The values of Dedekind zeta functions at 2

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Abstract

These notes contain the exposition of a paper of Don Zagier. The work entails Zagier's generalized expression for a particular special value of the Dedekind zeta function. The generalized expression was obtained by noting the appearance of the Dedekind zeta function in the expressions of the volumes of arithmetic hyperbolic 3-manifolds. This expression for volume, in turns comes out of realizing that the geometry of arithmetic hyperbolic manifolds is secretly the same as a certain quotient space of adéle groups.

In this text, we survey all the relevant background leading up to the proof of Theorem 1 from [1] and some surrounding material.

1 Introduction

In 1734, Euler resolved the famous Basel problem which asked the following question:

What does the sum of the following infinite series converge to?

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots \tag{1}$$

The answer was found to be $\frac{\pi^2}{6}$ by Euler. This has the following consequence in modern terminology.

We define the Riemann-zeta function for $1 < s \in \mathbb{R}$ to be the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

So rephrasing the Basel's problem simply amounts to saying that $\zeta(2) = \frac{\pi^2}{6}$. We will consider the generalization of the identity of Equation (2) in the case of the Dedekind zeta function.

The Dedekind zeta ζ_K associated to the number field K is defined as the following for a number field K and for $1 < s \in \mathbb{R}$.

$$\zeta_K(s) = \sum_{\substack{\mathcal{I} \text{ is a non-zero} \\ \text{integer ideal of K}}} N(\mathcal{I})^{-s}. \tag{3}$$

The Dedekind zeta function is an important function that has important connections in number theory. Putting $s = 2$ and K as the simplest number field $\mathbb{Q}$, the expression of Equation (3) turns into Equation (1). The generalization we want to focus at appears in Zagier's paper [1] in the form given below.

Theorem 1. *Let K be an arbitrary number field. Suppose $A(x)$ is a real valued function given by*

$$A(x) = \int_0^x \frac{1}{1+t^2} \log\left(\frac{4}{1+t^2}\right) dt. \tag{4}$$

Then we have that the value of $\zeta_K(2)$ is given by an expression as below.

$$\zeta_K(2) = \frac{\pi^{2r+2s}}{\sqrt{D}} \sum_{\nu} c_\nu A(x_{\nu,1}) A(x_{\nu,2}) \dots A(x_{\nu,n}). \tag{5}$$

where the sum above is finite, D is the discriminant of the number field, r is the number of real places and s is the number of complex places associated to the number field K. Moreover, the c_ν are rational and $x_{\mu,j}$ are real algebraic numbers.

This entire piece is devoted to understanding the proof of Theorem 1. The key idea of the proof is to realize that $\zeta_K(2)$ appears in the volume formula of arithmetic hyperbolic 3-manifolds. Once this is shown, we use the volume formulae of hyperbolic tetrahedra to show that the volume of the manifold itself can be written as $\sum_\nu c_\nu A(x_{\nu,1})A(x_{\nu,2})\dots A(x_{\nu,n})$ in the notation of Theorem 1. Therefore, the most crucial part of the picture is the volume formula of arithmetic hyperbolic 3-manifolds. This formula will be expressed as Theorem 8 in the penultimate section of this text.

Arithmetic hyperbolic 3-manifolds are manifolds that are the quotient space of the hyperbolic 3-space $\mathbf{H}^3$ by certain discrete subgroups of its isometry group. The "arithmetic" part in the name of this class of manifolds[1] comes from the fact that these discrete subgroups have a very arithmetic foundation.

To make a simple analogy, one could think of the hyperbolic 2-space and recall that the isometry group of the hyperbolic 2-space $\mathbf{H}^2$ is $PSL(2,\mathbb{R})$. Often times then one is interested in the discrete subgroup $PSL(2,\mathbb{Z})$ and the quotient space $\mathbf{H}^2/PSL(2,\mathbb{Z})$. In a similar taste, if suppose K is a number field with only one pair of complex embedding in $\mathbb{C}$, then the isometry group of $\mathbf{H}^3$ being $PSL(2,\mathbb{C})$, we get that $PSL(2,R)$ is a discrete subgroup of $PSL(2,\mathbb{C})$, where R is the ring of algebraic integers in K. In this case again one could talk about the quotient space $\mathbf{H}^3/PSL(2,R)$, which is an example of an arithmetic hyperbolic 3-manifold. The precise treatment of these object shall be in the text.

Endowed on these arithmetic hyperbolic 3-manifolds is the measure induced by the Riemannian metric induced from the hyperbolic 3-space. In this measure, we will find the volume of the arithmetic hyperbolic 3-manifolds by relating this measure to the Tamagawa measure on adéle groups. The Tamagawa measure on adéle groups comes through the product measure of a countable family of measures, each when multiplied together will give rise to the factor of $\zeta_K(2)$ in the volume according to the following Euler product of $\zeta_K(2)$.

$$\zeta_K(2) = \prod_{\substack{\mathcal{P} \text{ is a non-zero} \\ \text{prime ideal of K}}} (1 - N(\mathcal{P})^{-2})^{-1}. \tag{6}$$

Before beginning the proof, there are many preliminaries and results that will need to be established to give the full proof. We will start our discussion with some results and definitions about quaternion algebras and their ramifications, move to adéle and idéle groups and define the needful measures on these topological groups. After that, we will review some hyperbolic geometry and then move to the volume of arithmetics hyperbolic 3-manifolds. Throughout our treatment, we are only concerned with fields of characteristic 0.

2 Quaternion algebras and their ramification

Definition 1. *A quaternion algebra is an algebra B over a field K such that there is a basis $\{1, i, j, k\}$ for B as a K-vector space satisfying*

$$i^2 = a \tag{7}$$
$$j^2 = b \tag{8}$$
$$k = ij = -ji \tag{9}$$

This algebra B is denoted by $(\frac{a,b}{K})$.

Example 1. *The most useful example of a quaternion algebra is the Hamiltonian quaternion algebra denoted by $\mathbb{H}$. It is given by substituting $K = \mathbb{R}, a = -1, b = -1$ in the previous definition.*

Definition 2. *For a quaternion algebra $A = (\frac{a,b}{K})$, the norm map $n : A \to K$ is the map which when evaluated for $x = a_0 + a_1 i + a_2 j + a_3 k$ is given by*

$$n(x) = a_0^2 - aa_1^2 - ba_2^2 + aba_3^2. \tag{10}$$

Similarly we define the trace map $tr : A \mapsto K$ to be

$$tr(x) = 2a_0. \tag{11}$$

[1]It must be pointed out that arithmetic hyperbolic 3-manifolds are not necessarily manifolds, but orbifolds in general. We will however not get into this technicality, as it will pose no obstruction in our treatment

Definition 3. *Suppose B is a quaternion algebra over K and K is the field of fractions of a subring $R \subset K$. A complete R-lattice L of a K-vector V is a finitely generated R-module contained in V such that $K \otimes_R L = V$.*

Since B is a vector space of K, we can talk about complete R-lattices is B. A complete R-lattice $\mathcal{O}$ is called a R-order if it is also a subring of R. An R-order is maximal if it is not contained in any other R-order.

Definition 4. *A valuation υ on a field K is a mapping $\upsilon : K \to \mathbb{R}^+$, such that*

1. $\upsilon(x) > 0, \ \forall x \in K$,
2. $\upsilon(x) \Leftrightarrow x = 0$,
3. $\upsilon(xy) = \upsilon(x)\upsilon(y), \ \forall x, y \in K$,
4. $\upsilon(x + y) \leq \upsilon(x) + \upsilon(y), \ \forall x, y \in K$.

Two valuations υ, υ' on K are equivalent if there exists $a \in \mathbb{R}^+$ such that $\upsilon'(x) = [\upsilon(x)]^a, \ \forall x \in K$ (alternatively, they induce the same topology). We will not distinguish between equivalent valuations.[2]

Definition 5. *A valuation υ on a field K is said to be non-Archimedean or finite if it satisfies the property*

$$\upsilon(x + y) \leq \max\{\upsilon(x), \upsilon(y)\}, \ \forall x, y \in K. \tag{12}$$

A valuation that is not non-Archimedean is called infinite or Archimedean.

It is classically known (and given in [2]) that all finite valuations of a number field are $\mathcal{P}$-adic valuations of some prime ideal $\mathcal{P}$ of the ring of integers and all infinite valuations of a number field k are of the form $x \mapsto |\sigma(x)|$ where σ is an embedding of the field k in $\mathbb{R}$ or $\mathbb{C}$.

Proposition 1. *If A is a quaternion algebra over a field F, then A is either a division algebra or A is isomorphic to $M_2(F)$.*

Proof. We know from the Wedderburn's structure theorem (can be found in [2] for instance), that a quaternion algebra A is isomorphic to a full matrix algebra $M_n(D)$ (because it is simple), where D is a division algebra over F. Because $4 = \dim_F(A) = \dim_F(M_n(D)) = n^2 \dim_F(D)$, the only possibilities are $n = 1$ and A is the division algebra D or $n = 2$ and $D = F$. □

Determining whether or not a quaternion algebra over a field F is isomorphic to $M_2(F)$ is important. The next definition is in relation to this.

Definition 6. *Let A be a quaternion algebra over the number field k and let υ denote a valuation of the field k which may be finite or infinite but is not induced via a complex embedding. We denote the metric completion of k with respect to υ by k_υ. By A_υ we denote the quaternion algebra $A \otimes_k k_\upsilon$. Then A is said to be ramified at υ if A_υ is a division algebra over k_υ. Otherwise, we say that A splits over υ.*

Another important notion is that of the Eichler condition. We define the Eichler condition as

Definition 7. *A quaternion algebra A over a number field k is said to satisfy the Eichler Condition if there is at least one infinite valuation of k at which A is not ramified.*

We will find the upcoming proposition useful in the coming section.

Proposition 2. *Let A be a quaternion algebra over a number field k that has r_1 real embeddings and r_2 complex pair of embeddings (and hence $[k : \mathbb{Q}] = r_1 + 2r_2$). If A is ramified at s_1 of those r_1 real places, then*

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong s_1\mathbb{H} \oplus (r_1 - s_1)M_2(\mathbb{R}) \oplus r_2 M_2(\mathbb{C}) \tag{13}$$

Proof. Let $A = (\frac{a,b}{k})$. Suppose $\sigma_1, \sigma_2 \ldots \sigma_n$ are the embeddings such that the first s_1 are the real ramified embeddings, the next $r_1 - s_1$ are the remaining real places and the remaining are $2r_2$ complex embeddings. Let $A_i = (\frac{\sigma_i(a), \sigma_i(b)}{K_i})$, where $K_i = \mathbb{R}$ for $i = 1 \ldots r_1$ and $K_i = \mathbb{C}$ in all the other cases. Define the map $\hat{\sigma_i} : A \to A_i$ by $\hat{\sigma_i} : (a_0, a_1, a_2, a_3) \mapsto (\sigma_i(a_0), \sigma_i(a_1), \sigma_i(a_2), \sigma_i(a_3))$, where the coordinates are in the standard bases. With this, we define

[2] An equivalence class of valuations is generally called a place. However, when we will use valuation, we will always mean up to equivalence

$$\varphi : A \otimes_{\mathbb{Q}} \mathbb{R} \to \bigoplus_{i=1}^{n} A_i, \tag{14}$$

given as $\varphi(a \otimes b) = (b\hat{\sigma_1}(a), b\hat{\sigma_2}(a) \dots b\hat{\sigma_n}(a))$ so that φ is well-defined $\mathbb{R}$-algebra homomorphism.

Suppose σ_i, σ_j are two conjugate complex embeddings, then the projection on $A_i \oplus A_j$ of the image of φ lies in the diagonal embedding $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ given by $\Delta : x \mapsto (x, \bar{x})$. Hence, we can take the image to be lying in $s_1\mathbb{H} \oplus (r_1 - s_1)M_2(\mathbb{R}) \oplus r_2(\Delta M_2(\mathbb{C}))$.

We show that this is an isomorphism by counting the dimensions and showing that the map is injective. The image has dimensions $4s_1 + 4(r_1 - s_1) + 8r_2 = 4n$. Hence, it is sufficient to show that the map is injective. If we take the $\mathbb{Q}$-basis of k to be $\{1, t \dots t^{n-1}\}$, then this will induce an $\mathbb{R}$-basis on $A \otimes_{\mathbb{Q}} \mathbb{R}$ by tensoring with the standard basis of A. Writing the images of these basis elements and computing the matrix will yield the matrix $I \otimes D$ where I is the 4×4 identity matrix and $D = [\sigma_i(t^j)]$. This matrix is singular and hence we're done.

□

Now suppose Ω_∞ is the set of infinite valuations of a number field k and A is a quaternion algebra over k that satisfies the Eichler condition, and suppose $\mathrm{Ram}_\infty(A)$ is the set of all valuations on which A is ramified. Then we define the group G as

$$G = \bigoplus_{v \in \Omega_\infty \setminus \mathrm{Ram}_\infty(A)} A_v \cong \bigoplus M_2(k_v). \tag{15}$$

Because of the previous theorem, we have a clear embedding

$$\psi : A \to G. \tag{16}$$

3 Adéle and idéle groups

Before proceeding to some results, we'll need to delve into the definitions of adéle and idéle groups.

We start with defining the restricted product.

Definition 8. *Let Ω be an index set. Suppose, for each $v \in \Omega$, there exists a locally compact group G_v. Moreover, let there be a finite set $F \subset \Omega$ such that for each $v \in \Omega \setminus F$, there is a compact open subgroup $C_v \subset G_v$. Then we define the restricted product $G_{\mathcal{A}}$ of the collection $(\{G_v\}_{v\in\Omega}, \{C_v\}_{v\in\Omega\setminus F})$ as*

$$G_{\mathcal{A}} = \{x = (x_v) \in \prod G_v \mid x_v \in C_v \text{ for all but finitely many } v \in \Omega \setminus F\}. \tag{17}$$

Moreover, we assume that $G_{\mathcal{A}}$ is topologized by taking the basis of identity element to be $\prod U_v$, where U_v is an open set in G_v around identity, but $U_v = C_v$ for all but finitely many v.

Using the above definition, we will establish several important groups useful to us in the upcoming proofs. The following are some important examples of the restricted product of groups.

In all of the following cases, k is a number field, we choose Ω of Definition 8 to be the set of valuations of the number field k and we take the finite set F to be Ω_∞, the set of infinite valuations inside Ω. Also, A is a particular quaternion algebra over k and $\mathcal{O}$ is an R-order on A, R being the ring of algebraic integers inside k.

- Take G_v of Definition 8 to be the additive group of k_v, the v-completion of the number field k. Assign C_v to be the additive group of the ring of integers $R_v \subset k_v$, for $v \in \Omega \setminus \Omega_\infty$. This group is called the adéle group $k_{\mathcal{A}}$.
- Take G_v to be the multiplicative group k_v^* and for $v \in \Omega \setminus \Omega_\infty$, we take $C_v = R_v^*$, which is the group of units in k_v. This group is called the group of idéles, denoted by $k_{\mathcal{A}}^*$.
- Let $A_v = A \otimes_k k_v$. Then the additive group of A_v can be taken as G_v and for $v \in \Omega \setminus \Omega_\infty$, we take $C_v = \mathcal{O}_v$ where $\mathcal{O}_v = \mathcal{O} \otimes_R R_v$. This group is denoted by $A_{\mathcal{A}}$.
- Take $G_v = A_v^*$ and for $v \notin \Omega_\infty$, $C_v = \mathcal{O}_v^*$. This yields the group denoted by $A_{\mathcal{A}}^*$. Note that this group may not be abelian.

- Perhaps the most relevant example is when $G_v = A_v^1$, where $A_v^1 \subset A_v$ is the group of quaternions that have a reduced norm 1. For $v \notin \Omega_\infty$ we set $C_v = \mathcal{O}_v^1$, $\mathcal{O}_v^1 \subset \mathcal{O}_v$ given by $\mathcal{O}_v^1 = A_v^1 \cap \mathcal{O}_v$. We will denote this group by $A_{\mathcal{A}}^1$. This group too has no reason to be abelian.

One discussion however remains here which is it is to prove that in each case G_v and C_v chosen to fit inside the Definition 8 are indeed locally compact and the C_v are compact. However, proving these facts is not very difficult and these facts are known results in each case.

We have some important results before we proceed to the main result of this section.

Proposition 3. *Let R be the ring of integers in a number field k and let $\mathcal{O}$ be an R-order in the quaternion algebra A over k. Then $\mathcal{O}$ is a maximal order if and only if the orders $O_v = \mathcal{O} \otimes_k k_v$ in A_v is a maximal order for every every finite valuation v.*

The above stated Proposition 3 is a kind of a local-global principle. We will not go into the proof of this, but it is sufficient to state that this comes through the Hasse-Minkowski theorem. We will now look at the next part of the journey.

Lemma 1. *The topology on $A_{\mathcal{A}}^*$ is the induced topology given by the injection $i : A_{\mathcal{A}}^* \hookrightarrow A_{\mathcal{A}} \times A_{\mathcal{A}}$ given by $i : x \mapsto (x, x^{-1})$.*

Proof. To see this, consider a basic open set in $A_{\mathcal{A}}^*$. This open set is of the form xU where $x \in A_{\mathcal{A}}^*$ and $U = \Pi U_v$ where $U_v = A_v^*$ for all but finitely many finite valuations v. Now, simply observe that $U_v \subset A_v^*$ is an open set for some valuation v, and $x_v U_v$ is also an open set in A_v^* and therefore an open set in A_v. Moreover, since $U_v \subset A_v^*$, we know that $(xU_v)^{-1} \subset A_v^*$ and is therefore open in A_v. Hence, we observe that $i(U)$ is an open subset of $A_{\mathcal{A}} \times A_{\mathcal{A}}$.

□

Proposition 4. *A_k^1 is discrete in $A_{\mathcal{A}}^1$, where A_k^1 is the image of the diagonal embedding $A^1 \hookrightarrow A_{\mathcal{A}}^1$.*

Proof. To prove this, consider $x \in A_k^1$. We need to show that there exists an open set $U \subset A_{\mathcal{A}}^1$ such that $U \cap A_{\mathcal{A}}^1 = \{x\}$. Since $A_{\mathcal{A}}^1$ has the subspace topology of $A_{\mathcal{A}}^*$, it is sufficient to find an open set $U \subset A_{\mathcal{A}}^*$ such that $U \cap A_k^1 = \{x\}$. Moreover, since the topology of $A_{\mathcal{A}}^*$ is that induced by the embedding $i : A_{\mathcal{A}}^* \hookrightarrow A_{\mathcal{A}} \times A_{\mathcal{A}}$ given in Lemma 1, it is sufficient to consider an open set $U \subset A_{\mathcal{A}} \times A_{\mathcal{A}}$ such that $U \cap A_k^1 = \{x\}$.

We construct U by selecting any one particular finite valuation v_0 and take $U = \Pi U_v$ where $U_{v_0} = \{(x, x^{-1})\}$ and for all $U_v \neq v_0$ take $U_v = O_v \times O_v$. This U will satisfy the requirement.

□

We are also interested in making sense of the Tamagawa measure. The description of this measure is as follows.

4 Relevant measures

On the space of our adèle and idèle groups, we will be describing a measure which will be very useful for our result. The reader is assumed to be familiar with the concepts of abstract measure theory.

Consider a restricted product $G_{\mathcal{A}}$ of the collection $(\{G_v\}_{v\in\Omega}, \{C_v\}_{v\in\Omega\setminus F})$, as given in the definition 8. We construct a measure μ_v for every $v \in \Omega$. Also, we assume that the measure μ_v is such that for every $v \in \Omega \setminus F$, $\mu_v(C_v) = a_v$. This enables us to define a Borel-measure $\mu_{\mathcal{A}}$ on $G_{\mathcal{A}}$ which is given by

$$\mu_{\mathcal{A}} = \prod \mu_v. \tag{18}$$

If for some finite set $F' \subset \Omega$, the product $\prod_{v\notin F'} a_v$ exists and is non-zero, then this measure makes sense on the σ-algebra generated by open sets of the group $G_{\mathcal{A}}$ because every open set (around the identity) is of the form $U = \prod U_v$ where $U_v = C_v$ for all but finitely many $v \in \Omega$. Because the groups G_v already have a unique Haar measure up to scaling (as they're locally compact), μ_v is simply a rescaling of this Borel measure.

Therefore, it is sufficient to describe the measures μ_v on the groups G_v to be able to understand the Haar measure on $G_{\mathcal{A}}$. For this, we define the notion of a module of an automorphism.

Definition 9. *Suppose G is a locally compact topological group with a left-Haar measure μ, and $\alpha : G \to G$ is group homomorphism. We can now define a measure μ^α whose action on a Borel set $A \subset G$ is given by $\mu^\sigma(A) = \mu(\alpha(A))$. Since this is also a Haar measure, it must be a rescaling of the measure μ. Hence, we write that for some constant c_a*

$$\mu^\alpha = c_\alpha \mu. \tag{19}$$

When G is the additive group of a ring, then the module of the multiplication homomorphism for $h \in G$ (that is, the map $g \mapsto gh$) is given by $|h|$.

Remark 1. *Note that the module map $Hom(G, G) \to \mathbb{R}_+^*$ is a homomorphism.*

Observe that when G is the additive group of $\mathbb{R}$ then for $x \in \mathbb{R}^*$, $|x| = ||x||_\mathbb{R}$. Similarly, when G is the additive group $\mathbb{C}$, then for $x \in \mathbb{C}^*$, we have that $|x| = ||x||_\mathbb{C}^2$. Similarly, when G is the additive group of an order $\mathcal{O}$ inside the $\mathcal{P}$-adic ring $k_\mathcal{P}$, then the multiplication map of $x \in \mathcal{O}^*$ has module $|x| = (\#\frac{\mathcal{O}}{(x\mathcal{O})})^{-1} = N(x)^{-1}$ (this can be seen by writing $\mathcal{O}$ as a disjoint union of translates of $x\mathcal{O}$).

Now suppose that H is a locally compact abelian group. A character of H is a continuous homomorphism $G \to S^1$, where S^1 is the circle group (the set of complex numbers with norm 1). It is well known that there is a construction of a topological group $\hat{H}$ given by the group of all characters, endowed with the topology of uniform convergence on compact sets and point wise multiplication as the group law. Moreover, we also know that the group $\hat{H}$ is also locally compact and abelian, like H.

For certain choices of H, we denote the canonical character ψ_H to be the following:

- If H is the additive group of $\mathbb{R}$, then $\psi_\mathbb{R}$ is the map $x \mapsto e^{-2\pi i x}$.
- If H is the additive group $\mathbb{Q}_p$ then, we define ψ_H to be the map $x \mapsto e^{2\pi i \langle x \rangle}$ where $\langle x \rangle$ is the unique rational in the interval $(0, 1]$ of the form $\frac{a}{p^m}$ such that $x - \langle x \rangle \in \mathbb{Z}_p$.3
- If H is the additive group of a field that is a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$, then we simply define ψ_H to be the map $\psi_{H'} \circ T_H$ where $\psi_{H'}$ is the canonical character of the base field and T_H is the trace map onto the base field. In case H is the additive group of a quaternion algebra over a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$, then we right compose the trace map defined in 11 to get the canonical character.

These definitions are important in the context of the following theorem

Theorem 2. *Let H be a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$ or a quaternion algebra over such a finite extension. Then the map $f : H \to \hat{H}$ given by $(f(x))(y) = \psi_H(xy)$ is a topological isomorphism.*

The proof of this theorem requires analysis each of the possible cases of H separately. We will leave this for the reader to find the proof of each case in [2].

Now suppose G is a locally compact abelian group and has a Haar measure given by dx, then the Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined as

$$\hat{f}(\hat{x}) = \int_G f(x)\langle x, \hat{x}\rangle dx, \tag{20}$$

where $\langle x, \hat{x}\rangle$ denotes the value of character $\hat{x}$ on x. We know that there is a particular choice of the Haar measure $d'\hat{x}$ on the dual group $\hat{G}$ such that the following holds true

$$f(x) = \int_{\hat{G}} \overline{\langle x, \hat{x}\rangle} \hat{f}(\hat{x}) d'\hat{x}. \tag{21}$$

Now suppose G is the group H, then invoking our topological isomorphism $x \mapsto (y \mapsto \psi_H(xy))$ from Theorem 2, Equation (20) starts to look like

$$\hat{f}(y) = \int_H f(x)\psi_H(xy)dx. \tag{22}$$

However, this time the measure $d'\hat{x}$ that makes the following equation true may either be obtained from our favorite topological isomorphism from Theorem 2 or it may be normalized from the preexisting Haar measure found in Equation (22)

$$f(x) = \int_H \hat{f}(y)\psi_H(-yx)d'y. \tag{23}$$

The choice of scaling of the measure dx such that both these choices for the measure $d'x$ are the same gives rise to the Tamagawa measure on H.

Definition 10. *The Tamagawa measure on H is the additive measure on H which is self-dual in the sense described for the Fourier transform associated to the canonical character ψ_H.*

This definition is however not valid for our non-abelian groups (for example, when H is the multiplicative group of units of a quaternion algebra over a number field). We will have to explicitly state the Tamagawa measure in this case. We compile the measures on various groups below.

- If H is the additive group $\mathbb{R}$, then the Tamagawa measure dx_H is simply the Lebesgue measure.
- Suppose H is an additive group of a field extension or quaternion algebra over a field containing $\mathbb{R}$. In this case, we can get an $\mathbb{R}$-basis $\{e_1, \ldots e_k\}$ of H. For $x = \sum x_i e_i$, we get that the Tamagawa measure must be given by

$$dx_H = |\det(T_H(e_i e_j))|^{\frac{1}{2}} \prod dx_i. \tag{24}$$

  Here T_H means the most applicable definition of trace, which means the usual trace-map of number-fields if H is a field extension, or left-composed with the trace-map of quaternion algebra, if H is a quaternion algebra.
- Suppose H is an additive group of a number field or quaternion algebra over the p-adic field $\mathbb{Q}_p$. Moreover, $\mathcal{O}$ is a maximal order on H and we have a $\mathbb{Z}_p$-basis of $\mathcal{O}$, which is given by $\{e_1, \ldots e_k\}$ of H. For $x = \sum x_i e_i$, we take that scaling of the Haar measure which yields $\mathcal{O}$ to have a measure of 1 and call it dx_t. Then $dx_H = D_H^{1/2} dx_t$, where $D_H = |\det(T_H(e_i e_j))|_{\mathbb{Q}_p}^{-1}$ where $|\ |_{\mathbb{Q}_p}$ is the module of quantity and T_H is the most applicable version of the trace map $H \to \mathbb{Q}$ (refer to the previous case).
- When H is some multiplicative group of field extension or a quaternion algebra over the p-adic field $\mathbb{Q}_p$, and O is a maximal order like in the previous setting, then we take dx_H to be the measure given by $(1 - q^{-1})^{-1} ||x||_H^{-1} dx_t$, where dx_t is from the previous setting, $q = \#(\frac{R}{\mathcal{P}R})$, the order of the residual field, and $||x||_H$ is simply the module of x. One is invited to check that this measure is indeed translation invariant.

We remind the reader that in Proposition 4, it was proved that $A_k^1 \hookrightarrow A_{\mathcal{A}}^1$ is a discrete embedding. Hence, it is possible to induce the Tamagawa measure of $A_{\mathcal{A}}^1$ on the group $A_{\mathcal{A}}^1 / A_k^1$.

Proposition 5. *The volume of $\frac{A_{\mathcal{A}}^1}{A_k^1}$ with respect to the Tamagawa measure on $A_{\mathcal{A}}^1$ is 1.*

We will not be able to get into the proof of the above proposition, which involves constructing zeta functions. The details can be found in [3].

Proposition 6. *If A is a quaternion algebra, then $A_{\mathcal{A}}^1 / A_k^1$ is compact.*

The proof of this proposition can be found in [2], where it follows from the compactness of $A_{\mathcal{A}}^* / A_k^*$, which is established by the topological considerations of the quotient space.

Theorem 3. ***(Strong Approximation Theorem)*** *Let A be a quaternion algebra over the number field k and let S be a finite set of valuations of k such that $S \cap \Omega_\infty \neq \phi$ and for at least one $\upsilon_0 \in S$, $\upsilon_0 \notin Ram(A)$. Then $A_k^1 A_S^1$ is dense in $A_{\mathcal{A}}^1$.*

We point the reader towards [2] for a proof of Theorem 3.

In the proof of the upcoming theorem, we will refer to the vector space of Borel measurable real-valued functions with compact support on a differential manifold X by $\mathcal{K}(X)$. The complete theorem could be found in [4].

In order to build the theorem, we define the modular function as follows.

Definition 11. *Let G be a Lie group. Take a left-invariant Haar measure μ_G on G. Now for a fixed $g \in G$, define the Haar measure μ_H', whose value on a measurable set $A \subset G$ is given to be*

$$\mu_H'(A) = \mu_H(Ag^{-1}) \tag{25}$$

Since μ_G' is also a left-invariant measure, we get that $\mu_G' = c_g \mu_G$, where $c_g \in \mathbb{R}_+$ is a constant. We call the homomorphism $\Delta_G : G \to \mathbb{R}_+$ given by $g \mapsto c_g$, the modular function of G.

Theorem 4. *Let G be a Lie group and let H be a closed subgroup. Let Δ_G denote the modular function of G and let Δ_H denote the modular function of H. Then G/H admits a G-invariant measure iff $\Delta_G\big|_H = \Delta_H$. If such a measure exists, then it is unique up to scaling.*

Proof. **(Sketch)**

To construct a Borel measure on G/H simply amounts to constructing a functional $\Lambda : \mathcal{K}(G/H) \to \mathbb{C}$. In order to get the G-invariance, we want the functional to satisfy $\Lambda \circ L_g^* = \Lambda$ where L_g^* denotes pullback of functions on G/H. Now use Riesz representation theorem. So how does one construct this functional? Consider the map of integrating out H,

$$\lambda : \mathcal{K}(G) \to \mathcal{K}(G/H), \tag{26}$$

given by

$$\lambda(f)(gH) := \int_H f(gh)dh. \tag{27}$$

Now one checks that λ is surjective. Let Λ_G denote the positive functional corresponding to the Haar measure on G. Finally check that $\ker(\lambda) \subset \ker(\Lambda_G)$. It follows that the following diagram commutes

$$\begin{tikzcd} \mathcal{K}(G) \arrow[r, "\Lambda_G"] \arrow[dr, "\lambda"'] & \mathbb{C} \\ & \mathcal{K}(G/H) \arrow[u, "\Lambda"'] \end{tikzcd}$$

The required invariance property for Λ follows from that of Λ_G.

□

The following theorem will aid us in understanding the proof of Theorem 1. This Theorem is Theorem 8.1.2 taken from [2].

Theorem 5. *Let R be the ring of integers in a number field k. Let $\mathcal{O}$ be an R-order in a quaternion k-algebra A that satisfies the Eichler condition. Let $\mathcal{O}^1 = \{\alpha \in \mathcal{O} \mid n(\alpha) = 1\}$. Under the embedding $\psi : A \to G$ as given in Equation (16), $\psi(\mathcal{O}^1)$ is discrete, and has finite covolume as a subgroup of $G^1 = \bigoplus SL_2(k_v)$. Moreover, if A is a quaternion division algebra, then $\psi(\mathcal{O}^1)$ is cocompact.*

Proof. We invoke the idéle group $A^1_{\mathcal{A}}$ and consider the following open subset U of this group given by

$$U = G^1 \times \prod_{v \in \mathrm{Ram}_\infty(A)} A^1_v \times \prod_{v \in \Omega_f} \mathcal{O}^1_v \tag{28}$$

If A^1_k is the image of the diagonal embedding $A^1 \hookrightarrow A^1_{\mathcal{A}}$, then we claim that $A^1_k \cap U = \mathcal{O}^1$. To see this, we observe that $\mathcal{O}^1 \subset A^1_k$, because every element of $\mathcal{O}^1$ has norm 1 in every coordinate of completion of A. Conversely, using the local-global principle from Proposition 3, we observe that if $x \in A^1_k \cap U$ then $x \in \mathcal{O}_v$ for every $v \in \Omega_f$ and therefore $x \in \mathcal{O}$.

Secondly, we see that $A_{\mathcal{A}} = A^1_k U$. To see this, suppose $x = (x_v) \in A^1_{\mathcal{A}}$. Then there exists a finite set of places $S \supset \Omega_\infty$ such that $x_v \in \mathcal{O}^1_v$ whenever $v \notin S$. Then let $S = \Omega_\infty \sqcup T$ and let

$$V = \prod_{v \in \Omega_\infty} A^1_v \times \prod_{v \in T} x_v \mathcal{O}^1_v \times \prod_{v \in \Omega_f \setminus T} \mathcal{O}^1_v. \tag{29}$$

We note that V is open in the topology of $A^1_{\mathcal{A}}$ and by Strong Approximation Theorem as given in Theorem 3, we get that $A^1_k A^1_{\Omega_\infty} \cap V \neq \phi$ (because an open set has a non-empty intersection with a dense set). Take $x_0 \in A^1_k, y \in A^1_{\Omega_\infty}$ be such that $x_0 y \in V$. Then by construction, $x_o^{-1} x \in U$ which means that $A^1_k U = A^1_{\mathcal{A}}$.

We thus get a natural homeomorphism as captured by

$$\frac{A^1_{\mathcal{A}}}{A^1_k} = \frac{A^1_k U}{A^1_k} \to \frac{U}{U \cap A^1_k} = \frac{U}{\mathcal{O}^1}. \tag{30}$$

Thus $\mathcal{O}^1$ inside U is discrete by Proposition 4, has covolume 1 by Proposition 5. Furthermore, the quotient is compact if A is a division algebra by Proposition 6.

5 Hyperbolic geometry

We will now pass over from the mysterious geometry of the adéles and idéles to the more concrete world of hyperbolic geometry.

We will present our discussion following along the lines of the exposition given in [5].

5.1 Lorentzian space

On $\mathbb{R}^n$ we define the following inner product for vectors $x, y \in \mathbb{R}^n$

$$x \circ y = -x_1 y_1 + x_2 y_3 \cdots + x_n y_n. \tag{31}$$

The above is an inner product space called the Lorentzian vector space, we denote this space by the $\mathbb{R}^{1,n-1}$. We define the Lorentzian norm[3] on $\mathbb{R}^{1,n-1}$ as

$$||x|| = \sqrt{x \circ x}. \tag{32}$$

Note that we allow the Lorentzian norm to take positive, zero and imaginary with positive imaginary part. We may use $|||x|||$ to denote the absolute value of the norm. If by $\bar{x}$ we denote the vector $(x_2, x_3 \ldots, x_n) \in \mathbb{R}^n$ then it is clear that Equation (32) implies that

$$||x||^2 = |x_1|^2 - ||\bar{x}||^2. \tag{33}$$

We call the $C^{n-1} = \{x \in \mathbb{R}^{1,n-1}, ||x|| = 0\}$ the light-cone of $\mathbb{R}^{1,n-1}$. We call those vectors with $||x|| > 0$ space-like and $\Im(||x||) > 0$ as time-like.

Definition 12. *A linear transformation $\phi : \mathbb{R}^{1,n-1} \to \mathbb{R}^{1,n-1}$ is said to be a Lorentzian transformation if for every $x, y \in \mathbb{R}^{1,n-1}$ we have that*

$$x \circ y = \phi(x) \circ \phi(y). \tag{34}$$

Some facts about Lorentzian transformations are as follows.

Proposition 7. *The following statements are equivalent*

- *$\varphi : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ is a Lorentzian transformation.*
- *φ maps 'orthonormal basis' to 'orthonormal basis'.*
- *If A is a matrix associated to φ in the canonical basis of $\mathbb{R}^{1,n} \simeq \mathbb{R}^{n+1}$, then A satisfies $A^t J A = J$ where J is the matrix given by*

$$J = \begin{bmatrix} -1 & 0_{1\times n} \\ 0_{n\times 1} & I_{n\times n} \end{bmatrix}. \tag{35}$$

Note that here 'orthonormal basis' refers to a basis $\{e_1, e_2 \ldots e_{n+1}\}$ of $\mathbb{R}^{1,n} \simeq \mathbb{R}^{n+1}$ that satisfies the property that the matrix $[e_1 \circ e_{n+1}] = J$.

The proof of the above proposition can be found in [5].

We denote the group of Lorentzian transformations on $\mathbb{R}^{1,n-1}$ by $O(1, n-1)$. Note that because of Proposition 7, we get that $\varphi \in O(1, n-1) \to \det(\varphi) = \pm 1$. We denote the group $\{\varphi \in O(1, n-1), \det(\varphi) = 1\}$ by $SO(1, n-1)$. Lastly, note that any any $\varphi \in SO(1, n-1)$ will fix (as a set) the connected component of positive time-like vectors, or it will flip the light-cone. We denote the set of Lorentzian transformations of $SO(1, n-1)$ that preserve that time-like vectors by $PSO(1, n-1)$.

Lemma 2. *The action of $PO(1, n-1)$ is transitive on the set of time-like subspaces of the same dimension.*

Proof. **(Sketch)** The proof of this lemma follows from a generalization of the Grahm-Shmidt orthonormalization process for the case of the Lorentzian meaning of 'orthonormal basis'. Using this, we can show that every time-like subspace is spanned by an 'orthonormal basis', and therefore transitivity follows from Proposition 7. □

Theorem 6. *For time-like vectors $x, y \in \mathbb{R}^{1,n-1}$, we have*

$$x \circ y \leq ||x||||y||, \tag{36}$$

with equality if and only if x, y are linearly dependent.

[3]this function is actually not a norm in the usual sense. One must remember this

Proof. Because of Lemma 2, we can find some $\varphi \in PO(1, n-1)$ such that $\varphi(x) = te_1$ for some $t \in \mathbb{R}$ and e_1 being the vector $(1, 0, 0, \dots)$. Then replace x, y with $\varphi(x), \varphi(y)$ where $\varphi(y) = (y_1, y_2, \dots)$. This we have that

$$\begin{aligned} x \circ y &= \varphi(x) \circ \varphi(y) && (37) \\ &= te_1 \circ \varphi(y) && (38) \\ &= -ty_1. && (39) \end{aligned}$$

Whereas

$$\begin{aligned} ||x||^2||y||^2 &= ||A(x)||^2||A(y)||^2 && (40) \\ &= -t^2(y_1^2 + y_2^2 + \dots) && (41) \\ &\leq t^2y_1^2 && (42) \\ &= (x \circ y)^2. && (43) \end{aligned}$$

Also, when equality happens, $y_2^2 + y_3^2 \cdots = 0$. Hence, we see that x, y must be linearly dependent.

□

Definition 13. *Suppose $x, y \in \mathbb{R}^{1,n-1}$ are non-zero time-like vectors. We define the time-like angle between x and y to be the real positive number $\eta(x, y)$ defined by*

$$x \circ y = ||x||||y|| \cosh(\eta(x, y)). \tag{44}$$

5.2 Hyperbolic n-space

We define $\mathbf{H}^n = \{x \in \mathbb{R}^{1,n} \mid ||x|| = -1,\ \pi_1(x) > 0\}$. This space is called the hyperbolic n-space. Now glance upon the following proposition.

Proposition 8. *The function $d : \mathbf{H}^n \times \mathbf{H}^n \to \mathbf{H}^n$ is a metric on $\mathbf{H}^n$, where d is given by*

$$d(x, y) = \eta(x, y), \tag{45}$$

and η is from Equation (44).

Proof. Only the triangle inequality is worth proving. To see this, we will have to divert our discussion slightly and introduce the Lorentzian cross-product. Consider $x, y, z \in \mathbb{R}^{1,2} \in \mathbb{R}^3$ and define J_3 as

$$J_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{46}$$

and using this, we define $\bar{\times} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ as $x \bar{\times} y = J_3(x \times y)$ where $\times$ denotes the usual cross-product in $\mathbb{R}^3$.

After establishing some properties of $\bar{\times}$, we will be able to see the following

$$||x \bar{\times} y||^2 = (x \circ y) - ||x||^2||y||^2, \tag{47}$$

which implies that $||x \bar{\times} y|| = \sinh(\eta(x, y))||x||^2||y||^2$. With this, assuming $||x|| = ||y|| = ||z|| = -1$, we can write that

$$\begin{aligned} \cosh(\eta(x, y) + \eta(y, z)) &= \cosh(\eta(x, y)) \cosh(\eta(y, z)) + \sinh(\eta(x, y)) \sinh(\eta(y, z)), && (48) \\ &= (x \circ y)(y \circ z) + ||x \bar{\times} y||||y \bar{\times} z||, && (49) \\ &\geq (x \circ y)(y \circ z) + (x \bar{\times} z).(y \bar{\times} z), \text{ using the usual Cauchy-Shwartz ineq.} && (50) \\ &\geq (x \circ y)(y \circ z) + (x \circ z)(y \circ y) - (x \circ y)(y \circ z), \text{ because } a \bar{\times} y = J_3(a \times y), && (51) \\ &\geq -x \circ y = \cosh(\eta(x, y)). && (52) \end{aligned}$$

Using the monotonicity of cosh in the region of our interest, we get that $\eta(x, y)$ satisfies the triangle inequality in $\mathbf{H}^2 \subset \mathbb{R}^{1,2}$. For $x, y, z \in \mathbf{H}^n$, we make an isometry of the subspace generated by x, y, z in $\mathbb{R}^{1,n}$ with $\mathbb{R}^{1,2}$ to finish the proof. □

Since we know that Lorentzian transformations preserve the Lorentzian norm in $\mathbb{R}^{1,n}$, we can restrict a Lorentzian transformation to $\mathbf{H}^n$. Moreover, because the Lorentzian transformations preserve angles, they would restrict to become isometries of $\mathbf{H}^n$. Because of the next proposition, we know that the converse is also true.

Proposition 9. *Every positive isometry of* $\mathbf{H}^n \hookrightarrow \mathbb{R}^{1,n}$ *is the restriction of some* $\varphi \in PSO(1,n)$.

Proof. Suppose $\varphi : \mathbf{H}^n \to \mathbf{H}^n$ is an isometry. We will now extend it to an element $\varphi' \in PSO(1,n)$.

Let $x = \varphi(e_1)$, e_1 being the vector $(1,0,\dots) \in \mathbf{H}^n$. Take $B \in PSO(1,n)$ such that $B\varphi(e_1) = e_1$. Since it is sufficient to prove that $B\varphi \in PSO(1,n)$, we can assume without loss of generality that $\varphi(e_1) = e_1$.

Let $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map onto the last n coordinates. Since we know that p is bijective when restricted to $\mathbf{H}^n$, we can define $C : \mathbb{R}^n \to \mathbb{R}^n$ as $C = p\varphi p^{-1}$. Now clearly $C(p(x)) = p(\varphi(x))$ for any $x \in \mathbf{H}^n$. Since φ is an isometry, we get that for any $x, y \in \mathbf{H}^n$, $x \circ y = \varphi(x) \circ \varphi(y)$, and therefore

$$-x_1y_1 + C(p(x)).C(p(y)) = -x_1y_1 + \sum_{k=2}^{n+1} x_k y_k. \tag{53}$$

This tells us that C is a orthonormal linear transformation on $\mathbb{R}^n$ and we get that there is an orthonormal $n \times n$ matrix A corresponding to C on $\mathbb{R}^n$. We define the matrix $\hat{A}$ to be given by

$$A = \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & A \end{bmatrix}. \tag{54}$$

Now check that $\hat{A}$ is the required transformation φ'.

For uniqueness, simply observe that if φ extends to A and B in $PSO(1,n)$ then AB^{-1} fixes all of H^n. However, since H^n is not contained in any proper subspace of $\mathbb{R}^{1,n}$, $AB^{-1} = 1$ and we are done.

□

An important and straightforward corollary of the previous proposition is the following.

Corollary 1. *The group of orientation preserving isometries of* H^n *is isomorphic to* $PSO(1,n)$.

We end this section by giving the working models of our hyperbolic space.

Imagine the space $(\{0\} \times \mathbb{R}^n) \hookrightarrow \mathbb{R}^{1,n}$ and consider the unit open ball $B^n \hookrightarrow (\{0\} \times \mathbb{R}^n)$. For every point in $x \in H^n \hookrightarrow \mathbb{R}^{1,n-1}$ we construct a line-segment joining x to the point $-e_1$, e_1 being the one of the canonical basis elements of $\mathbb{R}^{1,n-1}$. This line intersects the ball B^n at a unique point which we will call $\tau(x)$. A little calculation gives us a formula below for $x = (x_0, \dots x_n)$.

$$\tau(x) = (\frac{x_1}{1+x_0}, \frac{x_2}{1+x_0}, \frac{x_2}{1+x_0}, \dots, \frac{x_n}{1+x_0}). \tag{55}$$

Having written down the formula explicitly, we see that this map is a continuous bijective homeomorphism. We can also convert B^n into a metric space by declaring the map above to be an isometry. Hence the metric on B^n is now given by

$$d(x,y) = \eta(\tau^{-1}(x), \tau^{-1}(y)), \tag{56}$$

where η is as defined in 44. Calculating this metric explicitly gives us the following proposition.

Proposition 10. *The metric* d *defined in Equation (56) is given by the following equation, (where* $||\ ||$ *is the* n*-dimensional Euclidean metric on* $B^n \hookrightarrow (\{0\} \times \mathbb{R}^n)$ *)*

$$\cosh(d(x,y)) = 1 + \frac{2||x-y||}{(1-||x||^2)(1-||y||^2)}. \tag{57}$$

Proof. It just follows from straightforward calculations. □

This setting of hyperbolic geometry is called the Poincarè disc model of hyperbolic geometry. We will be using the closely related model known as the upper half-plane model of hyperbolic geometry. As a set, the upper half-space, denoted by $\hat{U}$ is the space

$$\hat{U} = \{(x_1, x_2, \dots x_n) \in \mathbb{R}^n \ , \ x_1 > 0\}, \tag{58}$$

where ∞ is the point at infinity in $\hat{\mathbb{R}}$. There are many correspondences between $\hat{U}^n$ and B^n. One of them is the following: take by the bijective map $\beta : B^n \to \hat{U}$ described as a composition of three maps $a_1 \circ a_2 \circ a_3 = \beta$ where $a_3 : B^n \to B^n$, is the reflection about a plane passing though origin ($x_1 = 0$, for instance), $a_2 : B^n \to B_p(1/2)$ is just translation and orientation preserving scaling of B^n to $B_p(1/2)$ which is the ball of radius $1/2$ around $p = (1/2, 0, 0, \dots) \in \mathbb{R}^n$, and finally $a_3 : B_p(1) \to \hat{U}$ is the sphere inversion map.

The reason why the above correspondence is so convoluted is because of a number of reasons. It is because of the conformal nature of each of the maps $\alpha_1, \alpha_2, \alpha_3$. Such maps are called Möbius transformations. There is a lot more about this subject which can be found in [5], for instance.

We will be using the upper-half space model $\hat{U}^n$ for our considerations and we will call this space $\mathbf{H}^n$. And for all our needs, we will only need the dimension $n = 3$. We will label the coordinates of this set as

$$\hat{U}^3 = \mathbb{C} \times \mathbb{R}_+. \tag{59}$$

The following is a useful theorem, central to our use of geometry in this text.

Theorem 7. *The group $PSO(1,3)$ is isomorphic to $PSL(2,\mathbb{C})$, where $PSL(2,\mathbb{C})$ is the group $SL(2,\mathbb{C})/\{I, -I\}$ and $SL(2,\mathbb{C})$ is the group of matrices in $M_2(\mathbb{C})$ with determinant 1.*

Proof. We define the map $\gamma : PSL(2,\mathbb{C}) \to PSO(1,3)$ by it's action on the Lorentzian sphere $\mathbf{H}^n$. For a point $p = (x_1, x_2, x_3, x_4) \in \mathbf{H}^3$, we define the matrix A_p by

$$A_p = \begin{bmatrix} x_2 - x_1 & -(x_3 + ix_4) \\ (x_3 + ix_4) & x_2 + x_1 \end{bmatrix}. \tag{60}$$

with this we get that $det(A_p) = ||p|| = -1$. Thus, for any $P \in SL(2,\mathbb{C})$ we can consider the action of P as $p \mapsto PA_pP^*$, where P^* is the Hermitian transpose of P. By a little calculations, we can check that this map preserves norms, and therefore corresponds to an action of of a unique map from $PSO(1,3)$. Also, this map is independent of $\{I, -I\}$ symmetry and factors to become a map from $PSL(2,\mathbb{C})$.

Now we must prove that the kernel of this map is trivial. This is clear, as the images of any 100 points uniquely determine the element of $PSL(2,\mathbb{C})$ which is acting on $\mathbf{H}^n$. Hence we are done.

□

Some properties of hyperbolic 3-space are as follows, stated without proof. For more information, one can look at [5].

Proposition 11. *Every geodesic in the hyperbolic 3-space in the upper half space model can be extended to become a semi-circle perpendicular to the boundary. Every geodesic plane is of the form of a hemisphere whose centre is at the boundary.*

Here boundary means the space $(\mathbb{C} \times \{0\}) \cup \{\infty\}$. One must know that spheres with centres at ∞ look more like planes, in the usual sense.

Since the space $\mathbf{H}^3$ is a Riemannian manifold, it is possible to talk about volumes in this setting. The following formula is useful in finding volumes of "ideal tetrahedra" in hyperbolic 3-space. A tetrahedron is said to be an ideal tetrahedron if it has at least one ideal vertex in the upper-half space model, and one vertex at ∞, and the projection of the tetrahedron from ∞ is a right-angled triangle on the xy plane (See Figure 1). And ideal vertex in the upper-half space model is a vertex which is of the form of two vertical planes (spheres containing infinity), whose axis of intersection is intersecting with the unit hemispherical geodesic perpendicularly (and therefore, at the apex point of the hemisphere.

The volume of such a tetrahedron is given by the following formula in the upcoming proposition.

Proposition 12. *Suppose $T_{\alpha,\gamma}$ is the ideal tetrahedra that has a dihedral angle (the angle between the two vertical planes) as γ and has an acute angle (that is, the angle between the geodesic plane opposite to γ and the unit geodesic hemisphere) given by α (see Figure 1). Then the following is true.*

$$Vol(T_{\alpha,\gamma}) = \frac{1}{4}(\mathcal{L}(\alpha+\gamma) + \mathcal{L}(\alpha-\gamma) + 2\mathcal{L}(\frac{\pi}{2}-\alpha)). \tag{61}$$

Here $\mathcal{L}$ is the Lobachevski function, described in the proof.

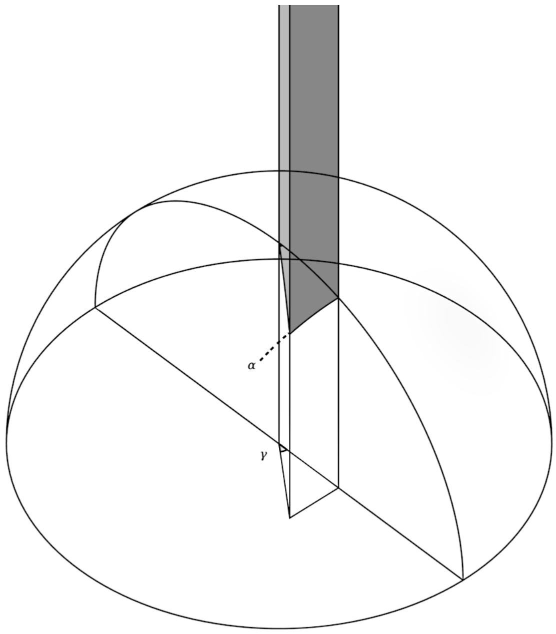

Figure 1: An ideal tetrahedron

Proof. After manipulations of the appropriately transforming the metric described in Equation (45), we obtain the following equation:

$$\text{Vol}(T_{\alpha,\gamma}) \int\int_T \int_{t\geq\sqrt{1-(x^2+y^2)}} \frac{dxdydz}{t^3}. \tag{62}$$

After some careful considerations, we obtain the equality given below

$$\text{Vol}(T_{\alpha,\gamma}) = \frac{1}{4}\int_{\gamma}^{\frac{\pi}{2}} \log\left(\left|\frac{\sin(\theta-\alpha)}{\sin(\theta-\alpha)}\right|\right). \tag{63}$$

We can conveniently express the integral in terms of the Lobachevski function. We define the Lobachevski function as

$$\mathcal{L}(\theta) = -\int_0^{\theta} \log|2\sin(u)|du. \tag{64}$$

After this, it is simply a matter of rewriting Equation (63) in terms of the Lobachevski function. □

The upper-half space is quite a commonly used model for hyperbolic geometry because of the ease of visualization. Moreover, many formulae become quite less cluttered in this setting. Hence, it is a model of preference for hyperbolic geometry.

6 Arithmetic hyperbolic 3-manifolds

After having defined the hyperbolic space $\mathbf{H}^3$ in entirety, we state the next definition.

Definition 14. *Let k be a number field with exactly one complex place and let A be a quaternion algebra over k which is ramified at all real places. Let ψ be a k-embedding of A into $M_2(\mathbb{C})$ (as provided in Equation (16)) and let $\mathcal{O}$ be an R_k-order of A. Then a subgroup $\Gamma \hookrightarrow SL(2,\mathbb{C})$ (or $PSL(2,\mathbb{C})$) is an arithmetic Kleinian group if it is conjugate to some $\psi(\mathcal{O}^1) \hookrightarrow SL(2,\mathbb{C})$.*

Hyperbolic manifolds and 3-orbifolds $\mathbf{H}^3/\Gamma$ will be referred to as arithmetic hyperbolic manifolds.

We immediately supplement our definition to provide some useful details to our arithmetic hyperbolic 3-manifolds.

Proposition 13. *Suppose Γ is a discrete subgroup of $SL(2,\mathbb{C})$ that has finite covolume in the Tamagawa measure (when looked at as a quotient of an adèle group), then $\mathbf{H}^3/\Gamma$ too has finite volume.*

Proof. To see this, simply observe that $SU(2,\mathbb{C}) \subset SL(2,\mathbb{C})$, is a compact subgroup and the quotient $SL(2,\mathbb{C})/SU(2,\mathbb{C}) \simeq \mathbf{H}^3$ (because $SU(2,\mathbb{C})$ is just the stabilizer of a point). Also, notice that $SU(2,\mathbb{C})$ is a compact space. Therefore, any continuous homomorphisms $\Delta : SU(2,\mathbb{C}) \to \mathbb{R}_+^*$ must be bounded, and if for some $t \in SU(2,\mathbb{C})$, $\Delta(t) > 1$, then $\lim_{n\to\infty} \Delta(t^n) \to \infty$ which leads to a contradiction. Hence Δ must be trivial.

Therefore, $\Delta_{SL(2,\mathbb{C})}|_{SU(2,\mathbb{C})}$ is trivial and is equal to $\Delta_{SU(2,\mathbb{C})}$ and the $SL(2,\mathbb{C})$-invariant measure induced by the Riemannian metric on $\mathbf{H}^3$ is some scaler multiple of the induced Tamagawa measure on $SL(2,\mathbb{C})/SU(2,\mathbb{C})$. Hence, we are done. ☐

As discussed before, we are mainly interested in finding a formula for the volume of an arithmetic hyperbolic manifold. To do this, we must determine the constant of proportionality between the Riemannian measure on $\mathbf{H}^3$ and the Tamagawa measure on the same, induced because of Equation (30).

Theorem 8. ***(Volume formula)***

Let k be a number field with exactly one complex place, A be a quaternion algebra over k, ramified and let $\mathcal{O}$ be a maximal order in A. Then if ψ is a k-representation of A to $M_2(\mathbb{C})$, then [4]

$$Vol(\mathbf{H}^3/\psi(\mathcal{O}^1)) = \frac{4\pi^2|\Delta_k|^{\frac{3}{2}}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]}}. \tag{65}$$

Proof. From the proof of Theorem 5, U is the open subgroup given as

$$U = SL(2,\mathbb{C}) \times \prod_{v\in \mathrm{Ram}_\infty(A)} A_v^1 \times \prod_{v\in\Omega_f} \mathcal{O}_v^1 \simeq SL(2,\mathbb{C}) \times \prod_{v \text{ real}} \mathbb{H}^1 \times \prod_{\mathcal{P}\in\Omega_f} \mathcal{O}_{\mathcal{P}}^1 \tag{66}$$

From Proposition 3, it follows that since $\mathcal{O}$ is maximal, all $\mathcal{O}_{\mathcal{P}}$ are maximal in $k_{\mathcal{P}}$ [5]. All the components of U apart from the first one are compact. Following the proof of Proposition 3, also tells us that the Tamagawa volume of $U/\mathcal{O}^1$ is 1. Since the Tamagawa measure is actually a product of measures on each component, the volume of $U/\mathcal{O}^1$ is the product of the local volumes of each component.

Using the descriptions of the measures provided in the discussion following Definition 10, we can conclude the following things (details can be found in [2]).

- $\mathrm{Vol}(\mathbb{H}^1) = 4\pi^2$.
- If $\mathcal{P} \notin \mathrm{Ram}_f(A)$, then $\mathrm{Vol}(\mathcal{O}_{\mathcal{P}}^1) = D_{k_{\mathcal{P}}}^{-3/2}(1 - N(\mathcal{P})^{-2})$.
- If $\mathcal{P} \in \mathrm{Ram}_f(A)$, then $\mathrm{Vol}(\mathcal{O}_{\mathcal{P}}^1) = D_{k_{\mathcal{P}}}^{-3/2}(1 - N(\mathcal{P})^{-2})(N(\mathcal{P})-1)^{-1}$.

[4] note the abuse of notation here, $\psi(\mathcal{O}^1)$ when seen as a subgroup of isometries of $\mathbf{H}^3$ should be taken modulo ± 1

[5] This is an abuse of notation and we are referring to the valuation associated with a prime ideal $\mathcal{P}$ by the symbol $\mathcal{P}$ itself

Here $N(\mathcal{P})$ is the norm of the ideal, given by the cardinality of $|R/\mathcal{P}|$ and $D_{k_\mathcal{P}}$ is the discriminant of the field extension $k_\mathcal{P}/\mathbb{Q}_p$, where $p = \mathcal{P} \cap \mathbb{Z}$. It is known that the product $\prod_\mathcal{P} D_{k_\mathcal{P}} = \Delta_k$, where Δ_k is the absolute determinant of k.

Therefore, multiplying all the terms gives us that

$$\mathrm{Vol}(SL(2,\mathbb{C})/\psi(\mathcal{O}^1)) = \prod_{v \text{ real}} \mathrm{Vol}(\mathbb{H}^1)^{-1} \prod_{\mathcal{P}} (\mathrm{Vol}(\mathcal{O}^1_\mathcal{P}))^{-1}, \tag{67}$$

$$= \frac{|\Delta_k|_{3/2} \zeta_k(2) \prod_{\mathcal{P} \in \mathrm{Ram}_f(A)} (N(\mathcal{P}) - 1)}{(4\pi^2)^{[k:\mathbb{Q}]-2}} \tag{68}$$

To arrive at Equation (65), we know that in accordance with Theorem 4, $Vol(\mathbf{H}^3/\psi(\mathcal{O}^1))$ is some constant multiple of $\mathrm{Vol}(SL(2,\mathbb{C}/\psi(\mathcal{O}^1))$. To determine the constant, we simply put $k = \mathbb{Q}[i]$. This makes $\psi(\mathcal{O}^1)$ the Picard group $SL(2,\mathbb{Z}[i])$, and it is known that one choice of the fundamental domain for $\mathbf{H}^3/PSL(2,\mathbb{Z}[i])$ is given by (see Figure 2)

$$\{(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \mid x^2 + y^2 + t^2 \geq 1, x \leq \frac{1}{2}, y \leq \frac{1}{2}, x + y \geq 0\}. \tag{69}$$

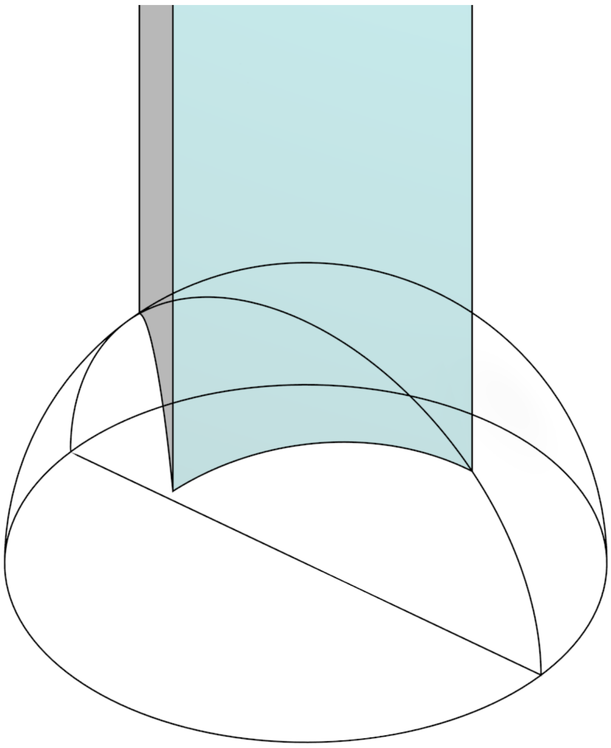

Figure 2: Fundamental region of $\mathbf{H}^3/PSL(2,\mathbb{Z}[i])$.

Evaluating the volume of this tetrahedron using our ideal tetrahedron formula, and evaluating it again using Equation (68) will complete the calculations to obtain Equation (65). These calculations will be performed towards the end as an example of the application of Theorem 1.

□

□

7 Proof of Theorem 1

Proof. We first assume that $s = 1$ before we prove the general version. Therefore, K is a $r + 2$ degree extension of $\mathbb{Q}$. We assume that B is a quaternion algebra over K such that it satisfies the following property.

$$B \otimes_K \mathbb{R} \simeq \mathbb{H} \tag{70}$$

Consider an R-order $\mathcal{O}$, where R is the ring of integers of K or some finite index subring of the ring of integers. Now consider the group $\mathcal{O}^1$ of units of $\mathcal{O}$ that have reduced norm 1 and consider Γ to be a further subgroup of $\mathcal{O}^1$ which is torsion-free and of finite index with respect to $\mathcal{O}^1$. Because $s = 1$, we have two possible embeddings of K into $\mathcal{C}$. We select one of the two embeddings. This gives us an identification of $B \otimes_K \mathbb{C}$ with $M_2(\mathbb{C})$ and we can embed Γ in $SL_2(\mathcal{C})$. This group must be discrete according to Theorem 5. Now $SL_2(\mathcal{C})/\{1, -1\}$ is the group of isometries of the hyperbolic 3-space $\mathbf{H}^3$ and thus we get a free and properly discontinuous action of Γ on $\mathbf{H}^3$. This gives us that the quotient $\mathbf{H}^3/\Gamma$ is a C^∞ manifold.

In accordance with the Theorem 5, the volume of this quotient manifold can be assumed to be finite [6]. Also, it is known that the volume of this compact-manifold is a rational multiple of $(\zeta_K(2)/\pi^{2r+2})\sqrt{|D|}$, according to the volume formula in Equation (65).

Putting these facts into perspective, in order to prove Theorem 1, it is sufficient to prove that the fundamental volume of $\mathbf{H}^3/\Gamma$ is a rational linear combination of values of $A(x)$ for algebraic arguments as x.

Choose an extension K_1 of K which is such that $B \otimes_K K_1 \simeq M_2(K_1)$.[7]. Choosing an embedding K_1 subset of $\mathbb{C}$ gives us an embedding $SL_2(K_1) \subset SL_2(\mathbb{C})$. This embedding is dense and countable and contains our discrete group Γ. The action of Γ on $\mathbf{H}^3$ is such that it preserves the dense set of points A_0 whose presentation as a coordinate in $\mathbf{H}^3 \simeq \mathbb{C} \times \mathbb{R}_+$ is in terms of K_1.

Because A_0 is a dense set, we can choose a triangulation with sufficiently small simplices such that the vertices of each simplices lie in A_0[8]. To prove the theorem for $s = 1$ then simply amounts to showing that each of these simplices has a volume that can be expressed in terms of rational linear combination of values of $A(x)$ where x are algebraic integers.

We prove it as follows. Consider a simplex with vertices $P_i \in A_0 = K_1 \times (K_1 \cap \mathbb{R}^+)$ for $i = 0, 1, 2, 3$. Connect P_0 and P_1 and extend this ray to the boundary of $\mathbf{H}^3$, i.e. $\mathbb{CP}^1$. Since the extended ray must be reaching the boundary at a point of $K_1\mathbb{P}^1 \subset \mathbb{CP}^1$. Therefore there must exist a volume preserving isometry of $SL_2(K_1)$ which sends this point to infinity. Hence, we see reorient all the points P_0, P_1, P_2, P_3 such that P_0 and P_1 are eclipsed when seen from ∞ while they still lie in A_0. Hence, the projection from ∞ on $\mathbb{C}$ would produce a triangle. Because this simplex is now simply the difference of two tetrahedrons with the apex point at infinity, it is sufficient for the proof of the theorem to prove that both of them have volumes that can be expressed in terms of rational linear combination of of values of $A(x)$ where x are algebraic integers.

Now such a tetrahedron is bounded by 3 vertical planes and a hemisphere orthogonal to the boundary of $\mathbf{H}^3$. One can imagine a triangle formed on the hemisphere by the 3 planes. Suppose P is the topmost point of the said hemisphere in Euclidean terms (that is, if each point in $\mathbf{H}^3$ has points of the form (z, t) where $z \in \mathbb{C}$ and $t \in \mathbb{R}_+$, then we want the point on the hemisphere that maximizes t). Draw 3 perpendicular along the said hemisphere. This splits up the triangle on the hemisphere into the sum or difference of 6 triangles, and therefore our tetrahedron is now split into the sum/difference of 6 different tetrahedrons. These tetrahedrons are nothing but the ideal tetrahedron. See Figure 3.

Now for proving the main theorem, it is further sufficient to show that each of these ideal tetrahedrons is of the required form. This is however an easier problem because the volume of such a tetrahedron is known according to the formula given below.

As noted earlier, an ideal tetrahedron having an acute angle α and the dihedral angle as γ has the volume given by

$$\mathrm{Vol}(T_{\alpha,\gamma}) = \frac{1}{4}(\mathcal{L}(\alpha + \gamma) + \mathcal{L}(\alpha - \gamma) + 2\mathcal{L}(\frac{\pi}{2} - \alpha)). \tag{71}$$

As noted earlier, $\mathcal{L}$ is the Lobachevski function. Through some easy manipulation, we discover the following identity.

$$A(x) = 2\mathcal{L}(\mathrm{arccot}(x)). \tag{72}$$

Through our insight, we quickly replace the terms in Equation (71) and substitute $\alpha = \tan(a), c = \tan(\gamma)$. This gives us access to the identity below.

[6] It took me many months to find out why

[7] In [1], it is said that such an extension K_1 of K is a splitting field of B

[8] I am wondering if there is more rigorous way to say this.

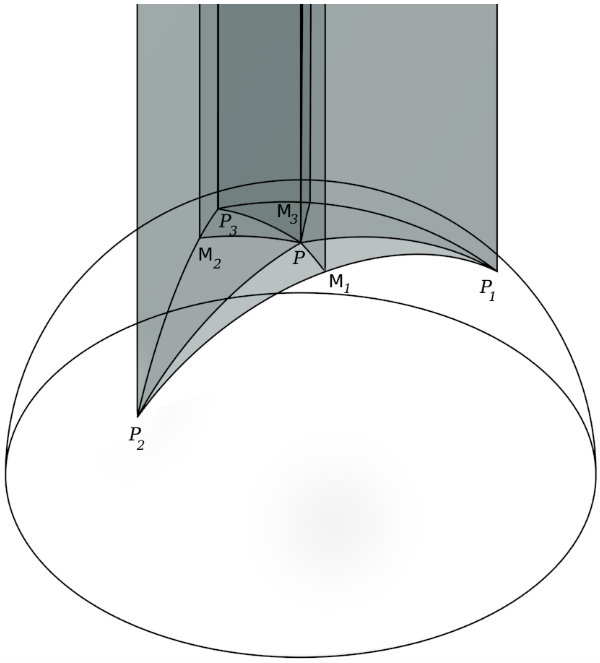

Figure 3: The splitting of the tetrahedron in ideal tetrahedra

$$\mathrm{Vol}\left(T_{\alpha,\gamma}\right) = \frac{1}{8}\left(A\left(\frac{1-ac}{a+c}\right) + A\left(\frac{1+ac}{a-c}\right) + 2A\left(a\right)\right). \tag{73}$$

So to arrive at the solution for the case where $s = 1$, we only need to show that a and c are algebraic integers. If we recall from the construction of the ideal triangles, the vertices of the tetrahedron that was split to make the ideal triangles were in the dense set of points A_0 whose presentation as a coordinate in $\mathbf{H}^3 \simeq \mathbb{C} \times \mathbb{R}_+$. Suppose the coordinates of the point P are (Z, R), then the points on the tetrahedron are at a fixed Euclidean distance r from the point $(Z, 0)$. So we have that, for each (z_i, r_i) having coordinates in K_1, $(i = 1, 2, 3)$,

$$|z_i|^2 + r_i^2 = r^2. \tag{74}$$

Solving these equations will give us that $Z \in K_1$ and $r^2 \in K_1$. Pursuing these calculations will also reveal that $\tan(\alpha)$ and $\tan(\gamma)$ lie in K_1. This will complete the proof for the case of $s = 1$.

When $s \neq 1$, then we take the same setting. B is a quaternion algebra over k which is ramified at all real places, $\mathcal{O}$ an order in B, and Γ is a torsion-free subgroup of finite index in the group $\mathcal{O}^1$ of units in $\mathcal{O}$ of reduced norm 1. We also assume that $\sigma_1 \dots \sigma_s : K \to \mathbb{C}$ are embeddings of K into $\mathbb{C}$.

Then from Equation (16), we have a discrete embedding $\Gamma \to M_2(\mathbb{C})^s$, which is discrete in accordance with Theorem 5. This gives us a properly discontinuous, free action of Γ on $(\mathbf{H}^3)^s$. Now consider the manifold $(\mathbf{H}^3)^s/\Gamma$. Then, it is sufficient to show that M can be decomposed as the union of sets that are of the form $\pi(\Delta_1 \times \Delta_2 \dots \Delta_s)$ where $\pi : (\mathbf{H}^3)^s$ is the canonical projection map and $\Delta_i \subset \mathbf{H}^3$ is a hyperbolic tetrahedron whose coordinates are in the set $\overline{\sigma_i}(K_1)$ where $\overline{\sigma_i} : K_1 \to \mathbb{C}$ is the extension of the field embedding $\sigma_i : K \to \mathbb{C}$. It will be sufficient to show this because the theorem has already been proved in the $s = 1$ case, therefore $\mathrm{Vol}(\Delta_i)$ is a rational linear combination of values $A(x)$ with x being algebraic over $\overline{\sigma_i}(K_1)$.

To show this is true, we know that $(\mathbf{H}^3)s/\Gamma$ is compact. This is true since because of Theorem 5, we have that Γ is cocompact subgroup of $SL(2,\mathbb{C})^s$, and there is a continuous map $\frac{SL(2,\mathbb{C})^s}{\Gamma} \to \frac{(\mathbf{H}^3)^s}{\Gamma}$. Because of this compactness, we can choose a compact sets $F_1, F_2 \dots F_s \subset \mathbf{H}^3$ large enough so that $F_1 \times F_2 \times \dots F_s$ contains a fundamental domain of $(\mathbf{H}^3)^s/\Gamma$ inside the interior.

Now suppose each F_j is triangulated by finitely many tetrahedra Δ_i^j such that each Δ_i^j has its vertices inside the set $\overline{\sigma_j}(K_1) \times (\overline{\sigma_j}(K_1) \cap \mathbb{R})$ and the image of each $\Delta_{i_1}^1 \times \Delta_{i_2}^2 \times \dots \Delta_{i_s}^s$ for any $(i_1, i_2, \dots i_s)$ via the map $(\mathbf{H}^3)^s \to \frac{(\mathbf{H}^3)^s}{\Gamma}$ are isomorphically embedded.[9].

[9]We are able to do this because Γ is a direct product of subgroups, each acting on a separate copy of $(\mathbf{H}^3)$ in $(\mathbf{H}^3)^s$

Therefore, once this is established, we write that $(\mathbf{H}^3)^s/\Gamma$ is covered in finitely many copies of form $\Delta_{\mathbf{i}} = \Delta^1_{i_1} \times \Delta^2_{i_2} \cdots \times \Delta^s_{i_s}$. Therefore, we have write the inclusion-exclusion principle as given below:

$$\mathrm{Vol}\left(\frac{(\mathbf{H}^3)^s}{\Gamma}\right) = \sum \mathrm{Vol}(\Delta_{\mathbf{i}}) - \sum \mathrm{Vol}(\Delta_{\mathbf{i}} \cap \Delta_{\mathbf{j}}) \dots \tag{75}$$

Because we know that $\Delta_{\mathbf{i}} \cap \Delta_{\mathbf{j}} = (\Delta_{i_1} \cap \Delta_{j_i}) \times (\Delta_{i_1} \times \Delta_{j_1}) \times \cdots \times (\Delta_{i_s} \times \Delta_{j_s})$, the volumes in the Equation (75) are of the same type as required, and hence the proof of the theorem is complete.

7.1 Example of Theorem 1

We consider the group $PSL(2, \mathbb{Z}[i])$ from the proof of Theorem 8. We will consider the application of Theorem 1 in this setting. We will evaluate the volume of the fundamental domain and justify the appearance of the Dedekind zeta in the volume, along with finding the value of $\zeta_{\mathbb{Q}(i)}(2)$.

Following the descriptions of the hyperbolic isometries and transforming the isometries of the Lorentzian model of hyperbolic geometry to elements of the group $PSL(2, \mathbb{C})$ acting on the Upper-half space gives us the following. The action of $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$ on $x = (z, t) \in \mathbf{H}^3 \simeq \mathbb{C} \times \mathbb{R}_+$ is given by [10]

$$A(x) = \begin{cases} \left(\frac{a}{c} - \frac{\overline{z + \frac{d}{c}}}{c^2(|z+\frac{d}{c}|^2 + t)}, \frac{t}{|c|^2(|z+\frac{d}{c}|^2+t^2)}\right) & c \neq 0 \\ \left(\frac{a}{d}(z + \frac{b}{a}), \frac{|a|}{|d|}t)\right) & c = 0. \end{cases} \tag{76}$$

Now we know that $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i & -1 \\ 0 & -i \end{smallmatrix}\right) \in PSL(2, \mathbb{Z}[i])$. The action of the last 2 matrices involves translation of the first coordinate by 1 and i (for the latter, we may have to combine the action of the second matrix). The first one is just sphere inversion with respect to the unit hemisphere, paired with conjugation of the first complex coordinate and the second matrix only performs negation of the complex coordinate. Therefore, we can conclude that by applications of these four matrices the orbit of any arbitrary z under $PSL(2, \mathbb{Z}[i])$ contains a point in the following set.

$$\mathcal{F} = \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \mid x^2 + y^2 + t^2 \geq 1, x \leq \frac{1}{2}, y \leq \frac{1}{2}, x + y \geq 0\}. \tag{77}$$

Now we will show that if there exist $x_1, x_2 \in \mathcal{F}$ such that there is some $A \in PSL(2, \mathbb{Z}[i])$ such that $A(x_1) = x_2$, then $x_1, x_2 \in \partial\mathcal{F}$. To show this, let $x_1 = (z_1, t_1)$ and $x_2 = (z_2, t_2)$. For this, assume that $t_2 \geq t_1$ and use Equation (76). Depending on $c = 0$ or $c \neq 0$, we get 2 conditions that we will explore.

If $c = 0$, then $t_2 = t_1 \frac{|a|}{|d|}$. Since $c = 0$, we get that $ad = 1$ and hence $a \in \{1, -1, i, -i\}$ which means that $t_2 = t_1$. In all of these cases, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is simply a translation, negation or both of the first coordinate, therefore $z_2 = \pm(z_1 + h)$ for some $h \in PSL(2, \mathbb{Z}[i])$. A brief look at Equation (77) should lead to the desired conclusion in this case.

When $c \neq 0$, $t_2 = \frac{t_1}{|c|^2(|z_1 + \frac{d}{c}|^2 + t_1^2)}$. Hence, we must have that $(|cz_1 + d|^2 + (ct_1)^2) \leq 1$. With this, we realize that $c \neq 0 \to |c| \geq 1$, and hence we must have $t_1 \leq 1$. Looking at Equation (69) convinces us that $t_1 \geq \frac{1}{\sqrt{2}}$, $|z_1| \leq \frac{1}{\sqrt{2}}$ and hence $|c| = 1$ or $\sqrt{2}$. Putting this information together leads to only finitely many possibilities of c and d. Chasing all the cases would take one to the desired conclusion.

Hence, it is accurate to say that $\mathrm{Vol}(\mathbf{H}^3/PSL(2, \mathbb{Z}[i])) = \mathrm{Vol}(\mathcal{F})$. We can accurately calculate this volume using our ideal tetrahedra formula of Proposition 12. Looking at Figure 2 should be convincing to realize that the region the volume of $\mathcal{F}$ can be written as the sum of 4 congruent ideal tetrahedra, each with the dihedral angle $\alpha = \pi/4$ and the acute angle $\gamma = \arccos(1/2) = \pi/3$. Plugging these values in Equation (71) gives us that

[10] It should be noted that the formula for the action of is not canonical and will depend on the correspondence of the Lorentzian model and the upper half-space model of hyperbolic space.

$$\mathrm{Vol}(T_{\alpha,\gamma}) = \frac{1}{4}\left[\mathcal{L}(\gamma+\alpha)+\mathcal{L}(\alpha-\gamma)+2\mathcal{L}(\frac{\pi}{2}-\alpha)\right] \tag{78}$$

$$, = \frac{1}{4}\left[\mathcal{L}\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+\mathcal{L}\left(\frac{\pi}{4}-\frac{\pi}{3}\right)+2\mathcal{L}\left(\frac{\pi}{4}\right)\right], \tag{79}$$

$$= \frac{1}{4}\left[\left(\mathcal{L}\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+\mathcal{L}\left(\frac{\pi}{4}\right)+\mathcal{L}\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\right)+\mathcal{L}\left(\frac{\pi}{4}\right)\right]. \tag{80}$$

We will leave this expression in this state so that we can bring the following lemma to our use.

Lemma 3.

$$\frac{\mathcal{L}(n\theta)}{n} = \sum_{j \ (mod\ n)} \mathcal{L}\left(\theta+\frac{\pi j}{n}\right) \tag{81}$$

Here is the sum is over all the residue classes modulo n.

Proof. By factoring the polynomial z^n-1 and substituting e^{iu}, one obtains the identity

$$2\sin(nu) = \prod_{j=0}^{n-1} 2\sin\left(u+\frac{\pi j}{n}\right). \tag{82}$$

Using the logarithm on both sides gives us the following.

$$\int_0^{n\theta} n\log|2\sin(nu)|du = n\sum_{j=0}^{n-1}\int_{\frac{\pi j}{n}}^{\frac{\pi j}{n}+\theta} \log(|2sin(u)|)du. \tag{83}$$

Using simple change of variables and Equation (64), we can come up with the following identity.

$$\frac{\mathcal{L}\left(n\theta\right)}{n} = \sum_{j=0}^{n-1}\mathcal{L}\left(\theta+\frac{\pi j}{n}\right)-\sum_{j=0}^{n-1}\mathcal{L}\left(\frac{\pi j}{n}\right). \tag{84}$$

Now note that because the Lobachevski function $\mathcal{L}$ is odd and has a period π (follows from Equation (64)), we get that the second term in Equation (84) must be 0. ☐

Using $n=3$ in Equation (80), we have a nice expression which says

$$\mathrm{Vol}(T_{\alpha,\gamma}) = \frac{1}{4}\left[\frac{1}{3}\mathcal{L}\left(\frac{3\pi}{4}\right)+\mathcal{L}\left(\frac{\pi}{4}\right)\right] = \frac{1}{6}\mathcal{L}\left(\frac{\pi}{4}\right). \tag{85}$$

This declares to us that, via a multiplication by 4

$$\mathrm{Vol}(\mathcal{F}) = \frac{2}{3}\mathcal{L}\left(\frac{\pi}{4}\right). \tag{86}$$

Now to connect the dots and put up the final picture, due to the classically known result about classification of Guassian primes, we get that for $s>1$

$$\zeta_{\mathbb{Q}[i]}(s) = \zeta_{\mathbb{Q}}(s)L(s,\chi), \tag{87}$$

where $L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$, $\chi(n)$ being the multiplicative character defined on a prime p as

$$\chi(p) = \begin{cases} 1 & p\equiv 1 \pmod 4 \\ -1 & p\equiv 3 \pmod 4 \end{cases} \tag{88}$$

Because of the uniformly convergent Fourier series expansion of $\mathcal{L}(\theta)$ given by

$$\mathcal{L}(\theta) = \frac{1}{2}\sum_{n=`}^{\infty}\frac{\sin(2n\theta)}{n^2}, \tag{89}$$

we get the pretty equation below.

$$2\mathcal{L}\left(\frac{\pi}{4}\right) = L(2,\chi). \tag{90}$$

This implies from Equation (87) that

$$\zeta_{\mathbb{Q}[i]}(2) = \frac{\pi^2}{3}\mathcal{L}\left(\frac{\pi}{4}\right). \tag{91}$$

Hence, we see the relation of $\zeta_{\mathbb{Q}[i](2)}$ and $\mathrm{Vol}(\mathbf{H}^3/PSL(2,\mathbb{Z}[i]))$ very explicitly.

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$\square$

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