EECS 127 Final Notes Yash Pansari

Duality:

- $p^* = \min f_0(\vec{x})$
  $f_i(\vec{x}) \le 0 \ \forall i \in \{1,..,m\}$
  $h_j(\vec{x}) = 0 \ \forall j \in \{1,...,p\}$
  $\Rightarrow \mathcal{L}(\vec{x}, \vec{\lambda}, \vec{\nu}) = f_0(\vec{x}) + \vec{\lambda}^T \vec{f}(\vec{x}) + \vec{\nu}^T \vec{h}(\vec{x})$
  $g(\vec{\lambda}, \vec{\nu}) = \min_{\vec{x} \in \mathbb{R}^n} \mathcal{L}(\vec{x}, \vec{\lambda}, \vec{\nu})$ [g is always concave]
- $\Rightarrow p^* = \min_{\vec{x} \in \mathbb{R}^n} \max_{\vec{\lambda} \in \mathbb{R}^m, \lambda_i \ge 0, \vec{\nu} \in \mathbb{R}^l} f_0(\vec{x}) + \vec{\lambda}^T \vec{f}(\vec{x}) + \vec{\nu}^T \vec{h}(\vec{x})$
  $\Rightarrow d^* = \max_{\vec{\lambda} \in \mathbb{R}^m, \lambda_i \ge 0, \vec{\nu} \in \mathbb{R}^l} \min_{\vec{x} \in \mathbb{R}^n} f(\vec{x}) + \vec{\lambda}^T \vec{f}(\vec{x}) + \vec{\nu}^T \vec{h}(\vec{x})$
  [dual problem is always convex]
- $p^* \ge d^*$ always. $p^* - d^* \ge 0$ is the duality gap.
  [weak duality]
- $p^* = d^*$ is called strong duality.
- Slater's condition: strong duality holds $\forall$ convex problems where there exists a strictly feasible point i.e. $\vec{x} \in \mathbb{R}^n$: $f_i(\vec{x}) < 0 \ \forall i$ where f_i is not affine.
- KKT conditions: Primal feasibility: $f_i(\vec{x}) \le 0$, $h_j(\vec{x}) = 0 \ \forall i, j$
  Dual feasibility: $\lambda_i \ge 0 \ \forall i$.
  Complementary Slackness: $\lambda_i f_i(\vec{x}) = 0 \ \forall i$
  Stationarity: $\nabla_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}, \vec{\nu}) = 0$.
- If strong duality holds, KKT are necessary for optimization.
  If convexity holds, KKT are sufficient for optimization.

Linear Programs:

- Standard form LP: $\boxed{p^*(\vec{c}, \vec{y}, A) = \min_{\vec{x} \ge 0 \,:\, A\vec{x} = \vec{y}} \vec{c}^T \vec{x}}$
- $d^* = \max_{\vec{\lambda} \ge 0, \vec{\nu} \in \mathbb{R}^m : \vec{c} - \vec{\lambda} + A^T \vec{\nu} = \vec{0}} -\vec{y}^T \vec{\nu}$
- Any LP is equivalent to some standard form LP.
- Any LP is convex.
- if p^* is finite and Ω has extreme points ($\vec{x} \in \Omega : \nexists \vec{y}, \vec{z} : \vec{x} = \theta \vec{y} + (1-\theta)\vec{z}$), $p^* = \vec{c}^T \vec{v}^*$ for some extreme point $\vec{v}^*$.

Quadratic Programs:

- Standard form QP: $\boxed{p^* = \min_{\vec{x} \in \mathbb{R}^n : A\vec{x} \le \vec{y}, C\vec{x} = \vec{z}} \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T \vec{x}}$ where $H \in S_n$. [P convex iff $H \succeq 0$].

Quadratic-Constrained Quadratic Programs:

- Standard form QCQP: $\boxed{p^* = \min_{\vec{x} \in \mathbb{R}^n} \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T \vec{x}}$
  $\frac{1}{2}\vec{x}^T P_i \vec{x} + \vec{b}_i^T \vec{x} + c_i \le 0 \ \forall i \in \{1...m\}$
  $\frac{1}{2}\vec{x}^T Q_i \vec{x} + \vec{d}_i^T \vec{x} + f_i = 0 \ \forall i \in \{1...p\}$.
  where $H, P_i, Q_i \in S_n$ [P convex iff $H, P_i \succeq 0$, $Q_i = 0 \ \forall i$]
  [convex problem cannot have non-linear equality constraints]

Second-Order Cone Programs:

- For any set $C \subseteq \mathbb{R}^n$: $\forall \vec{x} \in C, \alpha \ge 0 : \alpha\vec{x} \in C$, then C is a cone
  $\forall \vec{x}, \vec{y} \in C, \alpha, \beta \ge 0 : \alpha\vec{x} + \beta\vec{y} \in C$, then C is a convex cone.
- Second order cone: $\{(\vec{x}, t) \in \mathbb{R}^{n+1} : \|\vec{x}\|_2 \le t\}$
- Standard form SOCP: $\boxed{p^* = \min_{\vec{x} \in \mathbb{R}^n : \|A_i\vec{x} - \vec{y}_i\|_2 \le \vec{b}_i^T \vec{x} + z_i} \vec{c}^T \vec{x}}$ $\forall i \in \{1,..,m\}$ [convex problem]
  affine objective
  affine function in sec. ord. cone
- LPs are QPs, QPs are QCQPs, convex QCQPs are SOCPs: standard QCQP [convex $\Rightarrow Q_i = 0 \ \forall i$] $\xrightarrow{\text{SOCP}}$ $p^* = \min_{\vec{x} \in \mathbb{R}^n, t \in \mathbb{R}} t + \vec{c}^T \vec{x}$:
  $\left\| \begin{bmatrix} P_i^{1/2} \vec{x} \\ \frac{1}{2} + \vec{b}_i^T \vec{x} + c_i \end{bmatrix} \right\|_2 \le \frac{1}{2} - \vec{b}_i^T \vec{x} + c_i \ \forall i$
  $\left\| \begin{bmatrix} P_i^{1/2} \vec{x} \\ \frac{1}{2} - t \end{bmatrix} \right\|_2 \le \frac{1}{2} - t \ \forall i$
  $\|C\vec{x} - \vec{z}\|_2 \le 0$.

Regularization:

- $R : \Omega \to \mathbb{R}_+$ is a regularizer such that $p^* = \min_{\vec{x} \in \Omega} f_0(\vec{x})$
  becomes $p^*_\lambda = \min_{\vec{x} \in \Omega} \{f_0(\vec{x}) + \lambda R(\vec{x})\}$
- LASSO: $\min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{y}\|_2^2 + \lambda \|\vec{x}\|_1$
  - convex
  - if $rk(A) = n$, then μ-strongly convex, where $\mu = 2\sigma_n\{A\}^2$.
  - solution unique if $rk(A) = n$.
- $C(t) = \arg\min_{\vec{x} \in \mathbb{R}^n, R(\vec{x}) \le t} f_0(\vec{x})$, $R(\lambda) = \arg\min_{\vec{x} \in \mathbb{R}^n} f_0(\vec{x}) + \lambda R(\vec{x})$.
  $\forall t \ \exists \lambda : C(t) = R(\lambda)$, $\forall \lambda \ \exists t : C(t) = R(\lambda)$.

$\langle A, B \rangle = tr(A^T B)$.

EECS 126 Final Notes Yash Penseri

- $\beta(i) = \frac{1}{q_i} + \sum_j \frac{q_{ij}}{q_i}\beta(j)$
- $\pi(x) = \frac{1}{\mathbb{E}[T_x^+]}$
- $\boxed{X - \mathbb{E}[X|Y] \text{ is indep of } Y.}$
- long term time average fraction of transitions into state i for CTMC is $\psi(i)$, where ψ is stat. dist. for embedded chain.
- $\psi_i := \frac{\pi_i q_i}{\sum_j \pi_j q_j}$, where π is CTMC stat. dist, ψ is jump chain stat. dist.
- Let $A + B = C$. Then, $\mathbb{E}[A|C] = \frac{\sigma_A^2}{\sigma_A^2+\sigma_B^2}C$. $[A \sim N(0,\sigma_A^2),\ B \sim N(0,\sigma_B^2)]$

Let $X_1 \dots X_n$ iid
- $\min(X_1 \dots X_n) = U \Rightarrow F_U(u) = 1 - (1 - F_{X_i}(u))^n$.
- $\max(X_1 \dots X_n) = V \Rightarrow F_V(v) = F_{X_i}(v)^n$.
- $\mathbb{E}[\min(X_1 \dots X_n)$ where X_i iid $U(a,b)] = \frac{na+b}{n+1}$.
- $H(U(a,b)) = \log(b-a)$.
- $Var(X) = \mathbb{E}[Var(X|Y)] + Var(\mathbb{E}[X|Y])$.

Hypothesis Testing:

- Type I error / significance level / Prob. of False Alarm: $\alpha(A) = \mathbb{P}_{H_0}(x \notin A)$ — acceptance region.
  Type II error: $\beta(A) = \mathbb{P}_{H_1}(x \in A)$.
  PCD: $\mathbb{P}_{H_1}(x \notin A)$.
- Likelihood Ratio $L(x) = \frac{\mathbb{P}_{H_1}(x)}{\mathbb{P}_{H_0}(x)}$.
- Neyman-Pearson / Likelihood Ratio test: Accept if $L(x) < c$ w/ prob 1 or $L(x) = c$ w/ prob γ
  - for normal distribution, condition can be $y < c$.
  - most powerful test. if some test (t) gives $\alpha_t < \alpha_{NP}$, then $\beta_t > \beta_{NP}$.

Statistical Inference:

- $MAP(X|Y=y) = \arg\max_x \mathbb{P}(X=x|Y=y) = \arg\max_x \mathbb{P}[X=x]\,\mathbb{P}[Y=y|X=x]$
- $MLE(X|Y=y) = \arg\max_x \mathbb{P}(Y=y|X=x)$. $\therefore$ MLE = MAP for X uniform.
- $MSE = \mathbb{E}[(X-\phi(Y))^2] \Rightarrow MMSE = \arg\min_{\phi(y)} \mathbb{E}[(X-\phi(Y))^2] = \mathbb{E}[X|Y]$
- LLSE: $\boxed{\mathbb{L}[X|Y] = \mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}(Y - \mathbb{E}[Y])}$ $\boxed{\mathbb{L}[X|Y,Z] = \mathbb{L}[X|Y] + \mathbb{L}[X|Z - \mathbb{L}[Z|Y]]}$ for X, Y, Z zero mean

Jointly Gaussian RV:

- $(X_1, \dots X_n)$ are jointly Gaussian iff $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = A\vec{Z} + \vec{\mu}$ for some $A \in \mathbb{R}^{n\times l}$, $\vec{\mu} \in \mathbb{R}^n$, $\vec{Z} = \begin{bmatrix} N(0,1) \\ \vdots \\ N(0,1) \end{bmatrix}$
- Any linear combination, $\vec{u}^T\vec{X}$, is normally distributed.
- $f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$, where $\vec{\mu}$ is mean vector, Σ is cov. matrix.
  $\Sigma = \mathbb{E}[(\vec{x}-\vec{\mu})(\vec{x}-\vec{\mu})^T] = AA^T \succeq 0$. [$A, \vec{\mu}$ are same as from $\vec{X} = A\vec{Z} + \vec{\mu}$]
- JG RVs are independent iff. uncorrelated (Σ is diagonal).
- If X, Y are JG, then $\mathbb{L}[X|Y] = \mathbb{E}[X|Y] \Rightarrow LLSE = MMSE$

Kalman filter

- $x_n = a x_{n-1} + v_n$
  $y_n = x_n + w_n$

  $\sigma_{n|n}^2 = \mathbb{E}[(x_n - \hat{x}_{n|n})^2]$
  $\hat{x}_{n|m} = \mathbb{L}[x_n | y_1 \dots m]$
  $v_n = N(0, \sigma_v^2)$
  $w_n = N(0, \sigma_w^2)$

  $\Rightarrow \hat{x}_{n|n} = \hat{x}_{n|n-1} + k_n \tilde{y}_n = a\hat{x}_{n-1|n-1} + k_n(y_n - a\hat{x}_{n-1|n-1})$
  innovation
  $k_n = \frac{\sigma_{n|n-1}^2}{\sigma_{n|n-1}^2 + \sigma_w^2}$, $\sigma_{n|n-1}^2 = a^2\sigma_{n-1|n-1}^2 + \sigma_v^2$, $\sigma_{n|n}^2 = \sigma_{n|n-1}^2(1-k_n)$.

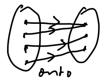
$X_n = AX_{n-1} + V_n$
$Y_n = CX_n + W_n$
$\Sigma_{n|n-1} = A\Sigma_{n-1|n-1}A^T + \Sigma_V$
$K_n = \Sigma_{n|n-1}C^T[C\Sigma_{n|n-1}C^T + \Sigma_W]^{-1}$
$\Sigma_{n|n} = (I - K_nC)\Sigma_{n|n-1}$
$\hat{X}_{n|n-1} = A\hat{X}_{n-1|n-1}$
$\tilde{Y}_n = Y_n - C\hat{X}_{n|n-1}$
$X_{n|n} = \hat{X}_{n|n-1} + K_n\tilde{Y}_n$.

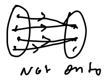
Random Vectors

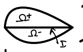
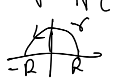
- $Var(AX) = A\,Var(X)A^T$
- $Cov(AX, BY) = A\,Cov(X,Y)B^T$
- $Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])^T] = \mathbb{E}[XY^T] - \mathbb{E}[X]\mathbb{E}[Y]^T$
- $tr(\mathbb{E}[\vec{a}\vec{b}^T]) = \mathbb{E}[\vec{a}^T\vec{b}]$

Hilbert Space:
- $\langle X, Y\rangle = \mathbb{E}[XY]$
  $\|X\|_2 = \sqrt{\mathbb{E}[X^2]}$

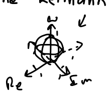
Math 185 Final Notes Yash Panseri



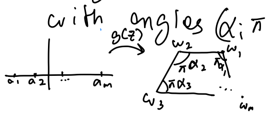


- $K \subset \mathbb{R}^n$ is compact iff:
  - closed and bounded
  - has an open cover w/ finite subcover
  - has any sequence of points with a convergent subsequence.
- continuous functions on compact sets are bounded.
- f holomorphic at point z_0 iff:
  - $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and converges.
  - $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial f}{\partial \bar{z}}\Big|_{z=z_0} = 0$ [Cauchy-Reimann eqns]
- Entire: hol'c on all of $\mathbb{C}$.
- if γ, γ^- are opposite orientation, $\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz$.
- Radius of convergence of $\sum_{n=0}^{\infty} |a_n| z^n$ is $R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$ [Hadamard's formula]
- within open disc of convergence, power series define hol'c functions. f, f', $f^{(n)}$ all have same radius of convergence.
- F is a primitive of f if $F'(z) = f(z)$ $\forall z \in \Omega$. $\Rightarrow \int_{w_1}^{w_2} f(z)\,dz = F(w_2) - F(w_1)$.
- if f is hol'c, and γ is a closed curve (in a simply connected open set), $\int_\gamma f(z)\,dz = 0$. [Cauchy's Theorem]
- if f is hol'c on Ω, then $\forall z \in \Omega$ $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{\omega - z}\,d\omega$ for some closed curve $C \in \Omega$. $\boxed{\text{Cauchy Integral formula}}$
- Let C be a circle in Ω. Then, $\forall z \in \text{int}(C)$: $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw$ [corollary of Cauchy formula].
- Let D be a disc in Ω with center z_0, radius R. Then, $|f^{(n)}(z_0)| \leq \frac{n!\,\|f\|_C}{R^n} = \frac{n!}{R^n}\sup_{z\in C}|f(z)|$ [Cauchy's inequality]
  [If hol'c on Ω, C is the boundary circle of D]
- Same disc D ⇑: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n = \frac{f^{(n)}(z_0)}{n!}$ $\forall n \geq 0$, $z \in D$ [Taylor Series].
- if f is entire & bounded, f is constant [Liouville's Theorem].
- Every polynomial of deg. $n \geq 1$ has precisely n roots in $\mathbb{C}$.
- Let (z_n) be a sequence of zeroes of hol'c f. Then, if (z_n) has a limit in Ω, then $f(z) = 0$ $\forall z \in \Omega$.
- if f is continuous in D, and for any triangle $T \subset D$, $\int_T f(z)\,dz = 0$. Then, f is holomorphic [Morera's Theorem].
- if (f_n) is a sequence of hol'c functions that converges uniformly to f in every compact subset of Ω, then f is hol'c in Ω.
  [(f_n') then also converge uniformly to f' on every compact subset of Ω].
- Let $\Omega = \Omega^+ \cup I \cup \Omega^-$ [eg. ]. Then, if f^+ hol'c on Ω^+, f^- on Ω^-, and $f^+ = f^-$ on I, then $f(z) = \begin{cases} f^+ & \text{on } \Omega^+ \\ f^+ = f^- & \text{on } I \\ f^- & \text{on } \Omega^- \end{cases}$ is hol'c on Ω. [Symmetry Principle]
- If f hol'c on Ω^+, extends to I, and is real valued on I, then there is an F hol'c on all of Ω such that $F = f$ on Ω^+. [Schwarz reflection Principle]
- If an integral is finite, you can switch up the order of integration.
- If f has pole of order n at z_0, $f = \sum_{i=1}^{n} \frac{a_i}{(z-z_0)^i} + G(z)$.
- $\boxed{\text{res}_{z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z).}$
  - z_0 is a removable singularity if f hol'c on $\Omega / \{z_0\}$.
  - f is meromorphic on Ω if (z_n) are poles with no limit point in Ω and f hol'c on $\Omega / \{z_1 \ldots z_n\}$.
- if $z_1 \ldots z_n$ are poles of f, on $\text{int}(\gamma)$, $\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} \text{res}_{z_i} f$ [Residue formula]
- $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$ = no. of zeroes − no. of poles [in $\text{int}(\gamma)$] [Argument Principle]
- if f, g hol'c in Ω, and $|f(z)| < |g(z)|$ $\forall z \in C$ for some circle $C \subset \Omega$, then f and $f+g$ have same number of zeroes in $\text{int}(C)$. [Rouche's Theorem]
- An open mapping maps open sets to open sets. f hol'c, non-constant in Ω $\Rightarrow$ f is an open mapping. [Open Mapping Theorem]
- If f is hol'c, non-constant in some open set Ω, it cannot attain a maximum in Ω. [maximum modulus principle]
- If f is hol'c in $D_R(z_0)$, then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$ $\forall r < R$.
  [Mean Value Theorem]
- $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\,dx$, $\xi \in \mathbb{R}$ [Fourier transform]
- $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\,d\xi$, $x \in \mathbb{R}$ [Fourier inversion].
- $|f(z)| \leq \frac{A}{1+\text{Re}(z)^2}$ is called the condition of moderate decrease.
- if $f(z)$ hol'c on $S_a = |\text{Im}(z)| < a$ and has moderate decrease in S_a, then $f \in \mathfrak{F}_a \Rightarrow |\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|}$ $\forall\, 0 \leq b < a$. for some B.
- if $f \in \mathfrak{F}$, then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$
- if $|\hat{f}(\xi)| < Ae^{-2\pi a|\xi|}$, then $f(z)$ hol'c for $|\text{Im}(z)| < a$.
- $A > 0$, $B \in \mathbb{R} \Rightarrow \int_0^{\infty} e^{-(A+iB)\xi}\,d\xi = \frac{1}{A+iB}$.
-  $\left|\int_\gamma f(z) e^{-2\pi i z\xi}\,dz\right| \leq \int_\gamma^R |f(z)| e^{2\pi R\sin\theta\,\xi}\,d\theta \xrightarrow{R\to\infty} 0$ for $\xi \leq 0$ [for $\xi > 0$, use contour below $\mathbb{R}$ line]
- if f is continuous and of moderate decrease, then it has an entire continuation to $\mathbb{C}$ with $|f(z)| \leq Ae^{2\pi M|z|}$ for some $A > 0$ iff $\hat{f}(\xi)$ vanishes for $|\xi| > M$ [$\xi \in \mathbb{R}$].
  [Paley-Weiner]

- Jensen's formula: $\sum_{n=1}^{N} \log\left|\frac{a_n}{R}\right| = \frac{-1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta + \log|f(0)|$ where $\{a_1, \dots a_N\}$ are zeros of f in D_R.
- $\int_0^R \frac{\eta_f(r)}{r}\,dr = \sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right|$, where $\eta_f(r)$ is no. of zeros of f in D_r (with mult.).
- For entire f, order of growth, $\rho_f = \inf\{\rho : \exists A, B > 0 : |f(z)| \le Ae^{B|z|^\rho}\}$
- $\eta_f(r) \le Cr^{\rho_f}$ for some $C > 0$, sufficiently large r. [$\eta_f(r) \le C(1+r^{\rho_f})$ $\forall r$]
- If $z_1, \dots z_N$ are zeros of f, $\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty$ for all $s > \rho_f$. [to check if $\rho < s$]
- If $\sum |a_n| < \infty$, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges.
- $\prod_{n=1}^{\infty} F_n(z) \to F(z)$ on Ω if $\sum |F_n(z) - 1| < \infty$ $\forall z \in \Omega$.
  $F(z)$ is hol'c at z if $F_n(z) \neq 0$ for any n, $\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$.
- Given a sequence (z_n) s.t. $\lim_{n\to\infty} |z_n| \to \infty$, there exists an entire function f with zeros only at (z_n).
  Every other function of this form is $f(z)e^{g(z)}$ for hol'c g.
  $\Rightarrow f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$, where m is multiplicity of zero at origin,
  $E_k(z) = (1-z)e^{z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \frac{z^k}{k}}$ [Weierstrass factorization]
- $|a_n| \geqslant 2|z| \Rightarrow \left|1 - E_k\left(\frac{z}{a_n}\right)\right| \leqslant \frac{c|z|^{k+1}}{|a_n|}$ for some $c > 0$. — $|a_n| \leqslant 2|z| \Rightarrow \left|E_k\left(\frac{z}{a_n}\right)\right| \geqslant |1-z|\,e^{-c|z|^k}$ for some $c > 0$.
- Let $k = \lfloor \rho_0 \rfloor$. Then $f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$, where $P(z)$ is a polynomial with degree $\le k$. [Hadamard Factorization]
- $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt$ for $Re(z) > 0$. [Gamma function] — $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ $\forall s \in \mathbb{C}$.
- $\Gamma(z+1) = z\Gamma(z)$ [for $Re(z) > 0$]. $\Rightarrow \Gamma(n+1) = n!$ for $n \in \mathbb{Z}^+$.
- Γ has poles at $z \in \mathbb{Z}^- \cup \{0\}$. with $res_{-n}\,\Gamma = \frac{(-1)^n}{n!}$ — $\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty}\left[\left(1 + \frac{s}{n}\right)e^{-s/n}\right]$
- $\left|\frac{1}{\Gamma(s)}\right| \leqslant c_1 e^{c_2|s|\log|s|} \Rightarrow \left|\frac{1}{\Gamma(s)}\right| \leqslant c(\varepsilon)\,e^{c_2|s|^{1+\varepsilon}}$ $\forall \varepsilon > 0$. $\Rightarrow$ $\frac{1}{\Gamma(s)}$ has virtually order 1.
- If f hol'c for $|z| < 1$, $|f(z)| \le 1$ $\forall |z| < 1$ and $f(0) = 0$, then $|f(z)| < |z|$ $\forall |z| < 1$. [Schwarz Lemma]
  - Also, $|f'(0)| \le 1$ with $|f'(0)| = 1$ iff $f(z) = \lambda z$ with $|\lambda| = 1$.
  - Further, if $|f(z)| = |z|$ for any $z \neq 0$ [$|z| < 1$], then $f(z) = \lambda z$ with $|\lambda| = 1$.
- Conformal self Map: One-to-one, onto, hol'c $f : \mathbb{D} \to \mathbb{D}$. [CSM]
- If $g(z)$ is a CSM such that $g(0) = 0$, then $g(z) = e^{i\varphi} z$ for $\varphi \in [0, 2\pi]$ [Rotation]
- CSM are of the form $f(z) = e^{i\varphi}\frac{z - a}{1 - \bar{a}z}$ for $|a| < 1$, $\varphi \in [0, 2\pi]$.
- If $f(z)$ analytic and $|f(z)| < 1$ for $|z| < 1$, then $|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2}$ $\forall |z| < 1$. [Pick's Lemma]
- If f is a CSM, then $|f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2}$
- Hyperbolic length of $\gamma = 2\int_\gamma \frac{|dz|}{1 - |z|^2}$; $\rho(z_0, z_1) = \inf\{\text{length}^{Hyp} \text{ of } \gamma \text{ for } \gamma \text{ piecewise smooth curve from } z_0 \text{ to } z_1\}$ [Hyperbolic dist.]
- Hyperbolic geodesics: Hyp. dist. minimizing curves. These are arcs of circles orthogonal to unit circle.
- Every analytic $f : \mathbb{D} \to \mathbb{D}$ has $\rho(f(z_0), f(z_1)) \le \rho(z_0, z_1)$ $\forall z_0, z_1 \in \mathbb{D}$.
  $\rho(f(z_0), f(z_1)) = \rho(z_0, z_1)$ iff f is CSM
- If D is a simply connected open strict subset of $\mathbb{C}$, then $\exists f : D \to \mathbb{D}$ s.t. f is conformal (One-to-one, onto, hol'c) [Reimann Mapping Theorem]
- Two domains are conformally equivalent if there is a conformal map between them.
- Every domain D has a unique map (up to a CSM) to $\mathbb{D}$, known as the Reimann Map of D.
- A simply connected domain in the Reimann sphere is either the whole sphere or it is conformally equiv. to either $\mathbb{C}$ or $\mathbb{D}$.



- Let $D \subset \mathbb{C}$ be simply connected with corner at $w_0 \in \partial D$ with interior angle $\alpha\pi$ [$\alpha \in (0,2]$]. Let $g : \mathbb{H} \to D$ be conformal. Then, there are two contiguous intervals [meeting at a_0] mapped analytically to the two sides of the corner, with $g(a_0) = w_0$. If $|a_0| < \infty$, $\frac{g''(z)}{g'(z)}$ has a simple pole at a_0 with residue $\alpha - 1$. If $|a_0| = \infty$, $\frac{g''(z)}{g'(z)}$ is analytic at a_0 and vanishes there.
- Let $g : \mathbb{H} \to D$ be conformal, where D is a bounded polygon. Then, let D have vertices (w_i) [mapped from (a_i) in $\mathbb{H}$] with angles $(\alpha_i\pi)$. Then, $\frac{g''(z)}{g'(z)} = \frac{\alpha_1 - 1}{z - a_1} + \dots + \frac{\alpha_m - 1}{z - a_m}$. And $\exists A, B$ [constants]: $g'(z) = A\prod_{i=1}^{m}(z - a_i)^{\alpha_i - 1}$ → let $a_1 < \dots < a_m$ for simplicity.



[Schwarz-Christoffel Formula] $g(z) = A\int_{z_0}^{z}\prod_{i=1}^{m}(t - a_i)^{\alpha_i - 1}\,dt + B$.
- f analytic for $r_1 < |z - z_0| < r_2$.
  $\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ where $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w - z_0)^{n+1}}\,dw = \frac{f^{(n)}(z_0)}{n!}$, and γ: any circle centered at z_0 with $r_1 < |w - z_0| < r_2$.
  [Laurent Expansion]
- $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$, $\ln(1+x) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}(-1)^n$, $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$

- fundamental parallelogram: $P_0 = \{z = a + b\tau : a, b \in [0,1)\}$ [$\tau = \frac{\omega_1}{\omega_2}$: $\mathrm{Im}(\tau) > 0$, ω_1 and ω_2 are periods of an $\boxed{\text{elliptical}}$ function] [$f(z+\omega_1) = f(z+\omega_2) = f(z)$]

Period parallelogram: $P_0 + h$ for any $h \in \mathbb{C}$.

[$\boxed{\text{Liouville}}$] An entire doubly periodic function is constant.

[$\boxed{\text{Elliptical}}$] No. of poles in P_0 is ≥ 2.
No. of zeros in P_0 = # poles in P_0.

- $\Lambda = \{n + m\tau : n, m \in \mathbb{Z}\}$ [Lattice] [disjoint covering of $\mathbb{C}$]

- $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda^* \\ [=\Lambda/\{0\}]}} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$ [$\boxed{\text{Weierstrass } \wp \text{ function}}$] [has periods $1, \tau$]

meromorphic w/ double poles at lattice points.

- $(\wp')^2 = 4(\wp - \frac{1}{2})(\wp - \frac{\tau}{2})(\wp - \frac{1+\tau}{2})$
- Every elliptical f w/ periods $1, \tau$ is a rational function of $\wp, \wp'$.

- $\boxed{\text{Eisenstein series of order } k}$ $E_k(\tau) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n+m\tau)^k}$.
  - converges if $k \geq 3$ to a func hol'c in $\mathbb{H}$.
  - $E_k(\tau) = 0$ if k is odd.
  - $E_k(\tau + 1) = E_k(\tau)$ - $E_k(\tau) = \tau^{-k} E_k(\frac{-1}{\tau})$.

- For z near 0, $\wp(z;\tau) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2}(\tau) \cdot z^{2k}$.

$\wp'(z;\tau) = 4\wp(z)^3 - 60 E_4(\tau)\wp(z) - 140 E_6(\tau)$

- $\zeta(k) = \sum_{n \in \mathbb{Z}/\{0\}} \frac{1}{n^k}$. [$\boxed{\text{Zeta function}}$]

- $E_k(\tau) = 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i \tau r}$, where $\sigma_k(r) = \sum_{d|r} d^k$ [d are divisors of r]

- $F(\tau) = \sum_m \sum_n \frac{1}{(n+m\tau)^2} = 2\zeta(2) - 8\pi^2 \sum_{r=1}^{\infty} \sigma_1(r) e^{2\pi i \tau r}$.
[$n, m \in \mathbb{Z} \times \mathbb{Z}/(0,0)$]

[$\boxed{\text{Forbidden Eisenstein series}}$]

- $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ [Laplacian] $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

- Harmonic iff $\Delta = 0$. $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

- Rational function: $f(z) = \frac{P(z)}{Q(z)}$ → polynomials

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