* Determinants :--

→ Possible for square matrices.

n^{th} order → $n \times n$ elements

→ Co factors ⇒ $(-1)^{i+j} A_{ij}$

* Properties of Determinants :-

[1] A determinant remains unchanged by changing its rows into columns and columns into rows (i.e. taking transpose)

Proof :-

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ [Expanding by R_1]

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2)$$

$$+ c_1(a_2b_3 - a_3b_2)$$

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ [Again expanding by R_1

$$= \underline{a_1(b_2c_3 - b_3c_2)} - a_2(b_1c_3 - b_3c_1)$$

$$+ a_3(b_1c_2 - b_2c_1)$$

=

$\underline{\Delta' = \Delta}$

[2] If two parallel lines (row or column) of a determinant are interchanged, the determinant retains its numerical value, but changes its sign.

Solution:-

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (\text{Expand by } R_1)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$\underline{\Delta_1 = -\Delta}$!

$$\Delta_1 = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \quad (\text{Expand by } R_1)$$

$\Delta_1 = -\Delta \qquad (-1)^1 \;;\; (-1)^2 \Rightarrow \Delta_1 = \Delta$

→ Consequently, if a line of Δ is passed over two parallel lines or more;

$\underline{\Delta_1 = (-1)^m \cdot \Delta}$; where m = no. of lines passed over

[3] A determinant vanishes if two parallel lines are identical.

[4] If each element of a line is multiplied by the same factor, the whole determinant is multiplied by that factor.

$$\begin{vmatrix} a_1 & x b_1 & c_1 \\ a_2 & x b_2 & c_2 \\ a_3 & x b_3 & c_3 \end{vmatrix} \Rightarrow x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$x \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad \begin{bmatrix} x a_1 & x b_1 \\ x a_2 & x b_2 \end{bmatrix}$$

→ Expanding by R_1;

$$LHS = a_1(x b_2 c_3 - x b_3 c_2) - x b_1 (a_2 c_3 - a_3 c_2)$$
$$+ c_1 (a_2 \cdot x b_3 - a_3 \cdot x b_2)$$
$$= x \left[a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2) \right]$$

$$= x \cdot \Delta = RHS.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ y a_2 & y b_2 & y c_2 \\ z a_3 & z b_3 & z c_3 \end{vmatrix} \Rightarrow (y \cdot z) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

NOTE :- If two parallel lines are such that the elements of one line are equi-multiple of elements of other, then the determinant vanishes.

$c_1 = p b_1$; $c_2 = p b_2$; $c_3 = p b_3$

$$\begin{vmatrix} a_1 & b_1 & p b_1 \\ a_2 & b_2 & p b_2 \\ \cdots \end{vmatrix} \Rightarrow p \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ \cdots & b_3 & b_3 \end{vmatrix} \rightarrow \because C_2 \equiv C_3 ; \; \underline{\Delta = 0.}$$

$$\begin{vmatrix} a_1 & \cdot & \cdot \\ a_2 & b_2 & pb_2 \\ a_3 & b_3 & pb_3 \end{vmatrix} \Rightarrow p\underbrace{\begin{vmatrix} \cdot & \cdot & \cdot \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix}} \rightarrow 0.$$

$$\begin{vmatrix} 1 & 2 & 6 \\ 3 & 4 & 12 \\ 5 & 7 & 21 \end{vmatrix} \qquad C_3 = 3 \times C_2 \qquad \begin{vmatrix} 0 & 0 & 0 \\ & & \\ & & \end{vmatrix}$$

$$\rightarrow 3\begin{vmatrix} 1 & 2 & 2 \\ 3 & 4 & 4 \\ 5 & 7 & 7 \end{vmatrix} = 0$$

[5] If in a line; there are m terms, the determinant can be expressed as the sum of m determinants :-

$m = 3$ terms

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 + d_1 - e_1 \\ a_2 & b_2 & c_2 + d_2 - e_2 \\ a_3 & b_3 & c_3 + d_3 - e_3 \end{vmatrix}$$

$m = 3$ terms

$= \Delta_1 + \Delta_2 + \Delta_3$

$$= \underbrace{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}_{\Delta_1} + \underbrace{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}_{\Delta_2} + \underbrace{\begin{vmatrix} a_1 & b_1 & -e_1 \\ a_2 & b_2 & -e_2 \\ a_3 & b_3 & -e_3 \end{vmatrix}}_{\Delta_3}$$

or $- \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$

Proof :-

Example :- If $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$; in which; a, b & c are different, then ... that $abc = 1$

show that $abc = 1$

Solution:- Splitting column 3 to result in two diff. determinants; we have;

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \quad (-1)^2 \Delta$$

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (abc-1) \underbrace{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}}_{\neq 0} = 0 \qquad \boxed{a \neq b \neq c}$$

Hence. $abc - 1 = 0$

$\therefore \boxed{abc = 1}$ Hence proved.

Prob. 2 Ex 2.1

If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$; then without expansion prove that

$xyz = -1$; where x, y & z are different

[6] If the line is added with multiplying a factor; to other line; (or substracted); the determinant remains unchanged.

to other ...

determinant remains unchanged.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \xrightarrow{\text{operate } C_1 - x.C_2} \Delta_1 = \begin{vmatrix} a_1 - xb_1 & b_1 & c_1 \\ a_2 - xb_2 & b_2 & c_2 \\ a_3 - xb_3 & b_3 & c_3 \end{vmatrix}$$

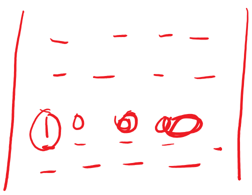
$\Delta_1 = \Delta$

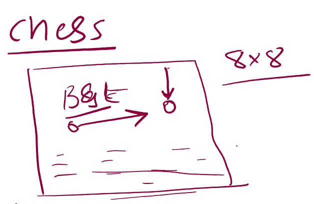
$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \underbrace{\begin{vmatrix} xb_1 & b_1 & c_1 \\ xb_2 & b_2 & c_2 \\ xb_3 & b_3 & c_3 \end{vmatrix}}_{\Delta_2 = 0}$$

$$\boxed{\Delta_1 = \Delta}$$

Ex. 1 Evaluate

$$\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix} = 0$$





a[3][3]

Getting zeros in a row or column so as expansion becomes simple.

If any row/column has all zeros, then $\Delta = 0$.

Solution:- Operating $R_1 - R_2 - R_4$ $(R_1 - (R_2 + R_4))$

$$\begin{vmatrix} -8 & -12 & 0 & -2 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Now operating $R_2 - 3R_3$

$$\begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

Now operating $R_3 - 2R_4$

$$\begin{array}{c} R_1 \rightarrow \\ \\ R_3 \rightarrow \\ \\ \end{array} \begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

$\hookrightarrow 1(-1)^{4+3}$

$R_1 \leftarrow (R_1 - (2R_3))$

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 5 & 7 & 1 & 2 \end{vmatrix} = 0$$

Ex. 2 Evaluate $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$.

$$\boxed{\Delta = 1}$$

Ex. 3 Solve the equation

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \end{vmatrix} = 0$$

Solve the equation

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0 ;$$

$\boxed{x = 9.}$

Solution:- Operating

$x = -1, -2$

$x = -1, -1, -2.$

$(x+1)(x+2)(x+1) = 0$; $(x+1)^2(x+2) = 0$

$x_1 = -1$
$x_2 = -1$
$x_3 = -2$

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Ex. 3.

Solve the equation $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0$

Solution:- $x = 0$; $x = -\frac{1}{2}$

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[7] Factor Theorem:-

If the elements of a determinant Δ are the function of x and two parallel lines become identical when $x = a$; then $(x-a)$ is a factor of Δ.

Let $\Delta = f(x)$; $x = a \Rightarrow \Delta = 0$; $f(a) = 0$

$x = a$ $(x-a)$

NOTE :- If k parallel lines of a determinant Δ become identical when $x=a$; then $(x-a)^{k-1}$ is a factor of Δ.

Ex. 1 Factorize

$$\Delta = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

Solution:- without Factorization Theorem :-

$$\Delta = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

$$= (a-b)(c-b)(d-b)(d-a)(c-a)(c-d)$$

Hint :- Performing $R_1 - R_2$; $R_2 - R_3$; $R_3 - R_4$;

we have

$$\begin{vmatrix} a^3-b^3 & a^2-b^2 & a-b & 0 \\ b^3-c^3 & b^2-c^2 & b-c & 0 \\ c^3-d^3 & c^2-d^2 & c-d & 0 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

$$= (a-b)(b-c)(c-d) \begin{vmatrix} a^2+ab+b^2 & a+b & 1 & 0 \\ b^2+bc+c^2 & b+c & 1 & 0 \\ c^2+cd+d^2 & c+d & 1 & 0 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

$*$

$$\begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$ fourth order

$a = b \Rightarrow R_1 \equiv R_2$; hence $(a-b)$ is a factor

$b = c \Rightarrow R_2 \equiv R_3$; " $(b-c)$ is a factor

$$\Delta = \underline{k}\, \underbrace{(a-b)(b-c)(c-d)(a-c)(a-d)(b-d)}$$

$k a^3 b^2 c = a^3 b^2 c \Rightarrow \boxed{k = 1}$

$\Delta =$

$a - b = 0$

$a = b$

$b - a = 0$

$b = a$

Ex 2.1

Prob. 1 Prove without expanding, that;

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$$ vanishes.

Solution:-

If we take $a = b$; then

$R_1 \equiv R_2$; hence $(a-b)$ is a factor.

Similarly we can find;

$b = c$; $R_2 \equiv R_3$ & when

$a = c$; $R_1 \equiv R_3$ hence $(b-c)$ & $(a-c)$

$a=b$; $b=c$; $a=c \Rightarrow a=b=c$ hence $\Delta = 0$

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[8] Multiplication of Determinants :-

- The product of two determinants of the same order is itself a determinant of that order.

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \& \quad \Delta_2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Then

$$\Delta_3 = \Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix}$$

Ex.:

Evaluate
$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$$

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ - & & \\ d_{31} & - & d_{33} \end{vmatrix}$$