# * Rank of a Matrix :-

Def. A matrix is said to be of rank $r$ when,

(i) it has at least one non-zero minor of order $r$

and (ii) every minor of order higher than $r$ is (zero). vanishes.

$$\begin{bmatrix} & \\ & \end{bmatrix}_{n\times n} \Rightarrow \begin{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \\ 0\ 0\ 0\ 0 \cdots 0 \end{bmatrix}$$

$(n-1)\times(n-1)$

$(n-1)\times n$

$r \neq n$

↳ All non zero, unique elements even after the operations, then it has rank $= \underline{n}$ [square]

possible ranks $\Rightarrow \underline{n}, (n-1), (n-2), \ldots, 1, \boxed{0}$ → NULL Matrix

Rectangular $\begin{bmatrix} & \\ & \end{bmatrix}_{m\times n} \Rightarrow$ max. rank can be minimum of either $m$ or $n$.

$A = \begin{bmatrix} & \\ & \end{bmatrix}_{3\times 4}$

$\rho(A)\Big|_{max} = 3$

$\rho(A) = \underline{3, 2, 1}, \boxed{0}$

# * Rank :-

↳ It gives us knowledge of no. of independent variables in the system (no. of independent vectors)

↳ Duplication can be identified

↳ Dimensionality - ( Independent directions)
↳ utilized to represent the data

↳ No of unique solution or it has infinite solutions

↳

Rank :— " Gives us linearly independent column vectors that can be used to construct all of the other column vectors".

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}_{[2\times 2]} \xrightarrow{C_2 - 2C_1} \sim \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 1$$

$n = 2$ ; 2, 1, 0

↳ $\rho(A) = 1$

$1 \leftarrow$

Raw operations

$$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \Rightarrow$$

⇒ Elementary Transformations of a Matrix :—

(1) The interchange of raws (columns)

$R_{ij} \Rightarrow \underbrace{R_i \Leftrightarrow R_j}$

$C_{ij} \Rightarrow C_i \Leftrightarrow C_j$

(2) The multiplication of any raw (column) by a non zero number

$R_i \leftarrow k R_i$ ; k is real No.

$C_i \leftarrow pC_i$

(3) The addition of a constant multiple of the element with respective elements

of other row (column)

$R_i \leftarrow R_i + k R_j$ $\quad \left( \begin{array}{l} -k \\ R_i \leftarrow (R_i \mp k R_j) \end{array} \right)$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 4R_1} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow (-1) R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\Downarrow R_1 \leftarrow R_1 - 2R_2$$

$A \sim B$ $\qquad \rho(A) = 2 \Leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 3 \\ 6 & 8 \end{bmatrix} \Rightarrow$$

$\rho(A) = 2 \Leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$

$\rho(A) = 2$ $\quad \overset{R_2 \leftarrow \frac{1}{10} R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

## * Methods to find Rank of a Matrix :-

(1) To evaluate minors and decide rank

(2) To perform elementary transformations

### (1) Rank of a Matrix :-

A matrix is said to have rank $r$ when

(i) it has at least one non-zero minor of order $r$

4 (ii) every minor of order higher than $r$ vanishes.

$[\ \ ]_{n \times n} \Rightarrow (n-1)(n-2) \cdots 1$

$4 \times 4 \Rightarrow 0$

$\boxed{4} \rightarrow \boxed{3 \times 3}$ $\ 2 \times 2$ $\ \boxed{1}$

$\neq 0 \Rightarrow \rho(A) = 3$

all $3 \times 3 = 0$

$4\times 4 \Rightarrow 0$ ... $\hookrightarrow$ all $3\times 3 = 0$

any one (a) $(2\times 2) \neq 0 \Rightarrow \rho(A) = 2$

**Ex. 1** Find rank of a Matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}_{\boxed{2}\times 3}$

**Solution:-** [Method of minors] $\Rightarrow$ $\rho(A)_{max} \Rightarrow 2$

$[R_2 \leftarrow R_2 - 2R_1]$ $A \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} \Rightarrow 6-6=0$

$\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} \Rightarrow 8-8=0$

$\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} \Rightarrow 24-24=0$

$\Rightarrow \rho(A) \neq 2$ $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 1$

**Ex. 2** Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

**Solution:-**

$\begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix} = 0$, $\begin{vmatrix} 1 & 4 & 5 \\ 1 & 6 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$, $\begin{vmatrix} 1 & 3 & 5 \\ 1 & 2 & 7 \\ 1 & 5 & 1 \end{vmatrix} = 0$

& $\begin{vmatrix} 3 & 4 & 5 \\ 2 & 6 & 7 \\ 5 & 0 & 1 \end{vmatrix} = 0$

$\rho(A) \neq 3$

order 2 $\Rightarrow$ $\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1 \neq 0$ hence $\boxed{\rho(A) = 2.}$

**Ex 3** Find the rank of matrix (using method of minors)

Ex 3

$$B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \Rightarrow \rho(B) = \underline{\underline{3}}$$

---

$6\times 6 \Rightarrow$ more large size

To find a rank using method of minors, will be difficult & time consuming.

## [2] Elemetary Transformation :-

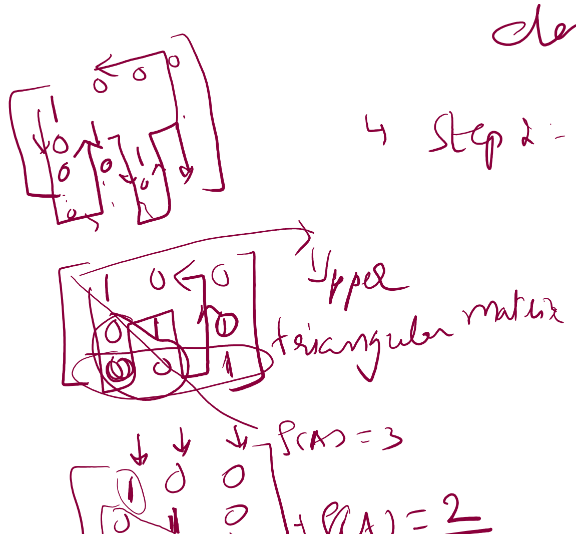
↳ only Row transformations throughout the problem, Elementary row transformation

↳ only Column transformations, throughout, it is known elementary column transformations.

⇒ Finding rank using elementary row transformations

↳ Step 1 :- arrange the matrix such that element $a_{11}$ becomes 1. (non zero)

↳ Step 2 :- Then utilize it to make all the other row elements of that column to zero.



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = \underline{2}$$

**Ex. 2** Find the rank of the matrix using elementary transformations :

$\boxed{\rho(A) = 2}$

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

**Solution :**

$R_2 \leftarrow R_2 - R_1$ ; $R_3 \leftarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) \neq 3$$

$R_2 \leftarrow (-1) \cdot R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I & \end{bmatrix}$$

$C_3 \leftarrow C_3 + 2C_2$ ; $C_4 \leftarrow C_4 + 2C_2$

$$\sim \begin{bmatrix} 1 & 3 & 10 & 11 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 - 3C_1$ ; $C_3 - 10C_1$ ; $C_4 - 11C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftarrow \text{Normal Form}$$

$$\begin{bmatrix} I_2 & 0 \\ \end{bmatrix} \quad \rho(A) = 2$$

$$\sim \begin{bmatrix} I_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\rho(A) = 2}$$

$\hookrightarrow$ Normal Form:

$\begin{bmatrix} & \\ & \end{bmatrix}_{4\times 4}$

$\rho(A) = 4, 3, 2, 1$

$|4\times 4| \neq 0$

$\rho(A) = 4$

else if $|4\times4| = 0$

& $|3\times 3| \neq 0$

$\rho(A) = 3$

else if $|4\times 4| = 0$

$|3\times 3| = 0$ & $|2\times 2| \neq 0$

then $\rho(A) = 2$

**Ex. 1** Find rank of matrix using elementary transformations

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

**Solution:-**

$\rho(A) = 2$

**Ex.**

$$\begin{bmatrix} 11 & 22 & 33 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

**Solution:-**

$R_1 \leftarrow \frac{1}{11} R_1$ ; $\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$\underline{\rho(A) = 3}$

$R_2 \leftarrow R_2 - 2R_1$ ; $R_3 \leftarrow R_3 - 3R_1$ ; $R_4 \leftarrow R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_4$ ;

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & +3 & -2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\underline{I_2}}$$

$\rho(A) \neq 4$

$C_2 \leftarrow C_2 - 2C_1$, $C_3 \leftarrow C_3 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \text{RRE Form} \\ \text{Raw reduced} \\ \text{Echelon Matrix} \end{bmatrix}$$

$\underline{\rho(A) = 3}$

* **Ex. B** Find the rank of the matrix using elementary transformations:

$$\begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$$

$$\begin{array}{c} \\ 0 \\ 0 \\ 0 \end{array}\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

[operations = 10]

Jinay

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ans :- $\rho(A) = 2$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} I_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* Nitish

$$\sim \begin{bmatrix} 1 & \cancel{3} & 4 & \cancel{5} \\ 1 & \cancel{4} & 5 & \cancel{6} \\ 1 & 5 & \cancel{6} & 7 \\ 1 & 10 & 11 & \cancel{12} \end{bmatrix} \times$$

* **Normal Form of a Matrix :-**

Every non zero matrix A of rank r, can be reduced by a sequence of elementary transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ which is called a Normal Form of a Matrix.

where; $I_r$ is the identity matrix of order r.

* Find the normal form of the matrix and hence rank of it :

(1) $$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

$\rho(A) = 3$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ \dots \end{bmatrix}$$

(2)

$$B = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution of Prob. 2

$R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 - R_1$; $R_4 \leftarrow R_4 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_4$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$C_2 \leftarrow C_2 - 2C_1$; $C_3 \leftarrow C_3 + C_1$; $C_4 \leftarrow C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$R_4 \leftarrow R_4 + 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_4\left(\frac{4}{5}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$(-1)C_4 \leftrightarrow C_2$ ; $\left(\cancel{R_4 \leftarrow \frac{R_4}{5}}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$\left(\frac{1}{5}\right) R_4 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} I_2 \Rightarrow & 0 \\ 0 & 0 \end{bmatrix}$$

* Derive the normal form of the matrix & hence find out the rank of the matrix.

Q. 25

Text Book

$$A = \begin{array}{c} 1 \longrightarrow \\ \left[\begin{array}{cccc} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{array}\right] \end{array} \Rightarrow \rho(A) = 3$$

Solution:-

**Theorem:-** Pre multiplying & Post-multiplying the matrix A
(Raw operations) (column operations)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}$$

$(R_1 - R_3)$

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \\ R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2+0-1 & 3+0-3 & 5+0-4 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 2R_2 - 3R_3 \longrightarrow \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}$$

$E_2$ $E_1$ A

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}}_{R_1 + 2R_2 - 3R_1} = \Big[$$

$E_2$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+2\times4-3\times1 & 0+2-3\times3 & 1+2\times9-3\times4 \\ 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 9 \\ 1 & 3 & 4 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 2R_1 + 3R_3 \Rightarrow -2R_1 + R_2 + 3R_3$$

$E_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftarrow R_3 + 3R_1 - 7R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -7 & 1 \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} R \end{bmatrix}$$

$E_4$ ← Left Hand Side

Pre multiplying

$$\underbrace{\begin{bmatrix} E_4 \end{bmatrix}\begin{bmatrix} E_3 \end{bmatrix}\begin{bmatrix} E_2 \end{bmatrix}\begin{bmatrix} E_1 \end{bmatrix}}_{\text{Elementary Raw operation matrix}}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} R \end{bmatrix}$$

$$\begin{bmatrix} E \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -7 & 1 \end{bmatrix}}_{E_4}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}$$

Ans.

$$\begin{bmatrix} 1 & 2 & -4 \\ -2 & -3 & 11 \\ 17 & 27 & \cancel{48} \; -98 \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

E.R.M.

Normal Form → $[P][A][\;]$

Normal Form $\rightarrow$ $\underset{E.R.M.}{[P]}[A][Q]$

---

$\rightarrow$ **Column Operations:**

$$\begin{array}{c} A \\ \begin{bmatrix} 2 & 3 & 5 \\ 1 & \cancel{2} & 9 \\ 4 & 6 & 8 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \\ 5 \end{array} \Rightarrow$$

$$\begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 7 \\ 4 & 6 & 2 \end{bmatrix}$$

$C_3 \leftarrow \underbrace{C_3 - {}^{1}_{\wedge}C_2}$

$$\underset{A}{\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 9 \\ 4 & 6 & 8 \end{bmatrix}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\longrightarrow$ RHS

$C_2 \leftarrow C_2 - 2C_1 + 3C_3$

$\leftarrow -2C_1 + C_2 + 3C_3$

$$\begin{bmatrix} 2 & 3 & -3+5 \\ 1 & 2 & -2+9 \\ 4 & 6 & -6+8 \end{bmatrix} \begin{array}{l} \rightarrow 2 \\ \rightarrow 7 \\ \rightarrow 2 \end{array}$$

$$\underset{A}{\begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 7 \\ 4 & 6 & 2 \end{bmatrix}}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix}\underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}\underset{E_2}{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}}\underset{E}{\begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}} = \begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix}$$

Row operation Left

Right column operations

Row

$\rightarrow$ Column

ope... operations

$\rightarrow$ $\underline{P}\underline{A}\underline{Q}$ (Row $\uparrow P$, $Q \rightarrow$ column)

**Ex. 1** For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Find non singular matrices P & Q such that $\underline{PAQ}$ is in the normal form. Hence find the rank.

Row: $\mathrm{I} \leftarrow A \rightarrow \mathrm{I}$ : column

**Solution:-**

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A = P A Q

$(C_2 - C_1 ;\ C_3 - 2C_1)$

P & Q are not unique.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftarrow (C_3 - C_2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_2$

$[1 \quad -1 \quad -1$ ...

$R_3 \leftarrow R_3 + R_2$

$(x, y), z$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 2$

← Like a solution on

$z \to (x, y)$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I_2 & [0]_{2\times 1} \\ [0]_{1\times 2} & 0 \end{bmatrix}$$

* **Q.2** Find non singular matrices P & Q such that PAQ is in the normal form for the matrices:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} & P & A & Q \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 3/4 & 1 & -1/4 \\ 3/4 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 1/2 & 1 & -1/2 \end{bmatrix} [A] \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$