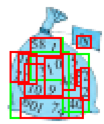
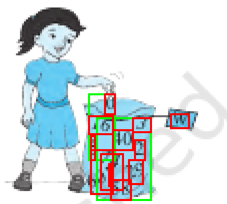
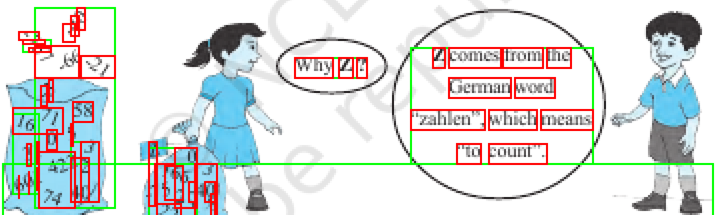
You might begin with picking up only natural numbers like 1, 2, 3, and so on. You know that this list goes on for ever. (Why is this true?) So, now your bag contains infinitely many natural numbers! Recall that we denote this collection by the symbol **N**.



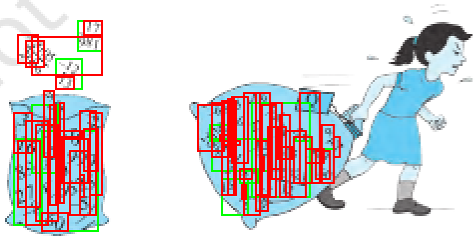
Now turn and walk all the way back, pick up zero and put it into the bag. You now have the collection of *whole numbers* which is denoted by the symbol **W**.



Now, stretching in front of you are many, many negative integers. Put all the negative integers into your bag. What is your new collection? Recall that it is the collection of all *integers*, and it is denoted by the symbol **Z**.



Are there some numbers still left on the line? Of course! There are numbers like $\frac{1}{2}$, $\frac{3}{4}$ or even $\frac{-2005}{2006}$. If you put all such numbers also into the bag, it will now be the



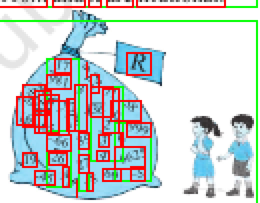
You already know that there are infinitely many rationals. It turns out that there are infinitely many irrational numbers too. Some examples are:

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, 0.10110111011110...$$

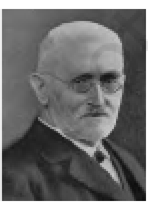
Remark : Recall that when we use the symbol $\sqrt{\ }$, we assume that it is the positive square root of the number. So $\sqrt{4} = 2$, though both 2 and –2 are square roots of 4.

Some of the irrational numbers listed above are familiar to you. For example, you have already come across many of the square roots listed above and the number π.

The Pythagoreans proved that $\sqrt{2}$ is irrational. Later in approximately 425 BC, Theodorus of Cyrene showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ and $\sqrt{17}$ are also irrationals. Proofs of irrationality of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc., shall be discussed in Class X. As to π, it was known to various cultures for thousands of years, it was proved to be irrational by Lambert and Legendre only in the late 1700s. In the next section, we will discuss why 0.10110111011110... and π are irrational.



Let us return to the questions raised at the end of the previous section. Remember the bag of rational numbers. If we now put all irrational numbers into the bag, will there be any number left on the number line? The answer is no! It turns out that the collection of all rational numbers and irrational numbers together make up what we call the collection of *real numbers*, which is denoted by **R**. Therefore, a real number is either rational or irrational. So, we can say that **every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number**. This is why we call the number line, the *real number line*.



R. Dedekind (1831-1916)

Fig. 1.4

In the 1870s two German mathematicians, Cantor and Dedekind, showed that : Corresponding to every real number, there is a point on the real number line, and corresponding to every point on the number line, there exists a unique real number.

G. Cantor (1845-1918)

Fig. 1.5

In the same way, you can locate $\sqrt{n}$ for any positive integer n, after $\sqrt{n-1}$ has been located.

EXERCISE 1.2

1. State whether the following statements are true or false. Justify your answers.
   (i) Every irrational number is a real number.
   (ii) Every point on the number line is of the form $\sqrt{m}$, where m is a natural number.
   (iii) Every real number is an irrational number.
2. Are the square roots of all positive integers irrational? If not, give an example of the square root of a number that is a rational number.
3. Show how $\sqrt{5}$ can be represented on the number line.
4. **Classroom activity (Constructing the 'square root spiral')** : Take a large sheet of paper and construct the 'square root spiral' in the following fashion. Start with a point O and draw a line segment OP_1 of unit length. Draw a line segment P_1P_2 perpendicular to OP_1 of unit length (see Fig. 1.9). Now draw a line segment P_2P_3 perpendicular to OP_2. Then draw a line segment P_3P_4 perpendicular to OP_3. Continuing in this manner, you can get the line segment $P_{n-1}P_n$ by drawing a line segment of unit length perpendicular to OP_{n-1}. In this manner, you will have created the points $P_2, P_3, \ldots, P_n, \ldots$, and joined them to create a beautiful spiral depicting $\sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$

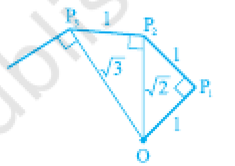


Fig. 1.9 : Constructing square root spiral

1.3 Real Numbers and their Decimal Expansions

In this section, we are going to study rational and irrational numbers from a different point of view. We will look at the decimal expansions of real numbers and see if we can use the expansions to distinguish between rationals and irrationals. We will also explain how to visualise the representation of real numbers on the number line using their decimal expansions. Since rationals are more familiar to us, let us start with them. Let us take three examples : $\frac{10}{3}, \frac{7}{8}, \frac{1}{7}$.

Pay special attention to the remainders and see if you can find any pattern.