$\boxed{\text{LTI MPC}}$

$$x(t) = \begin{bmatrix} SOC(t) \\ E_{fuel}(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} P_e(t) \\ P_m(t) \end{bmatrix}$$

$\rightarrow \in [0,1]$

$\rightarrow [J]$

Disturbance $\rightarrow P_{dem}(t) = P_{sh,req}(t)$

Constants $\rightarrow E_{batt,nom} = V_{nom} \cdot Q_{Ah} \cdot 3600 \, [J]$

$\boxed{\eta_e = 0.22 \quad | \quad \eta_m = 0.75}$

Differential Eqⁿs

Battery DE:

$$P_{cell,m}(t) = \frac{P_m(t)}{\eta_m^{\times \eta} \eta_{batt}}$$

$$E_{batt}(t) = SOC(t) \times E_{batt\_nom}$$

$$\frac{dE_{batt}(t)}{dt} = -P_{cell\_m}(t) \Rightarrow E_{batt\_nom} \frac{d(SOC(t))}{dt} = \frac{-P_m(t)}{\eta_m^{\times \eta} \eta_{batt}}$$

$$\dot{SOC}(t) = -\frac{1}{E_{batt\_nom} \eta_m^{\times \eta} \eta_{batt}} \times P_m(t)$$

Fuel DE:

$$P_{fuel\_in}(t) = \frac{P_e(t)}{\eta_e}$$

$\boxed{\text{Assuming fuel conversion efficiency + engine efficiency is lumped!}}$

$$\frac{d(E_{fuel})}{dt} = -P_{fuel\_in}(t) = \frac{-P_e(t)}{\eta_e}$$

$$\dot{E}_{fuel} = \frac{-1}{\eta_e} \times P_e(t)$$

Vector form / Matrix form

$$\dot{x}(t) = \begin{bmatrix} \dot{soc}(t) \\ \dot{E}_{fuel}(t) \end{bmatrix} \qquad x(t) = \begin{bmatrix} soc(t) \\ E_{fuel}(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} P_e(t) \\ P_m(t) \end{bmatrix} \qquad \begin{array}{l} \dot{soc} = -\dfrac{1}{E_{batt,nom}}\dfrac{P_m(t)}{\eta_{batt}} \\[2mm] \dot{E}_{fuel} = \dfrac{-1}{\eta_e} \times P_e(t) \end{array}$$

$$\underbrace{\begin{bmatrix} \dot{soc}(t) \\ \dot{E}_{fuel}(t) \end{bmatrix}}_{\dot{x}(t)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} soc(t) \\ E_{fuel}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -\dfrac{1}{E_{batt,nom}}\dfrac{\times \eta}{\eta_{batt}} \\ -\dfrac{1}{\eta_e} & 0 \end{bmatrix}}_{B} \begin{bmatrix} P_e(t) \\ P_m(t) \end{bmatrix}$$

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} soc(t) \\ E_{fuel}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} P_e(t) \\ P_m(t) \end{bmatrix}$$

$$y(t) = C\,x(t)$$

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$\frac{x_{k+1} - x_k}{T_s} = A x_k + B u_k \;\Rightarrow\; x_{k+1} = x_k + T_s A x_k + T_s B u_k$$

$$\boxed{x_{k+1} = (I_2 + T_s A)x_k + T_s B u_k}$$

$$x_{k+1} = A_d x_k + B_d u_k$$

$$A_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 & \dfrac{-T_s}{E^{\times \eta}_{batt,nom}} \\ \dfrac{-T_s}{\eta_e} & 0 \end{bmatrix}$$

Cost function

$$\underbrace{Q = \begin{bmatrix} q_{soc} & 0 \\ 0 & q_{fuel} \end{bmatrix}}_{\text{output weight}} \qquad \underbrace{R = \begin{bmatrix} r_e & 0 \\ 0 & r_m \end{bmatrix}}_{\text{input wt.}} \qquad \underbrace{S = \begin{bmatrix} S_{soc} & 0 \\ 0 & S_{fuel} \end{bmatrix}}_{\substack{\text{horizon end} \\ \text{S value}}}$$

$$\boxed{L(k) = \underbrace{(y(k) - y_{ref}(k))^\top Q\,(y(k) - y_{ref}(k))}_{\text{error}} + u(k)^\top R\,u(k)}$$

$$y_k = C_d x_k + D u_k$$

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Prediction Step

predicted horizon $\leftarrow h = N_p$

$$X = \begin{bmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+h) \end{bmatrix} \qquad U = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+h-1) \end{bmatrix}$$

$$x_{k+1} = x(k+1) = A_d\, x(k) + B_d\, u(k)$$
$$x_{k+2} = x(k+2) = A_d(A_d x_k + B_d u_k) + B_d u_{k+1}$$
$$x_{k+2} = A_d^2 + A_d B_d u_k + B_d u_{k+1}$$

$$x_{k+i} = A_d^i x_k + \sum_{j=0}^{i-1} A_d^{\,i-j-1} \cdot B_d\, u_{k+j}$$

initial

$$X = \begin{bmatrix} A_d x_k \\ \; \end{bmatrix} + \begin{bmatrix} B_d \, U \end{bmatrix}$$

predicted

$$\underbrace{\begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{k+h} \end{bmatrix}}_{(2h \times 1)} = \underbrace{\begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^h \end{bmatrix}}_{(2h \times 1)} x_k + \underbrace{\begin{bmatrix} B_d & 0 & \cdots & 0 \\ A_d B_d & B_d & & \\ \vdots & & \ddots & \\ A_d^{N-1} B_d & \cdots & & B_d \end{bmatrix}}_{(2h \times 2h)} \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{N-1} \end{bmatrix}}_{(2h \times 1)}$$

$$y_{k+1} = C_d \, x_{k+1}$$
$$y_{k+2} = C_d \, x_{k+2}$$
$$\vdots$$
$$y_{k+N} = C_d \, x_{k+N}$$

$$\left\{ \;\right\} \longrightarrow
\begin{bmatrix} y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+N} \end{bmatrix}_{2h\times 1}
=
\begin{bmatrix} C_d\,\Phi & & 0 \\ \Phi & \Phi & \\ 0 & \cdots & \Phi \end{bmatrix}_{2h\times 2h}
\begin{bmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \\ \vdots \\ x_{k+N} \end{bmatrix}_{2h\times 1}$$

$$\boxed{Y = C_H X + D_H U}$$
$$\boxed{D_H = 0}$$

$$y_{ref,k+i} = \begin{bmatrix} SOC(k) \\ E_{fuel}(k) \end{bmatrix} = \begin{bmatrix} 0.6 \\ E_{fuel}(k) \end{bmatrix}$$

$$SOC(x) < 0.25 \longrightarrow u$$

$$Y_{ref\_h} = \begin{bmatrix} y_{ref}(k+1) \\ y_{ref}(k+2) \\ \vdots \\ y_{ref}(k+h) \end{bmatrix}$$

$$Q_H = \begin{bmatrix} Q & & & & O \\ & Q & & & \\ & & Q & & \\ & & & Q & \\ O & & & & Q \cdots S \end{bmatrix}_{2h\times 2h}
\qquad
R_H = \begin{bmatrix} R & & & & \\ & R & & & \\ & & R & & \\ & & & R & R \cdots \end{bmatrix}$$

$$J = \underbrace{(Y - Y_{ref\_h})^T}_{1\times 2h} \; \underbrace{Q_H}_{2h\times 2h} \underbrace{(Y - Y_{ref\_h})}_{2h\times 1} + \underbrace{U^T}_{1\times 2h} \underbrace{R_H}_{2h\times 2h} \underbrace{U}_{2h\times 1}$$

$$X = A_H x_k + B_H U$$
$$Y = C_H X + D_H U$$

$$\boxed{Y = C_H \cdot X} \Rightarrow \boxed{Y = C_H A_H x_k + C_H B_H U}$$

$$J = (C_H X - Y_{ref,h})^T Q_H (C_H X - Y_{ref\_h}) + U^T R_H U$$
$$J = (C_H X - Y_{ref,h}^T) Q_H (C_H X - Y_{ref\_h}) + U^T R_H U$$

Substitute $X = A_H x_k + B_H U$

$$J =$$

$$J = (Y - Y_{ref,h})^T Q_H (Y - Y_{ref,h}) + U^T R_H U$$

$$J = (C_H A_H x_k + C_H B_H U - Y_{ref,h})^T Q_H (C_H A_H x_k + C_H B_H U - Y_{ref,h}) + U^T R_H U$$

$$J = (x_k^T A_H^T C_H^T + U^T B_H^T C_H^T - Y_{ref,h}^T) Q_H (C_H A_H x_k + C_H B_H U - Y_{ref,h}) + U^T R_H U$$

Assuming $\boxed{E = C_H A_H x_k - Y_{ref,h}}$ & $\boxed{C_H B_H = G}$

↳ Constant for one run of controller.

Substituting E in above eqⁿ:

$$J = (E^T - U^T G^T) Q_H (E - GU) + U^T R_H U$$

$$J = E^T Q_H E - E^T Q G U - U^T G^T Q_H E + U^T G^T Q_H G U + U^T R_H U$$

$$J = E^T Q_H E - E^T Q_H G U + U^T (G^T Q_H G + R_H) U$$

$$J = E^T Q_H E - E^T Q_H G U - U^T G^T Q_H E + U^T (G^T Q_H G + R_H) U$$

As Q_H is a symmetric matrix, $Q_H = Q_H^T$

$$J = E^T Q_H E - E^T Q_H G U - U^T G^T Q_H^T E + U^T (G^T Q_H G + R_H) U$$

$$J = E^T Q_H E - E^T Q_H G U - (E^T Q_H G U)^T + U^T (G^T Q_H G + R_H) U$$

As both are scalars then we can omitt transpose as the values after solving them will be one dimensional (1×1).

$$J = \underbrace{E^T Q_H E}_{Constant} - \underbrace{2 E^T Q_H G U}_{\substack{linear \ or \\ 2(U^T G^T Q_H E)}} + \underbrace{U^T (G^T Q_H G + R_H) U}_{Quadratic}$$

Substituting all values we found:

$$\boxed{J = (C_H A_H x_k - Y_{ref,h})^T Q_H (C_H A_H x_k - Y_{ref,h}) - 2(C_H A_H x_k - Y_{ref,h})^T Q_H \cdot C_H B_H \cdot U \\ + U^T ((C_H B_H)^T \cdot Q_H \cdot (C_H B_H) + R_H) U}$$

$$\boxed{J = a_0 + a_1 U + U^T a_2 U}$$

$$J = \frac{1}{2} a_0 + \frac{1}{2} a_1 U + \frac{1}{2} U^T a_2 U$$

Divide by 2

$$\frac{\partial J}{\partial U} = \frac{1}{2} a_1 + \underbrace{\frac{1}{2} \times 2 \times a_2 U^* = 0}_{} \quad \left\{ \frac{\partial \left(\frac{a_0}{2}\right)}{\partial U} = 0 \right\}$$

constant (Just an offset)

$$a_2 U^* = -\frac{1}{2} a_1$$

$$\boxed{U^* = -\frac{1}{2} a_2^{-1} a_1}$$

We can remove $\boxed{\frac{a_0}{2}}$ from cost function as it is just increasing computational time and processing power.

$$U^* = -\frac{1}{2} a_2^{-1} a_1 \qquad a_1 = -2(C_H A_H x_k - Y_{ref,h}) . Q_H . C_H B_H$$
$$a_2 = (C_H B_H)^T . Q_H . (C_H B_H) + R_H$$

$$U^* = -\frac{1}{2} \times (-2) \times \left[(C_H A_H x_k - Y_{ref,h}) . Q_H . C_H . B_H \right] \times \left[(C_H B_H)^T . Q_H . C_H . B_H + R_H \right]$$

$$U^* = (C_H A_H x_k - Y_{ref,h}) . Q_H . C_H . B_H \times \left[(C_H B_H)^T . Q_H . C_H . B_H + R_H \right]$$

$$[C_H]_{2N \times 2N} = \begin{bmatrix} I_2 & & & \\ & I_2 & & \\ & & I_2 & \\ & & & \ddots \\ & & & & I_2 \end{bmatrix}_{2N \times 2N}$$

$$B_H = \begin{bmatrix} B_d & & & \\ & B_d & & \\ & & B_d & \\ & & & \ddots \\ & & & & B_d \end{bmatrix}_{2N \times 2N}$$

$$A_H = \begin{bmatrix} A_d \\ A_d^2 \\ \vdots \\ A_d^N \end{bmatrix}_{2N \times 2} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

$$U^* = \begin{bmatrix} U_k^* \\ U_{k+1}^* \\ \vdots \\ U_{k+N-1}^* \end{bmatrix}_{2N \times 1}$$

Input $\leftarrow U_k = \begin{bmatrix} P_{e,k} \\ P_{m,k} \end{bmatrix}_{2 \times 1}$

$$x_k = \begin{bmatrix} SO C_k \\ E_{fuel\,K} \end{bmatrix}_{2 \times 1}$$

$$Y_{ref,h} = \begin{bmatrix} y_{ref,K+1} \\ y_{ref,K+2} \\ \vdots \\ y_{ref,K+N} \end{bmatrix}_{2N \times 1}$$

$$R_H = \begin{bmatrix} R & & & \\ & R & & \\ & & R & \\ & & & R \cdots \end{bmatrix}_{2N \times 2N}$$

$$Q_H = \begin{bmatrix} Q & & & & \\ & Q & & & \\ & & Q & & \\ & & & Q & \\ & & & & Q \cdots \end{bmatrix}_{2N \times 2N}$$

$$U^* = \left[(C_H A_H x_k - Y_{ref,h}) . Q_H . C_H . B_H \right]^{-1}$$

$$\left[(C_H B_H)^T . Q_H . C_H . B_H + R_H \right]$$

$$U^* = \left[Q_H . C_H . B_H . (C_H A_H x_k - Y_{ref,h}) \right]$$
$$\left[(C_H B_H)^T . Q_H . C_H . B_H + R_H \right]$$

$$U^* = -\frac{1}{2} \mathcal{E} a_2^{-1} a_1 \longrightarrow \boxed{\begin{aligned} a_1 &= -2(C_H A_H x_k - Y_{ref,h}) \cdot Q_H \cdot C_H \cdot B_H \\ a_2 &= (C_H B_H)^T \cdot Q_H \cdot (C_H \cdot B_H) + R_H \end{aligned}}$$

$$U^* = -\frac{1}{2} \times (-2) \cdot \left[(C_H B_H)^T \cdot Q_H \cdot (C_H \cdot B_H) + R_H \right]^{-1} \cdot \left[(C_H A_H x_k - Y_{ref,h}) \cdot Q_H \cdot C_H \cdot B_H \right]$$

$$\boxed{U^* = \left[(C_H B_H)^T \cdot Q_H \cdot (C_H \cdot B_H) + R_H \right]^{-1} \cdot \left[(C_H A_H x_k - Y_{ref,h}) \cdot Q_H \cdot C_H \cdot B_H \right]}$$

$$\left(\underbrace{(C_H B_H)^T}_{2N \times 2N} \cdot \underbrace{Q_H}_{2N \times 2N} \cdot \underbrace{(C_H \cdot B_H)}_{2N \times 2N} + \underbrace{R_H}_{2N \times 2N} \right)^{-1} \equiv \boxed{2N \times 2N}$$

$$(\underbrace{C_H}_{2N \times 2N} \underbrace{A_H}_{2N \times 2} \underbrace{x_k}_{2 \times 1} - Y_{ref,h})^T \cdot Q_H \cdot C_H \cdot B_H$$

$$\boxed{2N \times 1} \quad \boxed{2N \times 1} \qquad \boxed{2N \times 2N}$$

$$2N \times 1 \qquad 2N \times 2N$$

If I used $\mathcal{E}$ & $:$ $\quad J = \underbrace{\frac{1}{2} E^T Q_H E}_{constant} - \underbrace{\frac{1}{2} U^T G^T Q_M E}_{linear} + \underbrace{\frac{1}{2} U^T (G^T Q_H G + R_H) \cdot U}_{quadratic}$

$$f^T = \left[E^T Q_H G \right]$$

$$\frac{\partial J}{\partial U} = \underbrace{- G^T Q_H^T E}_{f^T} + \underbrace{(G^T Q_H G + R_H) \cdot U}_{H = G^T Q_H G + R_H}$$

as $f = \left[E^T Q_H G \right]$

$$\boxed{f = (C_H A_H x_k - Y_{ref,h})^T \cdot Q_H \cdot (C_H B_H)} \qquad \boxed{H = (C_H B_H)^T \cdot Q_H \cdot C_H \cdot B_H + R_H}$$

$$\frac{\partial J}{\partial U} = 0 \Rightarrow -f^T + H U^* = 0$$

$$\boxed{U^* = H^{-1} f^T}$$

But in control engineering we always do $(Y_{ref,h} - C_H A_H x_k)$ and not $(C_H A_H x_k - Y_{ref,h})$

• Therefore $\boxed{f = (Y_{ref,h} - C_H A_H x_k)^T \cdot Q_H \cdot (C_H \cdot B_H)} \quad \boxed{H = (C_H B_H)^T \cdot Q_H \cdot (C_H B_H)^T + R_H}$

$$\boxed{U^* = -H^{-1} f^T} \text{ Ans}$$

$[H] \rightarrow 2N \times 2N$

$[f] \rightarrow \underbrace{(Y_{stretch} - C_H A_H x_K)}^{T} \cdot \underbrace{Q_H \cdot C_H \cdot B_M}$

$\underbrace{(2N \times 1)^T = 1 \times 2N}_{} \qquad \underbrace{2H \times 2N}_{}$

$\boxed{[f] \rightarrow 1 \times 2N}$

$[f^T] \rightarrow 2N \times 1$

$[H^{-1}] \rightarrow 2N \times 2N$

$\boxed{\begin{array}{l} [H^{-1} f^T] \rightarrow 2N \times 1 \\ [U^*] \rightarrow 2N \times 1 \end{array}} \quad \boxed{\begin{array}{c} \text{Dimensionally} \\ \underline{\text{Satisfied !!}} \end{array}}$