MATH 110 Final Notes Yash Ransari

- Properties of complex arithmetic
  - commutative $\alpha+\beta = \beta+\alpha$
  - associative $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$; $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
  - identity $\lambda+0=\lambda$; $\lambda 1 = \lambda$
  - inverse $\forall \alpha, \exists \beta : \alpha+\beta=0$; $\forall \alpha, \exists \beta : \alpha\beta = 1$
  - distributive $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$

- Properties of linear maps
  - additivity $T(a+b) = T(a) + T(b)$
  - homogenity $T(\lambda a) = \lambda T(a)$
- $\mathcal{L}(V,W)$ = set of lin. maps $V \to W$.

- A vector space contains all linear combinations of its elements (closed under mult, addition)(contains $\vec{0}$).
- F^S is a set of all functions from set S to field F.
- sum of subspaces $V_1 + \dots V_m$ is smallest space V s.t. $V_i \subseteq V \ \forall i \in [m]$.
- $V_1 \oplus V_2$ means $V_1 + V_2$ is a direct sum $\Leftrightarrow$ each $v = v_1 + v_2$ has unique $v_1 \in V_1$, $v_2 \in V_2$ $\Leftrightarrow$ $V_1 \cap V_2 = \{\vec{0}\}$.
  $\Leftrightarrow$ each $v_i \in V_i$ is linearly independent to all $v_j \in V_j$ where $i \neq j$.
- span$(\vec{v}_1 .. \vec{v}_m) = \{\sum a_i \vec{v}_i \ \forall a_i \in F\}$ – $\mathcal{P}_m(F)$ is set of all polynomials w/ coefficients in F. (deg $\leq m$).
- basis(V) is linearly indep. list of vectors that span V. – $\dim(V) = \text{len}(\text{basis}(V))$
- $\dim(V_1+V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ – $\dim \text{range}\, T + \dim \text{null}\, T = \dim V$ (FTLA)
- $\text{null}(T) = \{\vec{v} \in V : T\vec{v} = \vec{0}\}$ – T injective $\Leftrightarrow$ $(Tu = Tv \Leftrightarrow u = v)$ – range $T = \{T\vec{v} : \vec{v} \in V\}$ – T surjective $\Leftrightarrow$ range $T = W$
- $\{\vec{v}_{1,\dots,n}\}$ = basis V, $\{\vec{w}_{1 \dots n}\}$ = basis W, $T \in \mathcal{L}(V,W)$. – $M(S+T) = M(S) + M(T)$
  $\Rightarrow$ $\boxed{T\vec{v}_k = \sum_{i=1}^{m} A_{i,k} \vec{w}_i}$ where $A = M(T, \{\vec{v}_i\}, \{\vec{w}_i\})$ is matrix of T.
- $F^{m,n}$ [set of $m \times n$ matrices] is a vector space w/ dim mn.
- $(AB)_{ij} = \sum_k A_{ik} B_{kj}$ $\Rightarrow$ $M(ST) = M(S)M(T)$
- $(AB)_{\cdot,k} = A(B_{\cdot,k})$
- $\forall A \in F^{m,n}$ with column rank c, $\exists C \in F^{m,c}$, $R \in F^{c,n}$: $A = CR$.
- columns of A lin indep $\Leftrightarrow$ $NS(A) = \{0\}$ $\Leftrightarrow$ injective
- $V' = \mathcal{L}(V,F)$ is the dual space of V.
- $T'(\varphi) = \varphi \circ T$ – $(ST)' = T'S'$

- invertible linear map = isomorphism
- V, W isomorphic $\Leftrightarrow$ $\| \in \mathcal{L}(V,W)$
- $\mathcal{L}(V,W)$ and $F^{m,n}$ isomorphic if $m = \dim V$, $n = \dim W$.
- $M(T)_{\cdot,k} = M(Tv_k)$
- $A = M(T, \{\vec{u}_i\})$, $B = M(T, \{\vec{v}_i\})$, $C = M(I, \{\vec{u}_i\}, \{\vec{v}_i\})$, $A = C^{-1}BC$
- $\{\vec{v}_i\}$ = basis V, $\{\varphi_i\}$ = basis V', $\varphi_j(v_k) = \text{int}(k==j)$,
  $\boxed{\vec{v} = \sum_{i=1}^{\dim V} \varphi_i(\vec{v}) \vec{v}_i}$ $\forall \vec{v} \in V$. [Dual basis]

- annihilator : $U^0 = \{\varphi \in V' : \varphi(u) = 0 \ \forall u \in U\}$
- $\dim U^0 = \dim V - \dim U$ – $\dim \text{null}\, T' = \dim W - \dim V + \dim \text{null}\, T$
- $\text{null}\, T' = (\text{Range}\, T)^0$ – T surjective $\Leftrightarrow$ T' injective and T' sur $\Leftrightarrow$ T inj
- $\dim \text{range}\, T' = \dim \text{range}\, T$ – $\text{range}\, T' = (\text{null}\, T)^0$ – $M(T') = M(T)^T$
- Change of Basis :
  if $\{x_{1,\dots,n}\}$ is β, and $\{y_{1,\dots,n}\}$ is β', then $M(I, \beta, \beta') = \left([x_1]_{\beta'} \dots [x_n]_{\beta'}\right)$. if $x_1 = y_1 - y_2$, $[x_1]_{\beta'} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix}$

- adjoint : $\langle T\vec{v}, \vec{w} \rangle = \langle \vec{v}, T^*\vec{w} \rangle$
  - $(S+T)^* = S^* + T^*$
  - $(\lambda T)^* = \bar{\lambda} T^*$
  - $(T^*)^* = T$
  - $(ST)^* = T^* S^*$
  - $T \in \mathcal{L}(V,W) \Rightarrow T^* \in \mathcal{L}(W,V)$
  - $(T^*)^{-1} = (T^{-1})^*$
  - $I^* = I$.
  - $\text{null}\, T^* = (\text{range}\, T)^\perp$
  - $\text{range}\, T^* = (\text{null}\, T)^\perp$
  - $A^* = \bar{A}^T$. $[A^*_{i,j} = \bar{A}_{j,i}]$

- eigenvalue $\Leftrightarrow$ $|T - \lambda I| = 0$. – # of λ w/ multiplicity = dim of space.
- invariant $\Leftrightarrow$ $T\vec{u} \in U \ \forall \vec{u} \in U$ – null $p(T)$, range $p(T)$ invariant under T.
- minimal polynomial : $\arg\min_p \deg p : p(T) = 0$, p monic. $\Rightarrow p_T(\lambda_i) = 0 \ \forall i$.
- $q(T) = 0$ $\Leftrightarrow$ q is multiple of p_T. – if U invariant under T, p_T multiple of $p_{T|_U}$.
- T not invertible $\Leftrightarrow$ p_T has constant term 0. – All complex operators have eigenvalues.
- $b^2 < 4c \Rightarrow \dim \text{null}\,(T^2 + bT + cI)$ is even. (because complex roots come in pairs). – All operator on odd space has an eigenvalue.
- $M(T)$ upper triangular $\Leftrightarrow$ $\prod_{i=1}^{n} (T - \lambda_i I) = 0$, $\lambda_i \in \text{diag}(M(T))$ $\Leftrightarrow$ $T\vec{v}_k \in \text{span}(\vec{v}_1 \dots \vec{v}_k) \ \forall k$.
  (diagonal elements are eigenvalues)
- T has UT matrix w.r.t. some basis of V $\Leftrightarrow$ $p_T = \prod_{i=1}^{m} (z - \lambda_i)$, $\lambda_i \in F \ \forall i$.
  [$\therefore$ all T on complex spaces always has some UT].
- Eigenspace : $E(\lambda, T) = \text{null}(T - \lambda I) = \{\vec{v} : V : T\vec{v} = \lambda\vec{v}\}$. – $\sum_{i=1}^{m} E(\lambda_i, T)$ is a direct sum with $\dim \leq \dim V$.
- T is diagonalizable $\Leftrightarrow$ Eigenvectors of T form a basis of V $\Leftrightarrow$ $V = \sum_{i=1}^{m} E(\lambda_i, T)$.
  [$\therefore$ $\dim V$ distinct eigenvalues $\Rightarrow$ T diagonalizable] $\Leftrightarrow$ $p_T = \prod_{i=1}^{m} (z - \lambda_i)$ for distinct λ_i.
- T diagonalizable on V $\Rightarrow$ $T|_U$ diag. on U if U invariant.
- T, S commute $\Leftrightarrow$ $ST = TS$ $\Leftarrow$ T, S diagonalizable w.r.t same basis of V.
  $\Rightarrow$ T, S share a common eigenvector
  $\Rightarrow$ T, S both UT for some basis of V,
  $E(\lambda, S)$ is invariant under T,
  every eigenval of $S+T$ = some λ_S + some λ_T,
  " " " ST = " · "
- $0 \subseteq \text{null}\, T \dots \subseteq \text{null}\, T^k \subseteq \text{null}\, T^{k+1} \dots$
- $\text{null}\, T^m = \text{null}\, T^{m+1} \Rightarrow \text{null}\, T^k = \text{null}\, T^m \ \forall k > m$.
- $\text{null}\, T^{\dim V} = \text{null}\, T^k \ \forall k > \dim V$.
- $\text{null}\, T^{\dim V} \oplus \text{range}\, T^{\dim V} = V$
- Generalized Eigenspaces : $G(\lambda, T) = \{\vec{v} \in V : \exists k \in \mathbb{Z}^+ \ (T - \lambda I)^k \vec{v} = \vec{0}\} = \text{null}\,(T - \lambda I)^{\dim V}$
- Generalized eigenspaces w.r.t different eigenvalues are linearly independent.

- nilpotent $\iff \exists k: T^k = 0 \iff T^{\dim V} = 0 \iff 0$ is the only eigenvalue of T. $\iff \exists m: p_T = z^m \iff$ w.r.t some basis of V, $M(T) = \begin{bmatrix} 0 & * \\ & \ddots \\ 0 & & 0 \end{bmatrix}$.
- To convert matrix to UT, use basis vectors from null space of T, T^2
- $G(\lambda, T)$ is invariant under T, $T|_{G(\lambda,T)}$ nilpotent, $V = G(\lambda_1, T) \oplus \dots G(\lambda_m, T)$
- multiplicity, $d_i = \dim \operatorname{null} (T - \lambda_i I)^{\dim V} = \dim G(\lambda_i, T)$, characteristic polynomial: $\prod_{i=1}^{m} (z-\lambda_i)^{d_i} = q_T(z)$
- Block-Diagonal Form: $\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$, $A_k = \underbrace{\begin{bmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}}_{d_k \times d_k}$. 1. choose bases for $G(\lambda_k, T)$ s.t. $(T-\lambda_k I)|_G$ is of form $\begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$ 2. combine bases to form BD basis for T.
- T nilpotent $\Rightarrow \sqrt{T-I}$ exists.
- T invertible $\iff \dim \operatorname{null} T = 0$ $\Rightarrow$ T has square root
- Jordan Basis: T is of form: $\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$, $A_k = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix}$ [combine some bases of $G(\lambda, T)$ to form Jordan basis] $\{\vec{v}, (T-\lambda I)\vec{v}, \vec{w}, \dots\}$, $\in G(\lambda,T)$, $\in G(\lambda,T)$
  - Nilpotent or complex $\Rightarrow$ Jordan exists.
  - $\boxed{T\vec{u}_k = \lambda_k \vec{u}_k + \vec{u}_{k-1}}$ [$u_{1,\dots,m}$ is Jordan Basis]
- Bilinear form, $\{\beta: V \times V \to F : \vec{v} \to \beta(\vec{v}, \vec{u})$ and $\vec{v} \to \beta(\vec{u}, \vec{v})$ both linear functionals$\} = V^{(2)}$
- $M(\beta)_{j,k} = \beta(\vec{e}_j, \vec{e}_k)$. The map $\beta \to M(\beta)$ is an isomorphism of $V^{(2)}$ onto $F^{n,n}$. $\dim V^{(m)} = (\dim V)^m$
- $M(\vec{u}, \vec{v} \to \beta(T\vec{u}, \vec{v})) = M(\beta) M(T)$. $-$ $M(\vec{u}, \vec{v} \to \beta(\vec{u}, T\vec{v})) = M(T)^T M(\beta)$.

$V^{(2)} = V^{(2)}_{alt} \oplus V^{(2)}_{sym}$

- If $A = M(\beta, \text{basis}_1)$; $B = M(\beta, \text{basis}_2)$; $C = M(I, \text{basis}_1, \text{basis}_2)$, then $A = C^T B C$
- Symmetric BF, $V^{(2)}_{sym} = \{\rho \in V^{(2)} : \rho(\vec{u}, \vec{\omega}) = \rho(\vec{\omega}, \vec{u})\ \forall \vec{u}, \vec{\omega}\}$ [$\iff M(\rho)$ symmetric for some basis $\iff M(\rho)$ '' '' every basis $\iff M(\rho)$ diagonal for some orthonormal basis]
- Alternating BF, $V^{(2)}_{alt} = \{\rho \in V^{(2)} : \rho(\vec{u}, \vec{u}) = 0\ \forall \vec{u}\}$ [$\iff \rho(\vec{u}, \vec{\omega}) = -\rho(\vec{\omega}, \vec{u})\ \forall \vec{u}, \vec{\omega} \in V$]
- Quadratic Form: $\exists \beta: q_\beta(\vec{v}) = \beta(\vec{v}, \vec{v}) \iff q(2\vec{v}) = 4q(\vec{v})$ and
  $\iff \exists A: q(\vec{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} x_i x_j\ \forall \vec{x}$. $\iff q(\lambda \vec{v}) = \lambda^2 q(\vec{v})$ and $[(\vec{u}, \vec{\omega}) \to q(\vec{u} + \vec{\omega}) - q(\vec{u}) - q(\vec{\omega})] \in V^{(2)}_{sym}$
  $\Rightarrow \exists$ some basis of V, and some $\lambda_1 \dots \lambda_n \in F$ $\forall \lambda \in F$ $q(\vec{x}) = \sum_{i=1}^{n} \lambda_i x_i^2$ [$F = \mathbb{R} \Rightarrow$ this basis can be chosen orthonormal]
- $V^{(m)}_{alt} = \{\alpha \in V^{(m)} : \alpha(\vec{v}_1 \dots \vec{v}_m) = 0$ whenever $\vec{v}_1 \dots \vec{v}_m$ linearly dependent$\}$ [$= \{0\}$ if $m > \dim V$]
  [$\iff \alpha(\vec{v}_{1 \dots k}, \vec{v}_{k+1 \dots m}) = -\alpha(\vec{v}_{1 \dots k-1}, \vec{v}_{k+1}, \vec{v}_k, \vec{v}_{k+2 \dots m})\ \forall k \in [m]$]
- perm m = set of all permutations of m elements.
- $\operatorname{sign}(j_1 \dots j_m) = (-1)^N$, $N = |\{a, b: a > b$, but $j_a < j_b\}|$ [swapping 2 entries in any perm multiplies sign by -1].
- $\alpha \in V^{(m)}_{alt} \Rightarrow \alpha(v_{j_1} \dots v_{j_m}) = \operatorname{sign}(j_{1 \dots m}) \cdot \alpha(v_{1 \dots m})$, $-$ $\dim V^{(\dim V)}_{alt} = 1$.
  $\alpha(v_{1 \dots n}) = \alpha(e_{1 \dots n}) \sum_{(j_1 \dots n) \in \text{perm } n} (\operatorname{sign}(j_1 \dots n)) (v_1)_{j_1} \cdots (v_n)_{j_n}$, $\alpha(v_{1 \dots n}) \neq 0 \iff v_{1 \dots n}$ independent (only for $n = \dim V$)
- Determinant: $\forall \alpha \in V^{(m)}_{alt}$, $\alpha_T(\vec{v}_{1 \dots m}) = \alpha(T\vec{v}_1, \dots T\vec{v}_m) = \det(T)\, \alpha(\vec{v}_{1 \dots m})$
  - $(\vec{v}_1, \dots \vec{v}_n \to |[\vec{v}_1 \cdots \vec{v}_n]|) \in V^{(n)}_{alt}$ $-$ $\boxed{|A| = \sum_{(j_1 \dots j_n) \in \text{perm } n} \operatorname{sign}(j_{1 \dots n}) A_{j_1,1} \cdots A_{j_n,n}}$. $-$ $|T| = \prod_{i=1}^{m} \lambda_i^{d_i}$
  - $|ST| = |S||T|$ $-$ $|T^{-1}| = \frac{1}{|T|}$ $-$ $|T - \lambda I| = 0 \iff \lambda$ is eigenvalue of T. $-$ $|A^T| = |A|$
- $|T'| = |T|$, $|T^*| = \overline{|T|}$ $-$ $M(T)$ not full rank $\iff |T| = 0$. $-$ equivariant to scalar mult of rows or cols.
  - $|A| = -|B|$ if B switches two rows/collumns in A.
  - invariant to adding scalar mult of row/col to another row/col.
  - Unitary operator $\iff S^* S = I \Rightarrow |\det S| = 1$. $-$ S positive $\Rightarrow |S| \geq 0$.
- $|\det T| = \prod_{i=1}^{n} \sigma_i$. $-$ $|zI - T| = \prod_{i=1}^{m} (z - \lambda_i)^{d_i} = q_T(z)$. $-$ Hadamard: $|\det A| \leq \prod_{k=1}^{M} \|v_k\|$.
- inner product: $V \times V \to F$. $-$ $\langle v, v \rangle \geq 0$ $-$ $\langle v, v \rangle = 0 \iff v = 0$ $-$ $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
  - $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ $-$ $\langle u, v \rangle = \overline{\langle v, u \rangle}$. $-$ $\langle v, 0 \rangle = \langle 0, v \rangle = 0$. $-$ $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
  - $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$ $-$ $\boxed{\|v\| = \sqrt{\langle v, v \rangle}}$ $-$ $\|\lambda v\| = |\lambda| \|v\|$. $-$ orthogonal: $\langle u, v \rangle = 0 \Rightarrow$ lin indep.
  - projection: $c = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}$, $\vec{\omega} = \vec{u} - \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \Rightarrow \vec{u} = c\vec{v} + \vec{\omega}$ $-$ Cauchy-Schwartz: $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$. [equality $\iff \vec{u} = c\vec{v}$]
  - $\sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} |\langle \vec{v}, \vec{e}_i \rangle|^2 = \|\vec{v}\|^2$ for orthonormal basis $\vec{e}_{1 \dots n}$ $-$ $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} \langle \vec{u}, \vec{e}_i \rangle \overline{\langle \vec{v}, \vec{e}_i \rangle}$
  - Gram-Schmidt: any list can be made orthonormal preserving span. $-$ $M(T)$ is UT w.r.t an orthonormal basis $\iff p_T = \prod_{i}^{n} (z - \lambda_i)^{d_i}$.
  - $G \subseteq H \Rightarrow H^\perp \subseteq G^\perp$ $-$ $V = U \oplus U^\perp\ \forall U$ (subspace) $\subseteq V$. $-$ $U = (U^\perp)^\perp$ $-$ $P_U \vec{v} = \vec{v} - P_{U^\perp} \vec{v}$
  - range $P_U = U$, null $P_U = U^\perp$, $P_U^2 = P_U$. $-$ span$\{\vec{e}_{1 \dots m}$ (orthonormal)$\} = U \Rightarrow \boxed{P_U \vec{v} = \sum_{i=1}^{m} \langle \vec{v}, \vec{e}_i \rangle \vec{e}_i}$, $\|P_U \vec{v}\| \leq \|\vec{v}\|$.
  - $\vec{v} \to \varphi_v : \varphi_v(\vec{u}) = \langle \vec{u}, \vec{v} \rangle$ is a bijection from V to V'. $-$ T maps $(\operatorname{null} T)^\perp$ to range T one to one.
- Self-adjoint $\iff T = T^* \iff \langle T\vec{v}, \vec{v} \rangle \in \mathbb{R}\ \forall \vec{v} \in V \Rightarrow$ all eigenvalues real $\Rightarrow p_T(z) = \prod_{i=1}^{m} (z - \lambda_i)$, $\lambda_i \in \mathbb{R}$.
- T self-adjoint in any F, or normal in $F = \mathbb{C}$ $\iff M(T)$ diag w.r.t some orthonormal basis $\iff$ eigenvectors of T form orthonormal basis of V.
- normal $\iff T^* T = T T^* \iff \|T^* \vec{v}\| = \|T\vec{v}\| \iff T = A + iB$, $AB = BA$, $A^* = A$, $B^* = B$. $-$ $T = T^*$, $b^2 < 4c \Rightarrow T^2 + bT + cI$ invertible
  $\Rightarrow$ null T = null T^*, $V = \operatorname{null} T \oplus \operatorname{range} T$, $(T\vec{v} = \lambda \vec{v} \iff T^* \vec{v} = \bar{\lambda} \vec{v}\ \forall \lambda \in F)$, range T = range T^*, $T - \lambda I$ normal $\forall \lambda \in F$, eigenvectors orthogonal.

Real/Complex Spectral Theorem