* Infinite Series :-

Common difference | follows specific patterns

→ Sequences :- Arithmetic Progression. [Natural numbers] $\mathbb{N}_0$

↳

→ An ordered set of numbers is called a sequence.

$$\{u_n\} = \{u_1, u_2, u_3, \ldots\ u_n\}$$

$u_n \to n^{th}$ term of the sequence

$$u_n = \frac{1}{n} \Rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}\}$$

→ Bounded sequence :-

- A sequence $\{u_n\}$ is said to be lower bounded if there exists a number k such that $\underline{k} \le u_n$ for all n.
- A sequence $\{u_n\}$ is said to be upper bounded $k \geq u_n$ ($u_n \le k$)
- If a sequence is lower & as well upper bounded, its known as a "Bounded sequence"

* Infinite series :-

- An infinite series is formed by sum of all terms of a sequence.

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

- Partial sum $(S_n) = u_1 + u_2 + u_3 + \ldots + u_n$

$n = 100000$

case

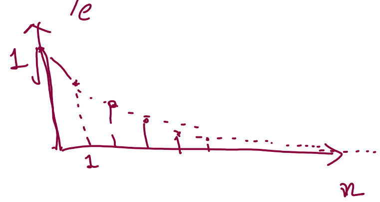
(1) If $\lim\limits_{n\to\infty} S_n = \underline{k}$; The series is convergent. [k is a finite number]

(2) If $\lim\limits_{n\to\infty} S_n = \pm\infty$; The series is divergent.

(3) If $\lim\limits_{n\to\infty} S_n =$ more than one finite value ; Oscillatory series

e/e^n $\quad e$

case (1) $\quad \frac{e}{e^n} \Rightarrow \frac{1}{2.718}$



$$\left[1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} \ldots\right] \to \text{Convergent}$$

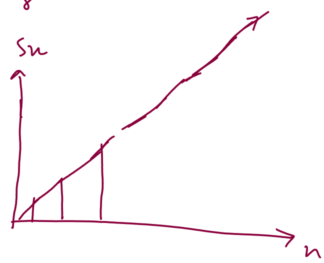
∞

$\left[1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\cdots\right] \to$ Convergent $\quad n \quad \infty$

Case-(2) $\quad 1+2+3+4+5\cdots+\infty$

$\lim_{n\to\infty} S_n = \frac{n(n+1)}{2} = +\infty$

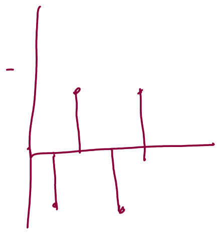
Divergent.



Case-(3) $\quad \lim_{n\to\infty} (-1)^n = -1+1-1+1-1+1\cdots$

$n=$ even $\Rightarrow S_n = 1$

$=$ odd $\Rightarrow S_n = -1$



$\sin n\theta$

$\cos n\theta$

$\times$ For a Geometric series : $1+r+r^2+r^3+\cdots+r^n+\cdots$

Case I $\quad S_n = 1+r+r^2+\cdots+r^{n-1}$

$|r|<1$ $\quad\quad |r|<1.$

$$S_n = \frac{1-r^n}{1-r}$$

$\lim_{n\to\infty} S_n = \frac{1}{1-r}$; Convergent.

Case-2 $|r|>1$:

$\lim_{n\to\infty} S_n = \frac{r^n-1}{r-1} \quad r^n \to \infty$

$=\pm\infty$; Divergent

Case 3 $\quad r=1$: $\quad 1+(1)^2+(1)^3+(1)^4+\cdots$

$= 1+1+1+\cdots$ to ∞

$= \infty$ as $n\to\infty$; Divergent

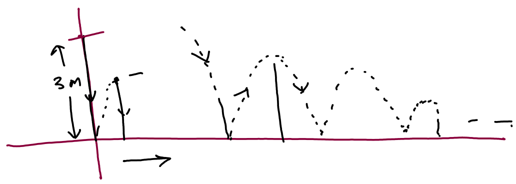
Case-4 : $r=-1$; $\quad 1+(-1)^1+(-1)^2+(-1)^3+(-1)^4\cdots$

$= 1-1+1-1$

results in either 0 or 1.

Oscillatory in nature

Ex. 1 A ball is dropped from a height h, Each time the ball hits the ground, it rebounds a distance r times the distance from where it falls, If h = 3 meters & r = 2/3; find the total vertical distance travelled by the ball till steady state.



(1) $rh = \frac{2}{3} \times 3 = 2$

(2) $= \frac{2}{3} \times 2 = \frac{4}{3}$

$r \cdot rh$

(3) $r \cdot (r^2 h) - - -$

$= h + 2rh + 2r \cdot rh + 2r \cdot r \cdot rh + \cdots$

$= h + 2rh + 2r^2h + 2r^3h + \cdots$

$= [2h + 2rh + 2r^2h + 2r^3h + \cdots] - h$

$= 2h[1 + r + r^2 + r^3 + \cdots] - h$

$|r| < 1$

$= 2h\left[\frac{1}{1-r}\right] - h$

$h = 3 \Rightarrow 15m;$ $\quad = 2 \times 3\left[\frac{1}{1 - 2/3}\right] - 3 = 15$ meters

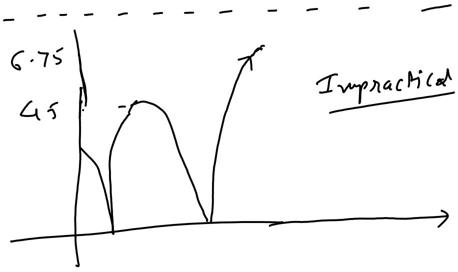
$\rightarrow h = 5; \quad h = 10;$

dist = 25m; dist = 50 m

$r = 1/3$; $h = 3, 5 \,\&\, 10.$

$\hookrightarrow$ 6m; 10m; 20m

$\left[r = 3/2 > 1\right]$



Impractical

Ex. 2 Express the repeating decimal 6.333333 - - - - as the ratio of two integers.

Solution:

$6 + 0.333\ldots = 6 + \frac{1}{3} = \frac{19}{3}$

$6 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

$= 6 + 3(0.1)^1 + 3(0.1)^2 + 3(0.1)^3 + \cdots$; $r = 0.1$

$= 6 + \frac{3}{10}\left[1 + 0.1 + (0.1)^2 + (0.1)^3 + \cdots\right]$

$= 6 + \frac{3}{10}\left(\frac{1}{1-0.1}\right) = 6 + \frac{3}{10} \times 10/9 = 6 + \frac{1}{3} = \frac{19}{3}$

E.3 5.23 23 23 2323 - - - - To be represented as ratio of two integers.

$= \frac{518}{99}$

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* General Properties of Behaviour of Series:-

(1) The behaviour of infinite series remains unaffected by the addition or removal of a finite number of terms.

(2) The behaviour of infinite series remains unaffected if each term is multiplied by a finite non zero constant.

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* A necessary condition for convergence :-

Theorem: If $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$

Suppose that $\sum_{n=1}^{\infty} u_n$ is convergent as given,

Partial sum S_n ; then $u_n = S_n - S_{n-1}$

↳ $(1 + + - - -) u_n$ $\lim u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$\hookrightarrow (1 + + - - -) u_n$ $\qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$

$s_{n-1}(- - - - u_{n-1})$

$$\boxed{\lim_{n\to\infty} u_n \neq 0 \Rightarrow \text{Divergent.}}$$

$= k - k$

$= 0$

Observations:-

Consider the series

$n \to \infty$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + - - - + \cdots 0$$

Partial sum $\Rightarrow (n \to \infty)$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \text{Divergent.}$$

$$\geq \left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + - - - + \frac{1}{\sqrt{n}}\right]$$

n $\Rightarrow [\quad]_n$

$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\lim_{n\to\infty} = \infty \to \text{divergent}$$

Case (2)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$$

Harmonic series:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Nicole Oresme

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) - -$$

$$\left(\frac{1}{2}\right)^n$$

$$= \underline{1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)} + - - -$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + - - -$$

$\Rightarrow \infty$; Divergent.

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* Positive Term Series:-

" An infinite series in which all terms after some particular term are positive, that series is called Positive Term series."

$$-3 - 2 \; -1 \; \overrightarrow{+2 + 4 + 6 - \; . \; -}$$

- Such series can be either convergent or divergent but can not be oscillatory.

Eg. 1
$$\sum \frac{n}{\sqrt{n^2+1}-1} \quad ; \quad \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}-1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\sqrt{1+1/n^2}-1/n\right)}$$

$$= \frac{1}{\sqrt{1}-0} = 1 \neq 0$$

The series is divergent.

Eg. 2
$$2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots$$

$$u_n = \frac{n+1}{n}$$

$\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$; Series is divergent.

* Integral Test:-

" A positive term series $f(1)+f(2)+f(3)+\cdots+f(n)$ where $f(n)$ decreases as n increases is convergent or divergent according to the integral Test i.e. The integral

$\int_1^\infty f(x)\,dx$ is finite $\to$ convergent

E(M)

$\int_1^\infty f(x)\,dx$ is infinite $\to$ divergent."

P-Series:-

Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \infty$ is convergent if $p>1$ otherwise divergent. $(p \le 1)$

Proof:- The proof will be given using integrations

$$f(n) = \frac{1}{n^p} \Rightarrow f(x) = \frac{1}{x^p}$$

$$I = \int_1^\infty \frac{dx}{x^p}$$

$$= \lim_{u\to\infty} \int^u x^{-p}\,dx$$

$$= \lim_{u\to\infty} \int_1^u x^{-p}\, dx$$

$$= \lim_{u\to\infty} \left[\frac{x^{1-p}}{1-p} \right]_1^u$$

$$= \lim_{u\to\infty} \frac{1}{1-p} \left[\frac{1}{x^{p-1}} \right]_1^u$$

$$= \lim_{u\to\infty} \frac{1}{1-p} \left[\frac{1}{u^{p-1}} - 1 \right]$$

$= \frac{1}{p-1}$ = finite → Leads to convergence of series. if $p > 1$

$\lim_{u\to\infty} \frac{1}{1-p} \left[\frac{1}{u^{p-1}} - 1 \right] = \infty$ if $p < 1$

↳ Leads to divergence of series.

$I = \int_1^\infty \frac{dx}{(x)^1} = [\log x]_1^\infty = \infty$ = Divergence of series.

$p > 1$; Convergent
$p \leq 1$; Divergent.

} Important result from Integral Test of P-Series.

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* <u>Comparision Test</u> :-

Let $\sum u_n$ & $\sum v_n$ are two positive term series, then:

(1) If the series $\sum v_n$ is convergent, then the series $\sum u_n$ is also convergent provided $u_n \leq v_n$ for all values of n or for all values of $n > m$.

(2) If the series $\sum v_n$ is divergent, and $u_n > v_n$ ~~for~~ all values of $n > m$; then $\sum u_n$

is also divergent.

(3) If $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = k \neq 0$; then both the series Σu_n & Σv_n either converge or diverge.

Test the convergence of following series.

E.1

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots$$

Solution:-

$u_n = \frac{1}{n(n+1)}$; So lets take $v_n = \frac{1}{n^2}$

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n(n+1)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)}$$

$= 1 \neq 0$; The both series either converge or diverge.

$\Sigma v_n = \Sigma \frac{1}{n^2}$ = Compared with p-series,

$p = 2 > 1$; hence

Σv_n is convergent. Hence Σu_n is also convergent.

E.2

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(2n-1)}{n(n+1)(n+2)} = \frac{2}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}$$

$$\sum_{n=1}^{\infty} \frac{n(2-1/n)}{n^3(1+1/n)(1+2/n)} = \sum_{n=1}^{\infty} \frac{(2-1/n)}{n^2(1+1/n)(1+2/n)}$$

$\Sigma v_n = \frac{1}{n^2}$

$\therefore$ the $\lim \frac{(2-1/n)}{\qquad} = 2 \neq 0$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2 - 1/n)}{(1+1/n)(1+2/n)} = 2 \neq 0$$

Hence either both converge or both diverge.

$\Sigma v_n = \Sigma \frac{1}{n^2}$; comparing with p series

$p = 2 > 1$;

Hence Σv_n is convergent.

So Σu_n is also convergent.

Ex 3

(i) $u_n = \sqrt{\dfrac{3^n - 1}{2^n + 1}}$

Divergent.

Solution :-

$$u_n = \left(\frac{3}{2}\right)^{n/2} \sqrt{\frac{1 - 1/3^n}{1 + 1/2^n}}$$

$$\Sigma v_n = \left(\frac{3}{2}\right)^{n/2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(3/2)^{n/2} \sqrt{\dfrac{1 - 1/3^n}{1 + 1/2^n}}}{(3/2)^{n/2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1 - 1/3^n}{1 + 1/2^n}}$$

$$= 1 \neq 0$$

- Hence either both converge or both diverge.

$\Sigma v_n = \Sigma \left(\frac{3}{2}\right)^{n/2} \Rightarrow$ GP series with $r > 1$

So it is divergent.

(ii) $\dfrac{\sqrt{2} - 1}{3^3 - 1} + \dfrac{\sqrt{3} - 1}{4^3 - 1} + \dfrac{\sqrt{4} - 1}{5^3 - 1} + \cdots \infty$

Convergent.

$$u_n = \frac{\sqrt{n+1} - 1}{(n+2)^3 - 1}$$

$$= \frac{\sqrt{n}\left[\sqrt{1 + 1/n} - 1/\sqrt{n}\right]}{n^3\left[(1 + 2/n)^3 - 1/n^3\right]}$$

$$v_n = \frac{1}{n^{5/2}}$$

Hence Σu_n is also divergent.

Ex. 4 $\sum_{n=1}^{\infty} \sin(1/n) \Rightarrow \lim_{n\to\infty} \sin(1/n) = 0 \to$ Zero Test fails.

↳ whether it is convergent or divergent.

$\sum_{n=1}^{\infty} \cos(1/n) \Rightarrow \lim_{n\to\infty} \cos(1/n) = \cos(0)$

$= 1 \neq 0$

Hence $\cos(1/n)$ is divergent.

$\sum_{n=1}^{\infty} \sin(1/n)$

$$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right]$$

$$\left[\sin x = \sqrt{1-\cos^2 x}\right]$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{3!}\cdot\left(\frac{1}{n}\right)^3 + \frac{1}{5!}\left(1/n\right)^5 - \frac{1}{7!}\cdot\left(1/n\right)^7 + \cdots\right]$$

$$\Sigma u_n = \sum_{n=1}^{\infty} \frac{1}{n}\left[1 - \frac{1}{3!}\left(\frac{1}{n}\right)^2 + \frac{1}{5!}\left(1/n\right)^4 - \frac{1}{7!}\left(1/n\right)^6 + \cdots\right]$$

$\underline{\Sigma v_n = 1/n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left[1 - \frac{1}{3!}\left(\frac{1}{n}\right)^2 + \frac{1}{5!}\left(1/n\right)^4 - \frac{1}{7!}\left(1/n\right)^6 + \cdots\right]$$

$= 1 \neq 0$

Hence either both converge or both diverge.

but $\Sigma v_n = \frac{1}{n}$ is divergent by p-series test.

Hence $\Sigma u_n = \Sigma \sin(1/n)$ is divergent.

17/06/21

(1) $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$

↳ Convergent

(2) $\sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{1}{n}\right)$ zero test $= 0 \Rightarrow$ fails

↳ Divergent

$v_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n}$

$= \lim_{n\to\infty} \cos(1/n)$

$= 1 \neq 0$

Divergent.

- Comparison Test requires a reference series whose nature can be known easily. (convergence / Divergent)
- In some cases, if the power of n is same in both numerator & denominator, then also it becomes difficult to apply comparision test.

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* <u>D' Alembert's Ratio Test</u>: -

Let $\sum u_n$ be a positive series; & u_n be the n^{th} term, u_{n+1} be the $(n+1)^{th}$ term, then after a finite term; if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k \neq 0$ then

(1) if $k > 1$; the series is divergent.

(2) if $k < 1$; the series is convergent

(3) if $k = 1$; ratio test fails hence need to check with other methods.

$\Rightarrow$ If the series $\sum u_n$; has all terms ratio after a term m; that results in ratio less than 1. So ignoring first m terms & for rest of the terms, we rewrite the series as

$u_1 + u_2 + u_3 + \cdots + \infty$

$\frac{u_2}{u_1} < 1$; $\frac{u_3}{u_2} < 1$; $\frac{u_4}{u_3} < 1$ - - - -

$\frac{u_2}{u_1} < 1$

$\frac{u_2}{u_1} < 1$, $\frac{\ }{u_1} < 1$, $\frac{\ }{u_2} < 1$, $\frac{\ }{u_3} < 1$

$\frac{u_2}{u_1} \le r$; $= r$ $= r$

$u_2 < u_1$

$\frac{u_3}{u_2} < 1$ $\quad u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \cdots \right]$

$u_3 < u_2 < u_1$

$\frac{2}{3} < 1 = 0.66$ $= u_1 \left[1 + r + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \frac{u_4}{u_3} \cdot \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \cdots \right]$

$u_1 = 3$; $u_2 = 2$; $u_3 = 1$

$\frac{u_3}{u_2} = \frac{1}{2} = 0.5$ $= u_1 \left[1 + r + r^2 + r^3 + \cdots \right]$ GP series ..

$(r < 1)$

$= \frac{u_1}{1-r}$ which will be a finite value.

Hence the

Ex. 1 Test the convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

Solution: Here ; $u_n = \frac{2^n}{n^2}$

$$u_{n+1} = \frac{2^{n+1}}{(n+1)^2}$$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}$$

$$= \lim_{n \to \infty} \frac{2}{(1 + 1/n)^2}$$

$$= 2 > 1$$

Hence the series is divergent.

Ex. 2 $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ $\Rightarrow$ Convergent.

Ex. 3 $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \frac{4}{1+2^4} + \cdots$

$\Rightarrow$ Convergent.

Ex. 4 $x \quad x^2 \quad x^3 + x^4 + \cdots$

Ex. 4 $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \cdots$

Solution:

Here, $u_n = \frac{x^n}{n(n+1)}$

$u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

Now, Applying Ratio Test, we have,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{\cancel{(n+1)}(n+2)} \cdot \frac{n\cancel{(n+1)}}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{(1+2/n)}$$

$$= x \quad ; \text{ if } x>1; \text{ divergent}$$

if $x<1$; convergent.

But if $x=1$; Ratio Test fails.

So, let us rewrite the series with $x=1$;

$$\sum u_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$$

$\Rightarrow \sum u_n = \frac{1}{n(n+1)}$ | $x=1$;

$v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2(1+1/n)}$$

$= 1 \neq 0$ Hence either both converge or both diverge. But $\sum v_n = \sum 1/n^2$ is convergent as per p-series, hence $\sum u_n$ is convergent if $x=1$.

* Test the convergence of the series (i) $\sum 1/n$ (ii) $\sum 1/\sqrt{n}$ (iii) $\sum 1/n^2$

using ratio test.

Integral Test

$\sum 1/n \Rightarrow$ ratio test $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1 \Rightarrow$ Divergent

$\sum 1/\sqrt{n} \Rightarrow$ ratio test $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1 \Rightarrow$ Divergent

$\sum 1/n^2 \Rightarrow$ ratio test $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = 1 \Rightarrow$ Convergent

$\Rightarrow$ Let us check, for some of the terms of the given series for observations: -

$\dfrac{u_{n+1}}{u_n} \Downarrow$	$\sum 1/n$	$\sum 1/\sqrt{n}$	$\sum 1/n^2$	$\sum \dfrac{2^n}{n!}$	$\sum \dfrac{n!\,2^n}{n^n}$
$\dfrac{u_5}{u_4}$	0.8 ↓	0.8944... ↓	0.64 ↓	↓ 0.4	↓ 0.8192
$\dfrac{u_6}{u_5}$	0.83333...	0.9128..	0.69444...	0.3333	0.8037..
$\dfrac{u_7}{u_6}$	0.857....	0.9257..	0.734..	0.2857	0.793139..
$\dfrac{u_8}{u_7}$	0.875...	0.9354....	0.7656...	0.25	0.785

Ex Test the convergence of the series

$$\sum \frac{n!\,2^n}{n^n}$$

Solution:

$$u_n = \frac{n!\,2^n}{n^n}\ ; \quad u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$$

Applying ratio test,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\,2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\,\cancel{n!}\,\cancel{2^n}\cdot 2}{\cancel{(n+1)}\,(n+1)^n} \cdot \frac{n^n}{\cancel{n!}\,\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{2}{(1+1/n)^n} \longrightarrow$$ L' Hospital's Rule $\hookrightarrow e$

$$= \frac{2}{e} < 1 \Rightarrow \text{Convergent.}$$

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* Test the convergence of the series:

(1) $$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

(2) $$2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \cdots$$

(1) $\Rightarrow$ $x^2 < 1 \Rightarrow$ Convergent $\left[u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \right]$

$x^2 > 1 \Rightarrow$ Divergent

$x^2 = 1 \Rightarrow$ Convergent.

(2) $x < 1 \Rightarrow$ Convergent $\quad u_n = x^{n-1} \cdot (1 + 1/n)$

$x > 1 \Rightarrow$ Divergent $\quad = x^{n-1}\left(\frac{n+1}{n}\right)$

$x = 1 \Rightarrow$ Divergent

Ex. 3

$$\frac{1}{} + \frac{2!}{4} + \frac{3!}{27} + \frac{4!}{256} + \frac{5!}{3125} + \cdots$$

$$u_n = \left\{ \frac{n!}{n^n} \right\}$$