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CS-512 - A0

A. $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

1> $2A - B = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2-4 \\ 4-5 \\ 6-6 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

2> $\|A\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$\|A\|_1 = |1| + |2| + |3| = 6$

$\|A\|_\infty = \max(|1|, |2|, |3|) = 3$

$\|A\|_p = \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} = \left[|1|^p + |2|^p + |3|^p \right]^{1/p}$

The angle of A relative to the positive x-axis,

$\cos\theta_x = \frac{a_x}{\|A\|} = \frac{1}{\sqrt{14}}$

$\theta_x = \cos^{-1} \frac{1}{\sqrt{14}} = \cos^{-1}(0.2673) = 1.3002 = 74.49^\circ$

3> $\hat{A}$, a unit vector in the direction of A,

$$\hat{A} = \frac{A}{\|A\|} = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}$$

4> The direction cosines of A,

$$\alpha = \cos\theta_x = \frac{a_x}{\|A\|} = \frac{1}{\sqrt{14}}$$

$$\beta = \cos\theta_y = \frac{a_y}{\|A\|} = \frac{2}{\sqrt{14}}$$

$$\gamma = \cos\theta_z = \frac{a_z}{\|A\|} = \frac{3}{\sqrt{14}}$$

5> $$A \cdot B = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} = (1\times4)+(2\times5)+(3\times6)$$
$$= 4 + 10 + 18$$
$$= 32$$

$$B \cdot A = \begin{bmatrix}4\\5\\6\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = (4\times1)+(5\times2)+(6\times3)$$
$$= 4 + 10 + 18$$
$$= 32$$

6> The angle b/w A and B,

$$\theta = \cos^{-1}\left[\frac{A\cdot B}{\|A\|\,\|B\|}\right] = \cos^{-1}\left[\frac{32}{\sqrt{14}\cdot\sqrt{77}}\right] = \cos^{-1}\left[\frac{32}{7\sqrt{22}}\right]$$

$$= \cos^{-1}(0.9746) = 0.2259 = 12.941^\circ$$

7> a vector which is perpendicular to A,

Suppose a vector is $D = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$

$A \cdot D = 0$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = 0$$

$d_x + 2d_y + 3d_z = 0$

Take $d_x = 1$, $d_y = 1$, $d_z = -1$

$1 + 2 - 3 = 0$

$$D = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

8> $A \times B = \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

$$= \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \hat{x} + \begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix} \hat{y} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \hat{z}$$

$$= (12 - 15)\hat{x} + (12 - 6)\hat{y} + (5 - 8)\hat{z}$$

$$= -3\hat{x} + 6\hat{y} - 3\hat{z}$$

$B \times A = \begin{vmatrix} x & y & z \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$

9> a vector which is perpendicular to both A and B.

A × B gives a vector which is perpendicular to a plan created by A and B.

$$A \times B = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

10> The linear dependency b/w A, B, C.

$$aA + bB + cC = 0$$

$$\begin{bmatrix} a + 4b - c \\ 2a + 5b + c \\ 3a + 6b + 3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$a + 4b - c = 0$ — (I)
$2a + 5b + c = 0$ — (II)
$3a + 6b + 3c = 0$ — (III)

(I) + (II), $3a + 9b = 0 \Rightarrow a + 3b = 0$ — (IV)
3(II) − 2(III), $3b - 3c = 0 \Rightarrow b - c = 0 \Rightarrow b = c$ — (V)
3(I) − 2(II), $+3a - 9c = 0 \Rightarrow a + 3c = 0$ — (VI)

from eq^n IV, V, VI, take $b = c = 1 \Rightarrow a = -3$

$$-3A + B + C = 0 \Rightarrow 3A = B + C$$

$$3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

13

11> $A^TB = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18$
$= 32$

$AB^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$

B. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$

1> $2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-1 & 4-2 & 6-1 \\ 8-2 & -4-1 & 6+4 \\ -3 & 10+2 & -2-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$

2> $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$

3) $(AB)^T = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}^T$

$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$

4) $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} = 1\cdot(2-15) + 2(0+4) + 3(20-0)$
$= -13 + 8 + 60$
$= 55$

$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = 1\cdot(15-6) + 2(-6-12) + 3(4+5)$
$= 9 - 36 + 27$
$= 0$

5) only in B, the row vectors form an orthogonal set,

for A, $\langle 1,2,3 \rangle \cdot \langle 4,-2,3 \rangle = 9 \neq 0$
for B, $\langle 1,2,1 \rangle \cdot \langle 2,1,-4 \rangle = 0$
$\langle 2,1,-4 \rangle \cdot \langle 3,-2,1 \rangle = 0$
$\langle 3,-2,1 \rangle \cdot \langle 1,2,1 \rangle = 0$
for C, $\langle 1,2,3 \rangle \cdot \langle 4,5,6 \rangle = 32 \neq 0$

6> $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{55} adj(A)$

0

$$= \frac{1}{55} adj \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$$

$$= \frac{1}{55} \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -13/55 & 17/55 & 12/55 \\ 4/55 & -1/55 & 9/55 \\ 20/55 & -5/55 & -10/55 \end{bmatrix}$$

$B^{-1} = \frac{1}{|B|} adj(B)$

$$|B| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{vmatrix} = 1(1-8)+2(-12-2)+1(-4-3)$$
$$= -7-28-7$$
$$= -42$$

$$B^{-1} = \frac{1}{-42} adj \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \frac{1}{-42} \begin{bmatrix} -7 & -4 & -9 \\ -14 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 7/42 & 4/42 & 9/42 \\ 14/42 & 2/42 & -6/42 \\ 7/42 & -8/42 & 3/42 \end{bmatrix}$$

C. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

1> eigenvalues and eigenvectors of A.

$$Ax = \lambda x \quad , \quad x \neq 0$$
$$\lambda x - Ax = 0 \quad , \quad x \neq 0$$
$$(\lambda I - A)x = 0 \quad , \quad x \neq 0$$

$$|(\lambda I - A)| = 0$$

$$\begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 2) - 6 = 0$$
$$\lambda^2 - 2\lambda - \lambda + 2 - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0$$
$$\lambda = 4 \text{ or } \lambda = -1$$

eigenvalues are $\lambda = 4$ and $\lambda = -1$.

- Take $\lambda = 4$,

$$\begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x_1 - 2x_2 \\ -3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 = 2x_2$$

Take $x_2 = 3 \Rightarrow x_1 = 2$

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- Take $\lambda = -1$

$$\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x_1 - 2x_2 \\ -3x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = -x_2$$

Take $x_2 = 1 \Rightarrow x_1 = -1$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

eigenvectors are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\lambda = 4$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $\lambda = -1$.

2) $V^{-1} A V = ?$

$$V = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{|V|} \operatorname{adj}(V)$$

$$|V| = 2 + 3 = 5 \qquad = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\operatorname{adj}(V) = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \qquad = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix}$$

$$V^{-1} A V = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix} \begin{bmatrix} 8 & 1 \\ 12 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

3> dot product b/w eigenvectors of A

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2+3 = 1$$

4> dot product b/w eigenvectors of B

$$Bx = \lambda x, \quad x \neq 0$$
$$\lambda x - Bx = 0, \quad x \neq 0$$
$$(\lambda I - B)x = 0, \quad x \neq 0$$

$$|(\lambda I - B)| = 0$$

$$\begin{vmatrix} \lambda-2 & 2 \\ 2 & \lambda-5 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda-5) - 4 = 0$$
$$\lambda^2 - 7\lambda + 10 - 4 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$(\lambda-6)(\lambda-1) = 0$$
$$\lambda = 6 \quad \text{or} \quad \lambda = 1$$

eigen values are $\lambda = 6$ and $\lambda = 1$

- Take $\lambda = 6$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

Take $x_2 = 2 \Rightarrow x_1 = -1$

$$x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

- Take $\lambda = 1$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x_1 + 2x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 2x_2$$

Take $x_2 = 1 \Rightarrow x_1 = 2$

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

dot product $\Rightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -2 + 2 = 0$

5) property of eigenvectors of B

here, B is a symmetric Matrix.

$B = B^T$.

First property, all eigenvalues of B are real.

$\lambda = 6 \in R$ and $\lambda = 1 \in R$

Second property, the eigenvectors of B are orthonormal i.e the composed matrix of eigenvectors is an orthogonal matrix.

eigen vectors => $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

divide both vectors with their determinant

=> $\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

These are orthonormal.

dot product => $-\frac{2}{5} + \frac{2}{5} = 0$

$\left| \begin{matrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{matrix} \right| = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1$

$\left| \begin{matrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{matrix} \right| = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{1} = 1$

composed matrix $= \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = U$

This is orthogonal!

$U^T U = U U^T = I$ | $\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$U^T = U^{-1}$

$U^T = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$U^{-1} = \frac{1}{|U|} \operatorname{adj}(U) = -1 \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$= U^T$

$|U| = \frac{-1}{5} - \frac{4}{5} = -1$

$\operatorname{adj}(U) = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

D. $f(x) = x^2 + 3, \quad g(x, y) = x^2 + y^2$

1> $f'(x) = \frac{d f(x)}{dx} = \frac{d}{dx}(x^2+3) = \frac{dx^2}{dx} + \frac{d3}{dx}$

$= 2x \frac{dx}{dx} + 0 = 2x$

$f''(x) = \frac{d f'(x)}{dx} = \frac{d}{dx}(2x) = \frac{2dx}{dx} = 2$

1) $\frac{\partial g}{\partial x} = \frac{\partial g(x,y)}{\partial x} = \frac{\partial (x^2+y^2)}{\partial x} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial x}$

$= 2x\frac{\partial x}{\partial x} + 0 = 2x$

$\frac{\partial g}{\partial y} = \frac{\partial g(x,y)}{\partial y} = \frac{\partial (x^2+y^2)}{\partial y} = \frac{\partial x^2}{\partial y} + \frac{\partial y^2}{\partial y}$

$= 0 + 2y\frac{\partial y}{\partial y} = 2y$

3) $\nabla g(x,y) = \begin{bmatrix} \frac{\partial g(x,y)}{\partial x} \\ \frac{\partial g(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

4) The probability density function of a univariate gaussian (normal) distribution,

$p(x \mid \mu, \sigma^2) = N(x; \mu, \sigma^2) = f(x)$

$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

μ = mean = expection of distribution
σ = standard deviation
σ^2 = variance
x = random variable.