Math 104 mt 2 Theorems and definitions

[All S are the set in the metric space (S, d) for these theorems]

same goes for compact sets

D13.1 ~~xxx~~ d is a metric (distance function on S) iff

D1. $d(x, x) = 0 \; \forall x \in S$ and $d(x, y) > 0 \; \forall x, y \in S$ where $x \neq y$.

D2. $d(x, y) = d(y, x) \; \forall x, y \in S$

D3. $d(x, z) \leq d(x, y) + d(y, z) \; \forall x, y, z \in S$.

[cont. func. go from connected sets to connected sets]

(S, d) is a metric space.

D13.2 (i) (s_n) converges to s in S if $\lim_{n \to \infty} d(s_n, s) = 0$.

(ii) (s_n) is cauchy if $\forall \varepsilon > 0, \exists N : (m, n > N \Rightarrow d(s_m, s_n) < \varepsilon)$

(iii) (S, d) is complete if every cauchy (s_n) in S converges to an s in S.

T13.3 (i) $(\vec{x}_n)$ converges in $\mathbb{R}^k$ iff (x_{jn}) converges in $\mathbb{R}$ $\forall j \in \{1, 2 \ldots k\}$

(ii) $(\vec{x}_n)$ is cauchy in $\mathbb{R}^k$ iff (x_{jn}) is cauchy in $\mathbb{R}$ $\forall j \in \{1, 2, \ldots k\}$.

T13.4 Euclidean $\mathbb{R}^k$ is complete. $[(\mathbb{R}^k, d_E)]$

T13.5 [Bolzano-Weierstrauss] Every bounded sequence in $\mathbb{R}^k$ has a conv. subseq

T13.6 $s_0 \in E \subset S$ is an interior point of E if for some $r > 0$,

$\{s \in S : d(s, s_0) < r\} \subseteq E$.

$\boxed{E^\circ}$ is the set of interior points of E. E is open iff $E = E^\circ$.

T13.7 (i) S is open in S.

(ii) $\emptyset$ is open in S.

(iii) The union of any collection of open sets is open.

(iv) The intersection of finitely many open sets is open.

D13.8 (i) $S / E = \boxed{E^c}$. E is closed iff E^c is open.

(ii) $\boxed{E^-}$ is the closure of E. It ~~xxxx~~ is the intersection of all closed sets in S containing E.

(iii) $\boxed{E^- / E^\circ}$ is the boundary of E.

(iv) intersection of any collect of closed sets is closed.

T13.9 a) E is closed iff $E = E^-$.

b) E is closed iff all (s_n) in E that converge have a limit in E.

c) E^- is the set of all limits of all convergent sequences in E.

d) $E^- / E^\circ = E^- \cap E^{c-}$

T13.10 Let (F_n) be a sequence such that $[F_1 \supseteq F_2 \supseteq \ldots]$ and F_n is closed and bounded and nonempty in $\mathbb{R}^k$ for all n. Then, $F = \bigcap_{n=1}^{\infty} F_n$ is also closed, bounded and nonempty.

D13.11 A family $\mathcal{U}$ of open sets is an open cover of E iff $E \subseteq \bigcup \{U : U \in \mathcal{U}\}$

A subcover is any subfamily of $\mathcal{U}$ that still covers E.

E is compact if every open cover of E has a finite subcover of E.

T13.12 [Heine-Borel] If $E \subseteq \mathbb{R}^k$, it is compact iff it is closed and bounded.

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D17.1 f is continuous at x_0 if for every sequence $(x_n) \subseteq \mathrm{dom}(f)$, $\lim_n (x_n) = x_0$ implies $\lim_n f(x_n) = f(x_0)$.

T17.2 f is continuous at x_0 iff $\forall \varepsilon > 0, \exists \delta > 0 : (x \in \mathrm{dom}(f) \wedge |x - x_0| < \delta) \Rightarrow |f(x) - f(x_0)| < \varepsilon$

T18.1 If f is continuous & defined on a closed interval $[a, b]$, then f is bounded and assumes its maximum and minimum values on $[a, b]$. $[\exists x_0, y_0 \in [a,b] : f(x_0) \leq f(x) \leq f(y_0) \ \forall x \in [a, b]]$

T18.2 Intermediate Value Theorem.
If f is continuous and real valued on interval I, and $a, b \in I$ such that $a < b$ and $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then $\exists x \in (a, b) : f(x) = y$

T18.3 If f is cont. and real val. on I, then $f(I)$ is an interval or single point.

T18.4 If f is cont and real val, $I = dom(f)$ (interval). Then $f(I) = J$ and $dom(f^{-1}) = J$. If f is strictly increasing, on I, and continuous, then f^{-1} is str. inc. and cont. on J.

T18.5 If g is strictly increasing on interval J and $g(J) = I$ is an interval, g is cont. on J.

T18.6 All one-to-one cont. functions on intervals are str. inc. or str. dec.

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D28.1 f is differentiable at a if $[f'(a) =] \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite.

T28.2 f is diff. at $a \Rightarrow f$ is cont. at a.

[All connected subsets of $\mathbb{R}$ are intervals or single points]

T28.3 if f, g diff at a, then:
(i) $(cf)'(a) = c \cdot f'(a)$ (ii) $(f+g)'(a) = f'(a) + g(a)$
(iii) $(fg)'(a) = f(a) g'(a) + f'(a) g(a)$ [prod. rule]
(iv) $(f/g)'(a) = \frac{g(a) \cdot f'(a) - f(a) g'(a)}{g(a) \cdot g(a)}$ if $g(a) \neq 0$.

T28.4 if f is diff at a and g is diff at $f(a)$ then $(g \circ f)'(a) = g'(f(a)) f'(a)$
[chain rule]

T29.1 If f is defined on I where $x_0 \in I$ and f assumes its max or min on x_0, then $f'(x_0) = 0$ if f is diff. at x_0.

T29.2 [Rolle's Theorem] If is cont on $[a, b]$, diff. on (a, b), and $f(a) = f(b)$, then $\exists x \in (a, b) : f'(x) = 0$.

T29.3 [Mean Val. The.] If f is cont. on $[a, b]$, diff. on (a, b), then $\exists x \in (a, b) : f'(x) = \frac{f(b) - f(a)}{b - a}$.

T29.4 If $\forall x \in (a, b)$, $f'(x) = 0$, f is a constant function.

T29.5 if $f' = g'$ on (a, b), then $f(x) = g(x) + C \ \forall x \in (a, b)$.

T29.7(i) f is strictly inc. on (a, b) iff $f'(x) > 0 \ \forall x \in (a, b)$
(ii) " " dec " " " $f'(x) < 0 \ \forall x \in (a, b)$
(iii) f is inc. if $f'(x) \geq 0$ (iv) dec if $f'(x) \leq 0$.

T29.8 IVT for derivatives.

T29.9 If f is diff. at $x_0 \in I$ and f is cont on open int. I, then if $f'(x_0) \neq 0$, $y_0 =$
$(f^{-1})'(y_0) = 1/f'(x_0)$.

COMPACT SETS

2.31 Definition By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

2.32 Definition A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

2.33 Theorem *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

2.34 Theorem *Compact subsets of metric spaces are closed.*

2.35 Theorem *Closed subsets of compact sets are compact.*

Corollary *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty K_n$ is not empty.*

2.37 Theorem *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof If no point of K were a limit point of E, then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely, q, if $q \in E$). It is clear that no finite subcollection of $\{V_q\}$ can cover E; and the same is true of K, since $E \subset K$. This contradicts the compactness of K.

2.38 Theorem *If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty I_n$ is not empty.*

4.14 Theorem *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

4.15 Theorem *If $\mathbf{f}$ is a continuous mapping of a compact metric space X into R^k, then $\mathbf{f}(X)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded.*

4.16 Theorem *Suppose f is a continuous real function on a compact metric space X, and*

$$M = \sup_{p \in X} f(p), \qquad m = \inf_{p \in X} f(p). \tag{14}$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

4.17 Theorem *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) = x \qquad (x \in X)$$

is a continuous mapping of Y onto X.

4.18 Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon \tag{15}$$

for all p and q in X for which $d_X(p, q) < \delta$.

4.25 Definition Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the definition of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).

It is clear that any point x of (a, b), $\lim_{t \to x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

4.26 Definition Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity*, at x. Otherwise the discontinuity is said to be of the *second kind*.

There are two ways in which a function can have a simple discontinuity: either $f(x+) \neq f(x-)$ [in which case the value $f(x)$ is immaterial], or $f(x+) = f(x-) \neq f(x)$.

7.7 Definition We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, \ldots$, converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon \tag{12}$$

for all $x \in E$.

It is clear that every uniformly convergent sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this: If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\varepsilon > 0$, and for every $x \in E$, there is an integer N, depending on ε *and* on x, such that (12) holds if $n \geq N$; if $\{f_n\}$ converges uniformly on E, it is possible, for each $\varepsilon > 0$, to find *one* integer N which will do for *all* $x \in E$.

We say that the series $\Sigma f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

The Cauchy criterion for uniform convergence is as follows.

7.8 Theorem *The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies*

$$|f_n(x) - f_m(x)| \leq \varepsilon. \tag{13}$$

7.9 Theorem *Suppose*

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

7.10 Theorem *Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose*

$$|f_n(x)| \leq M_n \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Then Σf_n converges uniformly on E if ΣM_n converges.

7.13 Theorem *Suppose K is compact, and*

(a) *$\{f_n\}$ is a sequence of continuous functions on K,*
(b) *$\{f_n\}$ converges pointwise to a continuous function f on K,*
(c) *$f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$.*

Then $f_n \to f$ uniformly on K.

Theory cheat sheet:

Here is a short recap of relevant theoretical materials for this homework.

Throughout this section let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $\alpha : [a, b] \to \mathbb{R}$ be an increasing function.

Definition 1. *A partition P of $[a, b]$ is a finite collection of points $\{x_0, x_1, \dots, x_n\}$ with $a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b$. For a fixed partition P define*

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \quad i = 1, \dots, n.$$

Definition 2. *Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$, put $M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$, $m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$. Define*

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i,$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \cdot \Delta\alpha_i.$$

Remark 3. Clearly, $L(P, f, \alpha) \leq U(P, f, \alpha)$.

Claim 4. *If $P^* \supset P$ (i.e. if P^* is a **refinement** of P) then*

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

Corollary 5. *For any partitions P_1, P_2*

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha).$$

Definition 6. *We define $\overline{\int}_a^b f d\alpha = \inf_P U(P, f, \alpha)$, and $\underline{\int}_a^b f d\alpha = \sup_P L(P, f, \alpha)$. We say that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if $\overline{\int}_a^b f d\alpha = \underline{\int}_a^b f d\alpha$. In this case we denote the two equal integrals simply by $\int_a^b f d\alpha$.*

Remark 7. If $\alpha(x) = x$, we write simply $\mathcal{R}$ instead of $\mathcal{R}(\alpha) = \mathcal{R}(x)$.

Remark 8. If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ for any partitions P_1, P_2

$$L(P_1, f, \alpha) \leq \int_a^b f d\alpha \leq U(P_2, f, \alpha).$$

Theorem 9. . *We have $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for any ε there exists a partition P, such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon. \tag{9.1}$$

Claim 10. *Suppose the condition 9.1 holds for $P = \{x_0, \dots, x_n\}$ then:*

1. *The condition 9.1 holds for any P^* with $P \subset P^*$.*
2. *For any $s_i, t_i \in [x_{i-1}, x_i]$*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon.$$

3. *If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, for any $t_i \in [x_{i-1}, x_i]$*

$$|\int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i) \Delta\alpha_i| < \varepsilon.$$

Theorem 11. 1. *If f is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*

2. *If f is monotonic on $[a, b]$ and α is continuous, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*
3. *If the set $E = \{x, \text{ s.t } f \text{ is discontinuous at } x\}$ is finite and α is continuous at every $x \in E$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*

23.1 Theorem.

For the power series $\sum a_n x^n$, let

$$\beta = \limsup |a_n|^{1/n} \quad \textit{and} \quad R = \frac{1}{\beta}.$$

[If $\beta = 0$ we set $R = +\infty$, and if $\beta = +\infty$ we set $R = 0$.] Then

(i) *The power series converges for $|x| < R$;*

(ii) *The power series diverges for $|x| > R$.*

7.16 Theorem *Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and*

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha. \tag{23}$$

(The existence of the limit is part of the conclusion.)

7.17 Theorem *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \le x \le b). \tag{27}$$

Corollary If $f_n \in \mathscr{R}(\alpha)$ *on $[a, b]$ and if*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \le x \le b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\,d\alpha.$$

In other words, the series may be integrated term by term.

Properties of the integral.

Theorem 12. *1. If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$, and*

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.$$

2. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $c \in \mathbb{R}$ then $c \cdot f \in \mathcal{R}(\alpha)$ on $[a,b]$, and

$$\int_a^b (c \cdot f) d\alpha = c \cdot \int_a^b f d\alpha.$$

3. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$ then

$$|\int_a^b f d\alpha| \leq M \cdot (\alpha(b) - \alpha(a)).$$

4. If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then $f_1 \cdot f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$.

5. Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $m \leq f(x) \leq M$ for all $x \in [a,b]$. If $\phi : [m,M] \to \mathbb{R}$ is continuous then $h = \phi \circ f \in \mathcal{R}(\alpha)$ on $[a,b]$.