Inference in First-Order Logic
Chapter 3

Outline

- Inference Rules in Propostional Logic
- Reducing first-order inference to propositional inference
- Unification
- Generalized Modus Ponens
- Forward and backward chaining
- Resolution

Inference Rules in Propositional Logic

Modus Ponens or Law of Detachment

(Modus Ponens = mode that affirms)

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \land (p \to q)) \to q$$

Proof using Truth Table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

aka Denying the Consequent

$$\begin{array}{c} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

Proof using Truth Table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

aka Transitivity of Implication or Chain Argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let *p* be "it snows."

Let *q* be "I will study discrete math."

Let *r* be "I will get an A."

"If it snows, then I will study discrete math."

"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

aka Disjunction Elimination or OR Elimination

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$((p \vee q) \wedge \neg p) \rightarrow q$$

Example:
Let *p* be "I will study discrete math."
Let *q* be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

aka Disjunction Introduction

$$\frac{p}{\therefore (p \vee q)}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let p be "I will study discrete math."

Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

aka Conjunction Elimination

$$\frac{p \wedge q}{\therefore\ p}$$

Corresponding Tautology:

$$(p \wedge q) \rightarrow p$$

Example:

Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

aka Conjunction Introduction

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$$((p) \wedge (q)) \rightarrow (p \wedge q)$$

Example:

Let p be "I will study discrete math."

Let q be "I will study English literature."

"I will study discrete math."

"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Resolution plays an important role in Artificial Intelligence and is used in the programming language Prolog.

Corresponding Tautology:

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

Example:

Let *p* be "I will study discrete math."
Let q be "I will study databases."
Let *r* be "I will study English literature."

"I will study discrete math or I will study databases."
"I will not study discrete math or I will study English literature."

"Therefore, I will study databases or I will English literature."

Reduction to Prepositional Inference

Universal Instantiation

- Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall v \quad \alpha}{\textsc{Subst}(\{v/g\}, \alpha)}$$

for any variable v and ground term g

- E.g., $\forall x \; King(x) \wedge greedy(x) \implies Evil(x)$ yields

$$King(John) \wedge greedy(John) \implies Evil(John)$$
$$King(Richard) \wedge greedy(Richard) \implies Evil(Richard)$$
$$King(Father(John)) \wedge greedy(Father(John)) \implies Evil(Father(John))$$
$$\vdots$$

Existential Instantiation

- For any sentence α, variable v, and constant symbol k that does not appear elsewhere in the knowledge base:

$$\frac{\exists v \;\; \alpha}{\textsc{Subst}(\{v/k\}, \alpha)}$$

- E.g., $\exists x \;\; Crown(x) \wedge OnHead(x, John)$ yields

$$Crown(C_1) \wedge OnHead(C_1, John)$$

provided C_1 is a new constant symbol, called a Skolem constant

Instantiation

- Universal Instantiation
  - can be applied several times to **add** new sentences
  - the new KB is logically equivalent to the old

- Existential Instantiation
  - can be applied once to **replace** the existential sentence
  - the new KB is **not** equivalent to the old
  - but is satisfiable iff the old KB was satisfiable

Reduction to Propositional Inference

- Suppose the KB contains just the following:

$$\forall x \; King(x) \wedge greedy(x) \Rightarrow Evil(x)$$
$$King(John)$$
$$greedy(John)$$
$$Brother(Richard, John)$$

- Instantiating the universal sentence in **all possible** ways, we have

$$King(John) \wedge greedy(John) \Rightarrow Evil(John)$$
$$King(Richard) \wedge greedy(Richard) \Rightarrow Evil(Richard)$$
$$King(John)$$
$$greedy(John)$$
$$Brother(Richard, John)$$

- The new KB is propositionalized: proposition symbols are

  $King(John)$, $greedy(John)$, $Evil(John)$, $Brother(Richard, John)$, etc.

Reduction to Propositional Inference

- Claim: a ground sentence* is entailed by new KB iff entailed by original KB
- Claim: every FOL KB can be propositionalized so as to preserve entailment
- Idea: propositionalize KB and query, apply resolution, return result
- Problem: with function symbols, there are infinitely many ground terms, e.g., $Father(Father(Father(John)))$
- Theorem: Herbrand (1930). If a sentence α is entailed by an FOL KB, it is entailed by a **finite** subset of the propositional KB
- Idea: For $n = 0$ to ∞ do
  create a propositional KB by instantiating with depth-n terms
  see if α is entailed by this KB
- Problem: works if α is entailed, loops if α is not entailed
- Theorem: Turing (1936), Church (1936), entailment in FOL is semidecidable

Practical Problems with Propositionalization

- Propositionalization seems to generate lots of irrelevant sentences.

- E.g., from

$$\forall x\ King(x) \wedge greedy(x) \implies Evil(x)$$
$$King(John)$$
$$\forall y\ greedy(y)$$
$$Brother(Richard, John)$$

it seems obvious that $Evil(John)$, but propositionalization produces lots of facts such as $greedy(Richard)$ that are irrelevant.

Unification

Plan

- We have the inference rule
  - $\forall x \; King(x) \wedge greedy(x) \implies Evil(x)$
- We have facts that (partially) match the precondition
  - $King(John)$
  - $\forall y \; greedy(y)$
- We need to match them up with substitutions: $\theta = \{ x/John, y/John \}$ works
  - unification
  - generalized modus ponens

Unification

- UNIFY$(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Mary)$	$\{x/Mary, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, Mary)$	$fail$

- Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, Mary)$

$Knows(John, x)$ $\quad Knows(z_{17}, Mary)$ $\quad \{z_{17}/John, x/Mary\}$

Generalized Modus Ponens

Generalized Modus Ponens

- Generalized modus ponens used with KB of **definite clauses** (exactly one positive literal)
- All variables assumed universally quantified

$$\frac{p_1{}',\ p_2{}',\ \ldots,\ p_n{}',\ (p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q)}{q\theta} \qquad \text{where } p_i{}'\theta = p_i\theta \text{ for all } i$$

- Rule: $King(x) \wedge greedy(x) \implies Evil(x)$
- Precondition of rule: p_1 is $King(x)$ $\quad p_2$ is $greedy(x)$
- Implication: q is $Evil(x)$
- Facts: $p_1{}'$ is $King(John)$ $\quad p_2{}'$ is $greedy(y)$
- Substitution: θ is $\{x/John, y/John\}$

$\Rightarrow$ Result of modus ponens: $q\theta$ is $Evil(John)$

Forward Chaining

Example Knowledge

- *The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.*

- Prove that Col. West is a criminal

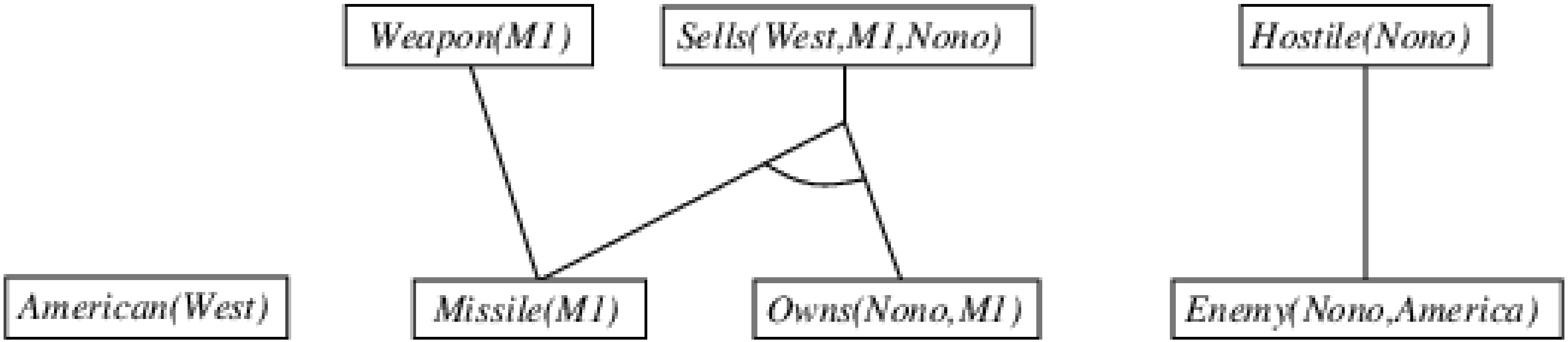
Example Knowledge Base

- ... it is a crime for an American to sell weapons to hostile nations:
  $American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \implies Criminal(x)$
- Nono ... has some missiles, i.e., $\exists\, x\ Owns(Nono, x) \wedge Missile(x)$:
  $Owns(Nono, M_1)$ and $Missile(M_1)$
- ... all of its missiles were sold to it by Colonel West
  $\forall\, x\ \ Missile(x) \wedge Owns(Nono, x) \implies Sells(West, x, Nono)$
- Missiles are weapons:
  $Missile(x) \Rightarrow Weapon(x)$
- An enemy of America counts as "hostile":
  $Enemy(x, America) \implies Hostile(x)$
- West, who is American ...
  $American(West)$
- The country Nono, an enemy of America ...
  $Enemy(Nono, America)$

Forward Chaining Proof

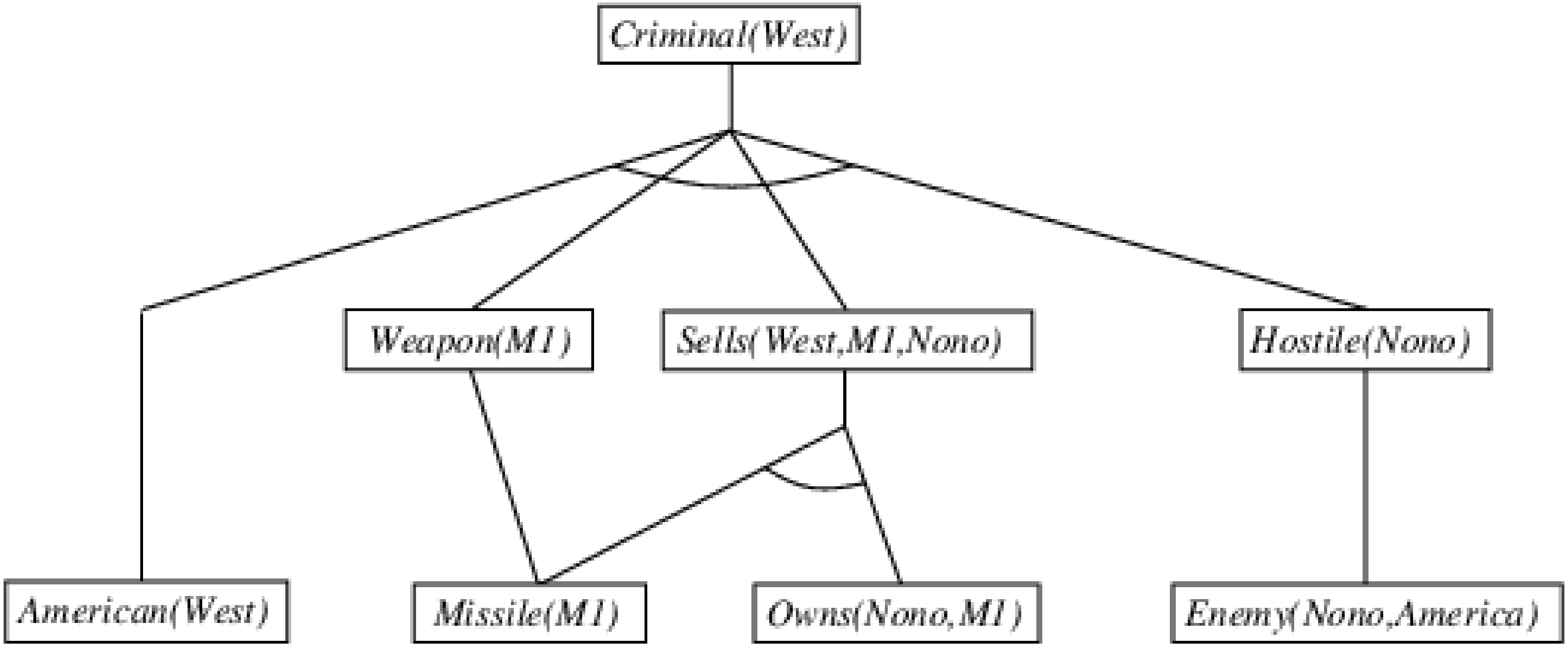
American(West) *Missile(M1)* *Owns(Nono,M1)* *Enemy(Nono,America)*

Forward Chaining Proof



(Note: $\forall x \;\; Missile(x) \wedge Owns(Nono, x) \implies Sells(West, x, Nono)$)

Forward Chaining Proof



(Note: $American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \implies Criminal(x)$)

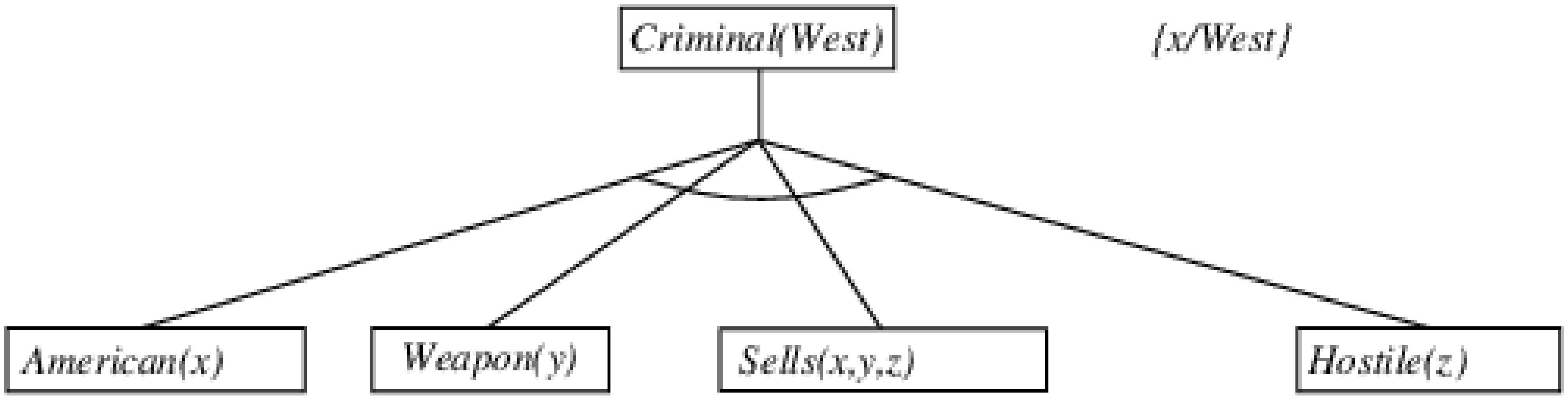
Backward Chaining

Backward Chaining

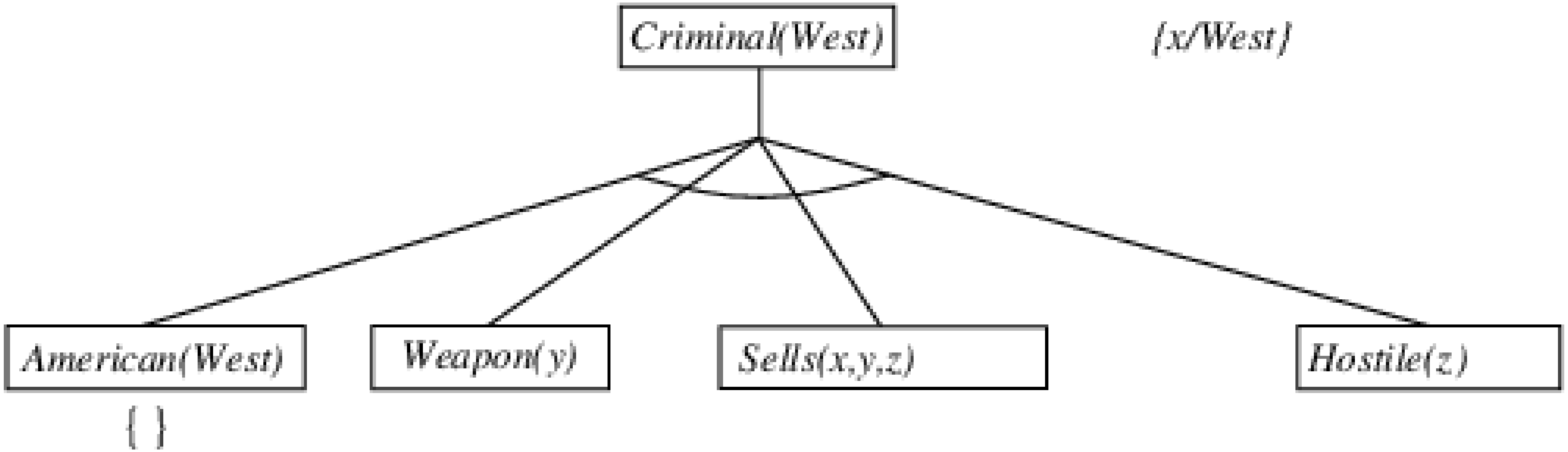
- Start with query
- Check if it can be derived by given rules and facts
  - apply rules that infer the query
  - recurse over pre-conditions

Backward Chaining Example

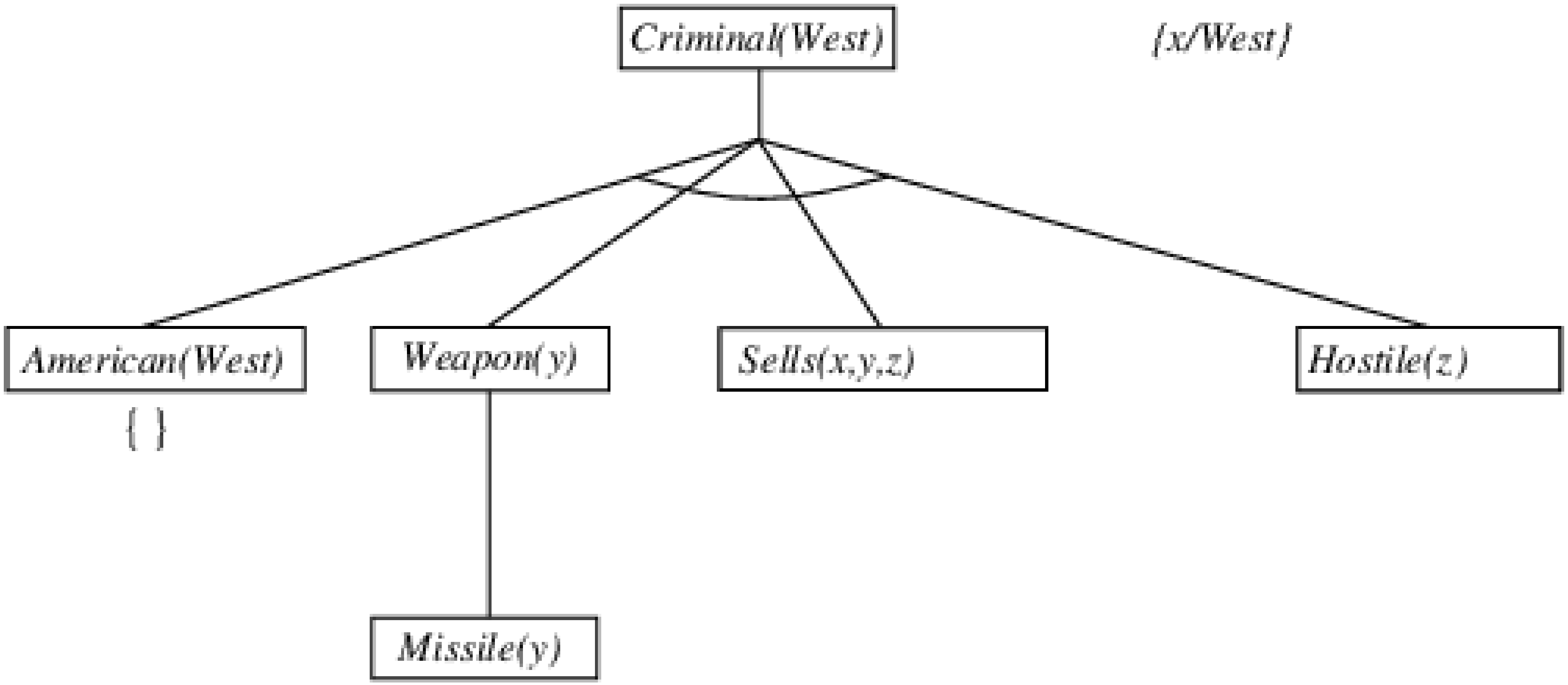
Backward Chaining Example



Backward Chaining Example



Backward Chaining Example



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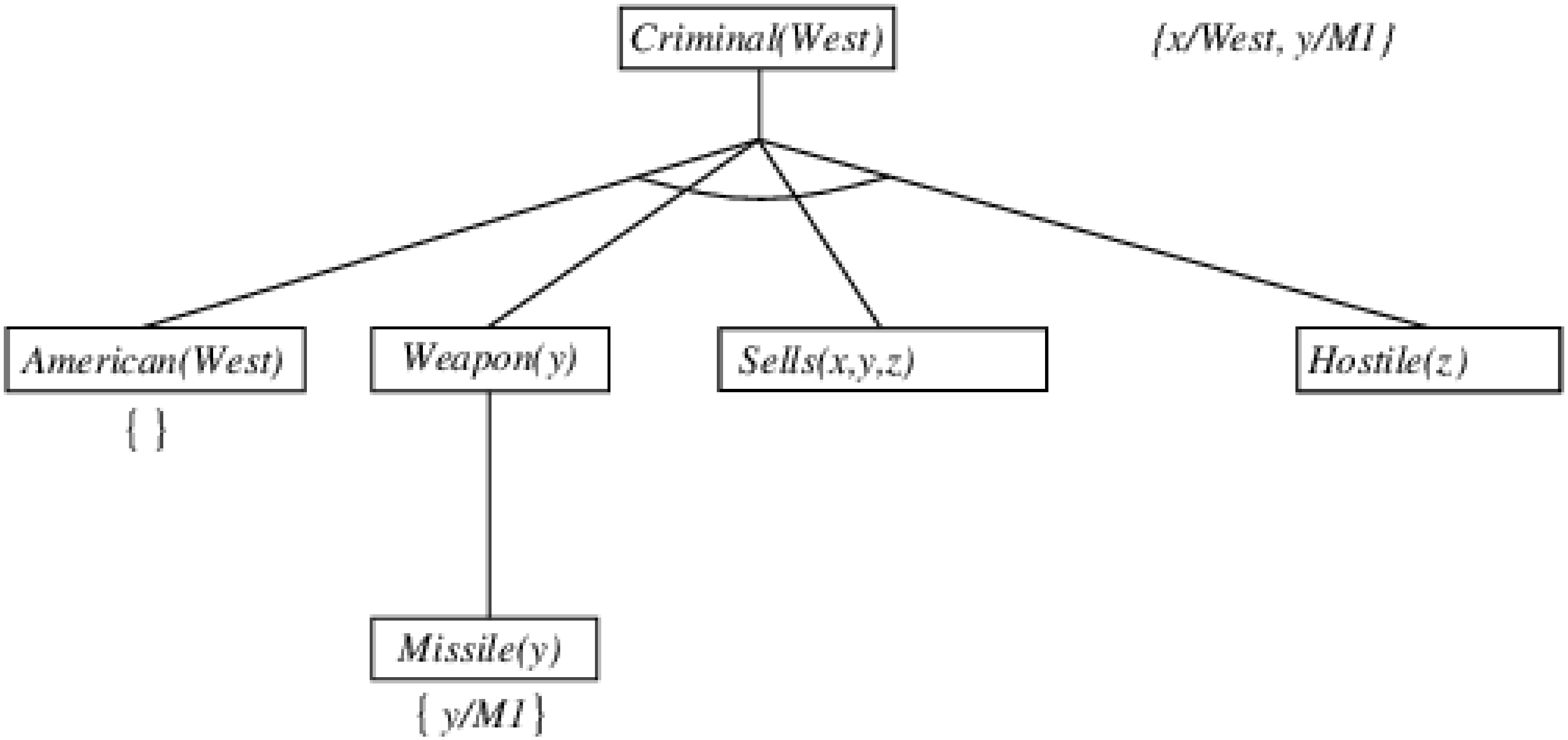

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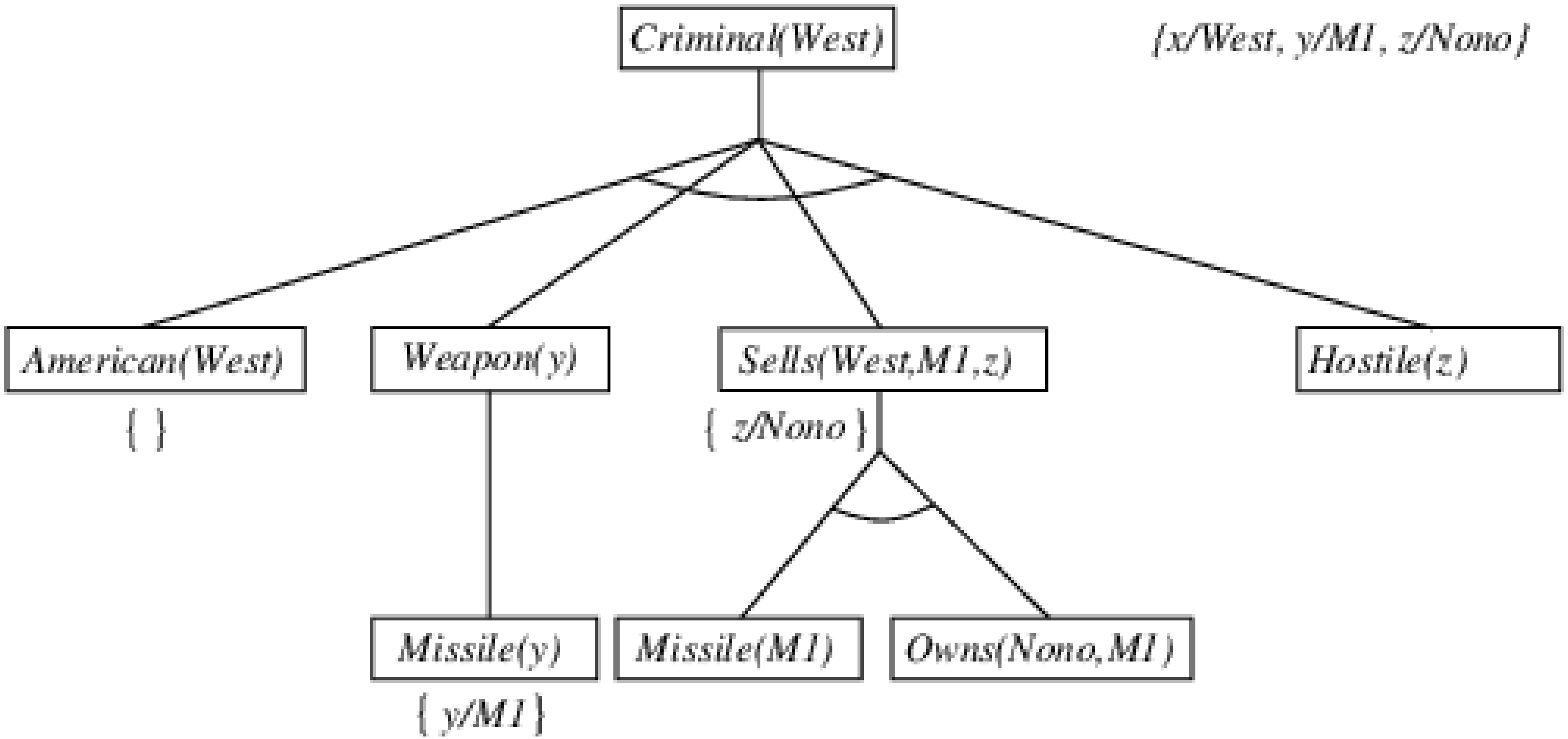

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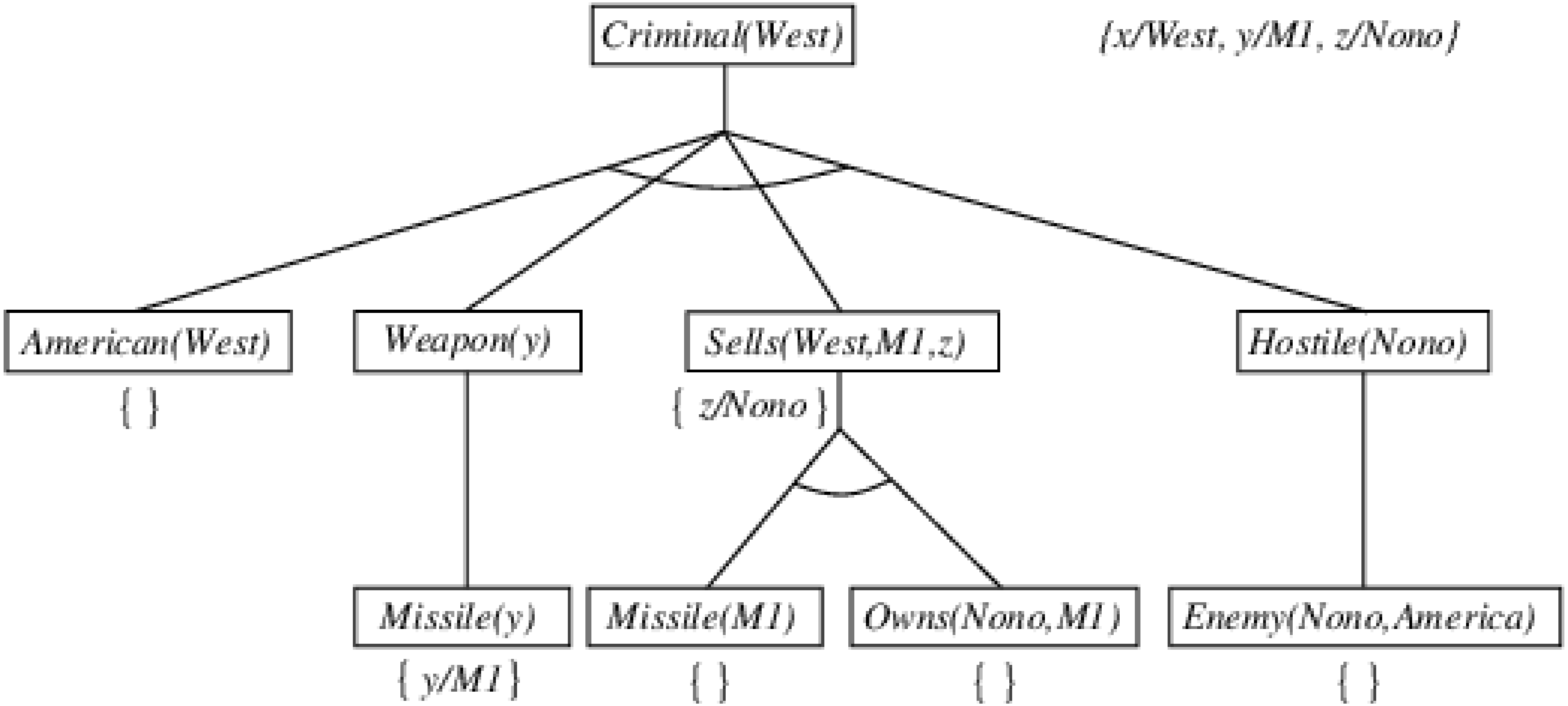

Resolution

Resolution: Brief Summary

- Full first-order version:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{(\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n)\theta}$$

where $\text{UNIFY}(\ell_i, \neg m_j) = \theta$.

Conversion to CNF

Everyone who loves all animals is loved by someone:

$$\forall x\ [\forall y\ Animal(y) \implies Loves(x,y)] \implies [\exists y\ Loves(y,x)]$$

1. Eliminate biconditionals and implications

$$\forall x\ [\neg\forall y\ \neg Animal(y) \lor Loves(x,y)] \lor [\exists y\ Loves(y,x)]$$

2. Move $\neg$ inwards: $\neg\forall x, p \equiv \exists x\ \neg p$, $\neg\exists x, p \equiv \forall x\ \neg p$:

$$\forall x\ [\exists y\ \neg(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y\ Loves(y,x)]$$
$$\forall x\ [\exists y\ \neg\neg Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ Loves(y,x)]$$
$$\forall x\ [\exists y\ Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ Loves(y,x)]$$

Conversion to CNF

3. Standardize variables: each quantifier should use a different one

$$\forall x \ [\exists y \ \ Animal(y) \wedge \neg Loves(x,y)] \vee [\exists z \ \ Loves(z,x)]$$

4. Skolemize: a more general form of existential instantiation.
   Each existential variable is replaced by a Skolem function
   of the enclosing universally quantified variables:

$$\forall x \ [Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee Loves(G(x), x)$$

5. Drop universal quantifiers:

$$[Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee Loves(G(x), x)$$

6. Distribute $\wedge$ over $\vee$:

$$[Animal(F(x)) \vee Loves(G(x), x)] \wedge [\neg Loves(x, F(x)) \vee Loves(G(x), x)]$$

Our Previous Example

- Rules
  - $American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \implies Criminal(x)$
  - $Missile(M_1)$ and $Owns(Nono, M_1)$
  - $\forall x \;\; Missile(x) \wedge Owns(Nono, x) \implies Sells(West, x, Nono)$
  - $Missile(x) \Rightarrow Weapon(x)$
  - $Enemy(x, America) \implies Hostile(x)$
  - $American(West)$
  - $Enemy(Nono, America)$
- Converted to CNF
  - $\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x, y, z) \vee \neg Hostile(z) \vee Criminal(x)$
  - $Missile(M_1)$ and $Owns(Nono, M_1)$
  - $\neg Missile(x) \vee \neg Owns(Nono, x) \vee Sells(West, x, Nono)$
  - $\neg Missile(x) \vee Weapon(x)$
  - $\neg Enemy(x, America) \vee Hostile(x)$
  - $American(West)$
  - $Enemy(Nono, America)$
- Query: $\neg Criminal(West)$

Resolution Proof