Math 124 - Programming for Mathematical Applications

UC Berkeley, Spring 2024

Project 5 - Neural Networks for Character Recognition

Due Friday, April 26

In [2]:
```
using PyPlot, Random, LinearAlgebra, Optim   # Packages needed
```

```
[ Info: Precompiling PyPlot [d330b81b-6aea-500a-939a-2ce795aea3ee]
[ Info: Precompiling Optim [429524aa-4258-5aef-a3af-852621145aeb]
WARNING: method definition for show at /srv/julia/pkg/packages/Optim/rES5
7/src/univariate/printing.jl:7 declares type variable T but does not use i
t.
```

Description

In this project, you will implement an artificial neural network for *Optical Character Recognition* (OCR). We will use a so-called *Multilayer Perceptron* (MLP) with a single hidden layer.

Preliminaries

First we define the characters that we will use to train the network. Each character will be an image of size 7-by-6, represented as a vector of 42 values, and we will use the 4 characters "MATH". In the network, these characters will be encoded using two output variables $y = (y_1, y_2)$, where $y = (0, 0)$ represents "M", $y = (0, 1)$ represents "A", and so on. The code below defines these images and returns the following variables:

- `training`, size 42-by-4 array containing the 42 pixel-values for each of the 4 characters
- `target`, size 2-by-4 array containing the desired output for each of the 4 characters
- `mapstr`, the string "MATH" which is the characters that each `target` output corresponds to

In [3]:
```
charstr = """
          000000  000000  000000  00..00
          000000  000000  000000  00..00
          0.00.0  00..00  ..00..  00..00
          0.00.0  00..00  ..00..  000000
          0....0  000000  ..00..  00..00
          0....0  00..00  ..00..  00..00
```

```
        0... 0  00..00  ..00..  00..00
        """

training = reshape(collect(charstr), :, 7)
training = Int.(training[[1:6;9:14;17:22;25:30],:] .== 'O')
training = reshape(training', 7*6, 4)
target = [0 0; 0 1; 1 0; 1 1]'
mapstr = "MATH";
```

We also define the plotting function below, which takes an array `images` with 42 rows and one column for each image, and shows the images in a grid:

In [4]:

```
function plot_chars(images)
    gray()
    n_images = size(images,2)
    for j = 1:n_images
        subplot(ceil(Int, n_images/4), 4, j)
        im = 1 .- reshape(images[:,j], 7, 6)
        imshow(im); axis("off");
    end
end
plot_chars(training)
```

Problem 1 - Generating noisy test characters

To test our trained OCR code, we need noisy or perturbed characters. Here we will artificially produce these by modifying the true character images in `training`.

Write a function `make_testdata(training)` which returns an array `testdata` of size 42-by-20. The first 4 images (columns) of `testdata` are identical copies of the `training` array. Generate the next 4 images by randomly choosing 2 pixels in each training image, and flip their values (that is, 0 becomes 1, 1 becomes 0). For the next 4 images you choose 2x2=4 random pixels to flip, then 2x3=6 pixels, and finally 2x4=8 pixels. This gives a total of 16 new perturbed images, and 20 columns total including the original images.

Plot your test data using the commands below. This should show a 5-by-4 array of successively worse letters MATH in each row.

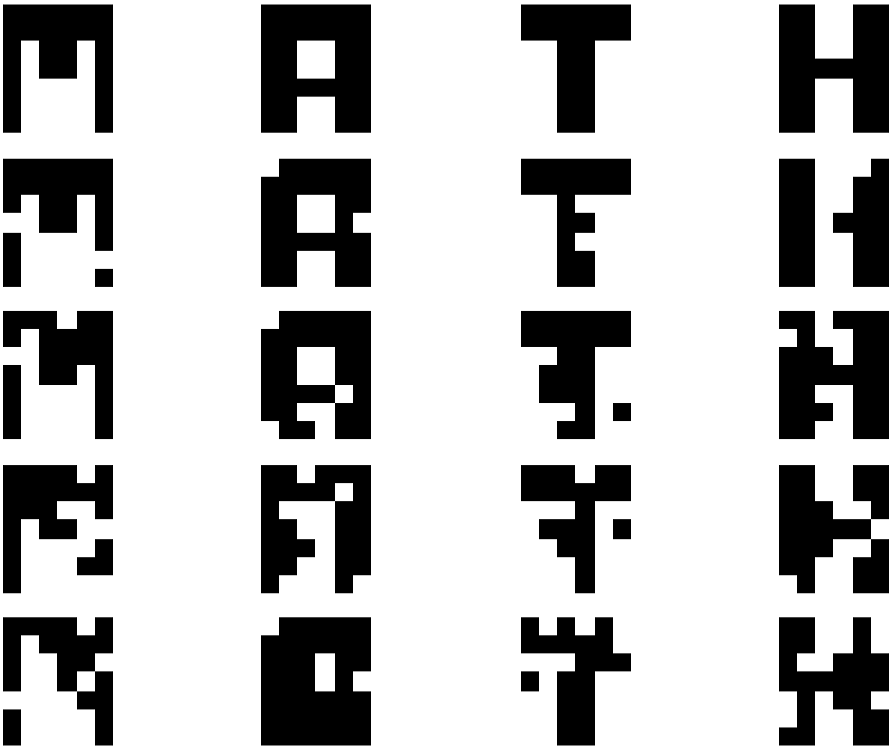
In [5]:

```
function make_testdata(training)
    testdata = zeros(42, 20)
    testdata[:, 1:4] = training
    for i = 1:4
        testdata[:, 4*i+1:4*i+4] = training[:, :]
        for j = 1:4
            ids = shuffle(1:42)[1:2*i]
```

```
                testdata[ids, 4*i+j] = 1 .- testdata[ids, 4*i+j]
            end
        end
    return testdata
end
```

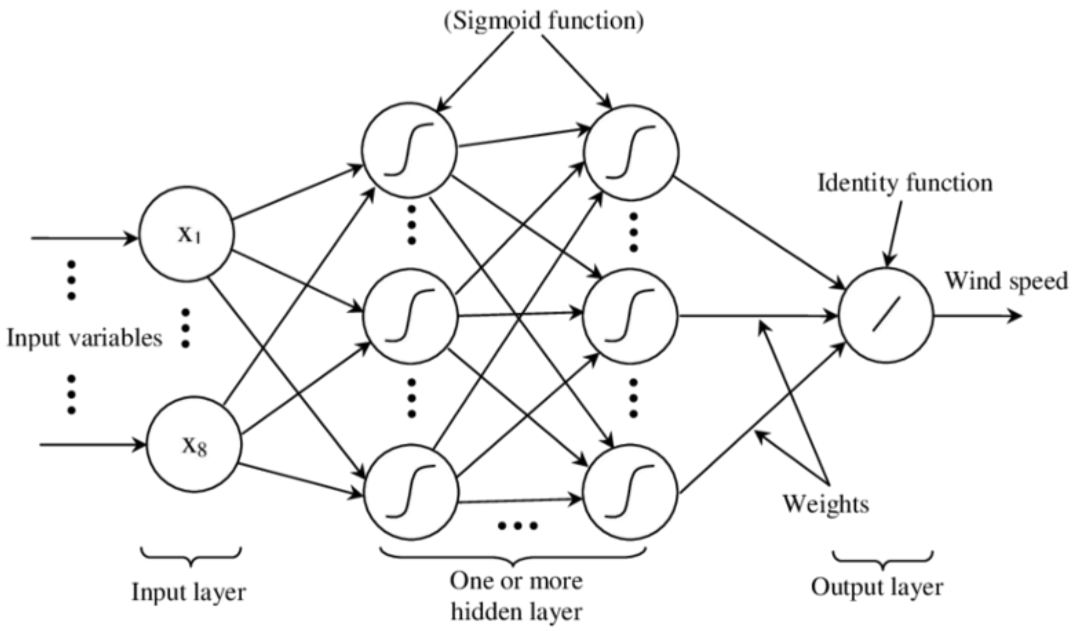
Out[5]: make_testdata (generic function with 1 method)

In [6]:
```
testdata = make_testdata(training);
plot_chars(testdata)
```



Machine learning using a multilayer perceptron

Our network is illustrated below. The input $x = (x_1, \ldots, x_{42})$ is in our case an image represented as a vector of length 42, and the output $y = (y_1, y_2)$ is a vector of length 2, with the predicted character encoded as described above.



(by V. Palaniappan, https://medium.com/engineer-quant/multilayer-perceptron-4453615c4337)

Here, the *sigmoid function* is defined by

$$\sigma(x) = \frac{1}{1 + e^{0.5-x}}$$

The *weights* (that is, the arrows between the layers) can be represented in a linear form as a matrix multiplication. The goal of the training is to determine these weights, that is, the entries of the matrices.

We will use a hidden layer with 10 neurons. This means that the first layer can be written as a matrix-vector product Vx for some weight matrix V of size 10-by-42. Next we apply the sigmoid function (element-wise) to get $r = \sigma(Vx)$, apply another weight matrix W of size 2-by-10, and finally apply the sigmoid function again to predict the output y. In total this gives $10\cdot 42

- 2\cdot 10 = 440$ weights that need to be determined during the training. The entire network can be written in a compact form as

$$y(x) = \sigma(W\sigma(Vx))$$

We will train our network by solving the following optimization problem: Find $V \in \mathbb{R}^{10\times 42}$, $W \in \mathbb{R}^{2\times 10}$ that minimize the following so-called *misfit function*

$$E(V, W) = \frac{1}{2}\sum_{j=1}^{4} \|y(x_j) - t_j\|_2^2$$

Here, x_j is the jth training vector (from the `training` array), t_j is the desired target output for the input x_j (from the `target` array), and $y(x)$ is the network described above.

Stochastic Gradient Descent training

Finding the true gradient of E is somewhat involved, and instead we will use the so-called Stochastic Gradient Descent method. At each iteration, we pick:

- A random number $j \in \{1, 2, 3, 4\}$ to decide which training vector x_j and target output t_j to use
- A random number $k \in \{1, 2\}$ to decide which component of the vector $y(x_j) - t_j$ to consider

We then only compute the gradient direction for the part of $E(V, W)$ that contribute to output k for training vector j. The gradient calculations get simplified with these assumptions, and they can be computed with the following sequence of computations:

$$\begin{aligned}
r &= \sigma(Vx_j) && \text{column-vector (matrix-vector product)} \\
y &= \sigma(W_k \cdot r) && \text{scalar (dot-product, } W_k \text{ is } k\text{th row of } W) \\
q &= (y - t_{kj})y(1 - y) && \text{scalar (} t_{kj} \text{ is component } k \text{ of } t_j) \\
u &= W_k^T r(1 - r) && \text{column-vector (all products are element-w} \\
\nabla_{W_k} E(V, W) &= qr^T && \text{row-vector (scalar times row-vector)} \\
\nabla_V E(V, W) &= qux_j^T && \text{matrix (scalar times outer product)}
\end{aligned}$$

Using these gradients, we update V and the kth row of W using standard gradient descent iterations for a given stepsize α.

Problem 2 - Training using SGD

Implement a function

```
function train_sgd(; maxiter=10000, rate=1)
```

which initializes V, W to normal-distributed random numbers and performs `maxiter` stochastic gradient descent iterations as described above. Use the step-size α set to `rate` (the so-called *learning rate*). The function returns V, W, and can be tested using the code below.

In [21]:

```
σ(x) = 1/(1 + exp(0.5 - x))
function train_sgd(; maxiter=10000, rate=1)
    V = randn(10, 42)
    W = randn(2, 10)
    for i = 1:maxiter
        j = rand([1,2,3,4])
        k = rand([1,2])
        r = σ.(V * training[:, j])
        y = σ.(transpose(W[k, :]) * r)
        q = (y - target[k, j]) * y * (1-y)
        u = W[k, :] .* r .* (1 .- r)
        V -= rate * q * u * transpose(training[:, j])
        W[k, :] -= reshape(rate * q * transpose(r), 10)
    end
```

```
    return V, W
end
```

Out[21]: train_sgd (generic function with 1 method)

In [22]:
```
V,W = train_sgd()
```

Out[22]:
```
([-0.14185414147046105 0.9159433511541616 … -1.8551798420932357 -1.79380
5273054716; -0.1554520138840822 -0.30645274531496647 … 0.508427126961835
9 -0.8321591498417067; … ; -1.6865845493887386 -0.6465274118931721 … -0.
06367953013335319 -1.0657486410788821; 0.46673437639661103 -1.1933341182
94236 … 0.00044142997094382834 0.31733979363197684], [4.532503131603402
-2.9260672541249493 … 1.6846222524867487 2.407407492519523; -1.005868236
8355252 -1.2311685481356864 … 4.919407022941263 -1.368674718645008])
```

Problem 3 - Predict output for noisy characters

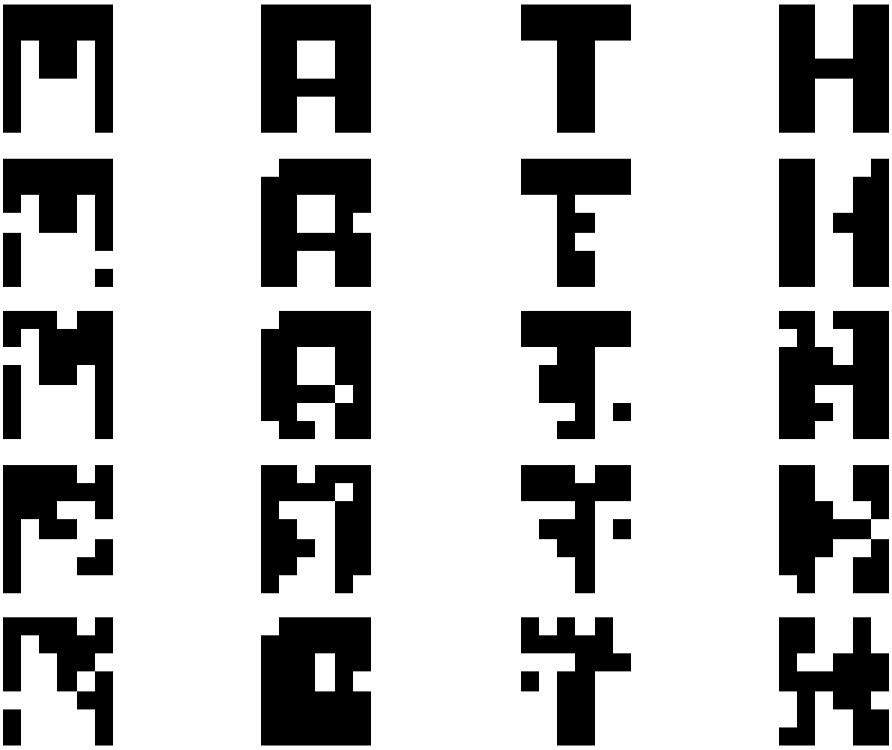
We can now apply the trained network V, W to the noisy characters in `testdata`, and see how well they match. We do this by simply computing $y(x_j)$ for each image x_j in `testdata`, rounding the outputs y_1, y_2 to integers, and mapping the four possibilities to the letters in `mapstr`.

Write a function `predict(testdata, V, W)` which performs these operations, and returns a 5-by-4 character array with the predicted characters. The ideal output would be 5 rows of `'M', 'A', 'T', 'H'`, but we do not of course expect to be perfect for the highly noisy cases.

In [23]:
```
function predict(testdata, V, W)
    res = ["" for i=1:5, j = 1:4]
    for i = 1:5
        for j = 1:4
            y = round.(σ.(W * σ.(V * testdata[:, (i-1)*4 + j])))
            res[i, j] = (y == [0, 0]) ? "M" : ((y == [0, 1]) ? "A" : ((y
        end
    end
    return res
end
```

Out[23]: predict (generic function with 1 method)

In [24]:
```
plot_chars(testdata)
predict(testdata, V, W)
```



Out[24]:
```
5×4 Matrix{String}:
 "M"  "A"  "T"  "H"
 "M"  "A"  "T"  "H"
 "M"  "A"  "T"  "H"
 "M"  "A"  "T"  "H"
 "M"  "H"  "T"  "H"
```

Problem 4 - True gradient descent using the `Optim` package

Since the `Optim` package can perform automatic differentiation, we can try to solve the optimization problem using true gradient descent. Write a function `train_optim()` which does this and returns V, W like before. Use the solver `GradientDescent()`, and remember to set `autodiff=:forward`.

One tricky thing with this is that `optimize` expects a single input vector x, but we have two matrices V, W. This is a quite common problem, and you can get around it by converting the 420+20=440 numbers in V, W to a vector. In the objective function, you also have to convert back to matrices in order to evaluate the misfit function $E(V, W)$.

Run your function, then predict the output using the resulting network V, W with the code below.

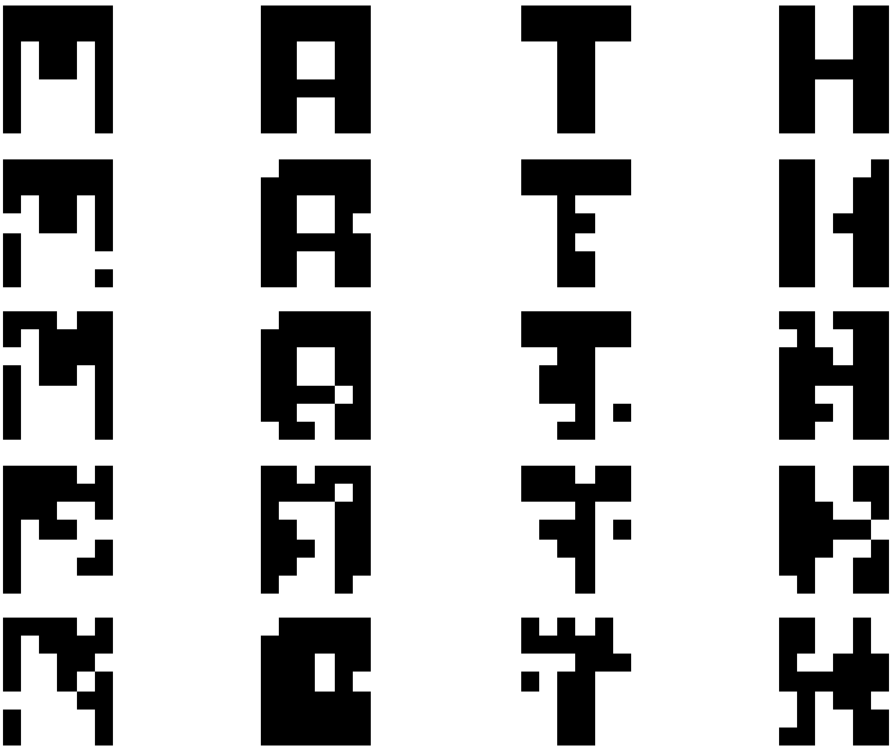
In [47]:
```julia
combine_vectors(V, W)                     = vcat(vec(V), vec(W))
reconstruct_vectors(combined_vector) = (reshape(combined_vector[1:420], 1
                                         reshape(combined_vector[421:end],

function E(w)
    V, W = reconstruct_vectors(w)
    return 0.5 * sum([norm(σ.(W * σ.(V * training[:, j])) - target[:, j])
end
```

Out[47]: E (generic function with 1 method)

In [61]:
```
function train_optim()
    V = randn(10, 42)
    W = randn(2, 10)
    w = combine_vectors(V, W)
    res = optimize(E, w, GradientDescent(); autodiff=:forward)
    return reconstruct_vectors(Optim.minimizer(res))
end
```

Out[61]: train_optim (generic function with 1 method)

In [62]:
```
plot_chars(testdata)
V,W = train_optim()
predict(testdata, V, W)
```



Out[62]:
```
5×4 Matrix{String}:
 "M"  "A"  "T"  "H"
 "M"  "A"  "T"  "H"
 "M"  "A"  "T"  "H"
 "M"  "A"  "T"  "H"
 "M"  "H"  "T"  "H"
```