Representation of the Symmetric Group

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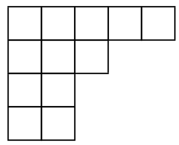
Contents

1 Notation

The set $\{1, 2, \ldots, m\}$ of the first m positive integers is denoted by $[m]$.

A **Young diagram** is a collection of boxes, or cells, arranged in left-justified in rows, with a weakly decreasing number of boxes in each row. Listing the number of boxes in each row gives a partition of an integer n, which is the total number of boxes in the Young diagram. Conversely, every partition of n corresponds to a Young diagram. For example, a partition of 12 into $5+3+2+2$ corresponds to the Young diagram given below.



We usually denote a partition by a lowercase Greek letter, such as λ. It is given by a sequence of weakly decreasing positive integers, sometimes written $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. One sometimes writes $\lambda = (d_1^{a_1}, \ldots, d_s^{a_s})$ to denote the partition that has a_i copies of positive integer d_i, $1 \leq i \leq s$. The notation $\lambda \vdash n$ is used to say that λ is a partition of n, and $|\lambda|$ is used for the number partitioned by λ.

Any way of putting positive numbers in the boxes of a Young diagram will be called a **numbering** of the diagram. A **Young tableau** is a numbering which is

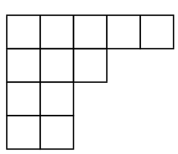
1. weakly increasing along each row
2. strictly increasing down each column

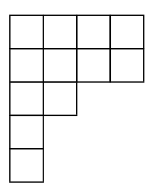
We say that λ is the *shape* of the tableau. A **standard tableau** is a tableau in which the entries are the numbers from 1 to n, each occurring once. Examples of Young tableau and standard tableau, for the partition $\lambda = (5, 3, 2, 2)$ of 12, respectively, are given below.

1	2	2	4	5
2	3	4		
4	6			
5	7			

1	3	4	6	7
2	5	9		
8	11			
10	12			

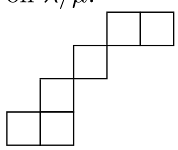
Entries of a tableau can be taken from any **alphabet**(totally ordered set), but we usually take positive integers. The diagram obtained after flipping the given diagram (corresponding to λ) over its main diagonal is called its **conjugate** diagram and is denoted by $\tilde{\lambda}$. So, for $\lambda = (5, 3, 2, 2)$, its conjugate $\tilde{\lambda} = (4, 4, 2, 1, 1)$ (see the diagram below).





Any numbering T of a diagram determines a numbering of the conjugate, called the **transpose**, and denoted T^τ. The transpose of a standard tableau is a standard tableau but, the transpose of a tableau need not be a tableau.

A **skew diagram** or **skew shape** is a diagram obtained by removing a smaller Young diagram from a larger one that contains it. If two diagrams correspond to partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, we write $\mu \in \lambda$ if the Young diagram of μ is contained in that of λ, or equivalently, $\mu_i \leq \lambda_i$ for all i. Th resulting skew tableau is denoted by λ/μ. For example, if $\lambda = (5, 3, 2, 2)$ and $\mu = (3, 2, 1)$, the following shows the skew diagram λ/μ and a skew tableau on λ/μ:



			4	5
		4		
	6			
5	7			

2 Tableaux Calculus

There are two fundamental operations on tableaux from which most of the combinatorial properties can be deduced: the *Schensted* “bumping” algorithm and the *Schutzenberger* “sliding” algorithm. When repeated, the first leads to Robinson-Knuth-Schensted correspondence, and the second to ”jeu de taquin”. We will briefly describe the first operation and mention the properties that we will need to understand the representations of symmetric group. We will also have a brief look at what the Littlewood-Richardson rule is. The reader is referred to [3] for a detailed survey on tableaux calculus.

2.1 Bumping and Sliding

Bumping: The bumping operation, takes a tableau T and a positive integer x and returns a new tableau, denoted $T \leftarrow x$. This new tableau will have an additional box, and its entries will be those from tableau T with an additional entry labeled x, but there is some moving around. The following procedure is followed: if x is as large as all the entries in the first row of T, then simply add x at the end of the first row. If not, then find the left-most entry in the first row, which is strictly larger than x. Put x in the box of this entry and “bump”(remove) the original entry. Take this entry which was bumped from the first row, and repeat the process on the second row. Keep going until the bumped entry can be put at the end of the row it is bumped into, or until it is bumped out of the last row, in which case it forms a new row with one entry.

For example, to row insert 2 in this tableau,

1	2	2	3	5
2	3	4		
4	6			
5				

the 2 bumps the 3 from the first row, which then bumps the 4 from the second row, which bumps the 6 from the third row, which can be put at the end of the fourth row:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & \blacksquare & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 5 \\ \hline 2 & 3 & \blacksquare \\ \cline{1-3} 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 5 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & \blacksquare \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 5 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2}\end{array}$$

One can see that the operation is indeed invertible. If we are given the resulting tableau together with the location of the box that has been added to the diagram, we can recover the original tableau T and the element x. The algorithm is simply to run backwards.

A row insertion $T \leftarrow x$ determines a collection R of boxes, which are those where an element is bumped from a row, together with the box where the last bumped element lands. This is called as the **bumping route** of row insertion, and the box added to the diagram for the last element is called as the **new box**. We say that a route R is **strictly left** (resp. **weakly left**) of a route R' if in each row which contains a box of R', R has a box which is left of (resp. left of or equal to) the box in R'.

Lemma 2.1 (Row bumping lemma)**.** *Consider two successive row insertions, first row inserting x in the tableau T and then row inserting x' in the resulting tableau $T \leftarrow x$, giving rise to two routes R and R', and two new boxes B and B'.*

1. *If $x \leq x'$, then R is strictly left of R', and B is strictly left of or weakly below B'.*

2. *If $x > x'$, then R' is weakly left of R and B' is weakly left of and strictly below B.*

This lemma has many important consequences. We now use the row bumping operation to define the **product tableau** $T \centerdot U$ from any two tableaux T and U. If we list the entries of U in order from left to right, and from bottom to top, getting a sequence $x_1, x_2, \ldots, x_s$, then

$$T \centerdot S = ((\ldots((T \leftarrow x_1) \leftarrow x_2) \leftarrow \ldots) \leftarrow x_{s-1}) \leftarrow x_s).$$

Another very important property is:

Lemma 2.2. *The product operation makes the set of tableaux into an associative monoid. The empty tableau is a unit in the monoid: $\phi \centerdot T = T \centerdot \phi = T$.*

Using the Schutzenberger sliding algorithm, there is another way of defining a product on the tableau monoid. We refer the reader to [3] for that.

2.2 Words

Given a tableau T, on listing the numbers in the Tableau from left to right, and from bottom to top, gives us a sequence $x_1, x_2, \ldots, x_n$, then $w = x_1 x_2 \ldots x_n$ is called the word of the tableau. Also, given the word w of a tableau T, we can get back the tableau from w, by simply breaking the word whenever one entry is strictly greater than the next, and the pieces are the rows of T, read from bottom to top.

Given two words w and w', their product is obtained by simply adding the word w' at the end of w. Knuth defined two elementary operations on words analogous to the algorithm for row bumping a new element into a tableau. The two operations called as **elementary Knuth transformations** are

- $yzx \mapsto yxz$ $(x < y \leq z)$ (K')
- $xzy \mapsto zxy$ $(x \leq y < z)$ (K'')

An **elementary Knuth transformation** on a word applies one of the transformations (K') or (K''), or their inverses, to three consecutive letters in the word. We call two words **Knuth equivalent** if they can be changed into each other by a sequence of elementary Knuth transformations, and we write $w = w'$ to denote that the words w and w' are Knuth equivalent.

Proposition 2.1. *For any tableau T and positive integer x,*

$$w(T \leftarrow x) = w(T).x.$$

Since the product of two tableaux is defined by successive row insertions, we have:

Corollary 2.3. *If $T \centerdot U$ is the product of two tableaux T and U, constructed by row-inserting the word of U into T, then*

$$w(t \centerdot U) = w(T) \centerdot w(U).$$

So, given two tableaux, in order to get their product, we can multiply their words. Then, we can apply elementary Knuth transformations on this word and get the corresponding tableau.

Theorem 2.4. *Every word is Knuth equivalent to the word of a unique tableau.*

The above theorem says that all the words in an equivalence class correspond to a unique tableau. If the word is $w = x_1 x_2 \ldots x_n$, one way to get the corresponding tableau is to apply the following series of row-bumping operations:

$$(((\ldots((\boxed{x_1} \leftarrow x_2) \leftarrow x_3) \leftarrow \ldots) \leftarrow x_{n-1}) \leftarrow x_n).$$

We will denote the unique tableau obtained above by $P(w)$. We skip the proof that the tableau obtained above is unique.

2.3 The Robinson-Knuth-Schensted Correspondence

The row bumping algorithm can be used to give a remarkable one-to-one correspondence between matrices with nonnegative integer entries and a pair of tableaux of the same shape, known as the Robinson-Knuth-Schensted correspondence.

We have seen that the Schensted algorithm of row inserting a letter into a tableau is reversible, provided one knows which box has been added to the diagram. This means that we can recover the word w from the tableau $P(w)$ together with the numbering of the boxes that arise in the canonical procedure. At the same time that we construct the tableau $P(w)$ we construct another tableau with the same shape, denoted $Q(w)$, called the **recording tableau** (or **insertion tableau**, whose entries are the integers $1, 2, ..., r$. The integer k is simply placed in the box that is added in the k^{th} step of the construction of $P(w)$.

By reversing the steps in he Schensted algorithm, we can obtain our word from the pair of tableaux (P, Q). To go from (P_k, Q_k) to (P_{k-1}, Q_{k-1}), take the largest numbered box in Q_k, and apply the reverse row-insertion algorithm to P_k with that box. The resulting tableau is P_{k-1}, and the element that is bumped out of the top row of P_k is the k^{th} element of the word w. Remove the largest element (which is k) from Q_k to form Q_{k-1}.

Knuth generalized the above procedure by allowing P to have entries from the alphabet $[n]$ and Q from the alphabet $[m]$. One can still perform the above reverse processes, to get a sequence of pair of tableaux

$$(P, Q) = (P_r, Q_r), (P_{r-1}, Q_{r-1}), \dots, (P_1, Q_1).$$

To construct (P_{k-1}, Q_{k-1}) from (P_k, Q_k), one finds the box in which Q_k has the largest entry; if there are several equal entries, the box that is farthest to the right is selected. Let u_k be the entry removed from Q_k, and let v_k be the entry that is bumped from the top row of P_k. One gets from this a two-rowed array $\begin{matrix} u_1 & u_2 & \cdots & u_r \\ v_1 & v_2 & \cdots & v_r. \end{matrix}$ By construction, the u_i's are in weakly decreasing order (taken from $[m]$):

$$u_1 \leq u_2 \leq \ldots \leq u_r. \tag{1}$$

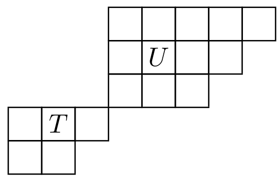
The following claim follows from the row-bumping lemma discussed earlier.

$$u_{k-1} = u_k \Rightarrow v_{k-1} \leq v_k. \tag{2}$$

We say that a two-rowed array is in lexicographic order if (1) and (2) hold. Given a two-rowed array in lexicographic ordering, we can construct a pair of tableaux (P, Q) with the same shape, by essentially the same procedure as before. Thus, we get a correspondence between a pair of tableaux of the same shape and the two rowed arrays.

3 Littlewood-Richardson Rule

Fix three Young diagrams (or partitions) as λ, μ and ν with $|\lambda| = n, |\mu| = m$ and $|\nu| = r$. We would now like to ask the question, what are the number of ways a tableau V of shape ν can be written as a product of two tableaux T and U of shape λ and μ, respectively? This basically means to evaluate the number of tableaux of the given form that rectify to a tableau V.



We define the following set

$$S(\nu/\lambda, U_o) = \{\text{skew tableaux } S \text{ of shape } \nu/\lambda : \mathit{Rect}(S) = U_o\} \tag{3}$$

and also the set

$$T(\lambda, \mu, V_o) = \{(T, U) : T \text{ is a tableau on } \lambda,\ U \text{ is a tableau on } \nu \text{ with } T.U = V_o\}. \tag{4}$$

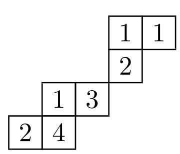
It turns out that the cardinality of the above two sets are independent of the contents of U_o and V_o and it can also be established that the cardinality of the sets is the same by establishing a bijection between the two.

$$T(\lambda, \mu, V_o) \longleftrightarrow S(\nu/\lambda, U_o). \tag{5}$$

The cardinality of either of the sets is defined as the Littlewood-Richardson number $c^{\nu}_{\lambda\mu}$

3.1 Reverse lattice words

A reverse lattice word or a *Yamanouchi* word is a word which satisfies that each right prefix has more 1s than 2s, more 2s than 3s, more 3s than 4s, and so on. We call a tableau T a Littlewood-Richardson skew Tableau with the property that the row-word of the tableau is a reverse lattice word. For example, the following is a Littlewood-Richardson skew tableau on $(5, 4, 3, 2)/(3, 3, 1)$.



A skew tableau is said to have a type $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ if the number of 1's in the tableau is μ_1, number of 2's is μ_2 and so on. With this we have the following theorem.

Theorem 3.1. *The number of Littlewood-Richardson skew tableaux of shape ν/λ and type μ is equal to the Littlewood-Richardson number $c^{\nu}_{\lambda\mu}$.*

The above fact gives us an alternate way to define the Littlewood-Richardson numbers.

4 Representation theory

A group can be represented as a set of matrices. Such a representation is called a linear representation.

Definition 4.1. Suppose G is a group and V is a vector space. Then, $GL(V)$ is the set of all invertible linear operators from V to V. Now, we define (π, V), where π is a homomorphism $\pi : G \mapsto GL(V)$, to be a linear representation of G.

Definition 4.2. Suppose G is a group having a linear representation π afforded by a subspace V and suppose W is a subspace of V such that $\pi(g)(w) \in W$, for all $w \in W$ and for all $g \in G$. Then, W is called a G-invariant subspace of V.

After this, we establish using Maschke's theorem that the decomposition of reducible representation into irreducible is clean. This is followed by the definition of characters of a group.

Definition 4.3. Let G be a finite group with a given representation (π, V). Then, we define $\chi_\pi : G \mapsto \mathbb{C}$ by $\chi_\pi(g) = trace(\pi(g))$. The map χ_π is called the character of π. A character is an irreducible character if it arises from an irreducible representation.

Since a character is a group function (or more specifically, a class function), we can define an inner product $< , >$ between group functions as given by

$$< f, f' >= \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)} \tag{6}$$

Under this inner product, it can be proved that characters form an orthonormal basis of the set of all class functions. Moreover, the product of a reducible character with an irreducible character is an integer which gives the multiplicity of the irreducible character in the reducible character when the reducible character is decomposed into irreducible characters.

Then, there is a definition of the induced representation given as follows.

Definition 4.4. Suppose G is a group having a subgroup H having index n and let $t_1H, t_2H, t_3H, \ldots, t_nH$ be the distinct cosets of H in G. Also, suppose that a representation of H is known to be (π_H, V), where V is some vector space. Then, the induced representation of G obtained from (π_H, V) is the representation (π^G_H, V') which is given by the following composition in terms of block matrices.

$$\pi_H^G(g) = \begin{pmatrix} \pi'_H(t_1^{-1}gt_1) & \pi'_H(t_1^{-1}gt_2) & \cdots & \pi'_H(t_1^{-1}gt_n) \\ \pi'_H(t_2^{-1}gt_1) & \pi'_H(t_2^{-1}gt_2) & \cdots & \pi'_H(t_2^{-1}gt_n) \\ \vdots & \vdots & \ddots & \vdots \\ \pi'_H(t_n^{-1}gt_1) & \pi'_H(t_n^{-1}gt_2) & \cdots & \pi'_H(t_n^{-1}gt_n) \end{pmatrix}, \tag{7}$$

where $\pi'_H(g) = \pi_H(g)$ if $g \in H$ or 0 otherwise.

We define an induced character as

Definition 4.5. The character of the induced representation (π_G^H, V') is called the induced character of G from (π_H, V) and is denoted by $\phi_H^G(g) = trace(\pi_H^G(g))$

We subsequently focus on the induced characters of the symmetric group.

5 Induced characters of the Symmetric group

We focus on the induced characters of the trivial representation of the Young subgroup on the Symmetric groups. These numbers appear as coefficients of some polynomials according to the two given theorems.

Theorem 5.1. *For the induced characters ϕ^λ, the following holds.*

$$\phi_\rho^\lambda = \sum_{(\rho_{ji})} \prod_{j=1}^{n} \frac{\rho_j!}{\rho_{j1}!\rho_{j2}!\rho_{j3}! \cdots \rho_{jn}!} \tag{8}$$

where ρ_{ij} are elements of a matrix satisfying

$$\rho_i = \rho_{1i} + \rho_{2i} + \rho_{3i} + \cdots \tag{9}$$

and

$$\lambda_j = \rho_{i1} + 2\rho_{i2} + 3\rho_{i3} + \cdots + \lambda_j \rho_{i\lambda_j}. \tag{10}$$

Theorem 5.2. *Let α_i's be variables and $s_{(\rho)}$ be a symmetric polynomial given by the formal polynomial $s_{(\rho)} = \prod_i (\sum_j \alpha_j^i)^{\rho_i}$. Let k_λ denote the polynomial $\sum \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\cdots$ which is the sum of all distinct monomials of the form as shown in the summation. Then,*

$$s_{(\rho)} = \sum_{\lambda \vdash n} \sum_{(\rho_{ji})} \prod_{j=1}^{n} \frac{\rho_j!}{\rho_{j1}!\rho_{j2}!\rho_{j3}! \cdots \rho_{jn}!} k_\lambda \tag{11}$$

where ρ_{ij} are elements of a matrix satisfying

$$\rho_i = \rho_{1i} + \rho_{2i} + \rho_{3i} + \cdots \tag{12}$$

and

$$\lambda_j = \rho_{i1} + 2\rho_{i2} + 3\rho_{i3} + \cdots + \lambda_j \rho_{i\lambda_j} \tag{13}$$

These two theorems connect the induced characters of the symmetric group with the theory of symmetric polynomials. Using the above two relations, we can conclude that

$$s_{(\rho)} = \sum_{\lambda \vdash n} \phi_{\rho}^{\lambda} k_{\lambda}. \tag{14}$$

References

[1] V.Krishnamurthy. Combinatorics, Theory and Application. New Delhi:Affiliated East-West Press, 1985.

[2] R.P. Stanley. Enumerative Combinatorics, volume 2. Cambridge University Press, 1999

[3] William Fulton. Young tableaux, Cambridge University Press, Cambridge, 1997.