* Gauss-Jordan Method:-

↳ Method to find inverses.

The elementary row transformations which reduce a given square matrix A to a unit matrix, when applied to a unit matrix I, results in inverse of A.

$$\underbrace{R_i \cdot R_{i-1} \cdots R_3 R_2 R_1}\, A \to I$$

$$R_i \cdot R_{i-1} \cdots R_3 \cdot R_2 R_1 \underbrace{A \cdot A^{-1}} = I\, A^{-1}$$

$$R_i \cdot R_{i-1} \cdots R_3 R_2 R_1 \cdot I = A^{-1}$$

→ $[A \,\vdots\, I]$ ← Apply same row transformations on both A & I.

- when A results in (I), $I \Rightarrow$ results in A^{-1}!

* Caution: Only row transformations are to be used.

Ex. 1

Using Gauss-Jordan method; find inverse of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution:

$$\begin{array}{cccccc} & A & & & I & \\ \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \end{array}\right.$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$[I] \qquad [A^{-1}]$$

$R_2 \leftarrow R_2 - R_1$; $R_3 \leftarrow R_3 + 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$$A^{-1} \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$R_2 \leftarrow 1/2\, R_2$

$R_3 \leftarrow 1/2\, R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_2$;

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

$R_2 \leftarrow R_2 + 3R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/2 & 1/2 & 0 \\ 0 & -2 & 0 & 5/2 & 1/2 & 3/2 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & -2 & 0 & 5/2 & 1/2 & 3/2 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

$R_2 \leftarrow \frac{R_2}{-2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

$R_2 \leftarrow \frac{1}{-2} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$\underbrace{}_{I}$ $\quad\quad \underline{\underline{A^{-1}}}$

$$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Ex. 2

$$A = \begin{bmatrix} 7 & 6 & 2 \\ -1 & 2 & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 7 & 6 & 2 & 1 & 0 & 0 \\ -1 & 2 & 4 & 0 & 1 & 0 \\ 3 & 6 & 8 & 0 & 0 & 1 \end{array}\right]$$

~~$R_1 \leftrightarrow R_2$~~

$R_1 \leftarrow R_1 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -14 & 1 & 0 & -2 \\ -1 & 2 & 4 & 0 & 1 & 0 \\ 3 & 6 & 8 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 + R_1$; $R_3 \leftarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -14 & 1 & 0 & -2 \end{array}\right.$$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -14 & 1 & 0 & -2 \\ 0 & -4 & -10 & 1 & 1 & -2 \\ 0 & 24 & 50 & -3 & 0 & 7 \end{array}\right]$$

$$A^{-1} \Rightarrow \begin{bmatrix} -1/5 & -9/10 & 1/2 \\ 1/2 & 5/4 & -3/4 \\ -3/10 & -3/5 & 1/2 \end{bmatrix}$$

* Find A^{-1} using Gauss-Jordan Method.

Also, find two non-singular matrix P & Q such that $PAQ = I$.

$$\begin{array}{cc} P & Q \\ \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow A^{-1} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

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* Solution of Linear System of Equations:

"Cramer's rule"

→ Matrix Inversion Method:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\begin{array}{ccc} A & X & = D \\ \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \end{array}$$

$$A^{-1} A X = A^{-1} D$$

$$IX = A^{-1} D$$

$$X = A^{-1}D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_x \\ b_y \\ c_z \end{bmatrix} \Rightarrow \begin{array}{l} x = a_x \\ y = b_y \\ z = c_z \end{array}$$

<u>Ex. 1</u> Solve the equations $3x + y + 2z = 3$; $2x - 3y - z = -3$; $x + 2y + z = 4$ using matrix inversion method.

<u>Solution</u>

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{D}{\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}}$$

Co-efficient matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$\Rightarrow$ Using Gauss - Jordan Method to solve it :-

$$\begin{array}{ccc|ccc} & A & & & I & \end{array}$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\rightarrow$ $R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & -1 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\rightarrow$ $R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & -1 & 0 \\ 0 & -11 & -7 & -2 & 3 & 0 \\ 0 & -2 & -2 & -1 & 1 & 1 \end{array}\right]$$

- $R_2 \leftarrow R_2 + (-6) R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & -1 & 0 \\ 0 & 1 & 5 & 4 & -3 & -6 \\ 0 & -2 & -2 & -1 & 1 & 1 \end{array}\right]$$

- $R_1 \leftarrow R_1 - 4R_2$; $R_3 \leftarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -17 & -17 & 11 & 24 \\ 0 & 1 & 5 & 4 & -3 & -6 \\ 0 & 0 & 8 & 7 & -5 & -11 \end{array}\right]$$

- $R_3 \leftarrow \frac{1}{8} R_3$;

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -17 & -17 & 11 & 24 \\ 0 & 1 & 5 & 4 & -3 & -6 \\ 0 & 0 & 1 & 7/8 & -5/8 & -11/8 \end{array}\right]$$

$\rightarrow$ $R_1 \leftarrow R_1 + 17 R_3$; $R_2 \leftarrow R_2 - 5R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 + \frac{17\times 7}{8} & 11 - \frac{17\times 5}{8} & 24 - \frac{17}{8} \\ 0 & 1 & 0 & 4 - 5\times 7 & -3 + 5\times 5 & -6 + 5 \end{array}\right.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17+\frac{17\times 7}{8} & 11-\frac{[illegible]}{8} & -[illegible]\ \frac{[illegible]}{8} \\ 0 & 1 & 0 & 4-\frac{5\times 7}{8} & -3+\frac{5\times \cancel{7}5}{8} & -6+5/8 \\ 0 & 0 & 1 & 7/8 & -5/8 & -11/8 \end{array}\right]$$

$$\Rightarrow \begin{array}{cc} I & A^{-1} \\ \multicolumn{2}{c}{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & 3/8 & 5/8 \\ 0 & 1 & 0 & -3/8 & 1/8 & 7/8 \\ 0 & 0 & 1 & 7/8 & -5/8 & -11/8 \end{array}\right]} \end{array}$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$X = A^{-1}D \Rightarrow \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\underline{x = 1}$; $\underline{y = 2}$; $\underline{z = -1}$