* Matrix :-

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad [m \times n]$$

$a_{ij} \Rightarrow$ i^{th} row & j^{th} column

- A matrix is to be treated as a single entity with number of components rather than considering collection of numbers.
- Unlike a determinant, a matrix cannot be reduced to a single number. Hence, there is no question of finding value of a matrix.
- An interchange of rows & columns does not alter the determinant but gives entirely a different matrix.

[2] Special Matrices :-

(a) Row Matrix :- $\rightarrow$ $\begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix}$
$1 \times n$

(b) Column Matrix $\rightarrow$ $\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}$
$m \times 1$

[3] Square Matrix :-

↳ $n \times n$ if it is $\Rightarrow$ n^{th} order matrix

↳ A determinant is possible only for square matrix - There is a matrix [A] ⇒ determinant

↳ |A| (Δ)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

↳ The diagonal of the matrix containing elements is known as Leading diagonal or Principal diagonal.

↳ The sum of the diagonal elements of the square matrix is called as "Trace" of A. ⇒ $1+5+9 = 15$ ⇒ Trace

↳ A square matrix is said to be "singular" if its determinant is zero. (i.e. $|A| = 0$) ; otherwise the matrix is "non-singular".

[4] Diagonal Matrix :-

↳ A square matrix whose all elements are zero except the leading diagonal ; is said to be diagonal matrix.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \Leftarrow \text{Diagonal matrix.}$$

- If a diagonal matrix having equal elements in leading diagonal, then it is called a "Scalar Matrix".

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[5] Unit / Identity Matrix :-

↳ A diagonal matrix of order n, having 1 as all elements, it is called an Identity matrix or unit matrix. $I_n \Rightarrow [\quad]_{n\times n} \Rightarrow I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Additive Identity

$x + 0 \Rightarrow x$

$x \times 1 = x$

Multiplicative Identity

$[I][A] = [A]$ | $|I| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1\times1) - 0$
$= 1$

[6] Null matrix :

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[7] Symmetric Matrix :-

↳ A square matrix A is said to be symmetric if

$a_{ij} = a_{ji}$ for all i, j.

$a_{12} = a_{21}$

$a_{13} = a_{31}$

For example :
$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

[8] Skew Symmetric Matrix :-

→ A square matrix 'A' is said to be skew-symmetric if $a_{ij} = -a_{ji}$ for all i, j.

$i = j = 1$

$(a_{11} = -a_{11}) \Rightarrow a_{11} - a_{11} = \boxed{0}$

→ All leading diagonal elements are zero.

For example :-

$$\begin{bmatrix} 0 & h & -g \\ -h & 0 & d \\ g & -d & 0 \end{bmatrix}$$

[9] Triangular Matrix :- ↳ Upper ⇒ U, ↳ Lower ⇒ L

↳ A square matrix whose all elements below leading diagonal are zero, is known as Upper Triangular matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

→ Lower Triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow$$

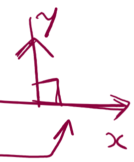
[10] Orthogonal Matrix :

↳ A square matrix A is said to be orthogonal if $AA^T = I$

→ A square ...

orthogonal if $AA^T = I$

$\sin 90^\circ$ $\cos 90^\circ$



$(AA^T = I)$

↳ orthogonal matrix

$(A \cdot A^{-1} = I)$ | if A^{-1} is possible

For orthogonal matrix ⇒ $A^T = A^{-1}$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

↳ All Identity matrix are orthogonal.

$AA^T = I$ ← $(AA^{-1} = I)$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$ ← Is this orthogonal ?

"Yes"

$$A' \Rightarrow A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} I_2$$

* Check that whether the matrix

$$A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$ is orthogonal or not ?

$$A^T = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ $\rightarrow \cos^2\phi + \sin^2\phi$

* 3×3 matrix → Orthogonality?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$ Exercise?

* Matrix Operations:-

(1) Equality of Matrices:

$A = B$ only if all elements of A are same as respective element of B

$a_{ij} = b_{ij}$

(2) Addition & subtraction of matrices:-

→ Possible for same size of matrix

$A + B = C$ | $C_{ij} = a_{ij} + b_{ij}$

$$A \begin{array}{c} a_{11} \\ \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ 3\times 2 \end{array} + \begin{array}{c} b_{11} \\ \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} \\ 3\times 2 \end{array} = \begin{array}{c} c_{11} \\ \begin{bmatrix} a_1+c_1 & b_1+d_1 \\ a_2+c_2 & b_2+d_2 \\ a_3+c_3 & b_3+d_3 \end{bmatrix} \\ 3\times 2 \end{array}$$

Similarly to have $A - B$;

$$D = \underset{(A-B)}{} \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \\ a_3 - c_3 & b_3 - d_3 \end{bmatrix}_{3\times 2}$$

$$A = \begin{bmatrix} a & a^2-1 \\ b & b^2-1 \\ c & c^2-1 \end{bmatrix} \Rightarrow \begin{bmatrix} a & a^2 \\ b & b^2 \\ c & c^2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} a & a^2 \\ b & b^2 \\ c & c^2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

NOTE:

(1) Same size (same order) is required

(2) Addition is commutative:

$$A+B = B+A$$

subtraction is not

$$A-B \neq B-A$$

(3) Addition & subtraction of matrices are associative:

$$\underbrace{(A+B)\boxed{-C}} = A + \underbrace{(B\boxed{-C})} = B + (A\boxed{-C})$$

(4) Multiplication of a Matrix by a scalar:-

$$kA = k\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} ka_1 & kb_1 \\ ka_2 & kb_2 \\ ka_3 & kb_3 \end{bmatrix}$$

:- Distributive law

$$k(A+B) = kA + kB$$

[4] Matrix Multiplication :-

↳ if number of columns of first matches with number of rows of second matrix / Matrices are conformable

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \Rightarrow \underset{m \times p}{C}$$

$\underline{m = p}$

↳ Condition

$$\underset{n \times p}{B} \cdot \underset{m \times n}{A} \Rightarrow$$

$$\underset{\substack{2 \times 3 \\ m \times n}}{A} \quad \underset{\substack{3 \times 2 \\ n \times p}}{B} \Rightarrow \underset{\substack{m \times p \\ 2 \times 2}}{C}$$

$$\underset{\substack{n \times p \\ 3 \times 2}}{B} \quad \underset{\substack{m \times n \\ 2 \times 3}}{A} \Rightarrow \underset{\substack{n \times n \\ 3 \times 3}}{D}$$

Conformability is to be satisfied

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \quad \underset{p \times q}{C} \Rightarrow \underset{m \times q}{D}$$

$$\underbrace{\underset{m\times n}{A}\ \underset{n\times p}{B}}\ \underset{p\times r}{C} \quad\quad \underset{m\times r}{D}$$

$$\underbrace{\underset{m\times p}{E}\ \underset{p\times r}{C}} \Rightarrow \underset{m\times r}{D}$$

NOTE :-

(1) Multiplication is associative (provided the conformability is satisfied) i.e.

$$A(BC) = (AB)C$$

(2) Multiplication is distributive over Addition & subtraction

$$A(B+C) = AB + AC$$

(3) For a square matrix, powers of matrix $A^2, A^3, A^4 \ldots$

if $\underbrace{A^2 = A}$, then the matrix A is called Idempotent.

$$\underset{2\times 3}{\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} \underbrace{2a+3c+4e} & 2b+3d+4f \\ 5a+6c+7e & 5b+6d+7f \end{bmatrix}}$$

Ex :- Prove that $A^3 - 4A^2 - 3A + 11I = O$

where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

A^2

A^3

Ex 2 If $A = \begin{bmatrix} 3 & 2 & 4 \\ -3 & 1 & 5 \\ 1 & 8 & 6 \end{bmatrix}$; find a matrix B such that

$$AB = \begin{bmatrix} 39 & 48 & 57 \\ 36 & 39 & 42 \\ 75 & 90 & 105 \end{bmatrix}$$

*Condition :- No use of Inverses.

Solution : A B

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \text{Solve equations}$$

a

3x3

$a = 1$	$b = 2$	$c = 3$
$d = 4$	$e = 5$	$f = 6$
$g = 7$	$h = 8$	$i = 9$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \_\_\_\_\_\_$$

Prob. 2 If $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ find matrix B

such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$

$a = 1 \quad b = 0 \quad c = 0$

$d = 0 \quad e = 2 \quad f = 0$

$g = 0 \quad h = 0 \quad i = 1$

$$\begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$$

$\times 3 \quad \times 2 \quad \times 1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 4 & 2 \\ 3 & 6 & 1 \\ 15 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 5 & 9 \\ 4 & 2 & 1 \end{bmatrix} \underline{B} = \begin{bmatrix} 1 & 8 & 6 \\ 1/3 & 10 & 27 \\ 4/3 & 4 & 3 \end{bmatrix}$$

$\times 1/3 \quad \times 2 \quad \times 3$

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Ex:- Matrix A has x rows & $x+5$ columns & Matrix B has y rows & $11-y$ columns. If AB & BA both exists, then find x & y.

Solution:-

$$\underset{(x \times (x+5))}{A} \quad \underset{(y \times (11-y))}{B} \Rightarrow B \quad A$$

$$\left.\begin{array}{l} x+5 = y \\ x = 11-y \end{array}\right\} \Rightarrow \begin{array}{l} x = 3 \\ y = 8 \end{array}$$

Ex. 4 If $A+B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ & $A-B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$

find AB.

$$AB = \begin{bmatrix} -2 & -2 \\ 0 & -6 \end{bmatrix} = \boxed{-2\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} ; \quad B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

Ex. 8 Prove that product of two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta \cdot \sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix when θ & ϕ differ by an odd multiple of $\pi/2$. $\quad \theta = 0 ; \ \phi = \pi/2$

$$A \cdot B \quad \begin{bmatrix} \overset{0}{\cos^2\theta \cos^2\phi} + \cos\theta \overset{0}{\sin\theta} \cdot \overset{0}{\cos\phi} \sin\phi \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

* Related Matrices:-

(1) Transpose of a Matrix:-

→ changing rows into columns.

$A' ; A^T$

$(A')' = A$

→ For a symmetric matrix, $A' = A$ 4

for a skew-symmetric matrix; $A' = -A$.

NOTE:-

(1) The transpose of a product of the two matrices, is the product of their transposes taken in reverse order.

$$\underbrace{(AB)^T = B^T A^T}, \quad \Big| \quad (AB)' = B'A'$$

$$\text{LHS} \Rightarrow \left(\underset{m\times n}{A} \quad \underset{n\times p}{B}\right) = \overset{C}{(m \times p)}$$

$$\text{RHS} \Rightarrow \underset{n\times m}{A^T} \quad \underset{p\times n}{B^T} = B^T . A^T = \underset{(p\times m)}{C^T}$$

[2] Every square matrix can be uniquely expressed as a sum of a symmetric and a skew symmetric matrices.

Prob. 1 For a square matrix A;

$$A = \frac{1}{2}\underbrace{(A + A')} + \frac{1}{2}\underbrace{(A - A')}$$

Proof:- Let us $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A + A' \Rightarrow$ Symmetric

$A - A' \Rightarrow$ Skew-symmetric

[3] Adjoint of a Matrix :-

↳ " The adjoint of a matrix A is a Transposed matrix of co-factors of A".

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \Rightarrow \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

co factor of $\quad \overset{a_{ij}}{a_1} = (-1)^{i+j} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

$$A_1 = (b_2 c_3 - b_3 c_2)$$

$$A_2 = (-1)^{i+j}$$

$\vdots$

C_3

$$\text{Co-factor Matrix} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \Rightarrow \text{Transpose it; results in}$$

$$\text{Adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

Ex. 1 Find the Adj of a matrix A; where

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 1 & 4 \\ 7 & 2 & 9 \end{bmatrix}$$

$$\text{Adj. } A = \begin{bmatrix} 1 & -41 & 18 \\ 19 & 13 & -10 \\ -5 & 29 & -2 \end{bmatrix}$$

[4] Inverse of a Matrix :-

↳ If A be any matrix, then a matrix B, "if it exists", such that $AB = BA = I$; then B is known as inverse of A. $B = A^{-1}$

$$AA^{-1} = I$$

$$A^{-1} = \frac{Adj.\ A}{|A|} \quad ; |A| \neq 0$$

$|A| = 0$; Inverse matrix is not possible; hence matrix A will be "Singular" matrix.

$\boxed{|A| \neq 0}$; Inverse matrix is possible, hence matrix A is a "Non singular" matrix

Q.2 Find inverse of a matrix A; where

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow |A| = 1$$

Solution:-

$$A^{-1} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$|A| = 1$

$A_1 = (-3 + 4) = 1$ ✓

$A_1 = (-1)(2 - 0) = -2$