $-1 \leq \rho \leq \frac{2}{3}$

b.) $\rho = -\frac{1}{4}$ $\quad \mu_\pi = \alpha\mu_1 + (1-\alpha)\mu_2 =$

$\alpha = 2 - 10\mu_\pi$

$\sigma_\pi^2 = 0.04(2 - 10\mu_\pi)^2$
$+ 0.09(10\mu_\pi - 1)^2$
$+ 0.12(-\frac{1}{4})(2 - 10\mu_\pi)(10$

$\sigma_\pi^2 = 16\mu_\pi^2 - 4.3\mu_\pi + 0.31$

$.1\alpha + 0.2 - 0.2\mu_2$

$- 0.1\alpha + 0.2$

$\mu_m^{-1})$

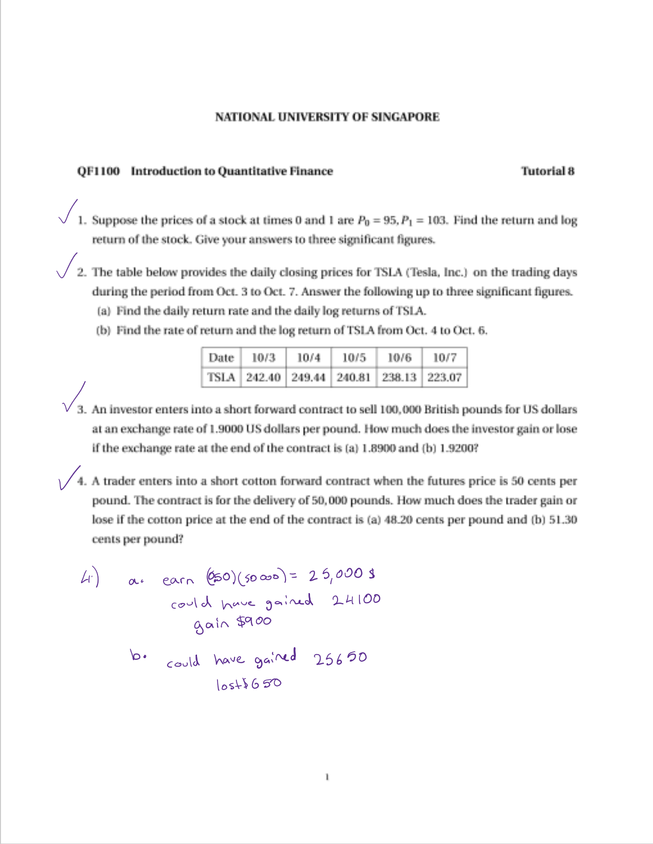
Reference Ans:

1. (i) $\alpha = 1/5$
   (ii) The mean is 1 and variance is 2/5
2. (ii) GMVP mean is 1/6 and variance is 0.
3. (i) $-1 \le \rho \le \frac{2}{3}$.
4. (i) $(5/9, 4/9)^T$
   (ii) $(5/9, 4/9)^T$
   (iii) $(11/20, 9/20)^T$
5. (i) 7/4
   (ii) $k = 19$
   (iii) $(3/4, 1/4)^T$
   (iv) $\sigma_A^2 = 1, \sigma_B^2 = 4, \sigma_{AB} = -1/2$,
6. (i) $B = -16$
   (ii) $C = 3$
   (iii) 5/3
   (iv) 1/3
   (v) $(2, -1)^T$
   (vi) 7/3
   (vii) $\frac{1}{4}$
   (viii) $(-2, 3)^T$

QF1100-Tutorial8

Wednesday, October 25, 2023 2:11 PM

QF1100-Tutorial8



NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 8**

✓ 1. Suppose the prices of a stock at times 0 and 1 are $P_0 = 95, P_1 = 103$. Find the return and log return of the stock. Give your answers to three significant figures.

✓ 2. The table below provides the daily closing prices for TSLA (Tesla, Inc.) on the trading days during the period from Oct. 3 to Oct. 7. Answer the following up to three significant figures.
   (a) Find the daily return rate and the daily log returns of TSLA.
   (b) Find the rate of return and the log return of TSLA from Oct. 4 to Oct. 6.

Date	10/3	10/4	10/5	10/6	10/7
TSLA	242.40	249.44	240.81	238.13	223.07

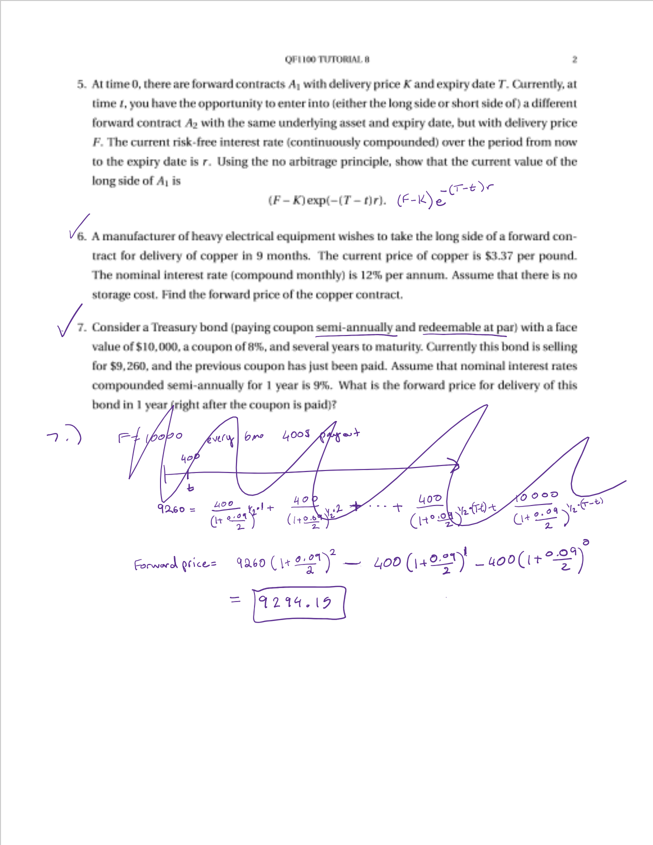
✓ 3. An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.9000 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.8900 and (b) 1.9200?

✓ 4. A trader enters into a short cotton forward contract when the futures price is 50 cents per pound. The contract is for the delivery of 50,000 pounds. How much does the trader gain or lose if the cotton price at the end of the contract is (a) 48.20 cents per pound and (b) 51.30 cents per pound?

4.) a. earn (0.50)(50 000) = 25,000 \$
could have gained 24100
gain \$900

b. could have gained 25650
lost \$650

1



QF1100 TUTORIAL 8 2

5. At time 0, there are forward contracts A_1 with delivery price K and expiry date T. Currently, at time t, you have the opportunity to enter into (either the long side or short side of) a different forward contract A_2 with the same underlying asset and expiry date, but with delivery price F. The current risk-free interest rate (continuously compounded) over the period from now to the expiry date is r. Using the no arbitrage principle, show that the current value of the long side of A_1 is

$$(F-K)\exp(-(T-t)r). \quad (F-K)e^{-(T-t)r}$$

✓ 6. A manufacturer of heavy electrical equipment wishes to take the long side of a forward contract for delivery of copper in 9 months. The current price of copper is \$3.37 per pound. The nominal interest rate (compound monthly) is 12% per annum. Assume that there is no storage cost. Find the forward price of the copper contract.

✓ 7. Consider a Treasury bond (paying coupon semi-annually and redeemable at par) with a face value of \$10,000, a coupon of 8%, and several years to maturity. Currently this bond is selling for \$9,260, and the previous coupon has just been paid. Assume that nominal interest rates compounded semi-annually for 1 year is 9%. What is the forward price for delivery of this bond in 1 year (right after the coupon is paid)?

7.) F = 10000 every 6mo 400\$ payout
400
t

$$9260 = \frac{400}{(1+\frac{0.09}{2})^{1/2 \cdot 1}} + \frac{400}{(1+\frac{0.09}{2})^{1/2 \cdot 2}} + \cdots + \frac{400}{(1+\frac{0.09}{2})^{1/2 \cdot (T-t)}} + \frac{10000}{(1+\frac{0.09}{2})^{1/2 \cdot (T-t)}}$$

$$\text{Forward price} = 9260\left(1+\frac{0.09}{2}\right)^2 - 400\left(1+\frac{0.09}{2}\right)^1 - 400\left(1+\frac{0.09}{2}\right)^0$$

$$= \boxed{9294.15}$$

1.)
$95(1+r) = 103$
$r = 0.0842$
$R = \ln\left(\frac{103}{95}\right) = 0.0809$

2.) a.

b. $238.13 = 249.44(1+r)$
$r = -0.045$
$\underset{\uparrow \text{ log}}{r} = \ln\left(\frac{238.13}{249.44}\right) = -0.046$

3.) a. sell 100,000 £ = receive 1.9·100k = 190,000 \$
on open market receive 189,000
gain 1000

b. on open market, 192000
lose 2000

5.)
$K = S_0 e^{rT}$
$F = S_0 e^{r(T-t)}$
$S_0 = \frac{K}{e^{rT}}$
$S_0 = \frac{F}{e^{r(T-t)}}$
$K = \frac{F}{e^{r(T-t)}} \cdot e^{rt} = \frac{Fe^{rt \cdot 2}}{e^{rT}}$

long side = spot pric − forward

$S_0 - K = \frac{F}{e^{r(T-t)}} - K$

long A_1: t: $-P$; T: $-K+S(T)$

~~Long~~ Short A_2: t: 0; T: $-F+S(t)$

cash: t: P; T: $-Pe^{r(T-t)}$

$\Pi =$ t: 0; T: $-F-K-Pe^{r(T-t)} > 0$

no abitrage if $F-K-Pe^{r(T-t)} = 0$

$P = F - Ke^{r(t-T)}$

6.) * no carrying costs
\$3.37/lb ← spot price(0)
T = 9 mo
12% / yr
mounthly

$F = (3.37)\left(1+\frac{0.12}{12}\right)^9 = \boxed{3.6857}$

long copper: 0: -3.37; 3/4: $S(3/4)$

short forward: 0: 0; 3/4: $K - S(3/4)$

or u can also short copper long forward

cash: 0: 3.37; 3/4: $-3.37\left(1+\frac{12\%}{12}\right)^9$

Π as described above → 0: 0; 3/4: $K - 3.37\left(1+\frac{12\%}{12}\right)^9$ ← must be positive = 0

I pretended yield is interest rate

long: 0: −9260; 400; $400 + S(1)$

short: 0; 0; $K - S(1)$

cash: 9260; −400; $-9260\left(1+\frac{9\%}{2}\right)^2 + 400\left(1+\frac{9\%}{2}\right.$

Reference Ans:

1. Return is 0.0842 and log return is 0.0809.

2. (a)

Date	TSLA	daily return rate	daily log raturn
10/3	242.40		
10/4	249.44	2.90%	0.0286
10/5	240.81	−3.46%	-0.0352
10/6	238.13	−1.11%	-0.0112
10/7	223.07	−6.32%	-0.0653

   (b) The rate of return is −4.53% and log return is −0.0464.

3. (a) Gain 1000 USD; (b) Lose 2000 USD.
4. (a) Gain $900 = 90000 cents; (b) Lose $650 = 65000 cents.
5.
6. $3.69 per pound.
7. $9294.15

QF1100-Tutorial9

Wednesday, November 1, 2023 10:42 PM

QF1100-Tutorial9

NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 9**

1. Suppose that an investor has the opportunity to engage in a REPO transaction in which Gilt-Edged stock worth \$800 today can be exchanged for \$820 in one year. Suppose also that the investor can borrow or lend at risk-free effective interest rate of 4%. Find an arbitrage portfolio.

2. Assume that IBM stock is currently traded at \$100 per share. IBM pays quarterly dividend of \$1 per share in the next 5 years (no change), and the dividend for this quarter was just paid. The continuously compounding interest rate is 5%. Today, you take a long position on a forward contract on the IBM stock that matures in two years.
   (i) If the forward price is \$103, then find an arbitrage opportunity.
   (ii) If the forward price is \$99, then find an arbitrage opportunity.
   (iii) Compute the forward price assuming the no arbitrage principle.

3. Consider a two-year forward contract on the dollar price of pound. The current exchange rate is \$1.90 per pound. The continuously compounding US dollar interest rate is at 4%, and the corresponding pound rate is at 5%. What should be the forward price at the two-year maturity?

4. A trader buys two futures contracts on orange juice. Each contract is for the delivery of 15000 pounds. The current futures price is 160 cents per pound; the initial margin is \$6000 per contract, and the maintenance margin is \$4500 per contract. Assume that the value of trader's margin account initially is the initial margin, and interest rates are 0%. What price change would lead to a margin call?

5. An investor shorts 10 futures contracts on day 1 for \$500 each. The initial margin is 40% and the maintenance margin is 30%. Assume that the interest rates are 0. If the pattern of prices is as given in the table below work out the margin account of the investor for each day until the position is closed out on day 7.

Day	Price
1	500
2	480
3	490
4	530
5	580
6	520
7	490

6. The standard deviation of monthly changes in the spot price of live cattle is (in cents per pound) 1.2. The standard deviation of monthly changes in the futures price of live cattle is 1.4. The correlation between the futures price changes and the spot price changes is 0.7. It is now October 15. A beef producer is committed to purchasing 200,000 pounds of live cattle on November 15. The producer wants to use the December live-cattle futures contracts to hedge its risk. Each contract is for the delivery of 40,000 pounds of cattle. What strategy should the beef producer follow?

7. A company wishes to hedge its exposure to a new fuel whose price changes have a 0.6 correlation with gasoline futures price changes. The company will lose $1 million for each 1 cent increase in the price per gallon of the new fuel over the next three months. The new fuel's price change has a standard deviation that is 50% greater than price changes in gasoline futures prices. Each gasoline futures contract is on 42,000 gallons. If gasoline futures are used to hedge the exposure, what should the hedge ratio be? What strategy should the company follow?

Reference Ans:

2(iii). $102.16.

3. $1.86.
4. Price lower than 150 cents per pound
5.

Day	Price	Margin Account
1	500	$2000
2	480	$2200
3	490	$2100
4	530	$1700
5	580	$1200 top up to $1500
6	520	$2100
7	490	$2400

6. The beef producer should take a long position in three December contracts closing out the position on November 15.
7. The hedge ratio is 0.9. The company should take a long position of 2143 gasoline futures contract.

ch5

Saturday, November 4, 2023 5:46 PM

ch5

Introduction to Quantitative Finance (QF1100)
Chapter 5: Options

Zhang Tengren

October 30, 2023

Contents

1 Options **2**
- 1.1 Call Options Payoffs and Profits 3
- 1.2 Put Options Payoffs and Profits 6

2 Options Trading Strategy **9**

3 Arbitrage-Based Restrictions on Option Prices **15**
- 3.1 Bounds on Call Option Prices 15
- 3.2 Bounds on Put Option Prices 17

4 Options with Different Strike Prices **19**
- 4.1 Prices of Call Options 19
- 4.2 Prices of Put Options 21

5 Put-Call Parity **23**

6 Binomial Option Pricing **26**
- 6.1 One-step Binomial Model 27

1 Options

In a forward contract, the buyer (respectively, the seller) is obligated to exchange the underlying asset at the forward price at expiration, even if the spot price of the underlying is less (respectively greater) than the forward price. Are there contracts where the buyer (respectively, the seller) is allowed to choose whether or not to buy (respectively, sell) the underlying asset at expiration?

The answer is yes! These derivatives are called options.

A *call option (respectively, put option)* is a contract that gives its holder the right to buy (respectively, sell) a specific quantity of a specified asset at a specified price (called the *strike price*) on or before a specified date (called the *expiration date* or *maturity date*) from the writer of the option. *American options* are those that can be exercised at anytime before the expiration date, while *European options* are those that can only be exercised on the expiration date. **All options are assumed to be European unless otherwise stated.**

The holder is not obligated to exercise the option. On the other hand, the writer of the contract is obligated to sell (respectively, buy) the asset at the specified price as long as the holder decides to exercise the option. The holder of the option pays the writer an up-front fee known as the *option premium* or *option price*. Unless stated otherwise, **all option prices in this chapter are based on one unit of the underlying asset**.

The holder of the contract is said to be in the *long position*, while the writer of the contract is said to be in the *short position*.

1.1 Call Options Payoffs and Profits

We will now discuss the profits of both the long and short side of call options.

Let T denote the maturity date of a call option, let S_T denote its spot price at maturity, and let K denote its strike price. The call option is said to be

- *out of the money* if $S_T < K$;
- *at the money* if $S_T = K$;
- *in the money* if $S_T > K$.

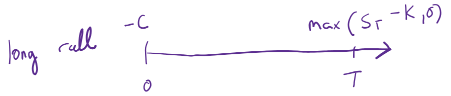
At maturity, it will be in the interest of the holder of the call option to exercise it if and only if the call option is in the money, that is $S_T > K$; if the call option is at the money or out of the money, the holder of the call option can get a better price buying the underlying asset at the spot price. Thus, the (gross) payoff on the long side of the call option is

$$\max(S_T - K, 0) = \begin{cases} 0 & \text{if } S_T < K, \\ S_T - K & \text{if } S_T \geq K. \end{cases}$$

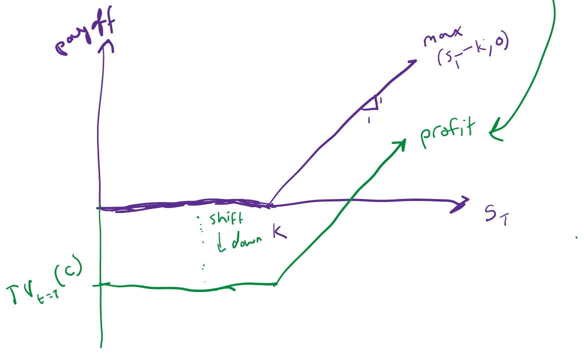
Since the holder pays a sum C for the option at time $t = 0$, the profit is defined as

$$\max(S_T - K, 0) - TV_{t=T}(C).$$

Figure 4.1 below illustrates the above payoffs.



$\text{profit} = \max(S_T - K, 0) - TV_{t=T}(C)$



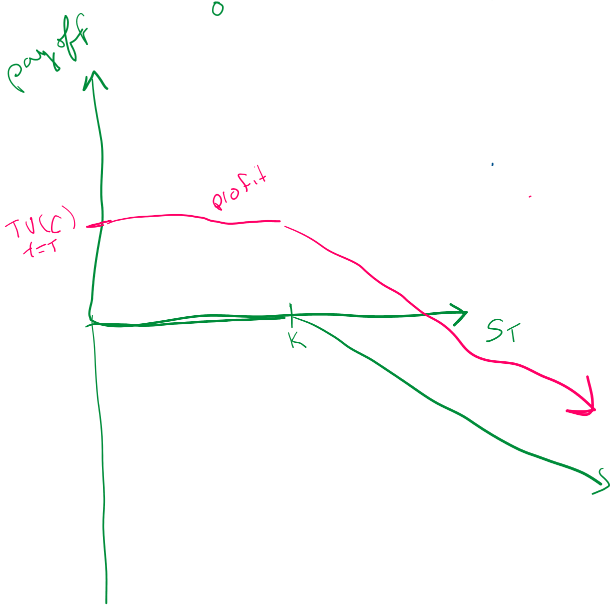
On the other hand, the (gross) payoff from the short side of a ~~put~~ call option is

$$-\max\left(S_T - K, 0\right) = \min\left(K - S_T, 0\right) = \begin{cases} 0 & \text{if } S_T < K, \\ K - S_T & \text{if } S_T \geq K. \end{cases}$$

Since the writer of the option is paid a sum C for the option, the profit is defined as

$$TV_{t=T}(C) + \min\left(K - S_T, 0\right).$$

short of call option



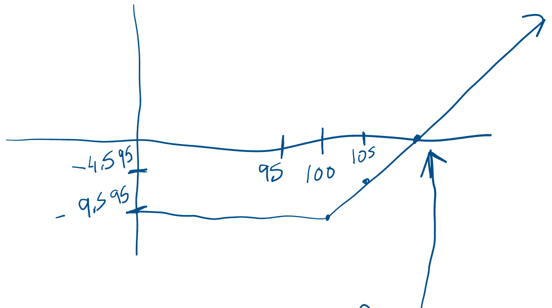
Exercise 1.1. A call option on a stock has a strike price of \$100 and expires in 6 months. The risk-free rate is 2% compounded continuously and the option premium is \$9.50. Find the profit of the option holder when the stock price at expiration is

1. \$105
2. \$95

Draw a payoff diagram for this option and determine the break-even value of S_T (the value at which profit is zero).

1.) $\text{profit} = (105-100) - 9.5 \cdot e^{0.02 \cdot \frac{1}{2}} = -4.595$

2.) $\text{profit} = (0) - 9.5e^{0.02 \cdot \frac{1}{2}} = -9.595$



break even @ $S_T = 109.595$

1.2 Put Options Payoffs and Profits

We will now discuss the profits of both the long and short side of put options. Let T denote the maturity date of a put option, let S_T denote its spot price at maturity, and let K denote its strike price. A put option is said to be

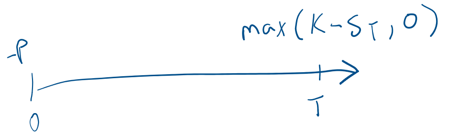
- *in-the-money* if $S_T < K$;
- *at-the-money* if $S_T = K$;
- *out-of-the-money* if $S_T > K$.

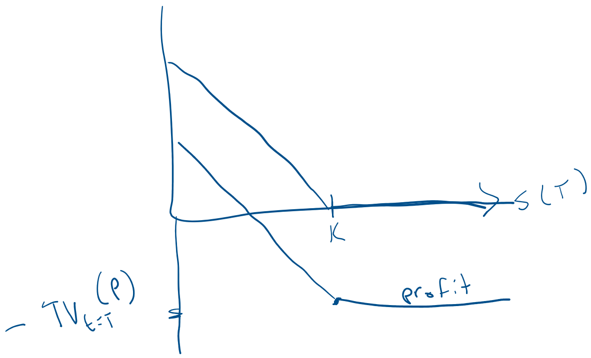
At maturity, it will be in the interest of the holder of the put option to exercise it if and only if the put option is in the money, that is $S_T < K$; if the put option is at the money or out of the money, the holder of the put option can get a better price selling the underlying asset at the spot price. Thus, the (gross) payoff on the long side of the put option is

$$\max(K - S_T, 0) = \begin{cases} K - S_T & \text{if } S_T < K, \\ 0 & \text{if } S_T \geq K. \end{cases}$$

Since the holder pays a sum P for the option at time $t = 0$, the profit is defined as

$$\max(K - S_T, 0) - TV_{t=T}(P).$$





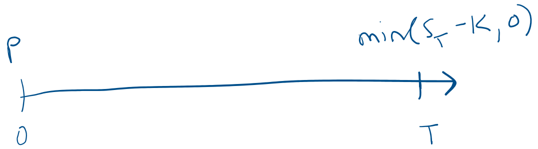
On the other hand, the (gross) payoff from the short side of a call option is

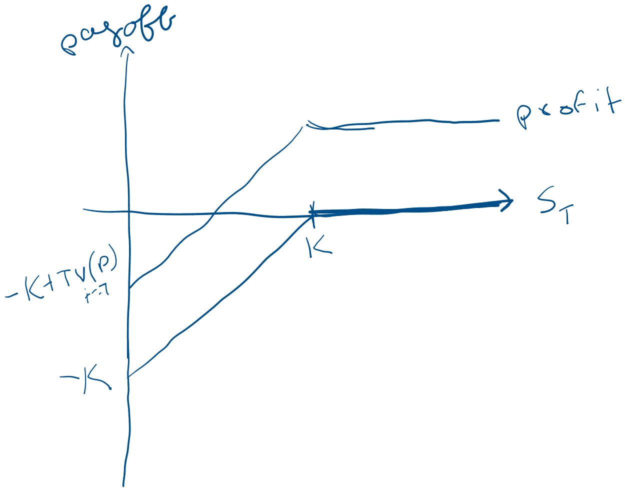
$$-\max(K - S_T, 0) = \min(S_T - K, 0) = \begin{cases} S_T - K & \text{if } S_T < K, \\ 0 & \text{if } S_T \geq K. \end{cases}$$

Since the writer of the option is paid a sum P for the option, the profit is defined as

$$TV_{t=T}(P) + \min(S_T - K, 0).$$

short put





Remark 1.2. 1. The holder of a call (put) option has the view that the price of the underlying will increase (decrease) beyond the strike price at maturity date, i.e. the holder is *bullish (bearish) on direction.*

2. The writer of a call (put) option has the view that the price of the underlying will not increase (decrease) beyond the strike price at maturity date, i.e. the holder is *not bullish (not bearish) on direction.*

3. The writer of a call option faces unlimited downside while the writer of a put option and holder of an option (call or put) faces limited downside.

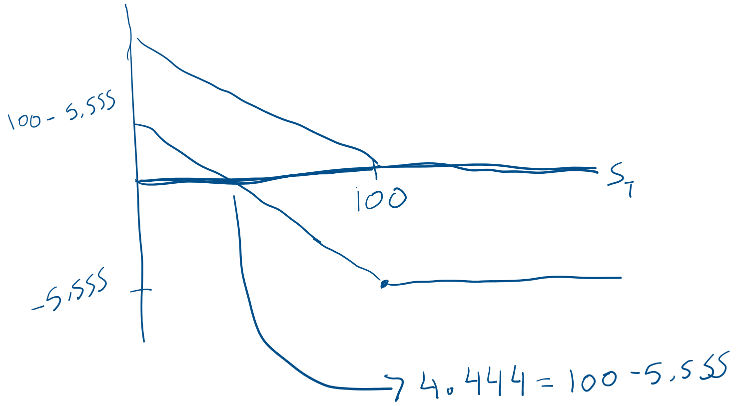
Exercise 1.3. A put option on a stock has a strike of 100 and expires in 6 months. The risk-free rate is 2% compounded continuously and the option premium is \$5.50. Find the profit of the option holder when the stock price at expiration (S_T) is

1. \$110
2. \$90

Draw a payoff diagram for this option and determine the break-even value of S_T (the value at which profit is zero).

1.) profit $= (0) - 5.5e^{0.02\cdot\frac{1}{2}} = -5.555$

2.) profit $= (100-90) - 5.50e^{0.02\cdot\frac{1}{2}} = 4.444$



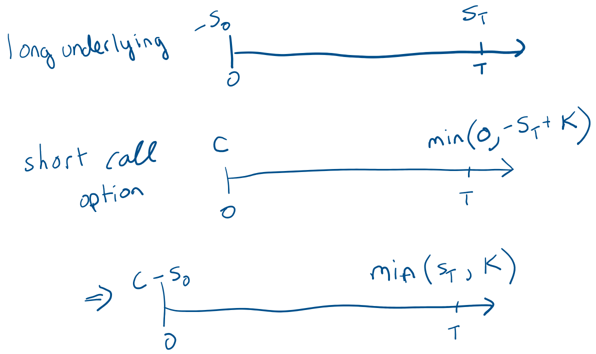
2 Options Trading Strategy

An *option trading strategy* is a portfolio made up of one or more options on a risky asset and possibly, the risky asset and cash (risk-free asset). We consider several well-known strategies and analyse their payoff structures. Throughout this section, we assume that the interest rate is $r\%$ compounded continuously.

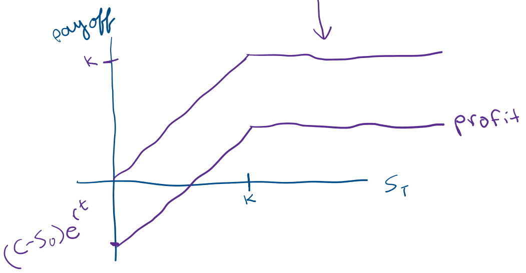
Example 2.1. (Covered Calls) A covered call consists of a long position in an underlying asset and a short position in a call option on the same asset. Determine and illustrate graphically the payoff for a covered call.

C - price of call option

K- strike price

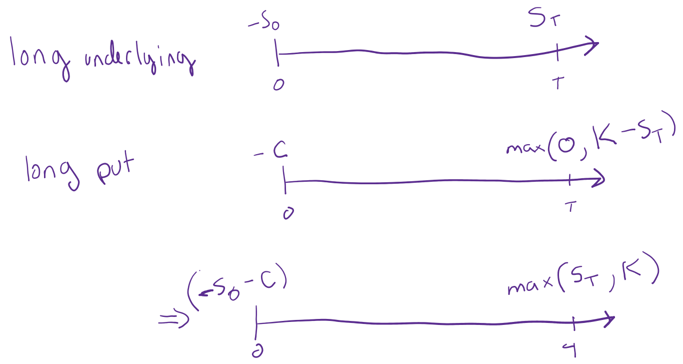


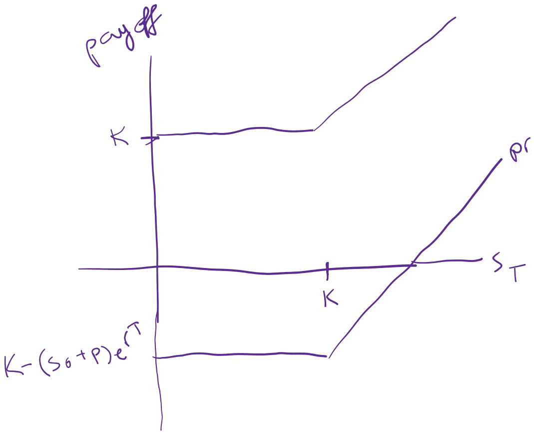
profit = $\min(S_T, K) + (C-S_0)e^{rt}$



The long position on the underlying asset "covers" the writer of the call option in the event that the option is exercised.

Exercise 2.2. (Protective Puts) A protective put consists of a long position in an underlying asset and a long position in a put option on the same asset. Determine and illustrate graphically the payoff for a protective put.





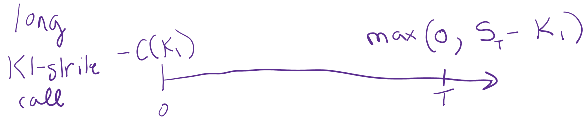
profit $= -(S_0+P)e^{rT} + \max(S_T, K)$

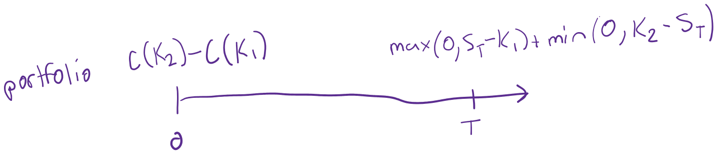
$(S_0+P)e^{rT} > K$ ← for no arbitrage

Protective puts are examples of the use of options as an insurance against the downside risk of the underlying asset.

Example 2.3. (Bull Spreads Using Calls) Consider a portfolio made up of a long position in one K_1-strike call on an underlying asset and a short position in a K_2-strike call on the same underlying asset, where $K_2 > K_1$. Determine and illustrate graphically the payoff for this strategy.

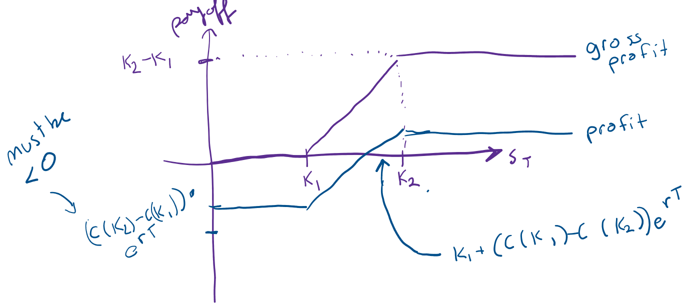
$C(K_i)$ = price of K_i strike call option





$$\max(0,-K_1+S_T)+\min(0,K_2-S_T) = \begin{cases} 0+0=0 & S_T < K_1 \\ -K_1+S_T & K_1 \leq S_T < K_2 \\ -K_1+K_2 & S_T \geq K_2 \end{cases}$$

$$\text{Profit} = \max(0,-K_1+S_T)+\min(0,K_2-S_T)+(C(K_2)-C(K_1))e^{rT}$$

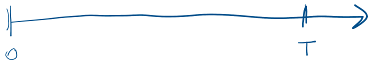


K_1 is typically chosen to be at or close to the current value of the underlying.

☆ in finals

Exercise 2.4. (Symmetric Butterfly Spreads Using Calls) Consider a portfolio made up of a long position in a K_1-strike call, a long position in a K_3-strike call and two short positions in a K_2-strike call, all on the same underlying asset, where $K_3 > K_1$ and $K_2 = \frac{1}{2}(K_1 + K_3)$. Determine and illustrate graphically the payoff for this strategy.

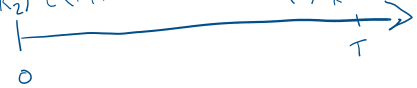
long K_1-strike call $-C(K_1)$ $\max(0, S_T - K_1)$



long K_3 strike call $-C(K_3)$ $\max(0, S_T - K_3)$

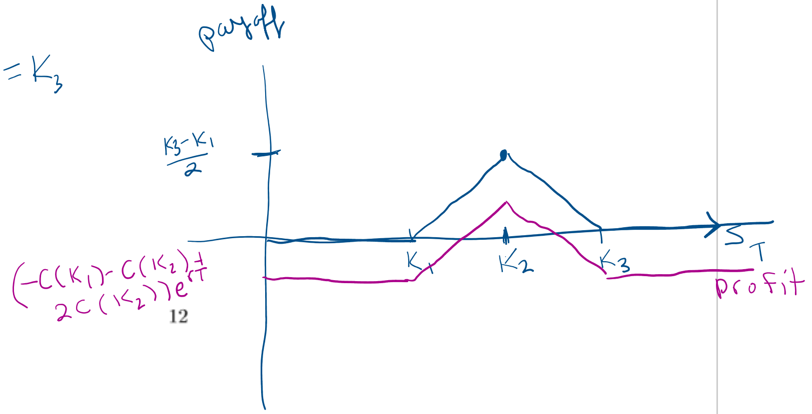
short K_2 strike call $C(K_2)$ $\min(0, K_2 - S_T)$

portfolio $2C(K_2) - C(K_1) - C(K_3)$ $\max(0, S_T - K_1) + \max(0, S_T - K_3) + 2\min(0, K_2 - S_T)$

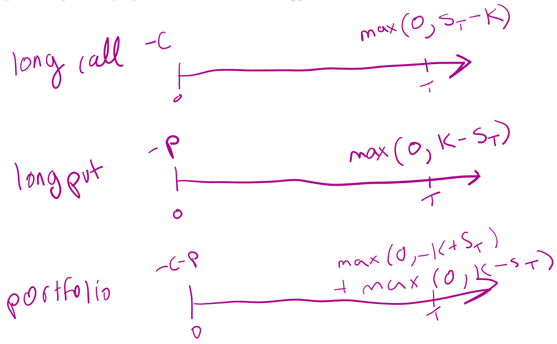


$$\text{gross payoff} = \begin{cases} 0 & K_1 > S_T \\ S_T - K_1 & K_1 < S_T < K_2 & S_T - K_1 + 0 + 0 \\ K_3 - S_T & K_2 < S_T < K_3 & S_T - K_1 + 0 + 2K_2 - 2S_T \\ 0 & S_T > K_3 & S_T - K_1 + S_T - K_3 + 2K_2 - 2S_T \end{cases}$$

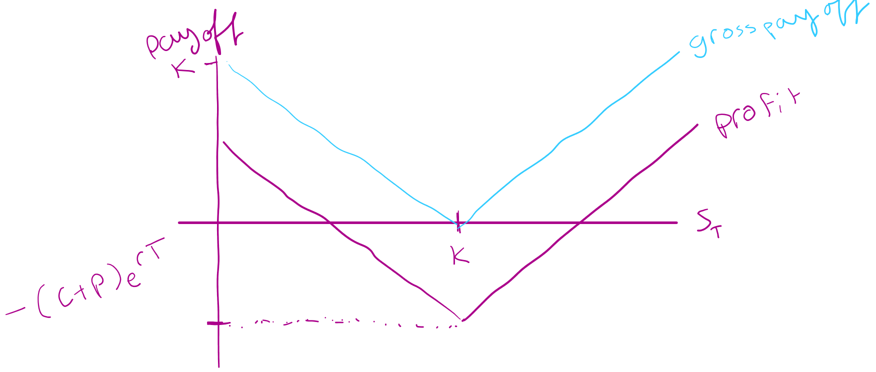
$2K_2 - K_1 = K_3$



Example 2.5. (Long Straddles) Consider a portfolio consisting of long positions in a call and a put with the same maturity and strike price. Determine and illustrate graphically the payoff for this strategy.



S_T	$\max(0, -K+S_T)$ $\max(0, K-S_T)$
$S_T \le K$	$0 + K - S_T$
$S_T > K$	$-K + S_T$



Exercise 2.6. An option trader forms a portfolio of options by taking the following positions:

Maturity date $= T$

1. a long position in three call options with strike price 50,
2. a short position in two call options with strike price 60,
3. a long position in three put options with strike price 80.

Draw the payoff table and sketch the payoff diagram for the trader's portfolio at maturity.

3 long call @ $K_1 = 50$ $\quad -3(C_{K_1})$ $\quad 3\max(0, S_T - 50)$

0 $\longrightarrow$ T

2 short call $K_2 = 60$ $\quad 2C_2$ $\quad \cancel{3}2\min(0, 60 - S_T)$

0 $\longrightarrow$ T

3 long put $K_3 = 80$ $\quad -3C_3$ $\quad 3\max(0, -S_T + 80)$

0 $\longrightarrow$ T

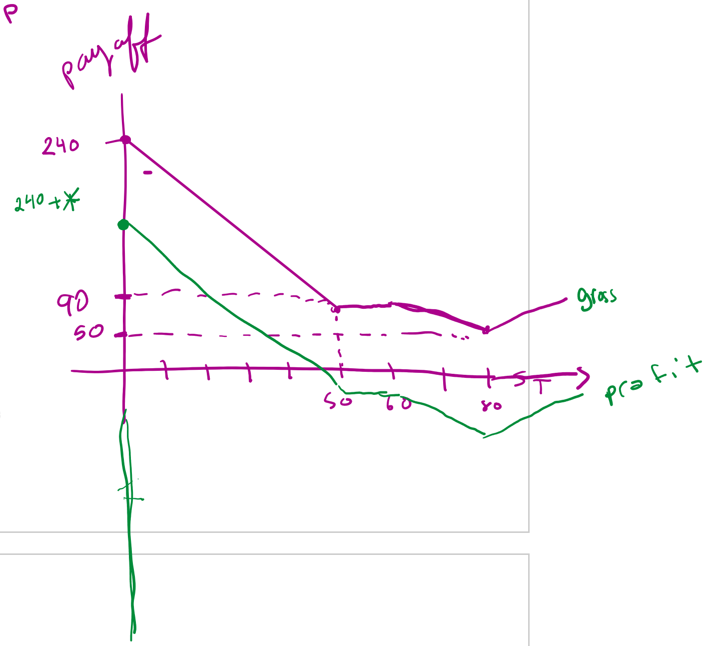
$\Rightarrow$ $-3C_1 + 2C_2 - 3C_3$ $\quad \left\{ \begin{array}{l} 3\max(0, S_T - 50) + \\ 2\min(0, 60 - S_T) + \\ 3\max(0, -S_T + 80) \end{array} \right.$

0 $\longrightarrow$ T

Gross payoff

$$\text{profit} = \text{gross payoff} + \overbrace{(-3C(50) + 2C(60) - 3P}^{*}$$

S_T	gross payoff
$S_T < 50$	$3(80 - S_T)$
$50 < S_T < 60$	$3S_T - 150 + 240 - 3S_T = 90$
$60 < S_T < 80$	$90 + 120 - 2S_T = 210 - 2S_T$
$S_T > 80$	$210 - 2S_T - (240 - 3S_T)$ $= S_T - 30$



3 Arbitrage-Based Restrictions on Option Prices

We will now use the no arbitrage assumption to find bounds on option prices.

3.1 Bounds on Call Option Prices

Suppose that we have an American call option and an European call option with the same underlying asset, same maturity, and same strike price. Let C_A be the price of

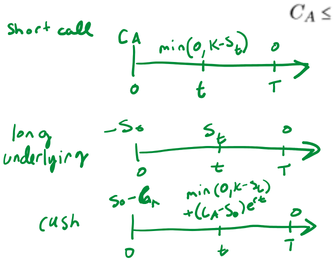
3 Arbitrage-Based Restrictions on Option Prices

We will now use the no arbitrage assumption to find bounds on option prices.

3.1 Bounds on Call Option Prices

Suppose that we have an American call option and an European call option with the same underlying asset, same maturity, and same strike price. Let C_A be the price of the American call option and C_E be the price of the European call option. Then

$$C_A \leq S_0.$$



$$\min(K, S_t) + (C_A - S_0)e^{rt} \geq 0 + (C_A - S_0)\cdot 1 = C_A - S_0$$

if $C_A - S_0 > 0$ then its arbitrage

so $C_A \leq S_0$

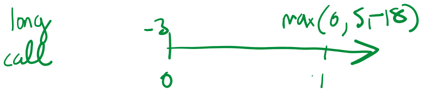
On the other hand,

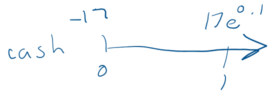
$$C_E \geq \max(0, S_0 - Ke^{-rT})$$

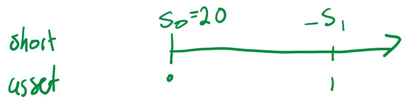
The above inequalities can be summarised as

$$\max\left(0, S_0 - Ke^{-rT}\right) \le C_E \le C_A \le S_0$$

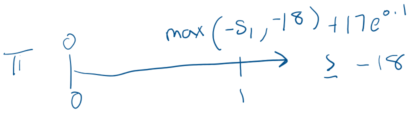
Exercise 3.1. A European call option with strike \$18 and maturity of 1 year is priced at \$3. The underlying non-dividend paying stock is currently trading at \$20. The continuously compounded interest rate is 10%. Construct an arbitrage strategy.







$\max(0, 20 - 18e^{-0.1\cdot 1}) \le 3 \le 20$



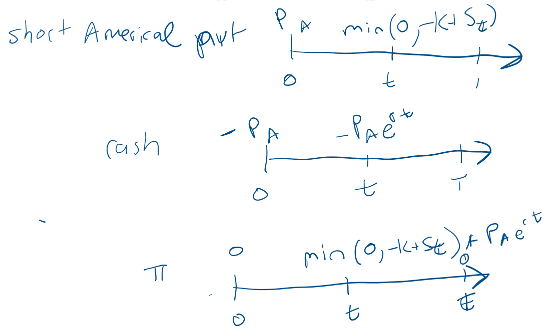
$\ge -18 + 17e^{0.1} > 0$

it is an arbitrage portfolio

3.2 Bounds on Put Option Prices

Suppose that we have an American put option and an European put option with the same underlying asset, same maturity, and same strike price. Let P_A be the price of the American put option and P_E be the price of the European put option. Then

$$P_A \leq K \quad \text{and} \quad P_E \leq Ke^{-rT}.$$



$\min(0, -K + S_t) + P_A e^{rt} \geq -K + P_A$

if $-K + P_A > 0$, then π is arbitrage portfolio

$-K + P_A \leq 0$ so $P_A \leq K$

for european

$\min(0, -K + S_T) + P_E e^{rT} \geq -K + P_E e^{rT}$

$\therefore -K + P_E e^{rT} \leq 0$

so $P_E \leq Ke^{-rT}$

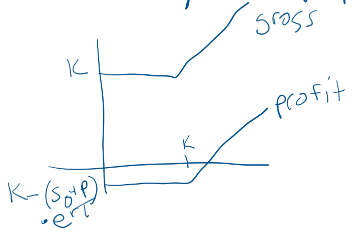
On the other hand,

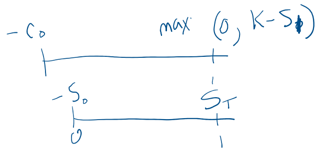
$$P_E \geq \max(Ke^{-rT} - S_0, 0).$$

Protected put

long ~~short~~ put

long underlying





No arbitrary principle

$K - (S_0 + P)e^{rT} \leq 0$

$K \leq (S_0 + P)e^{rT}$

$Ke^{-rT} \leq S_0 + P$

$P_E \geq Ke^{-rT} - S_0$

$P_E \geq \max(Ke^{-rT} - S_0, 0)$

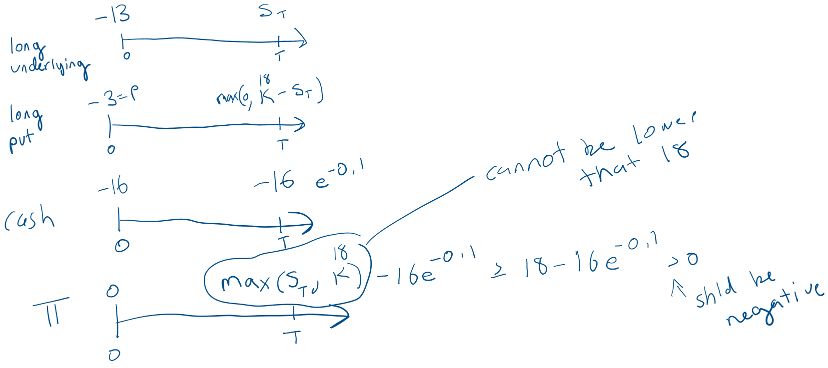
The above inequalities can be summarised as

$$\max\left(0, Ke^{-rT} - S_0\right) \le P_E \le Ke^{-rT} \quad \text{and} \quad \max\left(0, Ke^{-rT} - S_0\right) \le P_A \le K.$$

Exercise 3.2. A European put option with strike \$18 and maturity of 1 year is priced at \$3. The underlying non-dividend paying stock is currently trading at \$13. The continuously compounded interest rate is 10%. Construct an arbitrage strategy.

put option $\max(0, 18e^{-0.1} - 13) \le 3 \le 18e^{-0.1}$

$18e^{-0.1} - 13 \le 3$



∴ Π is arbitrage portfolio

4 Options with Different Strike Prices

We now compare the prices of (European) options with the same underlying asset and maturity, but different strike prices.

4.1 Prices of Call Options

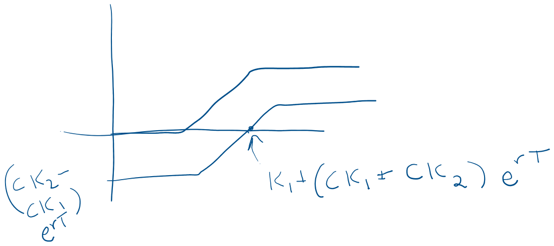
Suppose that we have two call options with the same underlying asset and maturity, but whose strike prices are $K_1 < K_2$. Let $C(K_1)$ and $C(K_2)$ be the price of the call option with strike price K_1 and K_2 respecitvely. Then,

$$C(K_2) \leq C(K_1).$$

must

$(C(K_2) - C(K_1))e^{rt} \geq 0$

$K_2 - K_1 + (C(K_2) - C(K_1))e^{rt} > 0$



On the other hand,

$$C(K_1) - C(K_2) \leq e^{-rT}(K_2 - K_1).$$

The above inequalities can be summarised as

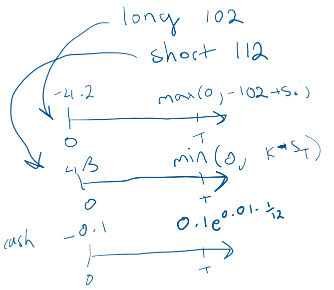
$$0 \le C(K_1) - C(K_2) \le e^{-rT}(K_2 - K_1).$$

Exercise 4.1. The current price of a stock is 100. A one-month European call option with a strike price of \$102 costs \$4.20 and a one-month European call option with a strike price of \$112 costs ~~\$14.30~~ 4.3. The one-month nominal rate of interest is 1%. Construct an arbitrage strategy.

$K_1 = 102 \qquad K_2 = 112$

$$\underbrace{0 \le 4.2 - 4.3}_{\text{lower bound violated}} \le (112-102)e^{-0.01 \cdot 1/12}$$

lower bound violated → long smaller cost
short higher cost



$$\underbrace{\max(0, -102 + S_{1/12}) + \min(0, 112 - S_{1/12})} + \underbrace{0.1e^{0.01 \cdot 1/12}}$$

$S_{1/12}$	gross payoff
$S_{1/12} < 102$	0
$102 < S_{1/12} < 112$	$-102 + S_{1/12} \ge 0$
$S_{1/12} > 112$	10

$\Rightarrow$ profit is $\ge 0.10e^{0.01 \cdot 1/12} > 0$

Π is arbitrage portfolio

4.2 Prices of Put Options

Suppose that we have two put options with the same underlying asset and maturity, but whose strike prices are $K_1 < K_2$. Let $P(K_1)$ and $P(K_2)$ be the price of the put option with strike price K_1 and K_2 respectively. Then,

$$P(K_2) \geq P(K_1).$$

On the other hand,

$$P(K_2) - P(K_1) \leq e^{-rT}(K_2 - K_1).$$

The above inequalities can be summarised as

$$0 \leq P(K_2) - P(K_1) \leq e^{-rT}(K_2 - K_1).$$

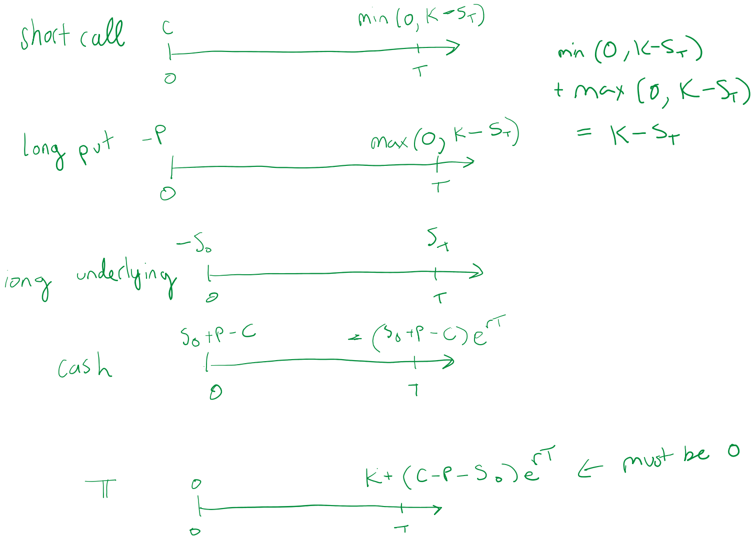
Exercise 4.2. Suppose two put options have the same underlying asset and maturity, but with strike prices $K_1 < K_2$. Let $P(K_1)$ and $P(K_2)$ denote the price of the call option with with strike price K_1 and K_2 respectively. Use a no-arbitrage argument to show that, for any $\alpha \in (0,1)$,

$$P(\alpha K_1 + (1-\alpha)K_2) \leq \alpha P(K_1) + (1-\alpha)P(K_2).$$

5 Put-Call Parity

One important equation connecting the value of (European) call and put options written on the same non-dividend paying asset and having the same exercise price and maturity is the following **put-call parity** equation.

$$C + Ke^{-rT} = P + S_0 \tag{5.1}$$



$$\min(0, K-S_T) + \max(0, K-S_T) = K-S_T$$

$$K+(C-P-S_0)e^{rT} = 0$$

$$Ke^{-rT} + C - P - S_0 = 0$$

✓ $$Ke^{-rT} + C = P + S_0$$

Example 5.2. The current price of a stock is 31. A three-month European call with a strike of 30 costs $3 and a three-month European put with a strike of 30 costs $2.25. The risk-free interest rate is 10% compounded continuously. Construct an arbitrage strategy.

$C+Ke^{-rT}=P+S_0$

$3+30e^{-0.1\cdot 1/4} \boxed{=} 2.25+31$ ×

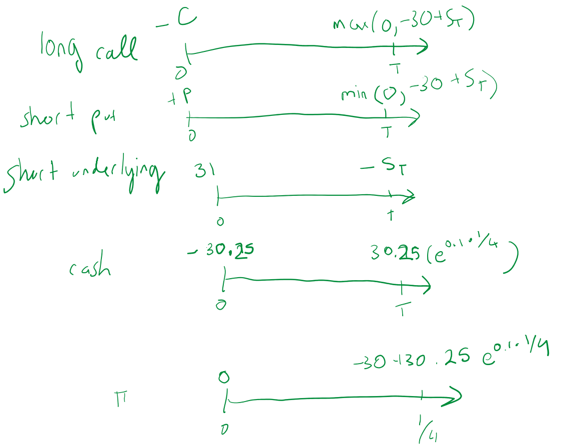
$<$

$P+S_0 > C+Ke^{-rT}$

$0 > (C-P-S_0)+Ke^{-rT}$

$0 > (C-P-S_0)e^{rT}+K$

so shorting results in arbitrage



> 0 should be $=0$

so Π is arbitrage portfolio

Exercise 5.3. Use put–call parity to show that the cost of a symmetric butterfly spread created from put options is identical to the cost of a symmetric butterfly spread created from call options.

6 Binomial Option Pricing

The Binomial Model is a model used to price options.

Suppose that an option on an underlying asset is valid during a fixed time interval $[0, T]$. Divide $[0, T]$ into N sub-intervals. Set $t_0 := 0$ and for each $i = 1, 2, \ldots, N$, let $t_i := t_{i-1} + \delta$, i.e.

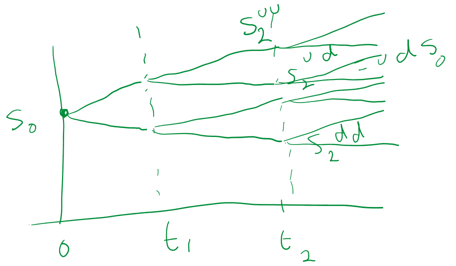
$$[t_0, t_1], \quad [t_1, t_2], \quad \ldots, \quad [t_{N-1}, t_N]$$

are the subintervals that we divide $[0, T]$ into. Let S_t be the asset price at time t (S_0 is a number that is determined, while S_t is a random variable for all $t > 0$).

In the binomial model, we assume that there are constants d and u with $0 < d < 1 < u$ such that for every $i = 1, \ldots, N$,

$$S_i = \begin{cases} S_i^u := uS_{i-1} & \text{with probability } p; \\ S_i^d := dS_{i-1} & \text{with probability } 1 - p. \end{cases}$$

More informally, this says that whenever the price changes, it either increases by a factor of u, or it decreases by a factor of d, and the probability that these events occur are the same at each t_i.



We also assume that the continuously compounded risk-free rate r satisfies

$$d < e^{r(t_i - t_{i-1})} < u$$

for all i. The above inequality ensures there are no arbitrage opportunities.

6.1 One-step Binomial Model

Suppose that we have an option whose price we want to calculate using the one-step binomial model, i.e. $N = 1$ and the option expires at t_1. Then the spot price S_1 of the underlying asset at t_1 has two possible values, $S_1^u := uS_0$ and $S_1^d := dS_0$. Notice that at t_1, the price (equivalently, the payoff) F_1 of the option has two possible values:

$$F_1^u = \begin{cases} \max(uS_0 - K, 0) & \text{if option is a call;} \\ \max(K - uS_0, 0) & \text{if option is a put.} \end{cases}$$

and

$$F_1^d = \begin{cases} \max(dS_0 - K, 0) & \text{if option is a call;} \\ \max(K - dS_0, 0) & \text{if option is a put.} \end{cases}$$

To price this option, we construct a *replicating portfolio*, namely a portfolio whose end-of-period value is the same as that of the option, consisting of

- Δ units of the underlying asset,
- an amount B invested in the risk-free asset.

Specifically, we want to choose Δ and B such that

$$\begin{cases} \Delta uS_0 + Be^{rt_1} = F_1^u; \\ \Delta dS_0 + Be^{rt_1} = F_1^d. \end{cases}$$

From the above equations, we get

$$\Delta = \frac{F_1^u - F_1^d}{S_0(u-d)} \quad \text{and} \quad B = \frac{uF_1^d - dF_1^u}{e^{rt_1}(u-d)}.$$

Based on the no-arbitrage principle, the initial value of the portfolio must be equal to that of the option. Hence,

$$\begin{aligned} F_0 &= \Delta S_0 + B \\ &= \frac{F_1^u - F_1^d}{u-d} + \frac{uF_1^d - dF_1^u}{e^{rt_1}(u-d)} \\ &= e^{-rt_1}\left[\frac{e^{rt_1} - d}{u-d}F_1^u + \frac{u - e^{rt_1}}{u-d}F_1^d\right]. \end{aligned}$$

Thus, if we let $q := \frac{e^{rt_1}-d}{u-d}$ then

$$F_0 = e^{-rt_1}\left[qF_1^u + (1-q)F_1^d\right].$$

We call q the *risk-neutral probability.*

Remark 6.1. 1. The above option pricing formula does not involve the spot price S_0, nor the probability p for which $S_1 = S_1^u$.

2. It is easy to verify that

$$qu + (1-q)d = e^{rt_1}.$$

Thus, it follows that

$$S_0 = e^{-rt_1}(qu + (1-q)d)S_0 = e^{-rt_1}(qS_1^u + (1-q)S_1^d).$$

Exercise 6.2. A stock is currently traded at \$20. At the end of three months, the price will be either \$22 or \$18. Take the annual risk-free rate to be 12% compounded continuously. Find the value of a 3-month call option with strike price of \$21. Find also the replicating portfolio.

$S_0 = 20$

$S_1^u = 22$

$$F_1^u = \max(uS_0 - K, 0) = \max(22-21, 0) = 1$$

$u = \frac{22}{20} = \frac{11}{10}$

$$F_1^d = \max(dS_0 - K, 0) = \max(18-21, 0) = 0$$

$d = \frac{18}{20} = \frac{9}{10}$

$$q = \frac{e^{rt_1} - d}{u-d} = \frac{e^{0.12\cdot 1/4} - 9/10}{\frac{11}{20} - \frac{9}{10}} = \frac{10e^{0.03} - 9}{2}$$

$$F_0 = e^{-0.12\cdot 1/4}(q\cdot 1 + (1-q)\cdot 0) = 0.633$$

$$\Delta = \frac{F_1^u - F_1^d}{S_0(u-d)} = \frac{1}{20\left(\frac{11}{10} - \frac{9}{10}\right)}$$

$$B = \frac{11/10 \cdot 0 - \frac{9}{10}\cdot 1}{e^{0.12\cdot 1/4}\left(\frac{11}{10} - \frac{9}{10}\right)} = -\frac{9}{2e^{0.03}}$$

Buy $\frac{1}{4}$ of stock

borrow $\frac{9}{2e^{0.03}}$ dollars

Two-step Binomial Model

Here, $N = 2$ and the option expires at t_2.

As before, at t_1, the spot price of the underlying asset can take two possible values, $S_1^u := uS_0$ and $S_1^d := dS_0$. Similarly, at t_2, the spot price of the underlying asset can take four possible values, $S_2^{uu} := u^2S_0$, $S_2^{du} = S_2^{ud} := udS_0$, and $S_2^{dd} := d^2S_0$.

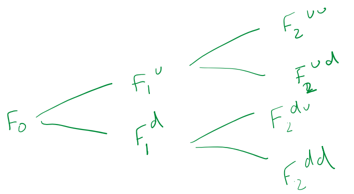
On the other hand, at t_1, the price F_1 of the option has two possible values, F_1^u and F_1^d, while at t_2, the price (equivalently, the payoff) F_2 of the option has four possible values:

$$F_2^{uu} := \begin{cases} \max(u^2S_0 - K, 0) & \text{if option is a call;} \\ \max(K - u^2S_0, 0) & \text{if option is a put,} \end{cases}$$

$$F_2^{ud} = F_2^{du} := \begin{cases} \max(udS_0 - K, 0) & \text{if option is a call;} \\ \max(K - udS_0, 0) & \text{if option is a put,} \end{cases}$$

and

$$F_2^{dd} := \begin{cases} \max(d^2S_0 - K, 0) & \text{if option is a call;} \\ \max(K - d^2S_0, 0) & \text{if option is a put.} \end{cases}$$



Using the same method that we did in the one step binomial model, by constructing an appropriate replicating portfolio, we may compute F_1^u and F_1^d. In the situation when we are at F_1^u, let the replicating portfolio for period 2 consist of Δ^u units of the underlying asset and B^u dollars invested the risk free asset. In the situation when we are at F_1^d, let the replicating portfolio for period 2 consist of Δ^d units of the underlying asset and B^d dollars invested the risk free asset.

Applying the same argument as we used in the one-step binomial model, but to period 2, we have

$$\begin{cases} \Delta^u u S_1^u + B^u e^{r(t_2-t_1)} = F_2^{uu}; \\ \Delta^u d S_1^u + B^u e^{r(t_2-t_1)} = F_2^{ud}. \end{cases}$$

and

$$\begin{cases} \Delta^d u S_1^d + B^d e^{r(t_2-t_1)} = F_2^{du}; \\ \Delta^d d S_1^d + B^d e^{r(t_2-t_1)} = F_2^{dd}. \end{cases}$$

which we can then solve to obtain

$$\Delta^u = \frac{F_2^{uu} - F_2^{ud}}{S_1^u(u-d)}, \quad B^u = \frac{u \cdot F_2^{ud} - d \cdot F_2^{uu}}{e^{r(t_2-t_1)}(u-d)},$$

$$\Delta^d = \frac{F_2^{du} - F_2^{dd}}{S_1^d(u-d)}, \quad B^d = \frac{u \cdot F_2^{dd} - d \cdot F_2^{du}}{e^{r(t_2-t_1)}(u-d)}.$$

These in turn give

$$F_1^u = e^{-r(t_2-t_1)}\left[q_2 F_2^{uu} + (1-q_2)F_2^{ud}\right] \quad \text{and} \quad F_1^d = e^{-r(t_2-t_1)}\left[q_2 F_2^{du} + (1-q_2)F_2^{dd}\right]$$

where $q_2 := \frac{e^{r(t_2-t_1)}-d}{u-d}$.

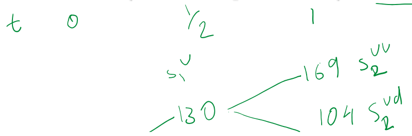
Applying the one-step binomial model once more to period 1, we have

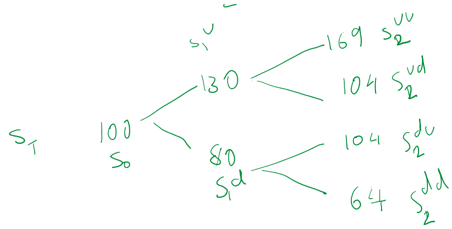
$$F_0 = e^{-rt_1}\left[q_1 F_1^u + (1-q_1)F_1^d\right],$$

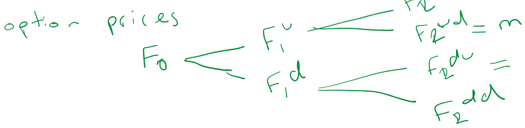
where $q_1 := \frac{e^{rt_1}-d}{u-d}$. Hence,

$$F_0 = e^{-rt_2}\left[q_1 q_2 F_2^{uu} + q_1(1-q_2)F_2^{ud} + q_2(1-q_1)F_2^{du} + (1-q_1)(1-q_2)F_2^{dd}\right].$$

Example 6.3. A stock is currently trading at \$100. At the end of 6 months, the price will either be up by 30% or down by 20%. The risk-free rate is 8% compounded continuously. Find the price of a 1-year call option with strike price \$95.







$F_2^{uu} = \max(0, -95+169) = 74$

$F_2^{ud} = \max(0, -95+104) = 9$

$F_2^{du} = \max(0, -95+104) = 9$

$F_2^{dd} = \max(0, -95+64) = 0$

$$F_0 = e^{-0.08(1)}\left[q^2 F_2^{uu} + 2q(1-q)F_2^{ud} + (1-q)^2 F_2^{dd}\right]$$

$$q_1 = \frac{e^{0.08\cdot\frac{1}{2}} - 0.8}{1.3-0.8} \qquad q_2 = \frac{e^{0.08\cdot\frac{1}{2}} - 0.8}{1.3-0.8}$$

$q_1 =$

$$F_0 = e^{-0.08}\left[(0.48162155)^2 \cdot 74 + 2q(1-q)\cdot 9\right]$$

$$= \boxed{19.994}$$

QF1100-Tutorial10

Wednesday, November 8, 2023 11:06 PM

QF1100-Tutorial10

NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 10**

1. Suppose that it is now January 15. A copper fabricator knows it will require 100,000 pounds of copper on May 15 to meet a certain contract. The spot price of copper is 340 cents per pound, and the futures price for May delivery is 320 cents per pound. The fabricator can hedge its position by taking a **long** position in four futures contracts (with expiry date years in the future) offered by the CME Group and closing its position on May 15. Each contract is for the delivery of 25,000 pounds of copper. Assume that interest rates are 0%.
   (a) Suppose that the future price of copper on May 15 proves to be 325 cents per pound. How much does the fabricator gain/loss on the futures contracts?
   (b) Suppose that the future price of copper on May 15 proves to be 305 cents per pound. How much does the fabricator gain/loss on the futures contracts?
   (c) Explain (by finding the cash flow) how the copper fabricator's hedging strategy has the effect of locking in the price of the required copper at close to 320 cents per pound.

✓ 2. A European option has a gross payoff function

$$\max(K_2 - \max(K_1, S_T), 0)$$

where $0 < K_1 < K_2$. Construct the payoff table and plot the gross payoff diagram for this option.

✓ 3. A put option on a stock has a strike of 100 and expires in 3 months. The risk-free rate is 1% compounded continuously and the option premium is \$7.89. Find the profit of the option
   (a) writer/seller
   (b) holder/buyer

if the stock price at expiration is (i) \$105 (ii) \$80. Draw the profit diagram for each of the cases (a) and (b).

1

copper: 0 at 0; $-S(5)$ at 5

$= 5$

a. futures: 0 at 0; $F(5)-F(0)$ at 5

$0.05(25000)(4) = \boxed{5000}$

b. $-0.15(25000)(4) = \boxed{-15000}$

c. payoff $= F(5) - F(0) - S(5)$

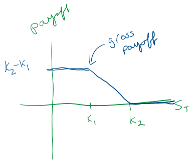
$-100000\, S(\tfrac{1}{3}) + F(\tfrac{1}{3}) \cdot 100000 - F(0) \cdot 100000$

$= 100000(F(\tfrac{1}{3}) - S(\tfrac{1}{3})) - 320000$

if $F(\tfrac{1}{3})$ is very close to $S(\tfrac{1}{3})$ then payoff is close to 320000

2.) a.

$$\begin{cases} K_2 - K_1 & \text{when } S_T < K_1 \\ K_2 - S_T & K_1 < S_T < K_2 \\ 0 & K_2 < S_T \end{cases}$$



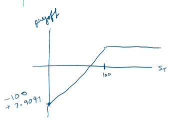
3.) a. writer: C at 0; $\min(0, -K+S_3)$ at 3

i. $S_3 = 105$ not excersized

profit $= 7.89 \cdot e^{0.01 \cdot 3/12} = 7.9097$

ii. $S_3 = 80$ excersized

profit $= 7.89e^{0.01 \cdot 1/4} + (-100+80) = -12.09$



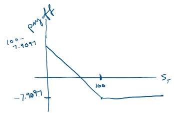
b. holder: $-C$ at 0; $\max(0, -S_3+K)$ at 3

i. $S_3 = 105$ not excersized

profit $= -7.89e^{0.01 \cdot 1/4} = 7.9097$

ii. $S_3 = 80$ excersized

profit $= -7.89e^{0.01 \cdot 1/4} + (-80+100) = 27.9097$



✓ 4. (**Bull spread using puts**) Let $0 < K_1 < K_2$. Consider a portfolio made up of a long position in one K_1-strike put and a short position in one K_2-strike put on the same underlying asset, both with maturity T years from now. Suppose that the continuously compounded nominal interest rate is r. Draw the payoff table and sketch the profit diagram for this strategy.

✓ 5. (**Symmetric butterfly spreads using puts**) Let $K_3 > K_1$ and let $K_2 := (K_1 + K_3)/2$. Consider a portfolio made up of a long position in a K_1-strike put, a long position in a K_3-strike put and two short positions in a K_2-strike put, all on the same underlying asset and with the same maturity T years from now. Suppose that the continuously compounded nominal interest rate is r. Draw the payoff table and sketch the profit diagram for this strategy.

6. (**Short Straddles**) Consider a portfolio consisting of short positions in a call and a put with the same underlying asset, same strike price K, and same maturity T years from now. Suppose that the continuously compounded nominal interest rate is r. Draw the payoff table and sketch the profit diagram for this strategy.

7. Sketch the gross profit diagram for each of the following portfolios.
   (a) A short position in one share of a stock and a short position in two K-call options whose underlying asset is the stock;
   (b) A long position in two shares of a stock and a long position in three K-put options whose underlying asset is the stock.

Assume that the calls and puts expire T years from now, and the continuously compounded nominal interest rate is r.

✓ 8. Let $K > 30$. The diagram below shows part of the gross payoff diagram for a portfolio created by writing m put options with strike price 20, holding n put options with strike price 30, and holding 2 call options with strike price K, all of which have the same expiry date T and same underlying asset. Let S_t denote the spot price of the underlying asset at time t.
   (i) Find the values of m, n and K.
   (ii) Sketch the payoff diagram for $20 < S_T < 50$.

← if $S_T < 20$

long n put @ K=30: 0; $n \max(0, 30 - S_T)$ at T

long 2 call @ K: 0; $2\max(0, -K+S_T)$ at T

short m put @ K=20: $m \cdot \min(0, -20+S_T)$

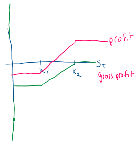
120 6

4.) K_1 long put: $-C_1$ at 0; $\max(0, K_1 - S_T)$ at T

short K_2 put: C_2 at 0; $\min(0, S_T - K_2)$ at T

gross profit: $\max(0, K_1 - S_T) + \min(0, S_T - K_2)$

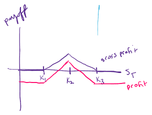
$S_T < K_1$	$K_1 - S_T + S_T - K_2 = K_1 - K_2$
$K_1 < S_T < K_2$	$0 + S_T - K_2$
$K_2 < S_T$	$0 + 0$



5.) long K_1 put: $-C$ at 0; $\max(0, -S_T+K_1)$ at T

long K_3 put: $-C$ at 0; $\max(0, -S_T+K_3)$ at T

2 short K_2 put: C at 0; $2\min(0, S_T - K_2)$ at T



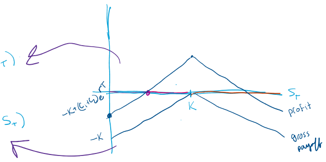
$S_T < K_1$	$-S_T + K_1 - S_T + K_3 + 2S_T - 2K_2 = K_1 + K_3 - 2K_2 = 0$
$K_1 < S_T < K_2$	$0 + -S_T + K_3 + 2S_T - 2K_2 = K_3 + S_T - 2K_2 = S_T - K_1$
$K_2 < S_T < K_3$	$0 + -S_T + K_3 + 0 = K_3 - S_T$
$K_3 < S_T$	$0 + 0 + 0 = 0$

6.) a. short call: C_1 at 0; $\min(0, K - S_T)$ at T

short put: C_2 at 0; $\min(0, -K+S_T)$ at T

if $S_T < K$ 0 +

$S_T > K$



S_T	gross payoff
$S_T < K$	$0 + -K + S_T$
$S_T \geq K$	$K - S_T + 0$

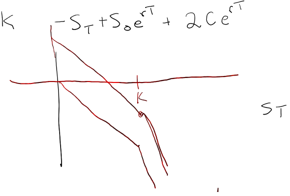
7.) a. short shares: S_0 at 0; $-S_T$ at T

Short 2 K-call: 2C at 0; $2\min(0, -K+S_T)$ at T

profit

$S_T < K$ $-S_T + S_0e^{rT} + 2(-K+S_T) + 2Ce^{rT}$

$S_T > K$ $-S_T + S_0e^{rT} + 2Ce^{rT}$

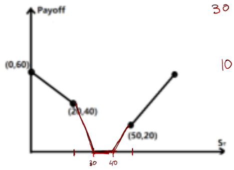


b. long 2 shares: $-2S_0$; $2S_T$

long 2 call@ K $\underset{0}{|}\xrightarrow{2\max(0,-K+S_T)}\underset{T}{|}$

short m put @ K=20 $\underset{0}{|}\xrightarrow{m\cdot\min(0,-20+S_T)}\underset{T}{|}$

120 6



at $S_T = 0$ $4\cdot 30 + m\cdot(-20) = 60$

30 + 20

at $S_T = 20$

$10n + 0$ $\boxed{n=4 \quad m=3}$

$S_T = 50$

$-2K + 100 = 20$

$\boxed{K=40}$

$20 < S_T < 30$ $4(30-S_T) = 120 - 4S_T$

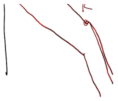
$30 < S_T \cancel{<} 40$ 0

$40 < S_T < 50$ $2(-40+S_T) = -80 + 2S_T$

Reference Ans:

1. (a) Gains $5,000 on the futures contract.
   (b) Loss $15,000 on the futures contract.
2. (a) (i) 7.90975, (ii) −12.09025
   (b) (i) −7.90975, (ii) 12.09025
8. (i) $n = 4, m = 3, K = 40$.

K - call 0

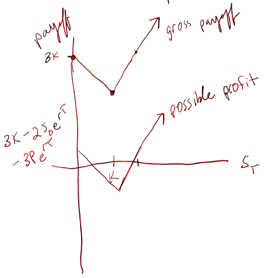


S_T

D. long 2 shares $\underset{0}{\overset{-2S_0}{|}}\xrightarrow{\hspace{3em}}\underset{T}{\overset{2S_T}{|}}$

long 3 K put $\underset{0}{\overset{-3C}{|}}\xrightarrow{\hspace{3em}}\underset{T}{\overset{3\max(0,K-S_T)}{|}}$

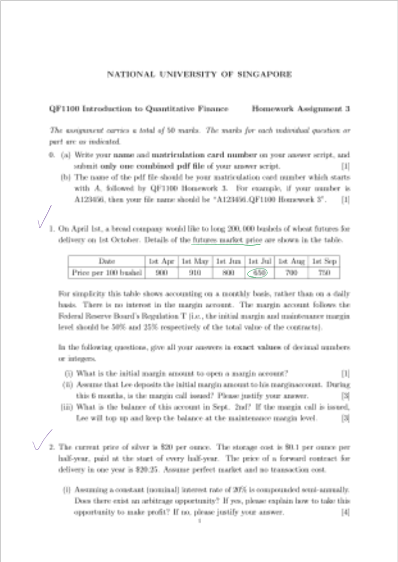
S_T	gross payoff
$S_T \le K$	$3K - S_T$
$S_T > K$	$2S_T$



QF1100-Assignment 3

Monday, November 13, 2023 6:14 PM

QF1100-Assignment 3



NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Homework Assignment 3**

The assignment carries a total of 50 marks. The marks for each individual question or part are as indicated.

0. (a) Write your **name** and **matriculation card number** on your answer script, and submit **only one combined pdf file** of your answer script. [1]
   (b) The name of the pdf file should be your matriculation card number which starts with A, followed by QF1100 Homework 3. For example, if your number is A123456, then your file name should be "A123456 QF1100 Homework 3". [1]

✓

1. On April 1st, a bread company would like to long 200,000 bushels of wheat futures for delivery on 1st October. Details of the futures market price are shown in the table.

Date	1st Apr	1st May	1st Jun	1st Jul	1st Aug	1st Sep
Price per 100 bushel	900	910	800	650	700	750

For simplicity this table shows accounting on a monthly basis, rather than on a daily basis. There is no interest in the margin account. The margin account follows the Federal Reserve Board's Regulation T (i.e., the initial margin and maintenance margin level should be 50% and 25% respectively of the total value of the contracts).

In the following questions, give all your answers in exact values of decimal numbers or integers.

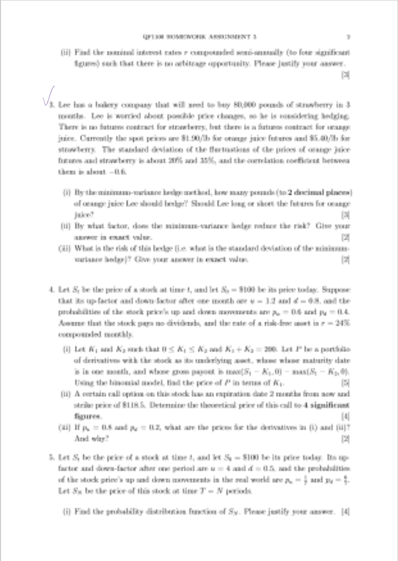
(i) What is the initial margin amount to open a margin account? [1]
(ii) Assume that Lee deposits the initial margin amount to his marginaccount. During this 6 months, is the margin call issued? Please justify your answer. [3]
(iii) What is the balance of this account in Sept. 2nd? If the margin call is issued, Lee will top up and keep the balance at the maintenance margin level. [3]

✓

2. The current price of silver is \$20 per ounce. The storage cost is \$0.1 per ounce per half-year, paid at the start of every half-year. The price of a forward contract for delivery in one year is \$20.25. Assume perfect market and no transaction cost.

(i) Assuming a constant (nominal) interest rate of 20% is compounded semi-annually. Does there exist an arbitrage opportunity? If yes, please explain how to take this opportunity to make profit? If no, please justify your answer. [4]

1



(ii) Find the nominal interest rates r compounded semi-annually (to four significant figures) such that there is no arbitrage opportunity. Please justify your answer. [3]

✓

3. Lee has a bakery company that will need to buy 80,000 pounds of strawberry in 3 months. Lee is worried about possible price changes, so he is considering hedging. There is no futures contract for strawberry, but there is a futures contract for orange juice. Currently the spot prices are \$1.90/lb for orange juice futures and \$5.40/lb for strawberry. The standard deviation of the fluctuations of the prices of orange juice futures and strawberry is about 20% and 35%, and the correlation coefficient between them is about −0.6.

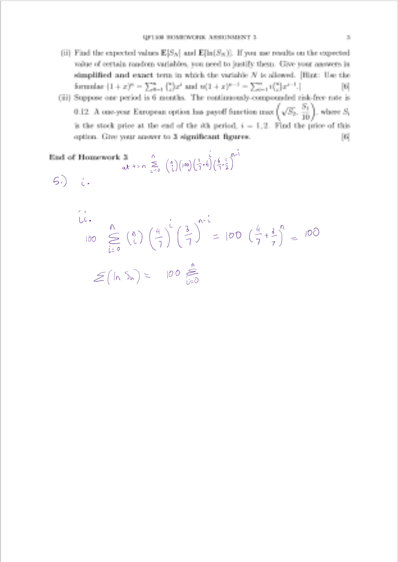
(i) By the minimum-variance hedge method, how many pounds (to **2 decimal places**) of orange juice Lee should hedge? Should Lee long or short the futures for orange juice? [3]
(ii) By what factor, does the minimum-variance hedge reduce the risk? Give your answer in exact value. [2]
(iii) What is the risk of this hedge (i.e. what is the standard deviation of the minimum-variance hedge)? Give your answer in exact value. [2]

4. Let S_t be the price of a stock at time t, and let $S_0 = \$100$ be its price today. Suppose that its up-factor and down-factor after one month are $u = 1.2$ and $d = 0.8$, and the probabilities of the stock price's up and down movements are $p_u = 0.6$ and $p_d = 0.4$. Assume that the stock pays no dividends, and the rate of a risk-free asset is $r = 24\%$ compounded monthly.

(i) Let K_1 and K_2 such that $0 \le K_1 \le K_2$ and $K_1 + K_2 = 200$. Let P be a portfolio of derivatives with the stock as its underlying asset, whose whose maturity date is in one month, and whose gross payout is $\max(S_1 - K_1, 0) - \max(S_1 - K_2, 0)$. Using the binomial model, find the price of P in terms of K_1. [5]
(ii) A certain call option on this stock has an expiration date 2 months from now and strike price of \$118.5. Determine the theoretical price of this call to **4 significant figures**. [4]
(iii) If $p_u = 0.8$ and $p_d = 0.2$, what are the prices for the derivatives in (i) and (ii)? And why? [2]

5. Let S_t be the price of a stock at time t, and let $S_0 = \$100$ be its price today. Its up-factor and down-factor after one period are $u = 4$ and $d = 0.5$, and the probabilities of the stock price's up and down movements in the real world are $p_u = \frac{1}{7}$ and $p_d = \frac{6}{7}$. Let S_N be the price of this stock at time $T = N$ periods.

(i) Find the probability distribution function of S_N. Please justify your answer. [4]



(ii) Find the expected values $\mathbf{E}[S_N]$ and $\mathbf{E}[\ln(S_N)]$. If you use results on the expected value of certain random variables, you need to justify them. Give your answers in **simplified and exact** term in which the variable N is allowed. [Hint: Use the formulae $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$ and $n(1+x)^{n-1} = \sum_{i=1}^{n} i\binom{n}{i} x^{i-1}$.] [6]
(iii) Suppose one period is 6 months. The continuously-compounded risk-free rate is 0.12. A one-year European option has payoff function $\max\left(\sqrt{S_2}, \frac{S_1}{10}\right)$, where S_i is the stock price at the end of the ith period, $i = 1, 2$. Find the price of this option. Give your answer to **3 significant figures**. [6]

End of Homework 3

at t = n $\sum_{i=0}^{n} \binom{n}{i}(100)\left(\frac{1}{7}\cdot 4\right)^{i}\left(\frac{6}{7}\cdot\frac{1}{2}\right)^{n-i}$

5.) i.

ii.

$$100 \sum_{i=0}^{n} \binom{n}{i}\left(\frac{4}{7}\right)^{i}\left(\frac{3}{7}\right)^{n-i} = 100\left(\frac{4}{7}+\frac{3}{7}\right)^{n} = 100$$

$$E(\ln S_n) = 100 \sum_{i=0}^{n}$$

1.)

i. $900(2000) \cdot 0.5 = \boxed{900000}$

ii. $900000 + 2000(650 - 900) = 400{,}000 <$ ↲

$900(2000) \cdot 0.25 = 450000$ ← maintence margin

margin call is issued

$900000 + 2000(x - 900) < 450000$

$x < 675$

iii. at July 1

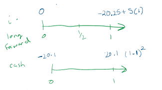
$900000 + 2000(650 - 900)$

$+ 50000$

at sept 2

$950000 + 2000(750 - 900) = \boxed{650000}$

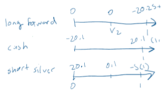
2.) i.



pay off = $-20.25 + 20.1(1.1)^2 - 0.1(1.1) > 0$

so arbitrage opportunity

ii.



$F_0 = S_0\left(1+\frac{r}{2}\right)^{2T} + 0.1\left(1+\frac{r}{2}\right) + 0.1$

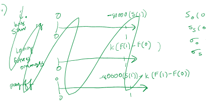
$20.25 = 20.1\left(1+\frac{r}{2}\right)^2 + \cancel{0.1\left(1+\frac{r}{2}\right)^2} + 0.1\left(1+\frac{r}{2}\right)$

$r = 0.00248$

0.2480% compounded semi annually

2

3.) i.



$S_o(0) = 1.9$

$S_s(0) = 5.4$

$\sigma_o = 0.2$

$\sigma_s = 0.35$

$\rho_{o,s} = -0.6$

$h = \frac{\sigma_S(T)}{\sigma_O(T)}\rho = \frac{0.35}{0.2} \cdot -0.6 = -1.05 = \frac{k'}{k} = \frac{k}{80000}$

long orange futures → $k = -84000$

ii. risk of MVH $= k\sigma_s\sqrt{1-\rho^2}$

$= 80000 \cdot (0.35)\sqrt{1-0.6^2}$ → iii = 22400

risk unhedged $= 0.35 \cdot 80000$

risk reduced factor $\frac{0.35 - 0.35\sqrt{1-0.6^2}}{0.35} = \boxed{0.2}$

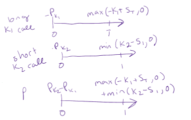
$K_1 + K_2 = 200$

if $120 < K_1$ ✗

$K_1 < 120$

$K_2 < 120$

4.) i. $-\max(S_1 - K_2, 0) = \min(K_2 - S_1, 0)$



$q = 0.6 \quad (1-q) = 0.4$

$k_1 < 80 \| \quad F_{0_1} = \left(0.6(120-K_1) + 0.4(80-K_1)\right)\left(1+\frac{0.24}{12}\right)^{-1}$

$100 > k_1 > 80 \quad F_{0_2} = \left(0.6(200-2K_1) + 0.4(120-K_1)\right)\left(1+\frac{0.24}{12}\right)^{-1}$

$F_{0_1} = (72 - 0.6K_1 + 32 - 0.4K_1)$

$F_{0_2} = (120 - 1.2K_1)$

$F_1^u = \max(-K_1 + 120, 0) + \min(K_2 - 120, 0)$ 20

$= -K_1 + 120$ if $K_1 < 80$ $120 - K_1$

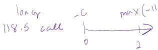
$-K_1 + K_2$ if $100 > K_1 > 80$ $-K_1 + (200 - K_1) = 200 - 2K_1$ 2

$F_1^d = \max(-K_1 + 80, 0) + \min(\overset{120-K_1}{K_2} - 80, 0)$

$= -K_1 + 80$ $K_1 < 80$ $80 - K_1$ $K_1 = 70$ $K_2 = 130$

~~K_2~~ 80 0 $K_1 > 80$ $120 - K_1$ $K_1 = 90$ $K_2 = 110$

ii.



$S_2^{uu} = 144$

$S_2^{ud} = S_2^{du} = 96$

$S_2^{dd} = 64$

$F_2^{uu} = \max(-118.5 + 144, 0) = 25.5$

$F_2^{du} = F_2^{ud} = \max(-118.5 + 96, 0) = 0$

$F_2^{dd} = \max(-118.5 + 64, 0) = 0$

$F_0 = \left((0.55)^2 \cdot (25.5)\right)\left(1+\frac{0.24}{12}\right)^{-2} =$

iii.

$F_{0_1} = \left(0.8(120-K_1) + 0.2(80-K_1)\right)(1.02)^{-1}$

$= \boxed{\frac{(112 + 1.6K_1)}{1.02}}$

$F_0 = \left((0.8)^2(25.5)\right)\left(1+\right.$

$= \boxed{15.686}$

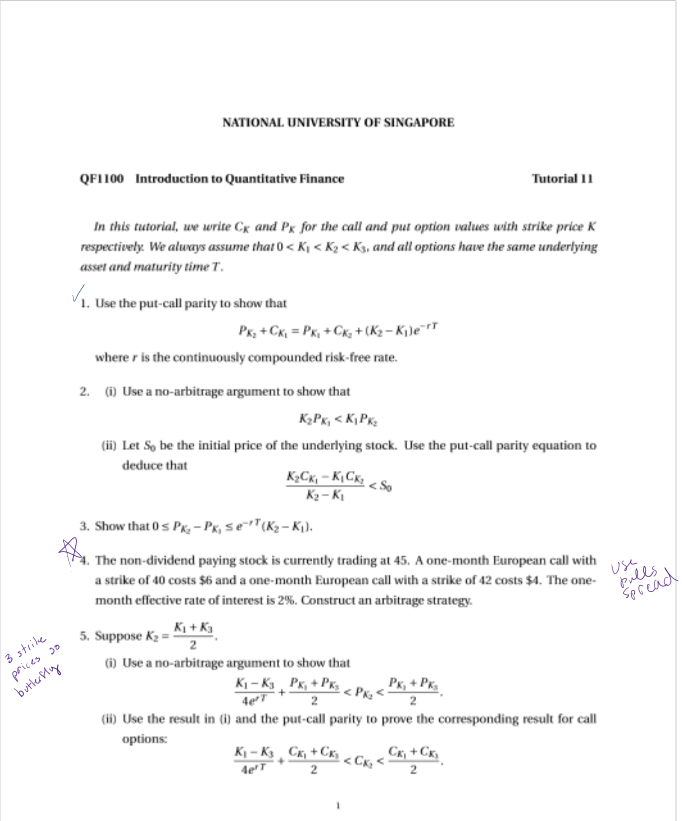
$$\frac{0.24}{12}\Big)^{-2}$$

$$\boxed{27}$$

QF1100-Tutorial11

Wednesday, November 15, 2023 6:28 PM

QF1100-Tutorial11



NATIONAL UNIVERSITY OF SINGAPORE

QF1100 Introduction to Quantitative Finance **Tutorial 11**

In this tutorial, we write C_K and P_K for the call and put option values with strike price K respectively. We always assume that $0 < K_1 < K_2 < K_3$, and all options have the same underlying asset and maturity time T.

✓ 1. Use the put-call parity to show that

$$P_{K_2} + C_{K_1} = P_{K_1} + C_{K_2} + (K_2 - K_1)e^{-rT}$$

where r is the continuously compounded risk-free rate.

2. (i) Use a no-arbitrage argument to show that

$$K_2 P_{K_1} < K_1 P_{K_2}$$

(ii) Let S_0 be the initial price of the underlying stock. Use the put-call parity equation to deduce that

$$\frac{K_2 C_{K_1} - K_1 C_{K_2}}{K_2 - K_1} < S_0$$

3. Show that $0 \le P_{K_2} - P_{K_1} \le e^{-rT}(K_2 - K_1)$.

☆ 4. The non-dividend paying stock is currently trading at 45. A one-month European call with a strike of 40 costs \$6 and a one-month European call with a strike of 42 costs \$4. The one-month effective rate of interest is 2%. Construct an arbitrage strategy.

Use bulls spread

5. Suppose $K_2 = \frac{K_1 + K_3}{2}$.

3 strike prices so butterfly

(i) Use a no-arbitrage argument to show that

$$\frac{K_1 - K_3}{4e^{rT}} + \frac{P_{K_1} + P_{K_3}}{2} < P_{K_2} < \frac{P_{K_1} + P_{K_3}}{2}.$$

(ii) Use the result in (i) and the put-call parity to prove the corresponding result for call options:

$$\frac{K_1 - K_3}{4e^{rT}} + \frac{C_{K_1} + C_{K_3}}{2} < C_{K_2} < \frac{C_{K_1} + C_{K_3}}{2}.$$

1

1.) $C + Ke^{-rT} = P + S_0$

$C_{K_1} + K_1 e^{-rT} = P_{K_1} + S_0$ $\quad C_{K_2} + K_2 e^{-rT} = P_{K_2} + S_0$

$C_{K_1} + K_1 e^{-rT} - P_{K_1} = C_{K_2} + K_2 e^{-rT} - P_{K_2}$

$C_{K_1} + P_{K_2} = (K_2 - K_1)e^{-rT} + C_{K_2} + P_{K_1}$

$S_T < K_1$ $\quad K_1 - S_T + -K_2 + S_T$

$K_1 < S_T < K_2$ $\quad -K_2 + S_T$

$K_2 > K_1$

long put $-P_1$ $\xrightarrow{\hspace{2em}}$ $\max(0, K_1 - S_T)$; $K_2 < S_T$

short put P_2 $\xrightarrow{\hspace{2em}}$ $\min(0, -K_2 + S_T)$

$\max(0, K_1 - S_T) + \min(0, -K_2 + S_T) - P_1 + P_2 < 0$

$K_2 P_{K_1}$

$K_1 P_{K_2}$

2.) i. show $K_2 P_{K_1} < K_1 P_{K_2} \Rightarrow K_2 P_{K_1} - K_1 P_{K_2} < 0$

long K_2 K_1-put: 0: $-P_{K_1} \cdot K_2$ $\quad T$: $K_2 \cdot \max(0, K_1 - S_T)$

short K_2 K_1 puts: 0: $P_{K_1} \cdot K_2$ $\quad T$: $-K_2 \max(0, K_1 - S_T)$

long K_1 K_2-put: 0: $-P_{K_2} \cdot K_1$ $\quad T$: $K_1 \max(0, K_2 - S_T)$

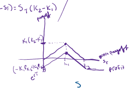
cash: 0: $K_1 P_{K_2} - K_2 P_{K_1}$ $\quad T$: $(-K_1 P_{K_2} + K_2 P_{K_1})e^{rT}$

No arbitrage principle

$(-K_1 P_{K_2} + K_2 P_{K_1})e^{rT} < 0$

$$\boxed{K_2 P_{K_1} < K_1 P_{K_2}}$$

S_T	gross payoff
$S_T <= K_1$	$-K_2(K_1 - S_T) + K_1(K_2 - S_T) = S_T(K_2 - K_1)$
$K_1 < S_T \le K_2$	$0 + K_1(K_2 - S_T)$
$K_2 < S_T$	$0 + 0 = 0$



ii. show $\dfrac{K_2 C_{K_1} - K_1 C_{K_2}}{K_2 - K_1} < S_0$

put-call parity

1 $\quad (C_{K_1} + K_1 e^{-rT} = P_{K_1} + S_0) \cdot K_2 \Rightarrow K_2 C_{K_1} + K_1 K_2 e^{-rT}$

2 $\quad (C_{K_2} + K_2 e^{-rT} = P_{K_2} + S_0) \cdot K_1 \Rightarrow$

$K_2 \text{①} - K_2 \text{②}$

$K_2 C_{K_1} - K_1 C_{K_2} + K_2 K_1 e^{-rT} - K_1 K_2 e^{-rT}$

$= K_2 P_{K_1} - K_1 P_{K_2} + K_2 S_0 - K_1 S_0$

$< K_2 S_0 - K_1 S_0$

$\Rightarrow \dfrac{K_2 C_{K_1} - K_1 C_{K_2}}{K_2 - K_1} < S_0$

3.) P_{K_2}: $0 \to T$: $\min(0, K_1 +$ $\quad$ $-P_{K_1}$: $0 \to$ $\max(0, K_2 - S_T)$

show $0 \le P_{K_2} - P_{K_1} \le e^{-rT}(K_2 - K_1)$

long K_1 put

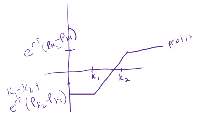
short K_2 put

No arbitrage principle

$\Rightarrow e^{rT}(P_{K_2} - K_1) > 0$ and $K_1 - K_2 + e^{rT}(P_{K_2} - P_{K_1}) < 0$

$0 < P_{K_2} - P_{K_1}$ $\quad P_{K_2} - P_{K_1} < e^{-rT}(K_2 - K_1)$

$$\boxed{0 < P_{K_2} + P_{K_1} < e^{-rT}(K_2 - K_1)}$$



4.) Stock @ 45

call @ K=40 ~~\$6~~

@ K= 42 \$4

short underlying: 0: 45 $\quad T$: $-S(T)$

long call: 0: -4 $\quad T$: $-40 + S(T)$

0: -41 $\quad T$: $41e^{0.02 \cdot 1/12}$

0: 0 $\quad 1/12$: $-40 + 41e^{0.02 \cdot 1/12} > 0$

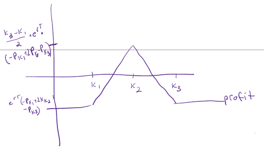
$0 \le C_{K_1} - C_{K_2} \le (1.02)(K_2 - K_1)$

5.) i. long K_1 put: 0: $-P_{K_1}$ $\quad T$: $\max(0, K_1 - S_T)$

short $2K_2$ puts: 0: $+2P_{K_2}$ $\quad T$: $2\min(0, -K_2 + S_T)$

long K_3 put: 0: $-P_{K_3}$ $\quad T$: $\max(0, K_3 - S_T)$

cash: 0: $P_{K_1} + P_{K_3} - 2P_{K_2}$ $\quad T$: $(P_{K_1} + P_{K_3} - 2P_{K_2})e^{rT}$



No arbitrage principle

$\dfrac{K_3 - K_1}{2} + e^{rT}(-P_{K_1} + 2P_{K_2} - P_{K_3}) > 0$ and $e^{rT}(-P_{K_1} + 2P_{K_2} - P_{K_3}) < 0$

$P_{K_2} > \dfrac{\left(\frac{K_1 - K_3}{2}e^{-rT} + P_{K_1} + P_{K_3}\right)}{2}$ $\quad P_{K_2} < \dfrac{P_{K_1} + P_{K_2}}{2}$

$P_{K_2} > \dfrac{K_1 - K_3}{4e^{rT}} + \dfrac{P_{K_1} + P_{K_2}}{2}$

ii. show $\dfrac{K_1 - K_3}{4e^{rT}} + \dfrac{C_{K_1} + C_{K_3}}{2} < C_{K_2} < \dfrac{C_{K_1} + C_{K_3}}{2}$

$\dfrac{K_1 - K}{4e^{rT}} < C_{K_2} - \dfrac{C_{K_1} + C_{K_3}}{2} < 0$

put call parity

$C_{K_1} + K_1 e^{-rT} = P_{K_1} + S_0$ ①

$C_{K_2} + K_2 e^{-rT} = P_{K_2} + S_0$ ②

$C_{K_3} + K_3 e^{-rT} = P_{K_3} + S_0$ ③

② $- \dfrac{\text{①} + \text{③}}{2}$

$C_{K_2} - \dfrac{C_{K_1} + C_3}{2} + \overbrace{K_2 e^{-rT} - \dfrac{K_1 e^{-rT} + K_3 e^{-rT}}{2}}^{=0} = P_{K_2} - \dfrac{P_{K_1} + P_{K_3}}{2} + \cancel{S_0 - \dfrac{S_0 + S_0}{2}}^{0}$

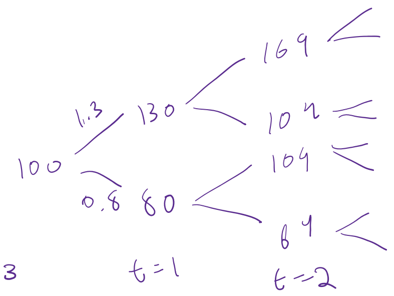
$C_{K_2} - \dfrac{C_{K_1} + C_{K_3}}{2} = P_{K_2} - \dfrac{P_{K_1} + P_{K_3}}{2}$

use 5 i.

Friday, November 17, 2023 6:50 PM

effective 5% interest

8 values



replicating portfolio

Δ^{uu} underlying

B^{uu} cash

$\Delta^{uu} = 169(1.3) + B^{uu}(1.05) = -170 + 169 \cdot 1.3$

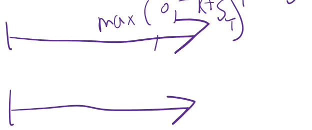
$\Delta^{du} = 169 \cdot 0.8 + B^{uu}(1.05) = 0$

$\Delta^{uu} \cdot 169 + B^{uu} = F_2^{uu}$

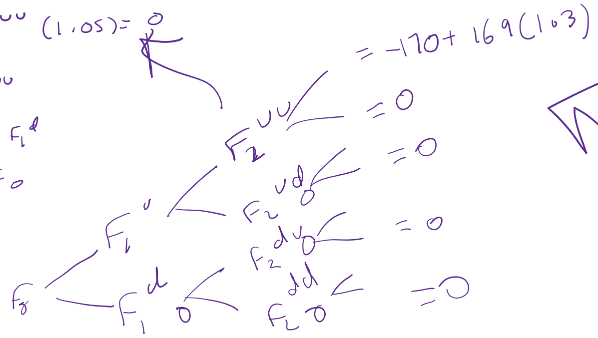
do same for F_1^u F_1^d

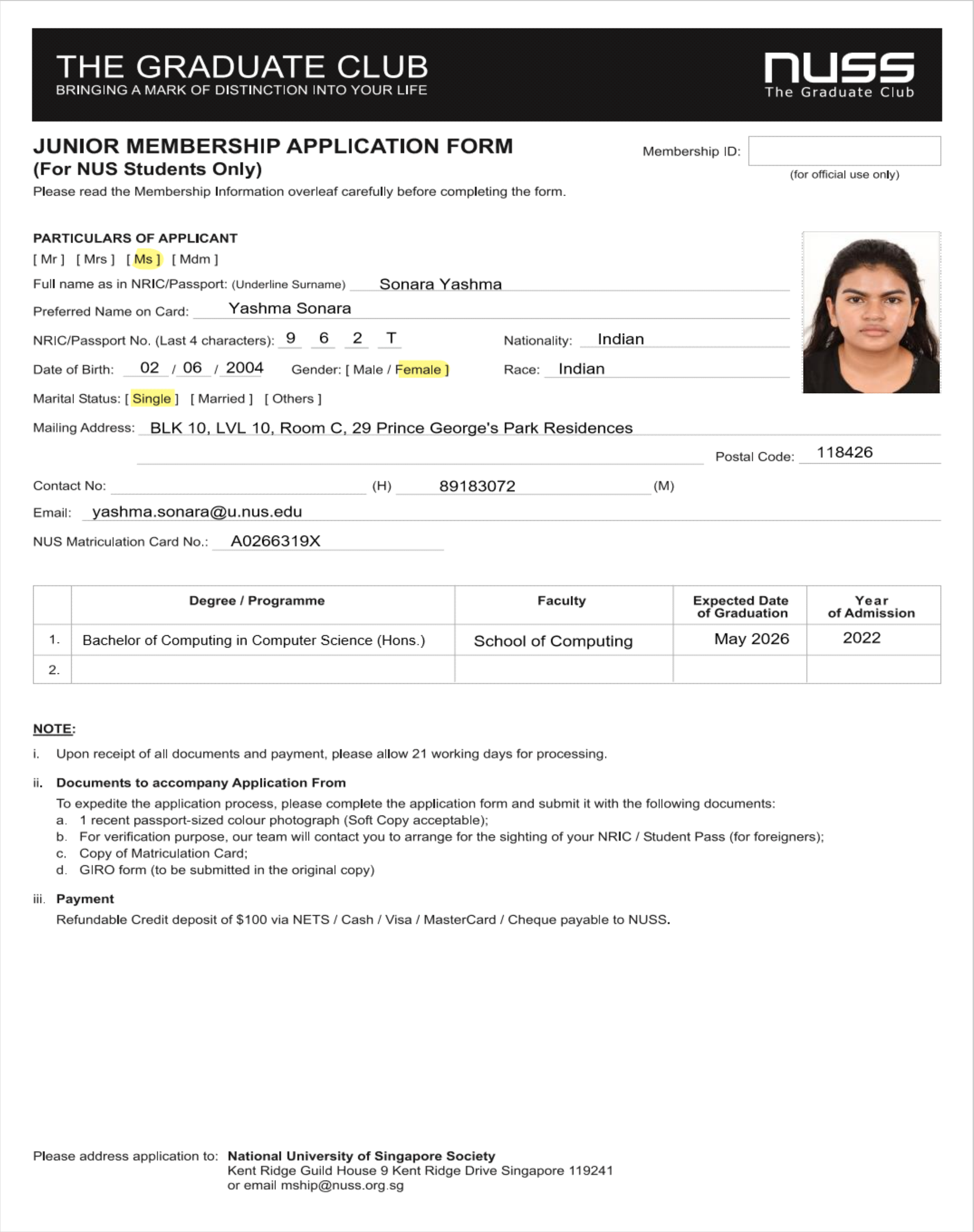
and for F_0

call optio $K = 170$



$F_1 = \max(0, -170 + S_T)$





THE GRADUATE CLUB
BRINGING A MARK OF DISTINCTION INTO YOUR LIFE

nuss
The Graduate Club

JUNIOR MEMBERSHIP APPLICATION FORM
(For NUS Students Only)

Membership ID: ______ (for official use only)

Please read the Membership Information overleaf carefully before completing the form.

PARTICULARS OF APPLICANT

[Mr] [Mrs] [✓ Ms] [Mdm]

Full name as in NRIC/Passport: (Underline Surname) Sonara Yashma

Preferred Name on Card: Yashma Sonara

NRIC/Passport No. (Last 4 characters): 9 6 2 T   Nationality: Indian

Date of Birth: 02 / 06 / 2004   Gender: [Male / [✓ Female]]   Race: Indian

Marital Status: [✓ Single] [Married] [Others]

Mailing Address: BLK 10, LVL 10, Room C, 29 Prince George's Park Residences

Postal Code: 118426

Contact No: ______ (H) 89183072 (M)

Email: yashma.sonara@u.nus.edu

NUS Matriculation Card No.: A0266319X

	Degree / Programme	Faculty	Expected Date of Graduation	Year of Admission
1.	Bachelor of Computing in Computer Science (Hons.)	School of Computing	May 2026	2022
2.				

NOTE:

i. Upon receipt of all documents and payment, please allow 21 working days for processing.

ii. **Documents to accompany Application From**

To expedite the application process, please complete the application form and submit it with the following documents:

a. 1 recent passport-sized colour photograph (Soft Copy acceptable);
b. For verification purpose, our team will contact you to arrange for the sighting of your NRIC / Student Pass (for foreigners);
c. Copy of Matriculation Card;
d. GIRO form (to be submitted in the original copy)

iii. **Payment**

Refundable Credit deposit of $100 via NETS / Cash / Visa / MasterCard / Cheque payable to NUSS.

Please address application to: **National University of Singapore Society**
Kent Ridge Guild House 9 Kent Ridge Drive Singapore 119241
or email mship@nuss.org.sg

Junior Membership Information (For NUS Students Only)

1. **Eligibility**

Type Of Membership	Age (Years Old)	Monthly Subscription Fee	Validity	Access to Facilities
NUS Undergraduate/ students pursuing graduate coursework programme in NUS	During Course Duration	$45*	Upon graduation of the degree course taken	All except age-specified outlets.

a. Membership is non-transferable.

2. **Rules and Regulations**

a. Junior members are required to comply with the rules and regulations as laid down in the Society's Constitution & Regulations.
b. There will be no constitutional rights and privileges for Junior members.
c. Junior members are allowed to sign in **2 guests per visit**.
d. Junior members are allowed to sign in **1 guest per visit** at the fitness centre during off-peak period (Monday to Friday: 10am to 5pm & 8pm to 10pm) and subject to prevailing guest fee.

3. **Monthly Subscription Fee**

a. NUSS reserves the right to change the monthly subscription fee and impose certain terms and conditions as it deems fit from time to time.
b. The monthly subscription fee billing commences once the application is approved.
c. The monthly subscription fee of $45* may be used to offset the entrance fee should the undergraduate member chooses to convert the Junior membership to an Ordinary membership (only for undergraduates pursuing their first degree or postgraduate degrees provided such postgraduate degrees commence within twelve(12) months after obtaining the first degree and subsequent full-time degree).
d. In the event that the subscription fees paid exceed the entrance fee, the surplus will be converted to F&B credits and credited into the member's account. Offsetting subscription fees against entrance fee is only allowed within 3 months upon degree conferment date.
e. The subscription fees are non-refundable if the undergraduate does not take up the offer to convert his/her Junior membership to Ordinary membership within 3 months upon degree conferment date.
f. Members joining NUSS at a concessionary entrance fee shall not be permitted to transfer their membership within 3 years from the date of joining.

DECLARATION BY APPLICANT

- I, the applicant, understand the above listed and declare that the particulars in this application are correct.
- For reinstatement: I agree that a fee of $20* applies.
- I understand that the monthly subscription fee is **non-refundable**.
- I am personally liable for the payment of the membership dues and charges incurred by me and my guests in the use of the Society's facilities.
- I agree to receive emails and other publicity materials from the Society and may choose to unsubscribe upon receipt of such emails.
- I agree to comply with and be bound by the Constitution, Rules and Regulations of the Society, as may from time to time be in force.
- I acknowledge that a 6-month lock-in period is required under the current promotion. A $45* fee will be charged for early termination/resignation of the Junior Membership.

Signature of Applicant: ____________________

Date: 18/11/2023

How do you get to know of NUSS Junior Membership (For NUS Students only)

☐ NUSS website
☐ NUS events, please state: ____________________
☐ NUSS Social Media
☐ Through friends
☐ Others, please state: ____________________

**Subject to prevailing GST and subject to change with prior notice. All information is correct at the time of print (February 2023).*

Please address application to: **National University of Singapore Society**
Kent Ridge Guild House 9 Kent Ridge Drive Singapore 119241
or email mship@nuss.org.sg