THE SUCCESS NOTES

DIFFERENTIAL EQUATION II

NOTES BY

Dr. Anil Maheshwari

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SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

By Dr. Anil Maheshwari

Assistant Professor, Mathematics

Engineering College, Ajmer

INTRODUCTION

INTRODUCTION

In general, a linear differential equation of second order with variable coefficients can be written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q, R are functions of independent variable x.

HOMOGENEOUS FORM

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

A second order linear differential equation of the form

$$\boxed{a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = X}$$

is called homogeneous,

where a_0, a_1, a_2 are constants, X is either a constant or a function of x only.

Working Procedure for finding solution of homogeneous differential equation of second order

1. Substitute $x = e^z$ or $z = \log x$ and successively

$$x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y\ ;\ D = \frac{d}{dz} \text{ in given differential equation}$$

$$a_0x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = X$$

2. Given differential equation reduces to $[a_0D(D-1) + a_1D + a_2]\,y = f(z)$ where $f(z)$ is obtained by substituting $x = e^z$ in X
3. Above differential equation is linear with constant coefficients in z and y, which can be solved by methods discussed in previous chapter.
4. Substitute z by $\log x$ in final step of solution to find the solution in given variables x and y

EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

A differential equation of the form

$$\boxed{a_0 (a + bx)^2 \frac{d^2y}{dx^2} + a_1(a + bx)\frac{dy}{dx} + a_2y = X,}$$

where a, b, a_0, a_1, a_2 are constants, X is either a constant or a function of x only, can be reduced to homogeneous form.

Working Procedure for finding solution of second order differential equation reducible to homogeneous form

1. Substitute $ax + b = e^z$ or $z = \log(ax + b)$ and successively

   $(ax + b)\dfrac{dy}{dx} = aDy$, $(ax + b)^2 \dfrac{d^2y}{dx^2} = a^2D(D - 1)y$; $D = \dfrac{d}{dz}$ in given

   differential equation $a_0(ax + b)^2 \dfrac{d^2y}{dx^2} + a_1(ax + b)\dfrac{dy}{dx} + a_2y = X$

2. Given differential equation reduces to $[a_0a^2D(D-1) + a_1aD + a_2]y = f(z)$ where $f(z)$ is obtained by substituting $ax + b = e^z$ in X
3. Above differential equation is linear with constant coefficients in z and y, which can be solved by methods discussed in previous chapter.
4. Substitute z by $\log(ax + b)$ in final step of solution to find the solution in given variables x and y

Exa. *Solve the following differential equations :*

(i) $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$

(ii) $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2) \dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Sol. (*i*) $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$

The given differential equation is a homogeneous linear differential equation of order two. Taking $x = e^z$ or $z = \log x$, we get

$$x \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2y}{dx^2} = D(D-1)\,y \text{ where } D = \frac{d}{dz}$$

then given differential equation becomes

$$\{D(D-1) - D + 4\}\,y = (\cos z + e^z \sin z)$$

$$\Rightarrow \qquad (D^2 - 2D + 4)\,y = (\cos z + e^z \sin z)$$

which is a linear differential equation of second order with constant coefficients.

Now, for C.F., auxiliary equation in m is given as

$$m^2 - 2m + 4 = 0$$

$$\Rightarrow \qquad m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$\Rightarrow \qquad m = \frac{2 \pm \sqrt{(4-16)}}{2}$$

$$\Rightarrow \qquad m = \frac{2 \pm \sqrt{-12}}{2}$$

$$\Rightarrow \qquad m = \frac{2 \pm \sqrt{12i^2}}{2}$$

$$\Rightarrow \qquad m = \frac{2 \pm 2i\sqrt{3}}{2}$$

$$\Rightarrow \qquad m = (1 \pm i\sqrt{3})$$

Thus, $\text{C.F.} = e^z (C_1 \cos \sqrt{3}\, z + C_2 \sin \sqrt{3}\, z)$

$$= x\, [C_1 \cos(\sqrt{3} \log x) + C_2 \sin(\sqrt{3} \log x)]$$

and P.I. $= \dfrac{1}{(D^2 - 2D + 4)} (\cos z + e^z \sin z)$

$$= \frac{1}{(D^2 - 2D + 4)} \cos z + \frac{1}{(D^2 - 2D + 4)} e^z \sin z$$

$$= \frac{1}{(-1 - 2D + 4)} \cos z + e^z \cdot \frac{1}{(D+1)^2 - 2(D+1) + 4} \sin z$$

$$= \frac{1}{(3 - 2D)} \cos z + e^z \cdot \frac{1}{(D^2 + 3)} \sin z$$

$$= \frac{(3 + 2D)}{(9 - 4D^2)} \cos z + e^z \cdot \frac{1}{(-1 + 3)} \sin z$$

$$= \frac{(3 + 2D)}{(9 + 4)} \cos z + \frac{e^z}{2} \sin z$$

$$= \frac{1}{18}(3 + 2D)\cos z + \frac{e^z}{2}\sin z$$

$$= \frac{1}{13}(3\cos z - 2\sin z) + \frac{e^z}{2}\sin z$$

$$= \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{x}{2}\sin(\log x)$$

Hence, the complete solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = x\,[C_1 \cos(\sqrt{3}\,\log x) + C_2 \sin(\sqrt{3}\,\log x)]$$

$$+ \frac{1}{13}[3\cos(\log x) - 2\sin(\log x)] + \frac{x}{2}\sin(\log x)$$

(*ii*) $$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = (3x^2+4x+1)$$

The given differential equation is reducible to a homogeneous linear differential equation of order two.

Taking $$(3x+2) = e^z$$

or $$z = \log(3x+2), \text{ we get}$$

$$(3x+2)\frac{dy}{dx} = 3Dy$$

and $$(3x+2)^2 \frac{d^2y}{dx^2} = 9D(D-1)\,y \text{ where } D = \frac{d}{dz}$$

then given differential equation becomes

$$\{9D(D-1)+3.3D-36\}\,y=3\left(\frac{e^{z}-2}{3}\right)^{2}+4\left(\frac{e^{z}-2}{3}\right)+1$$

$$\Rightarrow \qquad (9D^{2}-36)\,y=3\left(\frac{e^{2z}-4e^{z}+4}{9}\right)+\frac{4}{3}(e^{z}-2)+1$$

$$\Rightarrow \qquad 9(D^{2}-4)\,y=\frac{(e^{2z}-4e^{z}+4+4e^{z}-8+3)}{3}$$

$$\Rightarrow \qquad 9(D^{2}-4)\,y=\frac{1}{3}(e^{2z}-1)$$

$$\Rightarrow \qquad (D^{2}-4)\,y=\frac{1}{27}(e^{2z}-1),$$

which is a linear differential equation of second order with constant coefficients.

Now, for C.F., auxiliary equation in m is given as

$$m^2 - 4 = 0$$

$$\Rightarrow \quad (m+2)(m-2) = 0$$

$$\Rightarrow \quad m = 2, -2$$

Thus, $\text{C.F.} = C_1 e^{2z} + C_2 e^{-2z}$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2}$$

and $\quad \text{P.I.} = \dfrac{1}{27} \cdot \dfrac{1}{(D^2-4)} (e^{2z} - 1)$

$$= \frac{1}{27} \cdot \left[\frac{1}{(D^2-4)} e^{2z} - \frac{1}{(D^2-4)} . 1\right]$$

$$= \frac{1}{27}\left[\frac{1}{(D+2)(D-2)} e^{2z} - \frac{1}{(D^2-4)} e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{4(D-2)}e^{2z} - \frac{1}{(-4)}e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{4}\cdot\frac{z}{\lfloor 1}e^{2z} + \frac{1}{4}\right]$$

$$= \frac{1}{108}\,(ze^{2z} + 1)$$

$$= \frac{1}{108}\,[\log(3x+2)\,.\,(3x+2)^2 + 1]$$

Hence, the complete solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = C_1(3x+2)^2 + C_2(3x+2)^{-2} + \frac{1}{108}\,[(3x+2)^2\log(3x+2) + 1]$$

EXERCISE

Solve the following differential equations :

1. $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 2y = x \log x$

2. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = x^2 \sin(\log x)$

3. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = x^2 e^x$

4. $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x) \dfrac{dy}{dx} + 16y = 8(1+2x)^2$

5. $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x) \dfrac{dy}{dx} + 8y = 0$

ANSWERS

1. $y = [C_1 \cos(\log x) + C_2 \sin(\log x)]\, x + x \log x$

2. $y = x^2 \{C_1 \cos(\log x) + C_2 \sin(\log x)\} - \dfrac{x^2}{2} \log x \cos(\log x)$

3. $y = C_1 x + C_2 x^{-1} + e^x (1 - x^{-1})$

4. $y = (1 + 2x)^2 [C_1 + C_2 \log(1 + 2x) + \{\log(1 + 2x)\}^2]$

5. $y = C_1 (5 + 2x)^2 \cosh [\sqrt{2}\ \{\log(5 + 2x)\} + C_2]$

EXACT FORM

EXACT DIFFERENTIAL EQUATION OF SECOND ORDER

Let us consider the differential equation

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q$$

where P_0, P_1, P_2 and Q are functions of x.

If $P_2 - P_1' + P_0'' = 0$ is obtained for above differential equation, then it is called exact.

Working Procedure for solving exact differential equation of second order

1. Find the values of P_0, P_1, P_2, Q by comparing given differential equation with $P_0 \dfrac{d^2y}{dx^2} + P_1 \dfrac{dy}{dx} + P_2 y = Q$
2. Using P_0, P_1, P_2 check the condition of exactness *i.e.* $P_2 - P_1' + P_0'' = 0$
3. If condition of exactness is satisfied, then given differential equation

   reduces to a first order differential equation

   $P_0 \dfrac{dy}{dx} + (P_1 - P_0')y = \int Q dx + C$ which is easily solvable by the methods discussed earlier.

Equations though not exact, but can be brought to the exact form

If the equation $P_0 \dfrac{d^2y}{dx^2} + P_1 \dfrac{dy}{dx} + P_2 y = Q$ does not satisfy the condition of exactness *i.e.* if $P_2 - P_1' + P_0'' \neq 0$, then it is not exact.

Integrating Factor. If the given equation which is not an exact one becomes exact on multiplying by a function $f(x)$, then $f(x)$ is called its integrating factor.

(1) If P_0, P_1, P_2 all are of the form $(Ax^m + Bx^n + \ldots)$, then x^m is assumed to be an integrating factor. Multiply the given equation by x^m and apply the condition of exactness which will give a particular value of m. Thus, the exact value of the desired integrating factor will be known to us. The rest of the method is same as discussed earlier to solve exact differential equation.

(2) If P_0, P_1, P_2 are trigonometric functions, then integrating factor will also be a trigonometric function which is usually determined by the trial and error method.

Exa. *Solve the following differential equations :*

(i) $(2x^2 + 3x)\dfrac{d^2y}{dx^2} + (6x + 3)\dfrac{dy}{dx} + 2y = (x + 1)\,e^x$

(ii) $\sqrt{x}\,\dfrac{d^2y}{dx^2} + 2x\,\dfrac{dy}{dx} + 3y = x$

(iii) $\sin^2 x\,\dfrac{d^2y}{dx^2} - 2y = 0$

Sol. (i) $(2x^2 + 3x)\dfrac{d^2y}{dx^2} + (6x + 3)\dfrac{dy}{dx} + 2y = (x + 1)e^x$ ...(1)

On comparing equation (1) with standard exact differential equation of second order

$P_0 \dfrac{d^2y}{dx^2} + P_1 \dfrac{dy}{dx} + P_2 y = Q$, we get

$$P_0 = (2x^2 + 3x), P_1 = (6x + 3), P_2 = 2, Q = (x + 1)e^x$$

So, $P_2 - P_1' + P_0'' = 2 - 6 + 4 = 0$

Hence, the condition of exactness is satisfied.

Therefore the first integral of (1) is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int Q\,dx + C$$

$$\Rightarrow \quad (2x^2 + 3x)\frac{dy}{dx} + [(6x+3) - (4x+3)]\, y = \int (x+1)\, e^x\, dx + C$$

$$\Rightarrow \quad (2x^2 + 3x)\frac{dy}{dx} + 2xy = xe^x + C$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{(2x+3)}\, y = \frac{e^x}{(2x+3)} + \frac{C}{x(2x+3)} \qquad \ldots(2)$$

which is a linear differential equation of first order and first degree.

So, $\quad \text{I.F.} = e^{\int \frac{2}{(2x+3)}\,dx} = e^{\log(2x+3)} = (2x+3)$

Hence, the solution of equation (2) is given as

$$y(2x+3) = \int \left[\frac{e^x}{(2x+3)} + \frac{C}{x(2x+3)}\right](2x+3)\, dx + C_1$$

$$\Rightarrow \quad y(2x+3) = \int e^x\, dx + C\int \frac{1}{x}\, dx + C_1$$

$$\Rightarrow \quad y(2x+3) = e^x + C\log x + C_1$$

(ii) $\sqrt{x}\,\dfrac{d^2y}{dx^2} + 2x\,\dfrac{dy}{dx} + 3y = x$ ...(1)

On comparing equation (1) with standard exact differential equation of second order

$$P_0\,\frac{d^2y}{dx^2} + P_1\,\frac{dy}{dx} + P_2y = Q, \text{ we get}$$

$$P_0 = \sqrt{x},\ P_1 = 2x,\ P_2 = 3,\ Q = x$$

So, $P_2 - P_1' + P_0'' = 3 - 2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) x^{-3/2} \neq 0$

Therefore the given differential equation is not an exact one. Thus, multiplying both the sides by x^m, we get

$$x^{m+\frac{1}{2}}\,\frac{d^2y}{dx^2} + 2x^{m+1}\,\frac{dy}{dx} + 3x^m y = x^{m+1}$$

If the above equation is an exact equation,we must have

$$P_2 - P_1' + P_0'' = 0$$

$$\Rightarrow\quad 3x^m - 2(m+1)\,x^m + \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)x^{m-\frac{3}{2}} = 0$$

$$\Rightarrow\quad x^m - 2mx^m + \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)x^{m-\frac{3}{2}} = 0$$

$$\Rightarrow\quad -2x^m\left(m-\frac{1}{2}\right) + \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)x^{m-\frac{3}{2}} = 0$$

which gives $\quad m = \dfrac{1}{2}$

So, the integrating factor is $x^{1/2}$

Multiplying equation (1) by $x^{1/2}$, we get

$$x\frac{d^2y}{dx^2}+2x^{3/2}\frac{dy}{dx}+3x^{1/2}y=x^{3/2} \qquad \text{...(2)}$$

which is obviously an exact differential equation.

Thereore the first integral of (2) is

$$P_0\frac{dy}{dx}+(P_1-P_0')y=\int Q\,dx+C$$

$$\Rightarrow \quad x\frac{dy}{dx}+(2x^{3/2}-1)y=\int x^{3/2}\,dx+C$$

$$\Rightarrow \quad x\frac{dy}{dx}+(2x^{3/2}-1)y=\frac{2}{5}x^{5/2}+C$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(2x^{1/2}-\frac{1}{x}\right)y=\frac{2}{5}x^{3/2}+\frac{C}{x} \qquad \text{...(3)}$$

which is a linear differential equation of first order and first degree.

So, $\quad \text{I.F.}=e^{\int\left(2x^{1/2}-\frac{1}{x}\right)dx}$

$$=e^{\frac{4}{3}x^{3/2}}\cdot e^{-\log x}$$

$$=\frac{e^{\frac{4}{3}x^{3/2}}}{x}$$

Hence, the solution of equation (3) is given as

$$y\frac{e^{\frac{4}{3}x^{3/2}}}{x}=\int\left[\frac{2}{5}x^{3/2}+\frac{C}{x}\right]\frac{e^{\frac{4}{3}x^{3/2}}}{x}\,dx+C_1$$

$$\Rightarrow \quad y\frac{e^{\frac{4}{3}x^{3/2}}}{x}=\frac{2}{5}\int x^{1/2}e^{\frac{4}{3}x^{3/2}}\,dx+C\int x^{-2}e^{\frac{4}{3}x^{3/2}}\,dx+C_1$$

(*iii*) $\sin^2 x \dfrac{d^2y}{dx^2} - 2y = 0$ ...(1)

On comparing equation (1) with standard exact differential equation of second order

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q, \quad \text{...(2)}$$

we get

$$P_0 = \sin^2 x, P_1 = 0, P_2 = -2, Q = 0$$

So, $P_2 - P_1' + P_0'' = -2 - 0 + 2\cos 2x \neq 0$

Therefore the given differential equation is not an exact one. Thus, dividing the equation (1) by $\sin^2 x$, we get

$$\frac{d^2y}{dx^2} - \frac{2}{\sin^2 x} y = 0$$

Multiplying both the sides by $\cot x$, we get

$$\cot x \frac{d^2y}{dx^2} - 2y \cot x \operatorname{cosec}^2 x = 0 \quad \text{...(3)}$$

On comparing equation (3) with equation (2), we get

$$P_0 = \cot x, P_1 = 0, P_2 = -2\cot x \operatorname{cosec}^2 x, Q = 0$$

So, $P_2 - P_1' + P_0'' = -2\cot x \operatorname{cosec}^2 x - 0 + 2\cot x \operatorname{cosec}^2 x = 0$

Hence, the condition of exactness is satisfied.

Therefore the first integral of (3) is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = \int Q\, dx + C$$

$$\Rightarrow \quad \cot x \frac{dy}{dx} + (0 + \operatorname{cosec}^2 x)\, y = C$$

$$\Rightarrow \quad \cot x \frac{dy}{dx} + y \operatorname{cosec}^2 x = C$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{\operatorname{cosec}^2 x}{\cot x}\, y = \frac{C}{\cot x} \qquad \ldots(4)$$

which is linear differential equation of first order and first degree.

So,

$$\text{I.F.} = e^{\int \frac{\operatorname{cosec}^2 x}{\cot x} dx} = e^{-\log \cot x}$$
$$= \tan x$$

Hence, the solution of equation (4) is given as

$$y \tan x = \int \frac{C}{\cot x} \tan x\, dx + C_1$$

$$\Rightarrow \quad y \tan x = C \int \tan^2 x\, dx + C_1$$

$$\Rightarrow \quad y \tan x = C \int (\sec^2 x - 1)\, dx + C_1$$

$$\Rightarrow \quad y \tan x = C(\tan x - x) + C_1$$

EXERCISE

Solve the following differential equations :

1. $(x^3 - x)\dfrac{d^2y}{dx^2} + (5x^2 - 2)\dfrac{dy}{dx} + 4xy = \dfrac{1}{x^2}$

2. $x\dfrac{d^2y}{dx^2} + 2x\sqrt{x}\dfrac{dy}{dx} + 3\sqrt{x}\,y = x\sqrt{x}$

3. $x^4\dfrac{d^2y}{dx^2} + x^2(x-1)\dfrac{dy}{dx} + xy = x^3 - 4$

4. $\dfrac{d^2y}{dx^2} + 2\tan x\dfrac{dy}{dx} + 3y = \tan^2 x \sec x$

ANSWERS

1. $xy\sqrt{(x^2-1)} = \operatorname{cosec}^{-1} x + C_1 \log\left\{x + \sqrt{(x^2-1)}\right\} + C$

2. $\dfrac{y}{x} e^{\left(\frac{4}{3} x^{3/2}\right)} = \dfrac{1}{5} e^{\left(\frac{4}{3} x^{3/2}\right)} + C_1 \displaystyle\int \frac{e^{\left(\frac{4}{3} x^{3/2}\right)}}{x^2}\, dx + C_2$

3. $y\, e^{\frac{1}{x}} = \displaystyle\int \left(1 + \frac{C_1}{x}\right) e^{\frac{1}{x}}\, dx - 2\left(\frac{1}{x} - 1\right) e^{\frac{1}{x}} + C_2$

4. $y \sec^3 x = \dfrac{2}{3} \tan^2 x + \dfrac{1}{4} \tan^4 x - x \tan x + \dfrac{2}{3} \log \sec x$

$$- \frac{x}{3} \tan^3 x + C_1 \tan x + \frac{C_1}{3} \tan^3 x + C_2$$

CHANGE OF DEPENDENT VARIABLE / NORMAL FORM / REMOVING FIRST ORDER DERIVATIVE

CHANGE OF DEPENDANT VARIABLE / NORMAL FORM / REMOVING FIRST ORDER DERIVATIVE

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ is said to be a differential equation solvable with change of dependent variable, if we change its dependent variable from y to some other variable and in this process we have to normalize the given differential equation by removing its first order derivative.

Working Procedure for solving differential equation of second order using normal form

1. Find the values of P, Q, R by comparing given differential equation with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

2. Using P and Q confirm that $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$ becomes a constant or $\frac{\text{constant}}{x^2}$ so that normal form of given differential equation can be utilized.

3. Suppose that the complete solution of given differential equation is $y = uV$, where u is determined by removing the first order derivative as $e^{-\frac{1}{2}\int P dx}$

4. In the process of removing the first order derivative, given differential equation reduces to

$\frac{d^2V}{dx^2} + IV = S$ where $S = \frac{R}{u}$ and I has been obtained in step 2. Solution of this differential equation provides V.

5. Now, put the values of u and V in the supposed solution $y = uV$. This is the desired complete solution of the given differential equation.

Exa. *Solve the following differential equations :*

(i) $$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$$

(ii) $$x^2\frac{d^2y}{dx^2} - 2(3x^2 - 2x)\frac{dy}{dx} + 3x(3x - 4)y = e^{3x}$$

Sol. (i) $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2-1)\,y = -3e^{x^2}\sin 2x$ ...(1)

On comparing equation (1) with

$\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = -4x,\ Q = (4x^2-1),\ R = -3e^{x^2}\sin 2x$$

Let the complete solution of given differential equation be $y = uV$. Then its normal form will be

$$\frac{d^2V}{dx^2} + IV = S \text{ where } I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \text{ and } S = \frac{R}{u}$$

Now, $I = (4x^2-1) - \dfrac{1}{4}(-4x)^2 - \dfrac{1}{2}(-4) = 4x^2 - 1 - 4x^2 + 2 = 1$,

which is a constant.

Thus, to remove first order derivative, we choose

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -4x\,dx} = e^{x^2}$$

So, $S = \dfrac{R}{u} = -\dfrac{3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$

Hence, normal form is

$$\frac{d^2V}{dx^2} + V = -3\sin 2x$$

$\Rightarrow$ $(D^2+1)\,V = -3\sin 2x\ ;\ D = \dfrac{d}{dx}$ ...(2)

which is a linear differential equation with constant cofficients.

Now, for C.F., A.E. is $m^2 + 1 = 0$

$\Rightarrow$ $$m = \pm i$$

$\Rightarrow$ $$\text{C.F.} = (C_1 \cos x + C_2 \sin x)$$

and $$\text{P.I.} = \frac{1}{(D^2+1)} \cdot (-3 \sin 2x)$$

$$= -3 \cdot \frac{1}{(D^2+1)} \sin 2x$$

$$= -3 \cdot \frac{1}{(-4+1)} \sin 2x$$

$$= \sin 2x$$

Hence, solution of (2) is

$$V = \text{C.F.} + \text{P.I.}$$

$\Rightarrow$ $$V = C_1 \cos x + C_2 \sin x + \sin 2x$$

Thus, the complete solution of equation (1) is given as $y = uV$

$\Rightarrow$ $$y = e^{x^2} (C_1 \cos x + C_2 \sin x + \sin 2x)$$

(ii) $x^2 \dfrac{d^2y}{dx^2} - 2(3x^2 - 2x)\dfrac{dy}{dx} + 3x(3x-4)\,y = e^{3x}$ ...(1)

On comparing equation (1) with

$\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = \frac{-2(3x^2-2x)}{x^2},\quad Q = \frac{3x(3x-4)}{x^2},\quad R = \frac{e^{3x}}{x^2}$$

Let the complete solution of given differential equation be $y = uV$. Then its normal form will be

$$\frac{d^2V}{dx^2} + IV = S \text{ where } I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \text{ and } S = \frac{R}{u}$$

Now, $$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$I = \frac{3(3x-4)}{x} - \frac{1}{2}\left[-2\left(+\frac{2}{x^2}\right)\right] - \frac{1}{4}\cdot\frac{4(3x^2-2x)^2}{x^4}$$

$\Rightarrow$ $$I = 3\left(3 - \frac{4}{x}\right) + \frac{2}{x^2} - \frac{(3x-2)^2}{x^2}$$

$\Rightarrow$ $$I = 9 - \frac{12}{x} + \frac{2}{x^2} - 9 - \frac{4}{x^2} + \frac{12}{x} = -\frac{2}{x^2},$$

which is in form of $\dfrac{\text{Constant}}{x^2}$

Thus, to remove first order derivative, we choose

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int \frac{-2(3x^2-2x)}{x^2}dx} = e^{\int\left(3-\frac{2}{x}\right)dx}$$

$\Rightarrow$ $$u = e^{3x - 2\log x} = \frac{e^{3x}}{x^2}$$

So, $$S = \frac{R}{u} = \frac{\frac{e^{3x}}{x^2}}{\frac{e^{3x}}{x^2}} = 1$$

Hence, normal form is

$$\frac{d^2V}{dx^2} - \frac{2}{x^2} = 1$$

$$\Rightarrow \quad x^2 \frac{d^2V}{dx^2} - 2V = x^2 \qquad \ldots(2)$$

which is a homogeneous linear differential equation of second order. To solve it, we put $z = \log x$ or $x = e^z$, then

$$x\frac{dV}{dx} = DV \text{ and } x^2\frac{d^2V}{dx^2} = D(D-1)V \text{ where } D = \frac{d}{dz}$$

Thus, above homogeneous differential equation changes to

$$[D(D-1)-2]\,V = e^{2z}$$

$$\Rightarrow \quad (D^2 - D - 2)\,V = e^{2z}$$

$$\Rightarrow (D+1)(D-2)\,V = e^{2z}$$

Its auxilliary equation is

$$(m+1)(m-2) = 0$$

$$\Rightarrow \quad m = -1, 2$$

So, $$\text{C.F.} = C_1e^{-z} + C_2e^{2z}$$

$$= (C_1x^{-1} + C_2x^2)$$

and $$\text{P.I.} = \frac{1}{(D+1)(D-2)}\,e^{2z}$$

$$= \frac{1}{3}\,\frac{1}{(D-2)}\,e^{2z}$$

$$= \frac{1}{3}\,.\,ze^{2z} = \frac{x^2}{3}\,(\log x)$$

Hence, solution of (2) is

$$V = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad V = \frac{C_1}{x} + C_2x^2 + \frac{x^2}{3}\,(\log x)$$

Thus, the complete solution of equation (1) is given as $y = uV$

$$\Rightarrow \quad y = \frac{e^{3x}}{x^2}\left(\frac{C_1}{x} + C_2x^2 + \frac{x^2}{3}\log x\right)$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} - 2\tan x \dfrac{dy}{dx} + 5y = e^x \sec x$

2. $\left(\dfrac{d^2y}{dx^2} + y\right) \cot x + 2\left(\dfrac{dy}{dx} + y \tan x\right) = \sec x$

3. $\dfrac{d^2y}{dx^2} - \dfrac{1}{x^{1/2}} \dfrac{dy}{dx} + \dfrac{1}{4x^2}(x + x^{1/2} - 8)\, y = 0$

4. $x^2 (\log x)^2 \dfrac{d^2y}{dx^2} - 2x \log x \dfrac{dy}{dx} + [2 + \log x - 2(\log x)^2]\, y = x^2 (\log x)^3$

ANSWERS

1. $y = \left(C_1 \cos \sqrt{6}x + C_2 \sin \sqrt{6}x + \frac{1}{7} e^x \right) \sec x$

2. $y = \frac{1}{2} \sin x + (C_1 x + C_2) \cos x$

3. $y = e^{\sqrt{x}} \left(C_1 x^2 + \frac{C_2}{x} \right)$

4. $y = \left(C_1 x^2 + \frac{C_2}{x} + \frac{x^2 \log x}{3} \right) \log x$

CHANGE OF INDEPENDENT VARIABLE

SOLUTION OF THE DIFFERENTIAL EQUATION BY CHANGING THE INDEPNDENT VARIABLE

The given differential equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

where P, Q and R are functions of x. Let the independent variable be changed from x to z *i.e.* we assume $z = f(x)$

Working Procedure for solving differential equation of second order by changing independent variable

1. Find the values of P, Q, R by comparing given differential equation with
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

2. Let the independent variable be changed from x to z *i.e.* we assume $z = f(x)$

3. using above step-2, the differential equation given in step-1 becomes
$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1, \quad \ldots (1)$$
where
$$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

4. Let $Q_1 =$ constant. If the relation between z and x so obtained makes P_1 also a constant, then equation (1) becomes a linear differential equation with constant coefficient, which can be solved by methods discussed earlier.

Exa. *Solve the following differential equations :*

(i) $x^6 \dfrac{d^2y}{dx^2} + 3x^5 \dfrac{dy}{dx} + a^2 y = \dfrac{1}{x^2}$

(ii) $\dfrac{d^2y}{dx^2} + (3 \sin x - \cot x) \dfrac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x$

Sol. (i) $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2y = \frac{1}{x^2}$

or $$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6}y = \frac{1}{x^8} \quad ...(1)$$

On comparing equation (1) with

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$$P = \frac{3}{x}, Q = \frac{a^2}{x^6}, R = \frac{1}{x^8}$$

Changing the independent variable from x to z by a relation of the form $z = f(x)$, the given equation is transformed into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \quad ...(2)$$

where $$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Choosing z in such a way that $Q_1 = a^2$ (constant)

So, $$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \Rightarrow a^2 = \frac{\frac{a^2}{x^6}}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = \frac{1}{x^6}$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x^3}$$

$$\Rightarrow z = -\frac{1}{2x^2}$$

Aslo, $$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2} = \frac{\left(-\frac{3}{x^4} + \frac{3}{x^4}\right)}{\left(\frac{1}{x^3}\right)^2} = 0$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{1}{x^8}}{\left(\frac{1}{x^3}\right)^2} = \frac{1}{x^2} = -2z$$

Thus, transformed equation (2) is written as

$$\frac{d^2y}{dz^2} + a^2y = -2z$$

$\Rightarrow \quad (D^2 + a^2)\,y = -2z \;;\; D = \dfrac{d}{dz}$

Its auxilliary equation is

$$m^2 + a^2 = 0$$

$\Rightarrow \quad m = \pm ai$

So, $\quad \text{C.F.} = (C_1 \cos az + C_2 \sin az)$

$$= C_1 \cos\left(-\frac{a}{2x^2}\right) + C_2 \sin\left(-\frac{a}{2x^2}\right)$$

and $\quad \text{P.I.} = \dfrac{1}{(D^2 + a^2)}\,(-2z) = -\dfrac{2}{a^2}\cdot\dfrac{1}{\left(1+\dfrac{D^2}{a^2}\right)}\,z$

$$= -\frac{2}{a^2}\cdot\left(1+\frac{D^2}{a^2}\right)^{-1} z$$

$$= -\frac{2}{a^2}\left(1-\frac{D^2}{a^2}\right) z$$

$$= -\frac{2z}{a^2} = \frac{1}{a^2x^2}$$

Hence, the complete solution of given differential equation (1) is given as

$$y = C_1 \cos\left(-\frac{a}{2x^2}\right) + C_2 \sin\left(-\frac{a}{2x^2}\right) + \frac{1}{a^2x^2}$$

$\Rightarrow \quad y = C_1 \cos\left(\dfrac{a}{2x^2}\right) - C_2 \sin\left(\dfrac{a}{2x^2}\right) + \dfrac{1}{a^2x^2}$

(ii) $\frac{d^2y}{dx^2} + (3 \sin x - \cot x) \frac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x$ ...(1)

On comparing equation (1) with

$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$, we get

$$P = (3 \sin x - \cot x),\ Q = 2 \sin^2 x,\ R = e^{-\cos x} \sin^2 x$$

Changing the independent variable from x to z by a relation of the form $z = f(x)$, the given equation is transformed into

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{...(2)}$$

where $$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P \frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2},\ Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2},\ R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Choosing z in such a way that $Q_1 = 2$ (constant)

So, $$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \Rightarrow 2 = \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2} \Rightarrow \left(\frac{dz}{dx}\right)^2 = \sin^2 x$$

$\Rightarrow$ $$\frac{dz}{dx} = \sin x$$

$\Rightarrow$ $$z = -\cos x$$

Also, $$P_1 = \frac{\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x + (3\sin x - \cot x)(\sin x)}{\sin^2 x}$$

$\Rightarrow$ $$P_1 = \frac{\cos x + 3\sin^2 x - \cos x}{\sin^2 x} = 3$$

and $$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x} = e^{-\cos x} = e^z$$

Thus, transformed equation (2) is written as

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^z$$

$$\Rightarrow \quad (D^2 + 3D + 2)\,y = e^z \;;\; D = \frac{d}{dz}$$

Its auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0$$

$$\Rightarrow \quad m = -1, -2.$$

So,

$$\text{C.F.} = (C_1e^{-z} + C_2e^{-2z})$$
$$= (C_1e^{\cos x} + C_2e^{2\cos x})$$

and

$$\text{P.I.} = \frac{1}{(D^2+3D+2)}\,e^z$$

$$= \frac{1}{(1)^2+3(1)+2}\,e^z = \frac{e^z}{6} = \frac{1}{6}\,e^{-\cos x}$$

Hence, the complete solution of given differential equation (1) is given as

$$y = C_1e^{\cos x} + C_2e^{2\cos x} + \frac{1}{6}\,e^{-\cos x}$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} + \cot x \dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$

2. $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y \cos^3 x = 2 \cos^5 x$

3. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x) \dfrac{dy}{dx} + y = 4 \cos \log (1+x)$

ANSWERS

1. $y = C_1 \cos\left(2 \log \tan \frac{x}{2} + C_2\right)$

2. $y = C_1 e^{\sqrt{2} \sin x} + C_2 e^{-\sqrt{2} \sin x} + \sin^2 x$

3. $y = C_1 \cos\{\log(1+x)\} + C_2 \sin\{\log(1+x)\} + 2\log(1+x) \,.\, \sin\{\log(1+x)\}$

METHOD OF VARIATION OF PARAMETERS

METHOD OF VARIATION OF PARAMETERS

This is a very important tool for getting particular integral of a linear differential equation of second order when both the parts of complementary function are known. Particularly this method is used only when the determination of the particular integral of a linear differential equation is too hard to calculate.

Remarks : (1) The method of variation of parameters is used when you are asked to solve a problem by this method.

(2) If one part of complementary function is u, then other part v can be obtained by $v = u \int \frac{1}{u^2} e^{-\int P dx} dx$

(3) In this method, the complete solution is obtained by varying the arbitrary constants of the complementary function, so that this method is known as method of "Variation of Parameters".

Working Procedure for solving linear differential equation of second order using method of variation of parameters

1. Put the given differential equation in standard form

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ and find the values of P, Q, R

Please note that the coefficient of $\frac{d^2y}{dx^2}$ must be unity.

2. Find parts of C.F. of the given differential equation as u and v.

If it is difficult to find both parts of C.F., then find one part of C.F. using values of P and Q **using the following conclusions :**

(i) $y = x$ is a part of C.F. if $P + Qx = 0$

(ii) $y = x^2$ is a part of C.F. if $2 + 2Px + Qx^2 = 0$

(iii) $y = x^m$ is a part of C.F. if $m(m-1) + Pmx + Qx^2 = 0$

(iv) $y = e^x$ is a part of C.F. if $1 + P + Q = 0$

(v) $y = e^{-x}$ is a part of C.F. if $1 - P + Q = 0$

(vi) $y = e^{ax}$ is a part of C.F. if $a^2 + Pa + Q = 0$

and hence find other part of C.F. using above Remark (2).

3. Let $y = Au + Bv$ be the complete solution where A and B are functions of x.

4. Solve the simultaneous equations

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \text{ and } \frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} = R$$

to find the values of $\frac{dA}{dx}$ and $\frac{dB}{dx}$

5. Find the values of A and B by integrating $\frac{dA}{dx}$ and $\frac{dB}{dx}$ respectively.

6. Putting the values of A and B in $y = (Au + Bv)$, the complete solution of given differential equation is obtained.

Exa. *Solve the following differential equations using method of variation of parameters :*

(i) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{(1+e^x)}$

(ii) $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = x^2 e^x$

(iii) $\dfrac{d^2y}{dx^2} + (1 - \cot x) \dfrac{dy}{dx} - y \cot x = \sin^2 x$

Sol. (i) $$\frac{d^2y}{dx^2} - y = \frac{2}{(1+e^x)} \qquad \ldots(1)$$

The given differential equation is a second order linear differential equation with constant coefficients.

Thus, for C.F.

Its auxilliary equation is

$$m^2 - 1 = 0$$

$$\Rightarrow \qquad m = \pm 1$$

So, C.F. $= (C_1 x + C_2 e^{-x}) \Rightarrow u = e^x, v = e^{-x}$ are two parts of C.F.

Let $y = (Ae^x + Be^{-x})$ be the complete solution of the given differential equation (1), where A and B are functions of x to be determined.

Choosing A and B in such a way that

$$\left.\begin{aligned} u\frac{dA}{dx} + v\frac{dB}{dx} &= 0 \\ \text{and} \quad \frac{dA}{dx}\frac{du}{dx} + \frac{dB}{dx}\frac{dv}{dx} &= R \end{aligned}\right\} \Rightarrow \begin{aligned} & e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \\ \text{and} \quad & e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \frac{2}{(1+e^x)} \end{aligned}$$

Solving these two equations, we get

$$\frac{dA}{dx} = \frac{1}{e^x(1+e^x)} \text{ and } \frac{dB}{dx} = -\frac{e^x}{(1+e^x)}$$

$$\Rightarrow \qquad A = \int \frac{1}{e^x(1+e^x)}\,dx = \int\left(\frac{1}{e^x} - \frac{1}{(1+e^x)}\right)dx$$

$$\Rightarrow \qquad A = -e^{-x} + \log(1+e^{-x}) + C_1$$

and $\qquad B = -\int \frac{e^x}{(1+e^x)}\,dx = -\log(1+e^x) + C_2$

Hence, the complete solution of given differential equation (1) is given as

$$y = Ae^x + Be^{-x}$$

$$\Rightarrow \qquad y = [C_1 - e^{-x} + \log(1+e^{-x})]\,e^x + [C_2 - \log(1+e^x)]\,e^{-x}$$

$$\Rightarrow \qquad y = C_1e^x + C_2e^{-x} - 1 + e^x\log(1+e^{-x}) - e^{-x}\log(1+e^x)$$

(ii) $$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^2e^x \qquad ...(1)$$

To find C.F., we have the equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0 \qquad ...(2)$$

which is a homogeneous linear differential equation of second order.

Taking $z = \log x$ or $x = e^z$, we get

$$x \frac{d}{dx} = \frac{d}{dz} = D \text{ and } x^2 \frac{d^2}{dx^2} = D(D-1)$$

then equation (2) becomes

$[D(D-1) + D - 1]\, y = 0$

$\Rightarrow \qquad (D^2 - 1)\, y = 0$

Its auxilliary equation is

$$m^2 - 1 = 0$$

$\Rightarrow \qquad m = \pm 1$

So, $\qquad \text{C.F.} = C_1 e^t + C_2 e^{-t}$

$$= C_1 x + \frac{C_2}{x}$$

$\Rightarrow \qquad u = x,\ v = \dfrac{1}{x}$ are two parts of C.F.

Let $y = \left(Ax + \dfrac{B}{x}\right)$ be the complete solution of the given differential equation

(1), where A and B are functions of x to be determined.

Choosing A and B in such a way that

$$\left.\begin{aligned} u\frac{dA}{dx} + v\frac{dB}{dx} &= 0 \\ \text{and} \quad \frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} &= R \end{aligned}\right\} \Rightarrow \begin{aligned} x\frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx} &= 0 \\ \text{and} \quad \frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} &= e^x \qquad (\because \quad R = e^x) \end{aligned}$$

Solving these two equations, we get

$$\frac{dA}{dx} = \frac{1}{2} e^x \text{ and } \frac{dB}{dx} = -\frac{1}{2} x^2 e^x$$

$$\Rightarrow \quad A = \frac{1}{2}\int e^x \, dx \text{ and } B = -\frac{1}{2}\int x^2 e^x \, dx$$

$$\Rightarrow \quad A = \frac{e^x}{2} + C_1 \text{ and } B = -\frac{1}{2}(x^2e^x - 2xe^x + 2e^x) + C_2$$

Hence, the complete solution of given differential equation (1) is given as

$$y = Ax + \frac{B}{x}$$

$$\Rightarrow \quad y = \left(\frac{e^x}{2} + C_1\right)x + \left[-\frac{x^2e^x}{2} + xe^x - e^x + C_1\right]\frac{1}{x}$$

$$\Rightarrow \quad y = C_1x + \frac{C_2}{x} + \frac{xe^x}{2} - \frac{xe^x}{2} + e^x - \frac{e^x}{x}$$

$$\Rightarrow \quad y = C_1x + \frac{C_2}{x} + e^x - \frac{e^x}{x}$$

(*iii*) $$\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x \qquad \ldots(1)$$

On comparing equation (1) with

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$$P = (1-\cot x),\ Q = -\cot x,\ R = \sin^2 x$$

Since, $1 - P + Q = 1 - 1 + \cot x - \cot x = 0$

so $u = e^{-x}$ is a part of C.F.

Using $v = u\int \frac{1}{u^2} e^{-\int P dx}\, dx$, we get

$$v = \sin x - \cos x$$

$\Rightarrow$ $u = e^{-x}$, $v = \sin x - \cos x$ are two parts of C.F.

Let $y = Ae^{-x} + B(\sin x - \cos x)$ be the complete solution of the given differential equation (1), where A and B are functions of x to be determined.

Choosing A and B in such a way that

$$\left.\begin{aligned} u\frac{dA}{dx} + v\frac{dB}{dx} &= 0 \\ \text{and}\quad \frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} &= R \end{aligned}\right\} \Rightarrow \begin{aligned} &e^{-x}\frac{dA}{dx} + (\sin x - \cos x)\frac{dB}{dx} = 0 \\ \text{and}\quad &-e^{-x}\frac{dA}{dx} + (\cos x + \sin x)\frac{dB}{dx} = \sin^2 x \end{aligned}$$

Solving these two equations, we get

$$\frac{dA}{dx} = \frac{1}{2} e^x (\sin x \cos x - \sin^2 x) \text{ and } \frac{dB}{dx} = \frac{1}{2} \sin x$$

$$\Rightarrow \quad A = -\frac{e^x}{4} + \frac{e^x}{20} (3 \sin 2x - \cos 2x) + C_1 \text{ and } B = \left(-\frac{1}{2} \cos x + C_2\right)$$

Hence, the complete solution of given differential equation (1) is given as

$$y = Ae^{-x} + B(\sin x - \cos x)$$

$$\Rightarrow \quad y = \left[C_1 + \frac{e^x}{20}(3 \sin 2x - \cos 2x) - \frac{e^x}{4}\right] e^{-x} + \left(-\frac{1}{2} \cos x + C_2\right) (\sin x - \cos x)$$

$$\Rightarrow \quad y = C_1 e^{-x} + C_2(\sin x - \cos x) - \frac{1}{2} \cos x (\sin x - \cos x) + \frac{1}{20} (3 \sin 2x - \cos 2x) - \frac{1}{4}$$

$$\Rightarrow \quad y = C_1 e^{-x} + C_2(\sin x - \cos x) - \frac{1}{10} (\sin 2x - 2 \cos 2x)$$

EXERCISE

Solve the following differential equations using method of variation of parameters :

1. $\dfrac{d^2y}{dx^2} + 4y = 4\tan 2x$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x$

3. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \dfrac{1}{x^2}$

4. $(1-x)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = (1-x)^2$

ANSWERS

1. $y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

2. $y = C_1 + C_2 e^{2x} - \dfrac{1}{2} e^x \sin x$

3. $y = C_1 \cos(\log x) + C_2 \sin(\log x) + \dfrac{1}{5x^2}$

4. $y = C_1 x + C_2 e^x + x^2 + x + 1$

MORE PROBLEMS ON SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

Solve the following differential equations :

1. $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

2. $x^5 \dfrac{d^2y}{dx^2} + 3x^3 \dfrac{dy}{dx} + (3 - 6x)\, x^2 y = x^4 + 2x - 5$

3. $(ax - bx^2) \dfrac{d^2y}{dx^2} + 2a \dfrac{dy}{dx} + 2by = x$

4. $\dfrac{d^2y}{dx^2} + (1 - \cot x) \dfrac{dy}{dx} - y \cot x = \sin^2 x$

5. $\dfrac{d^2y}{dx^2} - 2 \tan x \dfrac{dy}{dx} + 3y = 2 \sec x$

6. $\dfrac{d^2y}{dx^2} + (\tan x - 1)^2 \dfrac{dy}{dx} - n(n-1)\, y \sec^4 x = 0$

7. $(a^2 - x^2) \dfrac{d^2y}{dx^2} - \dfrac{a^2}{x} \dfrac{dy}{dx} + \dfrac{x^2}{a}\, y = 0$

8. $4x^2 \dfrac{d^2y}{dx^2} + 4x^5 \dfrac{dy}{dx} + (x^8 + 6x^4 + 4)\, y = 0$

ANSWERS

1. $y = c_1x^2 + c_2x^{-1} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right) \log x$

2. $\frac{y}{x^3} e^{-3/x} = \int\left[\frac{1}{3} + \frac{2}{x^3} \log x + \frac{5}{x^4} + \frac{c_1}{x^3}\right] \frac{1}{x^3} e^{-3/x} dx + c_2$

3. $xy = \frac{x^3}{6a} + \frac{c_1}{3b} + c_2(a - bx)^3$

4. $y = \frac{c_1}{2} (\sin x - \cos x) + c_2e^{-x} - \frac{1}{10} (\sin 2x - 2 \cos 2x)$

5. $y = \sec x \left(c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}\right)$

6. $y = c_1e^{-n \tan x} + c_2e^{(n-1) \tan x}$

7. $y = c_1 \cos\left(\sqrt{\frac{a^2 - x^2}{a}}\right) - c_2 \sin\left(\sqrt{\frac{a^2 - x^2}{a}}\right)$

8. $y = e^{-x^4/8} \sqrt{x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2} \log x + c_2\right)\right]$

$+ 3)^2 + c_2 (x + 3)^3 + \frac{1}{2} (x + 2)$

TO SELECT AN APPROPRIATE METHOD TO SOLVE ANY LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH VARIABLE COEFFICIENTS

Following steps will make it very simple to select an appropriate method to solve such differential equations :

1. The method of variation of parameters must be used only when it is instructed in the problem.
2. Use simple technique of observation to identify homogeneous differential equations and differential equations reducible to homogeneous form.
3. Compare given differential equation with $P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = Q$ and check the condition of exactness *i.e.* $P_2 - P_1' + P_0'' = 0$. If it is not satisfied, then apply next step.
4. Change the given differentialequation in standard form $\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$ and find the value of $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$. If it is a constant or in form of $\frac{\text{Constant}}{x^2}$, then given differential equation is reducible to normal form. If I is neither a constant nor in form of $\frac{\text{Constant}}{x^2}$, then apply next step.

5. Try method of change of independent variable by choosing new independent variable z such that $\dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$ = a suitable constant (which we choose) and hence

$\dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$ becomes a constant. If this method fails, then apply next step.

6. Try to make given differential equation exact by multiplying it with x^m if coefficients of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ are algebric or multiply given differential equation with any suitable non-algebric function of x if at least one of the coefficients of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ is non-algebric.

THANKS