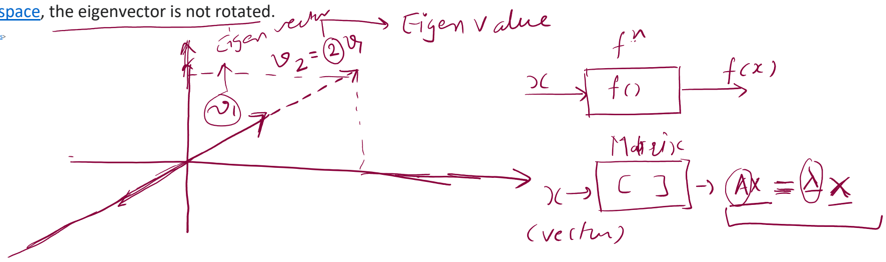
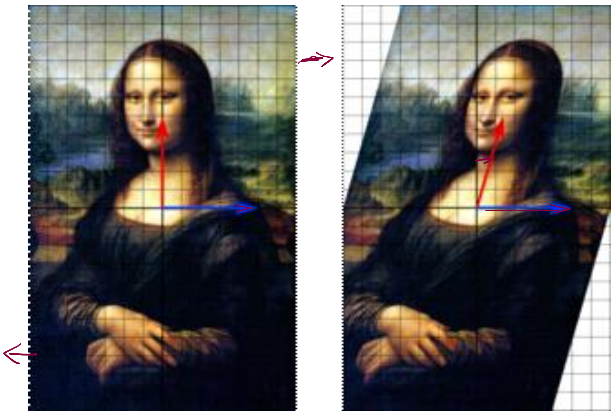
* Wikipedia :-

In linear algebra, an **eigenvector** (/ˈaɪɡən ˌvɛktər/) or **characteristic vector** of a linear transformation is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding **eigenvalue**, often denoted by ,[1] is the factor by which the eigenvector is scaled.

Geometrically, an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction in which it is stretched by the transformation and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed.[2] Loosely speaking, in a multidimensional vector space, the eigenvector is not rotated.

From <https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors>





$\lambda = 1$

" Eigen vector is non-zero vector that chages at most by a scalar factor when a linear transformation is applied on it."

" Eigen values are the factors by which the eigen vector is scaled". ($\lambda \neq 0$)

$$Ax = \lambda x$$
↳ (non zero)

$$Ax - \lambda x = 0$$

$$[A - \lambda I]\, x = 0 \quad \text{- - - (1)}$$
↳ $x \neq 0$

$$|A - \lambda I| = 0 \quad \text{- - - (2)}$$
↳ Singular

Eq.(2) Characteristic Equation, or Eigen value equation. ⇒ By Solving it, we will get λ. (eigen values)

$\lambda_1, \lambda_2 \Rightarrow \underbrace{X_1, X_2}$

↳ Eigen vectors

- Every $\underset{(n\times n)}{n^{th}}$ order matrix, has n eigen vectors. → values → vector

⇒ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow$ $\lambda_1 = 1$, $\lambda_2 = -1$, 0

$A X_1 = \lambda X_1$ → $\lambda = 1$

$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow A X \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow A X_2 \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underset{\lambda_2}{(-1)}\underset{X_2}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$\lambda_2 = (-1)$

- Sum of Eigen values is equal to the Trace of the square matrix.

Trace → [sum of principal diagonal elements]

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

X_1: $A X_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_1 = 4$

X_2

$\begin{bmatrix} \lambda_1 = 1 \;+\; \lambda_2 = -1 \\ \;\;+3 \qquad\quad +3 \\ \lambda_1 = 4 \;+\; \lambda_2 = 2 \end{bmatrix}$ $= 0$, $= 6$

$A X_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2.$

$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$A_2 = [A_1 + \underset{(\lambda)}{3} I] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$A_2 \rightarrow [A_1 + 3I]$

$A_2 X = \lambda_2 X$

$\boxed{[A_1 + 3I]\,X = (\lambda_1 + 3)\cdot X}$

↳ Vector direction remains same.

E.1 Find eigenvalues & eigen vectors for the given matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

given matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ 1

$\det [A - \lambda I] \Rightarrow \quad \lambda I \Rightarrow \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A - \lambda I \quad = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Rightarrow \lambda^n$

$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$(\lambda - 3)^2 = 0$

↳ repeated roots

$\det [A - \lambda I] = (3-\lambda)^2 - 1$

$= 9 - 6\lambda + \lambda^2 - 1$

$= \lambda^2 - 6\lambda + 8$

$= (\lambda - 4)(\lambda - 2) = 0$

$\lambda_1 = 4 \;;\; \lambda_2 = 2$ } Eigen values

→ Eigen Vectors :-

(1) For $\lambda_1 = 4$:- $\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$[A - \lambda I] X = 0$

$\begin{bmatrix} 3-\lambda_1 & 1 \\ 1 & 3-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 = 0$

$\therefore x_1 = x_2$

$\frac{x_1}{1} = \frac{x_2}{1} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ - - (a)

X_1

(2) For $\lambda_2 = 2$:-

$$\begin{bmatrix} 3-\lambda_2 & 1 \\ 1 & 3-\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} +1 & 1 \\ 1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \quad \text{--- (a)}$$

$$\therefore x_1 = -x_2 \qquad 3x_1 = -4x_2$$

$$\therefore \frac{x_1}{-4} = \frac{x_2}{3} \qquad \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\lambda_2 = 2$$

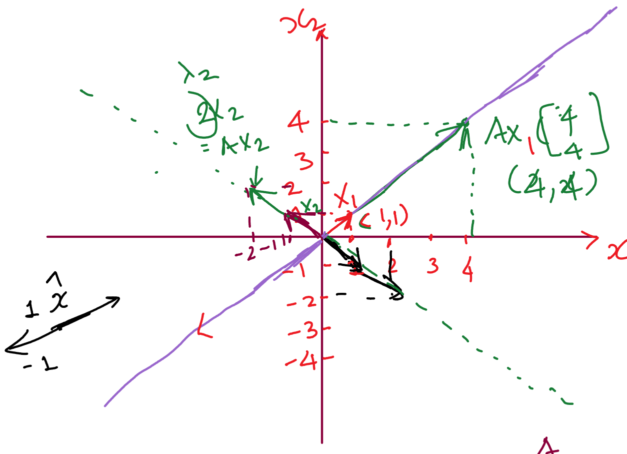
$$\frac{x_1}{1} = \frac{x_2}{-1} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \frac{x_1}{-1} = \frac{x_2}{1}$$

$$X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For λ_1 & $x_1 \Rightarrow$ $\underbrace{AX_1}_{} = \underbrace{\lambda_1 X_1}_{} \Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow Ax$

$$\lambda_1 x_1 \quad 4 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow \lambda x$$



$$X_1 \longrightarrow \boxed{A} \longrightarrow AX_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow$ Eigen $\underline{\text{Vector}}$

(Direction)

$$5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

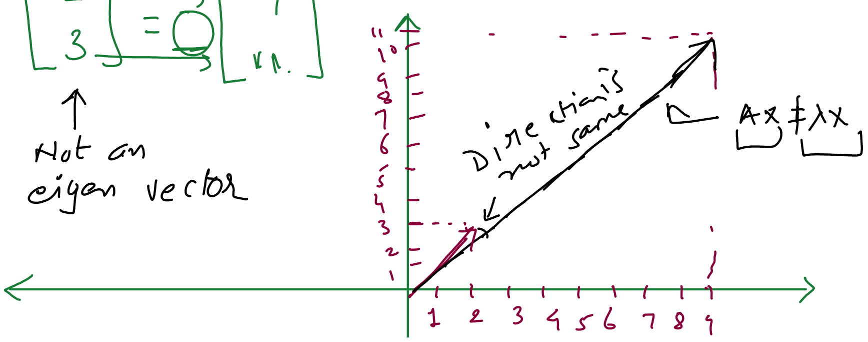
For λ_2 & $X_2 \Rightarrow$

$$\begin{array}{cc} A & X_2 \\ \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\begin{array}{cc} \lambda_2 & X_2 \\ 2 & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

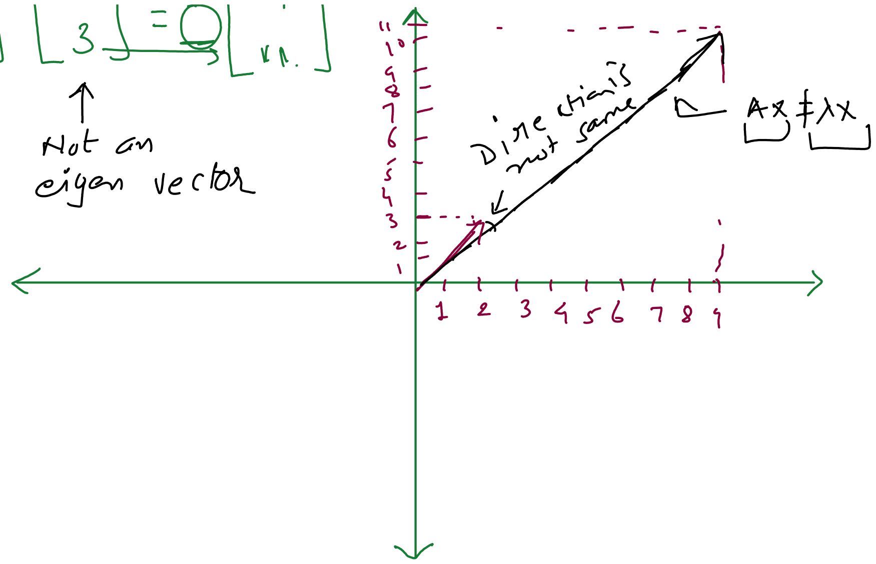
$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$$

Not an eigen vector



$[1 \quad 3] \; [3] = 0 \; [v_1]$

Not an eigen vector



* Ex. 2 Find the eigen values & corresponding eigen vectors for the given matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Solution:

$\lambda_1 = 5 \; ; \; \lambda_2 = -2 \quad \Big| \quad \lambda = 5, -2 \; ;$

$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; ; \; X_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Ex. 3

$A = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} \; ; \; \lambda = 2, -1$

$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow$ Transform matrix $\Rightarrow \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$

$[A - \lambda I]$

$|A - \lambda I| = 0$

(Det. = 0)

* Properties of Eigen Values :-

1) Any square matrix A and its transpose A' have the same eigen values.

$$A, A' \Rightarrow \underline{|A| = |A'|}$$

$$|(A-\lambda I)'| \Rightarrow (A-\lambda I)' \Rightarrow (A' - \lambda I') \quad (\text{But } I' = I)$$

$$(A' - \lambda I)$$

$$(A-\lambda I)' = (A' - \lambda I)$$

$$|(A-\lambda I)'| = |A' - \lambda I|$$

$$|(A-\lambda I)'| = |A - \lambda I| \qquad |A - \lambda I| = \underbrace{|A' - \lambda I|}_{} = 0 \quad \text{--- (1)}$$

$$|A - \lambda I| = 0$$

2) "The eigen values of a triangular matrix are just the diagonal elements of that matrix."

$$A = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ 0 & a_{22} & - & - & a_{2n} \\ 0 & 0 & a_{33} & - & a_{3n} \\ \vdots & & & & \\ 0 & 0 & - & 0 & a_{nn} \end{bmatrix} \Rightarrow A_U$$

square matrix $m = n$

$$\lambda \Rightarrow a_{11}, a_{22}, a_{33} \;--\; a_{nn}$$

Proof :-

$$A_U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$[A_U - I\lambda] = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

$$|A_U - \lambda I| = (a_{11}-\lambda)\left[(a_{22}-\lambda)\cdot(a_{33}-\lambda) - 0\right]$$

$$- a_{12}(0-0) + a_{13}(0-0)$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$$

$$\lambda_1 = a_{11}\ ;\ \lambda_2 = a_{22}\ ;\ \lambda_3 = a_{33}$$

$$\lambda_1+\lambda_2+\lambda_3 = a_{11}+a_{22}+a_{33} \Rightarrow \underline{\text{Trace}}$$

4) The sum of eigen values of a matrix is equal to its Trace: → (Addition of the diagonal elements)

Cosider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$(A-\lambda I) = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (a_{11}-\lambda)\begin{vmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$- a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22}-\lambda \\ a_{31} & a_{32} \end{vmatrix}$$

$A \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{bmatrix}$

$\to (C_{11}+C_{22}+C_{33})$

$$= -\lambda^3 + \lambda^2 \underbrace{(a_{11}+a_{22}+a_{33})}_{} - \lambda\,(a_{11}.a_{22} + a_{11}.a_{33}$$

$$+ a_{22}\,a_{33} - a_{23}.a_{32} - a_{12}.a_{21} - a_{13}.a_{31})$$

$x = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \to C_{33}$

$$\left\{ \begin{array}{l} + a_{11}.a_{22}.a_{33} + a_{13}\,a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} \\ - a_{12}\,a_{21}.a_{33} + a_{12}\,a_{23}\,a_{31} - a_{13}\,a_{31}.a_{22} \end{array} \right.$$

$+ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & 33 \end{vmatrix} \to C_{22}$

rearranging :-

$$= a_{11}\,(a_{22}.a_{33} - a_{23}\,a_{32})$$
$$- a_{12}\,(a_{21}.a_{33} - a_{23}.a_{31})$$
$$+ a_{13}\,(a_{21}.a_{32} - a_{31}.a_{22})$$

$+ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \to C_{11}$

$$= \underline{\Delta}(A)\ \text{Det}\cdot A$$

$$= -\lambda^3 + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda\,(C_{11}+C_{22}+C_{33}) + \Delta(A) \quad \text{--- (1)}$$

--- (1)

It will have $\Rightarrow \lambda_1, \lambda_2$ & $\lambda_3 \Rightarrow$ Three eigen values

Hence $|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$

$$(-1)\left[\lambda^3 - \lambda^2(\lambda_1+\lambda_2+\lambda_3) + \lambda(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \lambda_1.\lambda_2.\lambda_3\right]$$

(LHS)

$$-\cancel{\lambda^3} + \lambda^2(\lambda_1+\lambda_2+\lambda_3) - \lambda(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + \underline{\lambda_1.\lambda_2.\lambda_3} = -\cancel{\lambda^3} + \lambda^2(a_{11}+a_{22}+a_{33}) - \lambda(C_{11}+C_{22}+C_{33}) + \Delta(A) \quad \text{--- (2)}$$

$\lambda_1+\lambda_2+\lambda_3 = a_{11}+a_{22}+a_{33} \rightarrow$ Proof of property (4)

$$\left[C_{11}+C_{22}+C_{33} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right] \text{ - -}$$

Prop (5) Product of eigen values is equal to the determinant of the matrix.

From eq. (2)

$$\lambda_1.\lambda_2\lambda_3 = \Delta(A)$$

Hence proved.

6) If λ is eigen value a matrix A, then $1/\lambda$ is the eigen value of A^{-1}.

7) If A is orthogonal matrix, and λ is the eigen value of A, then $1/\lambda$ is also the eigen value of A.

$$(A' = A^{-1})$$

20/05/2021

Ex. 1 Find the eigen values & eigen vectors for matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow [A - \lambda I] \Rightarrow |A - \lambda I| = 0$$

→ Symmetric matrix

Solution :-

$\lambda_1 = 1$; $\lambda_2 = 2$; $\lambda_3 = 3$

Finding eigen vectors :-

(1) $\lambda_1 = 1$; $[A - \lambda I] = \begin{bmatrix} 2-\lambda_1 & 0 & 1 \\ 0 & 2-\lambda_1 & 0 \\ 1 & 0 & 2-\lambda_1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$[A - \lambda I] X = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \cancel{1} & 0 & \cancel{1} & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$; $x_1 + x_3 = 0$

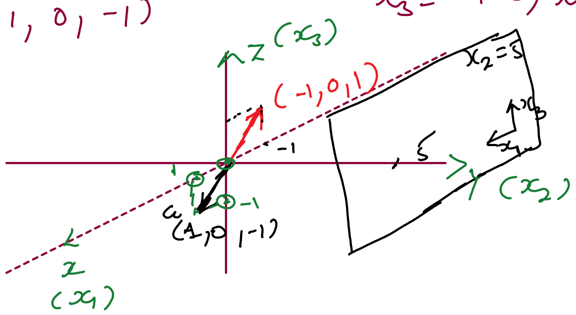
$\therefore x_1 = -x_3$

$x_3 = 1 \Rightarrow x_1 = -1$

$x_3 = -1 \Rightarrow x_1 = 1$

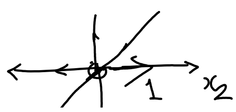
v_1 (−1, 0, 1)

a (1, 0, −1)

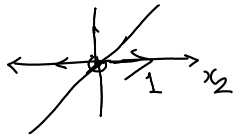


For $\lambda_2 = 2$:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix}$$



For $\lambda_2 = 2$:



$$[A-\lambda I] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_2 \neq 0$?

$x_1 = 0$; $x_3 = 0$; $x_2 = 1$ / -1

$[0, 1, 0]$

For $\lambda_3 = 3$:

$[1, 0, 1] \Leftrightarrow [-1, 0, -1]$

$-x_1 + x_3 = 0 \Rightarrow x_1 = x_3$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} ; \quad x_2 = 0$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1 \Rightarrow X_1 = [-1, 0, 1]'$

$\lambda_2 = 2 \Rightarrow X_2 = [0, 1, 0]$

$\lambda_3 = 3 \Rightarrow X_3 = [1, 0, 1]$

Ex. 2 Find the eigen values & eigen vectors for the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \Rightarrow \text{Symmetric Matrix}$$

Solution:-

$\lambda_1 = 0$; $\lambda_2 = 3$; $\lambda_3 = 15$

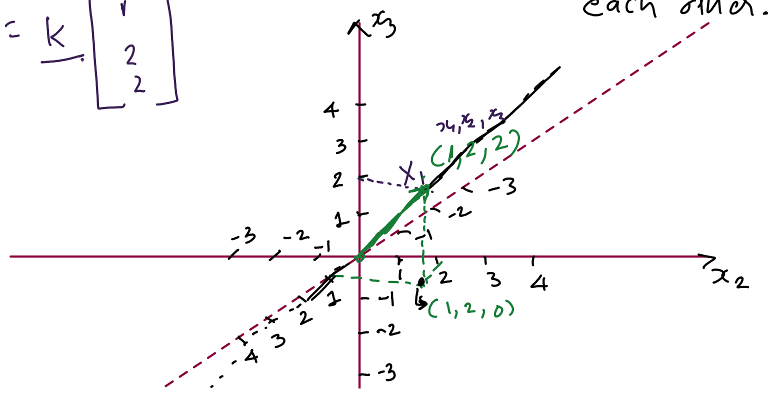
$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} ; \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} ; \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

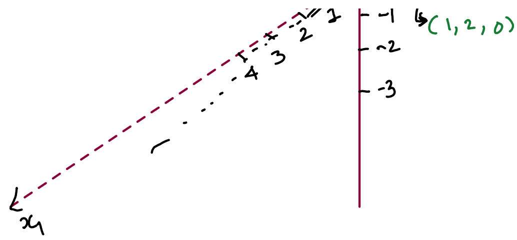
$X_1 . X_2 = 0$

$X_1 \perp X_2$

$$X_1 = k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Obs. The vectors are orthogonal to each other.





Prob. 1

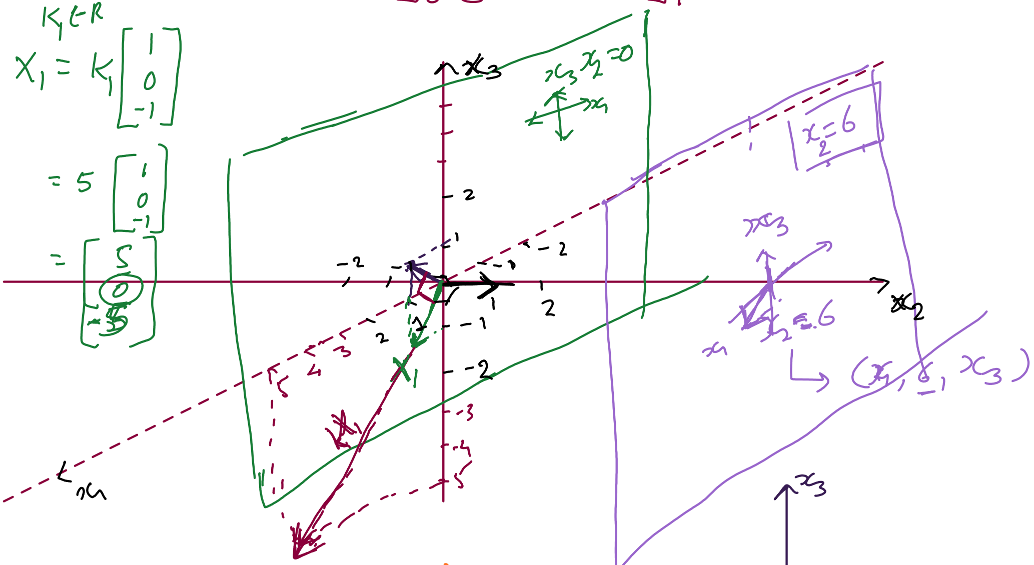
$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow x_2 = 0$

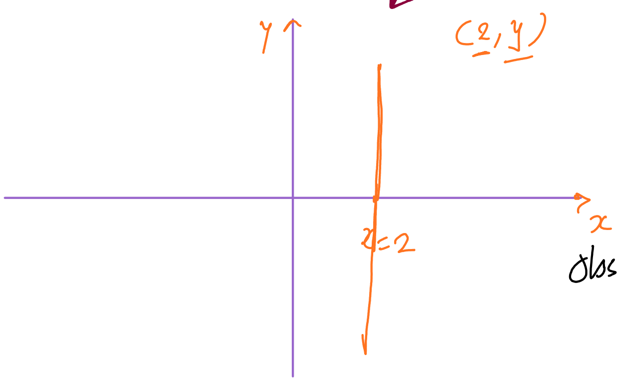
$k_1 \in R$

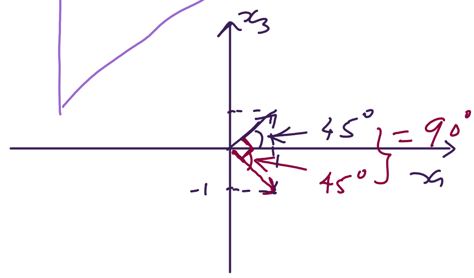
$X_1 = k_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$= 5 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$







Obs. (1) The vectors are found to be orthogonal to each other.

* Symmetric matrix:

↳ having independent vectors, they are orthogonal to each other.

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*
Q.3

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$; Eigen values & eigen vectors.

$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = -1 - \sqrt{10} \\ \lambda_3 = -1 + \sqrt{10} \end{matrix} \right\}$

?

$\lambda_1 = 5$

$\lambda_1 = -3$

$\lambda_1 = 6$

$\lambda_2 = -3$

$\lambda_3 = -3$

$X_1 \Rightarrow \lambda_1 = 5 ; \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$X \Rightarrow \underline{\lambda_2 = -3} ; \quad X_2 = \begin{bmatrix} \\ \\ \end{bmatrix}$

$$\begin{bmatrix} 2-(-3) & 2 & -3 \\ 2 & 1-(-3) & -6 \\ -1 & -2 & 0-(-3) \end{bmatrix} \Rightarrow [A - \lambda I]$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & +3 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1 ; \quad R_3 \leftarrow R_3 + R_1$

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ;$$

Two free variables, x_2, x_3

verify for $\boxed{x_1 = 0}$

$$x_1 + 2x_2 - 3x_3 = 0 \quad \cdots \text{(a)}$$

(*) At once; Let $\underline{x_2 = 0}$;

$$x_1 - 3x_3 = 0 \Rightarrow x_1 = 3x_3 \Rightarrow \frac{x_1}{3} = \frac{x_3}{1}$$

one vector

$\lambda_2 = -3 ; \quad X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

(*) In other case; Let $\underline{x_3 = 0}$;

$\lambda_3 = -3 ; \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} ; \quad x_1 + 2x_2 = 0 \quad \therefore x_1 = -2x_2$

or $= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$\lambda_2 \,\&\, \lambda_3 = -3$; two free variables $\Rightarrow$ two independent eigen vectors

one free var → two eigen vectors for $n=3$ matrix

25/05/21

Ex Find the eigen values and eigen vectors of the matrix:

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution:- $\lambda_1 = 2$; $\lambda_2 = 3$; $\lambda_3 = 5$

$$x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} ; \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Ex:

Batch 21

$$\begin{bmatrix} 6 & 2 & 1 \\ 3 & 4 & 2 \\ 2 & 8 & -1 \end{bmatrix} \begin{matrix} \Rightarrow 9 \\ 9 \\ 9 \end{matrix}$$

$\lambda_1 = 9 \Rightarrow X_1 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Batch 22

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{bmatrix} \begin{matrix} =6 \\ =6 \\ =6 \end{matrix}$$

$\lambda_1 = 1$

$\lambda_2 = -1$

$\boxed{\lambda_3 = 6} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Batch 23

$$\begin{matrix} 5 \leftarrow \\ 5 \leftarrow \\ 5 \leftarrow \end{matrix} \begin{bmatrix} 5 & 0 & 0 \\ 2 & 4 & -1 \\ 8 & -2 & -1 \end{bmatrix}$$

$\boxed{\lambda_1 = 5} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = \frac{3+\sqrt{33}}{2}$

$\lambda_3 = \frac{3-\sqrt{33}}{2}$

$$\begin{bmatrix} 7 & 5 & 1 \\ 8 & 2 & 3 \\ 4 & 4 & 5 \end{bmatrix} \begin{matrix} =13 \\ =13 \\ =13 \end{matrix} \Rightarrow \lambda = 13 ; \Rightarrow X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

All rows are having equal sum.

Proof:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$\begin{vmatrix} \cdots & \cdots & \cdots \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$C_1 \leftarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} a_{11}+a_{12}+a_{13}-\lambda & a_{12} & a_{13} \\ a_{21}+a_{22}+a_{23}-\lambda & a_{22}-\lambda & a_{23} \\ a_{31}+a_{32}+a_{33}-\lambda & a_{32} & a_{33}-\lambda \end{vmatrix}$$

Since all rows having equal sum;

$$a_{11}+a_{12}+a_{13} = a_{21}+a_{22}+a_{23} = a_{31}+a_{32}+a_{33} = n$$

$$= \begin{vmatrix} n-\lambda & a_{12} & a_{13} \\ n-\lambda & a_{22}-\lambda & a_{23} \\ n-\lambda & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$= \underline{(n-\lambda)} \underbrace{\begin{vmatrix} 1 & a_{12} & a_{13} \\ 1 & a_{22}-\lambda & a_{23} \\ 1 & a_{32} & a_{33}-\lambda \end{vmatrix}} = 0$$

$\lambda = n$ is one of the eigen value.

$$(9-\lambda) \begin{vmatrix} 1 & 2 & 1 \\ 1 & 4-\lambda & 2 \\ 1 & 8 & -1-\lambda \end{vmatrix}$$

[Refer. Note-2 (Eigen values & vectors) for remaining part of this Topic]