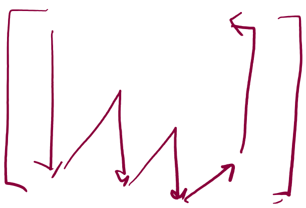
Class A - Solution to Linear System of Equations

Friday, April 23, 2021 10:24 AM

Ex. 2

Solve the following system of equations by matrix inverse method (Gauss-Jordan).

$$\left.\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= -1 \\ x_4 &= 3 \end{aligned}$$



$$\begin{bmatrix} -24/78 & -1/13 & \frac{13.5}{117} & 8/26 \\ 30/39 & -3/13 & -15/117 & 1/13 \\ 23/39 & -1/13 & -5/117 & -4/13 \\ 17/39 & 3/13 & -2/117 & -1/13 \end{bmatrix}$$

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* Consistency of the system:—
(Linear Equations)

↳ Non Homogeneous → $\overset{RHS}{[D]} \neq 0$; will have atleast one non zero element

↳ Homogeneous system → $[D] = 0$; All elements are zero.

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0 \quad \Rightarrow AX = [0]$$
$$a_3x + b_3y + c_3z = 0$$

$x = y = z = 0 \Rightarrow$ Trivial Solution

$$a_1x + b_1y + c_1z = d_1$$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\underline{X}} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\underline{D}}$$

→ Unique solution } consistent
→ Infinite solutions }
→ (No solution)
↳ inconsistent

Mathematical modelling systems

⇒ [A] → Co-efficient matrix

[A : D] → Augmented matrix

$$\hookrightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$\rho(A)$; $\rho(A:D)$

* Conditions for consistency of a non-homogeneous system of Linear equations ($\underline{AX=D}$) :-

(1) If $\underline{\rho(A)} = \rho(A:D) = \underline{n}$ (n = no. of unknowns); then the system of equations is consistent & has unique solution. $\underline{3 \times 3}$

(2) If $\overset{2}{\rho(A)} = \overset{2}{\rho(A:D)} = \underline{\overset{2}{r}} \overset{3}{<} n$; then the system of equation is consistent & has infinite solutions.

(3) If $\rho(A) \neq \rho(A:D)$; then the system of equations is inconsistent & has no solutions.

is inconsistent & has no solutions.

Ex. 1 Test the consistency of the system & solve it if it is consistent.

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Solution:-

$$AX = D$$

$$\underbrace{\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}}_{D}$$

Hence, $[A:D] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$ $(3 \times 4) \Rightarrow$ $\rho(A)_{max} = 3$, $\rho(A:D)_{max} = 3$

$\Rightarrow$ Perform <u>Only row</u> operations & make the lower triangle 0.

$$\left(R_1 \leftarrow \frac{1}{5} R_1\right) \sim \left[\begin{array}{ccc|c} 1 & 3/5 & 7/5 & 4/5 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$[D]$

$$R_2 \leftarrow (R_2 - 3R_1) \; ; \; R_3 \leftarrow (R_3 - 7R_1)$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 0 & 2 \\ 0 & -11 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$[A]$

$\rho(A) = 2$; $\rho(A:D) = 2 = r < n(3)$

Infinite solutions

$-11y + z = -3$; $\underline{z = k}$; $k \in R$

$-11y = -3 - k$

$\therefore y = \dfrac{-3-k}{-11}$

$= \dfrac{3+k}{11}$;

using first row,

$x + 5y = 2$

$\therefore x = 2 - 5 \times \dfrac{(3+k)}{11}$

$= \dfrac{22 - 15 - 5k}{11}$

$x = \dfrac{7-5k}{11}$

$$\underline{(x, y, z) = \left(\frac{7-5k}{11}, \frac{3+k}{11}, k\right)}$$

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Ex.2 Examine the consistency of the Linear System of equations defined as:

$2x - 3y + 7z = 5$

$3x + y - 3z = 13$

$2x + 19y - 47z = 32$

Solution:-

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ \end{bmatrix} \quad \begin{bmatrix} 5 \\ \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

$$A \quad X = D$$

$$[A:D] = \left[\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

⋮

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2/11 & 0 \\ 0 & 1 & -27/11 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

$\rho(A) = 2$

$\rho(A:D) = 3$

$\rho(A) \neq \rho(A:D)$

Hence system is inconsistent & has no solutions.

$[A:D]$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & 11 \\ 0 & 0 & 0 & 27 \end{array}\right]$$

$(0 \cdot x + 0 \cdot y + 0 \cdot z = (k))$; impractical thing

Ex 3 Examine the consistency of the system of linear equations and if consistent, solve the system.

$$\begin{aligned} 3x+3y+2z&=1\\ x+2y&=4\\ 10y+3z&=-2\\ 2x-3y\boxed{-z}&=5 \end{aligned}$$

Solution :-

$$[A:D] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

Answer :→ $\rho(A) = \rho(A:D) = 3 = n =$ no. of unknowns

Hence it has a unique solution.

$x = \underline{2}$

$y = \underline{1}$

$z = \underline{-4}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z=-4$; $y=1$; $x+2y=4$

$\therefore x = 4-2(1) = 2$

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Ex. 4 Find the values of λ & μ so that the equations : $2x+3y+5z=9$

$7x+3y-2z=8$

$$2x + 3y + \lambda z = \mu$$

have :

(1) No solution.

(2) Unique solution.

(3) Infinite solutions.

solution:-

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

(1) No solution $\Rightarrow \lambda = 5$ & $\mu \neq 9$

$$(A:D) \quad \left[\begin{array}{ccc|c} 0 & 3 & 32/5 & 46/5 \\ 5 & 0 & -7 & -1 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

Case - 3

$\lambda = 5 \Rightarrow \lambda - 5 = 0 \;:\; \rho(A) = 2$

$\mu = 9 \Rightarrow \mu - 9 = 0 \;:\; \rho(A:D) = 2$

} Infinite solutions

Case.1

if $\lambda = 5 \Rightarrow \rho(A) = 2$

& $\mu \neq 9 \Rightarrow \rho(A:D) = 3$

} $\rho(A) \neq \rho(A:D)$ Hence no solution

Case 2

if $\lambda \neq 5 \Rightarrow \rho(A) = 3$

$\mu \in R \Rightarrow \rho(A:D) = 3 = n$

} Unique solution.

Ex.5

$$3x + 3x + 2z = 1$$
$$x + 2y = 4$$

} check the consistency of the system.

$$\left.\begin{aligned} x + 2y \quad &= 4 \\ 10y + 3z &= -2 \\ -2x - 3y\ \boxed{} &= 5 \end{aligned}\right\}$$ consistency of the system.

Inconsistent system $\Rightarrow$

$\rho(A) \neq \rho(A:D)$

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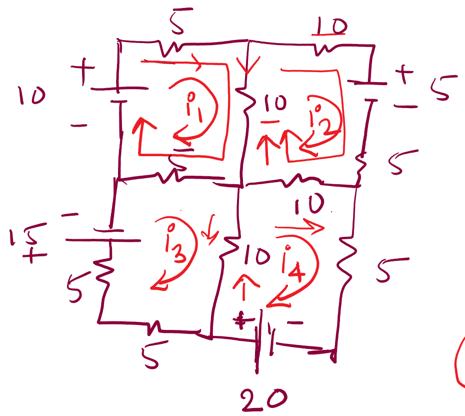
* **Mesh Analysis & Nodal Analysis :-**



KVL $\Rightarrow$ 4 equations

$$A\underbrace{\begin{bmatrix} & \end{bmatrix}}_{4\times4}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

$$(A|D) = \begin{bmatrix} & \end{bmatrix}_{4\times5}$$

Mesh 1:

$5i_1 + 10(i_1 - i_2) + 5(i_1 - i_3) - 10 = 0$

$\therefore 20i_1 - 10i_2 - 5i_3 = 10$

$\therefore 4i_1 - 2i_2 - i_3 = 2$ --- (1)

Mesh 2:

$10i_2 + 5 + 5i_2 + 10(i_2 - i_4)$

$+ 10(i_2 - i_1) = 0$

$35i_2 - 10i_1 - 10i_4 = -5$

$-2i_1 + 7i_2 - 2i_4 = -1$ --- (2)

$5i_3 - i_1 - 2i_4 = -3$ --- (3)

$-i_1 + 5i_3 - 2i_4 = -3$ --- (3)

$-2i_2 - 2i_3 + 3i_4 = 4$ (4)

$i_1 =$

$i_2 =$

$i_3 =$

i_4

$$\begin{bmatrix} 4 & -2 & -1 & 0 \\ -2 & 7 & 0 & -2 \\ -1 & 0 & 5 & -2 \\ 0 & -2 & -2 & 3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 4 \end{bmatrix}$$

$\rightarrow$

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* Homogeneous systems:-

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}\right\}$$

↳ Minimum Solution (if no. of equations (m) = no. of unknowns (n)) is a Trivial Solution

$$x = y = z = 0$$

Case (1) If rank $r = n$; system has unique solution & that is Trivial only.

Case (2) If rank $r < n$; system has infinite solutions (including Trivial)

⟹ Determinant = 0 (co-efficient matrix)

↳ "Eliminant"

Case (3) No. of equations $(m) < n$; Infinite Solutions

Ex.1

$$\left.\begin{aligned} 5x + 2y - 3z &= 0 \\ 3x + y + z &= 0 \\ 2x + y + 6z &= 0 \end{aligned}\right\}$$

Solution:-

$\rho(A) = 3 \Rightarrow$ Trivial solution:-

Ex.2

$$\begin{aligned} 2x - y + 3z &= 0 \\ 3x + 2y + z &= 0 \\ x - 4y + 5z &= 0 \end{aligned}$$

—

$x - 4y + \ldots 0$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ \cancel{1} & -4 & 5 \end{bmatrix} \Rightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow \text{Duplicate}$$

$x = k$;

$x + z = 0$

$\therefore z = -x = -k$

$\therefore -y + z = 0$

$y = z = -k$

$(k, -k, -k) \longleftrightarrow$

Non Trivial Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = -k$

$y = z = -k$

$x + 3y - 2z = 0$

$\therefore x = -3y + 2z$

$x = k$

$(k, -k, -k)$