PARTIAL DIFFERENTIATION

DEFINITION OF PARTIAL DIFFERENTIATION

A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are kept constant.

Consider a function z of two independent variables $z = f(x, y)$ the partial derivative of z with respect to x is denoted by $\frac{\partial z}{\partial x}$ and is defined as the limit

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}.$$

and is also denoted by z_x or $f_x(x, y)$.

Similarly, partial derivative of z w.r.t. y is

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y},$$

also denoted by z_y or $f_y(x, y)$.

Higher order partial derivatives :

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Similarly third and higher order partial derivatives can be found.

Remark

(1) The order of differentiation is immaterial in partial differentiation if the derivatives involved are continuous. Thus, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ or $f_{xy}=f_{yx}$

(2) A function of 'm' independent variables will have m^n, n^{th} order partial derivatives.

Example : *Find the first order partial derivatives of*

(a) $u=\tan^{-1}\left(\frac{y}{x}\right)$, *(b)* $u=e^{2x}\cos y$.

Solution: (a) Given $u=\tan^{-1}\left(\frac{y}{x}\right)$

Differentiating u w.r.t x, keeping y constant, we get

$$u_x=\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$

Similarly, $$u_y=\frac{\partial u}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

(b) Here $u=e^{2x}\cos y$

$$\therefore\ u_x=\frac{\partial u}{\partial x}=2e^{2x}\cos y \quad \text{and} \quad u_y=\frac{\partial u}{\partial y}=-e^{2x}\sin y$$

HOMOGENEOUS FUNCTION

An expression of the form $f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} x y^{n-1} + a_n y^n$ is a polynomial in x, y such that the degree of each of the term is same

Thus, f is a homogenous function of degree n.

$$f_{(x,y)} = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_{n-1}\left(\frac{y}{x}\right)^n + a_n\left(\frac{y}{x}\right)^n\right] = x^n \phi\left(\frac{y}{x}\right)$$

Thus, $z = f(x, y)$, which can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ is a homogeneous function of degree n, where n can be positive, negative or zero real number.

Also, a function $f(x, y)$ of two variables x and y is said to be homogeneous function of degree n for any $\lambda > 0$, if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

Remark : A function $u = f(x, y, z)$ is a homogeneous function of degree n if

$$u = f(x, y, z) = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$$

Example : *Show that* $z = \dfrac{\sqrt{y} + \sqrt{x}}{y + x}$, *is a homogeneous function.*

Solution: $z = f(x, y) = \dfrac{\sqrt{y} + \sqrt{x}}{y + x} = \dfrac{\sqrt{x}\left(1 + \sqrt{\frac{y}{x}}\right)}{x\left(1 + \frac{y}{x}\right)} = x^{-1/2}\left(\dfrac{1 + \sqrt{\frac{y}{x}}}{1 + \frac{y}{x}}\right);$

$$= x^{-1/2} \phi\left(\frac{y}{x}\right)$$

Thus, according to the definition, $z = f(x, y)$ is a homogeneous function of degree $(-1/2)$.

EULER'S THEOREM FOR HOMOGENEOUS FUNCTIONS

Theorem : If $u = f(x, y)$ is a homogeneous function of degree n. Then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof : Since f is a homogeneous function of degree n,

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right)$$

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) - x^n\phi'\left(\frac{y}{x}\right)\cdot\frac{y}{x^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = x^n\phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) - x^{n-1}y\phi'\left(\frac{y}{x}\right) + x^{n-1}y\phi'\left(\frac{y}{x}\right)$$

$$= nx^n\phi\left(\frac{y}{x}\right) = nf$$ which proves the result.

Corollary : *If f is a homogeneous function of degree n, then*

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Proof : Differentiating $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ partially w.r.t x and y, we get

$$\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y \partial x} = n\frac{\partial f}{\partial x} \quad \text{and} \quad x\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y}$$

Multiplying the above equation by x and y respectively, we get

$$x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = (n-1)(xf_x + yf_y)$$

$$= n(n-1)f \quad \text{[Using Euler's theorem]}$$

NOTE: If $u = f(x, y, z)$ is a homo function of three variables of degree n, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$

Example : *Verify Euler's theorem for* $u = \frac{x^2 y^2}{x + y}$.

Solution: Here $u = \frac{x^2 y^2}{x + y} = \frac{x^4}{x}\frac{(y/x)^2}{1 + (y/x)} = x^3\phi\left(\frac{y}{x}\right)$

Thus, by applying Euler's theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

$$\frac{\partial u}{\partial x} = \frac{2xy^2}{x + y} - \frac{x^2 y^2}{(x + y)^2} = \frac{xy^2(x + 2y)}{(x + y)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{2x^2 y}{x + y} - \frac{x^2 y^2}{(x + y)^2} = \frac{2x^3 y + x^2 y^2}{(x + y)^2}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^3 y^2 + 2x^2 y^3 + 2x^3 y^2 + x^2 y^3}{(x + y)^2} = \frac{3x^2 y^2}{x + y} = 3u$$

which verifies the Euler's theorem.

Ex: *Find the first order partial derivatives of* $u = \ln\left(x + \sqrt{x^2 - y^2}\right)$.

Solution: $u_x = \dfrac{\partial u}{\partial x} = \dfrac{1}{x + \sqrt{x^2 - y^2}}\left(1 + \dfrac{1}{2\sqrt{x^2 - y^2}}.2x\right) = \dfrac{1}{\sqrt{x^2 - y^2}}$

and $u_y = \dfrac{\partial u}{\partial y} = \dfrac{1}{x + \sqrt{x^2 - y^2}}\left(0 + \dfrac{1}{2\sqrt{x^2 - y^2}}(-2y)\right) = \dfrac{-y(x^2 - y^2)^{-1/2}}{x + \sqrt{x^2 - y^2}}$

Ex : *Find* $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ *if* $u = e^{xyz}$.

Sol : Differentiating u partially w.r.t. z, we get $u_z = \dfrac{\partial u}{\partial z} = yxe^{xyz}$

Again differentiating partially w.r.t. y,

$$u_{yz} = \frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + x^2 yze^{xyz} = (x + x^2 yz)e^{xyz}$$

Again differentiating partially w.r.t. x,

$$u_{xyz} = \frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 2xyz)e^{xyz} + (x + x^2 yz)yze^{xyz}$$

$$= (1 + 3xyz + x^2 y^2 z^2)e^{xyz}$$

Ex: *If* $u = (x^2 + y^2 + z^2)^{-1/2}$, *show that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Sol: Here, $u = (x^2 + y^2 + z^2)^{-1/2}$

Differentiating u partially w.r.t x,

$$\frac{\partial u}{\partial x} = -\frac{1}{2(x^2 + y^2 + z^2)^{3/2}} .2x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3x}{2(x^2 + y^2 + z^2)^{5/2}} .2x$$

$$= \frac{3x^2 - (x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly, by symmetry, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{3y^2 - (x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = \frac{3z^2 - (x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}.$$

Adding, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0$$

Ex: *If* $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, *show that* $u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$.

Solution: Differentiating the given relation partially w.r.t. x, we get

$$\frac{2x}{a^2+u}-\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}=0 \quad \text{...(i)}$$

By Symmetry, $$\frac{2y}{b^2+u}-\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y}=0 \quad \text{...(ii)}$$

and $$\frac{2z}{c^2+u}-\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z}=0 \quad \text{..(iii)}$$

Assuming $\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}=\lambda$,

from (i), (ii) and (iii), $\lambda\frac{\partial u}{\partial x}=\frac{2x}{a^2+u}$, $\lambda\frac{\partial u}{\partial y}=\frac{2y}{b^2+u}$ and $\lambda\frac{\partial u}{\partial z}=\frac{2z}{c^2+u}$ $\Bigg]$ (iv)

Squaring and adding, we have

$$\lambda^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]=4\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=4\lambda \quad \text{--(v)}$$

Also, multiplying the equations of (iv) by x, by y and by z and adding, we get

$$\lambda\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=2\left(\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+4}\right)=2 \quad \text{--(vi)}$$

Eliminating λ from (v) and (vi), we get

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{\lambda}=2\left(\frac{2}{\lambda}\right)=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Example: *If* $\theta = t^n e^{-r^2/4t}$, *then find the value of n such that* $\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 \frac{\partial \theta}{\partial r}\right\} = \frac{\partial \theta}{\partial t}$.

Solution: Given

$$\theta = t^n e^{-r^2/4t} \qquad \text{...(i)}$$

Differentiating partially w.r.t r, we get

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\left(-\frac{2r}{4t}\right) = -\frac{1}{2}t^{n-1} r e^{-r^2/4t}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 \frac{\partial \theta}{\partial r}\right\} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{1}{2}t^{n-1} r^3 e^{-r^2/4t}\right)$$

$$= -\frac{\frac{1}{2}t^{n-1}}{r^2}\left\{3r^2 e^{-r^2/4t} - \frac{2r}{4t} r^3 e^{-r^2/4t}\right\}$$

$$= \frac{1}{4}t^{n-2} e^{-r^2/4t}(r^2 - 6t) \qquad \text{...(ii)}$$

Again differentiating (i) partially w.r.t. t, we get

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\frac{r^2}{4t^2} = \frac{1}{4}t^{n-2} e^{-r^2/4t}(r^2 + 4nt) \quad \text{...(iii)}$$

Using (ii) and (iii), in the given relation and solving, we have

$$r^2 + 4nt = r^2 - 6t \quad \Rightarrow \quad n = -\frac{3}{2}$$

Example : *If* $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right)$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}\tan u$.

Solution: Here $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right)$

$$\Rightarrow \qquad \sin u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} = V(say)$$

$$\Rightarrow \qquad V = \frac{x^{1/4}}{x^{1/5}} \cdot \frac{\left(1 + \left(\dfrac{y}{x}\right)^{1/4}\right)}{\left(1 + \left(\dfrac{y}{x}\right)^{1/5}\right)} = x^{1/20}\phi\left(\frac{y}{x}\right)$$

Thus, by applying Euler's theorem, we get $x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} = \dfrac{1}{20}V$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{20}\sin u$$

$$\Rightarrow \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{20}\sin u \quad \Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u.$$

Example : *If* $u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, *then show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$.

Solution: Here $u=\tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)\Rightarrow \tan u=\frac{x^3+y^3}{x+y}=V$

$$\Rightarrow \qquad V=x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)}\right]=x^2\phi\left(\frac{y}{x}\right)$$

Thus, on applying Euler's theorem

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=2V\Rightarrow \quad x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u$$

$$\Rightarrow \qquad x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u\cos^2 u=2\sin u\cos u=\sin 2u$$

Example : *If* $u=\sin^{-1}\left(\dfrac{5x+2y+3z}{x^6+y^6+z^6}\right)$ *find the value of* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$.

Solution: Here,
$$u=\sin^{-1}\left(\frac{5x+2y+3z}{x^6+y^6+z^6}\right)$$

$$\Rightarrow \qquad \sin u=\frac{1}{x^5}\left[\frac{5+2\left(\dfrac{y}{x}\right)+3\left(\dfrac{z}{x}\right)}{1+\left(\dfrac{y}{x}\right)^6+\left(\dfrac{z}{x}\right)^6}\right]=\frac{1}{x^5}\phi.\left(\frac{y}{x},\frac{z}{x}\right)=V \quad \text{(say)}$$

Thus by Euler's theorem, we have
$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=(-5)V$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)+z\frac{\partial}{\partial z}(\sin u)=-5\sin u \qquad [\because V=\sin u]$$

$$\Rightarrow \quad \cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=-5\sin u \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-5\tan u.$$

EXERCISE

1. *If* $u = xy\, f\left(\dfrac{y}{x}\right)$. *Then prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$.

2. *Verify Euler's theorem for the following functions*

   (a) $\cos^{-1}\left(\dfrac{x}{y}\right)$ (b) $(ax+by)^{3/2}$

   (c) $\cos^{-1}\left(\dfrac{x}{y}\right) + \cot^{-1}\left(\dfrac{y}{x}\right)$ (d) $\dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2 + y^2}$

3. *If* $u = \sec^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$, *show that* $xu_x + yu_y = 2\cot u$

4. *If* $u = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, *show that* $xu_x + yu_y = -\dfrac{1}{2}\cot u$

5. *If* $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, *show that* $xu_x + yu_y = \tan u$

   and $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \tan^3 u$.

6. *If* $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$, *then prove that* $xu_x + yu_y = \dfrac{\sin 2u}{2}$ *and*

   $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = -\sin 2u \sin^2 u$.

7. *Verify Euler's theorem for* $f = \dfrac{z}{x+y} + \dfrac{y}{z+x} + \dfrac{x}{y+z}$.

8. *If* $r^2 = x^2 + y^2$ *and* $\tan\theta = \dfrac{y}{x}$ *show that*

   $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$ *and* $\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{r^2}$, $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2}$.

TOTAL DERIVATIVE

Let $u = f(x, y)$. then $du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ is total derivative of u.

In general if $u = f(x_1, x_2, \ldots x_n)$, function of n variables, then

$$du = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

Chain rule for partial differentiation :

Case I : *When x and y are functions of single variable :*

Let $u = f(x, y)$ and x and y are functions of a single variable t, $x = \phi(t)$, $y = \psi(t)$.

Now the total derivative of f w.r.t t is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt}$$

(i) If $u = f(x, y)$ and $y = \phi(x)$, then total derivative of f w.r.t. x is given by

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad \text{or} \quad \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

(ii) if $u = f(x, y)$ and $x = \psi(y)$ then total derivative of f, w.r.t. y is given by

$$\frac{df}{dy} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y}$$

Case II : *When x and y are function of two variables* :

Let $u = f(x, y)$ and x, y are functions of two independent variables say s and t i.e. $x = \phi(s,t)$ and $y = \psi(s,t)$.

Now the partial derivatives of f w.r.t. t and s are

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \qquad \text{(keeping } s \text{ constant)}$$

and $$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \text{(keeping } t \text{ constant)}$$

Case III : *Differentiation of an implicit function* :

An implicit function of x and y is an equation of the form $f(x, y) = 0$, which cannot be solved for one of the variables in terms of other variable.

e.g. $x^3 + y^3 = 3axy$

Defining $y = y(x)$, Using chain rule, we get

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{f_x}{f_y}, \text{ provided } f_y \neq 0 .$$

Example : *If* $x^y + y^x = a^b$, *then find* $\dfrac{dy}{dx}$.

Solution: Here $u = x^y + y^x - a^b = 0$ $\therefore$ $\dfrac{dy}{dx} = -\dfrac{\partial u/\partial x}{\partial u/\partial y}$

Now $\dfrac{\partial u}{\partial x} = yx^{y-1} + y^x \log y$ and $\dfrac{\partial u}{\partial y} = x^y \log x + xy^{x-1}$

Thus, $\dfrac{dy}{dx} = -\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$.

Example : *If* $u = f(y-z, z-x, x-y)$, *then prove that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Solution: Let $y - z = t_1$, $z - x = t_2$, $x - y = t_3$

Then $u = f(t_1, t_2, t_3)$ $\therefore$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t_1} \cdot \dfrac{\partial t_1}{\partial x} + \dfrac{\partial u}{\partial t_2} \cdot \dfrac{\partial t_2}{\partial x} + \dfrac{\partial u}{\partial t_3} \cdot \dfrac{\partial t_3}{\partial x}$

$$= \frac{\partial u}{\partial t_1} \cdot 0 + \frac{\partial u}{\partial t_2} \cdot (-1) + \frac{\partial u}{\partial t_3}(1) = -\frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_3} \quad \text{...(i)}$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial t_1} \cdot \dfrac{\partial t_1}{\partial y} + \dfrac{\partial u}{\partial t_2} \cdot \dfrac{\partial t_2}{\partial y} + \dfrac{\partial u}{\partial t_3} \cdot \dfrac{\partial t_3}{\partial y} = \dfrac{\partial u}{\partial t_1} - \dfrac{\partial u}{\partial t_3}$ ...(ii)

and $\dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial t_1} \cdot \dfrac{\partial t_1}{\partial z} + \dfrac{\partial u}{\partial t_2} \dfrac{\partial t_2}{\partial z} + \dfrac{\partial u}{\partial t_3} \cdot \dfrac{\partial t_3}{\partial z} = -\dfrac{\partial u}{\partial t_1} + \dfrac{\partial u}{\partial t_2}$ ..(iii)

Adding (i), (ii) and (iii), we have $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

Example: *If* $u = f\left(\dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x}\right)$, *prove that* $xu_x + yu_y + zu_z = 0$.

Solution: Let $\dfrac{x}{y} = r,\ \dfrac{y}{z} = s,\ \dfrac{z}{x} = t$. $\therefore\ u = f(r, s, t)$

Now $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\cdot\dfrac{\partial t}{\partial x} = \dfrac{1}{y}\dfrac{\partial u}{\partial r} + 0\cdot\dfrac{\partial u}{\partial s} - \dfrac{z}{x^2}\dfrac{\partial u}{\partial t}$

Thus, $$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} + \dfrac{\partial u}{\partial t}\cdot\dfrac{\partial t}{\partial y} = -\dfrac{x}{y^2}\dfrac{\partial u}{\partial r} + \dfrac{1}{z}\dfrac{\partial u}{\partial s} + 0\cdot\dfrac{\partial u}{\partial t}$

$\therefore$ $$y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \qquad \text{...(ii)}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= 0\cdot\frac{\partial u}{\partial r} - \frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t} \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

Example : *If* $x = \xi\cos\alpha - \eta\sin\alpha, y = \xi\sin\alpha + \eta\cos\alpha$, *show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}.$$

Solution: Here $u = f(x, y)$ and

$$\therefore \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi} = \frac{\partial u}{\partial x}\cdot(\cos\alpha) + \frac{\partial u}{\partial y}\cdot(\sin\alpha) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)u$$

and $$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta} = \frac{\partial u}{\partial x}(-\sin\alpha) + \frac{\partial u}{\partial y}(\cos\alpha) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)u$$

Thus, $\frac{\partial}{\partial \xi} \equiv \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}$; $\quad \frac{\partial}{\partial \eta} \equiv -\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}$

Now $$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right) = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y}\right)$$

$$= \cos\alpha\frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y}\right) + \sin\alpha\frac{\partial}{\partial y}\left(\cos\alpha\frac{\partial u}{\partial x} + \sin\alpha\frac{\partial u}{\partial y}\right)$$

$$= \cos^2\alpha\frac{\partial^2 u}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 u}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 u}{\partial y^2} \qquad \text{....(i)}$$

and $$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial u}{\partial x} + \cos\alpha\frac{\partial u}{\partial y}\right)$$

$$= -\sin\alpha\frac{\partial}{\partial x}\left(-\sin\alpha\frac{\partial u}{\partial x} + \cos\alpha\frac{\partial u}{\partial y}\right) + \cos\alpha\frac{\partial}{\partial y}\left(-\sin\alpha\frac{\partial u}{\partial x} + \cos\alpha\frac{\partial u}{\partial y}\right)$$

$$= \sin^2\alpha\frac{\partial^2 u}{\partial x^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 u}{\partial x\partial y} + \cos^2\alpha\frac{\partial^2 u}{\partial y^2} \qquad \text{..(ii)}$$

Adding (i) and (ii) $$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

EXERCISE

1. *If* $(\cos x)^y = (\sin y)^x$. *Find* $\frac{dy}{dx}$. [**Ans.** $\frac{y\tan x + \log \sin y}{\log \cos x - x \cot y}$]

2. *If* $z = f(x, y)$ *where* $x = e^u \cos v$, $y = e^u \sin v$ *show that* $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.

3. *If* $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$. *Find* $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.
   [**Ans.** $2(2x + y), -6x - 16y$]

4. *If* $V = f(2x - 3y, 3y - 4z, 4z - 2x)$. *Prove that* $6\frac{\partial V}{\partial x} + 4\frac{\partial V}{\partial y} + 3\frac{\partial V}{\partial z} = 0$.

5. *If* V *is a function of* u, v, *where* $u = x - y$ *and* $v = xy$. *Prove that*
$$x\frac{\partial^2 V}{\partial x^2} + y\frac{\partial^2 V}{\partial y^2} = (x + y)\left(\frac{\partial^2 V}{\partial u^2} + xy\frac{\partial^2 V}{\partial v^2}\right).$$

6. *If* $\cos z = u + v$ *and* $u = \sin x$, $v = \cos x$. *Find the total derivative of* z *w.r.t* x.
   [**Ans.** $\frac{dz}{dx} = \frac{u - v}{\sqrt{1 - (u + v)^2}}$]

7. *If* $x = r\cos\theta$, $y = r\sin\theta$, *prove that* $\left(\frac{\partial^2 r}{\partial x^2}\right)\left(\frac{\partial^2 r}{\partial y^2}\right) = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$.
   $\left[\textbf{Hint}: r_{xx} = \frac{1}{r} - \frac{x^2}{r^3}, r_{yy} = \frac{1}{r} - \frac{y^2}{r^3}, r_{xy} = \frac{-xy}{r^3}\right]$

8. *Find* $\frac{dy}{dx}$ *when* $y^{x^y} = \sin x$. [**Ans.** $\frac{-yx^{y-1}\ln y + \cot x}{x^y \ln x \cdot \ln y + x^y y^{-1}}$]

9. *Find* $\frac{d^2 y}{dx^2}$ *if* $x^5 + y^5 = 5a^3x^2$. [**Ans.** $\frac{6a^3x^2}{y^9}(a^3 + x^3)$]

10. *For the curve* $xe^y + ye^x = 0$, *find the equation of tangent line at the origin.*
    $\left[\textbf{Hint}: \text{Find}\left(\frac{dy}{dx}\right)_{(0,0)}\right]$ [**Ans.** $x + y = 0$]

THANK YOU

PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE

BY Mrs. SHALINI WALIA

ASSTT. PROF.

ENGINEERING COLLEGE AJMER

UNIT	CONTENTS
1	Matrices: Rank of a matrix, rank-nullity theorem; System of linear equations; Symmetric, skew-symmetric and orthogonal matrices; Eigenvalues and eigenvectors; Diagonalization of matrices; Cayley-Hamilton Theorem, and Orthogonal transformation.
2	First order ordinary differential equations: Linear and Bernoulli's equations, Exact equations, Equations not of first degree: equations solvable for p, equations solvable for y, equations solvable for x and Clairaut's type.
3	Ordinary differential equations of higher orders: Linear Differential Equations of Higher order with constant coefficients, Simultaneous Linear Differential Equations, Second order linear differential equations with variable coefficients: Homogenous and Exact forms, one part of CF is known, Change of dependent and independent variables, method of variation of parameters, Cauchy- Euler equation; Series solution: Power series solutions including Legendre differential equation and Bessel differential equations.
4	Partial Differential Equations – First order: Order and Degree, Formation; Linear Partial differential equations of First order, Lagrange's Form, Non Linear Partial Differential equations of first order, Charpit's method, Standard forms.
5	Partial Differential Equations– Higher order: Classification of Second order partial differential equations, Separation of variables method to simple problems in Cartesian coordinates including two dimensional Laplace, one dimensional Heat and one dimensional Wave equations.

Sr. No.	Objectives
CO1	To develop different operations and properties of matrices and various applications of matrices.
CO2	To solve the various ordinary differential equations of first order.
CO3	To solve (analytically or in terms of series) the various ordinary linear differential equations of higher order.
CO4	To solve the various partial differential equations of different order and degree.

PARTIAL DIFFERENTIAL EQUATION

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.

The partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ will be denoted by p and q respectively.

Examples
$$pz - qz = (x + y)^2 + z^2$$
$$p^2 - qy^2 = y^2 - x^2$$

ORDER AND DEGREE

The order of a partial differential equation (PDE) is defined as the order of the highest partial derivative occurring in the PDE.

The **degree** of a PDE is the degree of the highest order derivative which occur in it

The partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ are denoted by p,q,r,s and t respectively.

For example $$\frac{\partial z}{\partial x}+7\frac{\partial z}{\partial y}=9 \qquad \text{......(i)}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \qquad \text{.......(ii)}$$

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad \text{.......(iii)}$$

$$\frac{\partial^2 u}{\partial x \partial y}=\left(\frac{\partial u}{\partial z}\right)^2 \qquad \text{.......(iv)}$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)^2+\left(\frac{\partial^2 u}{\partial y^2}\right)^3+8u=0 \qquad \text{.......(v)}$$

The order of a partial differential equation is the order of the highest order partial derivative in the equation. The degree of an equation is the degree of the highest order derivative occurring in the equation. The equation (i) is of order one while (ii),(iii) and (iv) are of order 2. While (v) is of 3rd order and 2nd degree.

Linearity : A partial differential equation is linear if it is of the first degree in the dependent variable (the unknown function) and its partial derivatives and are not multiplied together.

	Partial differential equation	Order	Linear/non-linear
1.	$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z$	One	Linear
2.	$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$	One	Linear
3.	$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ (Laplace's Equation)	Two	Linear
4.	$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)=4xy$	One	Non-linear
5.	$\frac{\partial z}{\partial t}+c^2\frac{\partial^2 z}{\partial x^2}$ (Heat conduction equation)	Two	Linear
6.	$\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$ (Wave equation)	Two	Linear
7.	$\left(\frac{\partial^2 z}{\partial x^2}\right)^2=xy$	Two	Non-linear
8.	$\left(\frac{\partial^2 x}{\partial x\partial y}\right)^4=\left(\frac{\partial z}{\partial x}\right)^3$	Two	Non-linear

FORMATION OF A PARTIAL DIFFERENTIAL EQUATION

Partial differential equations can be formed by two method :

Elimination of Arbitrary Constants

Let z be a function of x and y such that $f(x, y, z, a, b) = 0$ ...(1)

where a and b are arbitrary constants.

Differentiating (1) partially with respect to x and y we get the relations

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0 \quad \text{...(2)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0 \quad \text{...(3)}$$

Using equation (1), (2) and (3), constants a and b can be eliminated and relation obtained is $\boxed{f(x, y, z, p, q) = 0}$

EXAMPLE

Form DE by eliminating a and b from $z = (x^2 + a)(y^2 + b)$.

SOLUTION : We have $\quad z = (x^2 + a)(y^2 + b) \quad \quad ...(1)$

Differentiating (1) partially wrt x and y, we have

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b) \text{ or, } \quad y^2 + b = \frac{p}{2x} \quad \quad ...(2)$$

and $$q = \frac{\partial z}{\partial y} = 2y(x^2 + a) \text{ or, } \quad x^2 + a = \frac{q}{2y} \quad \quad ...(3)$$

Substituting values of $(y^2 + b)$ and $(x^2 + a)$ in (1)

$$z = \frac{q}{2y}\left(\frac{p}{2x}\right) \quad \text{or,} \quad 4xyz = pq$$

which is required DE.

Elimination of Arbitrary Function Consider the relation between x, y and z of the type $f(u,v)=0$ or $u=\phi(v)$ where u and v are known function of x, y and z and f is an arbitrary function of u and v and ϕ is a arbitrary function

Here we assume z as dependent variable and x and y as independent variable so that $\frac{\partial z}{\partial x}=p$, $\frac{\partial z}{\partial y}=q$

Differentiating the given D.E. wrt independent variables x and y

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, we get Pp+Qq=R

where $P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}$, $Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$, $R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

EXAMPLE

Form a partial DE by eliminating the arbitrary function f from $f(x+y+z,\ x^2+y^2-z^2)=0$.

SOLUTION : We have $f(x+y+z,\ x^2+y^2-z^2)=0$ ...(1)

Let $$u = x+y+z \text{ and } v = x^2+y^2-z^2 \quad \text{...(2)}$$

Then (1) becomes $$f(u,v)=0 \quad \text{...(3)}$$

Differentiating eqn (3) partially wrt x, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial u}{\partial z}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right)=0 \quad \text{...(4)}$$

$$\frac{\partial f}{\partial u}(1+p)+\frac{\partial f}{\partial v}(2x-2pz)=0 \quad \text{...(5)}$$

or, $$\frac{\partial f}{\partial u}(1+p)=-2\frac{\partial f}{\partial v}(x-pz)$$

or, $$\frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{2(x-pz)}{1+p} \qquad \ldots(6)$$

Differentiating (3) partially wrt y, we get

$$\frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{2(y-qz)}{1+q} \qquad \ldots(7)$$

From (6) and (7), we get $\frac{x-pz}{1+p} = \frac{y-qz}{1+q}$

or, $$(1+q)(x-pz) = (1+p)(y-qz)$$

or, $$(y+z)p - (x+z)q = x-y$$

which is the required partial DE.

EXAMPLE

Form partial DE by eliminating the function ψ from $z = e^{ax+by}\psi(ax-by)$

SOLUTION : We have $\quad z = e^{ax+by}\psi(ax-by) \quad \ldots(1)$

Differentiating (1) partially wrt x and y, we get

$$p = \frac{\partial z}{\partial x} = ae^{ax+by}\psi(ax-by) + ae^{ax+by}\psi'(ax-by) \quad \ldots(2)$$

and $q = \dfrac{\partial z}{\partial y} = be^{ax+by}\psi(ax-by) - be^{ax+by}\psi'(ax-by) \ldots(3)$

multiply (2) by b and (3) by a and adding we get

$$bp + aq = 2abe^{ax+by}\psi(ax-by)$$

or, $$bp + aq = 2abz$$

which is required partial DE

LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER : LAGRANGE'S METHOD

A partial differential equation in which only the first order derivatives (i.e. p and q) appear, is called a partial differential equation of first order.

Lagrange's linear partial differential equation of first order is $Pp + Qq = R$, where P, Q and R are functions of x, y and z.

Lagrange's method of solving $Pp + Qq = R$

Lagrange's auxiliary or subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

solve these equations get two independent relation

$u = a$ and $v = b$, *where* u and v are functions of x,y,z and a and b are arbitrary constants.

The partial differential equation can be solved by two ways which are given below.

(a) Method of Grouping. In this , we take two terms from the auxiliary equation, say $\frac{dx}{P} = \frac{dy}{Q}$ and find a differential equation in x and y only. This can be solved and get one solution $u = a$.

Similarly, $\frac{dx}{P} = \frac{dz}{R}$ or $\frac{dy}{Q} = \frac{dz}{R}$ and find the second solution $v = b$.

(b) Method of Multipliers.

In this case, we use multipliers l, m, n (which are not always constant) and find

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$$

These multipliers can be so chosen that $lP+mQ+nR=0$ .Then $ldx+mdy+ndz=0$

After integration, we get one solution $u=a$.

Again by using another set of multipliers l,m,n we get another solution $v=b$.

The general solution is then written in one of the following forms,

$\phi(u,v)=0$ or $u=\phi(v)$ or $v=\phi(u)$, where ϕ is an arbitrary function.

EXAMPLE Solve $(x^2-yz)\,p+(y^2-zx)\,q=z^2-xy$.

SOLUTION : The Lagrange's AE's are $\dfrac{dx}{x^2-yz}=\dfrac{dy}{y^2-zx}=\dfrac{dz}{z^2-xy}$

Choosing $1,\ -1,\ 0$ and $0,\ 1,\ -1$ as multiplier of each fraction of eqn ,we get

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}=\frac{dx-dy}{x^2-yz-(y^2-zx)}=\frac{dy-dz}{y^2-zx-(z^2-xy)}$$

or, $$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}$$

or, $$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

Integrating above equation, we get $\log(x-y)-\log(y-z)=\log C_1$ or, $\dfrac{x-y}{y-z}=C_1$

Choosing x, y, z and 1, 1, 1 as multiplier, we get

$$\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

or, $$\frac{xdx+ydy+zdz}{x+y+z}=dx+dy+dz=d(x+y+z)$$

or, $(x+y+z)\,d(x+y+z)=xdx+ydy+zdz$

or, $2(x+y+z)\,d(x+y+z)$

$$-(2xdx+2ydy+2zdz)=0$$

$$(x+y+z)^2-(x^2+y^2+z^2)=C_2$$

or, $$xy+yz+zx=C_2$$

Now the required solution as $\dfrac{x-y}{y-z}=\psi(xy+yz+zx)$

where ψ is an arbitrary function.

EXAMPLE Solve $(y^2+z^2-x^2)\,p-2xyq=-2zx.$

SOLUTION : The Lagrange's AE's are

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2zx}$$

Taking last two fraction, we get $\dfrac{dy}{-2xy}=\dfrac{dz}{-2zx}$

or, $\dfrac{dy}{y}-\dfrac{dz}{z}=0$

On integrating eqn, we get $\log y-\log z=\log C_1$ or, $\boxed{\dfrac{y}{z}=C_1}$

Choosing x, y, z as multiplier of each fraction in eqn, we get

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2zx}=\frac{xdx+ydy+zdz}{x(y^2+z^2-x^2)-2xy^2-2z^2x}$$

$$=\frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}$$

Taking last fraction, we get

$$\frac{dz}{-2zx}=\frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}$$

or, $\dfrac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} - \dfrac{dz}{z} = 0$ On integrating , we get

$$\log(x^2 + y^2 + z^2) - \log z = \log C_2$$

or, $$\boxed{\frac{x^2 + y^2 + z^2}{z} = C_2}$$

the required general solution is $\dfrac{x^2 + y^2 + z^2}{z} = \psi\left(\dfrac{y}{z}\right)$ where ψ is an arbitrary function.

Ex : *Solve* $\dfrac{y - z}{yz}p + \dfrac{z - x}{zx}q = \dfrac{x - y}{xy}$.

Ex : *Solve* $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$.

Ex : Solve $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$.

Non Linear Partial Differential Equations of order 1

There are five ways of non-linear partial differential equations of first order and their method of solution as given below.

- Type I: $f(p,q)=0$
- Type II: $f(p,q,z)=o$
- Type III: $f_1(p\,,x)=\ f_3(y,q)$ (variable separable method)
- Type IV: Clairaut's Form
- CHARPIT'S METHOD

CHARPIT METHOD

The partial differential equation of first order and non-linear in p and q is $f(x, y, z, p, q) = 0$ ...(1)

Its solution will depend on the subsidiary equation.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{df}{0}$$

An integral of these equation, involving p or q or both, with the given differential eq. gives the values of p & q

Since z depends on x and y, we can write

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{or} \qquad dz = p\,dx + q\,dy$$

Substituting the values of p & q in the above and integrating we can get the required relation.

EX Find complete integrals of $2xz - px^2 - 2qxy + pq = 0$

SOL: We have $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq \quad \ldots(1)$

The subsidiary eq.

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

The 2nd fraction gives $dq = 0 \Rightarrow q = a$

Substituting $q = a$ in (1), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

Now $\quad dz = p\,dx + q\,dy$

or, $dz = \dfrac{2x(z - ay)}{x^2 - a}dx + a\,dy$

or, $\quad \dfrac{dz - a\,dy}{z - ay} = \dfrac{2x}{x^2 - a}dx$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or, $\quad z = ay + b(x^2 - a)$

which is a complete integral.

EXAMPLE Find complete integral of $pxy + pq + qy = yz$.

SOLUTION : $f(x, y, z, p, q) = pxy + pq + qy - yz \quad \ldots(1)$

The subsidiary eq.

$$\frac{dp}{py - py} = \frac{dq}{(px + q) - qy} = \frac{dz}{-p(xy + q) - q(p + y)}$$

$$= \frac{dx}{-(xy + q)} = \frac{dy}{-(p + y)}$$

From first fraction, we get $dp = 0 \Rightarrow p = a$

Substituting $p = a$ in the given eq., we get $q = y\dfrac{(z - ax)}{a + y}$

Now $\quad dz = pdx + qdy$

or, $\quad dz = adx + \dfrac{y(z - ax)}{a + y}dy$

or, $\quad \dfrac{dz - adx}{z - ax} = \dfrac{y}{a + y}dy = \left(1 - \dfrac{a}{a + y}\right)dy$

On integrating above equation, we get

$$\log(z - ax) = y - a\log(a + y) + \log b$$

or, $(z - ax)(a + y)^a = be^y$ which is a complete integral.

Complete Integral The relation which contains as many arbitrary constants and is a solution of a partial differential equation of the first order is called a complete integral

Particular Integral A particular integral can be obtained by giving particular values to arbitrary constants a and b in complete integral

Singular Integral The singular integral can be obtained by eliminating a and b between the three equation.

$$f=0,\ \frac{\partial f}{\partial a}=0 \text{ and } \frac{\partial f}{\partial b}=0$$

The General Integral If in a complete integral one of the constant is a function of the other, say $b=\phi(a)$ then complete integral becomes the general integral

$$f(x,y,z,a,\phi(a))=0$$

STANDARD FORMS

(I) Equation having only p and q $f(p, q) = 0 \quad \ldots(1)$

The complete integral is given by $z = ax + by + C \quad \ldots(2)$

From (3) $b = \phi(a)$ where $f(a, b) = 0 \quad \ldots(3)$

The complete integral is $z = ax + \phi(a)\, y + C$

General Integral Substituting $C = \psi(a)$,

$z = ax + \phi(a)\, y + \psi(a)$ where ψ arbitrary function

Differentiating with respect to a, we get

$$0 = x + \phi'(a)\, y + \psi'(a)$$

The general integral is obtained by eliminating a between above two equations.

Singular Integral is obtained by eliminating a and C between the complete integral and the equation formed by differentiating complete integral wrt a and C. We get

$z = ax + \phi(a)\, y + C \qquad 0 = x + \phi'(a)\, y$

and $0 = 1$ Since $0 = 1$ is inconsistent therefore in this case there is no singular integral.

EXAMPLE Solve $pq = k$ where k is a constant.

SOLUTION : We have $\quad pq = k \quad \ldots(1)$

Let solution is $\quad z = ax + by + c \quad \ldots(2)$

Now substituting $p = a$, and $q = b$ we get $ab = k \quad \ldots(3)$

Using (2) and (3) we get $z = ax + \dfrac{k}{a}y + c \quad \ldots(4)$

which is complete integral, a and c arbitrary constant.

For singular solution, differentiating eqn(4) partially wrt a and c, we have $0 = x - \dfrac{k}{a^2}y$ and $0 = 1$

But $0 \neq 1$. So, there is no singular solution of (1).

General solution :

Let $c = \phi(a)$, then (4) become $z = ax + \dfrac{k}{a}y + \phi(a) \quad \ldots(5)$

Differentiating eqn(5), partially wrt a, we get

$$0 = x - \frac{k}{a^2}y + \phi'(a) \quad \ldots(6)$$

Eliminating a from (5) & (6), we get the general solution.

EXAMPLE Solve $x^2 p^2 + y^2 q^2 = z^2$

SOLUTION : We have $x^2 p^2 + y^2 q^2 = z^2 \quad \ldots(1)$

or, $$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1$$

or, $$\left(\frac{x\,\partial z}{z\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{z\,\partial y}\right)^2 = 1 \qquad \ldots(2)$$

Substituting $$\frac{\partial x}{x} = \partial X \Rightarrow \log x = X$$

$$\frac{\partial y}{y} = \partial Y \Rightarrow \log y = Y$$

$$\frac{\partial z}{z} = \partial Z \Rightarrow \log z = Z$$

in eqn(2) we get $\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$

or, $$P^2 + Q^2 = 1 \qquad \ldots(3)$$
which is of the form $f(P, Q) = 0$. It solution is
$$Z = aX + bY + c \qquad \ldots(4)$$
where a and b are given by relation
$$a^2 + b^2 = 1 \qquad \ldots(5)$$
From eqn(5) and (4), we get
$$Z = aX + Y\sqrt{1 - a^2} + c \qquad \ldots(6)$$
which is complete integral for (3). Substituting X, Y and Z, we get
$$\log z = a \log x + \sqrt{1 - a^2} \log y + \log c'$$
or, $$z = c' x^a y^{\sqrt{1 - a^2}}$$
which is required complete integral.

(II) Equation involving only p, q and z $f(z,p,q)=0$

Let the trial solution is $z = f(x + ay)$

where a is an arbitrary constant.

$$z = f(X) \text{ where } X = x + ay$$

Now
$$p = \frac{\partial z}{\partial x} = \frac{dz}{dX}\frac{\partial X}{\partial x} = \frac{dz}{dX}$$
$$q = \frac{\partial z}{\partial y} = \frac{dz}{dX}\frac{\partial X}{\partial y} = a\,\frac{dz}{dX}$$

Now the given PDE becomes $f\left(z, \dfrac{dz}{dX}, a\dfrac{dz}{dX}\right) = 0$

which an ordinary DE of order one. Integrating it we may get the complete integral.

Example : *Solve* $p^3+q^3=27z$.

Solution: Taking $z=f(x+ay)=f(X)$

so that $p=\frac{\partial z}{\partial x}=\frac{dz}{dX}$, and $q=\frac{\partial z}{\partial y}=a\frac{dz}{dX}$

$\therefore$ The given equation reduces to $(1+a^3)\left(\frac{dz}{dX}\right)^3=27z$

or $(1+a^3)^{1/3}\frac{dz}{dX}=3z^{1/3}$

or $(1+a^3)^{1/3}\frac{2}{3}z^{-(1/3)}dz=2dX$

Integrating,

$$z^{2/3}(1+a^3)^{1/3}=2X+2c=2(x+ay+c)$$

or $(1+a^3)z^2=8(x+ay+c)^3$ This is the complete integral.

(III) Equation of the form $f(x,p) = g(y,q)$

Let us put side equals to an arbitrary constant. Then $f(x,p) = g(y,q) = a$ we obtain

$$p = f_1(x,a) \text{ and } q = f_2(y,a)$$

Now $dz = pdx + qdy$

$$dz = f_1(x,a)\,dx + f_2(y,a)\,dy$$

Integrating we get

$$z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$$

which is the complete integral.

EXAMPLE Find a complete integral of $yp = 2yx + \log q$.

SOLUTION : We have $yp = 2yx + \log q$ ...(1)

or, $$p = 2x + \frac{1}{y}\log q \quad ...(2)$$

or, $$p - 2x = \frac{1}{y}\log q \quad ...(3)$$

Equating each side to an arbitrary constant a, we have

$$p - 2x = a \Rightarrow p = a + 2x \quad ...(4)$$

$$\frac{1}{y}\log q = a \Rightarrow q = e^{ay} \quad ...(5)$$

We have $$dz = p\,dx + q\,dy \quad ...(6)$$

Substituting values of p and q in eqn6), we get

$$dz = (a + 2x)\,dx + e^{ay}\,dy \quad ...(7)$$

Integrating we get $z = ax + x^2 + \dfrac{e^{ay}}{a} + b$

EXAMPLE $z(p^2 - q^2) = x - y.$

SOLUTION : We have $z(p^2 - q^2) = x - y$ ...(1)

or, $$z\left(\frac{\partial z}{\partial x}\right)^2 - z\left(\frac{\partial z}{\partial y}\right)^2 = x - y$$

or, $$\left(\sqrt{z}\,\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\,\frac{\partial z}{\partial y}\right)^2 = x - y \quad ...(2)$$

Substituting $\sqrt{z}\,dz = dZ \Rightarrow \frac{2}{3} z^{3/2} = Z$ in eqn(2), we get

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

or, $$P^2 - x = Q^2 - y \quad ...(3)$$

$P^2 - x = a \Rightarrow P = \sqrt{a + x}$ $\quad Q^2 - y = a \Rightarrow Q = \sqrt{a + y}$

Now $dZ = P dx + Q dy$

Substituting values of P and Q, we get

$$dZ = \sqrt{a + x}\,dx + \sqrt{a + y}\,dy$$

Integrating above eqn, we get

$$Z = \frac{2}{3}(a + x)^{3/2} + \frac{2}{3}(a + y)^{3/2} + \frac{2}{3} b$$

or, $$z^{3/2} = (a + x)^{3/2} + (a + y)^{3/2} + b$$

(IV) Equation of the form $z = px + qy + f(p, q)$ are called clairaut's form

The complete integral of clairaut's equation is

$$z = ax + by + f(a, b)$$

EXAMPLE Solve $z = px + qy + pq$

SOLUTION : We have $z = px + qy + pq$ ...(1)

which is of clairaut's form. Thus its complete integral is

$z = ax + by + ab$ ...(2) a and b arbitrary constant.

Singular Integral :

Differentiating eqn (2) partially wrt a and b, we get

$$0 = x + b \text{ and } 0 = y + a \quad ...(3)$$

Eliminating a and b between (2) and (3), we get

$$z = -xy - xy + xy = -xy \quad ...(4)$$

which is required singular solution.

General Integral :

Let $b = \phi(a)$ where ϕ is arbitrary function, eq(2) becomes

$$z = ax + \phi(a)\,y + a\phi(a) \qquad \ldots(5)$$

Differentiating eqn(5) partially wrt a, we get

$$0 = x + y\phi'(a) + \phi(a) + a\phi'(a) \ldots(6)$$

Eliminating a from (5) & (6), we get the general solution.

EXAMPLE Solve $z = px + qy + p^2 q^2$.

SOLUTION : Its a Clairaut's form. Its complete solution is

$z = ax + by + a^2 b^2$ ...(1) a and b arbitrary constants.

Differentiating (1) partially wrt a and b, we get

$$0 = x + 2ab^2 \text{ and } 0 = y + 2a^2 b \qquad \text{...(2)}$$

From (2) we get $a = -\left(\dfrac{y^2}{2x}\right)^{1/3}$ and $b = -\left(\dfrac{x^2}{2y}\right)^{1/3}$

Substituting the values of a and b in (1) we get

$$z = -x\left(\frac{y^2}{2x}\right)^{1/3} - y\left(\frac{x^2}{2y}\right)^{1/3} + \left(\frac{x^2 y^2}{16}\right)^{1/3}$$

or, $$z = -3\left(\frac{xy}{y}\right)^{2/3}$$

which is required complete integral.

THANK YOU