Mathematics for Computer Science (III Semester)

(Subject code: BCS301)

Module 2: Joint probability distribution and Markov chain

Syllabus:

Joint probability distribution: Joint Probability distribution for two discrete random variables, expectation, covariance and correlation.
Markov Chain: Introduction to Stochastic Process, Probability Vectors, Stochastic matrices, Regular stochastic matrices, Markov chains, Higher transition probabilities, Stationary distribution of Regular Markov chains and absorbing states. **(12 Hours)**
(RBT Levels: L1, L2 and L3)

2.1 Joint Probability distribution

Introduction:

- Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two discrete random variables. Then $P(x, y) = J_{ij}$ is called joint probability function of X and Y if it satisfies the conditions:

$$(i)\ J_{ij} \geq 0 \quad (ii) \sum_{i=1}^{m}\sum_{j=1}^{n} J_{ij} = 1$$

- Set of values of this joint probability function J_{ij} is called joint probability distribution of X and Y.

X\Y	y_1	y_2	---	y_n	Sum
x_1	J_{11}	J_{12}	---	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	---	J_{2n}	$f(x_2)$
---	---	---	---	---	---
x_m	J_{m1}	J_{m2}	---	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	---	$g(y_n)$	$Total = 1$

- Marginal probability distribution of X

x_1	x_2	...	x_n
$f(x_1)$	$f(x_2)$	...	$f(x_n)$

Where $f(x_1) + f(x_2) + \cdots + f(x_n) = 1$

- Marginal probability distribution of Y

y_1	y_2	...	y_n
$g(y_1)$	$g(y_2)$	...	$g(y_n)$

Where $g(y_1) + g(y_2) + \cdots + g(y_n) = 1$

- The discrete random variables X and Y are said to be independent random variables if $f(x_i)g(y_j) = J_{ij}$.

Important results:

- Expectations:

$$E(x) = \sum_{i=1}^{m} x_i f(x_i) \quad E(y) = \sum_{j=1}^{n} y_j g(y_j) \quad E(xy) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j J_{ij}$$

- Covariance:

$$Cov(x, y) = E(xy) - E(x)E(y)$$

- Variance:

$$Var(x) = E(x^2) - [E(x)]^2 \quad Var(y) = E(y^2) - [E(y)]^2$$

- Standard deviation:

$$\sigma_x = \sqrt{Var(x)} \quad \sigma_y = \sqrt{Var(y)}$$

- Correlation of X and Y:

$$\rho(x, y) = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

- If X and Y are independent then $E(xy) = E(x)E(y)$.

1. The joint distribution of random variables X and Y is

X\Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Find (i) Marginal distribution of X and Y. (ii) $E(x), E(y)$ (iii) Are X and Y independent random variables? (iv) $Cov(x,y)$ (v) σ_x, σ_y (vi) $\rho(x,y)$

$x\backslash y$	-4	2	7	$f(x)$
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	1/2
$g(y)$	3/8	3/8	1/4	$Total = 1$

(i) Marginal probability distribution of X:

x	1	5
$f(x)$	1/2	1/2

Marginal probability distribution of Y:

y	-4	2	7
$g(y)$	3/8	3/8	1/4

(ii) $E(x) = \Sigma x f(x) = 1\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) = 3$

$E(y) = \Sigma y g(y) = -4\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 7\left(\frac{1}{4}\right) = 1$

(iii) $E(xy) = \Sigma\Sigma x_i y_j J_{ij}$

$= 1(-4)\left(\frac{1}{8}\right) + 1(2)\left(\frac{1}{4}\right) + 1(7)\left(\frac{1}{8}\right) + 5(-4)\left(\frac{1}{4}\right) + 5(2)\left(\frac{1}{8}\right) + 5(7)\left(\frac{1}{8}\right)$

$= -\frac{4}{8} + \frac{4}{8} + \frac{7}{8} - 5 + \frac{10}{8} + \frac{35}{8} = \frac{3}{2}$

$E(x)E(y) = 3(1) = 3$. Therefore, $E(xy) \neq E(x)E(y)$.

Therefore, x *and* y are not independent variables.

(iv) $Cov(x,y) = E(xy) - E(x)E(y) = \frac{3}{2} - 3(1) = -\frac{3}{2}$.

(v) $E(x^2) = \Sigma x^2 f(x) = 1^2\left(\frac{1}{2}\right) + 5^2\left(\frac{1}{2}\right) = 13$

$E(y^2) = \Sigma y^2 g(y) = (-4)^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 7^2\left(\frac{1}{4}\right) = \frac{79}{4}$

$\sigma_x = \sqrt{E(x^2) - [E(x)]^2} = \sqrt{13 - 3^2} = 2$

$\sigma_y = \sqrt{E(y^2) - [E(y)]^2} = \sqrt{\frac{79}{4} - 1^2} = 4.33$

(vi) $\rho(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = -\frac{1.5}{8.66} = -0.1732$

2. **Find the joint distribution of X and Y is as follows:**

x \ y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find (i) Marginal distribution of X and Y. (ii) $E(x), E(y)$ (iii) Are X and Y independent random variables? (iv) $Cov(x, y)$ (v) σ_x, σ_y (vi) $\rho(x, y)$

$x \backslash y$	-3	2	4	$f(x)$
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
$g(y)$	0.4	0.3	0.3	$Total = 1$

(i) Marginal probability distribution of X:

x	1	3
$f(x)$	0.5	0.5

Marginal probability distribution of Y:

y	-3	2	4
$g(y)$	0.4	0.3	0.3

(ii) $E(x) = \Sigma x f(x) = 1(0.5) + 3(0.5) = 2$

$E(y) = \Sigma y g(y) = -3(0.4) + 2(0.3) + 4(0.3) = 0.6$

(iii) $E(xy) = \Sigma\Sigma x_i y_j J_{ij} = 1(-3)(0.1) + 1(2)(0.2) + 1(4)(0.2)$

$+3(-3)(0.3) + 3(2)(0.1) + 3(4)(0.1)$

$= -0.3 + 0.4 + 0.8 - 2.7 + 0.6 + 1.2 = 0$

$E(xy) = 0$, $E(x)E(y) = 2(0.6) = 1.2$

Therefore, $E(xy) \neq E(x)E(y)$.

Therefore, x and y are not independent variables.

(iv) $Cov\,(x, y) = E(xy) - E(x)E(y) = 0 - 1.2 = -1.2$

(v) $E(x^2) = \Sigma x^2 f(x) = 1^2(0.5) + 3^2(0.5) = 5$

$E(y^2) = \Sigma y^2 g(y) = (-3)^2(0.4) + 2^2(0.3) + 4^2(0.3) = 9.6$

$\sigma_x = \sqrt{E(x^2) - [E(x)]^2} = \sqrt{5 - 2^2} = 1$

$\sigma_y = \sqrt{E(y^2) - [E(y)]^2} = \sqrt{9.6 - 0.6^2} = 3.0397$

(vi) $\rho(x, y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = -\frac{1.2}{3.0397} = -0.3948$

3. **Find the joint distribution of X and Y which are the independent random variables with the following respective distributions.**

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Since X and Y are independent random variables, $J_{ij} = f(x_i)g(y_j)$

Therefore,

$x\backslash y$	-2	5	8	$f(x)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y)$	0.3	0.5	0.2	$Total = 1$

4. **Consider the joint distribution of X and Y. Compute the following probabilities:**
(i) $P(X = 1, Y = 2)$ (ii) $P(X \geq 1, Y \geq 2)$ (iii) $P(X \leq 1, Y \leq 2)$ (iv) $P(X + Y \geq 2)$ (v) $P(X \geq 1, Y \leq 2)$.

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

(i) $X = \{0, 1\}, Y = \{0, 1, 2, 3, 4\}$

$P(X = 1, Y = 2) = P(1, 2) = \frac{1}{8}$

(ii) If $X \geq 1, X = \{1\}$. If $Y \geq 2, Y = \{2, 3\}$

$P(X \geq 1, Y \geq 2) = P(1, 2) + P(1, 3) = \frac{1}{8} + 0 = \frac{1}{8}$

(iii) If $X \leq 1, X = \{0, 1\}$. If $Y \leq 2, Y = \{0, 1, 2\}$

$P(X \leq 1, Y \leq 2) = P(0, 0) + P(0, 1) + P(0, 2) + P(1, 0) + P(1, 1) + P(1, 2)$

$= 0 + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

(iv) If $X + Y \geq 2$ then $X + Y = 0 + 2$ *or* $0 + 3$ *or* $1 + 1$ *or* $1 + 2$ *or* $1 + 3$

$P(X + Y \geq 2) = P(0, 2) + P(0, 3) + P(1, 1) + P(1, 2) + P(1, 3)$

$= \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + 0 = \frac{3}{4}$

(v) If $X \geq 1, X = \{1\}$. If $Y \leq 2, Y = \{0, 1, 2\}$

$P(X \geq 1, Y \leq 2) = P(1, 0) + P(1, 1) + P(1, 2) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$

5. **A fair coin is tossed thrice. The random variables X and Y are defined as follows: X=0 or 1 according as head or tail occur in the first toss. Y = Number of heads. Determine (i) The distribution of X and Y. (ii) The joint distribution of X and Y.(iii) The expectations of X and Y (iv) Standard deviation of X and Y (v) Covariance of X and Y (vi) Correlation of X and Y.**

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$X = \{0, 1\}$ and $Y = \{0, 1, 2, 3\}$

(i) Marginal distribution of X:

0	1
1/2	1/2

Marginal distribution of Y:

0	1	2	3
1/8	3/8	3/8	1/8

(ii) $J_{00} = P(First\ tail, no\ heads) = 1/8$

$J_{01} = P(First\ tail, 1\ head) = 2/8$

$J_{02} = P(First\ tail, 2\ heads) = 1/8$

$J_{03} = P(First\ tail, 3\ heads) = 0$

$J_{10} = P(First\ head, no\ heads) = 0$

$J_{11} = P(First\ head, 1\ head) = 1/8$

$J_{12} = P(First\ head, 2\ heads) = 2/8$

$J_{13} = P(First\ head, 3\ heads) = 1/8$

The joint distribution of X and Y:

X \ Y	0	1	2	3
0	1/8	2/8	1/8	0
1	0	1/8	2/8	1/8

(iii) $E(x) = \Sigma x f(x) = 0(1/2) + 1(1/2) = 1/2$

$E(y) = \Sigma y g(y) = 0(1/8) + 1(3/8) + 2(3/8) + 3(1/8) = 3/2$

(iv) $E(x^2) = \Sigma x^2 f(x) = 0^2(1/2) + 1^2(1/2) = 1/2$

$E(y^2) = \Sigma y^2 g(y) = 0^2(1/8) + 1^2(3/8) + 2^2(3/8) + 3^2(1/8) = 3$

$\sigma_x = \sqrt{E(x^2) - [E(x)]^2} = \sqrt{1/2 - (1/2)^2} = 1/2$

$\sigma_y = \sqrt{E(y^2) - [E(y)]^2} = \sqrt{3 - (3/2)^2} = \sqrt{3}/2$

(v) $E(XY) = 0 + 0 + 0 + 0 + 0 + 1(1)\left(\frac{1}{8}\right) + 1(2)\left(\frac{2}{8}\right) + 1(3)\left(\frac{1}{8}\right) = 1$

Covariance of X and Y: $Cov(X, Y) = E(XY) - E(X)E(Y) = 1 - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{1}{4}$

(vi) Correlation of X and Y: $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{1}{4} \times 2 \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 1.7321$

6. **The joint probability distribution of two discrete random variables X and Y is given by $f(x,y) = k(2x+y)$ for $0 \le x \le 2$, $0 \le y \le 3$. (i) Find the value of k. (ii) The marginal distribution of X and Y (iii) Show that X and Y are dependent.**

By data, $X = \{0,1,2\}$ and $Y = \{0,1,2,3\}$

$f(x,y) = k(2x+y)$

The joint probability distribution of X and Y:

X \ Y	0	1	2	3	$f(X)$
0	0	k	$2k$	$3k$	$6k$
1	$2k$	$3k$	$4k$	$5k$	$14k$
2	$4k$	$5k$	$6k$	$7k$	$22k$
g(Y)	$6k$	$9k$	$12k$	$15k$	42k

(i) Find the value of k:

$1 = \Sigma f(x,y) = 42k$, $\quad k = \frac{1}{42}$

(ii) Marginal probability distribution of X:

0	1	2
6/42	14/42	22/42

Marginal probability distribution of Y:

0	1	2	3
6/42	9/42	12/42	15/42

(iii) $J_{ij} = f(x_i, y_j) = f(0,1) = k$

$f(x_i) \times g(y_j) = f(0) \times g(1) = 6k \times 9k$

$J_{ij} \neq f(x_i) \times g(y_j)$. Therefore, X and Y are dependent.

7. **The joint probability distribution of X and Y is given by $f(x,y) = c(x^2 + y^2)$ for $x = -1, 0, 1, 3.$ and $y = -1, 2, 3.$ (i) Find the value of c. (ii) $P(x = 0, y \le 2)$ (iii) $P(x \le 1, y > 2)$ (iv) $P(x \ge 2 - y)$.**

By data, $X = \{-1, 0, 1, 3\}$ and $Y = \{-1, 2, 3\}$. $f(x,y) = c(x^2 + y^2)$

The joint probability distribution of X and Y:

X \ Y	-1	2	3	$f(X)$
-1	2c	$5c$	$10c$	$17c$
0	c	$4c$	$9c$	$14c$
1	$2c$	$5c$	$10c$	$17c$
3	$10c$	$13c$	$18c$	$41c$
g(Y)	$15c$	$27c$	$47c$	89c

(i) **Find c:** $1 = \Sigma f(x,y) = 89c$, $c = \frac{1}{89}$

(ii) $x = 0, y = \{-1, 2\}$

$P(x = 0, y \le 2) = P(0,-1) + P(0,2) = c + 4c = 5c = 5/89$

(iii) $x = \{-1, 0, 1\}$, $y = \{3\}$

$P(x \le 1, y > 2) = P(-1,3) + P(0,3) + P(1,3)$

$= 10c + 9c + 10c = 29c = 29/89$

(iv) $P(x \ge 2 - y) = P(x + y \ge 2)$

$= P(-1,3) + P(0,2) + P(0,3) + P(1,2) + P(1,3)$

$+ P(3,-1) + P(3,2) + P(3,3)$

$= 10c + 4c + 9c + 5c + 10c + 10c + 13c + 18c$

$= 79c = \frac{79}{89}$

Home work:

8. Two cards are selected from a box which contains 5 cards numbered 1, 1, 2, 2, 3. Find the joint distribution of X and Y, where X denote the sum and Y denote the maximum of two numbers drawn. Also determine $Cov(x,y)$.

9. The joint distribution of random variables X and Y is
Find Marginal distribution of X and Y.
Are X and Y independent random variables?

X\Y	1	3	6
1	1/9	1/6	1/18
3	1/6	1/4	1/12
6	1/18	1/12	1/36

10. X and Y are independent random variables. X takes values 2, 5, 7 with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. Y take values 3, 4, 5 with probabilities $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. (i) Find the joint probability distribution of X and Y. (ii) Find covariance of X and Y (iii) Find the probability distribution of $Z = X + Y$

2.2 Probability vector and stochastic matrix

Introduction:

- A vector $V = (v_1, v_2, \dots, v_n)$ is called a probability vector if each one of its components are non negative and their sum is equal to 1.

  Example: $u = (1,0), v = \left(\frac{1}{2}, \frac{1}{2}\right), w = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

- A square matrix P having every row in the form of a probability vector is called Stochastic matrix.

  Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

- A stochastic matrix is said to be regular stochastic matrix if all the entries of some power P^n is positive.A stochastic matrix is not regular if 1 occurs in the principal diagonal.

- Regular Stochastic matrix P has a fixed probability vector $V = (v_1, v_2, \dots, v_n)$ such that $VP = V$. Where $v_i = \frac{x_i}{\sum x_i}$.

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1. **Which of the following are probability vectors?**

   (i)$(1, 0)$ (ii) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (iii) $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ (iv) $\left(\frac{1}{4}, \frac{3}{2}, -\frac{1}{4}, \frac{1}{2}\right)$ (v) $\left(\frac{5}{2}, 0, \frac{8}{3}, \frac{1}{6}, \frac{1}{6}\right)$, (vi) $(3,0,2)$

   Ans : yes, yes, yes, no, no, no.

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2. **Which of the following are stochastic matrices?**

   (i) $\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$(ii) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$

   Ans: no (not square matrix), no $(0 - 1 \neq 1)$, yes, no $(\frac{1}{2} + \frac{3}{2} \neq 1)$.

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3. **Which of the stochastic matrices are regular? (i)** $\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

   (ii) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$ **(iii)** $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ **(v)** $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$

   (i) no (1 lies in the main diagonal)
   (i) no (a_{13}, a_{23} *are zero in* $A, A^2, A^3, \dots$)
   (ii) Yes, All entries of P^5 are positive.
   (iii) Yes, All entries of P^5 are positive.
   (iv) Yes, All entries of P^3 are positive.

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4. Find the unique fixed probability vector of the following regular stochastic matrices: (i) $\begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2/3 & 1/3 \\ 2/5 & 3/5 \end{pmatrix}$ (iii) $\begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix}$

(i) Let $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and V be the unique fixed probability vector.

<u>To find</u>: V

$VP = V, \quad where\ V = (x \quad y), \quad x + y = 1.$

$(x \quad y)\begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix} = (x \quad y), \quad x + y = 1.$

$\left(\frac{3x}{4} + \frac{y}{2} \quad \frac{x}{4} + \frac{y}{2}\right) = (x \quad y), \quad x + y = 1.$

Solve $\frac{3x}{4} + \frac{y}{2} = x\ and\ x + y = 1$, we get $x = \frac{2}{3},\ y = \frac{1}{3}$

Therefore, $V = (2/3 \quad 1/3)$.

(ii) Let $P = \begin{pmatrix} 2/3 & 1/3 \\ 2/5 & 3/5 \end{pmatrix}$ and V be the unique fixed probability vector.

<u>To find</u>: V

$VP = V, \quad where\ V = (x \quad y), \quad x + y = 1.$

$(x \quad y)\begin{pmatrix} 2/3 & 1/3 \\ 2/5 & 3/5 \end{pmatrix} = (x \quad y), \quad x + y = 1.$

$\left(\frac{2x}{3} + \frac{2y}{5} \quad \frac{x}{3} + \frac{3y}{5}\right) = (x \quad y), \quad x + y = 1.$

Solve $\frac{2x}{3} + \frac{2y}{5} = x,\ x + y = 1$ we get $x = \frac{6}{11},\ y = \frac{5}{11}$

Therefore, $V = (6/11 \quad 5/11)$

(iii) Let $P = \begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix}$ and V be the unique fixed probability vector.

<u>To find</u>: V

$VP = V, \quad where\ V = (x \quad y), \quad x + y = 1.$

$(x \quad y)\begin{pmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{pmatrix} = (x \quad y), \quad x + y = 1.$

$(0.2x + 0.5y \quad 0.8x + 0.5y) = (x \quad y), \quad x + y = 1.$

Solve $0.2x + 0.5y = x,\ \ x + y = 1$ we get $x = 5/13,\ y = 8/13$

Therefore, $V = (5/13 \quad 8/13)$

5. **Find the unique fixed probability vector of the following regular stochastic matrices:**

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

(i) Let $P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and V be the unique fixed probability vector.

To find: V

$VP = V, \quad where\, V = (x \quad y \quad z), \quad x + y + z = 1.$

$$(x \quad y \quad z)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (x \quad y \quad z), \quad x + y + z = 1.$$

$$\left(\frac{y}{3} \quad \frac{x}{2} + \frac{2y}{3} + z \quad \frac{x}{2}\right) = (x \quad y \quad z), \quad x + y + z = 1.$$

Solve $\frac{y}{3} = x,\ \frac{x}{2} = z\ and\ \ x + y + z = 1.$

We get $x = \frac{2}{9},\ y = \frac{6}{9}, z = \frac{1}{9}$

Therefore, $V = \left(\frac{2}{9} \quad \frac{6}{9} \quad \frac{1}{9}\right)$

(ii) Let $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$ and V be the unique fixed probability vector.

To find: V

$VP = V, \quad where\, V = (x \quad y \quad z), \quad x + y + z = 1.$

$$(x \quad y \quad z)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (x \quad y \quad z), \quad x + y + z = 1.$$

$$\left(\frac{y}{6} \quad x + \frac{y}{2} + z \quad \frac{y}{3} + \frac{z}{3}\right) = (x \quad y \quad z), \quad x + y + z = 1.$$

Solve $\frac{y}{6} = x,\ \frac{y}{3} + \frac{z}{3} = z\ and\ \ x + y + z = 1.$

We get $x = \frac{1}{10},\ y = \frac{6}{10}, z = \frac{3}{10}$

Therefore, $V = (1/10 \quad 6/10 \quad 3/10)$

(iii) Let $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$ and V be the unique fixed probability vector.

To find: V

$VP = V$, where $V = (x \quad y \quad z)$, $x + y + z = 1$.

$$(x \quad y \quad z)\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix} = (x \quad y \quad z), \qquad x + y + z = 1.$$

$$\left(\frac{y}{2}+\frac{z}{2} \quad x+\frac{z}{4} \quad \frac{y}{2}+\frac{z}{4}\right) = (x \quad y \quad z), \qquad x + y + z = 1.$$

Solve $\frac{y}{2}+\frac{z}{2}=x$, $x+\frac{z}{4}=y$ and $x+y+z=1$.

We get $x=\frac{5}{15}$, $y=\frac{6}{15}$, $z=\frac{4}{15}$.

Therefore, $V = (5/15 \quad 6/15 \quad 4/15)$

2.3 Markov Chain

Introduction:

- A stochastic process which is such that the generation of the probability distribution depend only on the present state is called a Markov process.
- If this state space is discrete, then Markov process is called Markov chain.
- Transition probability matrix of a Markov chain is a Stochastic matrix.
- n step transition matrix of $P = P^n$.
- A Markov chain is said to be regular if the associated transition matrix is regular.
- If a transition probability matrix is regular, then it is irreducible.
- Unique fixed probability vector is also called as stationary probability vector.
- In the long run, we get a stationary probability vector.

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1. **Prove that the Markov chain whose t.p.m.** $P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ **is irreducible. Also find the corresponding stationary probability vector.**

$$P^2 = \frac{1}{36}\begin{pmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{pmatrix}$$

$\therefore P$ is regular $\Rightarrow P$ is irreducible.

<u>To find:</u> V

$$VP = V, \quad where\ V = (x \quad y \quad z), \quad x + y + z = 1.$$

$$(x \quad y \quad z)\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (x \quad y \quad z), \quad x + y + z = 1.$$

$$\left(\frac{y}{2} + \frac{z}{2} \quad \frac{2x}{3} + \frac{z}{2} \quad \frac{x}{3} + \frac{y}{2}\right) = (x \quad y \quad z), \quad z = 1 - x - y.$$

$$\frac{y}{2} + \frac{z}{2} = x, \quad \frac{2x}{3} + \frac{z}{2} = y, \quad \frac{x}{3} + \frac{y}{2} = z \ and \ z = 1 - x - y.$$

By solving we get $x = \frac{1}{3}$, $y = \frac{10}{27}, z = \frac{8}{27}$. Therefore, $V = \begin{pmatrix} \frac{1}{3} & \frac{10}{27} & \frac{8}{27} \end{pmatrix}$.

This is the required stationary probability vector.

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2. **Three boys A, B, and C throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that after three throws (i) A has the ball (ii) B has the ball (iii) C has the ball.**

State space is $\{A, B, C\}$.

Transition probability matrix is $P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$

After three throws, 3 step transition matrix

$$P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Conclusion: Initially C has the ball. Therefore, After three throws,

(i) $P(A \text{ has the ball}) = \frac{1}{4}$

(ii) $P(B \text{ has the ball}) = \frac{1}{4}$

(iii) $P(C \text{ has the ball}) = \frac{1}{2}$

3. **Every year a man trade his car for a new car. If he has a 'Suzuki', he trades it for an 'Hyundai '. If he has an 'Tata', he trades it for a 'Hyundai'. If he has a 'Hyundai', he is just as likely to trade it for a new 'Hyundai' as to trade it for a 'Suzuki' or an 'Tata'. In 2000 he bought his first car which was a 'Hyundai'.**

(i)Find the probability that he has (a) 2002 Hyundai (b) 2002 Suzuki (c) 2003 Tata (d) 2003 Hyundai. (ii) In a long run, how often will he has a Hyundai?

Case (i)

Let A: Suzuki, B: Tata r, C: Hyundai

State space is $\{A, B, C\}$.

Transition probability matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array}.$$

After two years, 2 step transition matrix

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix}$$

After three years, 3 step transition matrix

$$P^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{27} & \frac{7}{27} & \frac{16}{27} \end{pmatrix}$$

Conclusion: Initially he has Hyundai. Therefore,

(i) P (he has Hyundai in 2002) = 4/9

(ii) P(he has Suzuki in 2002) = 1/9

(iii) P(he has Tata in 2003) = 7/27

(iv) P(he has Hyundai in 2003) = 16/27

Case (ii)

<u>To find</u>: V

$$VP = V, \quad where\ V = (x \quad y \quad z), \quad x + y + z = 1.$$

$$(x \quad y \quad z)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = (x \quad y \quad z), \quad x + y + z = 1.$$

$$\left(\frac{z}{3} \quad x + \frac{z}{3} \quad y + \frac{z}{3}\right) = (x \quad y \quad z), \quad z = 1 - x - y.$$

$$\frac{z}{3} = x, \quad x + \frac{z}{3} = y, \quad y + \frac{z}{3} = z\ and\ z = 1 - x - y.$$

$$z = 3x, \quad y = 2x, \quad z = 1 - 2x - 3x.$$

By solving we get $x = \frac{1}{6}$, $y = \frac{1}{3}$, $z = \frac{1}{2}$.

Therefore, $V = \left(\frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2}\right)$

<u>Conclusion</u>: In the long run, Probability that he has a Hyundai = 1/2.

4. A student's study habits are as follows: If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, 60% sure not to study next night. In the long run how often does he study?

Let A: Studying, B: Not studying

State space is $\{A, B\}$.

Transition probability matrix is $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$

To find: V

$VP = V, \quad where\ V = (x \quad y), \quad x + y = 1.$

$(x \quad y)\begin{pmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{pmatrix} = (x \quad y), \quad x + y = 1.$

$(0.3x + 0.4y \quad 0.7x + 0.6y) = (x \quad y), \quad y = 1 - x.$

$0.3x + 0.4(1 - x) = x, \quad y = 1 - x$

By solving we get $x = \frac{4}{11}$, $y = \frac{7}{11}$. Therefore, $V = \left(\frac{4}{11} \quad \frac{7}{11}\right)$

Conclusion: In the long run, 4/11 of the times he studies.

5. **A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2. On the other hand if smokes non filter cigarettes one week, there is a probability of 0.7 that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes?**

Let A: Filter cigarettes, B: Non filter cigarettes

State space is $\{A, B\}$.

Transition probability matrix is $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$

<u>To find</u>: V

$VP = V, \quad where\ V = (x \quad y), \quad x + y = 1.$

$(x \quad y)\begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} = (x \quad y), \quad x + y = 1.$

$(0.8x + 0.3y \quad 0.2x + 0.7y) = (x \quad y), \quad y = 1 - x.$

$0.8x + 0.3(1 - x) = x, \quad y = 1 - x$

By solving we get $x = \frac{3}{5}$, $y = \frac{2}{5}$. Therefore, $V = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \end{pmatrix}$

<u>Conclusion</u>: In the long run, 3/5 of the times he smokes filter cigarettes.