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Math 54 Notes

1) $(\underline{\underline{A}} \mid \underline{\underline{I}}) = (\underline{\underline{I}} \mid \underline{\underline{A}}^{-1})$

2) if $\{\vec{b}_1 \ldots \vec{b}_n\}$ is a basis for $\mathbb{R}^n$, $(\vec{b}_1 \ldots \vec{b}_n \mid \vec{v}) \xrightarrow{R.R.} (\underline{\underline{I}} \mid \vec{v}_B)$

3) $P_{I \leftarrow B} = (\vec{b}_1 \ldots \vec{b}_n)$, $P_{B \leftarrow I} = (\vec{b}_1 \ldots \vec{b}_n)^{-1}$ $\therefore$ $(\vec{c}_1 \ldots \vec{c}_n \mid \vec{b}_1 \ldots \vec{b}_n) \rightarrow (I_n \mid P_{C \leftarrow B})$

4) For $\vec{v}_i$ in some vector space V, span $(\vec{v}_1 \ldots \vec{v}_p) = V$ if and only if v_i are all linearly independent. Thus, $\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \vec{x} = \vec{0}$ only for $\vec{x} = \vec{0}$.

5) $\{\vec{v}_1 \ldots \vec{v}_p\}$ only make a basis for V if they are LI and spanning. for $\vec{v}_i$ as vectors in $\mathbb{R}^n$, $\{\vec{v}_1 \ldots \vec{v}_p\}$ LI for $p \leq n$, spanning for $p \geq n$. Thus, $\{\vec{v}_1 \ldots \vec{v}_p\}$ is a basis for $\mathbb{R}^n$ only if $p = n$.

6) $\text{Dim}(V)$ = size of any basis of V, where V is a finite dimensional V.S.

7) If U is a subspace of V, $\dim(U) \leq \dim(V)$

8) All inner products in $\mathbb{R}^n$, $\langle \vec{u}, \vec{v} \rangle$, are equivalent to $\vec{u}^T \underline{\underline{A}} \vec{v}$, where $\underline{\underline{A}}$ is an $n \times n$ symmetrical matrix with non-zero eigenvalues that are all positive.

9) $\text{Proj}_{col(\underline{\underline{A}})}(\vec{b}) = \sum_{i=1} \frac{\vec{a}_i \cdot \vec{b}}{\vec{a}_i \cdot \vec{a}_i} \vec{a}_i$ if $\underline{\underline{A}} = [\vec{a}_1 \ldots \vec{a}_n]$

10) Least squares solution to $\underline{\underline{A}}\vec{x} = \vec{b}$ is $\hat{x}$ where $\underline{\underline{A}}\hat{x} = \text{Proj}_{col(\underline{\underline{A}})}(\vec{b})$

11) $\text{Range}(T_{\underline{\underline{A}}}) = col(\underline{\underline{A}})$ where T is a linear transformation given by $\underline{\underline{A}}$.

12) If $T_{\underline{\underline{A}}}$ is onto, $\text{Range}(T_{\underline{\underline{A}}}) = \mathbb{R}^m$, so A has pivot in all rows.

13) Kernel of $T_{\underline{\underline{A}}}$ is $\ker(T_{\underline{\underline{A}}}) = \text{Null}(\underline{\underline{A}}) = \{\vec{x} \text{ such that } T_{\underline{\underline{A}}}(\vec{x}) = \vec{0}\}$

14) $T_{\underline{\underline{A}}}$ is one-to-one if $\ker(\underline{\underline{A}}) = \text{Null}(\underline{\underline{A}}) = \{\vec{0}\}$, $\underline{\underline{A}}$ has pivot in all columns.

15) If $\underline{\underline{A}}\vec{x}_n = \vec{0}$ and $\underline{\underline{A}}\vec{x}_p = \vec{b}$, then $\vec{x} = \vec{x}_p + \vec{x}_n$ is a general solution.

16) Rank(T_A) = dim(Range(T_A)). ⇒ Range(T_A) is the number of pivot collumns

17) Nullity(T_A) = dim(Ker(T_A)) ⇒ Nullity(T_A) is the number of free collumns

18) if $T_A : V \to W$, Rank(T_A) + Nullity(T_A) = dim(V)

19) If T is onto, Range(T) = W, Rank(T) = dim(W), dim(V) $\geq$ dim(W)

20) If T is one-to-one, $\ker(T) = \{\vec{0}\}$, Nullity(T) = 0, Rank(T) = dim(V), dim(V) $\leq$ dim(W)

21) If T is one-to-one AND onto, dim(V) = dim(W).

22) $A_{B,C}(\vec{v}_B) = (A\vec{v})_C$ ⇒ $A_{B,C} = P_{C \leftarrow I}\, A\, P_{I \leftarrow B} = (\vec{c}_1 \cdots \vec{c}_m)^{-1} A (\vec{b}_1 \cdots \vec{b}_n)$

23) Orthogonal set $\{\vec{v}_1 \cdots \vec{u}_n\}$ if $\vec{u}_i \cdot \vec{u}_j = 0 \;\forall\; i \neq j$

Orthonormal set is an orthogonal set with $\|\vec{v}_i\| = 1 \;\forall\; i$

24) If $U = [\vec{u}_1 \cdots \vec{u}_n]$, if $\{\vec{u}_1 \cdots \vec{u}_n\}$ orthogonal, $U^T U$ is diagonal

if $\{\vec{u}_1 \cdots \vec{u}_n\}$ orthonormal, $U^T U = I$, $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$,

$\|U\vec{x}\| = \|\vec{x}\|$, $U^T = U^{-1}$

25) For any vector $\vec{y}$ and any vector space W, $\vec{y} = \vec{z}_1 + \vec{z}_2$ where $\vec{z}_1$ in W and $\vec{z}_2$ in $W^\perp$ [$\vec{z}_1 = \text{Proj}_W \vec{y}$, $\vec{z}_2 = \text{Proj}_{W^\perp} \vec{y}$]

$\|\text{Proj}_W \vec{y} - \vec{y}\| \leq \|\vec{w} - \vec{y}\| \;\forall\; \vec{w} \in W$.

[if $\|\text{Proj}_W \vec{y} - \vec{y}\| = \|\vec{w} - \vec{y}\|$ for any $\vec{w} \in W$, then $\vec{w} = \text{Proj}_W \vec{y}$]

26) If $\{\vec{v}_1 \cdots \vec{u}_n\}$ is an orthonormal basis of W, then $\text{Proj}_W \vec{y} = U U^T \vec{y}$
[orthonormal]

27) Gram-Schmidt Process: [Given a basis $\{\vec{x}_1 \cdots \vec{x}_n\}$, construct $\{\vec{v}_1 \cdots \vec{v}_n\}$]

→ $W_{k-1} = \text{Span}(\vec{v}_1 \cdots \vec{v}_{k-1})$ → $\vec{v}_k = \vec{x}_k - \text{Proj}_{W_{k-1}}(\vec{x}_k) = \vec{x}_k - \frac{\vec{x}_k \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{x}_k \cdot \vec{v}_{k-1}}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}}\vec{v}_{k-1}$

→ Repeat for k from 2 to n, then divide all v by their magnitudes to normalised.

28) For an $n \times p$ matrix A, Q is an $n \times p$ matrix with orthonormal collumns [LI collumns] and $A = QR$ where R is an upper triangular matrix.
[$R = Q^T A$]

23) Algebraic multiplicity eigenvalue λ = number of linearly independent eigen ~~values~~ vectors with the same eigenvalue.

~~Algebraic~~ Geometric multiplicity of λ = free variables in $\text{rref}(\underline{A} - \lambda \underline{I})$.

$\underline{A}$ is only diagonalizable if $AM(\lambda) = GM(\lambda)\ \forall \lambda$ of $\underline{A}$.

24) $A_{B,C} = \left(T(\vec{b_1})_C , T(\vec{b_2})_C , T(\vec{b_3})_C \right)$

25) Eigen-space for matrix $\underline{A}$ with eigenvalue is the solution to $(\underline{A} - \lambda \underline{I})\vec{x} = \vec{0}$

26) If span $\{\vec{b_1} \dots \vec{b_p}\}$ = eigenspace of $\underline{A}$ then $\underline{A} = \underline{P}\,\underline{D}\,\underline{P}^{-1} \Rightarrow \underline{P}^{-1}\underline{A}\,\underline{P} = \underline{D}$

where $\underline{D} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}$, $\underline{P} = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix}$ [λ is in decreasing order]

27) $\text{Proj}_W(\vec{x})$ can only be done using orthogonal basis of W.

28) $P_{C \leftarrow B} = \left[(\vec{b_1})_C , (\vec{b_2})_C , (\vec{b_3})_C \right]$

29) Basis for Range $(T_{\underline{A}})$ is just the collumns of $\underline{A}$ where pivots occur.

Basis for $\ker(T_{\underline{A}})$ is null space of $\underline{A}$. [basis of nullspace]

30) $W^{\perp} = \text{Null}\begin{pmatrix} - \vec{b_1} - \\ - \vec{b_2} - \\ - \vec{b_3} - \end{pmatrix}$ where $(\vec{b_1}, \vec{b_2}, \vec{b_3})$ is the basis of W.

31) SVD: $\underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$ where $\underline{V} = \left[\vec{v_1} \cdots \vec{v_n} \right]$ and $\vec{v_i}$ are the eigenvectors of A^TA.

$\Sigma = \left[\begin{array}{cc|c} \sigma_1 & 0 & \\ 0 & \sigma_r & 0 \\ \hline & 0 & 0 \end{array} \right]$ where r is the rank of A.

$\sigma_i = \sqrt{\lambda_i}$, σ are reordered in descending order

$\vec{u_i} = \frac{1}{\sigma_i} A \vec{v_i}$ for $i \le r$

orthonormal basis for $\text{Null}(A^T)$ for $i > r$

32) Homogeneous Linear Second Order Diff Equation: $ay'' + by' + cy = 0$

If $ar^2 + br + c = 0$ has

distinct, real roots: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = \text{span}(e^{r_1 t}, e^{r_2 t})$

repeated, real roots: $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t} = \text{span}(e^{r_1 t}, t e^{r_1 t})$

complex roots: $y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) = \text{span}(e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t)$

$[r = \alpha \pm i\beta]$

33) Non-homogeneous Secondorder Linear Differential Equations:

for $ay'' + by' + cy = g(x)$ $\Rightarrow y = y_c + y_p$ where

$ay_c'' + by_c' + cy_c = 0$

$g(x) = p(x)e^{\alpha x}[\cos(\beta x) \text{ or } \sin(\beta x)]$

$y_p = x^s q(x)\, e^{\alpha x} \cos\beta x + x^s r(x)\, e^{\alpha x} \sin\beta x$

where p, q, r are polynomials of the same degree.

and $s = 0$ if $e^{\alpha x}\cos(\beta x)$ is not in y_c

1 if $e^{\alpha x}\cos(\beta x)$ is in y_c but $xe^{\alpha x}\cos(\beta x)$ isn't

2 if $e^{\alpha x}\cos(\beta x)$ and $xe^{\alpha x}\cos(\beta x)$ are both in y_c.

34) Linear Systems of Differential Equations:

$\underline{\underline{W}} [\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n](t) = \|\vec{x}_1(t) \ldots \vec{x}_n(t)\|$ [This is called the Wronskian]

If Wronskian is 0 for all t, $\vec{x}_1 \ldots \vec{x}_n$ are a linearly dependant set

If Wronskian is not 0 for all t, $\vec{x}_1 \ldots \vec{x}_n$ are linearly independant.

If $\vec{x}_1 \ldots \vec{x}_n$ are linearly independent, they are called a fundamental solution set.

35) Constant coefficient Linear Systems: $\vec{x}'(t) = \underline{A}\vec{x}(t)$

If $\{\vec{v}_1 \ldots \vec{v}_n\}$ is a basis of eigenvectors of A, then $\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 \ldots \vec{x}_n(t) = e^{\lambda_n t}\vec{v}_n$ is a fundamental solution set.

If $\lambda = \alpha + i\beta$, $\vec{v} = \vec{a} + i\vec{b}$, $\vec{x}_1(t) = e^{\alpha t}\cos\beta t\, \vec{a} - e^{\alpha t}\sin\beta t\, \vec{b}$

$\vec{x}_2(t) = e^{\alpha t}\sin\beta t\, \vec{a} + e^{\alpha t}\cos\beta t\, \vec{b}$

are real, linearly independent solutions to $\vec{x}'(t) = \underline{A}\vec{x}(t)$

36) $\text{Nul}(A)^{\perp} = \text{Col}(A^T) \Rightarrow \text{Col}(A)^{\perp} = \text{Nul}(A^T)$

37) Fourier series of f on $[-L, L]$: $\frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right]$

where $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$, $b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$

for $x = \pm L$, the series converges to $\frac{1}{2}(f(-L^+) + f(L^-))$

where $f(x^+) = \lim_{h\to 0^+} f(x+h)$

38) Fourier cosine series for f on $[0, L] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$ $\left[a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right.$

39) Fourier sine series for f on $[0, L] = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $\left[b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right.$