MATH 128A Notes

$f(t_{k-1}, y_{k-1}) = f(t_k, y_k) - h\frac{d}{dt}f(t_k, y_k) + O(h^2)$

extrapolate by subbing $h \leftarrow 2h$

Rolle's Theorem $f(a) = f(b) \Rightarrow \exists c \in (a,b): f'(c) = 0$

Mean Value Theorem $\exists c \in (a,b): f'(c) = \frac{f(b)-f(a)}{b-a}$

Extreme Value Theorem $\{\arg\max_{x\in[a,b]} f(x), \arg\max_{x\in[a,b]} f(x)\} \subseteq \{a,b\} \cap \{x : f'(x) = 0\}$

Generalized Rolle's Theorem f has $n+1$ zeroes, n derivatives $\Rightarrow \exists c \in (a,b): f^{(n)}(c) = 0$.

Weighted MVT for Integrals $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$ for some $c \in (a,b)$.

Taylor's Theorem $f(x) = R_n(x) + P_n(x)$, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$, $\exists \xi \in (x_0, x): R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$

Rate of Convergence $x_n = \alpha + O(\beta_n)$ if $|x_n - \alpha| \le k|\beta_n|$ for large n, some k.

Bisection Method solves for $f(x^*) = 0$, update: $x = a + \frac{b-a}{2}$.

$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
$= \cos^2\theta - \sin^2\theta$

Fixed-Point Iteration solves for $g(p^*) = p^*$, update: $p = g(p)$.

Fixed-Point Theorem $g(x) \in [a,b]\ \forall x \in [a,b]$, and $|g'(x)| \le k < 1\ \forall x \in (a,b) \Rightarrow$ Iteration converges from all points in $[a,b]$.

Newton's Method solves for $f(x^*) = 0$, update: $p = p - \frac{f(p)}{f'(p)}$ $\Big|$ $p \in (a,b)$, $f(p) = 0$, $f'(p) \neq 0$ $\Rightarrow \exists \delta > 0 : p_0 \in [p-\delta, p+\delta]$ converges to p.

Secant Method $p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$ [Newton's method fails when $f(x) = f'(x) = 0$. Modified: $p = p - \frac{f(p)f'(p)}{f'(p)^2 - f(p)f''(p)}$]

Aitken's Δ^2 Method $\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$ $\therefore \lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1 \Rightarrow \lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$.

Order of convergence $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} < 1 \Rightarrow p_n$ converges of order α.

Steffensen's method $p = p - \frac{(g(p) - p)^2}{(g^2(p) - 2g(p) + p)}$. Gives Quadratic convergence wherever iteration gives convergence

Horner's method

To evaluate the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - x_0)Q(x) + b_0$$

and its derivative at x_0:

INPUT degree n; coefficients $a_0, a_1, \ldots, a_n$; x_0.

OUTPUT $y = P(x_0)$; $z = P'(x_0)$.

Step 1 Set $y = a_n$; *(Compute b_n for P.)*
$z = a_n$. *(Compute b_{n-1} for Q.)*

Step 2 For $j = n-1, n-2, \ldots, 1$
set $y = x_0 y + a_j$; *(Compute b_j for P.)*
$z = x_0 z + y$. *(Compute b_{j-1} for Q.)*

Step 3 Set $y = x_0 y + a_0$. *(Compute b_0 for P.)*

Step 4 OUTPUT (y, z);
STOP.

[using $y[j]$ in place of y gives coefficients of Q]

Lagrange Interpolation

$P_{x_0\ldots n}(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$;

$L_{n,k}(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$;

$f(x) = P_{x_0\ldots n}(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$

Neville's method

$P_{x_0\ldots n} = \frac{(x - x_j)P_{x_0\ldots j-1, j+1 \ldots n}(x) + (x - x_i)P_{x_0 \ldots i-1, i+1 \ldots n}(x)}{(x_i - x_j)}$

Newton's Divided Difference Formula

$P_{x_0\ldots n} = f[x_0] + \sum_{i=1}^{n} f[x_0 \ldots x_i] \prod_{j=0}^{i-1}(x - x_j)$;

$f[x_i \ldots x_{i+k-1}] = \frac{f[x_{i+1} \ldots x_{i+k}] - f[x_i \ldots x_{i+k-1}]}{x_{i+k} - x_i}$

Hermite Polynomials

$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$

$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)] L^2_{n,j}(x)$; $\hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x)$

$\Rightarrow f(x) = H_{2n+1}(x) + \frac{\prod_{i=0}^{n}(x - x_i)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$.

$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0 \ldots k](x - z_0)\cdots(x - z_k)$

$[z_{2i} = z_{2i+1} = x_i,\ f[z_{2i}, z_{2i+1}] = f'(x_i)]$

Müller's Method

To find a solution to $f(x) = 0$ given three approximations, p_0, p_1, and p_2:

INPUT p_0, p_1, p_2; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $h_1 = p_1 - p_0$;
$h_2 = p_2 - p_1$;
$\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$;
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$;
$i = 3$.

Step 2 While $i \le N_0$ do Steps 3–7.

Step 3 $b = \delta_2 + h_2 d$;
$D = (b^2 - 4f(p_2)d)^{1/2}$. *(Note: May require complex arithmetic.)*

Step 4 If $|b - D| < |b + D|$ then set $E = b + D$
else set $E = b - D$.

Step 5 Set $h = -2f(p_2)/E$;
$p = p_2 + h$.

Step 6 If $|h| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 7 Set $p_0 = p_1$; *(Prepare for next iteration.)*
$p_1 = p_2$;
$p_2 = p$;
$h_1 = p_1 - p_0$;
$h_2 = p_2 - p_1$;
$\delta_1 = (f(p_1) - f(p_0))/h_1$;
$\delta_2 = (f(p_2) - f(p_1))/h_2$;
$d = (\delta_2 - \delta_1)/(h_2 + h_1)$;
$i = i + 1$.

Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(The procedure was unsuccessful.)
STOP.

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$ and $f'(x_0), \ldots, f'(x_n)$.

OUTPUT the numbers $Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1}$ where

(Hermite) $H(x) = Q_{0,0} + Q_{1,1}(x - x_0) + Q_{2,2}(x - x_0)^2 + Q_{3,3}(x - x_0)^2(x - x_1)$
$+ Q_{4,4}(x - x_0)^2(x - x_1)^2 + \cdots$
$+ Q_{2n+1,2n+1}(x - x_0)^2(x - x_1)^2 \cdots (x - x_{n-1})^2(x - x_n)$.

Step 1 For $i = 0, 1, \ldots, n$ do Steps 2 and 3.

Step 2 Set $z_{2i} = x_i$;
$z_{2i+1} = x_i$;
$Q_{2i,0} = f(x_i)$;
$Q_{2i+1,0} = f(x_i)$;
$Q_{2i+1,1} = f'(x_i)$.

Order of method = exponent on error

Step 3 If $i \neq 0$ then set

$$Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}.$$

Step 4 For $i = 2, 3, \ldots, 2n+1$
for $j = 2, 3, \ldots, i$ set $Q_{i,j} = \frac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$.

Step 5 OUTPUT $(Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1})$;
STOP.

Cubic Spline Interpolation

- $S_j(x)$ cubic polynomial on $[x_j, x_{j+1}]$
- $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1}) = S_{j+1}(x_{j+1})$
- $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$, $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$
- natural/free: $S''(x_0) = S''(x_n) = 0$

or clamped: $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$

Three-point Formulae

$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi_0)$ for $\xi_0 \in (x_0, x_0+2h)$ (endpoint)

$= \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f^{(3)}(\xi_1)$ for $\xi_1 \in (x_0-h, x_0+h)$ (midpoint)

Five-point

$= \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] + \frac{h^4}{30}f^{(5)}(\xi_1)$ for $\xi_1 \in (x_0-2h, x_0+2h)$

$= \frac{1}{12h}[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)] + \frac{h^4}{5}f^{(5)}(\xi_0)$

$f''(x_0) = \frac{1}{h^2}[f(x_0-h) - 2f(x_0) + f(x_0+h)] - \frac{h^2}{12}f^{(4)}(\xi)$

Rounding error $|e(x_0 \pm h)| = |f(x_0 \pm h) - \tilde{f}(x_0 \pm h)| < \varepsilon$, $|f^{(3)}(x_0) - \tilde{f}^{(3)}(x_0)| < M$, then

$\left|f'(x_0) - \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0-h)}{2h}\right| \le \frac{\varepsilon}{h} + \frac{h^2}{6}M$,

minimized at $h^* = \left(\frac{3\varepsilon}{M}\right)^{1/3}$

Newton's method converges w/ $\alpha = 2$ for simple roots, $\alpha = 1$ for double roots.

Quadrature formula:

$\int_a^b f(x)dx = \sum_{i=0}^{n} f(x_i)\int_a^b L_{n,i}(x)dx + E(f)$

$E(f) = \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x - x_i) f^{(n+1)}(\xi(x))dx$

Trapezoidal formula: $h = \frac{b-a}{n}$, $x_j = a + jh$, $\mu \in (a,b)$

$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{(b-a)}{12}h^2 f''(\mu)$

Simpson's rule: [default: $n = 2$, $h = \frac{b-a}{2}$]

$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{(b-a)}{180}h^4 f^{(4)}(\mu)$

Error $\le (b-a)\varepsilon$, where $\max_j |e(f(x_j))| \le \varepsilon$

Midpoint rule: n even, $h = \frac{b-a}{n+2}$, $x_j = a + (j+1)h$

$\int_a^b f(x)dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\mu)$.

Romberg Integration

$R_{1,1} = \frac{h}{2}(f(a) + f(b))$, $R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1} - 1}(R_{k,j-1} - R_{k-1,j-1})$ [of error $O(h^{2j})$]

Step 1 Set $h = b - a$;
$R_{1,1} = \frac{h}{2}(f(a) + f(b))$.

Step 3 For $i = 2, \ldots, n$ do Steps 4–8.

Step 4 Set $R_{2,1} = \frac{1}{2}\left[R_{1,1} + h\sum_{k=1}^{2^{i-2}} f(a + (k - 0.5)h)\right]$.

(Approximation from Trapezoidal method.)

Step 5 For $j = 2, \ldots, i$
set $R_{2,j} = R_{2,j-1} + \frac{R_{2,j-1} - R_{1,j-1}}{4^{j-1} - 1}$. *(Extrapolation.)*

Step 6 OUTPUT ($R_{2,j}$ for $j = 1, 2, \ldots, i$).

Step 7 Set $h = h/2$.

Step 8 For $j = 1, 2, \ldots, i$ set $R_{1,j} = R_{2,j}$. *(Update row 1 of R.)*

$R_{n,n}$ is stored as $R_{1,n}$ at the end, and this approximates I to $O(h^{2n})$.

Degree of Precision: n if formula is accurate for $f(x) = P_i(x)\ \forall i \le n$, but inaccurate for $f(x) = P_{n+1}(x)$

Bezier Curves

To construct the cubic Bézier curves $C_0, \ldots, C_{n-1}$ in parametric form, where C_i is represented by

$$(x_i(t), y_i(t)) = (a_0^{(i)} + a_1^{(i)}t + a_2^{(i)}t^2 + a_3^{(i)}t^3, b_0^{(i)} + b_1^{(i)}t + b_2^{(i)}t^2 + b_3^{(i)}t^3),$$

for $0 \le t \le 1$, as determined by the left endpoint (x_i, y_i), left guidepoint (x_i^+, y_i^+), right endpoint (x_{i+1}, y_{i+1}), and right guidepoint (x_{i+1}^-, y_{i+1}^-) for each $i = 0, 1, \ldots, n-1$:

INPUT n; $(x_0, y_0), \ldots, (x_n, y_n)$; $(x_0^+, y_0^+), \ldots, (x_{n-1}^+, y_{n-1}^+)$; $(x_1^-, y_1^-), \ldots, (x_n^-, y_n^-)$.

OUTPUT coefficients $\{a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, b_0^{(i)}, b_1^{(i)}, b_2^{(i)}, b_3^{(i)}$, for $0 \le i \le n-1\}$.

Step 1 For each $i = 0, 1, \ldots, n-1$ do Steps 2 and 3.

Step 2 Set $a_0^{(i)} = x_i$;
$b_0^{(i)} = y_i$;
$a_1^{(i)} = 3(x_i^+ - x_i)$;
$b_1^{(i)} = 3(y_i^+ - y_i)$;
$a_2^{(i)} = 3(x_i + x_{i+1}^- - 2x_i^+)$;
$b_2^{(i)} = 3(y_i + y_{i+1}^- - 2y_i^+)$;
$a_3^{(i)} = x_{i+1} - x_i + 3x_i^+ - 3x_{i+1}^-$;
$b_3^{(i)} = y_{i+1} - y_i + 3y_i^+ - 3y_{i+1}^-$;

Step 3 OUTPUT $(a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, b_0^{(i)}, b_1^{(i)}, b_2^{(i)}, b_3^{(i)})$.

$\frac{dy}{dx} + Py = Q$, multiply both sides with $I = e^{\int P dx}$.

Integrating Factor Method $\Rightarrow y = I^{-1} \int I Q \, dx$

Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{ (t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty \}$$

and that a constant M exists with

$$|y''(t)| \le M, \quad \text{for all } t \in [a, b],$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Let $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then, for each $i = 0, 1, 2, \ldots, N$,

$$\boxed{|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].} \tag{5.10}$$

Suppose $D = \{ (t, y) \mid a \le t \le b$ and $-\infty < y < \infty \}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is well posed.

Suppose the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + h\phi(t_i, w_i, h).$$

Suppose also that a number $h_0 > 0$ exists and that $\phi(t, w, h)$ is continuous and satisfies a Lipschitz condition in the variable w with Lipschitz constant L on the set

$$D = \{ (t, w, h) \mid a \le t \le b \text{ and } -\infty < w < \infty, 0 \le h \le h_0 \}.$$

Then

(i) The method is **stable**;

(ii) The difference method is **convergent** if and only if it is **consistent**, which is equivalent to

$$\phi(t, y, 0) = f(t, y), \quad \text{for all } a \le t \le b;$$

(iii) If a function τ exists and, for each $i = 1, 2, \ldots, N$, the local truncation error $\tau_i(h)$ satisfies $|\tau_i(h)| \le \tau(h)$ whenever $0 \le h \le h_0$, then

$$|y(t_i) - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}.$$

Modified Euler Method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.

The function $f(t, y_1, \ldots, y_m)$, defined on the set

$$D = \{(t, u_1, \ldots, u_m) \mid a \le t \le b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2, \ldots, m\},$$

is said to satisfy a **Lipschitz condition** on D in the variables $u_1, u_2, \ldots, u_m$ if a constant $L > 0$ exists with

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \le L \sum_{j=1}^{m} |u_j - z_j|, \tag{5.47}$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in D.

By using the Mean Value Theorem, it can be shown that if f and its first partial derivatives are continuous on D and if

$$\left|\frac{\partial f(t, u_1, \ldots, u_m)}{\partial u_i}\right| \le L,$$

Alternative equiv Lipschitz condition

for each $i = 1, 2, \ldots, m$ and all $(t, u_1, \ldots, u_m)$ in D, then f satisfies a Lipschitz condition on D with Lipschitz constant L (see [BiR], p. 141). A basic existence and uniqueness theorem follows. Its proof can be found in [BiR], pp. 152–154.

Suppose that A is a square matrix.

(i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(ii) If A is an $n \times n$ matrix, with $n > 1$, the **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

(iii) The **cofactor** A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iv) The **determinant** of the $n \times n$ matrix A, when $n > 1$, is given either by

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \ldots, n,$$

or by

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } j = 1, 2, \ldots, n.$$

Taylor method of order n

Euler's Method is Taylor of order 1

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1, \tag{5.17}$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i).$$

Local Truncation Error

The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

has **local truncation error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i), \approx O(h^n)$$

order of method

for each $i = 0, 1, \ldots, N-1$, where y_i and y_{i+1} denote the solution of the differential equation at t_i and t_{i+1}, respectively.

Definition 5.22 Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote the (not necessarily distinct) roots of the characteristic equation

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_1\lambda - a_0 = 0$$

associated with the multistep difference method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}).$$

If $|\lambda_i| \le 1$, for each $i = 1, 2, \ldots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the **root condition**.

Definition 5.23

(i) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly stable**.

(ii) Methods that satisfy the root condition and have more than one distinct root with magnitude one are called **weakly stable**.

(iii) Methods that do not satisfy the root condition are called **unstable**.

The initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{5.2}$$

is said to be a **well-posed problem** if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, in $(0, \varepsilon_0)$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0, \tag{5.3}$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a, b].$$

LU Factorization

To factor the $n \times n$ matrix $A = [a_{ij}]$ into the product of the lower-triangular matrix $L = [l_{ij}]$ and the upper-triangular matrix $U = [u_{ij}]$, that is, $A = LU$, where the main diagonal of either L or U consists of all 1s:

INPUT dimension n; the entries a_{ij}, $1 \le i, j \le n$ of A; the diagonal $l_{11} = \cdots = l_{nn} = 1$ of L or the diagonal $u_{11} = \cdots = u_{nn} = 1$ of U.

OUTPUT the entries l_{ij}, $1 \le j \le i$, $1 \le i \le n$ of L and the entries, u_{ij}, $i \le j \le n$, $1 \le i \le n$ of U.

Step 1 Select l_{11} and u_{11} satisfying $l_{11}u_{11} = a_{11}$.
If $l_{11}u_{11} = 0$ then OUTPUT ('Factorization impossible');
STOP.

Step 2 For $j = 2, \ldots, n$ set $u_{1j} = a_{1j}/l_{11}$; (*First row of* U.)
$l_{j1} = a_{j1}/u_{11}$. (*First column of* L.)

Step 3 For $i = 2, \ldots, n-1$ do Steps 4 and 5.

Step 4 Select l_{ii} and u_{ii} satisfying $l_{ii}u_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik}u_{ki}$.
If $l_{ii}u_{ii} = 0$ then OUTPUT ('Factorization impossible');
STOP.

Step 5 For $j = i+1, \ldots, n$

set $u_{ij} = \frac{1}{l_{ii}}\left[a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}\right]$; (*ith row of* U.)

$l_{ji} = \frac{1}{u_{ii}}\left[a_{ji} - \sum_{k=1}^{i-1} l_{jk}u_{ki}\right]$. (*ith column of* L.)

Step 6 Select l_{nn} and u_{nn} satisfying $l_{nn}u_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk}u_{kn}$.
(*Note: If* $l_{nn}u_{nn} = 0$, *then* $A = LU$ *but* A *is singular.*)

Step 7 OUTPUT (l_{ij} for $j = 1, \ldots, i$ and $i = 1, \ldots, n$);
OUTPUT (u_{ij} for $j = i, \ldots, n$ and $i = 1, \ldots, n$);
STOP.

The $n \times n$ matrix A is said to be **diagonally dominant** when

$$|a_{ii}| \ge \sum_{\substack{j=1, \\ j \ne i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \ldots, n. \tag{6.10}$$

A diagonally dominant matrix is said to be **strictly diagonally dominant** when the inequality in Eq. (6.10) is strict for each n, that is, when