INVENTORY OF OBJECTS EQUIVALENT FOR A GIVEN PERMUTATION

The following is an attempt to correct the proof given in V. Krishnamurthy's book for the given version of De Brujin's extension of PET of inventory of orbits that are unchanged with a given permutation of figures by avoiding the incorrect premises.

Suppose G is a group acting on the set D having cycle index given by $Z(G)$. R is the set of figures such that R^D becomes the set of all mappings from D to R. Let h be a single bijection (and hence a permutation) on R.

G can now be thought of acting on elements of R^D given by the action

$$g\phi = \phi g^{-1}$$

such that $\phi \in R^D$. If the figures have weights given by $w(x)$, $x \in R$, we know by Polya's Enumeration Theorem that the inventory of orbits of R^D under the action of G are given by the following Polya-Substitition.

$$\sum_{[\phi]} W(\phi) = Z(G, \sum_{r\in R} w(r), \sum_{r\in R} w(r)^2, \sum_{r\in R} w(r)^3, ...)$$

where $[\phi]$ is the orbit of ϕ and $W(\phi)$ denotes weight of the mapping ϕ and is given by $W(\phi) = \prod_{d\in D} w(\phi d)$. The weight of the mapping is invariant for all elements of the orbit.

The interest now lies in evaluating orbits such that modifying any mapping ϕ with h gives an element of the same orbit as that of ϕ. Note that if a single ϕ has this property with h then all mappings of $[\phi]$ will have this property, since $h(g\phi) = g(h\phi) = h\phi g^{-1}$, $g \in G$. Thus we need to find $\displaystyle\sum_{[\phi]=[h\phi]} W(\phi)$.

The expression for this can be obtained on similar lines as that of the initial one. We look at the set $A = \{(g, \phi) : g\phi = h\phi, g \in G, \phi \in R^D\}$. We try to evaluate $\sum_{(g,\phi)\in A} W(\phi)$ in two ways. One way is by fixing ϕ and the other way is by fixing g. First we have the former, given by the following.

$$\sum_{(g,\phi)\in A} W(\phi) = \sum_{\phi} \sum_{\substack{g \\ g\phi = h\phi}} W(\phi)$$

Now consider the stabilizer subgroup G_ϕ of ϕ. Suppose now there is a $g \in G$ such that $g\phi = h\phi$. Then for all elements of $g' \in gG_\phi$, this should be true.

Moreover, these are the only elements of G satisfying this property as $g'\phi = h\phi$ implies $g^{-1}g'$ stabilizes ϕ. Therefore, the following should hold.

$$\sum_{(g,\phi)\in A} W(\phi) = \sum_{\substack{\phi \\ [\phi]=[h\phi]}} W(\phi)|G_\phi|$$

By realizing the fact that $|G_\phi|$ is same for all members of the same orbit (because stabilizers are conjugates for elements of same orbit), and using the celebrated Orbit-Centralizer theorem gives the sum to be something desirable.

$$\sum_{(g,\phi)\in A} W(\phi) = \sum_{[\phi]=[h\phi]} W(\phi)|G_\phi||[\phi]|$$
$$\sum_{(g,\phi)\in A} W(\phi) = |G| \sum_{[\phi]=[h\phi]} W(\phi)$$

Calculating the other way, we get the following.

$$\sum_{(g,\phi)\in A} W(\phi) = \sum_{g} \sum_{\substack{\phi \\ \phi=h\phi g}} W(\phi)$$

For the evaluation of this sum, refer to Krishnamurthy's book. The final result comes out to be,

$$\sum_{[\phi]=[h\phi]} W(\phi) = Z(G, \mu_1, \mu_2, \mu_3 ...)$$

where μ_k is given by the following.

$$\mu_k = \sum_{\substack{r\in R \\ h^k r = r}} \prod_{i=0}^{k-1} w(h^i r)$$

References

[1] V.Krishnamurthy. Combinatorics, Theory and Application. New Delhi:Affiliated East-West Press, 1985.