* Skew symmetric Matrix :- [Eigen values & Eigen vectors]

Ex. 1 For a skew symmetric matrix :

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \qquad \det = \underline{0}$$

Find Eigen values :-

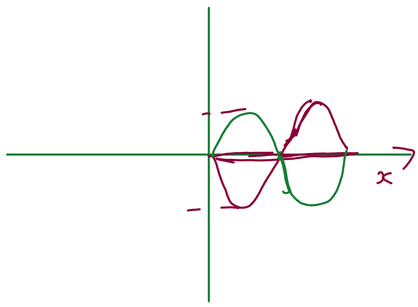
$$\lambda(\lambda^2 + 14) = 0$$

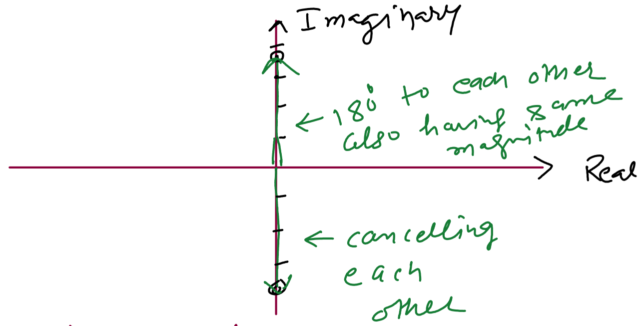
$\lambda = 0$;

$\lambda^2 = -14$

$\lambda = \pm\sqrt{-14}$

$= \pm i\sqrt{14}$ ⟶ Imaginary roots





$\sin\theta + \sin(180 - \theta)$

$= \sin\theta - \sin\theta$

$= 0$

$= \pm i3.7417$

$\lambda_2 = 0 + i3.7417$

$\lambda_3 = 0 - i3.7417$

Ex. 2

$$B = \begin{bmatrix} 0 & 4 & -5 \\ -4 & 0 & 7 \\ 5 & -7 & 0 \end{bmatrix} \qquad \det = \underline{0}$$

$\underline{\lambda_1 = 0}$;

$$\left.\begin{aligned} \lambda_2, \lambda_3 &= \pm\sqrt{-90} \\ &= \pm i\sqrt{90} \end{aligned}\right\} \text{imaginary roots.}$$

Obs. ⇒ If the order of a skew symmetric matrix is odd, then its determinant will be zero & the one eigen value will be zero (0) and others will be pure imaginary pairs.

Ex. 3

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \qquad \det = 0 + 4 = 4 \neq 0$$

$\boxed{\lambda = \pm i \cdot 2}$ ⇒ $\boxed{\lambda = 0}$ — Part is not present.

Ex-4

$$\begin{bmatrix} 0 & 1 & 3 & 6 \\ -1 & 0 & -4 & -7 \\ -3 & 4 & 0 & 2 \\ -6 & 7 & -2 & 0 \end{bmatrix}$$

det = ____

$\lambda^4 + 51\lambda^2 + 64\lambda + 1$

stuti ⇒ $\lambda^4 + 115\lambda^2 + 1$

$\lambda^4 + 106\lambda^2 - 36\lambda + 1$

eigen values will be pure imaginary pairs.

* Obs:- (1) If a skew symmetric matrix has an odd order, then one of the eigen value is zero & rest are pure imaginary (conjugate) pairs

(2) If the order is even than eigen values are in pairs a pure imaginary conjugates.

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* Eigen values & vectors for Identity matrix:-

$$A = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda = 1, 1, 1$$

$\lambda_1 = 1$

$$[A - \lambda I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ 3 variables are free;

$\Rightarrow r = 0 \; ; \; n = 3$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \; x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \; x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If k variables are free, then k-1 variables can be set to zero & 1 variable shdl be kept non zero. [generally select the unit value = 1]

Ex. 2

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \lambda = 5, 5, 5$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Ex. 3

$$\begin{bmatrix}5&0&0\\0&3&0\\0&0&2\end{bmatrix} \Rightarrow \lambda = 5, 3, 2$$

$$x_1 = \begin{bmatrix}1\\0\\0\end{bmatrix};\ x_2 = \begin{bmatrix}0\\1\\0\end{bmatrix};\ x_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\lambda_1 = 5$;

$$|A - \lambda I| \Rightarrow \begin{bmatrix}0&0&0\\0&-2&0\\0&0&-3\end{bmatrix}$$

$x_2 = 0$; $x_3 = 0$; $x_1 \Rightarrow$ free

$x_1 = 1$

Ex. 4

$$\begin{bmatrix}5&2&1\\0&3&0\\0&0&2\end{bmatrix} \Rightarrow \lambda = 5, 3, 2$$

$$x_1 = \begin{bmatrix}1\\0\\0\end{bmatrix};\ X_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix};\ X_3 = \begin{bmatrix}-1\\0\\3\end{bmatrix}$$

For $\lambda_3 = 2$;

$$\begin{bmatrix}3&2&1\\0&1&0\\0&0&0\end{bmatrix}$$

$x_2 = 0$;

$3x_1 + x_3 = 0$

$\therefore\ x_3 = 1 \Rightarrow x_1 = -1/3$

Ex. 5

$$\begin{bmatrix}5&0&0\\2&3&0\\1&2&4\end{bmatrix} \Rightarrow \lambda = 5, 3, 4$$

$$x_1 = \begin{bmatrix}1\\1\\3\end{bmatrix};\ x_2 = \begin{bmatrix}0\\-1\\2\end{bmatrix};\ X_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Ex. 6

$$\begin{bmatrix}5&0&2\\3&2&1\\1&0&3\end{bmatrix};\quad \lambda_1 = 2 \longrightarrow X_1 = \begin{bmatrix}0\\1\\0\end{bmatrix} ✓$$

$\lambda_2 = 4 + \sqrt{3}$;

$\lambda_3 = 4 - \sqrt{3}$

$$\begin{bmatrix}3&0&2\\3&0&1\\1&0&1\end{bmatrix}$$

$x_1 + x_3 = 0$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \quad x_4 = 0; \quad x_1 + x_3 = 0 \Rightarrow x_3 = 0; \quad x_2 = 1$$

→ Probability vector :→ It is a vector with non negative values (entries) that add up to 1.

For e.g. $X_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$
$\overline{}$
1

→ Stochastic Matrix → It is a square matrix having probability vectors as in columns.

→ Markov chains utilize stochastic type matrix (some times also known as markov matrix).

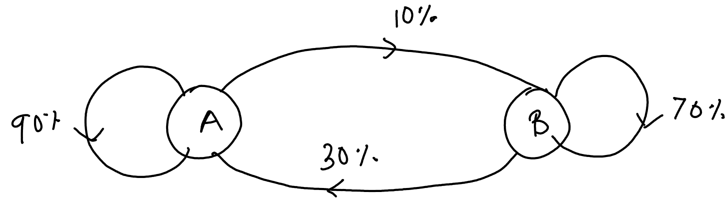
$$X_{n+1} = \underline{M}\, X_n$$

Q.1. There are two nearby cities in a state having some population. After every year, if it is observed that 10% of population (of A) drifts (moves) from city A to city B & 30% of population (of B) drifts from city B to city A.

- Whether there will be a stable state even though the transfer happens every year i.e. the population once reach to steady state, that figure don't change for both cities. ?
- Calculate the population after 5 year & 8 year.
- Assume that city A has 1000 people

& city B has 600 people initially.

Solution :-



$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 600 \end{bmatrix} \Bigg\} \text{ Initial condition}$$

$$A_1 = 0.9 A_0 + 0.3 B_0 =$$

$$B_1 = 0.1 A_0 + 0.7 B_0 =$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \end{bmatrix}$$

1600

$$= \begin{bmatrix} 900 + 180 \\ 100 + 420 \end{bmatrix}$$

$$= \begin{bmatrix} 1080 \\ 520 \end{bmatrix}$$

1600

$$A_2 = 0.9 A_1 + 0.3 B_1$$

$$B_2 = 0.1 A_1 + 0.7 B_1$$

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 1080 \\ 520 \end{bmatrix}$$

$$= \begin{bmatrix} 972 + 156 \\ 108 + 364 \end{bmatrix} = \begin{bmatrix} 1128 \\ 472 \end{bmatrix}$$

1600

$$\begin{bmatrix} A_5 \\ B_5 \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} A_4 \\ B_4 \end{bmatrix}$$

$$\begin{bmatrix} A_8 \\ B_8 \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} A_7 \\ B_7 \end{bmatrix}$$

→ we need to find something more that can help us to solve such problems with less iterations & of course correct or close to correct values.

⇒ Role of Eigen values & Eigen vectors:-

$$\underbrace{\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}}_{C} \begin{bmatrix} A_n \\ B_n \end{bmatrix}$$

$\lambda_1 + \lambda_2 = 1.6$

$\lambda_2 = 1.6 - 1 = 0.6$

$$|C - \lambda I| \Rightarrow \begin{bmatrix} \underline{0.9-\lambda} & 0.3 \\ 0.1 & \underline{0.7-\lambda} \end{bmatrix} \Rightarrow \underline{\lambda_1 = 1} \; ; \; \lambda_2 = 0.6$$

$\underbrace{1}$ $\underbrace{1}$

$(a_{11} + a_{22}) = (\lambda_1 + \lambda_2)$

$1 \; ; \; < 1$

$\underline{< 1}$ $\quad < 2$

$\lambda_2 < 1$

$$\therefore \lambda_1 = \underline{1} \; ; \quad \lambda_2 = \underline{0.6}$$

$$x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \underbrace{C_1}\, (\underset{1}{\bar{\lambda}_1})^{n+1} X_1 + \underbrace{C_2}\, (\underset{<1}{\underline{\lambda_2}})^{n+1} X_2 \quad --- (a)$$

$\lambda_1 = 1 \; ; \; \lambda_2 = 0.6$

$X_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \; ; \; X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 600 \end{bmatrix}$

→ Using initial condition, at $n+1 = 0$

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = C_1 (1)^0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 (0.6)^0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1000 \\ 600 \end{bmatrix} = C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$3C_1 - C_2 = 1000$$
$$C_1 + C_2 = 600$$
$$4C_1 = 1600$$
$$\boxed{C_1 = 400} \quad -(1)$$

$$\therefore C_2 = 600 - C_1 = 200 \quad --(2)$$

Less dominant & tends to zero as $n \to \infty$

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \underbrace{\underbrace{400}\, \underbrace{(1)^{n+1}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}} + \underbrace{200\, (0.6)^{n+1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \quad - (b)$$

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = 400 (1)^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 200 (0.6)^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1128 \\ 472 \end{bmatrix}$$

→ stable situation: Let $n+1 = 100$; then $\boxed{(0.6)^{100} \simeq 0}$

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \underline{400 (1)^{100} \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1200 \\ 400 \end{bmatrix}$$

→ $$\begin{bmatrix} A_5 \\ B_5 \end{bmatrix} = \begin{bmatrix} 1184.448 \\ 415.552 \end{bmatrix} \Rightarrow \begin{bmatrix} 1185 \\ 415 \end{bmatrix}$$

1600

$$\begin{bmatrix} A_8 \\ B_8 \end{bmatrix} = \begin{bmatrix} 1197 \\ 403 \end{bmatrix}$$

1600

$$\begin{bmatrix} A_{20} \\ B_{20} \end{bmatrix} = \begin{bmatrix} \begin{Bmatrix} 1200 \\ 400 \end{Bmatrix} \end{bmatrix} \leftarrow \simeq \left\{ \begin{bmatrix} 1199.992687 \\ 400.00731 \end{bmatrix} \right\}$$

$$\lambda_2 = (0.1)^{10} \Rightarrow 10^{-10} \Rightarrow 1000 \Rightarrow 10^{-7}$$

0.0000001

$$(0.1)^{10} \Rightarrow \frac{10}{10} \Rightarrow (1)$$

Set - 2

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 1600 \\ 0 \end{bmatrix}$$

Set - 3

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 800 \\ 800 \end{bmatrix}$$

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 1200 \\ 400 \end{bmatrix} ; \begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 1200 \\ 400 \end{bmatrix}$$

- C_1 remains unchanged. even if initial condⁿ is different.
- more time to reach to stability

$$\left.\begin{bmatrix} 1600 \longrightarrow 1200 \\ 0 \longrightarrow 400 \end{bmatrix}\right\}$$

Capacitor → 0% charged
↳ 50% charged

(Eigen values)

Property VIII

If $\lambda_1, \lambda_2 \ldots \lambda_n$ are eigen values & $x_1, x_2 \ldots x_n$ are corresponding eigen vectors; then

$$Ax_1 = \lambda_1 x_1 \quad \text{--- (1)}$$

$$A \cdot Ax_1 = A \cdot \lambda_1 x_1$$

$$= \lambda_1 Ax_1$$

$$= \lambda_1 (\lambda_1 x_1)$$

$$A^2 x_1 = \lambda_1^2 x_1 \quad \text{--- (2)}$$

$$A^n x_1 = \lambda_1^n x_1 \quad \text{--- (3)}$$

→ Consequently it can be said that λ_1^n is an eigen value of A^n & vector x_1 remains same.

* Diagonalization of a square matrix:

$$A \Rightarrow PDP^{-1}$$

Revision:

$$\Rightarrow \underset{A}{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 5 & 3 \end{bmatrix}} \begin{bmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} & \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 17 & 21 & 25 \\ 25 & 30 & 35 \\ 30 & 33 & 36 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \underset{x_1}{\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}} & A \underset{x_2}{\begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}} & A \underset{x_3}{\begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} 17 & 21 & 25 \\ 25 & 30 & 35 \\ 30 & 33 & 36 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 1 \rightarrow (3 \times 1)$

$$\begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix}$$

Suppose we have a square matrix of dimension $n \times n$, and it has eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ & corresponding eigen vectors $x_1, x_2, x_3 \ldots x_n$ ..

$AX_1 = \lambda_1 X_1$; $AX_2 = \lambda_2 X_2$; $AX_3 = \lambda_3 X_3 \cdots$

$A^2 X_1 = \lambda_1^2 X_1$; $A^2 X_2 = \lambda_2^2 X_2$; $A^2 X_3 = \lambda_3^2 X_3$

→ So lets take a matrix P having all eigen vectors as its columns for a given square matrix A,

$$P = [X_1 \; X_2 \; X_3 \; \cdots \; X_n]_{n\times n}$$

$\hookrightarrow$ size = $[\;]_{n\times 1}$

$$AP = [AX_1 \; AX_2 \; AX_3 \; \cdots \; AX_n]$$

$$= [\lambda_1 X_1 \; \lambda_2 X_2 \; \lambda_3 X_3 \; \cdots \; \lambda_n X_n]$$

$$= [$$

$$[3x_1 \;\; 4x_2 \;\; 5x_3] \Rightarrow [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= [3x_1 \;\; 4x_2 \;\; 5x_3]$$

$$= \underset{n\times n}{[X_1 \; X_2 \; \cdots \; X_n]} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdot & & 0 & \lambda_n \end{bmatrix}_{n\times n}}_{[D]}$$

$$AP = PD$$

$$\boxed{A = PDP^{-1}}$$ → Diagonalization with eigen values & eigen vectors

$\hookrightarrow$ ⓐ

$$A \cdot A = (PDP^{-1}) \cdot A$$

$$= PD\underbrace{P^{-1} \cdot P}D P^{-1}$$

$$A^2 = PD^2P^{-1}$$

$$A^n = PD^nP^{-1}$$ | P must be non singular.

$\underline{A^{n}} = \underline{P D^{n} P^{-1}}$ | P must be non singular.
eigen values
eigen vectors

$\Updownarrow$

$A_n = c_1 (\lambda_1)^n x_1 + c_2 (\lambda_2)^n x_2$

$B_n =$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}} \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} = \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}} \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}} \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix}$$

$$= C^{n} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \underset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$P \Rightarrow$ non singular $\quad P^{-1}$

$$A^2 = \underset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{D^2}{\begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\Rightarrow$ <u>Cayley Hamilton Theorem</u> :-

Statement : " Every square matrix A satisfies its characteristic equation i.e. A is one of the root of its characteristic equation"

$|A - \lambda I| = \lambda^2 + - - +$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \lambda^2 + \lambda(\text{Trace}) + \text{Determinant} = 0$$

Cayley Hamilton $\Rightarrow$ $\rightarrow A^2 + \lambda A + D.I = 0$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \lambda^3 + \lambda^2 (\text{Trace}) + \lambda \left(\text{cofactors of } a_{11}, a_{22} \,\&\, a_{33}\right)$$

$$+ \underline{\text{Det}} = 0$$

$$= A^3 + A^3 \quad + XA + \text{Det } I = 0$$

<u>Ex. 1</u> Verify Cayley Hamilton theorem for a matrix

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find its inverse.

Solution

$$\lambda^2 - \underline{4}\lambda - \underline{5} \qquad = \underline{0}$$

$$A^2 - 4A - 5I = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3/5 & 4/5 \\ 2/5 & -1/5 \end{bmatrix}$$