THE SUCCESS NOTES

DIFFERENTIAL EQUATION I

NOTES BY

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TOPIC INDEX

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

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DIFFERENTIAL EQUATION : DEFINITION

An equation containing derivaives is called a differential equation.

Examples:

(1) $\frac{dy}{dx} + 2xy = \sin x$

(2) $\frac{d^2 y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}$

(3) $\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = e^x$

(4) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} = 0$

Remark: In above examples, first 3 are ordinary differential equations and the fourth example contains a partial differential equation.

ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Example : In the differential equation $\frac{d^3y}{dx^3}+3\frac{dy}{dx}+2y=e^{2x}$,

the order of highest order derivative is 3.

So, it is a differential equation of order 3.

Remark: The order of a differential equation is always a positive integer.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.

Example : Consider the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = k\left(\frac{d^2y}{dx^2}\right)$$

when expressed as a polynomial in derivatives it becomes

$$k^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 0$$

clearly, the power of the highest order differential coefficient is 2.

CLASSIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS (LINEAR / NON-LINEAR)

Linear Differential Equation : A differential equation, in which the dependent variable and all its derivatives occur in the first degree only and there is no product of dependent variable and derivatives occur is called a linear differential equation.

Non-linear Differential Equation : A differential equation, which is not linear is called a non-linear differential equation.

S. No.	Ordinary Differential Equation	Order	Degree	Linear / Non - Linear
1	$\frac{d^2y}{dx^2}+2y=0$	2	1	Linear
2	$\frac{d^4y}{dx^4}=\left[k+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$	4	2	Non-Linear
3	$\frac{d^4y}{dx^4}=\cos\left(\frac{d^3y}{dx^3}\right)=0$	4	Undefined	Non-Linear
4	$\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$	2	Undefined	Non-Linear
5	$\frac{d^3y}{dx^3}+3\frac{dy}{dx}+y=e^{-x}$	3	1	Linear

LINEAR DIFFERENTIAL EQUATIONS

LINEAR DIFFERENTIAL EQUATION

In general, a linear differential equation of first order and first degree is written as

$$\frac{dy}{dx} + Py = Q \ ,$$

where P & Q are functions of x or may be constants.

Examples: (i) $\frac{dy}{dx} + xy = x^2$ (ii) $\frac{dy}{dx} + y\sin x = \cos x$

Following algorithm may be used to solve above linear differential equation:

Step 1. Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$ and obtain the values of P & Q.

Step 2. Find the integrating factor (I.F.) $= e^{\int P\,dx}$

Step 3. Solution of the given differential equation is obtained by simplifying the equation $y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

Remark: If we consider as dependent variable and as independent variable, then the above linear differentia equation is written as $\frac{dx}{dy} + Rx = S,$

where $R \& S$ are functions of y or may be constants.

Then the solution of the given differential equation is obtained by simplifying the equation $x\,(\text{I.F.}) = \int S\,(\text{I.F.})dy + c$, where (I.F.) $= e^{\int R dy}$

Exa. Solve: $x \log x \dfrac{dy}{dx} + y = 2 \log x$

Sol.
$$x \log x \frac{dy}{dx} + y = 2 \log x$$

The given differential equation can be written as

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x} \qquad \text{...(1)}$$

which is linear in y and x.

On comparing equation (1) with standard form of linear differential equation

$$\frac{dy}{dx} + Py = Q, \text{ we get}$$

$$P = \frac{1}{x \log x}, Q = \frac{2}{x}$$

So, $$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx}$$

$$= e^{\int \frac{1}{t} dt} \qquad \text{(Put } \log x = t\text{)}$$

$$= e^{\log t}$$

$$= t = \log x$$

Thus, the solution of equation (1) is given as

$$y \text{ (I.F.)} = \int Q \text{ (I.F.)} \, dx + c$$

$$\Rightarrow \qquad y \log x = \int \frac{2}{x} \log x \, dx + c$$

$$\Rightarrow \qquad y \log x = (\log x)^2 + c$$

Exa. Solve: $(x + 2y^3)\dfrac{dy}{dx} = y$

Sol.

$$(x + 2y^3)\frac{dy}{dx} = y$$

The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2 \qquad \text{...(1)}$$

which is linear in x and y.

On comparing equation (1) with $\dfrac{dx}{dy} + R\,x = S$, we get

$$R = -\frac{1}{y},\ S = 2y^2$$

$$\text{I.F.} = e^{\int R\,dy} = e^{-\int \frac{1}{y}\,dy} = e^{-\log y} = \frac{1}{y}$$

Thus, the solution of equation (1) is given as

$$x(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + c$$

$$\Rightarrow \qquad \frac{x}{y} = \int 2y^2 \cdot \frac{1}{y}\,dy + c$$

$$\Rightarrow \qquad \frac{x}{y} = 2\int y\,dy + c$$

$$\Rightarrow \qquad \frac{x}{y} = y^2 + c \quad \Rightarrow \quad x = y^3 + cy$$

Exa. Solve: $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

Sol.

$$(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$$

The given differential equation can be written as

$$\frac{dx}{dy} + \frac{1}{(1 + y^2)}\,x = \frac{\tan^{-1} y}{(1 + y^2)} \qquad \ldots(1)$$

which is linear in x and y.

On comparing equation (1) with $\dfrac{dx}{dy} + Rx = S$, we get

$$R = \frac{1}{(1 + y^2)},\ S = \frac{\tan^{-1} y}{(1 + y^2)}$$

So,

$$\text{I.F.} = e^{\int R\,dy} = e^{\int \frac{1}{(1 + y^2)}\,dy} = e^{\tan^{-1} y}$$

Thus, the solution of equation (1) is given as

$$x(\text{I.F.}) = \int S(\text{I.F.})\, dy + c$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{(1+y^2)} \cdot e^{\tan^{-1} y}\, dy + c$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = \int te^t\, dt + c \qquad (\text{Put } \tan^{-1} y = t)$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = (t-1)\, e^t + c$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + c$$

$$\Rightarrow \quad x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

EXERCISE

Solve the following differential equations:

1. $x \cos x \dfrac{dy}{dx} + y(x \sin x + \cos x) = 1$

2. $x(1 + x^2) \dfrac{dy}{dx} = y(1 - x^2) + x^3 \log x$

3. $y \log y \, dx + (x - \log y) \, dy = 0$

4. $(x + y + 1) \dfrac{dy}{dx} = 1$

ANSWERS

1. $xy \sec x = c + \tan x$

2. $y(x^2 + 1) = \frac{1}{2} x^3 \log x - \frac{1}{4} x^3 + cx$

3. $x \log y = \frac{1}{2} (\log y)^2 + c$

4. $x = ce^y - (y + 2)$

DIFFERENTIAL EQUATIONS RECUCIBLE TO LINEAR FORM

DIFFERENTIAL EQUATIONS REDUCIBLE TO LINEAR FORM

In general, a differential equation can be reduced in linear form, if it can be written as

$$f'(y)\,\frac{dy}{dx} + P\,f(y) = Q\,,$$

where P & Q are functions of x or may be constants.

OR

A differential equation can be reduced in linear form, if it can be written as

$$f'(x)\frac{dx}{dy} + R\,f(x) = S,$$

where R & S are functions of y or may be constants.

Exa. Solve: $x\dfrac{dy}{dx}+y=xy^3$

Sol. $x\dfrac{dy}{dx}+y=xy^3 \Rightarrow \dfrac{dy}{dx}+\dfrac{y}{x}=y^3$

$$\Rightarrow \frac{dy}{dx}-y^3=-\frac{y}{x}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}-y^2=-\frac{1}{x},$$

where $\dfrac{1}{y}$ is not derivative of y^2

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx}+\frac{1}{y^2}\frac{1}{x}=1,$$

where $\dfrac{1}{y^3}$ is representable as derivative of $\dfrac{1}{y^2}$

$$\text{Let } \frac{1}{y^2} = v \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

So, given differential equation reduces to

$$-\frac{1}{2}\frac{dv}{dx} + v\frac{1}{x} = 1 \Rightarrow \frac{dv}{dx} + v\left(\frac{-2}{x}\right) = -2,$$

which is linear, for which $\text{I.F.} = e^{\int\left(\frac{-2}{x}\right)dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$

Now, the final solution of the LDE is given as

$$vx^{-2} = \int -2x^{-2}\,dx + c \Rightarrow vx^{-2} = 2x^{-1} + c \Rightarrow \frac{1}{y^2}x^{-2} = 2x^{-1} + c$$

$$\Rightarrow xy^2(2 + cx) = 1$$

Exa. Solve: $\dfrac{dy}{dx} = \dfrac{1}{xy\,(x^2y^2 + 1)}$

Sol.

$$\frac{dy}{dx} = \frac{1}{xy\,(x^2y^2 + 1)} \qquad \text{...(1)}$$

The given equation can be written as

$$\frac{dx}{dy} = xy\,(x^2y^2 + 1)$$

$$\Rightarrow \qquad \frac{dx}{dy} - xy = x^3y^3$$

$$\Rightarrow \qquad \frac{1}{x^3}\frac{dx}{dy} - \frac{y}{x^2} = y^3 \qquad \text{...(2)}$$

Put

$$-\frac{1}{x^2} = v \qquad \text{...(3)}$$

$$\Rightarrow \qquad \frac{2}{x^3}\frac{dx}{dy} = \frac{dv}{dy} \qquad \text{...(4)}$$

Using (3) and (4) into (2), we get

$$\frac{1}{2}\frac{dv}{dy} + vy = y^3$$

$$\Rightarrow \qquad \frac{dv}{dy} + 2vy = 2y^3 \qquad ...(5)$$

which is linear differential equation in v and y.

So, $\qquad$ $\text{I.F.} = e^{\int 2y\,dy} = e^{y^2}$

Thus, the solution of (1) is given as

$$ve^{y^2} = \int 2y^3 e^{y^2}\,dy + c$$

$$\Rightarrow \qquad ve^{y^2} = \int y^2 e^{y^2}\, 2y\,dy + c$$

$$\Rightarrow \qquad ve^{y^2} = \int te^t\,dt + c \qquad (Put\ \ y^2 = t\ )$$

$$= (t-1)\,e^t + c$$

$$\Rightarrow \qquad ve^{y^2} = (y^2 - 1)\,e^{y^2} + c$$

$$\Rightarrow \qquad v = y^2 - 1 + ce^{-y^2}$$

$$\Rightarrow \qquad -\frac{1}{x^2} = y^2 - 1 + ce^{-y^2}$$

Exa. **Solve:** $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$

Sol. $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$

The given equation can be written as

$$\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$$

$$\Rightarrow \quad e^y \frac{dy}{dx} + e^x e^y = e^{2x} \qquad \ldots (1)$$

Put

$$e^y = v \qquad \ldots (2)$$

$$\Rightarrow \quad e^y \frac{dy}{dx} = \frac{dv}{dx} \qquad \ldots (3)$$

Using (2) and (3) into (1), we get

$$\frac{dv}{dx} + ve^x = e^x \cdot e^x \qquad \text{...(4)}$$

which is linear differential equation in v and x.

So, $\quad \text{I.F.} = e^{\int e^x \, dx} = e^{e^x}$

Thus, the solution of (1) is given as

$$ve^{e^x} = \int e^x \cdot e^x \cdot e^{e^x} \, dx$$

$$\Rightarrow \quad ve^{e^x} = \int e^t \, t dt \qquad (\text{Put } e^x = t)$$

$$\Rightarrow \quad ve^{e^x} = (t - 1)\, e^t + c$$

$$\Rightarrow \quad ve^{e^x} = (e^x - 1)\, e^{e^x} + c$$

$$\Rightarrow \quad v = e^x - 1 + ce^{-e^x}$$

$$\Rightarrow \quad e^y = e^x - 1 + ce^{-e^x}$$

EXERCISE

Solve the following differential equations :

1. $(x^2y^3 + xy)\, dy = dx$

2. $2\,\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$

3. $\sin y\,\dfrac{dy}{dx} = \cos y\,(1 - x\cos y)$

4. $2\sin x\,\dfrac{dy}{dx} - y\cos x = xe^x y^3$

ANSWERS

1. $$\frac{1}{x} = 2 - y^2 + c\,e^{-\frac{y^2}{2}}$$

2. $$x = y(1 + c\sqrt{x}\,)$$

3. $$\sec y = x + 1 + ce^x$$

4. $$y^{-2} \sin x = e^x (1 - x) + c$$

EXACT DIFFERENTIAL EQUATIONS

EXACT DIFFERENTIAL EQUATION

A differential equation is said to be exact, if it can be directly derived from its primitive (general solution) by differentiation, without any further operation of elimination or reduction.

NECESSARY AND SUFFICIENT CONDITION OF EXACTNESS FOR AN ODE OF FIRST ORDER AND FIRST DEGREE

The differential equation $Mdx + Ndy = 0$, **where** M, N **are functions of** x and y, **is said to be exact, if** $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

SOLUTION OF AN EXACT DIFFERENTIAL EQUATION

If $Mdx + Ndy = 0$ **is an exact differential equation, then its solution is given as**

$$\boxed{\int_{y=\text{constant}} M\,dx + \int (\text{Terms in } N \text{ not containing } x)\,dy = c}$$

Exa. Solve : $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Sol. $$(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0 \quad ...(1)$$

Here,we have

$$M(x,y) = 1 + e^{x/y} \Rightarrow \frac{\partial M}{\partial y} = \frac{-x}{y^2} e^{x/y}$$

and $N(x,y) = e^{x/y}\left(1 - \frac{x}{y}\right) \Rightarrow \frac{\partial N}{\partial x} = -\frac{x}{y^2} e^{x/y}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, eqn.(1) is exact and its solution is

$$\int_{(y=\text{constant})} (1 + e^{x/y})dx + \int (0)dy = c$$

or $\quad x + ye^{x/y} = c$

Exa. Solve : $(x^2+y^2)dx + 2xy\,dy = 0$

Sol.

$$(x^2+y^2)dx + 2xy\,dy = 0 \quad \ldots(1)$$

Here, we have

$$M(x,y) = x^2+y^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$$

and $N(x,y) = 2xy \Rightarrow \dfrac{\partial N}{\partial x} = 2y$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, eqn.(1) is exact and its solution is

$$\int_{(y=\text{constant})}(x^2+y^2)dx + \int (0)dy = c$$

or $\dfrac{x^3}{3} + y^2x = c$

Exa. $x\,dx + y\,dy = a^2\left(\dfrac{x\,dy - y\,dx}{x^2 + y^2}\right)$

Sol. $x\,dx + y\,dy = a^2\left(\dfrac{x\,dy - y\,dx}{x^2 + y^2}\right)$

The given differential equation can be written as

$$\left(x + \frac{a^2 y}{x^2 + y^2}\right) dx + \left(y - \frac{a^2 x}{x^2 + y^2}\right) dy = 0 \qquad \ldots(1)$$

Comparing the given differential equation with

$M\,dx + N\,dy = 0$, we get

$$M = x + \frac{a^2 y}{(x^2 + y^2)} \text{ and } N = y - \frac{a^2 y}{(x^2 + y^2)}$$

$$\Rightarrow \qquad \frac{\partial M}{\partial y} = a^2\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

and
$$\frac{\partial N}{\partial x} = -a^2\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, eqn.(1) is exact and its solution is

$$\int_{(y=\text{constant})}\left[x+\frac{a^2 y}{(x^2+y^2)}\right]dx + \int y\,dy = c$$

or
$$\frac{x^2}{2} + a^2\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = c$$

EXERCISE

Solve the following differential equations :

1. $(e^y + 1)\cos x dx + e^y \sin x dy = 0$

2. $(a^2 - 2xy - y^2)\, dx - (x + y)^2\, dy = 0$

3. $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]\, dy = 0$

4. $\cos x\,(\cos x - \sin \alpha \sin y)\, dx + \cos y\,(\cos y - \sin \alpha \sin x)\, dy = 0$

ANSWERS

1. $(e^y + 1)\sin x = c$
2. $(x + y)^3 = x^3 + 3a^2x + c$
3. $y(x + \log x) + x\cos y = c$
4. $2(x + y) + \sin 2x + \sin 2y - 4\sin\alpha\sin x\sin y = c$

DIFFERENTIAL EQUATIONS RECUCIBLE TO EXACT FORM

DIFFERENTIAL EQUATIONS REDUCIBLE TO EXACT FORM

Integrating Factor (I.F.)

Some of the differential equations, which are not exact can be made exact after multiplying them by some suitable function of x and y. Such a function is called an integrating factor.

For example, consider the differential equation

$$ydx - xdy = 0 \quad ...(1)$$

Here $M = y$ and $N = -x \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Therefore the differential equation is not exact.

$\Rightarrow$

(1) Multiplying by $\frac{1}{y^2}$, it becomes

$$\frac{ydx - xdy}{y^2} = 0, \text{ which is exact.}$$

(2) Multiplying by $\frac{1}{x^2}$, it becomes

$$\frac{ydx - xdy}{x^2} = 0, \text{ which is exact.}$$

(3) Multiplying by $\frac{1}{xy}$, it becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0, \text{ which is exact.}$$

Therefore $\frac{1}{y^2}$, $\frac{1}{x^2}$ and $\frac{1}{xy}$ are integrating factors of (1).

Remarks: (1) We have to use any one integrating factor for reducing a non - exact differential equation into exact form.

(2) For finding an integrating factor, different specific cases are to be discussed.

Rules for finding integrating factors

Rule - I: If the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$,

then integrating factor $= \dfrac{1}{Mx + Ny}$

Rule - II: If the differential equation $Mdx + Ndy = 0$ can be presented in the form

$f_1(xy)y\,dx + f_2(xy)x\,dy = 0$ and $Mx - Ny \neq 0$,

then integrating factor $= \dfrac{1}{Mx - Ny}$

Rule - III: If the differential equation $Mdx + Ndy = 0$ provides the values of M and N such that we obtain $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ or a constant k,

then integrating factor $= e^{\int f(x)dx}$ or $e^{\int k\,dx}$

Rule - IV: If the differential equation $Mdx + Ndy = 0$ provides the values of M and N such that we obtain $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ or a constant k,

then integrating factor $= e^{\int f(y)dy}$ or $e^{\int k\,dy}$

Rule - V: If the differential equation $Mdx+Ndy=0$ can be presented in the form

$$x^a y^b(mydx+nxdy)+x^c y^d(pydx+qxdy)=0\ ;\ a,b,c,d,m,n,p,q\in R,$$

then we assume $x^h y^k; h,k\in R$ as an integrating factor. In order to find the values of h and k, we use the condition of exactness on the given differential equation after multiplying it by the assumed integrating factor $x^h y^k$

Rule - VI: In many problems, integrating factor can be obtained by the method of inspection using various formulae, out of which several are given as:

(*i*) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(*ii*) $\dfrac{ydx - xdx}{y^2} = d\left(\dfrac{x}{y}\right)$

(*iii*) $\dfrac{xdy - ydx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$

(*iv*) $\dfrac{ydx - xdy}{xy} = d\left[\log\left(\dfrac{x}{y}\right)\right]$

(*v*) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

(*vi*) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left[\tan^{-1}\left(\dfrac{x}{y}\right)\right]$

(*vii*) $xdy + ydx = d(xy)$

(*viii*) $\dfrac{xdy + ydx}{xy} = d\,[\log (xy)]$

(*ix*) $xdx + ydy = d\left[\dfrac{1}{2}(x^2 + y^2)\right]$

(*x*) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log (x^2 + y^2)\right]$ etc.

Exa. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

Sol. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ ...(1)

Comparing the given differential equation with

$Mdx + Ndy = 0$, we get

$M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = x^2 - 4xy \text{ and } \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, given differential equation is non-exact, but it is homogeneous and

$$Mx + Ny = (x^2y - 2xy^2)\,x + (3x^2y - x^3)\,y = x^2y^2 \neq 0$$

Thus, $\quad$ I.F. $= \dfrac{1}{x^2y^2}$

Now, multiplying equation (1) by $\dfrac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx + \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 0$$

which must be exact.

Hence, the required solution is given as

$$\int\left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = c$$

(y = constant)

$$\text{or} \quad \frac{x}{y} - 2 \log x + 3 \log y = c$$

Exa. $(xy \sin xy + \cos xy)\, ydx + (xy \sin xy - \cos xy)\, xdy = 0$

Sol. $(xy \sin xy + \cos xy)\, ydx + (xy \sin xy - \cos xy)\, xdy = 0$ ...(1)

Comparing given differential equation with $Mdx + Ndy = 0$, we get

$$M = xy^2 \sin xy + y \cos xy \text{ and}$$

$$N = x^2y \sin xy - x \cos xy$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 2xy \sin xy + x^2y^2 \cos xy + \cos xy - xy \sin xy$$

$$= xy \sin xy + (x^2y^2 + 1) \cos xy$$

$$\text{and} \quad \frac{\partial N}{\partial x} = 2xy \sin xy + x^2y^2 \cos xy - \cos xy + xy \sin xy$$

$$= 3xy \sin xy + (x^2y^2 - 1) \cos xy$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, given differential equation is non-exact, but it can be written in the

form $\quad f_1(xy)\, ydx + f_2(xy)\, xdy = 0$

and $$Mx - Ny = (xy^2 \sin xy + y \cos xy)\,x - (x^2y \sin xy - x \cos xy)\,y$$
$$= 2xy \cos xy \neq 0$$

Thus, $$\text{I.F.} = \frac{1}{2xy \cos xy}$$

Now, multiplying equation (1) by $\dfrac{1}{2xy \cos xy}$, we get

$$\frac{1}{2}\left(y \tan xy + \frac{1}{x}\right) dx + \frac{1}{2}\left(x \tan xy - \frac{1}{y}\right) dy = 0$$

which must be exact.

Hence, the required solution is given as

$$\int \frac{1}{2}\left(y \tan xy + \frac{1}{x}\right) dx + \int \left[-\frac{1}{2y}\right] dy = c$$

(y = constant)

or $$\frac{y}{2}\left[\frac{1}{y} \log(\sec xy)\right] + \frac{1}{2} \log x - \frac{1}{2} \log y = c$$

$$\Rightarrow \quad \frac{1}{2} \log(\sec xy) + \frac{1}{2} \log x - \frac{1}{2} \log y = c$$

Exa. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2)\, dy = 0$

Sol. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2)\, dy = 0$ ...(1)

Comparing given differential equation with $Mdx + Ndy = 0$, we get

$$M = y + \frac{y^3}{3} + \frac{x^2}{2} \text{ and } N = \frac{1}{4}(x + xy^2)$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 1 + y^2 \text{ and } \frac{\partial N}{\partial x} = \frac{1}{4} + \frac{1}{4}y^2$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, given differential equation is non-exact, but

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x + xy^2}\left(1 + y^2 - \frac{1}{4} - \frac{1}{4}y^2\right)$$

$$= \frac{4}{x(1 + y^2)} \cdot \frac{3}{4}(1 + y^2)$$

$$= \frac{3}{x} \text{ (Function of } x \text{ only)}$$

Thus, I.F. $= e^{\int \frac{3}{x} dx} = x^3$

Now, multiplying equation (1) by x^3, we get

$$\left(x^3 y + \frac{x^3 y^3}{3} + \frac{x^5}{2}\right) dx + \frac{1}{4}\ (x^4 + x^4 y^2)\ dy = 0$$

which must be exact.

Hence, the required solution is given as

$$\int\left(x^3 y + \frac{x^3 y^3}{3} + \frac{x^5}{2}\right) dx + \int (0)\ dy = c$$

y = constant

or $$\frac{1}{4}\ x^4 y + \frac{1}{12}\ x^4 y^3 + \frac{1}{12}\ x^6 = c$$

$$\Rightarrow \quad 3x^4 y + x^4 y^3 + x^6 = c_1$$

Exa. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

Sol. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ ...(1)

Comparing given differential equation with $Mdx + Ndy = 0$, we get

$$M = 2xy^4e^y + 2xy^3 + y \text{ and } N = x^2y^4e^y - x^2y^2 - 3x$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1 \text{ and } \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, given differential equation is non-exact, but

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(2xy^3e^y + 2xy^2 + 1)}(2xy^4e^y - 2xy^2 - 3 - 8xy^3e^y - 2xy^4e^y - 6xy^2 - 1)$$

$$= \frac{1}{y(2xy^3e^y + 2xy^2 + 1)}(-4)(2xy^3e^y + 2xy^2 + 1)$$

$$= -\frac{4}{y} \text{ (Function of } y \text{ only)}$$

Thus, $\quad$ I.F. $= e^{\int -\frac{4}{y}dy} = \frac{1}{y^4}$

Now, multiplying equation (1) by $\frac{1}{y^4}$, we get

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) dx + \left(x^2 e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right) dy = 0$$

which must be exact.

Hence, the required solution is given as

$$\int \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) dx + \int (0)\, dy = c$$

$y = \text{constant}$

or $$x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Exa. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

Sol. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$ ...(1)

Comparing given differential equation with $Mdx + Ndy = 0$, we get

$$M = y^2 + 2x^2y \text{ and } N = 2x^3 - xy$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 2y + 2x^2 \text{ and } \frac{\partial N}{\partial x} = 6x^2 - y$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence, given differential equation is non-exact, but it can be written in the form

$x^a y^b\,(mydx + nxdy) + x^c y^d\,(pydx + qxdy) = 0$

because it is expressible as

$y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$

So, $\quad$ I.F. $= x^h y^k$,

where values of h and k can be obtained by assuming exactness of eqn. (1) after multiplying with $x^h y^k$

i.e. assume $(x^h y^{k+2} + 2x^{h+2} y^{k+1})\, dx + (2x^{h+3} y^k - x^{h+1} y^{k+1})\, dy = 0$ is exact

$$\Rightarrow \quad \frac{\partial}{\partial y}(x^h y^{k+2} + 2x^{h+2} y^{k+1}) = \frac{\partial}{\partial x}(2x^{h+3} y^k - x^{h+1} y^{k+1})$$

$$\Rightarrow \quad (k+2)\, x^h y^{k+1} + 2(k+1)\, x^{h+2} y^k = 2(h+3)\, x^{h+2} y^k - (h+1)\, x^h y^{k+1}$$

By equating the coefficients of $x^h y^{k+1}$ and $x^{h+2} y^k$, we get

$$k + 2 = -(h+1) \text{ and } 2(k+1) = 2(h+3)$$

$$\Rightarrow \qquad h + k = -3 \text{ and } h - k = -2$$

$$\Rightarrow \qquad h = -\frac{5}{2}, k = -\frac{1}{2}$$

Thus, $\qquad \text{I.F.} = x^{-\frac{5}{2}} y^{-\frac{1}{2}}$

Now, multiplying equation (1) by $x^{-\frac{5}{2}} y^{-\frac{1}{2}}$, we get

$$\left(x^{-\frac{5}{2}} y^{\frac{3}{2}} + 2x^{-\frac{1}{2}} y^{\frac{1}{2}}\right) dx + \left(2x^{\frac{1}{2}} y^{-\frac{1}{2}} - x^{-\frac{3}{2}} y^{\frac{1}{2}}\right) dy = 0$$

which must be exact.

Hence, the required solution is given as

$$\int \left(x^{-\frac{5}{2}} y^{\frac{3}{2}} + 2x^{-\frac{1}{2}} y^{\frac{1}{2}}\right) dx + \int (\ 0\ )\, dy = c$$

y = constant

or $\qquad -\frac{2}{3} x^{-\frac{3}{2}} y^{\frac{3}{2}} + 4x^{\frac{1}{2}} y^{\frac{1}{2}} = c$

Exa. $(1 + xy)\, xdy + (1 - xy)\, y\, dx = 0$

Sol. $(1 + xy)\, x\, dy + (1 - xy)\, ydx = 0$

Given differential equation can be written as

$$xdy + ydx + xy\,(xdy - ydx) = 0$$

$$\Rightarrow \qquad \frac{xdy + ydx}{x^2y^2} + \frac{xdy - ydx}{xy} = 0$$

$$\Rightarrow \qquad d\left(-\frac{1}{xy}\right) + d\left[\log\left(\frac{y}{x}\right)\right] = 0$$

Integrating it, we get the solution of given differential equation as

$$-\frac{1}{xy} + \log\left(\frac{y}{x}\right) = c$$

Exa. $y(axy + e^x)\,dx - e^x\,dy = 0$

Sol. $y(axy + e^x)\,dx - e^x\,dy = 0$

Given differential equation can be written as

$$axdx + \frac{ye^x\,dx - e^x\,dy}{y^2} = 0$$

$$\Rightarrow \qquad axdx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating it, we get the solution of given differential equation as

$$\frac{ax^2}{2} + \frac{e^x}{y} = c \quad \Rightarrow \quad ax^2y + 2e^x = 2cy$$

$$\Rightarrow \qquad ax^2y + 2e^x = c_1y$$

EXERCISE

Solve the following differential equations :

1. $x^2 y\,dx - \left(x^3 + y^3\right)dy = 0$
2. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$
3. $\left(x^2 + y^2 + 1\right)dx - 2xy\,dy = 0$
4. $\left(3xy + 8y^5\right)dx + \left(2x^2 + 24xy^4\right)dy = 0$
5. $y\,dx - x\,dy + \left(1 + x^2\right)dx + x^2 \sin y\,dy = 0$
6. $\left(x^3 + xy^2 + a^2 y\right)dx + \left(y^3 + yx^2 - a^2 x\right)dy = 0$

ANSWERS

1. $y = c\,e^{\frac{x^3}{3y^3}}$

2. $\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$

3. $x^2 - y^2 - 1 = cx$

4. $x^3y^2 + 4x^2y^6 = c$

5. $x^2 - y - 1 - x\cos y = cx$

6. $x^2 + y^2 + 2a^2\tan^{-1}\left(\frac{x}{y}\right) = c$

MORE PROBLEMS ON ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

1. $(x^4y^4 + x^2y^2 + xy)\,ydx + (x^4y^4 - x^2y^2 + xy)\,xdy = 0$
2. $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$
3. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0$
4. $3x\,(1 - x^2)\,y^2\,\dfrac{dy}{dx} + (2x^2 - 1)\,y^3 = ax^3$
5. $\sqrt{(a^2 + x^2)}\,\dfrac{dy}{dx} + y = \sqrt{(a^2 + x^2)} - x$
6. $\sin y\,\dfrac{dy}{dx} = \cos y\,(1 - x\cos y)$

ANSWERS

1. $$\frac{x^2y^2}{2} + \log x - \frac{1}{xy} - \log y = c$$

2. $$x^2y^2 (x^2 - y^2) = c$$

3. $$\frac{x}{y} + e^{x^3} = c$$

4. $$y^3 = ax + cx\sqrt{(1 - x^2)}$$

5. $$\left[\frac{\sqrt{(a^2 + x^2)} + x}{a}\right] y = a \sin h^{-1}\left(\frac{x}{a}\right) + c$$

6. $$\sec y = x - 1 + ce^x$$

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

INTRODUCTION AND VARIOUS CASES FOR FINDING COMPLEMENTORY FUNCTION (C.F.)

INTRODUCTION

Mathematically, a linear differential equation of order n with constant coefficients is represented as

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n y = X \qquad \ldots(1)$$

where $a_0, a_1, \ldots\ldots, a_n$ are constants and X is either a constant or function of independent variable x only.

If we take D to represent $\frac{d}{dx}$, equation (1) becomes

$$(a_0 D^n + a_1 D^{n-1} + \ldots\ldots + a_n)\, y = X$$

or $$f(D)\, y = X \qquad \ldots(2)$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + \ldots\ldots + a_n$

i.e. $f(D)$ is a polynomial of degree n in D.

COMPLETE SOLUTION OF DIFFERENTIAL EQUATION $f(D)y = X$

Complete solution of equation (2) is given by

$$\boxed{y = \text{C.F.} + \text{P.I.}}$$

Remark : If $X = 0$ occurs in equation (2), then its complete solution is given by

$$\boxed{y = \text{C.F.}}$$

AUXILIARY EQUATION (A.E.) FOR FINDING C.F.

In order to find C.F. for equation (2), consider

$$a_0 m^n + a_1 m^{n-1} + \ldots\ldots + a_n = 0 \qquad \ldots(3)$$

Equation (3), whose left hand side is obtained by substituting D by m in $f(D)$ is called auxilliary equation (A.E.) for equation (2)

RULES FOR FINDING C.F.

Case I : If all the roots of auxiliary equation are real and distinct

i.e. $m = m_1, m_2, \ldots\ldots, m_n,$

then $\boxed{C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}}$

Case II : If two roots of auxiliary equation are real and equal and remaining $(n-2)$ roots are real and distinct

i.e. $$m = m_1, m_1, m_3, \ldots\ldots, m_n,$$

then $\boxed{C.F. = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots.. + c_n e^{m_n x}}$

Also, if three roots of auxiliary equation are real and equal and remaining $(n-3)$ roots are real and different i.e. $m = m_1, m_1, m_1, m_4, \ldots.. m_n,$ then proceeding in similar fashion as discussed above

$$\boxed{C.F. = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots.. + c_n e^{m_n x}}$$

Remark: Above concept can be generalised upto any number of real and equal roots of auxiliary equation.

Case III : If two roots of auxiliary equation are real and occur as pairs in form of surds and remaining $(n-2)$ roots are real and distinct

i.e. $$m = m_1 \pm \sqrt{m_2}\ , m_3, m_4, \ldots\ldots, m_n,$$

then $$\boxed{\text{C.F.} = e^{m_1 x}\,[c_1 \cosh \sqrt{m_2}\, x + c_2 \sinh \sqrt{m_2}\, x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}}$$

Remark : Above concept can be generalised upto any number of real and surd roots of auxiliary equation.

Case IV : If two roots of auxiliary equation are imaginary and remaining $(n-2)$ roots are real and distinct

i.e. $m = m_1 \pm i m_2, m_3, \ldots\ldots, m_n$, (As such roots occur in pairs)

then $$\boxed{\text{C.F.} = e^{m_1 x}(c_1 \cos m_2 x + c_2 \sin m_2 x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}$$

Remark : Above concept can be generalised upto any number of imaginary roots of auxiliary equation.

Case V : If two pairs (4 roots) of imaginary roots of auxiliary equation are equal and remaining $(n-4)$ roots are real and distinct

i.e. $$m = m_1 \pm i m_2,\ m_1 \pm i m_2,\ m_5, \ldots, m_n,$$

then $$\boxed{\text{C.F.} = e^{m_1 x}\left[(c_1 + c_2 x)\cos m_2 x + (c_3 + c_4 x)\sin m_2 x\right] + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}}$$

Remark : While solving any differential equation, we can combine all these cases as required.

e.g. for $(D^4 - 3D^3 + 2D^2 - D - 1)\,y = X$,

auxilliary equation can be written as

$$m^4 - 3m^3 + 2m^2 - m - 1 = 0$$

$\Rightarrow \quad (m^2 - m + 1)(m^2 - 2m - 1) = 0$

Now, $\quad m^2 - m + 1 = 0$

$$\Rightarrow \quad m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$\Rightarrow \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

and $\quad m^2 - 2m - 1 = 0 \quad \Rightarrow \quad m = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4+4}}{2}$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{8}}{2}$$

$$\Rightarrow \quad m = \frac{2 \pm 2\sqrt{2}}{2}$$

$$\Rightarrow \quad m = (1 \pm \sqrt{2}\,)$$

$$\Rightarrow \quad m = \frac{1}{2} \pm \frac{i\sqrt{3}}{2},\ 1 \pm \sqrt{2}$$

Hence, $\quad \text{C.F.} = e^{x/2}\left[c_1 \cos\left(\dfrac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\dfrac{\sqrt{3}x}{2}\right)\right]$

$$+ e^{x}\,(c_3 \cosh \sqrt{2}\,x + c_4 \sinh \sqrt{2}\,x)$$

Exa. *Solve the following differential equations :*

(i) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

(ii) $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 2y = 0$

(iii) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

(iv) $\dfrac{d^4y}{dx^4} + a^4y = 0$

(v) $(D^2 + D + 1)^2\, y = 0\,;\, D = \dfrac{d}{dx}$

Sol. (i) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

or $(D^3 - 6D^2 + 11D - 6)\,y = 0\,;\, D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^3 - 6m^2 + 11m - 6 = 0$$

$\Rightarrow$ $m^2(m-1) - 5m(m-1) + 6(m-1) = 0$

$\Rightarrow$ $(m-1)(m^2 - 5m + 6) = 0$

$\Rightarrow$ $(m-1)((m-2)(m-3) = 0$

$\Rightarrow$ $m = 1, 2, 3$

Thus, $\text{C.F.} = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F. or } y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

(ii) $$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$$

or $$(D^3 - 4D^2 + 5D - 2)\,y = 0 \,;\, D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$\Rightarrow \quad m^2(m-1) - 3m(m-1) + 2(m-1) = 0$$

$$\Rightarrow \quad (m-1)(m^2 - 3m + 2) = 0$$

$$\Rightarrow \quad (m-1)(m-1)(m-2) = 0$$

$$\Rightarrow \quad (m-1)^2(m-2) = 0$$

$$\Rightarrow \quad m = 1, 1, 2$$

Thus, $$\text{C.F.} = (c_1 + c_2x).e^x + c_3e^{2x}$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F. or } y = (c_1 + c_2x)\,e^x + c_3e^{2x}$$

(iii) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

or $(D^2 - 4D + 1)y = 0 ; D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow \qquad m = \frac{4 \pm \sqrt{(16-4)}}{2} = \frac{4 \pm \sqrt{12}}{2} = (2 \pm \sqrt{3})$$

Thus, $\qquad \text{C.F.} = e^{2x}(c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x)$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F. or } y = e^{2x}(c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x)$$

(iv) $\dfrac{d^4y}{dx^4} + a^4y = 0$

or $(D^4 + a^4)y = 0 : D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^4 + a^4 = 0$$

$\Rightarrow$ $$m^4 + 2m^2a^2 + a^4 - 2m^2a^2 = 0$$

$\Rightarrow$ $$(m^2 + a^2)^2 - 2m^2a^2 = 0$$

$\Rightarrow$ $$(m^2 + a^2)^2 - (\sqrt{2}\, ma)^2 = 0$$

$\Rightarrow$ $$(m^2 + a^2 + \sqrt{2}\, ma)(m^2 + a^2 - \sqrt{2}\, ma) = 0$$

If $$m^2 + \sqrt{2}\, ma + a^2 = 0,$$

then $$m = \frac{-\sqrt{2}a \pm \sqrt{(2a^2 - 4a^2)}}{2} = \frac{-a\sqrt{2} \pm ia\sqrt{2}}{2}$$

$\Rightarrow$ $$m = \frac{a}{\sqrt{2}}(-1 \pm i)$$

and if $m^2 - \sqrt{2}\, ma + a^2 = 0$,

then $$m = \frac{a\sqrt{2} \pm \sqrt{(2a^2 - 4a^2)}}{2} = \frac{a\sqrt{2} \pm ia\sqrt{2}}{2}$$

$\Rightarrow$ $$m = \frac{a}{\sqrt{2}}(1 \pm i)$$

Thus, $$\text{C.F.} = e^{-\frac{ax}{\sqrt{2}}}\left[c_1 \cos\left(\frac{ax}{\sqrt{2}}\right) + c_2 \sin\left(\frac{ax}{\sqrt{2}}\right)\right] + e^{\frac{ax}{\sqrt{2}}}\left[c_3 \cos\left(\frac{ax}{\sqrt{2}}\right) + c_4 \sin\left(\frac{ax}{\sqrt{2}}\right)\right]$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.}$$

or $$y = e^{-\frac{ax}{\sqrt{2}}}\left[c_1 \cos\left(\frac{ax}{\sqrt{2}}\right) + c_2 \sin\left(\frac{ax}{\sqrt{2}}\right)\right] + e^{\frac{ax}{\sqrt{2}}}\left[c_3 \cos\left(\frac{ax}{\sqrt{2}}\right) + c_4 \sin\left(\frac{ax}{\sqrt{2}}\right)\right]$$

(v) $(D^2 + D + 1)^2 y = 0\ ;\ D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$(m^2 + m + 1)^2 = 0$$

$\Rightarrow$ $(m^2 + m + 1) = 0, (m^2 + m + 1) = 0$

If $m^2 + m + 1 = 0,$

then $$m = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\Rightarrow$ $$m = \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$$

Thus, $$\text{C.F.} = e^{-\frac{x}{2}}\left[(c_1 + c_2 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (c_3 + c_4 x)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.}$$

or $$y = e^{-\frac{x}{2}}\left[(c_1 + c_2 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (c_3 + c_4 x)\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} - 15\dfrac{d^2y}{dx^2} + 71\dfrac{dy}{dx} - 105y = 0$

2. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 8y = 0$

3. $(D^4 + 4D^3 - 5D^2 - 36D - 36)\,y = 0\,;\, D = \dfrac{d}{dx}$

4. $\dfrac{d^4y}{dx^4} - a^4y = 0$

5. $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)\,y = 0$

6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

7. $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0$

ANSWERS

1. $y = c_1 e^{3x} + c_2 e^{5x} + c_3 e^{7x}$
2. $y = (c_1 + c_2 x + c_3 x^2)\, e^{-2x}$
3. $y = c_1 e^{-3x} + c_2 e^{3x} + (c_3 + c_4 x)\, e^{-2x}$
4. $y = c_1 e^{-ax} + c_2 e^{ax} + c_3 \cos ax + c_4 \sin ax$
5. $y = e^{-px} (c_1 \cos qx + c_2 \sin qx)$
6. $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$
7. $y = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$

VARIOUS CASES FOR FINDING PARTICULAR INTEGRAL (P.I.)

RULES FOR FINDING PARTICULAR INTEGRAL

Considering the equation $f(D)\,y = X$

P.I. is given as $\boxed{\text{P.I.} = \dfrac{1}{f(D)}\,X}$

where $\dfrac{1}{f(D)}$ is an inverse operator of $f(D)$.

Now, we will find P.I. with different situations given for X as

(i) e^{ax}

(ii) $\sin ax$ or $\cos ax$

(iii) x^m ; $m \in N$

(iv) $e^{ax}V$; $V = \sin ax, \cos ax$ or x^m

(v) xV ; $V = \sin ax$ or $\cos ax$

(vi) *Any function of* x *(General case)*

P.I. WHEN

$$X = e^{ax}\ ; a \in R$$

P.I. WHEN $X = e^{ax}$

For differential equation with $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_n$

$$\boxed{\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \; ; f(a) \neq 0}$$

Remark : If $f(a) = 0$ occurs in above P.I., which is possible only if $(D - a)^r$ exists as a factor of $f(D)$ *i.e.*

if $f(D) = (D - a)^r \psi(D)$,

then $\text{P.I.} = \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{(D-a)^r \psi(D)} e^{ax}$

or $\text{P.I.} = \dfrac{x^r}{\lfloor r} \cdot \dfrac{e^{ax}}{\psi(a)}$

Exa. *Solve the following differential equations :*

(i) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$$

(ii) $$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$$

(iii) $$\frac{d^4y}{dx^4} - a^4y = \cosh ax$$

Sol. (i) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{5x}$

or $(D^2 - 3D + 2)\,y = e^{5x}\,;\; D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 3m + 2 = 0$$

$\Rightarrow$ $(m-1)(m-2) = 0$

$\Rightarrow$ $m = 1, 2$

Thus, $\text{C.F.} = (c_1 e^x + c_2 e^{2x})$

and
$$\text{P.I.} = \frac{1}{(D^2 - 3D + 2)} e^{5x}$$
$$= \frac{1}{(D-1)(D-2)} e^{5x}$$
$$= \frac{1}{(5-1)(5-2)} e^{5x}$$
$$= \frac{1}{12} e^{5x} \quad \left[\because \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \text{ when } f(a) \neq 0\right]$$

Hence, the complete solution of the given differential equation will be
$$y = \text{C.F.} + \text{P.I.}$$
or
$$y = (c_1 e^x + c_2 e^{2x}) + \frac{1}{12} e^{5x}$$

(ii) $$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$$

or $$(D^4 - 2D^3 + 5D^2 - 8D + 4)\,y = e^x \;;\; D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$$

$$\Rightarrow \quad m^3(m-1) - m^2(m-1) + 4m(m-1) - 4(m-1) = 0$$

$$\Rightarrow \quad (m-1)(m^3 - m^2 + 4m - 4) = 0$$

$$\Rightarrow \quad (m-1)[m^2(m-1) + 4(m-1)] = 0$$

$$\Rightarrow \quad (m-1)(m-1)(m^2+4) = 0$$

$$\Rightarrow \quad m = 1, 1, \pm 2i$$

Thus, $$\text{C.F.} = (c_1 + c_2x)\,e^x + (c_3 \cos 2x + c_4 \sin 2x)$$

and
$$\text{P.I.} = \frac{1}{(D^4 - 2D^3 + 5D^2 - 8D + 4)} e^x$$

$$= \frac{1}{(D-1)^2 (D^2+4)} e^x$$

$$= \frac{1}{(1^2+4)(D-1)^2} e^x$$

$$= \frac{1}{5} \cdot \frac{1}{(D-1)^2} e^x$$

$$= \frac{1}{5} \cdot \frac{x^2}{\lfloor 2} \cdot e^x$$

$$= \frac{1}{10} x^2 e^x$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or
$$y = (c_1 + c_2 x) e^x + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{10} x^2 e^x$$

(iii) $$\frac{d^4y}{dx^4} - a^4y = \cosh ax$$

or $$(D^4 - a^4)\,y = \left(\frac{e^{ax} + e^{-ax}}{2}\right); D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$\Rightarrow$ $$(m^2 - a^2)(m^2 + a^2) = 0$$

$\Rightarrow$ $$m = a, -a, \pm ai$$

Thus, $$\text{C.F.} = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$$

and $$\text{P.I.} = \frac{1}{(D^4 - a^4)}\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^4 - a^4)} e^{ax} + \frac{1}{(D^4 - a^4)} e^{-ax}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2 + a^2)(D^2 - a^2)} e^{ax} + \frac{1}{(D^2 + a^2)(D^2 - a^2)} e^{-ax}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2 + a^2)(D + a)(D - a)} e^{ax} + \frac{1}{(D^2 + a^2)(D + a)(D - a)} e^{-ax}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(a^2 + a^2)(a + a)} \cdot \frac{1}{(D - a)} e^{ax} + \frac{1}{(a^2 + a^2)(-a - a)} \cdot \frac{1}{(D + a)} e^{-ax}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4a^3} \cdot \frac{x}{\lfloor 1} e^{ax} - \frac{1}{4a^3} \cdot \frac{x}{\lfloor 1} e^{-ax}\right]$$

$$= \frac{1}{4a^3} \cdot x \left(\frac{e^{ax} - e^{-ax}}{2}\right) = \frac{x \sinh ax}{4a^3}$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax + \frac{x \sinh ax}{4a^3}$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = e^{-x}$

2. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^x$

3. $\dfrac{d^3y}{dx^3} - y = (1 + e^x)^2$

4. $(D^2 + 2D + 1)\,y = \sinh 3x \,;\, D = \dfrac{d}{dx}$

ANSWERS

1. $y = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] + e^{-x}$

2. $y = c_1 e^x + c_2 e^{2x} - x\, e^x$

3. $y = c_1 e^x + e^{-\frac{x}{2}}\left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] - 1 + \frac{2xe^x}{3} + \frac{e^{2x}}{7}$

4. $y = (c_1 + c_2 x)\, e^{-x} + \frac{1}{32}\, e^{3x} - \frac{1}{8}\, e^{-3x}$

P.I. WHEN

$$X = \sin ax \quad OR \quad \cos ax \; ; a \in R$$

P.I. WHEN $X = \sin ax$ OR $\cos ax$

For differential equation with $X = \sin ax$ or $\cos ax$

$$\text{P.I.} = \frac{1}{f(D)} \sin ax \text{ or } \frac{1}{f(D)} \cos ax$$

$$\text{P.I.} = \frac{1}{f(D)} \sin ax$$

$$= \frac{1}{f_1(D^2) + Df_2(D^2)} \sin ax$$

$$= \frac{1}{f_1(-a^2) + Df_2(-a^2)} \sin ax$$

$$= \frac{1}{A + DB} \sin ax$$

$$= \frac{(A - DB)}{(A^2 - D^2B^2)} \sin ax$$

$$= \frac{(A - DB)}{A^2 - (-a^2)B^2} \sin ax$$

$$= \frac{(A \sin ax - aB \cos ax)}{(A^2 + a^2B^2)}$$

Remarks : (1) If in above P.I., we obtain $f(-a^2) = 0$, which is possible only if $(D^2 + a^2)$ or its any higher power exists as a factor of $f(D)$ *i.e.*

if $$f(D) = (D^2 + a^2)^r,$$

$$\frac{1}{(D^2 + a^2)^r} \sin ax = \left(-\frac{x}{2a}\right)^r \frac{1}{\lfloor r} \sin\left(ax + \frac{r\pi}{2}\right)$$

and $$\frac{1}{(D^2 + a^2)^r} \cos ax = \left(-\frac{x}{2a}\right)^r \frac{1}{\lfloor r} \cos\left(ax + \frac{r\pi}{2}\right)$$

and if $$f(D) = (D^2 + a^2),$$

$$\frac{1}{(D^2 + a^2)} \sin ax = -\frac{x}{2a} \cos ax$$

and $$\frac{1}{(D^2 + a^2)} \cos ax = \frac{x}{2a} \sin ax$$

(2) Above **P.I.** remains unchanged if we replace $\sin ax$ or $\cos ax$ by $\sin(ax + b)$ or $\cos(ax + b)$ respectively.

Exa. *Solve the following differential equations :*

(i) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x} + \sin 2x$

(ii) $\dfrac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \dfrac{x}{2}$

(iii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = \sin(3x + 1)$

(iv) $\dfrac{d^2y}{dx^2} + a^2y = \sin ax$

(v) $(D^2 + 1)^2 y = \cos x \cos 2x$; $D = \dfrac{d}{dx}$

Sol. (*i*) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \sin 2x$$

or $$(D^2 - 4D + 4)\,y = e^{2x} + \sin 2x\;;\; D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 4m + 4 = 0$$

$\Rightarrow$ $$(m-2)^2 = 0$$

$\Rightarrow$ $$m = 2, 2$$

Thus, $$\text{C.F.} = (c_1 + c_2 x)\,e^{2x}$$

and
$$\text{P.I.} = \frac{1}{(D^2 - 4D + 4)} (e^{2x} + \sin 2x)$$

$$= \frac{1}{(D^2 - 4D + 4)} e^{2x} + \frac{1}{(D^2 - 4D + 4)} \sin 2x$$

$$= \frac{1}{(D - 2)^2} e^{2x} + \frac{1}{(-4 - 4D + 4)} \sin 2x$$

$$= \frac{x^2}{\lfloor 2} e^{2x} - \frac{1}{4} \frac{1}{D} \sin 2x$$

$$= \frac{x^2}{\lfloor 2} e^{2x} - \frac{1}{4}\left(-\frac{\cos 2x}{2}\right)$$

$$= \frac{1}{2} x^2 e^{2x} + \frac{1}{8} \cos 2x$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or
$$y = (c_1 + c_2 x) e^{2x} + \frac{1}{2} x^2 e^{2x} + \frac{1}{8} \cos 2x$$

(ii) $$\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \frac{x}{2}$$

or $$(D^3 + 1)\,y = \sin 3x - \cos^2 \frac{x}{2} \; ; D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^3 + 1 = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - m + 1) = 0$$

$$\Rightarrow \quad m = -1, \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$\Rightarrow \quad m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

Thus, $$\text{C.F.} = c_1 e^{-x} + e^{\frac{x}{2}} \left[c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right) \right]$$

and $$\text{P.I.} = \frac{1}{(D^3+1)}\left(\sin 3x - \cos^2\frac{x}{2}\right)$$

$$= \frac{1}{(D^3+1)}\sin 3x - \frac{1}{(D^3+1)}\cdot\cos^2\frac{x}{2}$$

$$= \frac{1}{(-9D+1)}\sin 3x - \frac{1}{(D^3+1)}\cdot\left(\frac{1+\cos x}{2}\right)$$

$$= \frac{(1+9D)}{(1-81D^2)}\sin 3x - \frac{1}{2}\left[\frac{1}{(D^3+1)}\cdot e^{0x} + \frac{1}{(D^3+1)}\cos x\right]$$

$$= \frac{(1+9D)}{[1-81(-9)]}\sin 3x - \frac{1}{2}\left[e^{0x} + \frac{1}{(-D+1)}\cos x\right]$$

$$= \frac{(1+9D)}{730}\sin 3x - \frac{1}{2}\left(1 + \frac{(1+D)}{(1-D^2)}\cos x\right)$$

$$= \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2}\left(1 + \frac{(1+D)}{2}\cos x\right)$$

$$= \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2}\left(1 + \frac{1}{2}\cos x - \frac{1}{2}\sin x\right)$$

$$= \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2} - \frac{1}{4}\cos x + \frac{1}{4}\sin x$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or

$$y = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2\cos\left(\frac{\sqrt{3}x}{2}\right) + c_3\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

$$+ \frac{1}{730}(\sin 3x + 27\cos 3x) - \frac{1}{2} + \frac{1}{4}(\sin x - \cos x)$$

(*iii*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = \sin(3x + 1)$

or $(D^2 - 2D + 2)\,y = \sin(3x + 1)\,;\; D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 2m + 2 = 0$$

$$\Rightarrow \qquad m = \frac{2 \pm \sqrt{4 - 4\,.\,1\,.\,2}}{2}$$

$$\Rightarrow \qquad m = \frac{2 \pm \sqrt{(4 - 8)}}{2}$$

$$\Rightarrow \qquad m = \frac{2 \pm 2i}{2} = 1 \pm i$$

Thus, $\quad$ C.F. $= e^x\,(c_1 \cos x + c_2 \sin x)$

$$\text{and P.I.} = \frac{1}{(D^2 - 2D + 2)} \sin(3x + 1)$$

$$= \frac{1}{(-9 - 2D + 2)} \sin(3x + 1)$$

$$= \frac{1}{(-7 - 2D)} \sin(3x + 1)$$

$$= -\frac{1}{(7 + 2D)} \sin(3x + 1)$$

$$= -\frac{(7 - 2D)}{(49 - 4D^2)} \sin(3x + 1)$$

$$= -\frac{(7-2D)}{[49-4(-9)]}\sin(3x+1)$$

$$= -\frac{(7-2D)}{85}\sin(3x+1)$$

$$= -\frac{1}{85}[(7\sin(3x+1)-6\cos(3x+1)]$$

$$= \frac{6\cos(3x+1)-7\sin(3x+1)}{85}$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = e^x(c_1\cos x + c_2\sin x) + \frac{6\cos(3x+1)-7\sin(3x+1)}{85}$$

(iv) $$\frac{d^2y}{dx^2} + a^2y = \sin ax$$

or $$(D^2 + a^2)\,y = \sin ax\ ;\ D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 + a^2 = 0$$

$\Rightarrow$ $$m = \pm ai$$

Thus, $$\text{C.F.} = (c_1 \cos ax + c_2 \sin ax)$$

and $$\text{P.I.} = \frac{1}{(D^2 + a^2)} \sin ax$$

$$= -\frac{x}{2a} \cos ax \qquad \left[\because \frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax\right]$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a} \cos ax$$

(v) $(D^2 + 1)^2 y = \cos x \cos 2x \; ; D = \dfrac{d}{dx}$

Now to find C.F., the auxiliary equation in m is given as

$$(m^2 + 1)^2 = 0$$

$$\Rightarrow \quad m = \pm i, \pm i$$

Thus, $\quad \text{C.F.} = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

and $\quad \text{P.I.} = \dfrac{1}{(D^2+1)^2} \cdot \cos x \cos 2x$

$$= \frac{1}{2} \cdot \frac{1}{(D^2+1)^2} (2 \cos 2x \cos x)$$

$$= \frac{1}{2} \frac{1}{(D^2+1)^2} (\cos 3x + \cos x)$$

$$= \frac{1}{2} \frac{1}{(D^2+1)^2} \cos 3x + \frac{1}{2} \frac{1}{(D^2+1)^2} \cos x$$

$$= \frac{1}{2} \frac{1}{(-9+1)^2} \cos 3x + \frac{1}{2} \frac{1}{(D^2+1)^2} \cos x$$

$$= \frac{1}{128} \cos 3x + \frac{1}{2} \frac{(-1)^2 x^2}{(2)^2 \lfloor 2} \cos (x+\pi)$$

$$= \frac{1}{128} \cos 3x - \frac{1}{16} x^2 \cos x$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + \frac{1}{128} \cos 3x - \frac{1}{16} x^2 \cos x$$

EXERCISE

Solve the following differential equations :

1. $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} = 3e^{2x} + 4\sin x$

2. $\frac{d^2y}{dx^2} + 4y = \sin^2 x + 4$

3. $\frac{d^2y}{dx^2} + 9y = \sin 2x \cos x$

4. $\frac{d^4y}{dx^4} - m^4 y = \sin mx$

ANSWER

1. $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-3x} + \frac{2}{5}(2\sin x + \cos x) + \frac{3}{20} e^{2x}$

2. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{9}{8} - \frac{1}{8} x \sin 2x$

3. $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{16} \sin x - \frac{1}{12} x \cos 3x$

4. $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3} \cos mx$

P.I. WHEN

$$X = x^m;\ m \in N$$

P.I. WHEN $X = x^m$; $m \in N$

$$\text{P.I.} = \frac{1}{f(D)} x^m = \frac{1}{(a_0 D^n + a_1 D^{n-1} + \ldots + a_n)} x^m$$

$$= \frac{1}{a_n [(1 + \psi(D)]} x^m$$

$$= \frac{1}{a_n} [1 + \psi(D)]^{-1} x^m$$

Now, expand $[1 + \psi(D)]^{-1}$ in ascending powers of D upto the term containing D^m only as $D^{m+1}(x^m) = D^{m+2}(x^m) = \ldots\ldots = 0$ and finally operate x^m by the various terms of expansion containing different powers of D.

Remark : If $f(D) = (D - \alpha_1)(D - \alpha_2) \ldots (D - \alpha_n)$,

then
$$\text{P.I.} = \frac{1}{(D - \alpha_1)(D - \alpha_2) \ldots (D - \alpha_n)} x^m$$

$$= \left(\frac{A_1}{D - \alpha_1} + \frac{A_2}{D - \alpha_2} + \ldots + \frac{A_n}{D - \alpha_n} \right) x^m$$

[By factorizing partially]

and then we can use the process similar to above article to find $\frac{A_1}{D - \alpha_1} x^m$, $\frac{A_2}{D - \alpha_2} x^m, \ldots, \frac{A_n}{D - \alpha_n} x^m$ separately.

Exa. *Solve the following differential equations :*

(i) $$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = (1 + x^2)$$

(ii) $$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$$

Sol. (i) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = (1 + x^2)$

or $(D^3 - D^2 - 6D)\,y = (1 + x^2)\ ;\ D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^3 - m^2 - 6m = 0$$

$\Rightarrow$ $m\,(m^2 - m - 6) = 0$

$\Rightarrow m\,(m^2 - 3m + 2m - 6) = 0$

$\Rightarrow$ $m\,(m - 3)\,(m + 2) = 0$

$\Rightarrow$ $m = 0, 3, -2$

Thus, $\text{C.F.} = c_1e^{0x} + c_2e^{3x} + c_3e^{-2x} = c_1 + c_2e^{3x} + c_3e^{-2x}$

and $$\text{P.I.} = \frac{1}{(D^3 - D^2 - 6D)}(1 + x^2)$$

$$= -\frac{1}{6D\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)}(1 + x^2)$$

$$= -\frac{1}{6D}\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} + \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(x^2+1) - \frac{1}{6}D(x^2+1) + \frac{7}{36}D^2(x^2+1) + \dots\right]$$

$$= -\frac{1}{6D}\left[(x^2+1) - \frac{2x}{6} + \frac{14}{36}\right]$$

$$= -\frac{1}{6D}\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right)$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or

$$y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right)$$

(ii) $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = (e^{2x} + x^2 + x)$

or $(D^3 + 2D^2 + D)\,y = (e^{2x} + x^2 + x)$; $D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^3 + 2m^2 + m = 0$$

$$\Rightarrow \quad m(m^2 + 2m + 1) = 0$$

$$\Rightarrow \quad m(m+1)^2 = 0$$

$$\Rightarrow \quad m = 0, -1, -1$$

Thus,

$$\text{C.F.} = c_1 e^{0x} + (c_2 + c_3 x)\,e^{-x}$$

$$= c_1 + (c_2 + c_3 x)\,e^{-x}$$

and $$\text{P.I.} = \frac{1}{(D^3 + 2D^2 + D)}\,(e^{2x} + x^2 + x)$$

$$= \frac{1}{(D^3 + 2D^2 + D)}\,e^{2x} + \frac{1}{(D^3 + 2D^2 + D)}\,(x^2 + x)$$

$$= \frac{1}{(2^3 + 2.2^2 + 2)}\,e^{2x} + \frac{1}{D(D+1)^2}\,(x^2 + x)$$

$$= \frac{1}{18}\,e^{2x} + \frac{1}{D}\,(1 + D)^{-2}\,(x^2 + x)$$

$$= \frac{1}{18}\,e^{2x} + \frac{1}{D}\,(1 - 2D + 3D^2 \ldots)\,(x^2 + x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} \{(x^2 + x) - 2(2x + 1) + 3(2)\}$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} (x^2 - 3x + 4)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{3} x^3 - \frac{3x^2}{2} + 4x$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = c_1 + (c_2 + c_3 x) e^{-x} + \frac{1}{18} e^{2x} + \frac{1}{3} x^3 - \frac{3x^2}{2} + 4x$$

EXERCISE

Solve the following differential equations :

1. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

2. $\frac{d^3y}{dx^3} + 8y = (x^4 + 2x + 1)$

3. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \cos x + x^2$

4. $\frac{d^2y}{dx^2} + y = e^{-x} + \cos x + x^3$

ANSWERS

1. $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{12}(2x^3 - 9x^2 + 21x)$

2. $y = c_1 e^{-2x} + e^{x}(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x) + \frac{1}{8}(x^4 - x + 1)$

3. $y = e^{2x}(c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x) + \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left(x^2 + \frac{4x}{3} + \frac{2}{9}\right)$

4. $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x} + \frac{1}{2} x \sin x + x^3 - 6x$

P.I. WHEN

$$X = e^{ax} V \; ; \; V = \sin ax, \cos ax \text{ or } x^m$$

P.I. WHEN $X = e^{ax}V$; $V = \sin ax, \cos ax$ or x^m

$$\text{P.I.} = \frac{1}{f(D)}(e^{ax}V)$$

$$= e^{ax}\frac{1}{f(D+a)}V$$

Exa. *Solve the following differential equations :*

(i) $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}$$

(ii) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cosh x \sin x$$

Sol. (i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2e^{3x}$

or $(D^2 - 2D + 1)\,y = x^2e^{3x}\,;\, D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 2m + 1 = 0$$

$\Rightarrow$ $$(m-1)^2 = 0$$

$\Rightarrow$ $$m = 1, 1$$

Thus, $$\text{C.F.} = (c_1 + c_2x)\,e^x$$

and $\quad$ P.I. $= \dfrac{1}{(D^2 - 2D + 1)} x^2 e^{3x} = \dfrac{1}{(D-1)^2} x^2 e^{3x}$

$$= e^{3x} \cdot \frac{1}{(D + 3 - 1)^2} x^2$$

$$= e^{3x} \cdot \frac{1}{(D+2)^2} x^2$$

$$= \frac{1}{4} e^{3x} \cdot \frac{1}{\left(1 + \frac{D}{2}\right)^2} x^2$$

$$= \frac{1}{4} e^{3x} \left(1 + \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4} e^{3x} \left(1 - D + \frac{3D^2}{4} + \ldots\right) x^2$$

$$= \frac{1}{4} e^{3x} \left(x^2 - 2x + \frac{3}{2}\right) = \frac{1}{8} e^{3x} (2x^2 - 4x + 3)$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or

$$y = (c_1 + c_2 x) e^x + \frac{1}{8} e^{3x} (2x^2 - 4x + 3)$$

(ii) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cosh x \sin x$$

or $(D^2 + 3D + 2)\, y = \cosh x \sin x \,;\, D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 + 3m + 2 = 0$$

$\Rightarrow \quad (m + 1)(m + 2) = 0$

$\Rightarrow \qquad m = -1, -2$

Thus, $\quad$ C.F. $= (c_1 e^{-x} + c_2 e^{-2x})$

and $\text{P.I.} = \dfrac{1}{(D^2 + 3D + 2)} \cdot \cosh x \sin x$

$$= \frac{1}{(D^2 + 3D + 2)} \cdot \left(\frac{e^x + e^{-x}}{2} \right) \sin x$$

$$= \frac{1}{2} \left[\frac{1}{(D^2 + 3D + 2)} \cdot e^x \sin x + \frac{1}{(D^2 + 3D + 2)} \cdot e^{-x} \sin x \right]$$

$$\frac{1}{2}\left[e^{x}\cdot\frac{1}{(D+1)^2+3(D+1)+2}\sin x+e^{-x}\cdot\frac{1}{(D-1)^2+3(D-1)+2}\sin x\right]$$

$$\frac{1}{2}\left[e^{x}\cdot\frac{1}{(D^2+5D+6)}\sin x+e^{-x}\cdot\frac{1}{(D^2+D)}\sin x\right]$$

$$\frac{1}{2}\left[e^{x}\cdot\frac{1}{(-1+5D+6)}\sin x+e^{-x}\cdot\frac{1}{(-1+D)}\sin x\right]$$

$$\frac{1}{2}\left[e^{x}\cdot\frac{1}{(5D+5)}\sin x+e^{-x}\cdot\frac{1}{(D-1)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{5}\cdot\frac{(D-1)}{(D^2-1)}\sin x + e^{-x}\cdot\frac{(D+1)}{(D^2-1)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{5}\cdot\frac{(D-1)}{(-1-1)}\sin x + e^{-x}\cdot\frac{(D+1)}{(-1-1)}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{e^x}{10}(D-1)\sin x - \frac{e^{-x}}{2}(D+1)\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{e^x}{10}(\cos x - \sin x) - \frac{e^{-x}}{2}(\cos x + \sin x)\right]$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or
$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{2}\left[\frac{e^x}{10}(\cos x - \sin x) + \frac{e^{-x}}{2}(\cos x + \sin x)\right]$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}(1+x)$

2. $\dfrac{d^2y}{dx^2} - y = \cosh x \cos x$

3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = e^x \cos 2x + \cos 3x$

4. $\dfrac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x \cos 2x$

ANSWERS

1. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \frac{1}{12}\left(x + \frac{17}{12}\right) e^{2x}$

2. $y = c_1e^{-x} + c_2e^{x} + \frac{2}{5} \sin x \sinh x - \frac{1}{5} \cos x \cosh x$

3. $y = c_1e^{x} + c_2e^{3x} - \frac{1}{8} e^{x} (\sin 2x + \cos 2x) - \frac{1}{3} (2 \sin 3x + \cos 3x)$

4. $y = c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x + \frac{1}{121} e^{3x} \left(11x^2 - 12x + \frac{50}{11}\right)$
$$+ \frac{1}{17} e^{x} (4 \sin 2x - \cos 2x)$$

P.I. WHEN

$$X = x^m V \; ; \; V = \sin ax \text{ or } \cos ax \text{ and } m \in N$$

P.I. WHEN $X = xV$; $V = \sin ax$ or $\cos ax$

$$\text{P.I.} = \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{f(D)^2}V$$

where $\frac{1}{f(D)}V$, $\frac{f'(D)}{[f(D)]^2}V$ can be evaluated by previously discussed methods, if V is given in form of $\sin ax$ or $\cos ax$

Remark : To find $\frac{1}{f(D)}(x^m V)$; $m \in N$ and $m \geq 2$ with $V = \sin ax$ or $\cos ax$,

we first obtain $\frac{1}{f(D)}(x^m e^{iax})$ by $\frac{1}{f(D)}(e^{iax}x^m) = e^{iax}\frac{1}{f(D+ia)}x^m$

and then by considering real or imaginary parts, we obtain

$\frac{1}{f(D)}(x^m \cos ax)$ or $\frac{1}{f(D)}(x^m \sin ax)$ respectively.

Exa. *Solve the following differential equations :*

(i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x \cos x$

(ii) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \cos x$

(iii) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8x^2 e^{2x} \sin 2x$

Sol. (i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x \cos x$

or $$(D^2 + D)\,y = x \cos x \;;\; D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 + m = 0$$

$\Rightarrow$ $$m(m+1) = 0$$

$\Rightarrow$ $$m = 0, -1$$

Thus, $$\text{C.F.} = c_1 e^{(0)x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$$

$$\text{and P.I.} = \frac{1}{(D^2 + D)} x \cos x$$

$$= x \cdot \frac{1}{(D^2 + D)} \cos x - \frac{(2D + 1)}{(D^2 + D)^2} \cos x$$

$$= x \cdot \frac{1}{(-1 + D)} \cos x - \frac{(2D + 1)}{(-1 + D)^2} \cos x$$

$$= x \cdot \frac{(D + 1)}{(D^2 - 1)} \cos x - \frac{(2D + 1)}{(D^2 - 2D + 1)} \cos x$$

$$= x \cdot \frac{(D + 1)}{(-1 - 1)} \cos x - \frac{(2D + 1)}{(-1 - 2D + 1)} \cos x$$

$$= -\frac{x}{2}(D+1)\cos x + \frac{1}{2D}(2D+1)\cos x$$

$$= -\frac{x}{2}(-\sin x + \cos x) + \frac{1}{2D}(-2\sin x + \cos x)$$

$$= \frac{x}{2}(\sin x - \cos x) + \frac{1}{2}(2\cos x + \sin x)$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = c_1 + c_2 e^{-x} + \frac{x}{2}(\sin x - \cos x) + \frac{1}{2}(2\cos x + \sin x)$$

(ii) $$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$$

or $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$; $D = \dfrac{d}{dx}$

Now, to find C.F., the auxiliary equation in m is given as

$$m^4 + 2m^2 + 1 = 0$$

$\Rightarrow$ $$(m^2 + 1)^2 = 0$$

$\Rightarrow$ $$m = \pm i, \pm i$$

Thus, $$\text{C.F.} = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$$

and $$\text{P.I.} = \frac{1}{(D^4 + 2D^2 + 1)} x^2 \cos x$$

$$= \text{R.P. of } \frac{1}{(D^2 + 1)^2} (x^2 e^{ix})$$

$$= \text{R.P. of } \left[e^{ix} \frac{1}{\{(D + i)^2 + 1\}^2} x^2 \right]$$

$$= \text{R.P. of} \left[e^{ix} \frac{1}{(D^2 + 2iD - 1 + 1)^2} x^2 \right]$$

$$= \text{R.P. of} \left[e^{ix} \frac{1}{(D^2 + 2iD)^2} x^2 \right]$$

$$= \text{R.P. of} \left[e^{ix} \frac{1}{(2iD)^2} \frac{1}{\left(1 + \frac{D}{2i}\right)^2} \right] x^2$$

$$= \text{R.P. of} \left[\ e^{ix} \left(- \frac{1}{4D^2} \right) \left(1 - \frac{D}{i} + \frac{3D^2}{4i^2} + \ldots \right) \ \right] x^2$$

$$= \text{R.P. of} \left[\ \left(- \frac{1}{4} e^{ix} \right) \frac{1}{D^2} \left(x^2 - \frac{2x}{i} - \frac{3}{2} \right) \ \right]$$

$$= \text{R.P. of} \left[\left(-\frac{1}{4}e^{ix}\right)\frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3x}{2}\right) \right]$$

$$= \text{R.P. of} \left[\left(-\frac{1}{4}e^{ix}\right)\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3x^2}{4}\right) \right]$$

$$= \text{R.P. of} \left[-\frac{1}{4}(\cos x + i\sin x)\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3x^2}{4}\right)\right]$$

$$= -\frac{1}{4}\left[\left(\frac{x^4}{12} - \frac{3x^2}{4}\right)\cos x - \frac{x^3}{3}\sin x\right]$$

$$= -\frac{1}{48}\left[(x^4 - 9x^2)\cos x - 4x^3 \sin x\right]$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or

$$y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x - \frac{1}{48}\left[(x^4 - 9x^2)\cos x - 4x^3\sin x\right]$$

(iii) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2e^{2x}\sin 2x$$

or $$(D^2 - 4D + 4)\,y = 8x^2e^{2x}\sin 2x\ ;\ D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow \quad (m-2)^2 = 0$$

$$\Rightarrow \quad m = 2, 2$$

Thus, $$\text{C.F.} = (c_1 + c_2x)\,e^{2x}$$

and $$\text{P.I.} = \frac{1}{(D-2)^2}\,8x^2e^{2x}\sin 2x$$

$$= 8e^{2x}\,.\,\frac{1}{(D+2-2)^2}\,x^2\sin 2x$$

$$= 8e^{2x}\,\frac{1}{D^2}\,x^2\sin 2x$$

$$= 8e^{2x}\text{ I.P. of }\left[\frac{1}{D^2}(x^2e^{2ix})\right]$$

$$= 8e^{2x} \text{ I.P. of } \left[e^{2ix} \frac{1}{(D+2i)^2} x^2 \right]$$

$$= 8e^{2x} \text{ I.P. of } \left[e^{2ix} \frac{1}{(2i)^2} \left(1 + \frac{D}{2i}\right)^{-2} x^2 \right]$$

$$= 8e^{2x} \text{ I.P. of } \left[-\frac{1}{4} e^{2ix} \left(1 - \frac{D}{i} + \frac{3}{4i^2} D^2 + \ldots\right) x^2 \right]$$

$$= 8e^{2x} \text{ I.P. of } \left[-\frac{1}{4} (\cos 2x + i \sin 2x) \left(x^2 + 2xi - \frac{3}{2} \right) \right]$$

$$= 8e^{2x}\left[-\frac{1}{4}\left\{2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right\}\right]$$

$$= -2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or $$y = (c_1 + c_2 x)\,e^{2x} - 2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} + 4y = x \sin x$

2. $\dfrac{d^2y}{dx^2} - y = x^2 \cos x$

3. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x$

ANSWERS

1. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{3}\, x \sin x - \dfrac{2}{9}\, \cos x$

2. $y = c_1 e^{-x} + c_2 e^{x} - x \sin x - \dfrac{1}{2}\,(1 - x^2) \cos x$

3. $y = (c_1 + c_2 x)\, e^{x} - e^{x}\,(x \sin x + 2 \cos x)$

P.I. WHEN

X is any function of x

P.I. WHEN X IS ANY FUNCTION OF x

To find $\frac{1}{f(D)} X$, we resolve $f(D)$ into linear factors (whether real or complex)

i.e.

$$\frac{1}{f(D)} X = \frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)} X$$

$$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \ldots + \frac{A_n}{D-\alpha_n}\right) X$$

$$= A_1 \frac{1}{D-\alpha_1} X + A_2 \frac{1}{D-\alpha_2} X + \ldots + A_n \frac{1}{D-\alpha_n} X$$

Now, $\frac{1}{D-\alpha_r} X = e^{\alpha_r x} \int X e^{-\alpha_r x}\, dx$

or $\frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_r)} X = A_1 e^{\alpha_1 x} \int X e^{-\alpha_1 x}\, dx + A_2 e^{\alpha_2 x} \int X e^{-\alpha_2 x}\, dx$

$+ \ldots\ldots\ldots\ldots + A_n e^{\alpha_n x} \int X e^{-\alpha_n x}\, dx$

Exa. *Solve the following differential equations :*

(i) $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$

(ii) $\dfrac{d^2y}{dx^2} - y = \dfrac{1}{(1-e^{-x})^2}$

<u>Sol.</u> (*i*) $$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

or $$(D^2 + a^2)\,y = \sec ax\ ;\ D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 + a^2 = 0$$

$\Rightarrow$ $$(m + ai)\,(m - ai) = 0$$

$\Rightarrow$ $$m = ai, -ai$$

Thus, $$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

and
$$\text{P.I.} = \frac{1}{(D^2 + a^2)} \sec ax$$
$$= \frac{1}{2ia}\left[\frac{1}{(D - ai)} - \frac{1}{(D + ai)}\right] \sec ax$$

Now, $\dfrac{1}{(D - ai)} \sec ax = e^{ixa} \int e^{-iax} \sec ax \, dx$
$$= e^{ixa} \int \frac{(\cos ax - i \sin ax)}{\cos ax} \, dx$$
$$= e^{ixa} \int (1 - i \tan ax) \, dx$$
$$= e^{ixa} \left[x + \frac{i}{a} \log (\cos ax)\right]$$

Similarly,
$$\frac{1}{(D + ai)} \sec ax = e^{-iax} \left[x - \frac{i}{a} \log (\cos ax)\right]$$

Thus,
$$\text{P.I.} = \frac{1}{2ia}\left[e^{ixa}\left\{x + \frac{i}{a}\log(\cos ax)\right\} - e^{-ixa}\left\{x - \frac{i}{a}\log(\cos ax)\right\}\right]$$

$$= \frac{x}{2ia}(e^{ixa} - e^{-ixa}) + \frac{1}{a^2}[\log(\cos ax)]\left(\frac{e^{iax} + e^{-iax}}{2}\right)$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \,.\, \log(\cos ax)$$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or
$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \,.\, \log(\cos ax)$$

(ii) $$\frac{d^2y}{dx^2} - y = \frac{1}{(1-e^{-x})^2}$$

or $$(D^2-1)y = \frac{1}{(1-e^{-x})^2} \quad ; D = \frac{d}{dx}$$

Now, to find C.F., the auxiliary equation in m is given as

$$m^2 - 1 = 0$$

$\Rightarrow$ $$(m+1)(m-1) = 0$$

$\Rightarrow$ $$m = 1, -1$$

Thus, $$\text{C.F.} = (c_1e^x + c_2e^{-x})$$

and $$\text{P.I.} = \frac{1}{(D^2-1)}\left[\frac{1}{(1-e^{-x})^2}\right]$$

$$= \frac{1}{(D-1)(D+1)} \cdot \frac{1}{(1-e^{-x})^2}$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)} - \frac{1}{(D+1)}\right] \cdot \frac{1}{(1-e^{-x})^2}$$

Now,

$$\frac{1}{(D-1)} \cdot \frac{1}{(1-e^{-x})^2} = e^x \int \frac{e^{-x}}{(1-e^{-x})^2}\, dx$$

$$= e^x \int \frac{1}{u^2}\, du \qquad [\text{Put } (1-e^{-x}) = u]$$

$$= -\frac{e^x}{u} = -\frac{e^x}{(1-e^{-x})}$$

and $\frac{1}{(D+1)} \cdot \frac{1}{(1-e^{-x})^2} = e^{-x} \int \frac{e^x}{(1-e^{-x})^2}\, dx$

$$= e^{-x} \int \frac{e^x . e^{2x}}{(e^x - 1)^2}\, dx$$

$$= e^{-x} \int \frac{(v+1)^2}{v^2}\, dv \qquad [\text{Put } (e^x - 1) = v]$$

$$= e^{-x} \int \left(1 + \frac{2}{v} + \frac{1}{v^2}\right) dv$$

$$= e^{-x} \left(v + 2\log v - \frac{1}{v}\right)$$

$$= e^{-x}\left[(e^x - 1) + 2\log(e^x - 1) - \frac{1}{(e^x - 1)}\right]$$

Thus, $\text{P.I.} = \frac{1}{2}\left[\frac{-e^x}{(1-e^{-x})} - e^{-x}\left\{(e^x - 1) + 2\log(e^x - 1) - \frac{1}{(e^x - 1)}\right\}\right]$

Hence, the complete solution of the given differential equation will be

$$y = \text{C.F.} + \text{P.I.}$$

or

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}\left[\frac{-e^x}{(1-e^{-x})} - e^{-x}\left\{(e^x - 1) + 2\log(e^x - 1) - \frac{1}{(e^x - 1)}\right\}\right]$$

EXERCISE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$

2. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

3. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin(e^x)$

ANSWERS

1. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log \left[\tan\left(x + \frac{\pi}{4}\right)\right]$

2. $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \,.\, \log(\sin x)$

3. $y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin(e^x)$

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THANKS