CSci 5551: Introduction to Intelligent Robotic Systems
Fall 2020

Homework 2

Due: Tuesday March 17, 2020, 23:59:00 hours

Answer all questions. As before, please show details of your work, not merely the answer.

1. [20+5=25 points]:

   (a) Show the derivation of the rotation matrix R representing the *axis-angle* rotation around an arbitrary unit vector $k = [k_x k_y k_z]^T$ by an angle θ, as shown below:

   $$\begin{bmatrix} k_x^2 v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y^2 v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z^2 v\theta + c\theta \end{bmatrix} \tag{1}$$

   where $v\theta = 1 - c\theta$.

   (b) What happens when $\theta = 0$?

2. [10+10=20 points]: Suppose a frame of reference F_0 is rotated first by $90°$ around z_0, followed by a rotation of $30°$ around y_0, finally followed by a rotation of $60°$ around x_0.

   (a) Find the resulting rotation matrix R.

   (b) The same effect can be achieved by rotating F_0 around a (unit) vector k by an angle θ. From R above find this *axis-angle* representation of this rotation. That is, find k and θ

3. [20 points] Imagine rotating a vector Q about a vector $\hat{K}$ by an amount θ to form a new vector Q'; that is

   $$Q' = R_K(\theta)$$

   Use **Equation 1 above to derive Rodriguez's formula**:

   $$Q' = Q cos\theta + sin\theta(\hat{K} \times Q) + (1 - cos\theta)(\hat{K} \cdot Q)\hat{K}$$

4. [15+5=20 points]

   (a) For rotations sufficiently small that the approximations $sin\theta = \theta$, $cos\theta = 1$, $\theta^2 = 0$ hold, derive the rotation-matrix equivalent to a rotation of θ about a general axis, $\hat{K}$. Start with Equation 1 for your derivation.

   (b) Using the above result, show that two infinitesimal rotations commute (*i.e.*, the order in which the rotations are performed is not important).

5. [15 points] Show that multiplying three quaternions $q_1 = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, $q_2 = e + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$, and $q_3 = s + t\mathbf{i} + u\mathbf{j} + v\mathbf{k}$ is associative; *i.e.*, $q_1(q_2 q_3) = (q_1 q_2) q_3$.