Characters of symmetry

Nihar Prakash Gargava
Department of Mathematics and Statistics, IIT Kanpur

Project Supervised by:
Professor B. Sury
Stat-Math Unit, I.S.I. Bangalore

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Contents

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Introduction

The following is an attempt to illustrate to touch the topic of characters of symmetric group. The following reading should be complete in itself.

1 Some talk on Linear Representation

Groups are abstract entities. There's no doubt about it. Nobody can own a group and the governments cannot tax it.

Apart from being unimaginably abstract, groups are mysterious and elusive. Starting with only four properties (usually), groups explode out to give so many new insights in themselves that it sometimes leads to a sheer disbelief of how little information was given about them to begin with.

When facts and theorems about groups are seen, one never has to look at the what the group operation in hand is. However, if one can represent the operation as something that one knows about, it becomes easier to understand and visualize a lot of properties.

Group characters are one such concept that become simplified when finite groups are seen as matrices. So we start out with first establishing what linear representations are.

Definition 1.1. *Suppose G is a group and V is vector space. Then we define (π, V), where π is a homomorphism $\pi : G \mapsto GL(V)$, to be a linear representation of G.*

Sometimes only π is said to be a linear representation, but the context is usually clear. Dimension of a representation is the dimension of the associated vector space. Since the map is a homomorphism, all the properties of homomorphisms should hold. Throughout the discussion, all the groups are assumed to be finite and all the matrices and vector spaces are finite dimensional. The base fields of vector spaces shall be $\mathbb{C}$. Sometimes we may abuse the notation and write $\pi(g)(v)$ as gv, when the context is clear.

Here are some easy examples to explain what sort of representations one could talk about:

- For S_n, a natural representation comes to be the set of all $n \times n$ permutation matrices acting on $\mathbb{C}^n$.
- Cyclic additive group $\mathbb{Z}_n$ can be represented by the rotation matrices acting on $\mathbb{C}^2$ given by $\pi(k) = \begin{pmatrix} cos(\frac{2k\pi}{n}) & -sin(\frac{2k\pi}{n}) \\ sin(\frac{2k\pi}{n}) & cos(\frac{2k\pi}{n}) \end{pmatrix}$
- The most evident representation of a group G is the trivial representation, that is $\pi(g) = 1 \ \forall g \in G$. This representation, although looking seemingly useless, turns out to have a lot to offer later by 'inducing' more representations.

- A frequently discussed representation of a group G can be given by thinking of the vector space $\mathbb{C}[G]$. This vector space consists of all formal sums $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{C}$. It is easy to see how this space behave like a vector space under the formal sum. Now each element G can act on the sum linearly through right multiplication as given by $\pi(g')(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g (g' g)$. So in a way, all the coefficients permute giving some sort of a permutation representation. This representation is sometimes called the God-given representation or the Standard representation of G.

Now it is important to note that if we have one representation π, we can have infinitely many more representations from π by taking an invertible linear transformation P and defining a representation as $\pi' = P\pi P^{-1}$. In fact, this establishes an equivalence relation among representations. We can have an equivalence relation between representation such that two representation π and σ are said to be equivalent if there exits an invertible linear map P such that $\sigma = P\pi P^{-1}$

There seems to be another way of creating new representations out of known ones. Suppose π_1 and π_2 are two representations over the spaces V_1 and V_2 respectively. Then one can have a representation $\pi_1 \oplus \pi_2$ acting on the space $V_1 \oplus V_2$ given by $(\pi_1, \pi_2)(g)(v_1, v_2) = (\pi(g)(v_1), \pi(g)(v_2))$.

The same question can in fact be asked in the other direction as well. Given a representation, when is it possible to reduce a representation into smaller ones? The answer turns out to be pretty neat. Given some favorable things, we can decompose representations into smaller irreducible atomic representations for a group, and that too uniquely, in a manner of speaking. To get into the details, we lay the following definitions and theorems.

Definition 1.2. *Suppose G is a group having a linear representation π afforded by subspace V and suppose W is a subspace of V such $\pi(g)(w) \in W, \quad \forall w \in W, g \in G$. Then W is called a G-invariant subspace of V.*

In a manner of speaking, a G-invariant subspace is a subspace which equals the union G-orbit of all the vectors residing in it.

Definition 1.3. *If a subspace V has a G-invariant non-trivial proper subspace W, then V is said to be reducible. Otherwise it is said to be irreducible.*

We present the following two theorems now.

Theorem 1.4. (Maschke) *Suppose G is a group having a reducible linear representation in V. We can find two G-invariant orthogonal (non-trivial proper) subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$.*

Proof. Since we already know that V is a reducible representation, we may directly find a G-invariant subspace in V. Let's call this space V_1. We need to now find another subspace V_2 such that $V = V_1 \oplus V_2$.

To do this, first we establish an inner product given by

$$< u, v >_G = \frac{1}{|G|} \sum_{g \in G} < gu, gv > \qquad u, v \in V \tag{1}$$

where $< u, v >$ is a natural inner product on V by some basis (remember, V is finite dimensional). It is easy to verify that $< u, v >_G = < gu, gv >_G$.

We now find the orthogonal compliment of the space V_1 under this inner product and call it V_2. It should be sufficient to prove that this subspace will do.

Suppose $v_2 \in V_2$ and $g \in G$. We need to prove that $gv_2 \in V_2$. Since V_2 is the orthogonal compliment of V_1, it should be sufficient to prove that $< gv_1, v_2 >_G$ vanishes for all $v_2 \in V$. This is true since $< gv_1, v_2 >_G = < g^{-1}gv_1, g^{-1}v_2 >_G = 0$ (because, $g^{-1}v_2 \in V_2$). $\square$

A word of caution here is that $\frac{1}{|G|}$ has to exist in whatever base-field we use. For $\mathbb{C}$, this is not a point to worry.

After looking at the above theorem, it is tempting to conclude that representations can be recursively broken down into smaller irreducible representations. A good way to directly make this happen is by looking at the above inner product $< \, , \, >_G$ and generate an orthonormal basis from it through the Gram-Schmidt routine. Now all endomorphisms under this inner product can now be defined to be finite matrices under this basis (because finite dimensional vector space). Not only that, for a suitable ordering of the basis, these matrices would be decomposable into block matrices of the sub representations as $diag\{D_1, D_2, D_3...\}$, where $D_1, D_2, D_3...$ are block matrices of irreducible representations. This is because orthogonal spaces under the inner product will actually become partitions of the basis set.

It is not too difficult to see that the above block representation is actually a unitary representation (that is, all $\pi(g)$ under that basis are unitary matrices). This notion is helpful when we think representations belong to equivalent classes. After all, equivalent are the same linear transformations in different choice of a basis. So unitary representations should be the preferred representative representations for the equivalence as defined above as they neatly decompose reducibles into irreducibles. Also, for unitary representations, inverse matrices are simply the conjugate transpose.

However, we need to know if the decompositions are unique. The following theorem is going to serve as a major tool in this journey.

Theorem 1.5. (Schur) *Let (π, V) and (π', V') be two irreducible representations of G such that there exists a linear map $P : V' \mapsto V$ such that $\pi(g)P = P\pi'(g), \quad \forall g \in G$. Then exactly one of the following will hold.*

- *V and V' have the same dimensions and P is an invertible matrix/map, and hence the two representations are actually equivalent.*
- *P is the null matrix (in any basis) of appropriate dimensions.*

Proof. We look at $ker(P)$, the kernel of P. Notice that $\pi(g)Pv = P\pi'(g)v = 0$. This means that $Pv = 0 \implies \pi'(v) \in ker(P)$. Hence, $ker(P)$ is actually a G-invariant subspace of the representation (π', V'). But since that representation was irreducible, it cannot have any non-trivial proper G-invariant subspaces. This means that $ker(P)$ actually either 0 or V' itself.

The rest of the facts intuitively follow. If $ker(P)$ is V' then P has to be the null-matrix. Otherwise, we already know that P is injective and using a similar argument as above, image of P turns out to be G-invariant in V. This means that $image(P) = V$ (as V is also irreducible and $image(V)$ cannot be 0 being injective).

□

For $\mathbb{C}$ as the base field, we actually have more to say.

Corollary 1.6. *For the P as defined in 1.5, $P = \lambda I$ where I is the identity map of appropriate dimensions and $\lambda \in \mathbb{C}$.*

Proof. By the virtue of Theorem 1.5, P is either an isomorphism or the null map. In the latter case, $\lambda = 0$ trivially. Since the base field is $\mathbb{C}$, there is at least one root of the characteristic polynomial hence there is at least one eigenvector of P in V'. Let λ be the corresponding eigenvalue. Now look at $ker(P - \lambda I)$. It isn't too difficult to see that this is a G-invariant subspace of V'. So $ker(P - \lambda I)$ is either V' or 0, since V' is irreducible. The latter is not possible since there's already a non-zero eigenvector that we know of which exists in the space. This leads to the happy conclusion that $P - \lambda I = 0$.

□

Having established all of the above, we now move on the most important aspect of the theory.

2 Some Character theory

Characters are the magical things that make group theory seem something else entirely. We start with the following definition of a character.

Definition 2.1. *Given a representation (π, V). Then we denote $\chi_\pi : G \mapsto \mathbb{C}$ defined as $\chi_\pi(g) = trace(\pi(g))$. This χ_π is called the character of π. A character is an irreducible character if it arises from an irreducible representation.*

But this definition seems slightly awkward. It seems to look like the character of a representation depends on the choice of basis.

But that's actually not true. *trace* is a property that remains constant over a class of similar matrices. Hence $trace(A) = trace(PAP^{-1})$. This could be readily verified by elementary methods. Hence character of a representation is independent of the choice of basis. In fact, more than that, the above fact also tells us that for any representation π of a group G, $\chi_\pi(g) = \chi_\pi(hgh^{-1}), \quad \forall h, g \in G$. Hence characters can now be looked upon as class-invariant functions on the group. This property is going to be very noteworthy.

To formalize the above, we define the space $\mathcal{H}$ to be the subspace of $\mathbb{C}[G]$, as mentioned above, as the vectors having equal coefficients for conjugate elements. If we imagine $\mathbb{C}[G]$ as a space of functions mapping G to $\mathbb{C}$ instead, $\mathcal{H}$ can also be thought of as the space of functions $\{f : G \mapsto \mathbb{C},\ \ f(g) = f(hgh^{-1}),\ \forall h, g \in G\}$ (the set of functions that remain invariant on classes, that is). On the space $\mathbb{C}[G]$, we define the inner product $< , >$ as given by

$$< f, f' >= \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)} \tag{2}$$

This above bilinear function follows all properties of an inner product. And now when we have the inner product, we can talk about the orthogonality of characters.

We start with establishing some basic facts.

Theorem 2.2. *For (π, V) and (π', V') being two inequivalent irreducible representations of G. We denote $\pi_{ij}(g)$ as the ij-th entry of the matrix $\pi(g)$. Let n be the dimension of V. Then, the following identities hold.*

- $$\sum_{g \in G} \pi_{ij}(g)\pi'_{mn}(g^{-1}) = 0, \quad \forall(i,j),\ \forall(m,n) \tag{3}$$

- $$\sum_{g \in G} \pi_{ij}(g)\pi_{mn}(g^{-1}) = \frac{1}{n}\delta_{ij}\delta_{mn}, \tag{4}$$

Proof. Let T be a linear map from V to V'. Consider the map T' given by the following.

$$T' = \frac{1}{|G|} \sum_{g \in G} \pi(g)T\pi'(g^{-1}) \tag{5}$$

It can be verified that $\pi(g)T' = T'\pi'(g)$ holds for all $g \in G$. Hence by Theorem 1.5, $T' = 0$. We know that $T'_{ij} = \frac{1}{|G|}\sum_{g \in G}\sum_{j,m} \pi_{ij}(g)T_{jm}\pi'_{mn}(g^{-1})$. This sum is zero for any arbitrary choice of T. Hence by some slight trickery with the summation, Equation (3) can be established.

For the other equation, We construct T'' as

$$T'' = \frac{1}{|G|} \sum_{g \in G} \pi(g)T\pi(g^{-1}) \tag{6}$$

Instead of becoming 0, Theorem 1.5 will cause T'' to become an isomorphism. Moreover, from the corollary, it can be further said that $T'' = \lambda I$ for some identity matrix I and constant λ. Now $n\lambda = trace(T'') = trace(T)$. Writing the ij-th term of $T'' = \lambda I$ and using the fact that T is arbitrary gives Equation (4). □

It can be trivially argued that the dimension of $\mathcal{H}$ as defined above is equal to the number of conjugacy classes in G. The following theorem is very important to character theory.

Theorem 2.3. *The set of irreducible characters form an orthonormal set of the inner product space $\mathbb{C}[G]$ coupled with the inner product $<\ ,\ >$. Hence $<\chi_\pi, \chi_{\pi'}>= 1$ if π and π' are equivalent or 0 otherwise.*

Proof. Without loss of generality, we can assume that the two representations are unitary. Now, $<\chi_\pi, \chi_{\pi'}>= \frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi'_\pi(g)} = \frac{1}{|G|}\sum_{g\in G}\sum_i \psi_{ii}(g)\overline{\sum_j \pi'_{jj}(g)})$. For a unitary representation, we can know that $\pi(g^-1) = \pi(g)^*$ (* being the conjugate transpose operator). Hence, $<\chi_\pi, \chi_{\pi'}>= \frac{1}{|G|}\sum_{g\in G}\sum_i \pi_{ii}(g)\sum_j \pi'_{jj}(g^{-1})$. Distributing the product over that sum and using Equations (3) and (4) proves the above result. □

This theorem has an immediate consequence that number of inequivalent irreducible representations of a group is finite as the characters form an orthonormal set of a finite dimensional space. Moreover, we have an equivalent condition for equivalence of irreducible representations from the above. So in this way, we can identify two different irreducible representations. Hence characters completely characterize irreducible representations. But what about reducibles?

We saw what representations are and how they can be broken down into irreducible representations with the help of Theorem 1.4. Because of the finiteness of dimensions, we know that it can only be decomposed to finitely many irreducibles, but this does not necessarily mean that there is a unique way to decompose it. But the following theorem shall prove so.

Theorem 2.4. *Let (Π, W) be a reducible representation and (π, V) be an irreducible one. Then, $<\chi_\Pi, \chi_\pi>$ is the number of times π will be seen in the decomposition of Π into irreducibles.*

Proof. Let $W \equiv V_1 \oplus V_2 \oplus V_3 ...$ be one such decomposition where $(\pi_1, V_1), (\pi_2, V_2), (\pi_3, V_3)...$ are irreducibles. Hence $\chi_\Pi = \chi_{\pi_1} + \chi_{\pi_2} + \chi_{\pi_3}...$ and by the linearity of the inner product and the last theorem, $<\chi_\Pi, \chi_{\pi_i}>=$number of times π_i appears in the decomposition. □

The above theorem hence holds the uniqueness of the decomposition up to equivalence of characters. It also shows that characters completely determine the representation. Characters are the face of all representations!

But there is one more interesting thing about characters, that needs attention.

Theorem 2.5. *Characters form an orthonormal basis of the inner product space $\mathcal{H}$ coupled with the inner product $<\ ,\ >$.*

Proof. Assume a class function $f : G \mapsto \mathbb{C}$, $f \in \mathcal{H}$ and such that $<f, \chi_\pi>= 0$ for any irreducible representation π. It should be sufficient to prove that in this case f should be equal to 0.

Let (π, V) be an irreducible representation of n dimensions. We look at the map $T_\pi : V \mapsto V$, given by $T_\pi = \sum_{g \in G} f(g)\pi(g)$. Since f is a class function, it remains invariant over all the conjugacy classes. Hence, it can be seen that $\pi(g)T_]i\pi(g^{-1}) = T_pi$ works out $\forall g \in G$. So using Theorem 1.5 and the Corollary, we can say that $T = \lambda I$. Also, we see that $n\lambda = trace(T_\pi) = \sum_{g \in G} f(g)\chi_\pi(g) = |G| < f, \chi_\pi >= 0$ implying $\lambda = 0$ and thus $T_\pi = 0$. Now if (Π, W) is an irreducible representation, T_Π, being the diagonal composition of block matrices of T_pi, where πs are constituent irreducible representations, we see that T_Π should also be the null map. In that case, we take Π to be the regular representation on $\mathbb{C}[G]$. $0 = T_\Pi e = \sum_{g \in G} f(g)g.e = \sum_{g \in G} f(g)g$ implying $f(g) = 0$ for all $g \in G$. $\square$

The above theorem proves that the number of irreducible characters, and hence the numbers of irreducible representations, which is equal to the dimension of the space $\mathcal{H}$ and hence equal to the number of conjugate classes in G.

Before moving on to the next section, there is an urge to introduce what are known as induced characters, which is actually a natural way to create group representations from representations of a particular subgroup of the group.

Definition 2.6. *Suppose G is a group having a subgroup H having index n and let t_1H, t_2H, t_3H ... t_nH be the cosets of H in G. A representation of H is known to be (π_H, V) where V. Then the induced representation of G from (π_H, V) is the representation (π_H^G, V') given by the following composition of block matrices.*

$$\pi_H^G(g) = \begin{pmatrix} \pi'_H(t_1^{-1}gt_1) & \pi'_H(t_1^{-1}gt_2) & \cdots & \pi'_H(t_1^{-1}gt_n) \\ \pi'_H(t_2^{-1}gt_1) & \pi'_H(t_2^{-1}gt_2) & \cdots & \pi'_H(t_2^{-1}gt_n) \\ \vdots & \vdots & \ddots & \vdots \\ \pi'_H(t_n^{-1}gt_1) & \pi'_H(t_n^{-1}gt_2) & \cdots & \pi'_H(t_n^{-1}gt_n) \end{pmatrix} \tag{7}$$

where $\pi'_H(g) = \pi_H(g)$ if $g \in H$ or 0 otherwise.

Now this raises the question whether the above representation is actually valid. It is, and below we prove that $\pi_H^G(u)\pi_H^G(v) = \pi_H^G(uv)$, $\forall u, v \in G$.

We need to show that

$$\sum_{r=1}^{n} = \pi'_H(t_i^{-1}ut_r)\pi'_H(t_r^{-1}ut_j) = \pi'_H(t_i^{-1}uvt_j) \tag{8}$$

Suppose, $t_i^{-1}uvt_j \notin H$. Then RHS should be 0. But then either $t_i^{-1}ut_r$ or $t_r^{-1}ut_j$ would be not be in H otherwise there product $t_i^{-1}uvt_j$ would be in H. So each term of the summation would become 0. However, if $t_i^{-1}uvt_j \in H$, we know that vt_j will belong to only one t_kH for some k and hence $t_r^{-1}vt_j \in H$ iff $r = k$ and hence we see that only one term in the summation remains, making $\sum_{r=1}^{n} = \pi'_H(t_i^{-1}ut_r)\pi'_H(t_r^{-1}ut_j) = \pi'_H(t_i^{-1}ut_k)\pi'_H(t_k^{-1}ut_j)$, but then

$\pi'_H(t_i^{-1}ut_k)\pi'_H(t_k^{-1}ut_j) = \pi'_H(t_i^{-1}uvt_j)$ is trivially true as π_H is a representation of H.

What we are interested in though, is the characters of the induced representation.

Definition 2.7. *The character of the representation (π_G^H, V') as above induced from H is called the induced character of G from (π_H, V) and is denoted by $\phi_H^G(g) = trace(\pi_H^G(g))$*

Now we have the basic result that $\phi_H^G(g) = \sum_{i=1}^n \phi_H(t_i^{-1}gt_i)$ in the above case. It seems that the induced characters seem to depend on the choice of the representatives of cosets, that is the t_is. However, it does not take much to show that the sum is independent of the choice (it takes a little argument but basically boils down to the character being invariant of conjugacy).

We move on to prove an important result of induced characters.

Theorem 2.8. *If C_g is the conjugacy class of g, then*

$$\phi_H^G(g) = \frac{|G|}{|H||C_g|} \sum_{w \in C_g \cap H} \phi_H(w), \quad \forall g \in C_g \tag{9}$$

Proof. We start with looking at the sum $\sum_{v \in G} \phi'_H(v^{-1}gv)$, where ϕ'_H is as defined in definition. This sum runs over all the conjugates of g. We see that v traverses over the cosets of H exacly $|H|$ times. Basically, if we fix a particular traversal t_1, t_2 ... t_n as in the definition, the the sum can be re-written as follows.

$$\sum_{v \in G} \phi'_H(v^{-1}gv) = \sum_{h \in H} \sum_i \phi'_H(h^{-1}t_i^{-1}gt_ih) = \sum_{h \in H} \phi_H^G(g) = |H|\phi_H^G(g) \tag{10}$$

Another way to see this sum is that the sum runs over all the conjugates of g and each conjugate appears exactly $|Z_g| = |G|/|C_g|$ times (by the orbit-centralizer theorem for the group action of conjugacy).

This quickly gives the result:

$$\phi_H^G(g) = \frac{|G|}{|H||C_g|} \sum_{w \in C_g} \phi'_H(w), \quad \forall g \in C_g \tag{11}$$

Realizing that $\phi'_H(g)$ is 0 for g outside H yields the equation of the theorem. □

We now move on to the next section about characters of symmetric groups.

3 Characters of Symmetric groups

We know that conjugacy classes of the symmetric group S_n are actually in bijection with the partitions of n in a very natural manner. This intuition appears when one realizes conjugate permutations have the same cycle structure (that is, for a permutation acting on $1, 2...n$, the distribution of the orbits by the action of the cyclic group generated by the permutation is same). This automatically leads us to the conclusion that the characters of S_n are $p(n)$ in number, where $p(n)$ is the number of partitions of n.

The vector λ when denoted as $\lambda \vdash n$ contains $\lambda = (\lambda_1, \lambda_2, \lambda_3...)$ such that $\lambda_1 + \lambda_2 + \lambda_3... = n$. This λ is a partition of n.

When a partition λ is being used to define a conjugacy class C_λ, it actually means that in the conjugacy class, all permutations have λ_1 1-cycles, λ_2 2-cycles, λ_3 3-cycles and so on.

We define now the Young subgroup of S_n.

Definition 3.1. *Let a partition $\lambda \vdash n$ be given and the symmetric group S_n be acting on on the set $1, 2, 3...n$. Given such a partition, the Young subgroup H_λ is the set of all permutations such that the first λ_1 elements permute among themselves, the next λ_2 elements permute among themselves, the next λ_3 elements permute among themselves and so on.*

Hence, through the above definition it should be clear that H_λ is isomorphic to $S_{\lambda_1} \oplus S_{\lambda_2} \oplus S_{\lambda_3}$.... It is the induced characters of the trivial representation these groups that we are interested in. The trivial representation is simply the representation $\pi(g) = 1,\ \forall g \in G$. Of course, $\chi_\pi(g) = 1,\ \forall g \in G$ and the representation is irreducible (one cannot reduce a one-dimensional representation).

We denote ϕ^λ with the induced character of the trivial representation of H_λ over S_n. We further use ϕ^λ_ρ to denote $\phi^\lambda(g)$ where g is a permutation of conjugacy class of type ρ. This leads on to this theorem.

Theorem 3.2. *For the induced characters ϕ^λ, the following holds.*

$$\phi^\lambda_\rho = \sum_{(\rho_{ji})} \prod_{j=1}^{n} \frac{\rho_j!}{\rho_{j1}!\rho_{j2}!\rho_{j3}!...\rho_{jn}!} \tag{12}$$

where ρ_{ij} are elements of a matrix satisfying

$$\rho_i = \rho_{1i} + \rho_{2i} + \rho_{3i}... \tag{13}$$

and

$$\lambda_j = \rho_{i1} + 2\rho_{i2} + 3\rho_{i3}...\lambda_j\rho_{i\lambda_j} \tag{14}$$

Proof. Applying Theorem 2.8 to the case of S_n quickly yields that

$$\phi^\lambda_\rho = \frac{|S_n|}{|H_\lambda||C_\rho|} \sum_{w \in C_\rho \cap H_\lambda} \phi_H(w) \tag{15}$$

Now, $\phi_H(w) = 1$ So the expression turns into

$$\phi^\lambda_\rho = \frac{n!}{\lambda_1!\lambda_2!...\lambda_n!|C_\rho|} |C_\rho \cap H_\lambda| \tag{16}$$

Now $|C_\rho|$ is a commonly known result, which arises by trying to count the number of ways an element of class C_ρ can be represented in the cyclic notation, and calculates $|C_\rho|$ to be $n!/(1^{\rho_1}2^{\rho_2}3^{\rho_3}...n^{\rho_n}\rho_1!\rho_2!\rho_3!...\rho_n!)$. Plugging this in gets rid of the $n!$ term to give

$$\phi_\rho^\lambda = \frac{1^{\rho_1}2^{\rho_2}3^{\rho_3}...n^{\rho_n}\rho_1!\rho_2!\rho_3!...\rho_n!}{\lambda_1!\lambda_2!...\lambda_n!}|C_\rho \cap H_\lambda| \tag{17}$$

Now all that remains is to calculate $|C_\rho \cap H_\lambda|$. This is basically the conglomerate of all those permutations of H_λ that have the type ρ.

Let us look at a typical element $h \in H_\lambda$. Now $h = h_1h_2h_3...$ where h_j is a permutation of λ_j terms. This gives us the opportunity to further think about the cyclic compositions of those permutations individually. Suppose each h_j is of type $(\rho_{j1}, \rho_{j2}, \rho_{j3}...)$. Now our current requirement is forcing us to implement the two conditions below.

We desire the total number of i-cycles to be equal to the solution ρ_i. Hence

$$\rho_i = \rho_{1i} + \rho_{2i} + \rho_{3i}... \tag{18}$$

Also, we desire that all the ρ_{ij} denote cycle structure of h_is. Hence

$$\lambda_j = \rho_{i1} + 2\rho_{i2} + 3\rho_{i3}...\lambda_j\rho_{i\lambda_j} \tag{19}$$

But each $(\rho_{j1}, \rho_{j2}, \rho_{j3}...)$ will only give the conjugacy class of h_j. Hence this yields the final result as

$$\phi_\rho^\lambda = \frac{1^{\rho_1}2^{\rho_2}3^{\rho_3}...n^{\rho_n}\rho_1!\rho_2!\rho_3!...\rho_n!}{\lambda_1!\lambda_2!...\lambda_n!}\sum_{(\rho_{ji})}\prod_{j=1}^{n}\frac{\lambda_j!}{1^{\rho_{j1}}2^{\rho_{j2}}3^{\rho_{j3}}...\rho_{j1}!\rho_{j2}!\rho_{j3}!...} \tag{20}$$

Performing some cancellations gives the result as stated above. □

The next theorem now forms a bridge between the characters and polynomials.

Theorem 3.3. *Let α_i be variables and $s_{(\rho)}$ be a symmetric polynomial given by the formal polynomial $s_{(\rho)} = \prod_i(\sum_j \alpha_j^i)^{\rho_i}$. Let k_λ denote the polynomial $\sum \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}$...which is the sum of all distinct monomials of the form as shown in the summation. Then*

$$s_{(\rho)} = \sum_{\lambda\vdash n}\sum_{(\rho_{ji})}\prod_{j=1}^{n}\frac{\rho_j!}{\rho_{j1}!\rho_{j2}!\rho_{j3}!...\rho_{jn}!}k_\lambda \tag{21}$$

where ρ_{ij} are elements of a matrix satisfying

$$\rho_i = \rho_{1i} + \rho_{2i} + \rho_{3i}... \tag{22}$$

and

$$\lambda_j = \rho_{i1} + 2\rho_{i2} + 3\rho_{i3}...\lambda_j\rho_{i\lambda_j} \tag{23}$$

Proof. We are entrusted here with the task of breaking down the power sum symmetric polynomial $s_{(\rho)}$ into monomial symmetric polynomials k_λ. We do this by trying to find a coefficient of k_λ in $s_{(\rho)}$. It should be find the coefficient of $\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}$... as similar terms of the k_λ can be simply done by permuting the index which will leave $s_{(\rho)}$ unchanged.

In the expansion of $s_{(\rho)} = \prod_i(\sum_j \alpha_j^i)^{\rho_i}$, let a monomial $\alpha_1^{\rho_{i1}}\alpha_2^{\rho_{i2}}$... from the term $(\sum_j \alpha_j^i)^{\rho_i}$. The coefficient of this term is the multinomial coefficient $\rho_i!/(\rho_{i1}!\rho_{i2}!\rho_{i3}!...)$. This yields terms filling up a matrix ρ_{ij} that needs to satisfy the required equations. This completes the proof. □

So as surprising as this looks, we have now arrived at a point where characters of symmetric groups and symmetric polynomials have mysteriously combined themselves to arrive at a strange turn. We combine the above two results in the most natrual way and say that

$$s_{(\rho)} = \sum_{\lambda \vdash n} \phi_\rho^\lambda k_\lambda \tag{24}$$

There is certainly more to add in this than that. Developing the theory after this will take certain definitions and notions dealing deeply in the theory of symmetric polynomials. The following will contain only an introductory account of that.

The space of symmetric polynomials is a vector space that has several basis polynomials. Two of such basis polynomials, namely the power sum symmetric function and the monomial symmetric function, were presented above. There is also a scaler product defined among them arises out of a notion of duality. Under that scaler product, there exists an orthonormal basis of symmetric polynomials called the Schur polynomials.

Schur polynomials also arise out of a definition given by ratio of alternants. There is another definition that relates it to inserting numbers into tables. For now, we just call them e_λ, where λ is a partition.

We then have a way of representing the monomial symmetric polynomials into Schur polynomials (as they form a basis). The following expression demonstrates it.

$$k_\lambda = \sum_{\mu \vdash n} H_{\lambda\mu} e_\mu \tag{25}$$

This will lead us to the simplification of the horrorshow expression developed earlier, yielding the result

$$s_{(\rho)} = \sum_{\mu \vdash n}(\sum_{\lambda \vdash n} \phi_\rho^\lambda H_{\lambda\mu}) e_\mu \tag{26}$$

And that turns this expression into something very interesting. The innocent looking coefficients $(\sum_{\lambda \vdash n} \phi_\rho^\lambda H_{\lambda\mu})$ are in reality the irreducible characters of S_n!

Hence all one does to find the irreducible characters of is to express the power sum symmetric polynomials as schur polynomials. And that is one hallmark result of character theory for symmetric groups. However going to the depth of this knowledge would need a long path to arrive.

With that this report stands concluded.

References

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