Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series.**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution: The Python code `codes/1_1.py` plots $x(t)$ below.

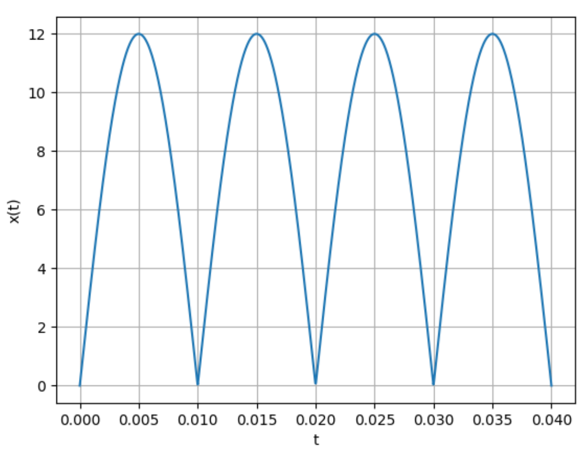

Fig. 1.1: $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.
Solution: From Fig. (1.1), we see that $x(t)$ is periodic. Further,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right)\right| \tag{1.2}$$

$$= A_0 \left|\sin\left(2\pi f_0 t + \pi\right)\right| \tag{1.3}$$

$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.4}$$

Hence the period of $x(t)$ is $\frac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: We have for some $n \in \mathbb{Z}$,

$$x(t) e^{-j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi (k-n) f_0 t} \tag{2.3}$$

But we know from the periodicity of $e^{j2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t}\, dt = \frac{1}{f_0} \delta\left(k\right) \tag{2.4}$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t}\, dt = \frac{c_n}{f_0} \tag{2.5}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t}\, dt \tag{2.6}$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.7}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos\left(2\pi n f_0 t\right)\, dt + \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \sin\left(2\pi n f_0 t\right)\, dt \tag{2.8}$$

$$= 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \cos\left(2\pi n f_0 t\right)\, dt \tag{2.9}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n+1\right) f_0 t\right)\right)\, dt - f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n-1\right) f_0 t\right)\right)\, dt \tag{2.10}$$

$$= A_0 \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.11}$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.12}$$

2.3 Verify (2.1) using python.
Solution: The Python code `codes/2_3.py` verifies (2.13).

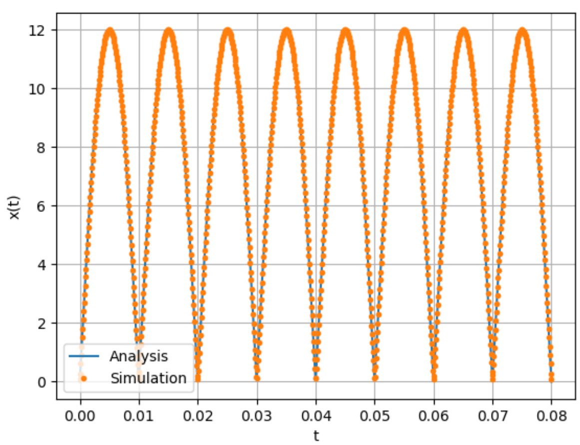

Fig. 2.3: Verification of (2.1).

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t\right) \tag{2.13}$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.14}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos\left(2\pi k f_0 t\right) + \sum_{k=0}^{\infty} \left(c_k - c_{-k}\right) \sin\left(2\pi k f_0 t\right) \tag{2.16}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.17}$$

$$b_k = c_k - c_{-k} \tag{2.18}$$

2.5 Find a_k and b_k for (1.1)
Solution: From (2.1), we see that since $x(t)$ is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

where we substitute $k := -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.17) and (2.18), for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k > 0,\ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

$$b_k = 0 \tag{2.23}$$

2.6 Verify (2.13) using python.
Solution: The Python code `codes/2_6.py` verifies (2.13).

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

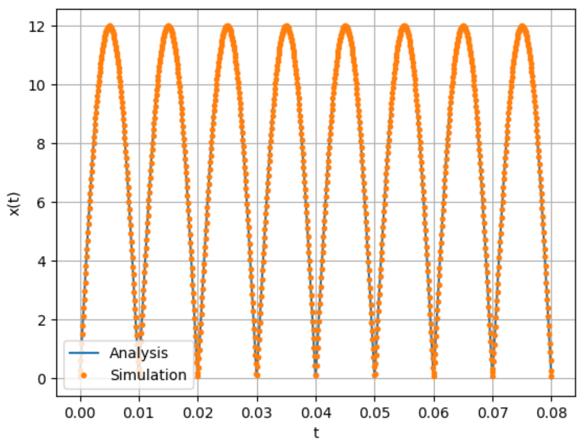

Fig. 2.6: Verification of (2.13).

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt \tag{3.3}$$

3.3 Show that

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi f t_0} \tag{3.4}$$

Solution: We write, substituting $u := t - t_0$,

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t-t_0)e^{-\jmath 2\pi ft}\, dt \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(u)e^{-\jmath 2\pi f(u+t_0)}\, du \tag{3.6}$$

$$= G(f)e^{-j2\pi f t_0} \tag{3.7}$$

where the last equality follows from (3.3).

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.8}$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi ft}\, df \tag{3.9}$$

Hence, setting $t := -f$ and $f := t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{-\jmath 2\pi ft}\, dt \tag{3.10}$$

$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.11}$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

Solution: We have, from the definition of $\delta(t)$,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t)e^{-\jmath 2\pi ft}\, dt \tag{3.12}$$

$$= \int_{-\infty}^{\infty} \delta(0)\, dt \tag{3.13}$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.14}$$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} ?$

Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then,

$$g(t)e^{\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi (f-f_0)t}\, dt \tag{3.15}$$

$$= F(f - f_0) \tag{3.16}$$

Using (3.9) in (3.14), $1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-f)$. Hence, applying (3.16),

$$e^{-\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(-(f+f_0)) = \delta(f+f_0) \tag{3.17}$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

Solution: Using the linearity of the Fourier Transform and (**??**),

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right) \tag{3.18}$$

$$\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left(\delta(f+f_0) + \delta(f-f_0)\right) \tag{3.19}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution: Substituting (2.12) in (2.1),

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f + kf_0) \tag{3.20}$$

$$= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f+2kf_0)}{1-4k^2} \tag{3.21}$$

The python code `codes/3_8.py` verifies (3.21).

3.9 Show that

$$\operatorname{rect} t \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{sinc} t \tag{3.22}$$

Verify using python.

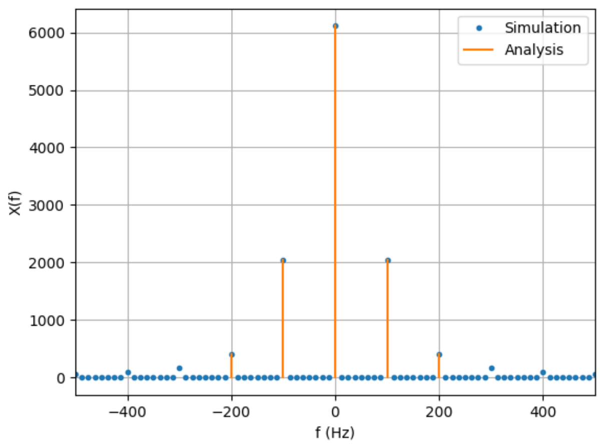

Fig. 3.8: Fourier Transform of $x(t)$.

Solution: We write

$$\operatorname{rect} t \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \operatorname{rect} t e^{-\jmath 2\pi f t} \, dt \tag{3.23}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t} \, dt \tag{3.24}$$

$$= \frac{e^{\jmath \pi f} - e^{-\jmath \pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \operatorname{sinc} f \tag{3.25}$$

The python code `codes/3_9.py` verifies (3.25).

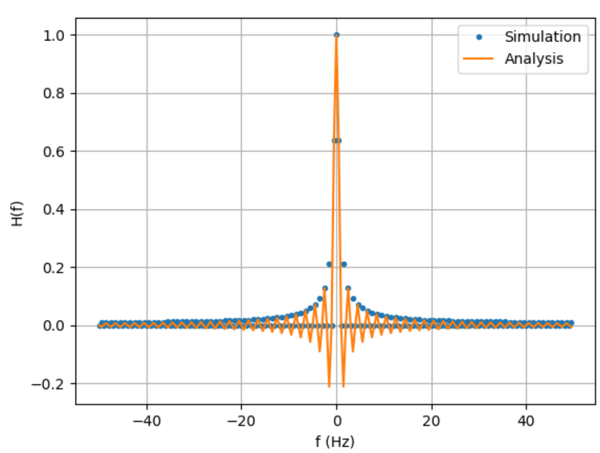

Fig. 3.9: Fourier Transform of rect(t).

3.10 $\operatorname{sinc} t \overset{\mathcal{F}}{\longleftrightarrow}$? Verify using python.
Solution: From (3.9), we have

$$\operatorname{sinc} t \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{rect}(-f) = \operatorname{rect} f \tag{3.26}$$

Since $\operatorname{rect} f$ is an even function. The python code `codes/3_10.py` verifies (3.26).

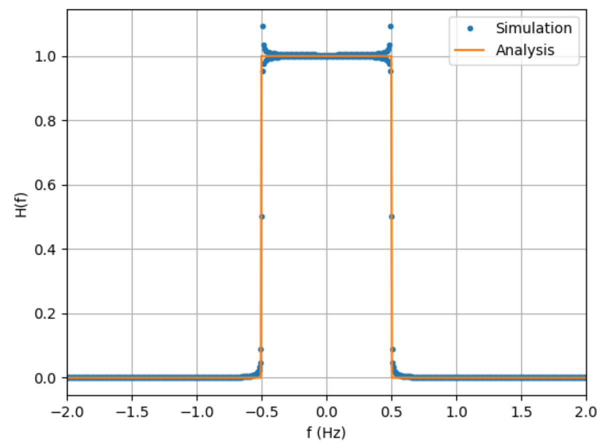

Fig. 3.10: Fourier Transform of sinc(t).

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \operatorname{rect} \frac{f}{2f_c} = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0} \operatorname{rect}\left(\frac{f}{2f_c}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.
Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then, for some nonzero $a \in \mathbb{R}$

$$g(at) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(at) e^{-\jmath 2\pi f t} \, dt \tag{4.3}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} g(u) e^{\left(-\jmath 2\pi \frac{f}{a} t\right)} \, dt \tag{4.4}$$

$$= \frac{1}{a} G\left(\frac{f}{a}\right) \tag{4.5}$$

where we have substituted $u := at$. Using (4.5) of the Fourier Transform in (4.1),

$$h(t) = 2 * f_c \operatorname{sinc}(2 * f_c t) \tag{4.6}$$

4.3 Verify your result using convolution.
Solution: The Python code `codes/4_3.py` verifies the result by plotting the graph below.

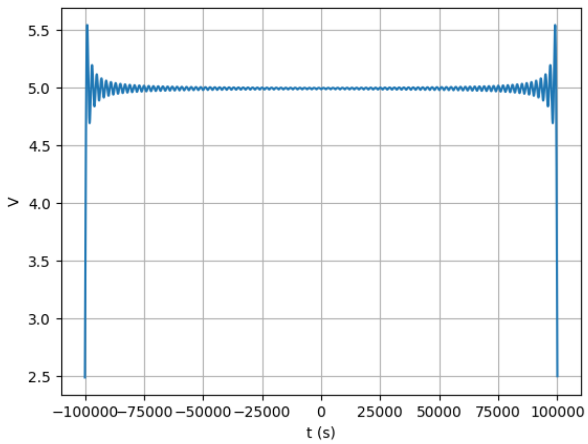

Fig. 4.3: Convolution of the two signals.

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
Solution: The Butterworth filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \left(\frac{f}{f_c}\right)^{2n}\right)} \tag{5.1}$$

where n is the order of the filter and f_c is the cutoff frequency. The loss at frequency f is given by

$$\text{Loss} = -10 \log_{10} |H(f)|^2 \tag{5.2}$$
$$= -20 \log_{10} |H(f)| \tag{5.3}$$

Assuming the cutoff frequency $f_c = 10$ Hz, the Python code `codes/5_1.py` plots the magnitude response for different values of n as shown below. At $f = 10 f_c = 2 f_0$, we obtain a 40 dB loss for $n = 2$, which is ideal. Hence, we choose $n = 2$. Further, note that the DC gain is $\frac{5}{12}$.

5.2 Design a Chebyshev filter for $H(f)$.

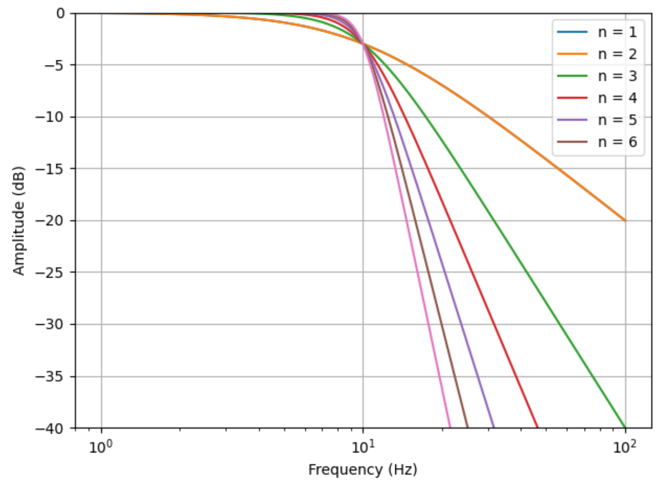

Fig. 5.1: Butterworth response for various n.

Solution: The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 C_n^2 \left(\frac{f}{f_c}\right)^2\right)} \tag{5.4}$$

where n is the order of the filter, ϵ is the ripple, f_c is the cutoff frequency and C_n denotes the n^{th} order Chebyshev polynomial. Assuming the cutoff frequency to be at f_0, the Python codes `codes/5_2.py` and `codes/5_3.py` plot the magnitude response for different values of n.

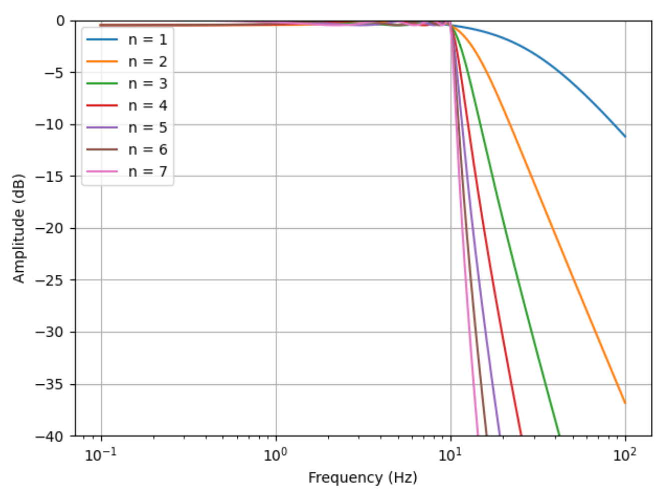

Fig. 5.2: Chebyshev response for various n.

At $f = 10 f_c$, we obtain close to 40 dB loss for $n = 2$, which is ideal. Hence, we choose $n = 2$. Further, note that the DC gain is $\frac{5}{12}$.

5.3 Design a circuit for your Butterworth filter.

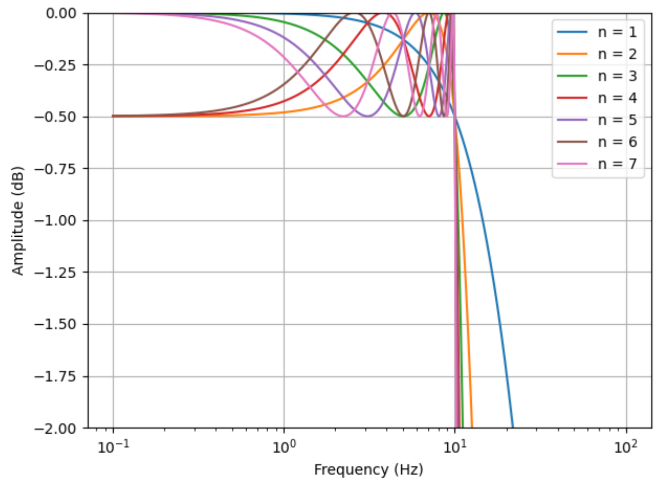

Fig. 5.2: Chebyshev response for various n, in detail.

Solution: Looking at the table of normalized g values of the Butterworth filter for order 2, we see that

$$g_1 = C_1 = 1.5774 \qquad g_2 = L_1 = 0.4226 \tag{5.5}$$

De-normalizing these values,

$$C_1' = \frac{C_1}{\omega_c} = 2.51 \times 10^{-2} \text{ F} \tag{5.6}$$

$$L_1' = \frac{L_1}{\omega_c} = 6.72 \text{ mH} \tag{5.7}$$

The L-C network is shown below

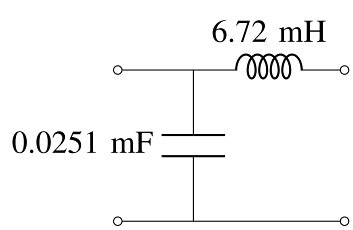

Fig. 5.3: L-C Butterworth Filter

5.4 Design a circuit for your Chebyshev filter.
Solution: Looking at the table of normalized g values of the Chebyshev filter for order 2 and 0.5 dB ripple, we see that

$$g_1 = C_1 = 1.4029 \qquad g_2 = L_1 = 0.7071 \tag{5.8}$$

De-normalizing these values,

$$C_1' = \frac{C_1}{\omega_c} = 2.23 \times 10^{-2} \text{ F} \tag{5.9}$$

$$L_1' = \frac{L_1}{\omega_c} = 1.12 \times 10^{-2} \text{ H} \tag{5.10}$$

The L-C network is shown below

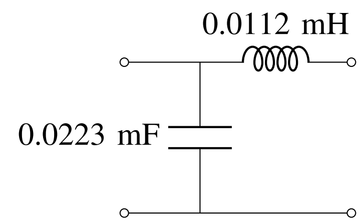

Fig. 5.4: L-C Chebyshev Filter