Yash M. Sawant

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Abstract

Ellipse is a very interesting closed curve. It has various properties such as reflection property. Also, we had to approximate the perimeter of an ellipse as we cannot deduce the closed form for the same. In this article, I will discuss another interesting property based on reflection property of the ellipse.

Equation of an ellipse is defined as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

Suppose a light beam start its journey from one of the focus say $S_1(-ae, 0)$ where $e = \sqrt{\frac{a^2-b^2}{a^2}}$.
It will reflect back to S_2 say from point on the ellipse at A_1 because of the reflection property. We can see the infinite sequence as follows

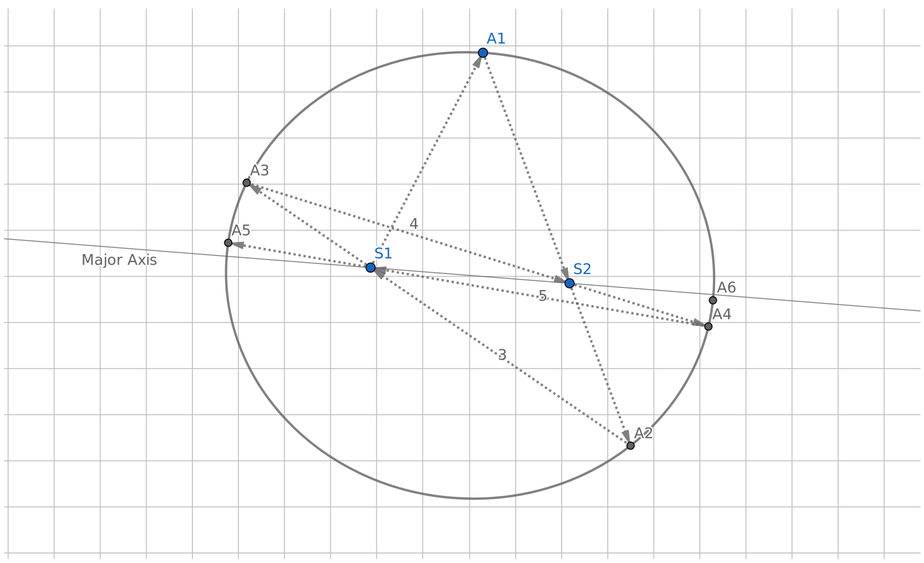

Figure 1: Convergence of Light Beam along Major-axis

$S_1, A_1, S_2, A_2, S_1, A_3, ...$
Also, we can use the parametric form of ellipse as $P(a\cos(\phi), b\sin(\phi))$

Consider a sequence S_1, A_1, S_2 and equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2}$$

Equation of a line joining points $A_1(a\cos(\phi_1), b\sin(\phi_1))$ and $S_2(ae, 0)$ is

$$y = (x - ae)\left(\frac{b\sin(\phi_1)}{a\cos(\phi_1) - ae}\right) \tag{3}$$

Using equation (2) and (3)

$$\frac{x^2}{a^2} + \frac{(x-ae)^2}{b^2}\left(\frac{b\sin(\phi_1)}{a\cos(\phi_1) - ae}\right)^2 = 1 \tag{4}$$

Line (3) will satisfies also point A_2 therefore putting $x = a\cos(\phi_2)$ of point A_2

$$\frac{(a\cos(\phi_2))^2}{a^2} + \frac{\{cos(\phi_2) - e\}^2}{\{cos(\phi_1) - e\}^2}\sin^2(\phi_1) = 1 \tag{5}$$

If we consider equation (5) as quadratic in $\cos(\phi_2)$, we can write it as

$$(1+S)\cos^2(\phi_2) - 2eS\cos(\phi_2) + e^2(S-1) \tag{A}$$

where $S = \frac{\sin^2(\phi_1)}{(\cos(\phi_1)-e)^2}$.

Also, we know that for a quadratic equation $ax^2 + bx + c = 0$
$\alpha + \beta = -\frac{b}{a}$ where α, β are roots of the quadratic equation
Therefore using (A) and as quadratic equation (A) satisfies (ϕ_1, ϕ_2) because line (3) will intersect (2) at only two points i.e. A_1 and A_2 going through S_2.

$$\cos(\phi_1) + \cos(\phi_2) = \frac{2eS}{1+S} \tag{6}$$

Simplifying equation (6), we will get

$$\cos(\phi_2) = -\cos(\phi_1)\frac{\left\{1 + e^2 - \frac{2e}{\cos(\phi_1)}\right\}}{\left\{1 + e^2 - 2e\cos(\phi_1)\right\}} \tag{7}$$

From equation (7) we can deduce similar equation going through another focus $S_1(-ae, 0)$

$$\cos(\phi_3) = -\cos(\phi_2)\frac{\left\{1 + e^2 + \frac{2e}{\cos(\phi_2)}\right\}}{\left\{1 + e^2 + 2e\cos(\phi_2)\right\}} \tag{8}$$

Notice sign change in term of $\frac{2e}{\cos(\phi_2)}$ and $2e\cos(\phi_2)$ in correspondence to equation (7).

Now, we can deduce the following, starting with S_1, A_1, S_2, A_2, S_1, A_3, ...

If the point ϕ_2 is in right half of the ellipse i.e on the positive side of x-axis, x-coordinate being +ve of the point and it is going to other point(of ellipse ϕ_3) going through other focus i.e. $S_1(-ae, 0)$ then equation (8) is used and also

$$1 + e^2 + \frac{2e}{\cos(\phi_2)} > 1 + e^2 + 2e\cos(\phi_2) \qquad \text{(as } \cos(\phi_2) > 0\text{)}$$

Therefore,

$$|cos(\phi_3)| > |\cos(\phi_2)| \tag{9}$$

Similarly, if the point ϕ_1 is in left half of the ellipse i.e. x-coordinate being -ve and it is going to other point(of ellipse ϕ_2) going through other focus i.e. $S_2(ae, 0)$ then equation (7) is used and also

$$1 + e^2 - \frac{2e}{\cos(\phi_1)} > 1 + e^2 - 2e\cos(\phi_1) \qquad \text{(as } \cos(\phi_1) < 0\text{)}$$

Therefore,

$$|\cos(\phi_2)| > |\cos(\phi_1)| \tag{10}$$

And,

$$|\cos(\phi_1)| < |\cos(\phi_2)| < |\cos(\phi_3)| < ... < |\cos(\phi_i)| < ... \tag{11}$$

We can conclude that equation (9) and (10) suggests that if we start at one focus that will go through another focus. There will be situation where a point ϕ_i will be close point $A(\pm a, 0)$. Suppose with some $\epsilon \sim 0$, we will send light parallel to x-axis with equation $x = 0 + \epsilon$ from focus S_1. Light will first ascend and then descend to x-axis.