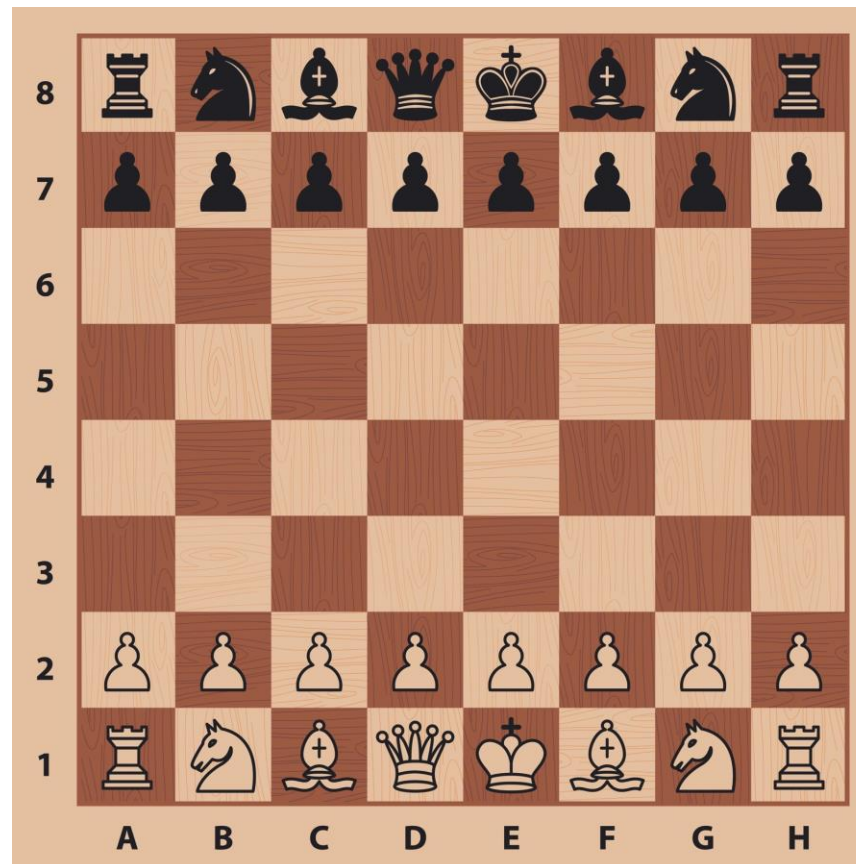


Artificial Intelligence

Minimax Search and Alpha-beta Pruning



Adversarial Search

- Adversarial search problems: games
- They occur in multiagent competitive environments
- There is an **opponent** we can't control: planning against us!
- Game vs. search: optimal solution is not a sequence of actions but a **strategy** (policy): If opponent does a, agent does b, else if opponent does c, agent does d, etc.
- Tedious and fragile if hard-coded (i.e., implemented with rules)
- Good news: Games are modeled as **search problems** and use **heuristic evaluation** functions.

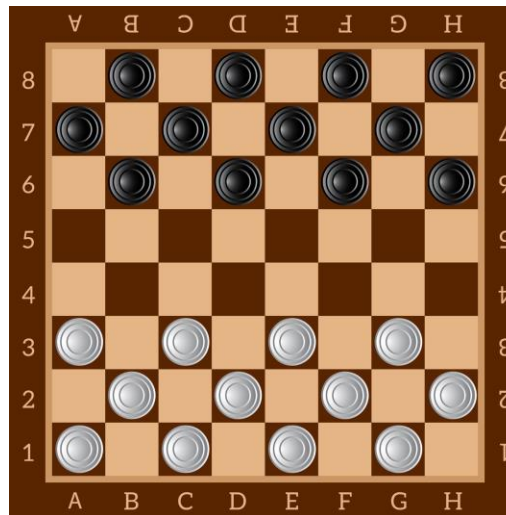
Games: hard topic

- Games are a big deal in AI
- Games are interesting to AI because they are too hard to solve
- Chess has a branching factor of 35, with 35^{100} nodes roughly equal to 10^{154}
- Need to make some decision even when the optimal decision is infeasible

Adversarial Search

Checkers:

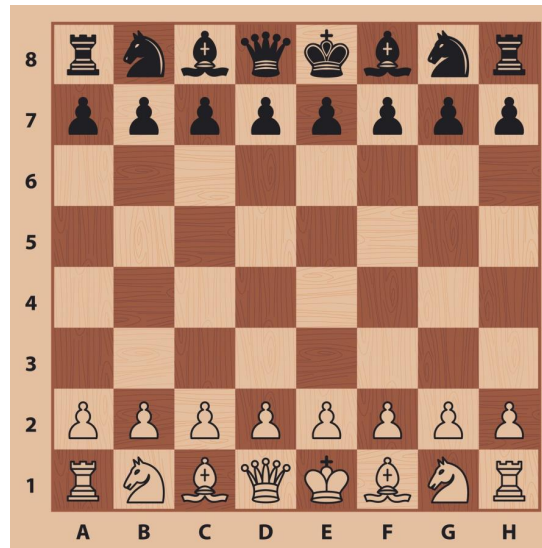
- Chinook ended 40-year-reign of human world champion Marion Tinsley in 1994.
- Used an endgame database defining perfect play for all positions involving 8 or fewer pieces on the board, a total of 443,748,401,247 positions.



Adversarial Search

Chess:

- In 1949, Claude E. Shannon in his paper "Programming a Computer for Playing Chess", suggested *Chess* as an AI problem for the community.
- Deep Blue defeated human world champion Gary Kasparov in a six-game match in 1997.
- In 2006, Vladimir Kramnik, the undisputed world champion, was defeated 4-2 by Deep Fritz.



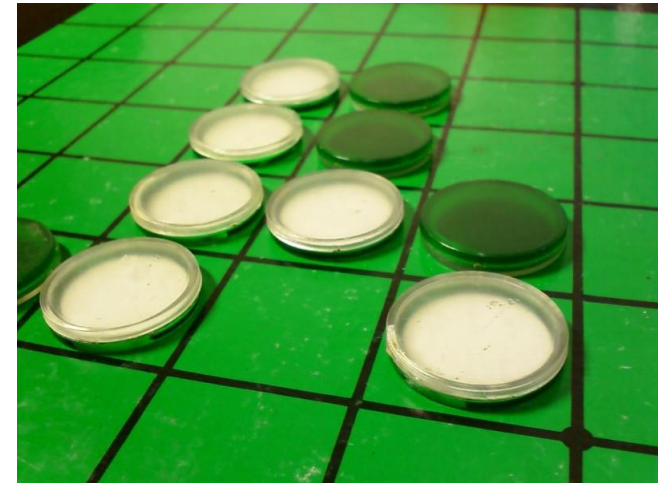
Adversarial Search

Go: $b > 300!$ Google Deep mind Project AlphaGo. In 2016, AlphaGo beat both Fan Hui, the European Go champion and Lee Sedol the worlds best player.

Othello: Several computer othello exists and human champions refuse to compete against computers, that are too good.



By Donarreiskøer



By Paul_012

via Wikimedia Commons

Types of games

	deterministic	chance
perfect information	chess, checkers, go, othello	backgammon monopoly
imperfect information	battleships, blind tictactoe	bridge, poker, scrabble nuclear war

We are mostly interested in deterministic games, fully observable environments, zero-sum, where two agents act alternately.

Zero-sum Games

- Adversarial: Pure competition.
- Agents have different values on the outcomes.
- One agent maximizes one single value, while the other minimizes it.

Zero-sum Games

- Adversarial: Pure competition.
- Agents have different values on the outcomes.
- One agent maximizes one single value, while the other minimizes it.
- Each move by one of the players is called a “ply.”

One function: one agents maximizes it and one minimizes it!

Embedded thinking...

Embedded thinking or backward reasoning!



- One agent is trying to figure out what to do.
- How to decide? He thinks about the consequences of the possible actions.
- He needs to think about his opponent as well...
- The opponent is also thinking about what to do etc.
- Each will imagine what would be the response from the opponent to their actions.
- This entails an embedded thinking.

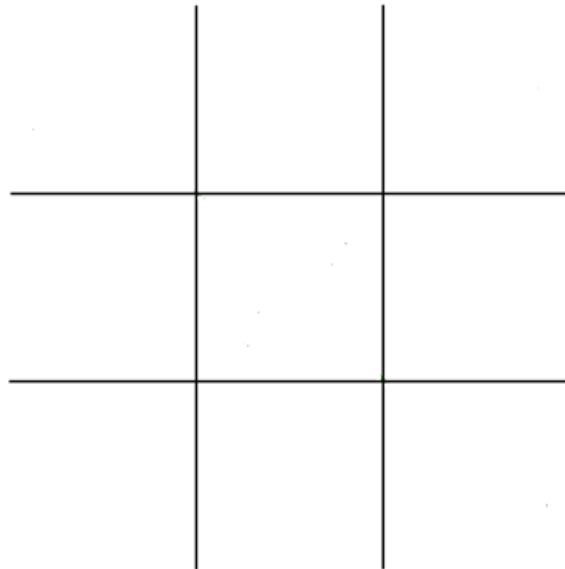
Formalization

- The **initial state**
- $\text{Player}(s)$: defines which player has the move in state s . Usually taking turns.
- $\text{Actions}(s)$: returns the set of legal moves in s
- **Transition** function: $s * A = s'$ defines the result of a move
- Terminal test: True when the game is over, False otherwise.
States where game ends are called **terminal states**
- $\text{Utility}(s, p)$: **utility function** or objective function for a game that ends in terminal state s for player p . In Chess, the outcome is a win, loss, or draw with values $+1, 0, 1/2$. For tic-tac-toe we can use a utility of $+1, -1, 0$.

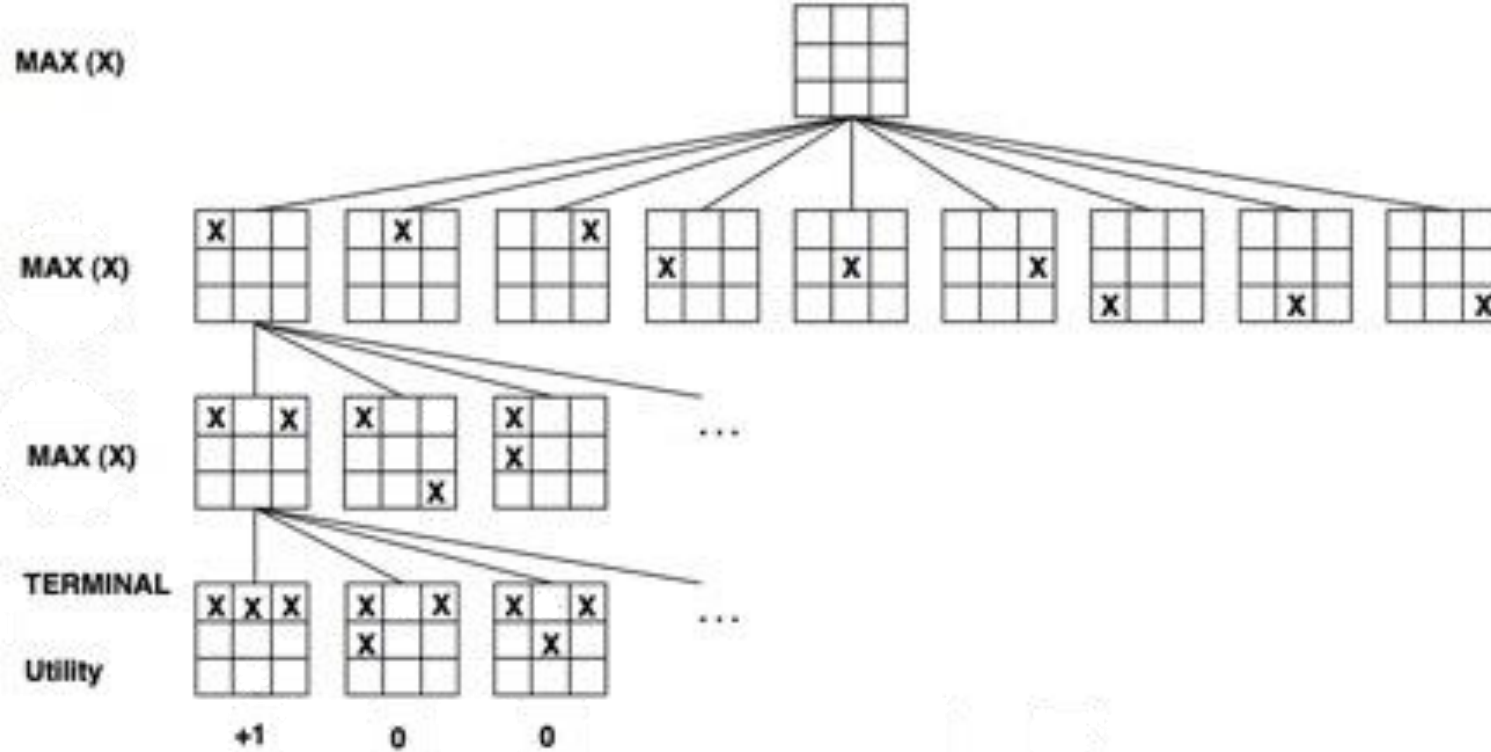
Single player...

Assume we have a tic-tac-toe with one player.

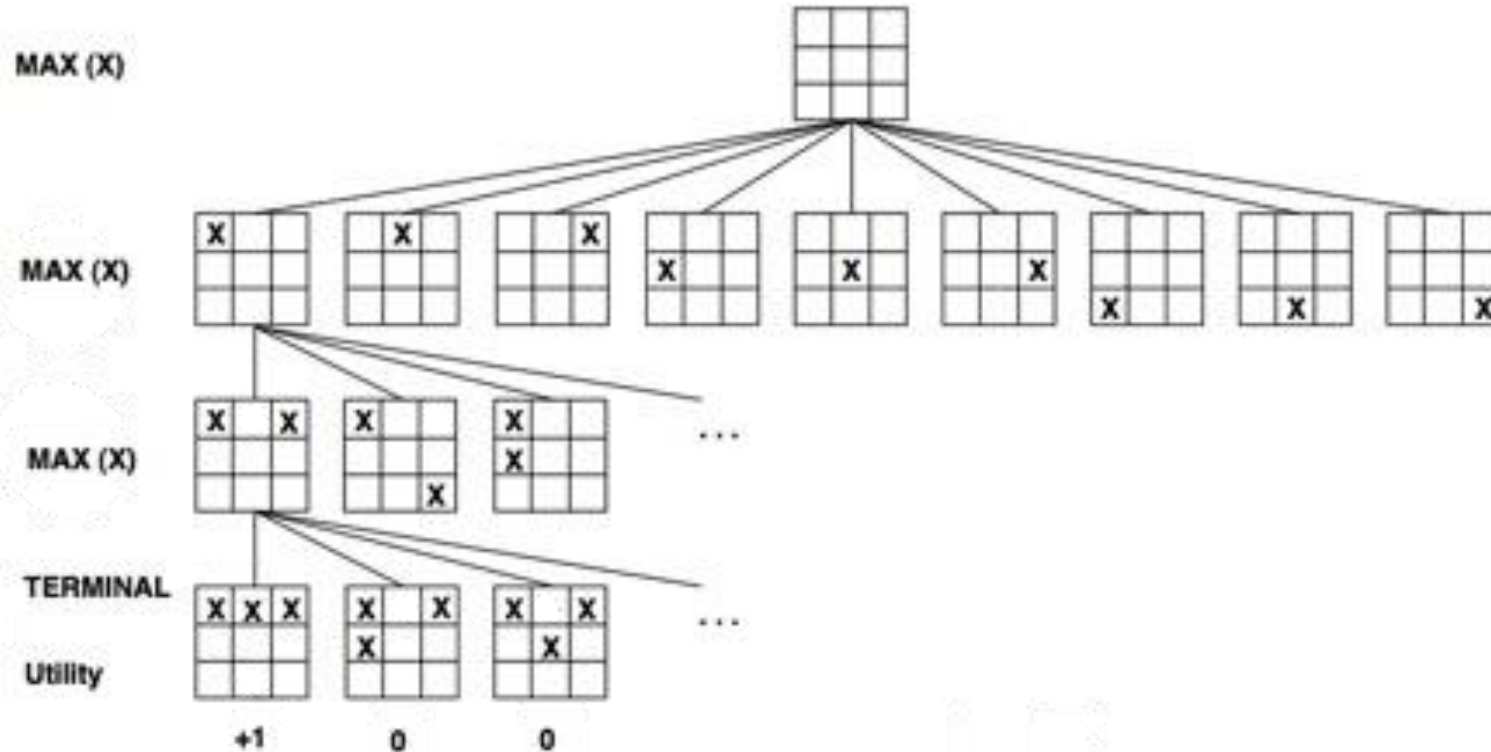
Let's call him Max and have him play three moves only for the sake of the example.



Single player...



Single player...

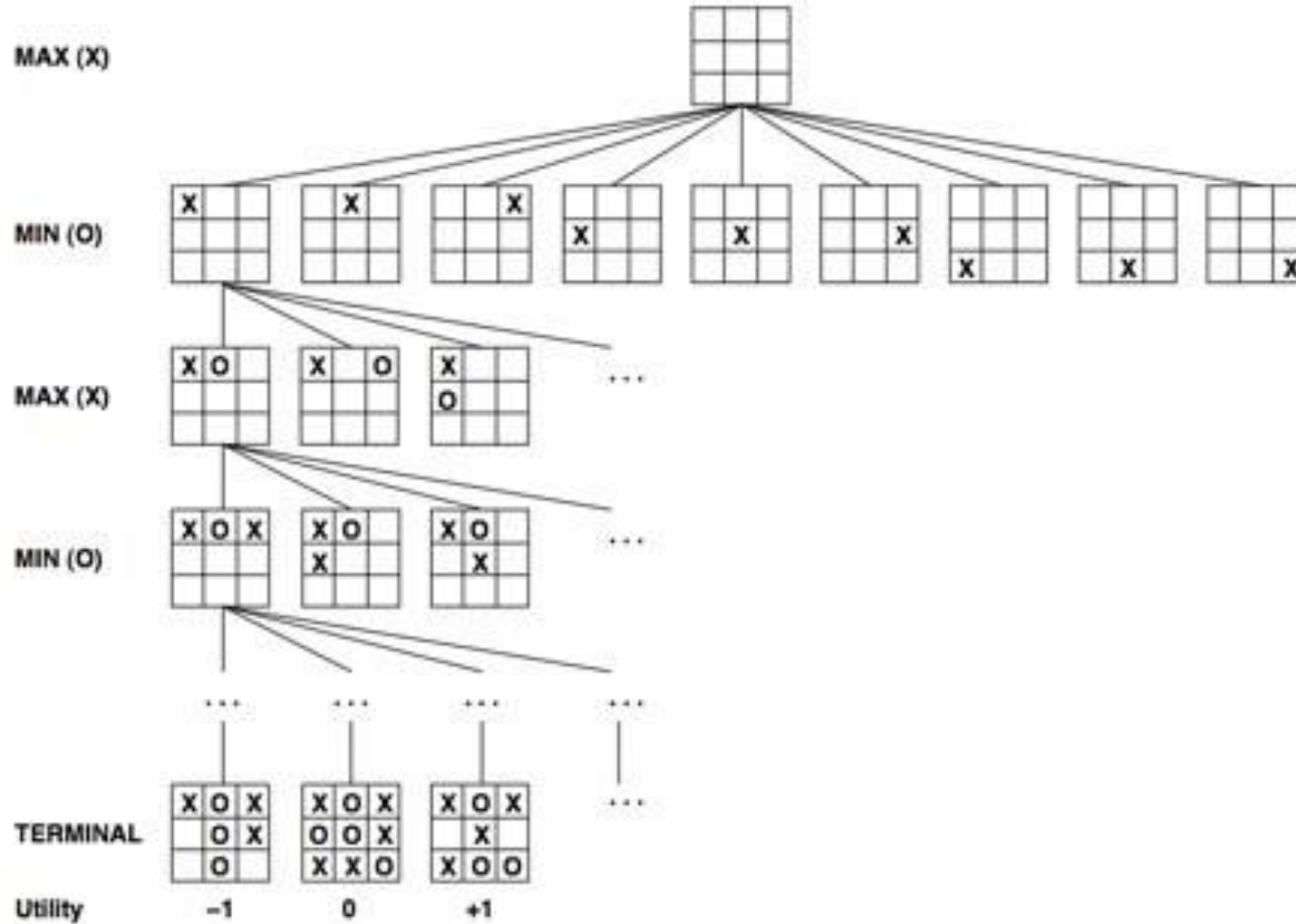


In the case of one player, nothing will prevent Max from winning (choose the path that leads to the desired utility here 1), unless there is another player who will do everything to make Max lose, let's call him Min (the Mean :))

Adversarial search: minimax

- Two players: Max and Min
- Players alternate turns
- Max moves first
- Max maximizes results
- Min minimizes the result
- Compute each node's minimax value's the best achievable utility against an optimal adversary
- Minimax value: best achievable payoff against best play

Minimax example



Adversarial search: minimax

- Find the optimal strategy for Max:
 - Depth-first search of the game tree
 - An optimal leaf node could appear at any depth of the tree
 - Minimax principle: compute the utility of being in a state assuming both players play optimally from there until the end of the game
 - Propagate minimax values up the tree once terminal nodes are discovered

Adversarial search: minimax

- If state is terminal node: Value is $utility(state)$
- If state is MAX node: Value is highest value of all successor node values (children)
- If state is MIN node: Value is lowest value of all successor node values (children)

Adversarial search: minimax

For a state s $\text{minimax}(s) =$

Utility(s)	if Terminal-test(s)
$\max_{a \in \text{Actions}(s)} \text{minimax}(\text{Result}(s,a))$	if Player(s) = Max
$\min_{a \in \text{Actions}(s)} \text{minimax}(\text{Result}(s,a))$	if Player(s) = Min

The minimax algorithm

/ Find the child state with the lowest utility value */*

function MINIMIZE(state)

returns **TUPLE** of $\langle \mathbf{STATE}, \mathbf{UTILITY} \rangle$:

if **TERMINAL-TEST**(state):

return $\langle \mathbf{NULL}, \mathbf{EVAL}(\text{state}) \rangle$

$\langle \text{minChild}, \text{minUtility} \rangle = \langle \mathbf{NULL}, \infty \rangle$

for child **in** state.children():

$\langle _, \text{utility} \rangle = \mathbf{MAXIMIZE}(\text{child})$

if utility < minUtility:

$\langle \text{minChild}, \text{minUtility} \rangle = \langle \text{child}, \text{utility} \rangle$

return $\langle \text{minChild}, \text{minUtility} \rangle$

/ Find the child state with the highest utility value */*

function MAXIMIZE(state)

returns **TUPLE** of $\langle \mathbf{STATE}, \mathbf{UTILITY} \rangle$:

if **TERMINAL-TEST**(state):

return $\langle \mathbf{NULL}, \mathbf{EVAL}(\text{state}) \rangle$

$\langle \text{maxChild}, \text{maxUtility} \rangle = \langle \mathbf{NULL}, -\infty \rangle$

for child **in** state.children():

$\langle _, \text{utility} \rangle = \mathbf{MINIMIZE}(\text{child})$

if utility > maxUtility:

$\langle \text{maxChild}, \text{maxUtility} \rangle = \langle \text{child}, \text{utility} \rangle$

return $\langle \text{maxChild}, \text{maxUtility} \rangle$

/ Find the child state with the highest utility value */*

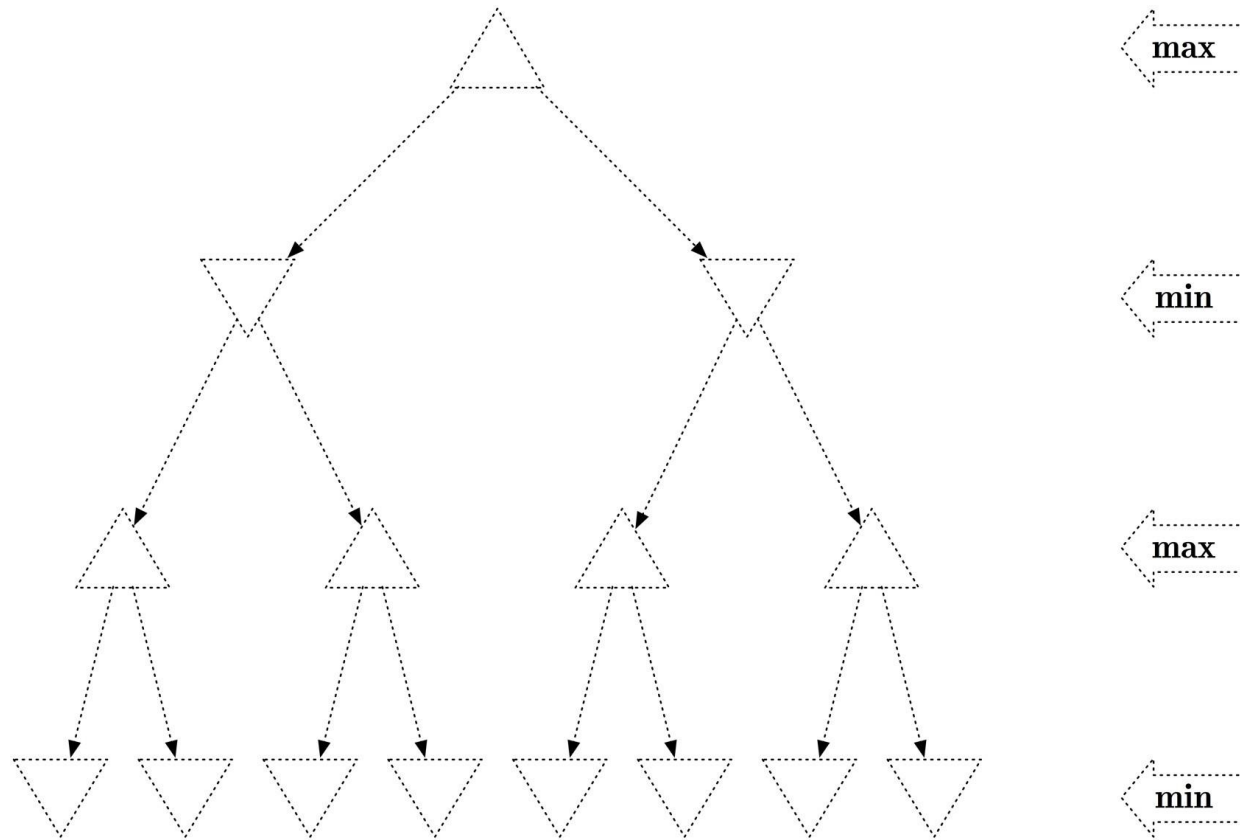
function DECISION(state)

returns **STATE** :

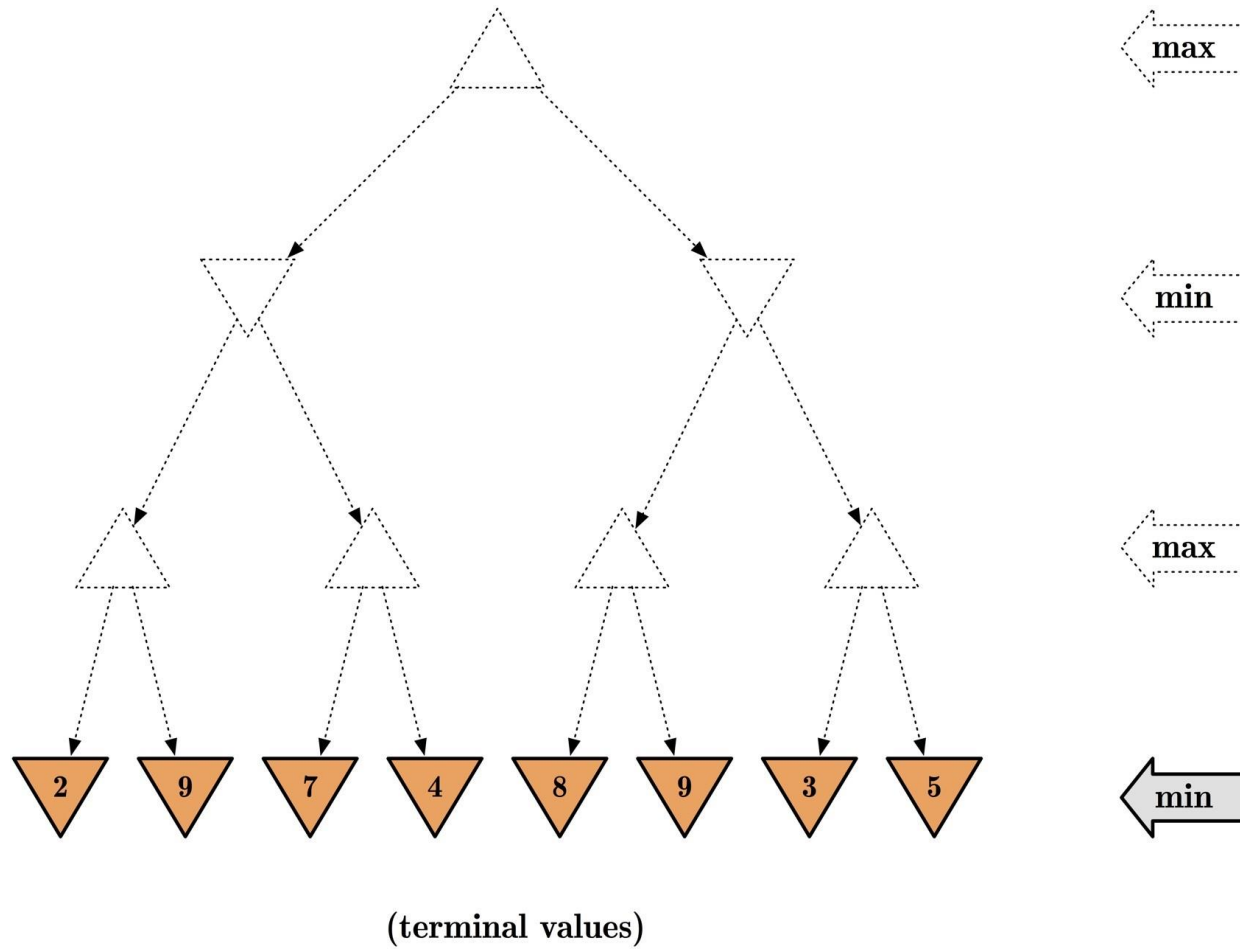
$\langle \text{child}, _ \rangle = \mathbf{MAXIMIZE}(\text{state})$

return child

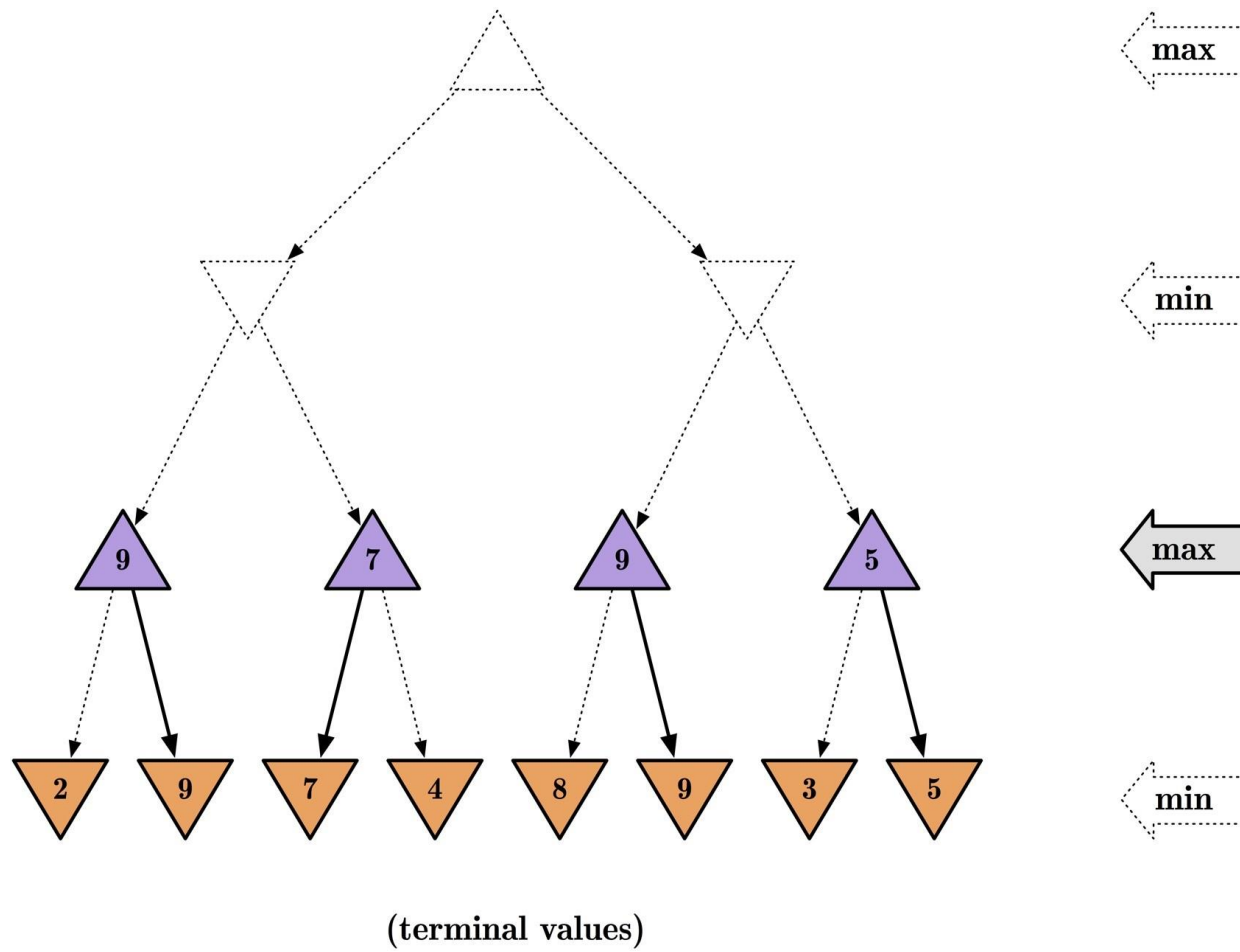
Minimax example



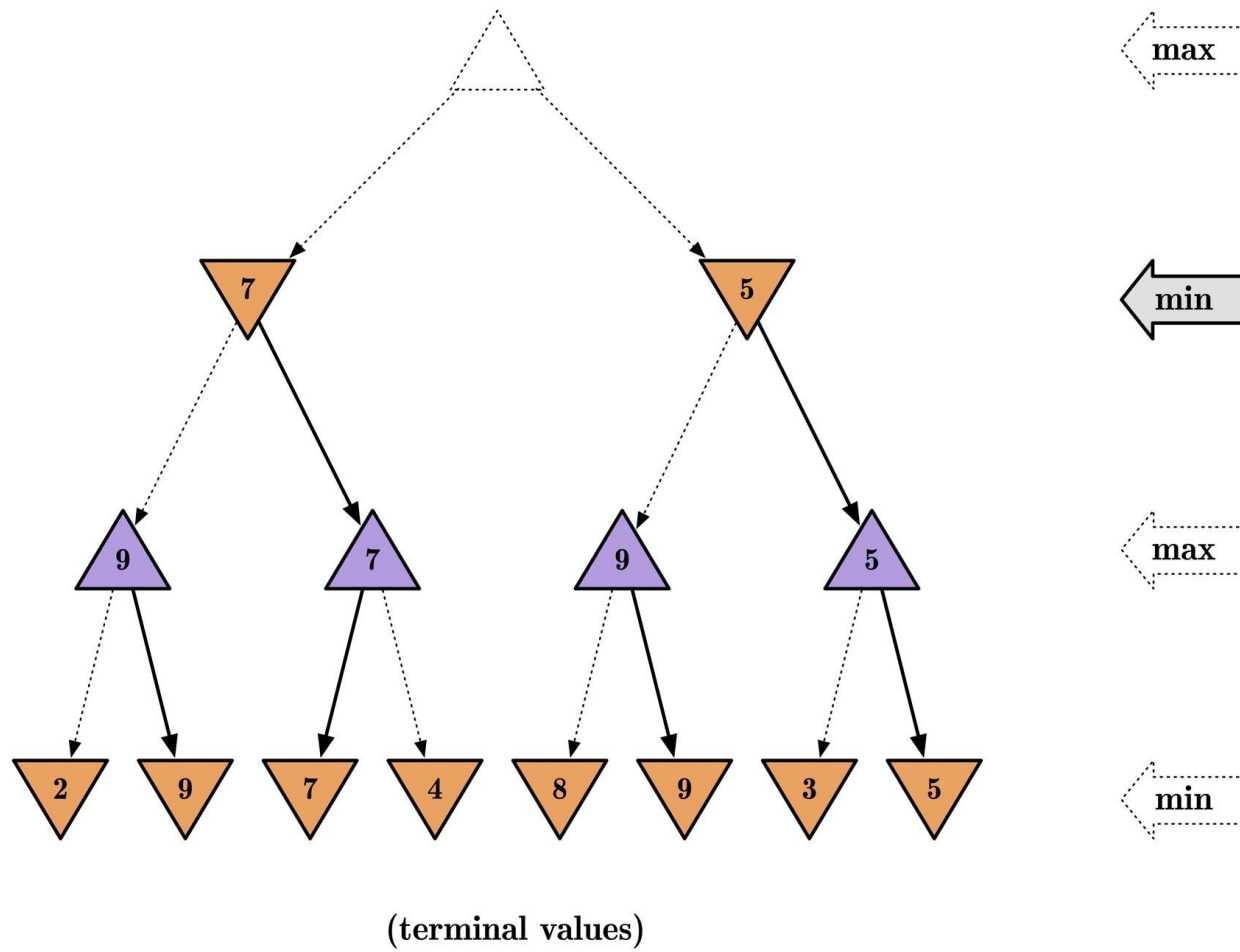
Minimax example



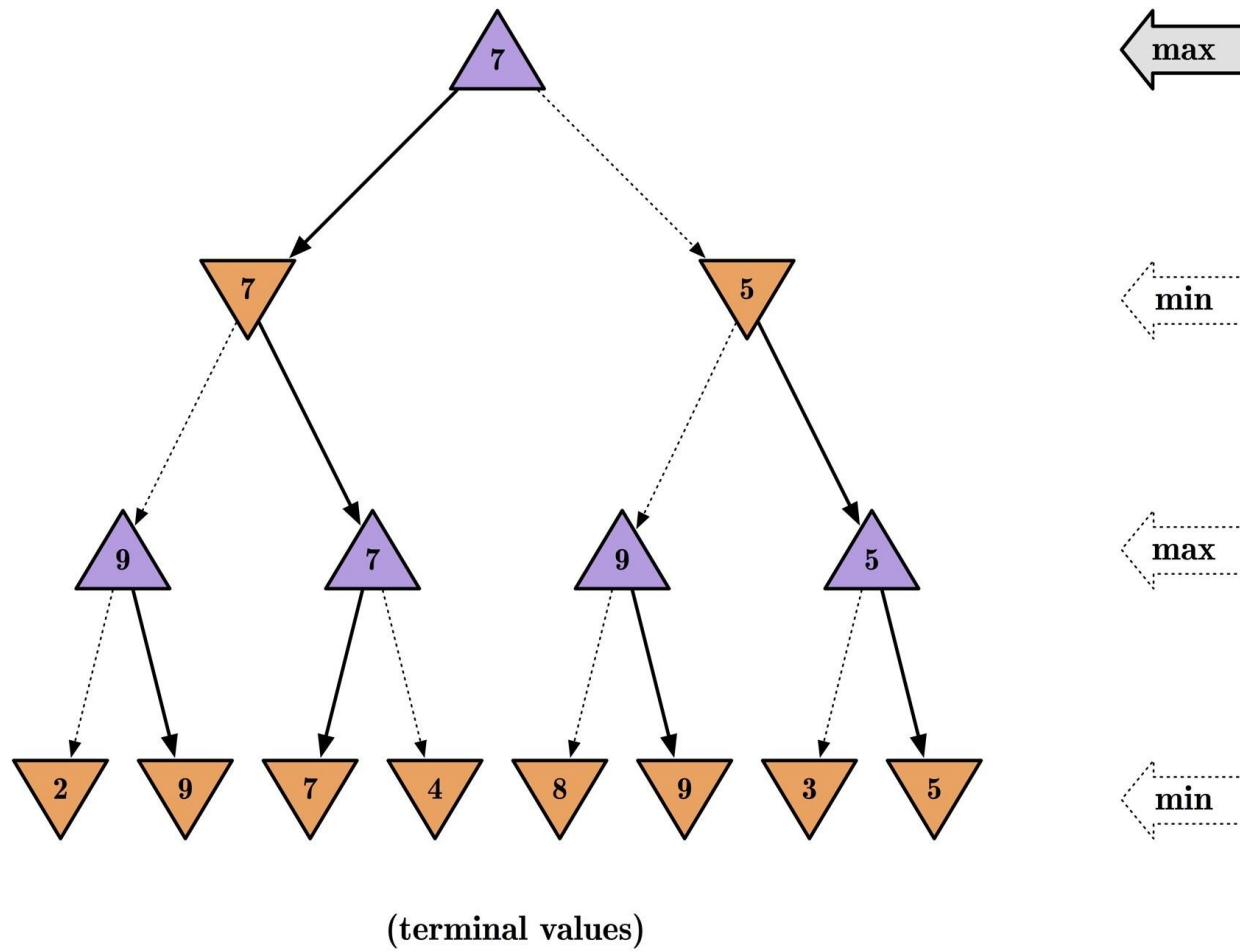
Minimax example



Minimax example



Minimax example



Properties of minimax

- Optimal (opponent plays optimally) and complete (finite tree)
- DFS time: $O(b^m)$
- DFS space: $O(bm)$

- **Tic-Tac-Toe**

5 legal moves on average, total of 9 moves (9 plies).

$$5^9 = 1,953,125$$

$$9! = 362,880 \text{ terminal nodes}$$

- **Chess**

$b = 35$ (average branching factor)

$d = 100$ (depth of game tree for a typical game)

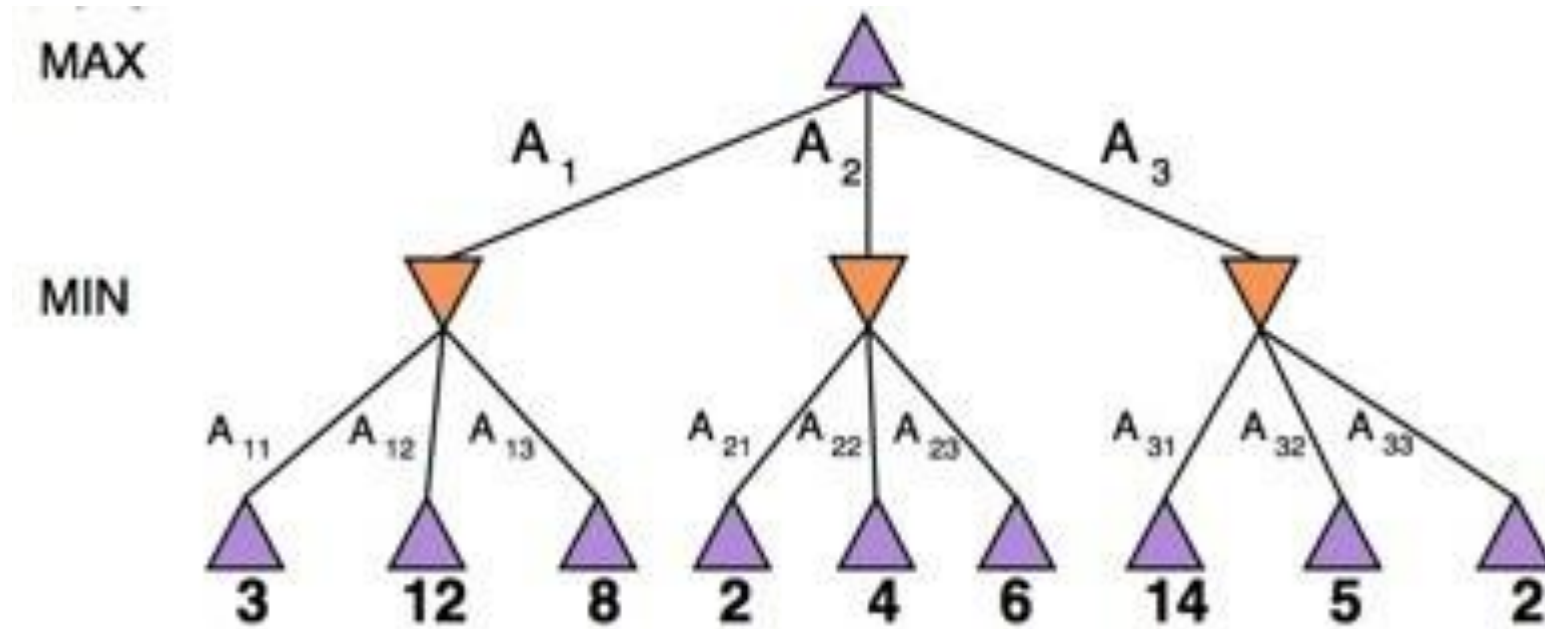
$$b^d = 35^{100} = 10^{154} \text{ nodes}$$

- **Go** branching factor starts at 361 (19X19 board)

Case of limited resources

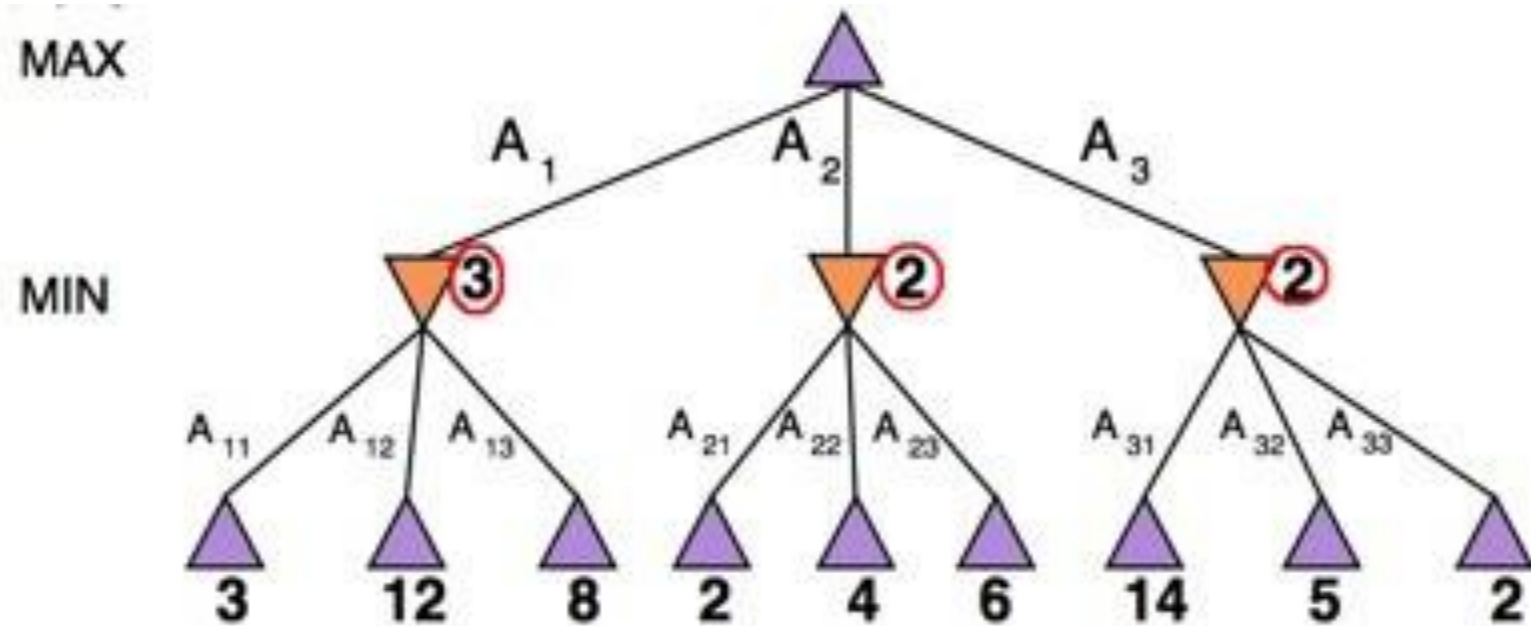
- **Problem:** In real games, we are limited in time, so we can't search the leaves.
- To be practical and run in a reasonable amount of time, min-max can only search to some depth.
- More plies make a big difference.
- **Solution:**
 1. Replace terminal utilities with an evaluation function for non-terminal positions.
 2. Use Iterative Deepening Search (IDS).
 3. Use pruning: eliminate large parts of the tree.

α - β pruning

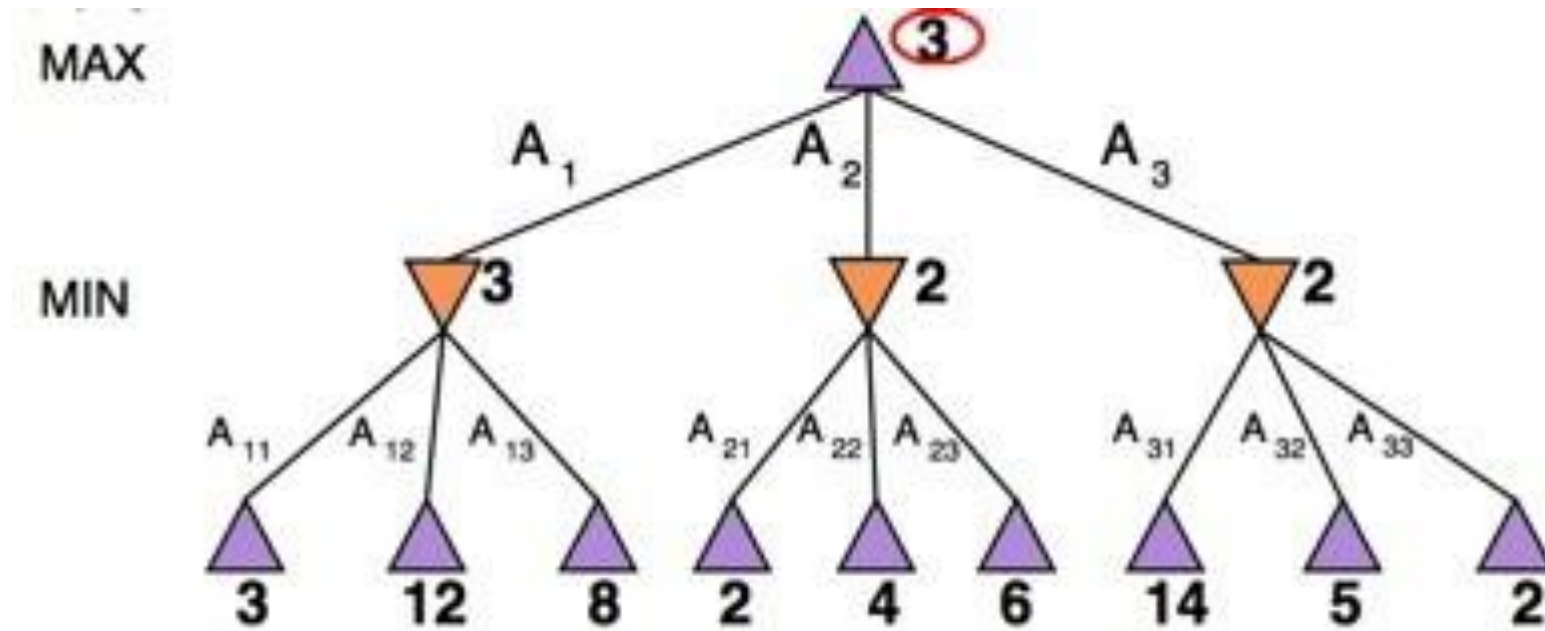


A two-ply game tree.

α - β pruning

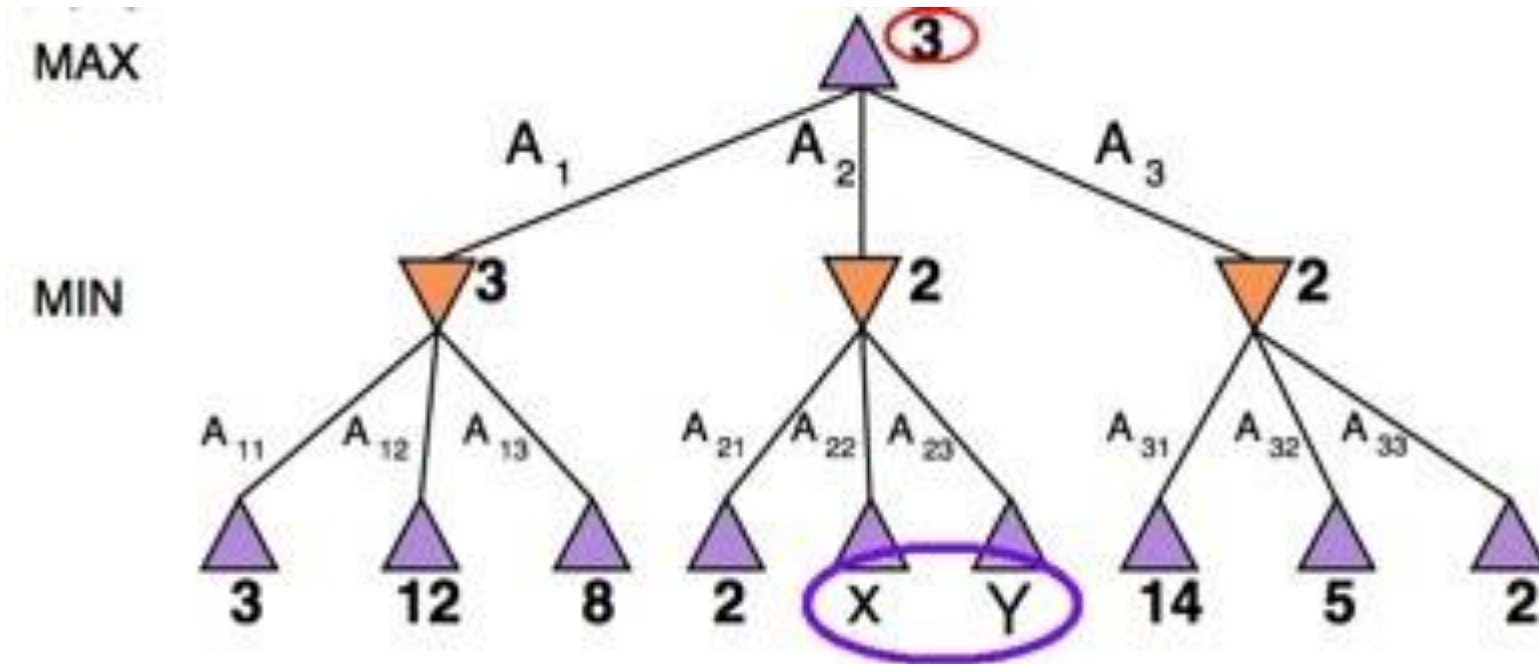


α - β pruning

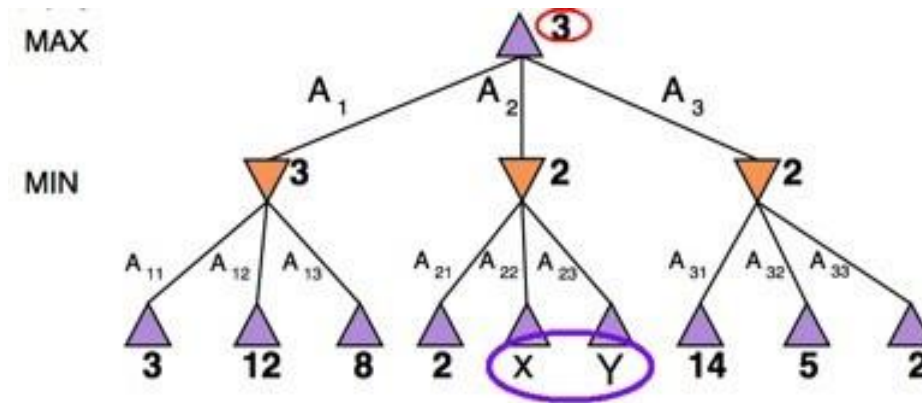


α - β pruning

Which values are necessary?

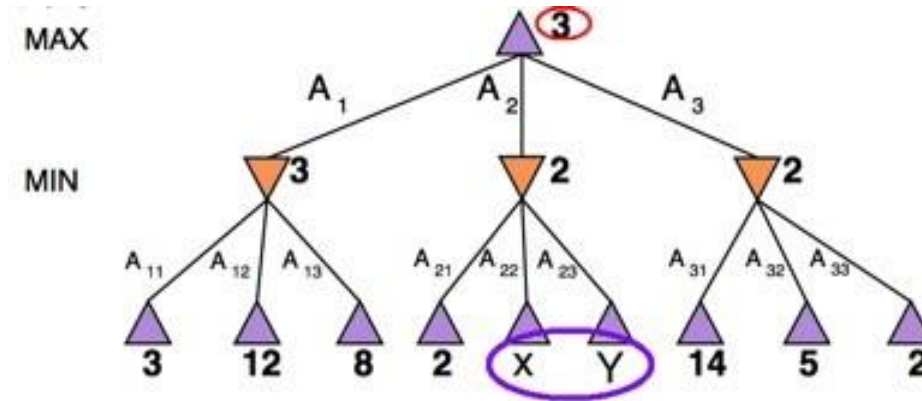


α - β pruning



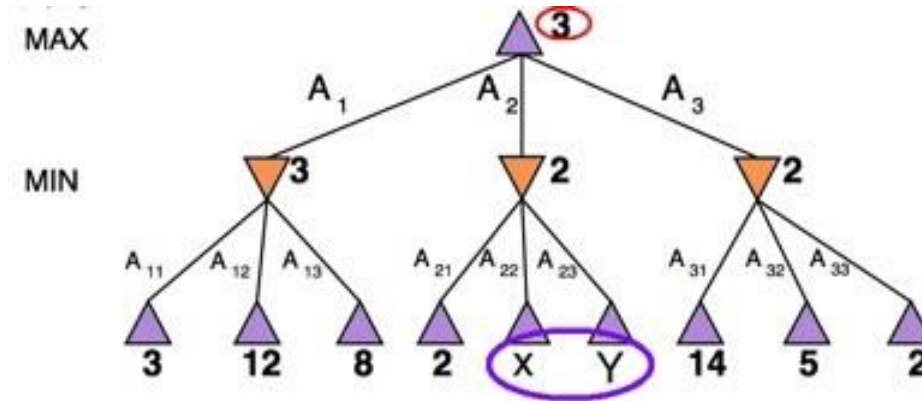
$$\text{Minimax}(\text{root}) = \max(\min(3, 12, 8), \min(2, X, Y), \min(14, 5, 2))$$

α - β pruning



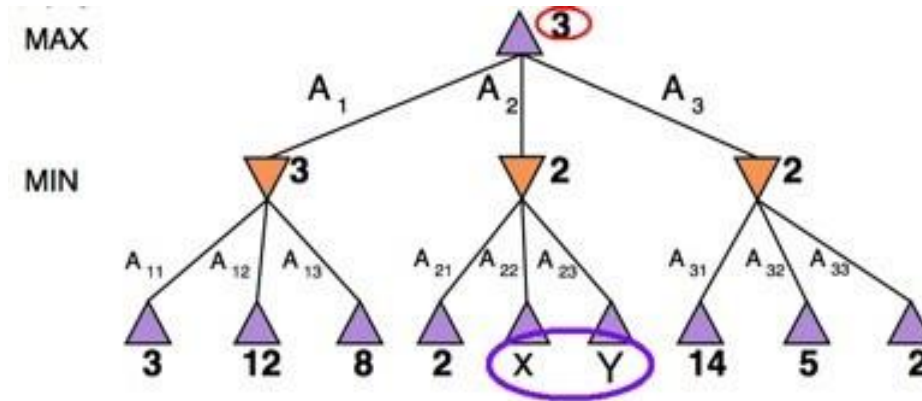
$$\begin{aligned}\text{Minimax}(\text{root}) &= \max(\min(3, 12, 8), \min(2, X, Y), \min(14, 5, 2)) \\ &= \max(3, \min(2, X, Y), 2)\end{aligned}$$

α - β pruning



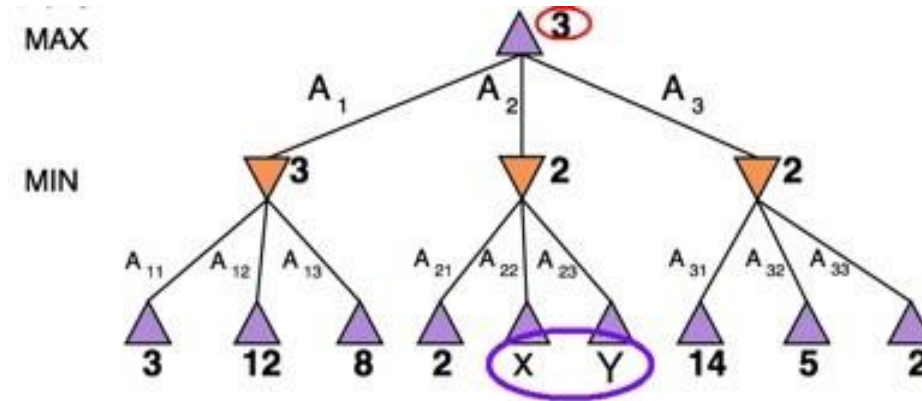
$$\begin{aligned}\text{Minimax}(\text{root}) &= \max(\min(3, 12, 8), \min(2, X, Y), \min(14, 5, 2)) \\ &= \max(3, \min(2, X, Y), 2) \\ &= \max(3, Z, 2) \quad \text{where } Z = \min(2, X, Y) \leq 2\end{aligned}$$

α - β pruning



$$\begin{aligned}\text{Minimax}(\text{root}) &= \max(\min(3, 12, 8), \min(2, X, Y), \min(14, 5, 2)) \\ &= \max(3, \min(2, X, Y), 2) \\ &= \max(3, Z, 2) \quad \text{where } Z = \min(2, X, Y) \leq 2 \\ &= 3\end{aligned}$$

α - β pruning



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Minimax decisions are independent of the values of X and Y.

α - β pruning

- **Strategy:** Just like minimax, it performs a DFS.

α - β pruning

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- **Parameters:** Keep track of two bounds
 - α : largest value for Max across seen children (current lower bound on MAX's outcome).
 - β : lowest value for MIN across seen children (current upper bound on MIN's outcome).

α - β pruning

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- **Initialization:** $\alpha = -1$, $\beta = 1$

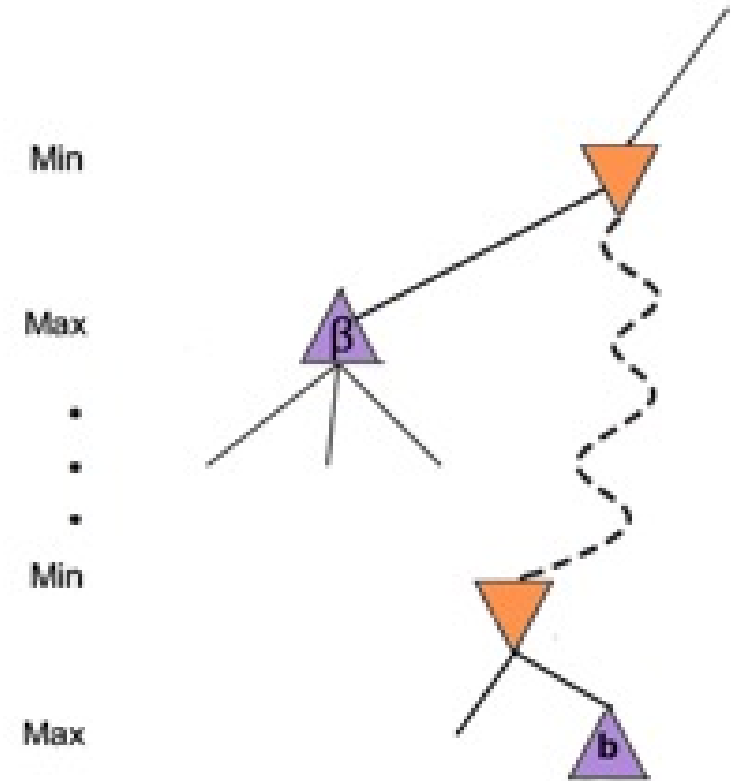
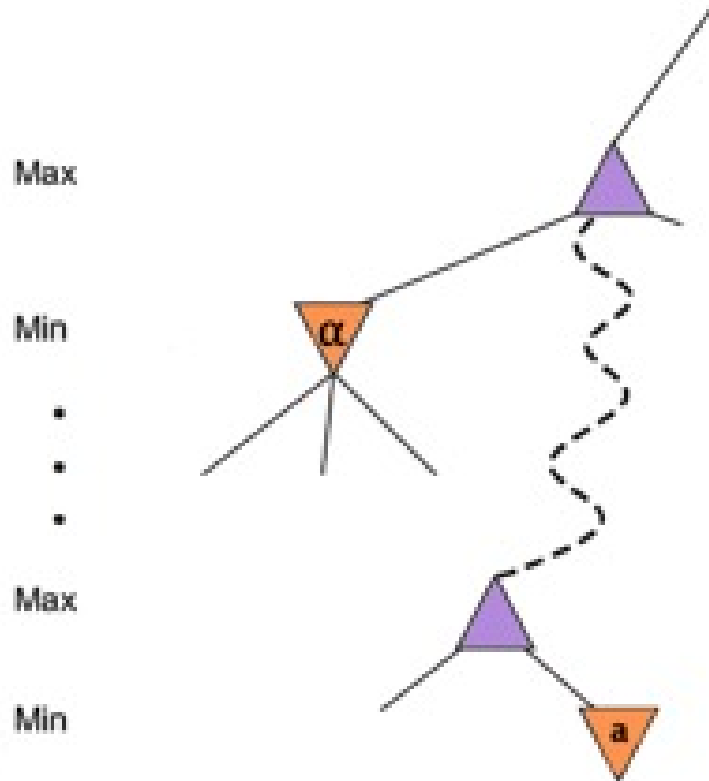
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- **Initialization:** $\alpha = -1$, $\beta = 1$
- **Propagation:** Send α , β values *down* during the search to be used for pruning.
 - Update α , β values by *propagating upwards* values of terminal nodes.
 - Update α only at Max nodes and update β only at Min nodes.

α - β pruning

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 - Update α , β values by *propagating upwards* values of terminal nodes.
 - Update α only at Max nodes and update β only at Min nodes.
- **Pruning:** Prune any remaining branches whenever $\alpha \leq \beta$.

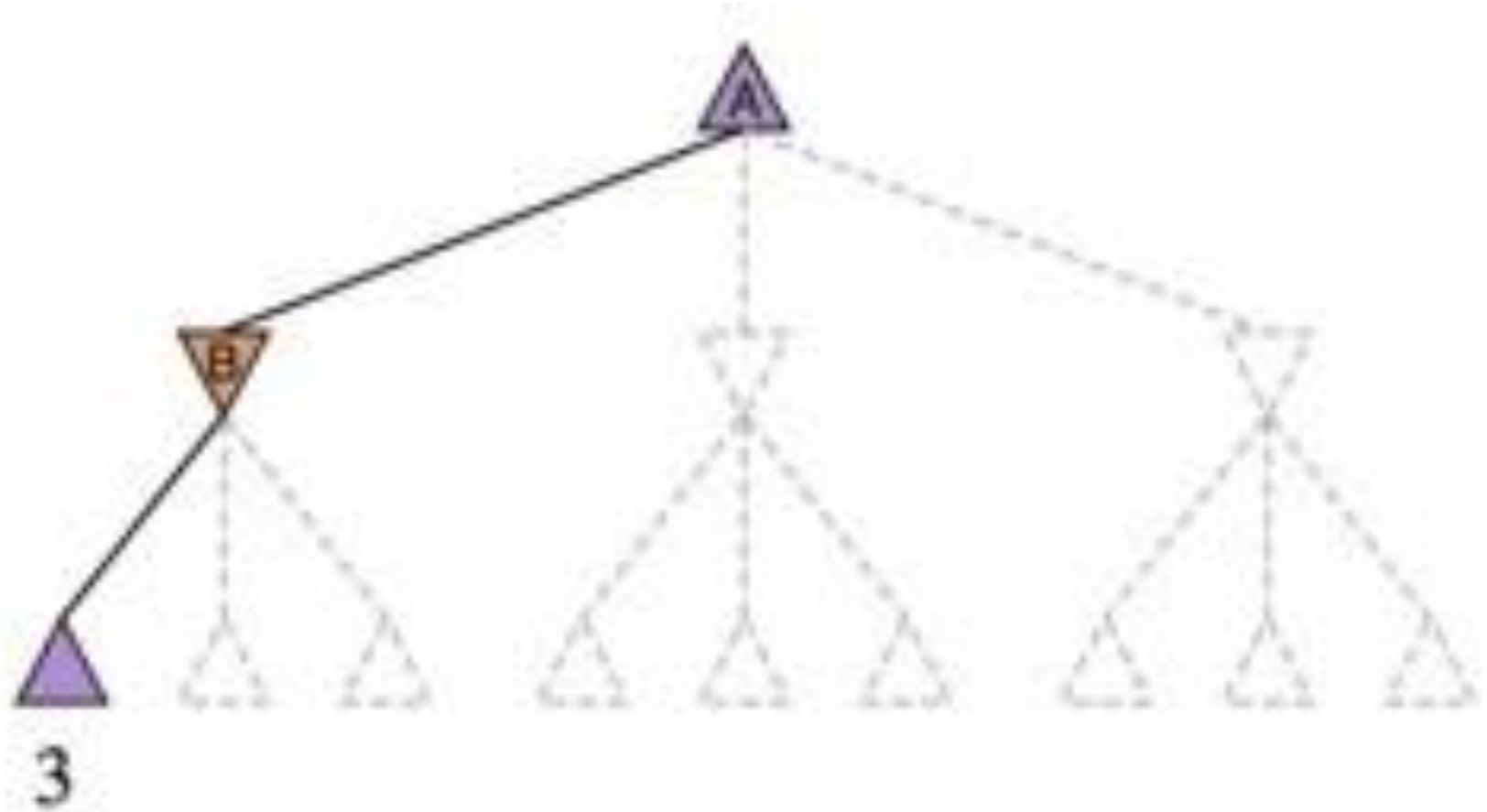
α - β pruning



- If α is better than a for Max, then Max will avoid it, that is prune that branch.
- If β is better than b for Min, then Min will avoid it, that is prune that branch.

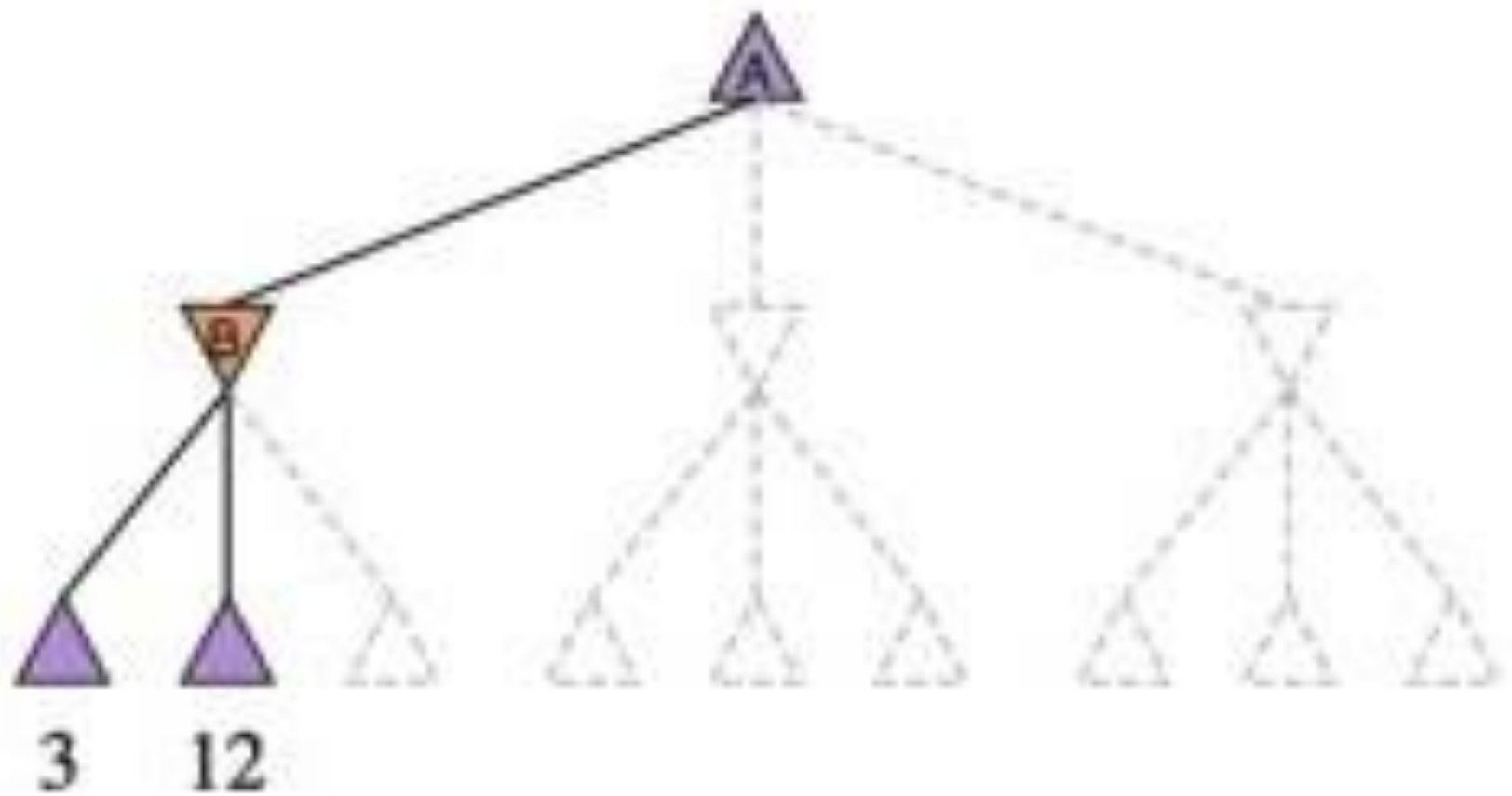
α - β pruning

(a)



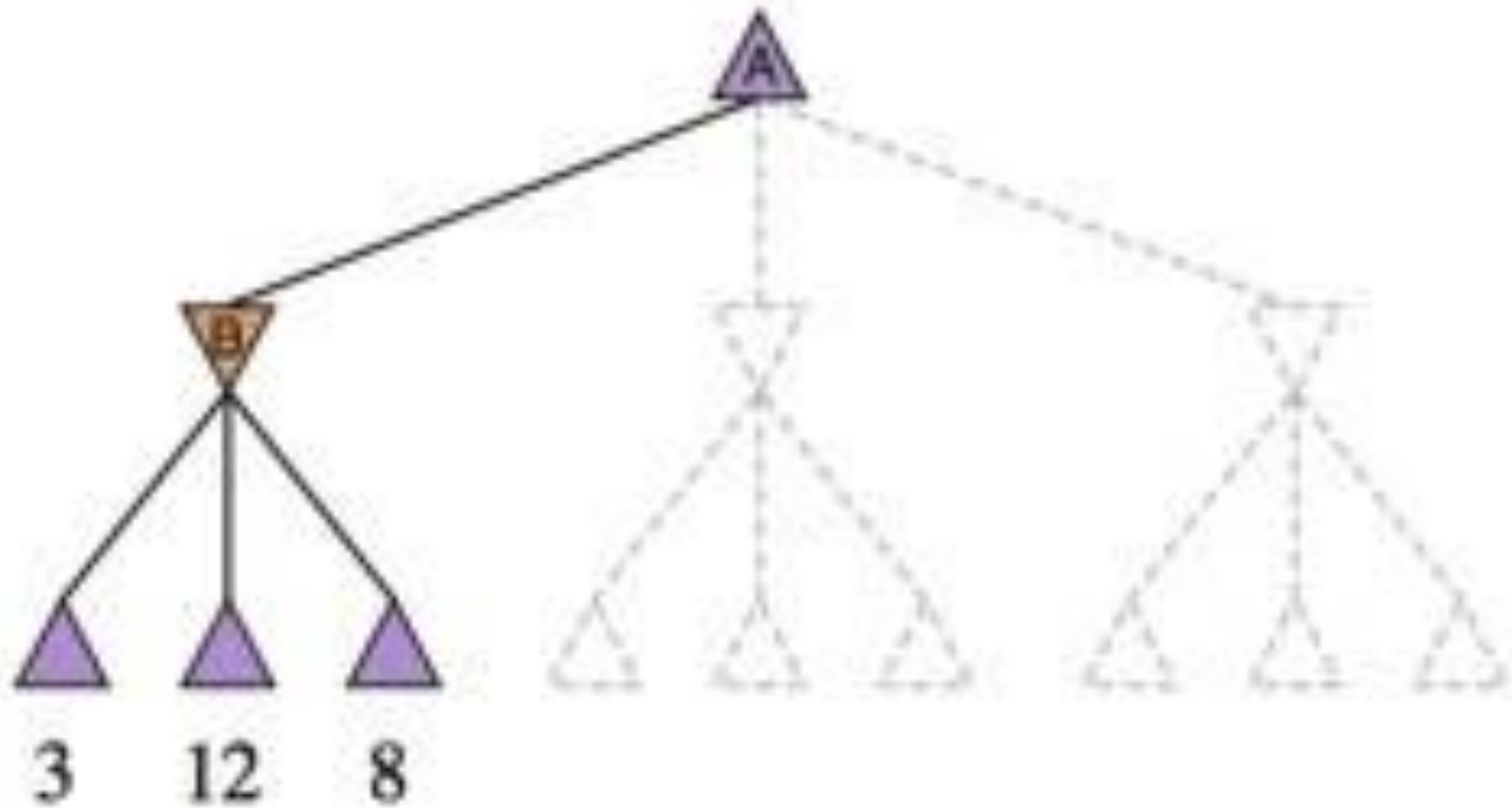
α - β pruning

(b)



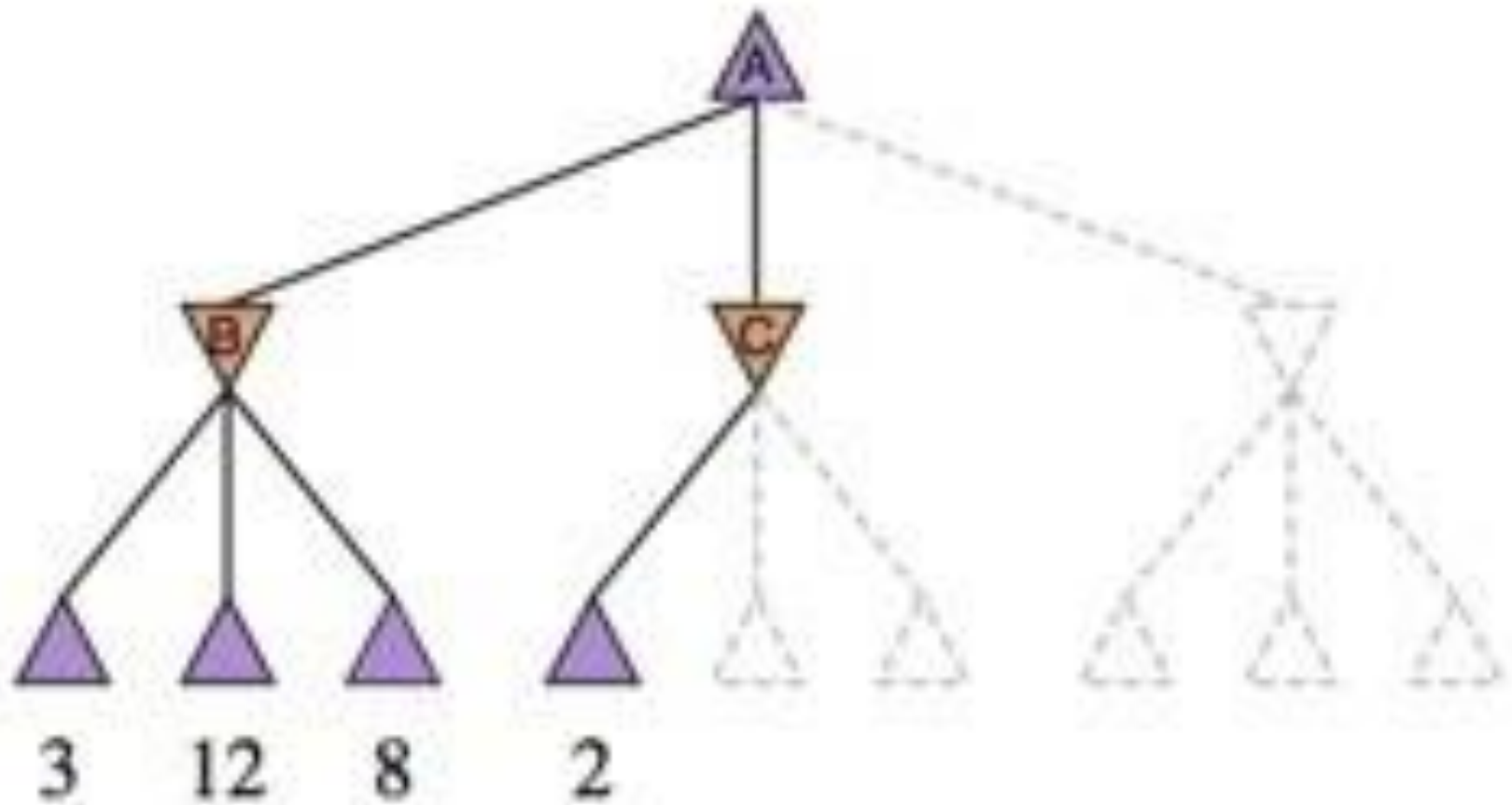
α - β pruning

(c)



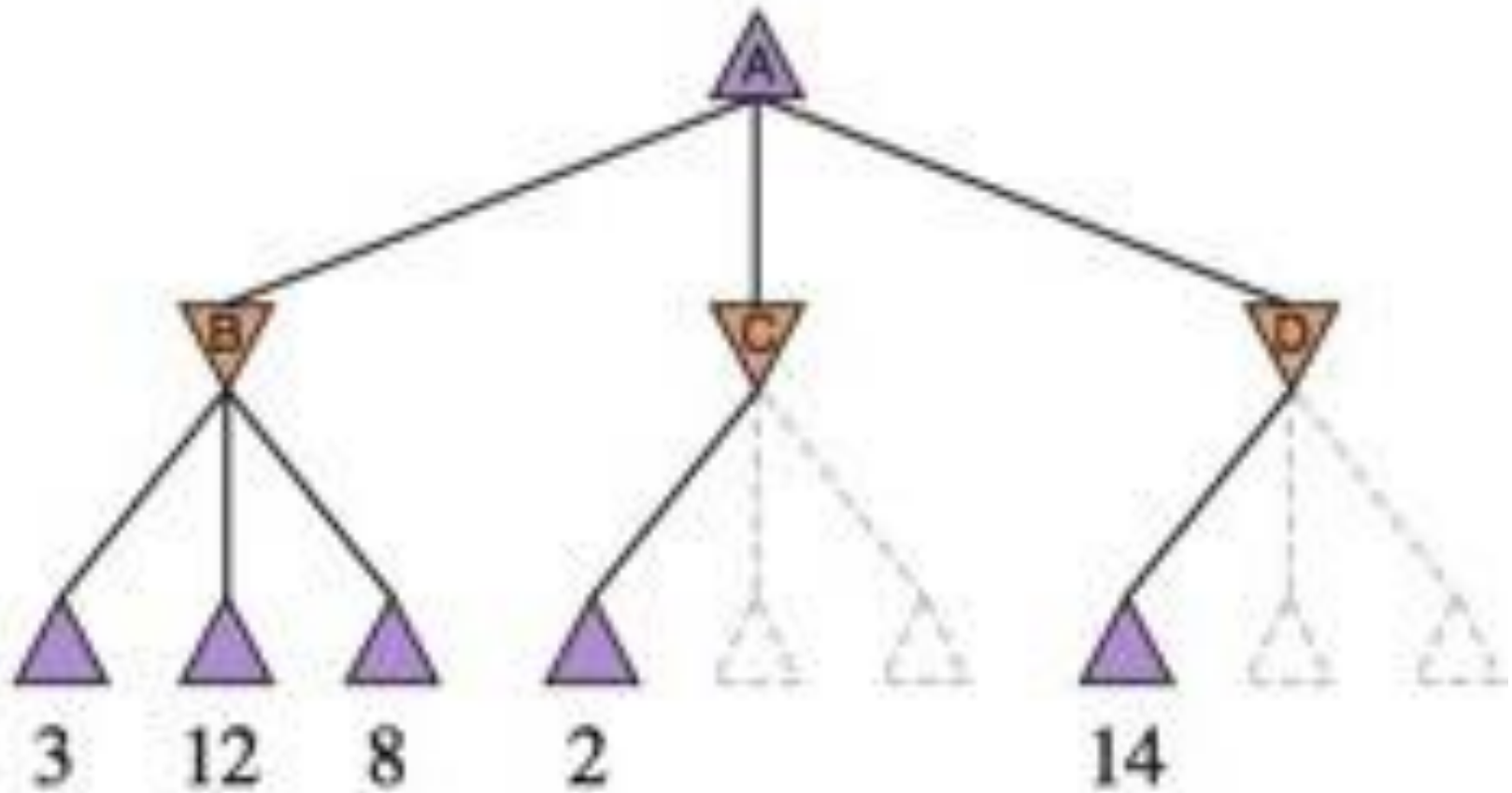
α - β pruning

(d)



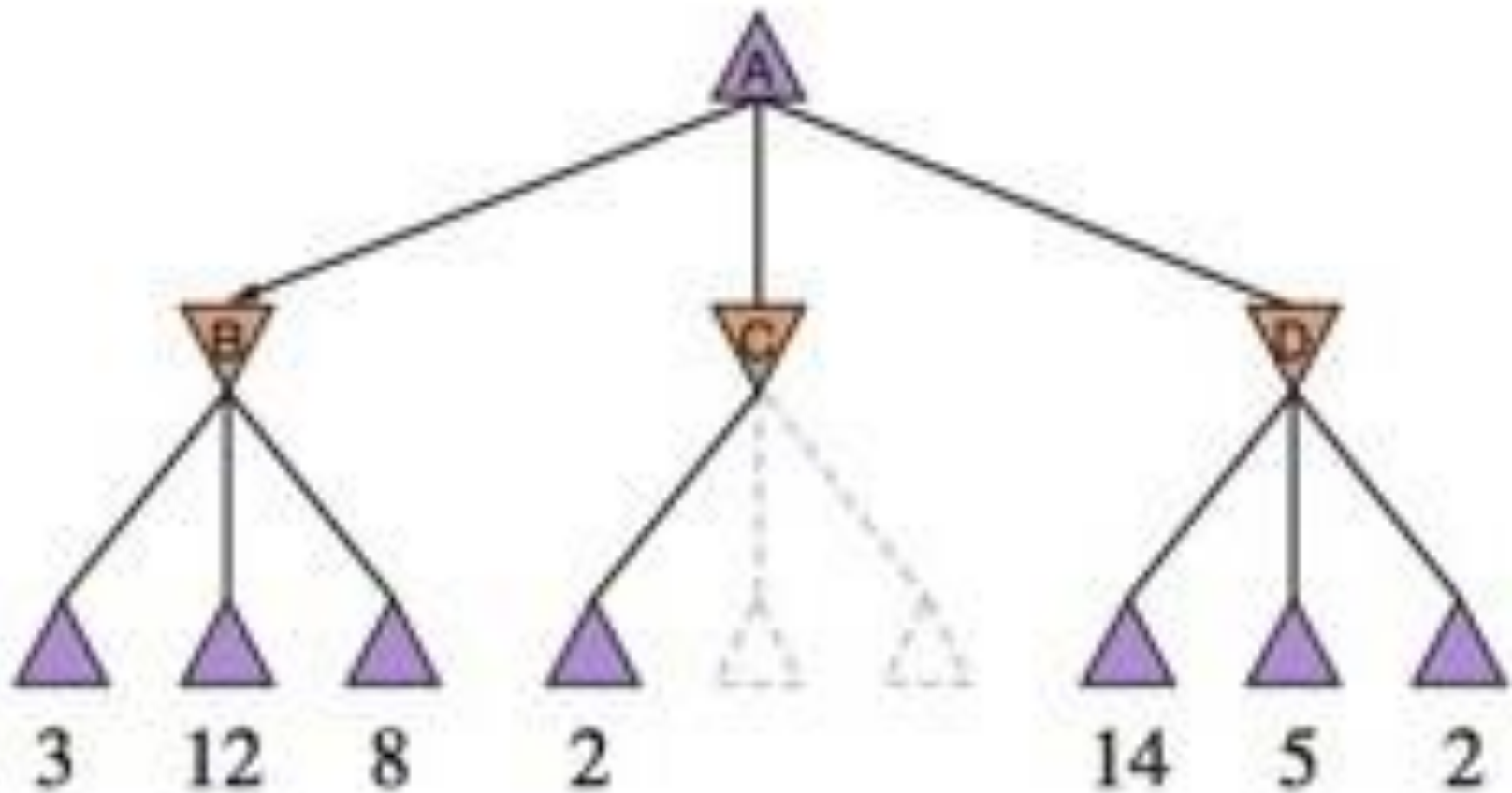
α - β pruning

(e)

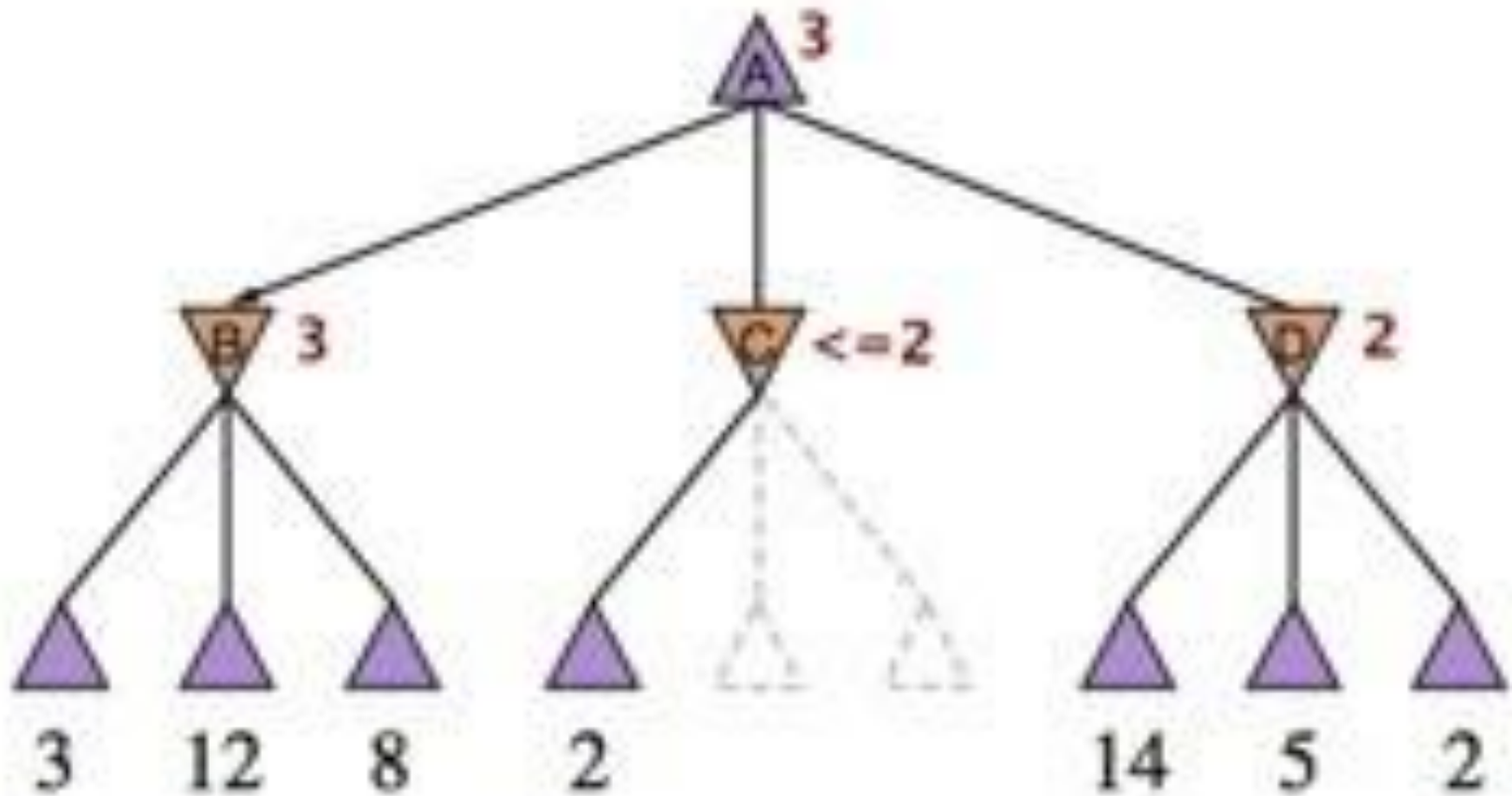


α - β pruning

(f)



α - β pruning



α - β pruning

/ Find the child state with the lowest utility value */*

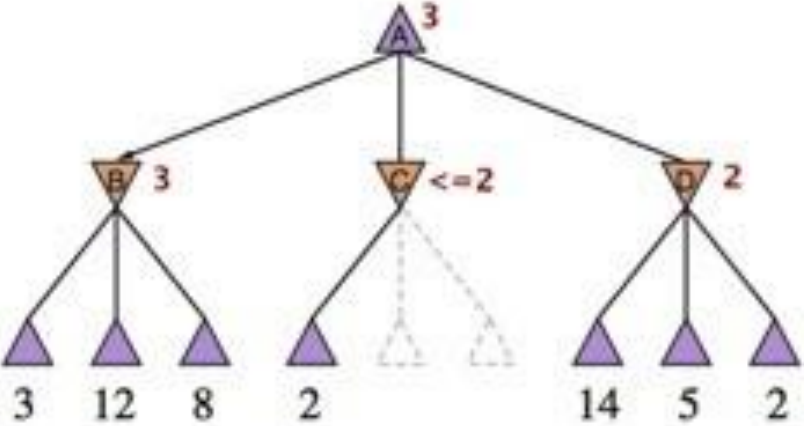
```
function MINIMIZE(state,  $\alpha$ ,  $\beta$ )  
  returns TUPLE of  $\langle$ STATE, UTILITY $\rangle$  :  
  
  if TERMINAL-TEST(state):  
    return  $\langle$ NULL, EVAL(state) $\rangle$   
  
   $\langle$ minChild, minUtility $\rangle$  =  $\langle$ NULL,  $\infty$  $\rangle$   
  
  for child in state.children():  
     $\langle$ _, utility $\rangle$  = MAXIMIZE(child,  $\alpha$ ,  $\beta$ )  
  
    if utility < minUtility:  
       $\langle$ minChild, minUtility $\rangle$  =  $\langle$ child, utility $\rangle$   
  
    if minUtility  $\leq$   $\alpha$ :  
      break  
  
    if minUtility <  $\beta$ :  
       $\beta$  = minUtility  
  
  return  $\langle$ minChild, minUtility $\rangle$ 
```

/ Find the child state with the highest utility value */*

```
function MAXIMIZE(state,  $\alpha$ ,  $\beta$ )  
  returns TUPLE of  $\langle$ STATE, UTILITY $\rangle$  :  
  
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    if utility > maxUtility:  
       $\langle$ maxChild, maxUtility $\rangle$  =  $\langle$ child, utility $\rangle$   
  
    if maxUtility  $\geq$   $\beta$ :  
      break  
  
    if maxUtility >  $\alpha$ :  
       $\alpha$  = maxUtility  
  
  return  $\langle$ maxChild, maxUtility $\rangle$ 
```

/ Find the child state with the highest utility value */*

```
function DECISION(state)  
  returns STATE :  
  
   $\langle$ child, _ $\rangle$  = MAXIMIZE(state,  $-\infty$ ,  $\infty$ )  
  
  return child
```



- It does matter as it affects the effectiveness of α - β pruning.
- Example: We could not prune any successor of D because the worst successors for Min were generated first. If the third one (leaf 2) was generated first we would have pruned the two others (14 and 5).
- Idea of ordering: examine first successors that are likely best.

Move ordering

- **Worst ordering:** no pruning happens (best moves are on the right of the game tree). Complexity $O(b^m)$.
- **Ideal ordering:** lots of pruning happens (best moves are on the left of the game tree). This solves tree twice as deep as minimax in the same amount of time. Complexity $O(b^{m/2})$ (in practice). The search can go deeper in the game tree.
- **How to find a good ordering?**
 - Remember the best moves from shallowest nodes.
 - Order the nodes so as the best are checked first.
 - Use domain knowledge: e.g., for chess, try order: captures first, then threats, then forward moves, backward moves.
 - Bookkeep the states, they may repeat!

Games: conclusion

- Games are modeled in AI as a search problem and use heuristic to evaluate the game.
- Minimax algorithm chooses the best move given an optimal play from the opponent.
- Minimax goes all the way down the tree which is not practical given game time constraints.
- Alpha-Beta pruning can reduce the game tree search which allows to go deeper in the tree within the time constraints.
- Pruning, bookkeeping, evaluation heuristics, node re-ordering and IDS are effective in practice.

Games: conclusion

- Games is an exciting and fun topic for AI.
- Devising adversarial search agents is challenging because of the huge state space.
- We have just scratched the surface of this topic.
- Further topics to explore include partially observable games (card games such as bridge, poker, etc.).
- Except for robot football (a.k.a. soccer), there was no much interest from AI in physical games.
(see <http://www.robocup.org/>).