NUMBER OF HEXAHEXAFLEXAGONS

This is an attempt to count the number of inequivalent hexahexaflexagon structures of n sides as indicated in [1]. Without taking the equivalence into consideration, the number becomes the C_{n-2}, the $(n-2)$th Catalan number. However this number overcounts the structures as indicated in the reference. This article will only focus on the combinatorial problem of counting the inequivalent classes.

Explained in the reference, there exists a bijection between hexahexaflexagon structures of $n-2$ sides. The triangulations of an n-polygon and the number can be evaluated by counting the number of orbits of the triangulations of an n-sided polygons that arise through the action of dihedral group. We heavily rely on Burnside's lemma for this problem. The accompanying figure should give the gist of the issue. Every triangulation can be understood as a plane graph on n vertices. Then the dihedral group, $\mathbf{D_n}$ can act on the vertices as a subgroup of $\mathbf{S}_{\binom{\mathbf{n}}{\mathbf{2}}}$, the symmetric group acting on the $\binom{n}{2}$ edges. [Figure 1] should give a better understanding.

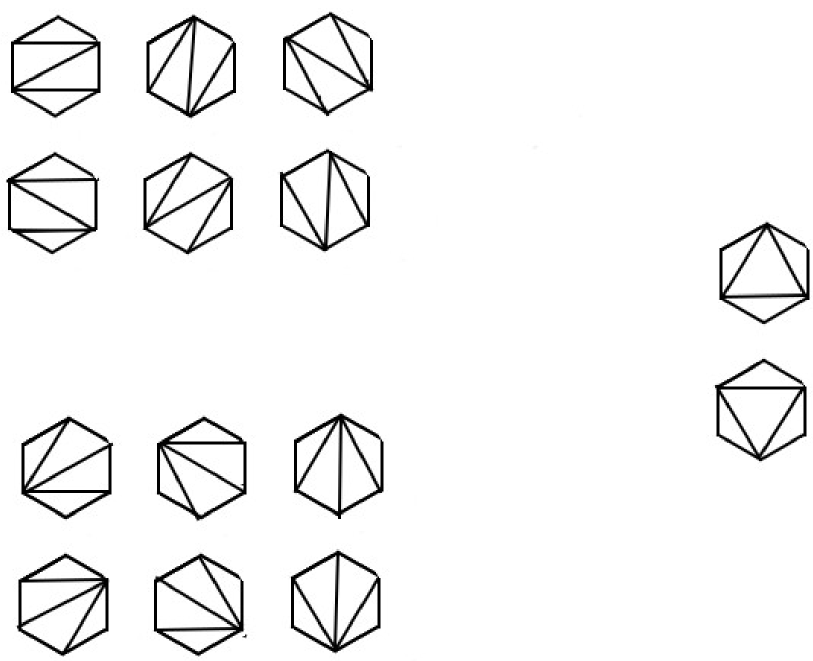

Figure 1: *The number of triangulations of a 6-sided polygon is $C_4 = 12$ which can be categorized into 3 inequivalent orbits under the dihedral group*

Let us start by stating the problem formally.

We denote $C_0, C_1, C_2, ...$ as the Catalan numbers (given by $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$). Some first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790 ...

Let V be the set of vertices $\{1, 2, 3...n\}$ of a regular polygon of n sides ($n > 2$). Let S be the set of the C_{n-2} triangulations of the polygons. Note that every $X \in S$ is a plane graph with non-intersecting edges. Our focus lies in calculating the orbits of S under the action of $\mathbf{D_n}$ (which, as shown in [Figure 1], is 3 for $n = 6$). We start with the following lemma.

Lemma 1

A convex polygon of n sides cannot have more than $n-3$ non intersecting diagonals and with $n-3$ diagonals the polygon is divided into $n-2$ triangles

Proof

Proof is trivial using induction on n. This lemma will be useful since if we are aware of a smaller polygonal area within the existing triangulation, no diagonal can enter into this area without intersections and we have a smaller polygonal area to triangulate independently.

Lemma 2

Suppose g is a permutation of V and suppose there exists a cycle of length $p > 3$ in g having a representation $(a_1, a_2, a_3, ..., a_p)$ such that $a_1 > a_2 > a_3... > a_n$ and suppose $X \in S$ is fixed under g, i.e g is an automorphism of X. Then there cannot be an edge between any two vertices a_i and a_j in X such that $|i-j| = 1$ or $|i-j| = p-1$, i.e. a_i and a_j are such that $ga_i = a_j$ or $ga_j = a_i$

Proof

Suppose there exists an edge (a_i, a_j). Now if this happens, then since $g, g^2, g^3...$ are automorphisms, $(ga_i, ga_j), (g^2a_i, g^2a_j), (g^3a_i, g^3a_j)...$ should also be edges in X. This means that there will have to be p edges if there is any edge. Now since all the a_k form an increasing sequence, we can imagine a polygon of p sides and $a_1, a_2, a_3..$ as vertices forming p diagonals. This possibility is however excluded by the previous lemma.

Lemma 3

Suppose a generator g of the cyclic action has the cycle $(1, 2, 3, 4, ..., n)$. For this subgroup of the dihedral action, g^k is an automorphism of some $X \in S$ if and only if $k \in B$, where B is the set $\{0, n/2, n/3, 2n/3\}$ if $6|n$, $\{0, n/2\}$ if $2|n$, $\{0, n/3, 2n/3\}$ if $3|n$ and $\{0\}$ otherwise.

Proof

The case of $k = 0$ is trivial. Hence we now assume $k \in \{1, 2, 3...n - 1\}$.

Consider the action of g^k where $gcd(k, n) = d$. We would expect the disjoint cycle structure of g^k to be the following.

$$(1, g^k 1, g^{2k} 1, ...)(2, g^k 2, g^{2k} 2, ...)(3, g^k 3, g^{2k} 3, ...)...(d, g^k d, g^{2k} d, ...)$$

Let us now divide the set of vertices into these d sets $V_1, V_2, V_3, ...V_d$ such that $V_i = \{i, g^k i, g^{2k} i, ...\}$ for all i. Note that each one of these d sets will have at least 2 elements as $k < n$ and thus $d < n$.

First observe that there has to be $n - 3$ diagonals and n edges in X. The n edges are fixed in any cyclic permutation of V. We must ensure the fixing of the $n - 3$ diagonals. By fixing, of course, the meaning is that edge (u, v) is an edge in X iff $(g^k u, g^k v)$ is an edge in X.

We now look for possibilities of fixing of the edges in the d sets of vertices created. Now if $d < n/3$, then each of those d sets will have more than 3 elements, so Lemma 2 can apply. Suppose (u, v) be a diagonal. We now systematically eliminate the possible places (u, v) can occur.

- Suppose both u and v belong to the same set V_i. Here two cases may arise.
  - $u \neq g^k v$ or $g^k u \neq v$. But this possibility is prevented through Lemma 2 as the elements of V_i form a cycle having a representation as increasing sequence and has length greater than 3.
  - $u = g^k v$ or $g^k u = v$. To disprove this, let's first assume the latter, i.e. $g^k u = v$. Now $(u, g^k u)$ is and edge. So $(g^k u, g^{2k} u)$, $(g^{2k} u, g^{3k} u)$, $(g^{3k} u, g^{4k} u)$... will all have to be edges. This means that all adjacent vertices in the cycle created by vertices of V_i have to be edges in X. But then if this happens, we have in X a closed polygon having the vertex set as V_i of more than 3 sides in which no diagonal can enter. Futher, creation of diagonals within the polygon is prevented by Lemma 2. Hence, this construction contradicts Lemma 1.
- Suppose $u \in V_i$ and $v \in V_j$ such that $i \neq j$. $d \geq 2$ obviously holds here. Here again two cases might arise.
  - u and v are adjacent vertices of the polygon. This type of construction of an edge is actually permitted. But as one can see, it is not a diagonal. It is a side edge.

- u and v are not adjacent. This construction will force us to introduce $(g^k u, g^k v)$,$(g^{2k} u, g^{2k} v)$,$(g^{3k} u, g^{3k} v)$... as edges too. Overall, this will contribute to n/d edges in X of this type. This type of an edge can however again be disqualified. We provide an induction-like argument below to show this.

  The smallest case where we can talk about such an event happening where $2 \le d < n/3$ is $n = 8$ and $k = d = 2$. Suppose $u = 1$, then (u, v)can be non-adjacent and from different sets if $v = 4$ or $v = 6$. Both cases lead to intersection of diagonals as shown in [Figure 2].

  For the general case, we try to find the first time a value of n allows this construction. Hence we add the assumption that n is the smallest number of sides for which such an edge (u, v) of the given type can be fixed (where u and v belong to different cycles). Now evidently, $n > 8$ and $2 \le d < n/3$ as discussed previously. Since g^k has to work as an automorphism, (u, v), $(g^k u, g^k v)$,$(g^{2k} u, g^{2k} v)$,$(g^{3k} u, g^{3k} v)$... will be n/d edges in total. We will now try to construct a closed loop within the polygon in which will again permit such construction, contradicting the newly added assumption. Here again we split our journey into two parts.

  * Any two of the n/d edges intersect. If this happens, we stop the argument here.
  * If no two edges intersect, then the edges can be connected in a natural way to produce a smaller polygon. All the n/d diagonals will give rise to n/d regions of a polygon, enclosed one side by the edges and the other side by the sides as shown in [Figure

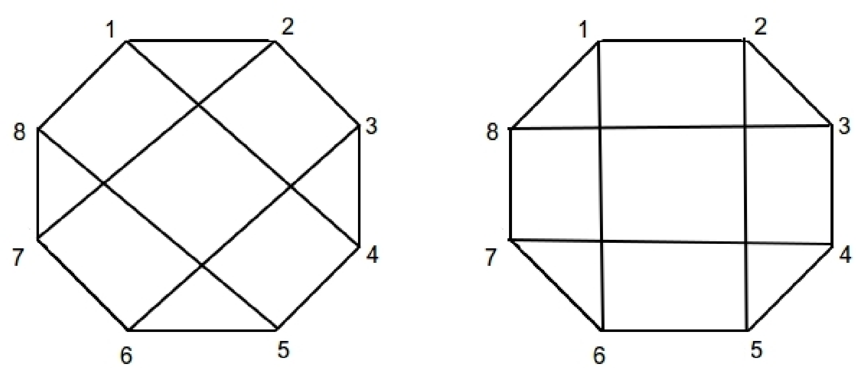

Figure 2: *For an octagon, for the permutation (1,3,5,7)(2,4,6,8), trying to have a diagonal (1,3) or (1,6) leads to unwanted intersections as shown in the left and right figure respectively.*

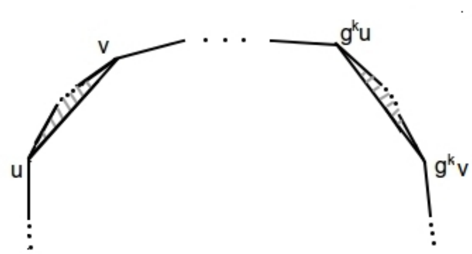

Figure 3: *Removing the shaded region, we get a smaller polygon of smaller number of sides. The number of elements in cycles of the sub-permutation acting on this smaller polygon is same as the complete one. Hence* $n/d = n'/d'$

3], and the central region will form a new polygon of side length equal to $n' < n$ as u and v are not adjacent by assumption. The value of d', which is analogous to d in the new polygon, will have to be such that $d \geq$ as u and v will belong to different cycles of the action of g on the new polygon and since $n'/d' = n/d > 3$, $d' < n'/3$. This is completely analogous to the original polygon. For the original polygon to be fixed, this new polygon will have to be fixed, and the only type of edges permissible are these. This contradicts the added assumption that the original polygon of n sides was the smallest with such a property.

The above has proved the "only if" part. Now we need to show the existence of fixable triangulations for the values of k in set B. This can be easily shown through [Figure 4]. However a closer look at the constructions of triangulations gives the image that these are the only ones possible when we try to maintain the symmetry.

Hence this leads us to the conclusion of the next theorem.

Theorem 1

Number of orbits of a triangulation of a regular convex n-polygon under the action of the cyclic group is $\frac{1}{n}C_{n-2} + \frac{1}{2}C_{n/2-1} + \frac{2}{3}C_{n/3-1}$ where only those terms are taken whose indices are integers.

Proof

We take it given that the total ways of triangulation for a regular convex n-polygon is C_{n-2}. Using the Burnside's lemma, we have the expression.

$$Number\ of\ orbits = |G|^{-1} \sum_{g \in G} |\{X \in S\ such\ that\ gX = X\}|$$

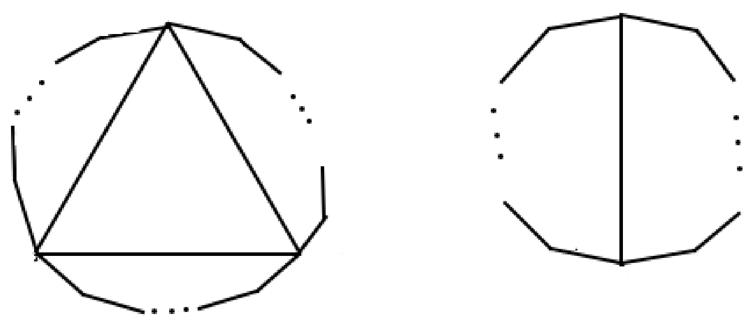

Figure 4: ***Left***:*To have a triangulation fixed by* $g^{n/3}$*, we first make the inner triangle as shown and triangulate the remaining regions symmetrically so that the final figure has 3-fold symmetry. Hence this reduces the problem to triangulation of a polygon of side* $(n/3)+1$. ***Right***:*To have a triangulation fixed by* $g^{n/2}$*, we draw the diametrical diagonal as shown and triangulate the remaining regions symmetrically so that the final figure has 2-fold symmetry. Hence this reduces the problem to triangulation of a polygon of side* $(n/2)+1$

We will only need to prove this for n that is divisible by 6. Now if G becomes the cyclic group, the only elements that fix any triangulations are 0th,$(n/2)$th, $(n/3)$th and $(2n/3)$th powers of $(1,2,3...,n)$ by [Lemma 3]. We list out the cases as what follows.

- Number of elements fixed by the 0th power, i.e. identity, is $|S|$, which is C_{n-2} in this case.
- Number of elements fixed by the $n/2$th power is the number of ways of triangulating a regular $(n/2+1)$-polygon as seen in [Figure 4], which is $\frac{n}{2}C_{n/2-1}$
- Number of elements fixed by the $n/3$th and $2n/3$th power is the number of ways of triangulating a regular $(n/3+1)$-polygon as seen in [Figure 4], which is $\frac{n}{3}C_{n/3-1}$.

The total number of ways thus, from Burnside's lemma, becomes $\frac{1}{n}(C_{n-2}+\frac{n}{2}C_{n/2-1}+\frac{n}{3}C_{n/3-1}+\frac{n}{3}C_{n/3-1})$ which on simplification yields the theorem's result. It can be easily seen that for cases where n is not divisible by 6, terms with non-integer indices will disappear.

Lemma 2

Suppose $a \in \mathbf{D_n}$ is the reflection permutation (i.e. $i \mapsto n+1-i$) and $b \in \mathbf{D_n}$ is the cycle $(1,2,...n)$ and $X \in S$, then

- **If,** $n \equiv i \pmod 2$**, then** $ab^iX = X$ **iff** $ab^{\frac{n+i}{2}}X = b^{\frac{n+i}{2}}X$
- **If,** $n \equiv i-1 \pmod 2$**, then** $ab^iX = X$ **iff** $abb^{\frac{n+i-1}{2}}X = b^{\frac{n+i-1}{2}}X$

Proof

Both can be proved by simple algebraic manipulation using the fact that $ab^i = b^{-i}a$. This lemma is useful because each fixed element of ab^i is equivalent to a fixed element of a or ab through rotation. Hence, finding fixed elements of a and ab should be sufficient.

And now we present the core theorem of this matter.

Theorem 2

Number of orbits of a triangulation of a regular convex n-polygon under the action of the dihedral group is

- $\frac{1}{2n}C_{n-2} + \frac{1}{3}C_{n/3-1} + \frac{1}{2}C_{(n-3)/2}$ **if n is odd**
- $\frac{1}{2n}C_{n-2} + \frac{1}{3}C_{n/3-1} + \frac{3}{4}C_{n/2-1}$ **if n is even.**

Proof

Here we apply the Burnside's lemma on the dihedral group. We again need to know the number of fixed triangulations for each of the member of the group. We already know this for the cyclic part, and hence that's n out of $2n$ elements taken care of. We need to find the numbers now for the involutions of the dihedral group.

By [Lemma 4], we do not need to find the fixed elements for each one of those involutions as each fixed triangulation has a corresponding fixed triangulation for either the action of a or ab, as defined in [Lemma 4]. Hence, we will try to find these for the cases of n being even and odd.

- For n being even, we first show that a will not be able to fix any triangulations. This part is even going to be important later for the case where n is odd.

  We first declare that a 4-polygon cannot be triangulated in such a way that a can fix it. This can be checked trivially. Now let n be the smallest even number of sides of a polygon where for a triangulation X, a fixes it. Note that no vertex is fixed by this map. We can see that 1 becomes to n through a. There has to be a diagonal coming out of vertex 1 or vertex n. Otherwise, $(1, n)$ cannot be the side of a triangle. But then $(1, j)$ and $(n, n+1-j)$, for some $j \neq n, 1, 2$, both have to be a diagonal. We let this be true. These diagonals cannot be intersecting. But then $j \to 1 \to n \to (n+1-j) \to j$ becomes a smaller polygon with even sides that needs to be fixed by a. This contradicts the hypothesis.

For the map ab, there are two fixed points, notice that the map actually takes all $i \neq n$ to $n-i$ and fixed n. For n even, this means that $n/2$ is also fixed. Suppose (n, j) is an edge. Now $(n, n-j)$ will also have to be an edge. This means that $n \to j \to n-j$ are three vertices of a triangle and $(j, n-j)$ is also an edge. Similarly $(n/2, j')$ being an edge will lead to a triangle $(n/2) \to j' \to n-j'$. But if $j \neq j'$, we have a smaller polygon excluding the two triangles mentioned and containing vertices $j, j', n-j, n-j'$ that has a triangulation that can be fixed be fixed by a permutation of type a. This is not possible.

Hence $j = j'$ has to be true. Now we iterate on j to let all triangulations fixed under this involution come to us. [Figure 5] tries to explain this idea for the case of an octagon. Without loss of generality, we assume that $1 \leq j < n/2$ For each such value of j, the possible values of smaller $(j+1)$-polygon and $((n/2)-j+1)$-polygon need to be triangulated. The number of ways this can be done is $C_{j-1}C_{(n/2)-j-1}$. So we evaluate, $\sum_{j=1}^{(n/2)-1} C_{j-1}C_{(n/2)-j-1} = C_{(n/2)-1}$. For $j = 0$ or $j = n/2$ (both are equivalent), this number is again $C_{(n/2)-1}$. So total $2C_{(n/2)-1}$ fixed elements exist for ab action.

By [Lemma 3], this solution gives solutions to ab^i for all even values of i. Hence in totality, for n even, the involutions contribute to $nC_{(n/2)-1}$ solutions.

- Assume that n is odd. Now we need to figure out elements of X that are

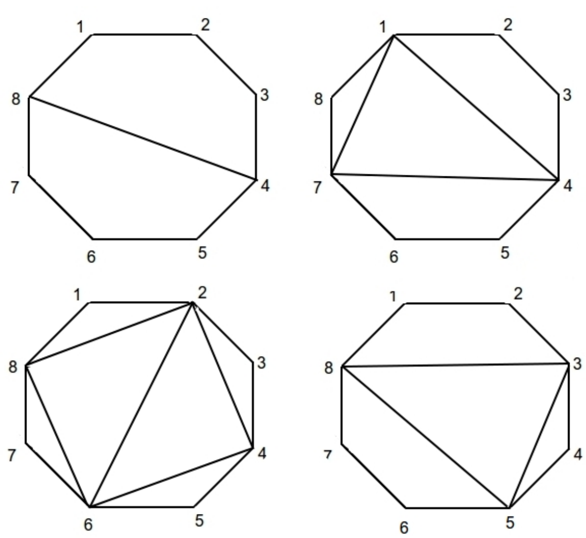

Figure 5: *Incomplete triangulations of an octagon that will can be made fixed under the action of* ab

fixed under a and ab. For a first, a triangulation would have to remain fixed under than map $i \mapsto n+1-i$. Here, $(n+1)/2$ is a fixed vertex. There has to be a diagonal incident from this edge in a fixed triangulation x , as if there is not, the vertex will be the peak of triangle created by adjacent sides. The base of this triangle, along with the rest of the polygon, will create a smaller polygon of even number of sides which can't be fixed by a as seen in the previous proof. Although, this construction will fail for $n = 3$, that case can be trivially handled separately.

Now if a diagonal $((n+1)/2, j)$ exists, then $((n+1)/2, n+1-j)$ is also an edge under this triangulation. These two diagonals are two edges of the triangle. By [Lemma 1], this demands $(j, n+1-j)$ to be the third edge of the triangle. This edge cannot be another diagonal since when n is odd, in a polygon of odd number of sides going from vertex j to vertex $n+1-j$ along this diagonal and then coming back along the sides will give a smaller polygon of even number of sides. This polygon, when further triangulated, cannot be fixed by a as seen in the previous case. Hence $(j, n+1-j)$ is in fact the side $(1, n)$. So the problem has been reduced to triangulating the symmetrically opposite smaller polygons. This can be done in $C_{(n-3)/2}$ ways. By [Lemma 3], this solution gives solutions to ab^i for all odd values of i.

For triangulations fixed by ab, we see that n is the only fixed vertex this time. This case is identical to the previous one with the map a. Using the same arguments, the number of ways is again $C_{(n-3)/2}$. By [Lemma 3], this solution gives solutions to ab^i for all even values of i. Hence in totality, for n odd, the involutions contribute to $nC_{(n-3)/2}$ solutions.

After arriving at the above results, we plug in these values to arrive at the values.

When calculated, the first few values of this sequence, starting from $n = 3$ are 1 1 1 3 4 12 27 82 228 733 2282 7528 24834 83898 285357 983244 3412420 11944614 42080170 149197152 531883768 1905930975 6861221666 24806004996 90036148954 327989004892

These values match the values given in the reference.

References

[1] David King, `www.drking.org.uk/hexagons/theory1.html`, 2001.