

# Fourier Series

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**Abstract**—This manual provides a simple introduction to Fourier Series

## 2 FOURIER SERIES

### 1 PERIODIC FUNCTION

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (1.1)$$

Consider  $A_0 = 12$  and  $f_0 = 50$  for all numerical calculations.

1.1 Plot  $x(t)$ .

**Solution:** The Python code `codes/1_1.py` plots  $x(t)$  in Fig. (1.1).

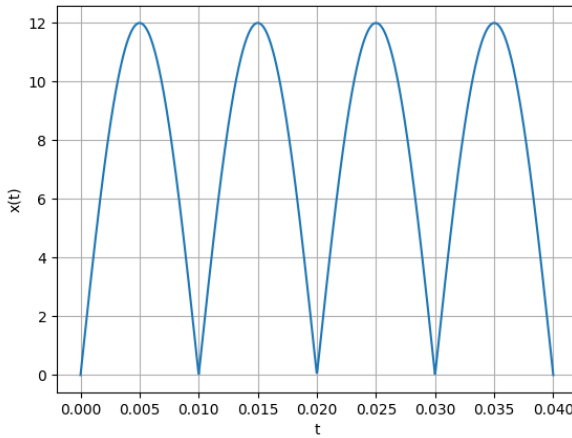


Fig. 1.1:  $x(t)$

1.2 Show that  $x(t)$  is periodic and find its period.

**Solution:** From Fig. (1.1), we see that  $x(t)$  is periodic. Further,

$$x\left(t + \frac{1}{f_0}\right) = A_0 \left| \sin\left(2\pi f_0 \left(t + \frac{1}{f_0}\right)\right) \right| \quad (1.2)$$

$$= A_0 |\sin(2\pi f_0 t + 2\pi)| \quad (1.3)$$

$$= A_0 |\sin(2\pi f_0 t)| \quad (1.4)$$

Hence the period of  $x(t)$  is  $\frac{1}{f_0}$ .

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.2)$$

**Solution:** We have for some  $n \in \mathbb{Z}$ ,

$$x(t) e^{-j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} \quad (2.3)$$

But we know from the periodicity of  $e^{j2\pi k f_0 t}$ ,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} dt = \frac{1}{f_0} \delta(k) \quad (2.4)$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt = \frac{c_n}{f_0} \quad (2.5)$$

$$\Rightarrow c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt \quad (2.6)$$

2.2 Find  $c_k$  for (1.1)

**Solution:** Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| e^{-j2\pi n f_0 t} dt \quad (2.7)$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \cos(2\pi n f_0 t) dt$$

$$+ j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \sin(2\pi n f_0 t) dt \quad (2.8)$$

$$= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t) dt \quad (2.9)$$

$$= f_0 A_0 \int_0^{\frac{1}{2f_0}} (\sin(2\pi(n+1)f_0 t)) dt - f_0 A_0 \int_0^{\frac{1}{2f_0}} (\sin(2\pi(n-1)f_0 t)) dt \quad (2.10)$$

$$= A_0 \frac{1 + (-1)^n}{2\pi} \left( \frac{1}{n+1} - \frac{1}{n-1} \right) \quad (2.11)$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad (2.12)$$

2.3 Verify (2.1) using python.

**Solution:** The Python code codes/2\_3.py verifies (2.13).

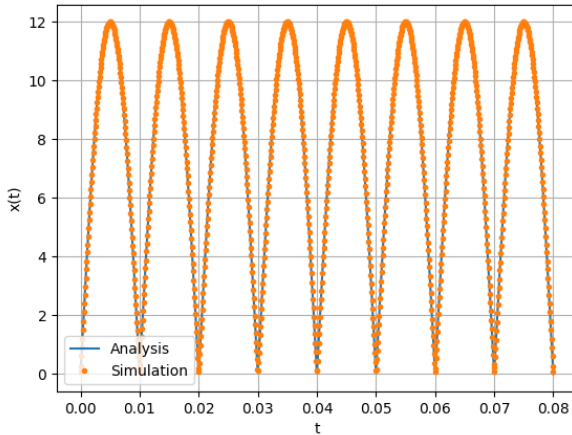


Fig. 2.3: Verification of (2.1).

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t) \quad (2.13)$$

and obtain the formulae for  $a_k$  and  $b_k$ .

**Solution:** From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.14)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t} \quad (2.15)$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$

$$+ \sum_{k=0}^{\infty} (c_k - c_{-k}) \sin(2\pi k f_0 t) \quad (2.16)$$

Hence, for  $k \geq 0$ ,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.17)$$

$$b_k = c_k - c_{-k} \quad (2.18)$$

2.5 Find  $a_k$  and  $b_k$  for (1.1)

**Solution:** From (2.1), we see that since  $x(t)$  is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k f_0 t} \quad (2.19)$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{j2\pi k f_0 t} \quad (2.20)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.21)$$

where we substitute  $k \mapsto -k$  in (2.20). Hence, we see that  $c_k = c_{-k}$ . So, from (2.18) and for  $k \geq 0$ ,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k > 0, k \text{ even} \\ 0 & \text{otherwise} \end{cases} \quad (2.22)$$

$$b_k = 0 \quad (2.23)$$

2.6 Verify (2.13) using python.

**Solution:** The Python code codes/2\_6.py verifies (2.13).

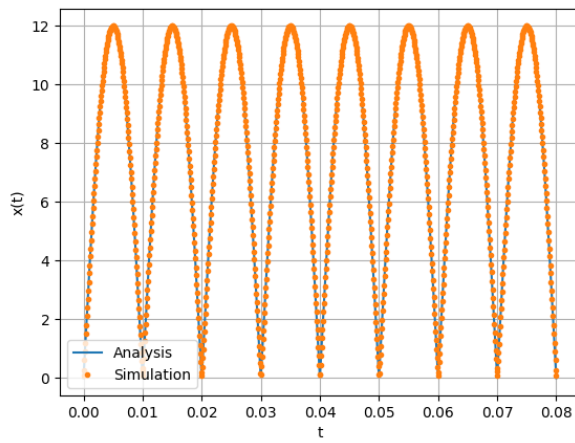


Fig. 2.6: Verification of (2.13).