PREDICATE LOGIC

WHAT PROPOSITIONAL LOGIC CAN'T DO

It can't explain the validity of the Socrates Argument

Because from the perspective of Propositional Logic, its premises and conclusion have no internal structure.

1. All men are mortal
2. Socrates is a man

---

3. Therefore, Socrates is mortal

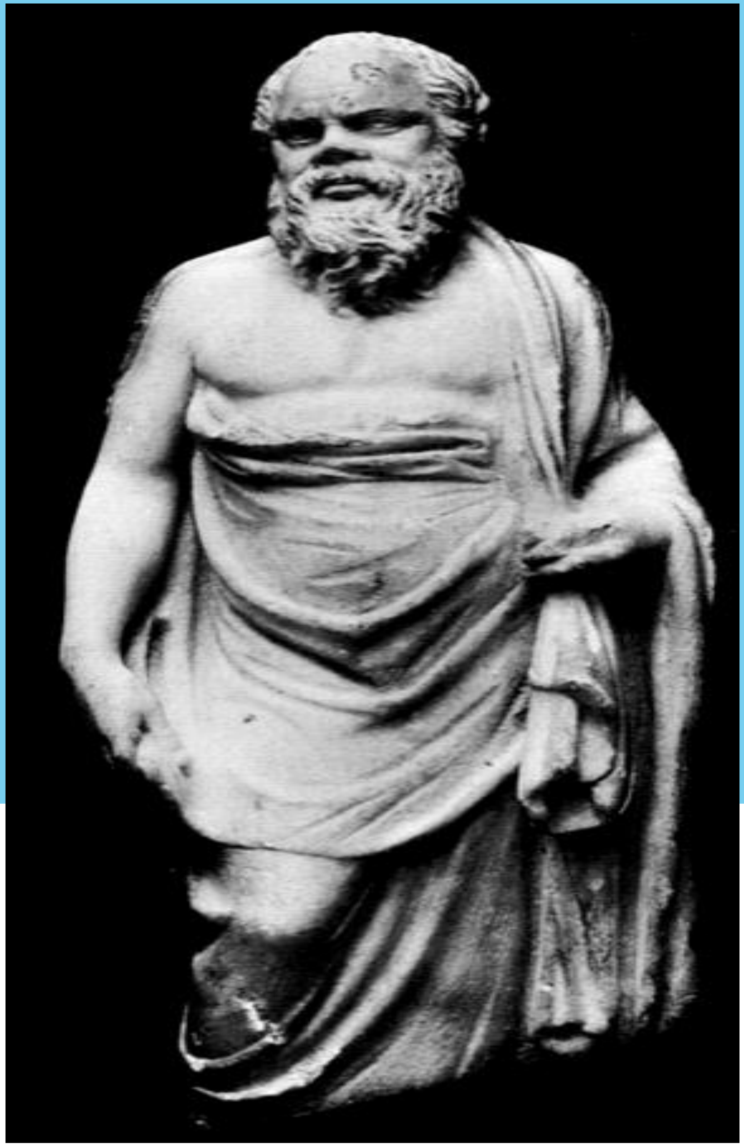
Translates as:

1. H
2. S

---

3. M

THE SOCRATES ARGUMENT



THE PROBLEM

The problem is that the validity of this argument comes from the internal structure of these sentences--which Propositional Logic cannot "see":

1. All men are mortal
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SOLUTION

We need to split the atom!

- To display the internal structure of those "atomic" sentences by adding new categories of vocabulary items

- To give a semantic account of these new vocabulary items and

- To introduce rules for operating with them.

PREDICATE LOGIC/FIRST-ORDER LOGIC

First-order logic (FOL) models the world in terms of

- **Objects,** which are things with individual identities
- **Properties** of objects that distinguish them from other objects
- **Relations** that hold among sets of objects
- **Functions,** which are a subset of relations where there is only one "value" for any given "input"

Examples:

- Objects: Students, lectures, companies, cars ...
- Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
- Properties: blue, oval, even, large, ...
- Functions: father-of, best-friend, second-half, one-more-than ...

USER PROVIDES

Constant symbols, which represent individuals in the world

- Mary
- 3
- Green

Function symbols, which map individuals to individuals

- father-of(Mary) = John
- color-of(Sky) = Blue

Predicate symbols, which map individuals to truth values

- greater(5,3)
- green(Grass)
- color(Grass, Green)

FOL PROVIDES

Variable symbols

- E.g., x, y, foo

Connectives

- Same as in PL: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\rightarrow$), if and only if (biconditional $\leftrightarrow$)

Equality

- $=$

Quantifiers

- Universal $\forall$**x** or **(Ax)**
- Existential $\exists$**x** or **(Ex)**

SENTENCES ARE BUILT FROM TERMS AND ATOMS

A **term** (denoting a real-world individual) is a constant symbol, a variable symbol, or an n-place function of n terms.

x and $f(x_1, \ldots, x_n)$ are terms, where each x_i is a term.

For example, LeftLeg(x). Here, x is a variable.

A term with no variables is a **ground term.**

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A **well-formed formula (wff)** is a sentence containing no “free” variables. That is, all variables are “bound” by universal or existential quantifiers.

SINGULAR STATEMENTS

Make assertions about persons, places, things or times

Examples:

- Socrates is a man.
- Athens is in Greece.
- Thomas Aquinas preferred Aristotle to Plato

To display the internal structure of such sentences, we need two new categories of vocabulary items:

- Individual constants ("names")
- Predicates (relations/functions)

INDIVIDUAL CONSTANTS & PREDICATES

Socrates is a man.

The name of a person—which we'll express by an individual constant

A predicate—assigns a property (being-a-man) to the person named

MORE VOCABULARY

So, to the vocabulary of Propositional Logic we add:

Individual Constants: lower case letters of the alphabet (a, b, c,…, u, v, w)

Predicates: upper case letters (A, B, C,…, X, Y, Z)

- These represent predicates like "__ is human," "__ the teacher of __," "__ is between __ and __," etc.
- Note: Predicates express *properties* which a single individual may have and *relations* which may hold on more than one individual

PREDICATES

1-place predicates assign properties to individuals:

__ is a man

__ is red

2-place predicates assign relations to pairs of individuals

__ is the teacher of __

__ is to the north of __

3-place predicates assign relations to triples of individuals

__ preferred __ to __

And so on…

TRANSLATION

Socrates is a man.

Hs

Socrates is mortal.

Ms

Athens is in Greece

Iag

Thomas preferred Aristotle to Plato.

Ptap

We always put the predicate first followed by the names of the things to which it applies.

PREDICATES DESIGNATE SETS!

Individual constants name individuals—persons, places, things, times, etc.

We resolve to understand properties and relations as *sets*—of individuals or ordered n-tuples of individuals (pairs, triples, quadruples, etc.)

Fido

is brown

{ brown things }

WHAT SINGULAR SENTENCES SAY

Singular sentences are about set membership.

Ascribing a property to an individual is saying that it's a member of the set of individuals that have that property, e.g.

- "Descartes is a philosopher" says that Descartes is a member of {Plato, Aristotle, Locke, Berkely, Hume, Quine…}

Saying that 2 or more individuals stand in a relation is saying that some ordered n-tuple is a member of a set so, e.g.

- "San Diego is north of Chula Vista" says that <San Diego, Chula Vista> is a member of {<LA, San Diego>, <Philadelphia, Baltimore>…}

SINGULAR SENTENCES

1. All men are mortal
2. Socrates is a man

---

3. Therefore, Socrates is mortal

We can now understand what 2 and 3 say:

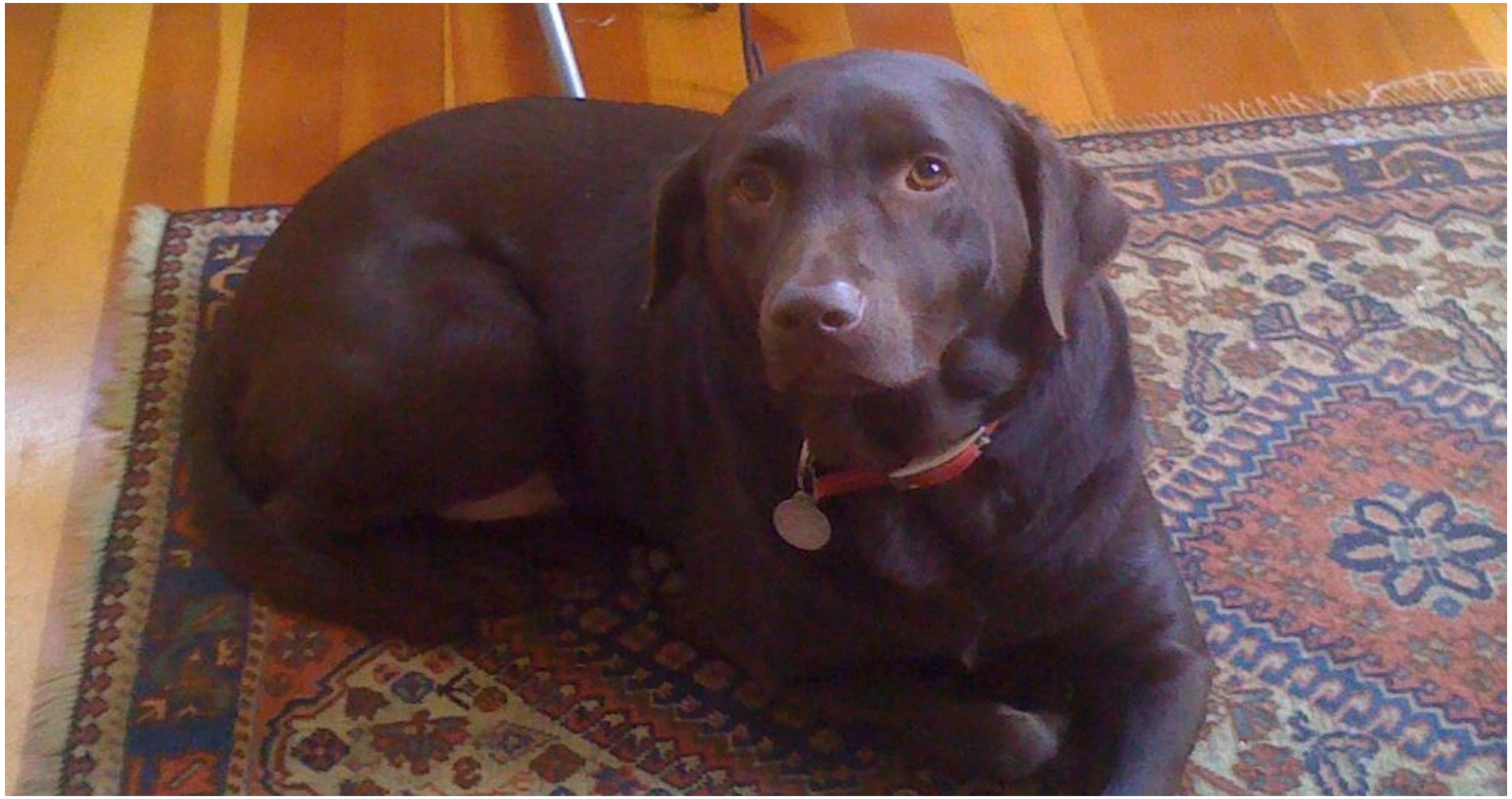
2 says that Socrates **ε** {humans}

3 says that Socrates **ε** {mortals}

SINGULAR STATEMENTS

Translation

DUCATI IS BROWN.



Bd

TWEETY BOPPED SYLVESTER

Bts

COMPLEX SENTENCES

Complex sentences are made from atomic sentences using connectives

$\neg S$, $S1 \wedge S2$, $S1 \vee S2$, $S1 \Rightarrow S2$, $S1 \Leftrightarrow S2$

E.g. ¬Brother(LeftLeg (Richard), John)

Brother(Richard, John) ∧ Brother(John, Richard)

¬King (Richard) ⇒ King (John)

Note: The last one is equivalent to:

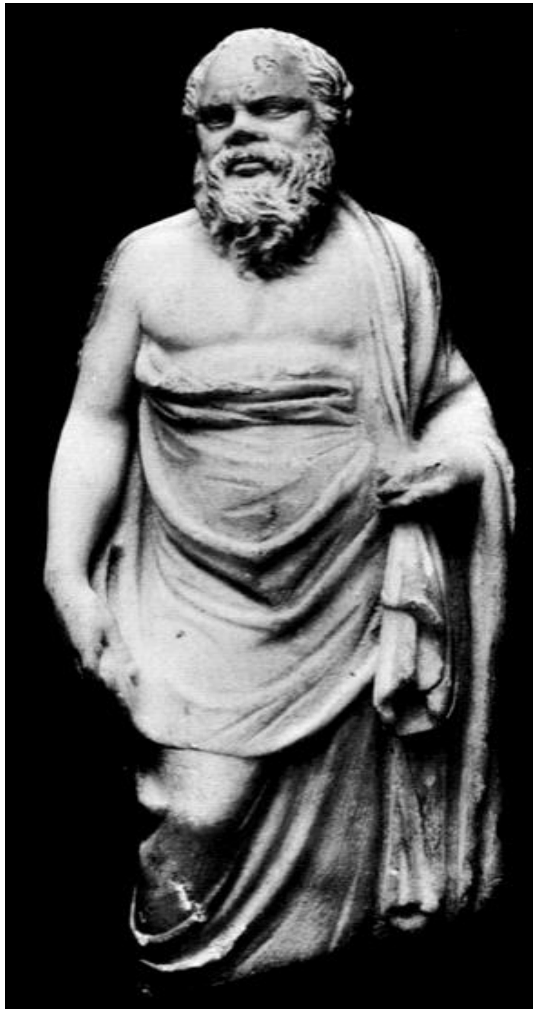
(¬King (Richard)) ⇒King (John)

and not:

¬(King (Richard) ⇒ King (John))

GENERAL STATEMENTS

GENERAL SENTENCES



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But we still don't have an account of what 1 says

Because it doesn't ascribe a property or relation to any individual or individuals.

GENERAL SENTENCES

1. All men are mortal
2. Socrates is a man

---

3. Therefore, Socrates is mortal

We can't treat "all men" as the name of an individual as, e.g. the sum or collection of all humans

Consider: "All men weigh under 1000 pounds."

We want to ascribe properties to every man *individually*.

QUANTIFIERS

Universal quantification

- $(\forall x)P(x)$ means that P holds for **all** values of x in the domain associated with that variable
- E.g., $(\forall x)$ dolphin(x) $\rightarrow$ mammal(x)

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Existential quantification

- $(\exists\ x)P(x)$ means that P holds for **some** value of x in the domain associated with that variable
- E.g., $(\exists\ x)$ mammal(x) $\wedge$ lays-eggs(x)
- Permits one to make a statement about some object without naming it

EXAMPLE

Consider the interpretation in which

- Richard → Richard the Lionheart
- John → the evil King John
- Brother → the brotherhood relation

Under this interpretation, Brother(Richard, John) is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

UNIVERSAL QUANTIFICATION

$\forall$ <variables> <sentence>

All kings are persons:

- $\forall x \; King(x) \Rightarrow Person(x)$

$\forall x \; P$ is true in a model m iff P is true with x being each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

- $(King(Richard) \Rightarrow Person(Richard))$
- $\land \; (King(John) \Rightarrow Person(John))$
- $\land \; (King(TheCrown) \Rightarrow Person(TheCrown))$
- $\land \; (King(LeftLeg(Richard)) \Rightarrow Person(LeftLeg(Richard)))$
- $\land \; . \; . \; .$

A COMMON MISTAKE TO AVOID

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$\forall$ x King(John) $\wedge$ Person(John) means “Everything is both a king and a person”

EXISTENTIAL QUANTIFICATION

∃ <variables> <sentence>

King John has a crown on his head:

$\exists\, x\ Crown(x) \land OnHead(x, John)$

$\exists\, x\ P$ is true in a model m iff P is true with x being some possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

- $(Crown(Richard) \land OnHead(Richard, John))$
- $\lor\ (Crown(John) \land OnHead(John, John))$
- $\lor\ (Crown(TheCrown) \land OnHead(TheCrown, John))$
- $\lor\ (Crown(LeftLeg(Richard)) \land OnHead(LeftLeg(Richard), John))$
- $\lor\ \ldots$

ANOTHER COMMON MISTAKE TO AVOID

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$\exists\, x\ Crown(x) \Rightarrow OnHead(x, John)$

is true if there is anything that is not a crown!

QUANTIFIERS

Universal quantifiers are often used with "implies" to form "rules":

$(\forall x)$ student(x) $\rightarrow$ smart(x) means "All students are smart"

Universal quantification is *rarely* used to make blanket statements about every individual in the world:

$(\forall x)$student(x)$\wedge$smart(x) means "Everyone in the world is a student and is smart"

Existential quantifiers are usually used with "and" to specify a list of properties about an individual:

$(\exists x)$ student(x) $\wedge$ smart(x) means "There is a student who is smart"

A common mistake is to represent this English sentence as the FOL sentence:

$(\exists x)$ student(x) $\rightarrow$ smart(x)

- But what happens when there is a person who is *not* a student?

QUANTIFIER SCOPE

Switching the order of universal quantifiers *does not* change the meaning:

- $(\forall x)(\forall y)P(x,y) \boxed{\leftrightarrow} (\forall y)(\forall x)\, P(x,y)$

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Similarly, you can switch the order of existential quantifiers:

- $(\exists x)(\exists y)P(x,y) \boxed{\leftrightarrow} (\exists y)(\exists x)\ P(x,y)$

Switching the order of universals and existentials *does* change meaning:

- Everyone likes someone: $(\forall x)(\exists y)$ likes(x,y)
- Someone is liked by everyone: $(\exists y)(\forall x)$ likes(x,y)

CONNECTIONS BETWEEN ALL AND EXISTS

We can relate sentences involving $\forall$ and $\exists$ using De Morgan's laws:

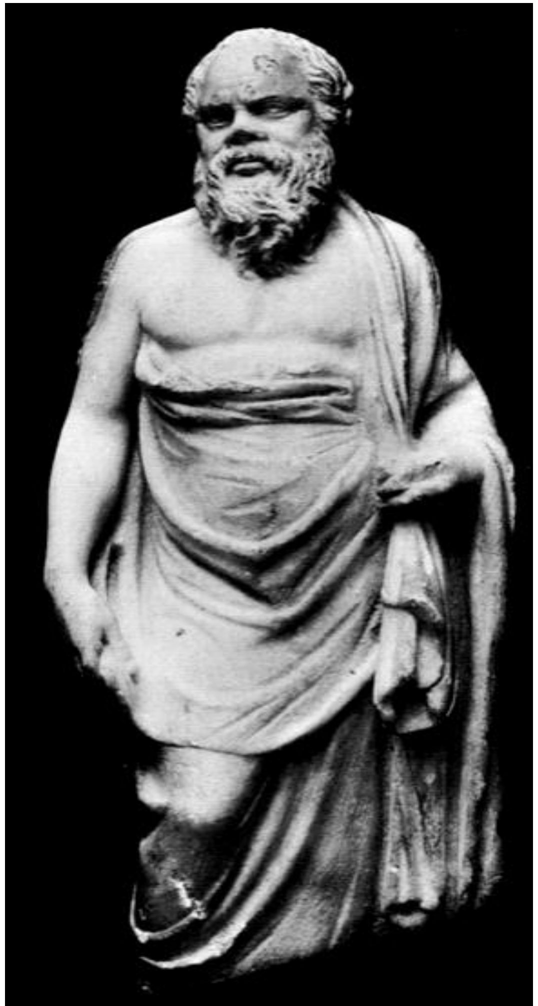
$(\forall x) \neg P(x) \; \boxed{\leftrightarrow} \; \neg(\exists x) P(x)$

$\neg(\forall x) P \; \boxed{\leftrightarrow} \; (\exists x) \neg P(x)$

$(\forall x) P(x) \; \boxed{\leftrightarrow} \; \neg (\exists x) \neg P(x)$

$(\exists x) P(x) \; \boxed{\leftrightarrow} \; \neg(\forall x) \neg P(x)$

THE SOCRATES ARGUMENT IS VALID!



1. All men are mortal
2. Socrates is a man

---

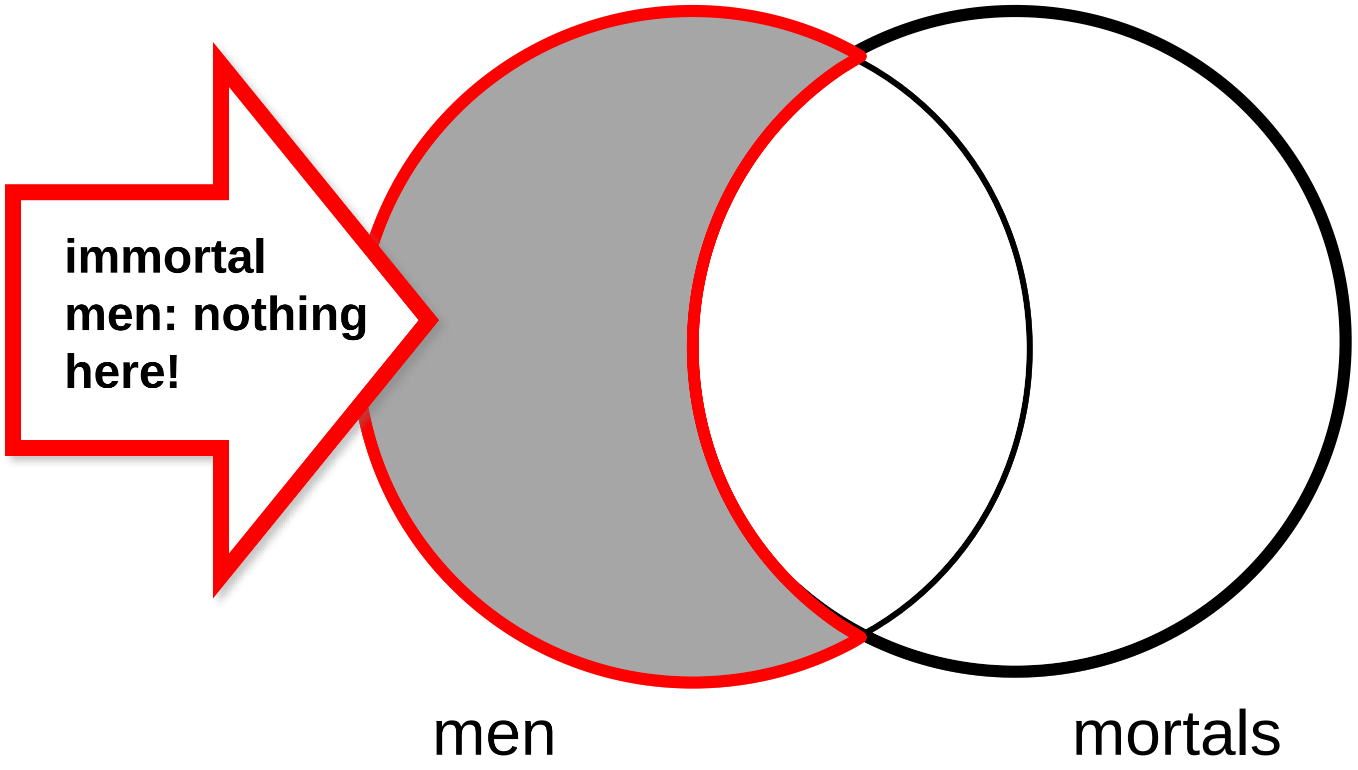
3. Therefore, Socrates is mortal

1 says that there are no men that don't belong to the set of mortals

2 says that Socrates ε {men}

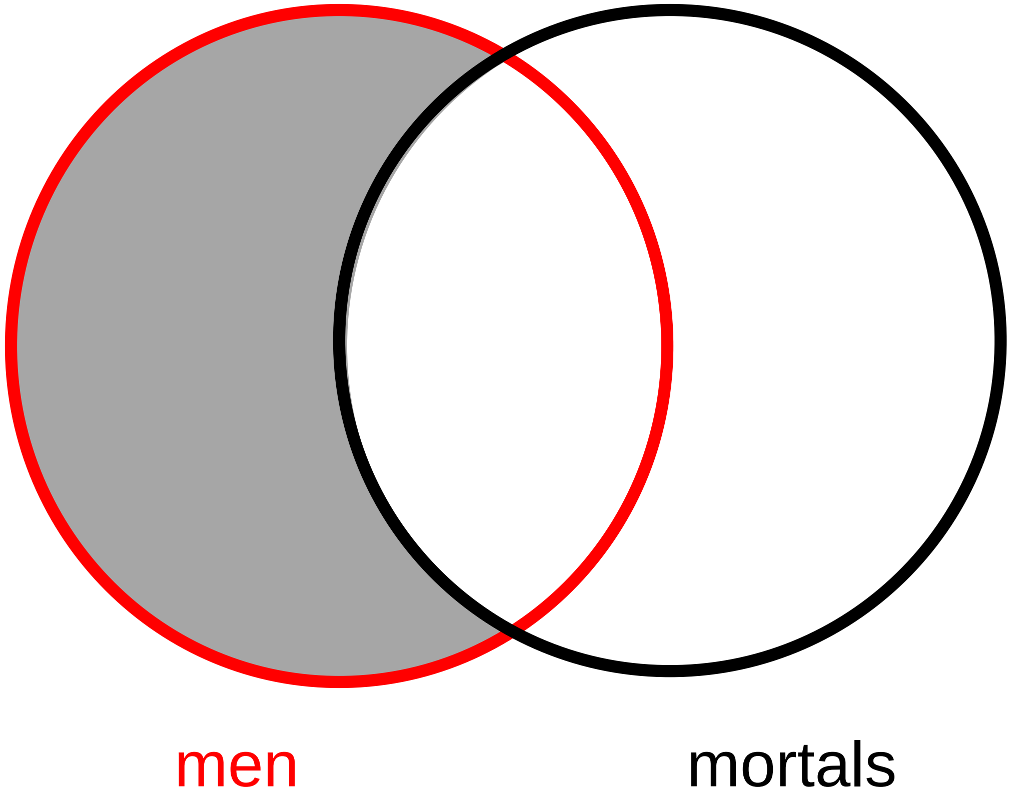
3 says that, therefore, Socrates ε {mortals}

ALL MEN ARE MORTALS



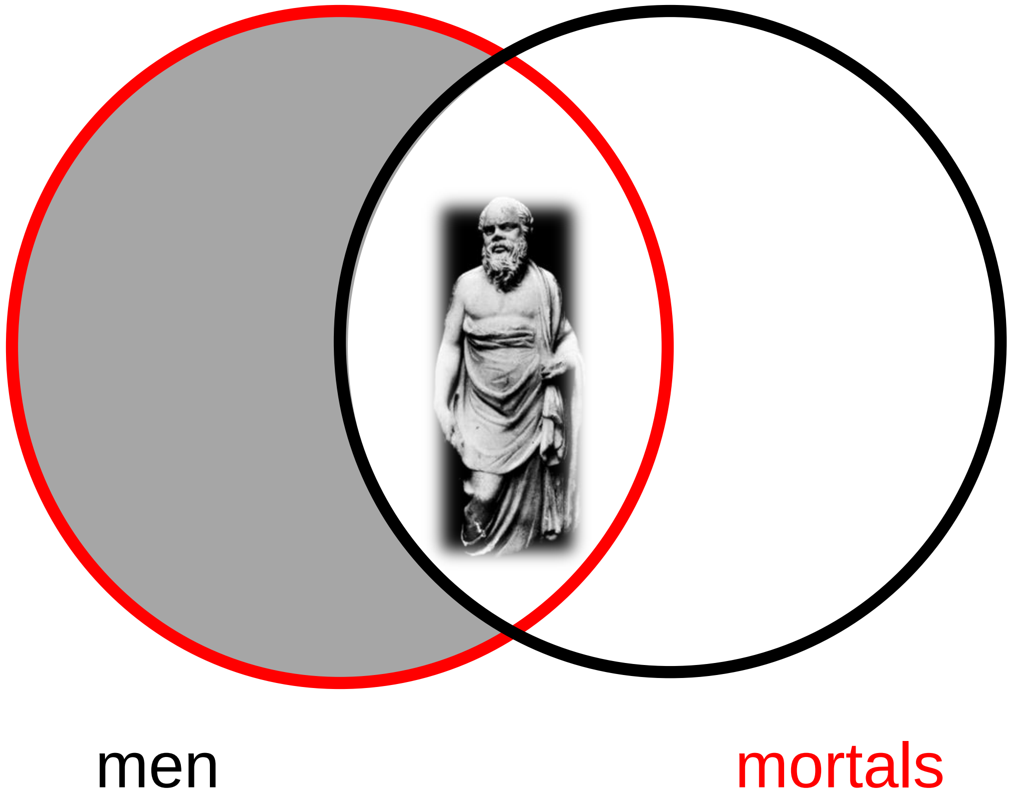
There's nothing in the set of men outside of the set of mortals; the set of immortal men is empty.

SOCRATES IS A MAN.



This is the only place in the men circle Socrates can go since we've already said that there are no immortal men.

THEREFORE, SOCRATES IS A MORTAL.



So Socrates automatically ends up in the mortals circle: the argument is, therefore, valid!

ARGUMENTS INVOLVING RELATIONS

1. All dogs love their people
2. Bo is Obama's Dog

___

3. Bo loves Obama

If we treat the predicates in these 3 sentences as one-place predicates, we can't explain the validity of this argument!

It looks like we're ascribing 3 different properties that don't have anything to do with one another: people-loving, being-Obama's-dog and Obama-loving

VALIDITY OF THE DOG ARGUMENT

1. All dogs love their people
2. Bo is Obama's Dog

———————————————

3. Bo loves Obama

We can consider *being the dog of* and *loves* as relations and designate them by the 2-place predicates "D" and "L"

Now the 3 sentences have something in common--they don't involve 3 different properties; they involve 2 relations: *being the dog of* and *loves*

So we can link them to show validity

VALIDITY OF THE DOG ARGUMENT

1. All dogs love their people
2. Bo is Obama's Dog

____

3. Bo loves Obama

1. (x)(y)(Dxy ⊃ Lxy)
2. Dbo

____

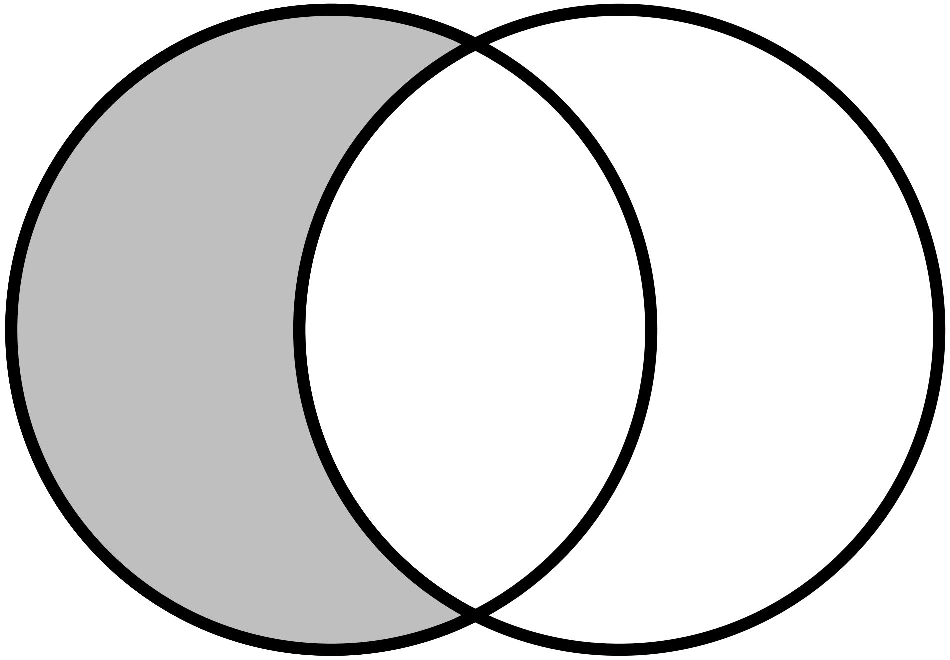
3. Lbo

1 says that for any x and y, if x is the dog of y then x loves y.

2 says that Bo is the dog of Obama.

So it follows that Bo loves Obama!

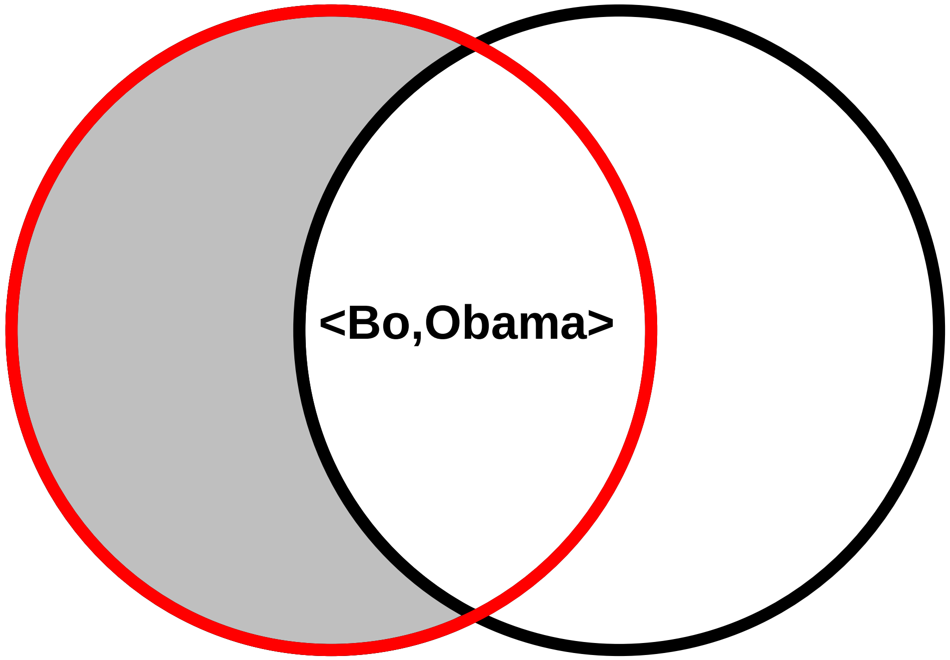
ALL DOGS LOVE THEIR PEOPLE



__ is the dog of __ __ loves __

$$(x)(y)(Dxy \supset Lxy)$$

BO IS OBAMA'S DOG

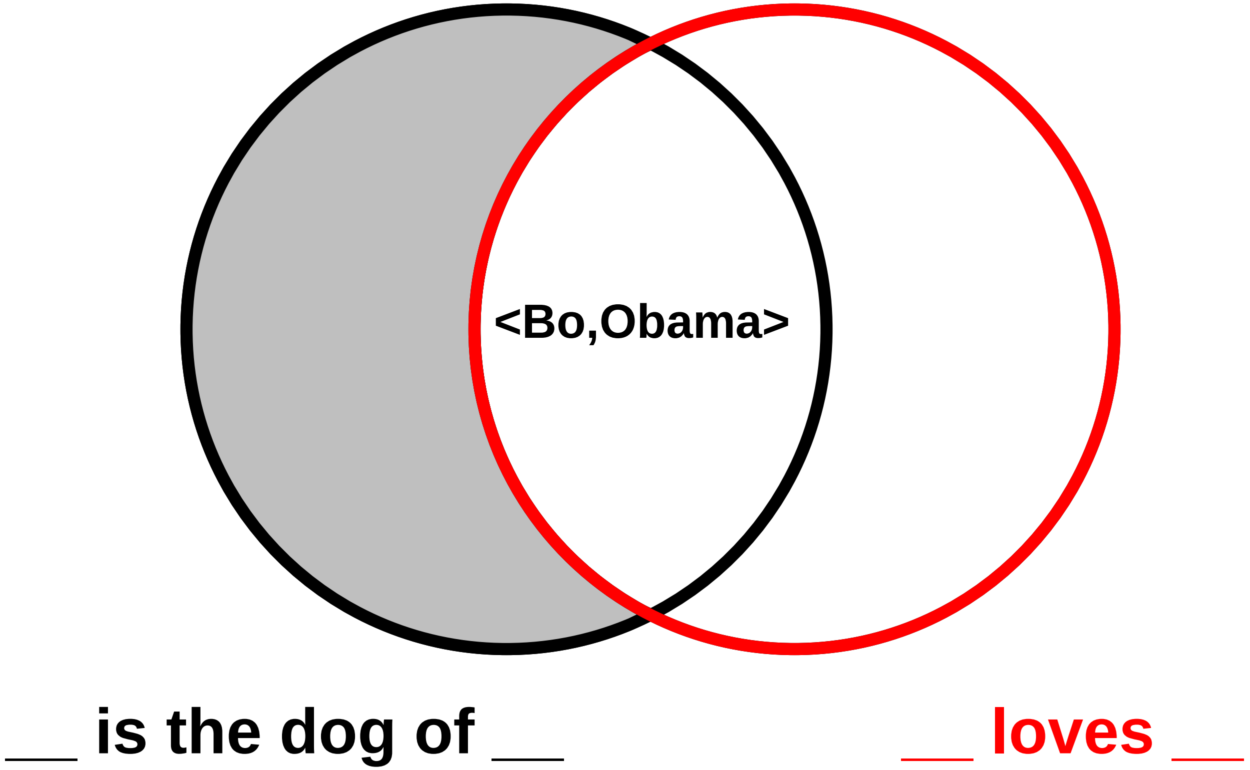


__ is the dog of __

__ loves __

Dbo

BO LOVES OBAMA



Lbo

SCOPE

THERE ARE DOGS AND THERE ARE CATS.

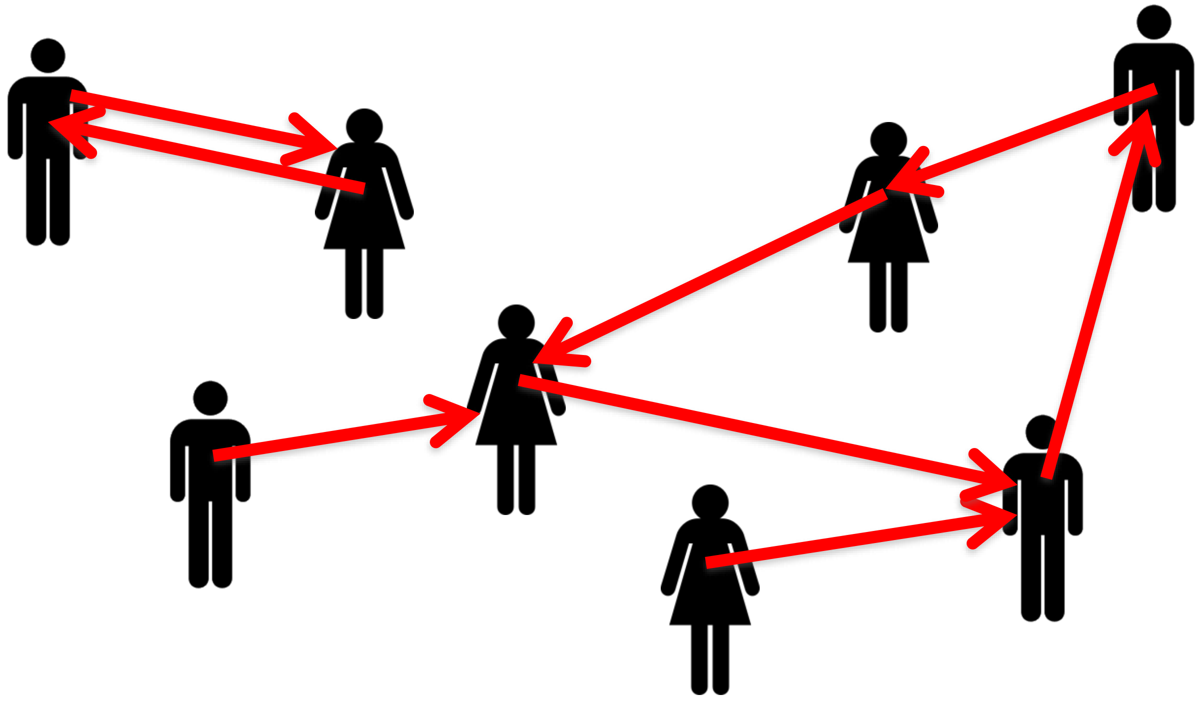
$(\exists x)Dx \bullet (\exists x)Cx$: There exists an x such that x is a dog and there exists an x such that x is a cat

THERE ARE DOG-CATS

$(\exists x)(Dx \bullet Cx)$: There exists an x such that x is a dog-and-a-cat

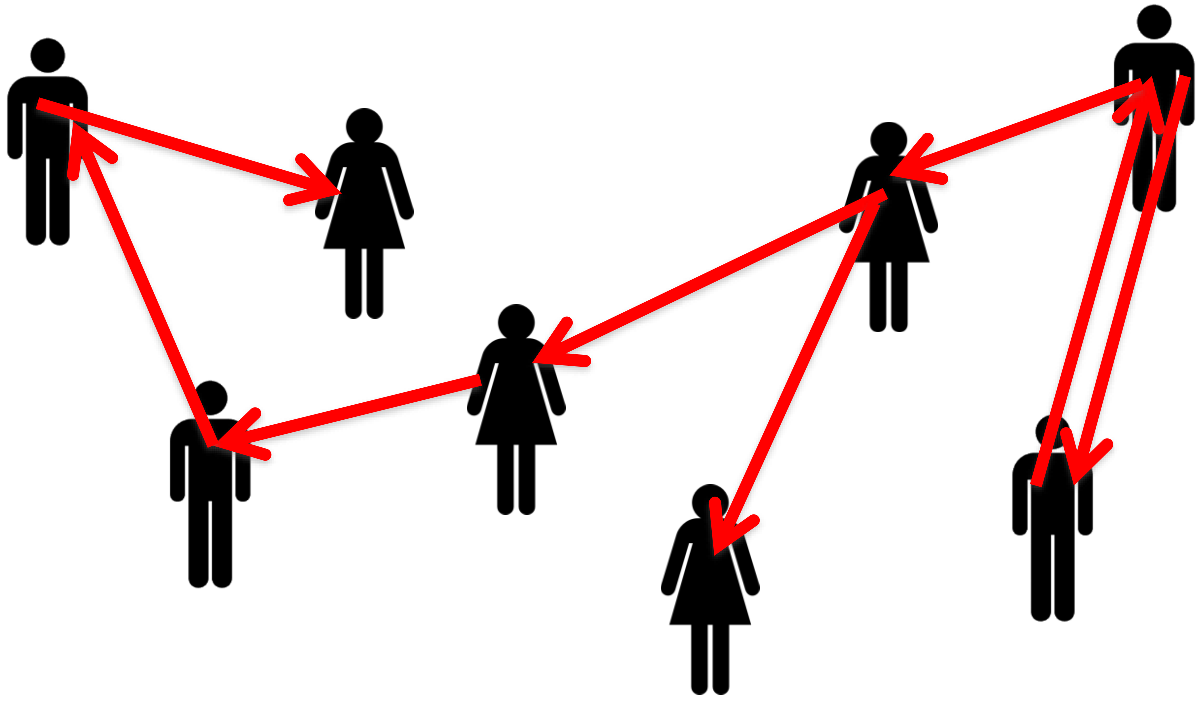
OVERLAPPING QUANTIFIERS

EVERYBODY LOVES SOMEBODY



$(\forall x)(\exists y)Lxy$: for all x, there exists a y such that x loves y

EVERYBODY IS LOVED BY SOMEBODY



$(\forall x)(\exists y)Lyx$: for all x, there exists a y such that y loves x

THERE'S SOMEBODY EVERYBODY LOVES

$(\exists x)(\forall y)Lyx$: There exists an x such that for all y, y loves x

THERE'S SOMEBODY WHO LOVES EVERYBODY

$(\exists x)(\forall y)Lxy$: There exists an x such that for all y, x loves y