ENGINEERING MATHEMATICS - I (I SEM)

ASYMPTOTES

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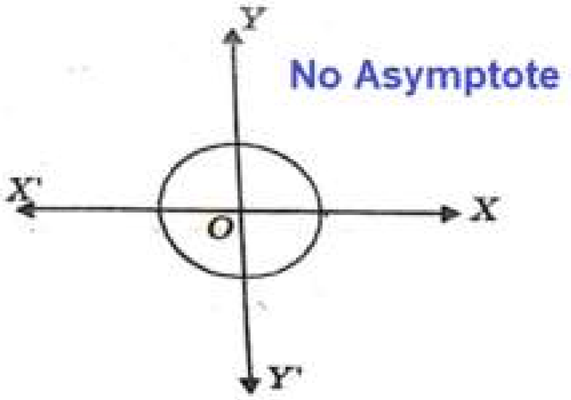
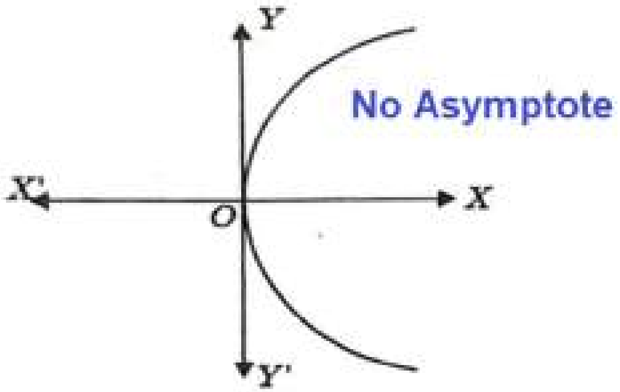
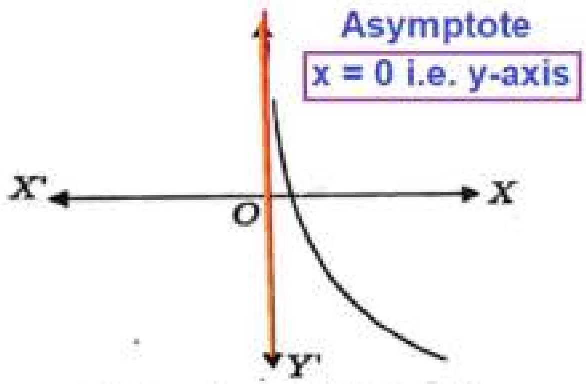


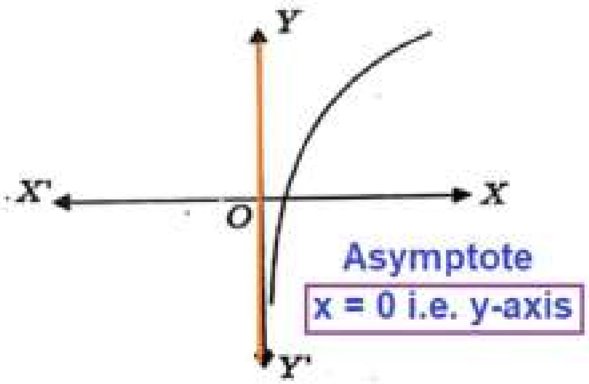
(a) $x^2 + y^2 = a^2$



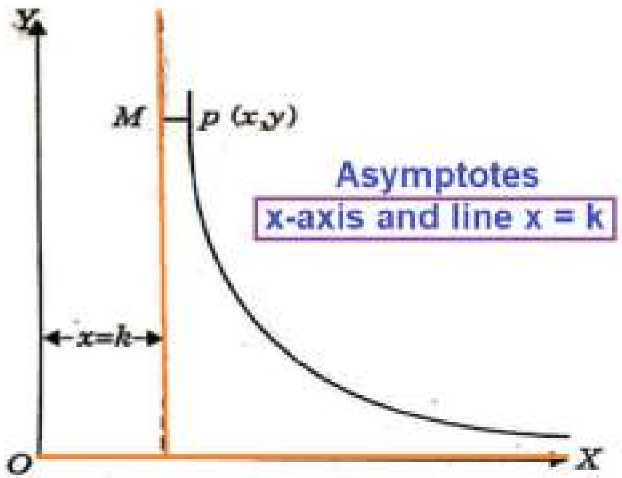
(b) $y^2 = 4ax$



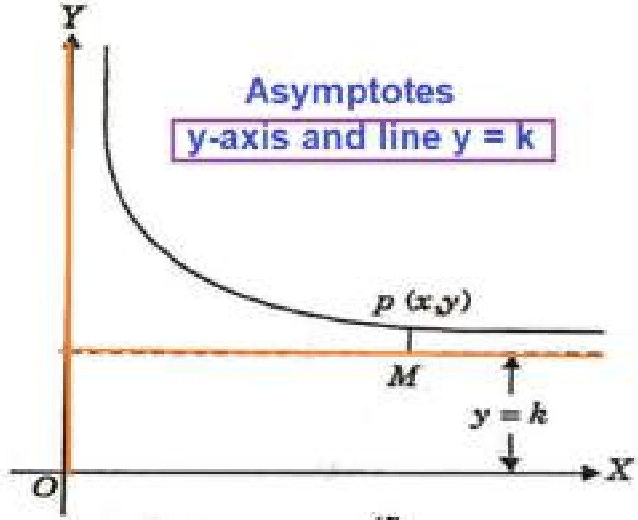
(c) $y = \log_a x \ (0 < a < 1)$



(d) $y = \log_a x \ (a > 1)$



(e)



(f)

ASYMPTOTE

Definition: In other words, a straight line is said to be an asymptote of a curve $y = f(x)$, if the perpendicular distance of the point $P(x, y)$ on the curve from the line tends to zero when x or y or both tend to infinity.

OR

Asymptote can be defined as a pseudo tangent at infinity.

ASYMPTOTES PARALLEL TO THE COORDINATE AXES

ASYMPTOTES PARALLEL TO THE COORDINATE AXES

Asymptotes parallel to the axes can be obtained by the usual method. Here we shall establish a direct method for finding such asymptotes :

(*i*) **Asymptotes parallel to x-axis :** The asymptotes parallel to x-axis can be obtained by equating to zero, the coefficients of the highest power of x in the equation of the curve, provided that it is not a constant term.

(*i*) **Asymptotes parallel to y-axis :** The asymptotes parallel to y-axis can be obtained by equating to zero, the coefficients of the highest power of y in the equation of the curve, provided that it is not a constant term.

Exa. *Find the asymptotes of the following curves which are parallel to x-axis.*

(i) $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) = 0$

(ii) $\dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = 1$

Sol. (i) $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) = 0$

Given curve can be rewritten as

$$x^2(y^2 - a^2) - a^2y^2 - a^3(x + y) = 0$$

Now, asymptotes parallel to x-axis are given by equating to zero the coefficients of highest power of x in the given curve

i.e. $\quad y^2 - a^2 = 0$

$\Rightarrow \quad y = \pm a$ are the desired asymptotes.

(ii) $$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$$

Given curve can be rewritten as

$$a^2y^2 - b^2x^2 = x^2y^2$$

$\Rightarrow$ $$x^2(y^2 + b^2) - a^2y^2 = 0$$

Now, asymptotes parallel to x-axis are given by equating to zero the coefficients of highest power of x in the given curve

i.e. $$y^2 + b^2 = 0$$

$\Rightarrow$ $$y = \pm ib$$

Hence for the given curve there is no real asymptote parallel to x-axis.

EXERCISE

Find the asymptotes parallel to coordinate axes of the following curves :

1. $y^2 = \dfrac{x^5}{(x^3 - a^3)}$

2. $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$

ANSWERS

1. $x = a$

2. $x = \pm a, y = \pm b$

GENERAL METHOD FOR FINDING ALL ASYMPTOTES OF A POLYNOMIAL CURVE IN CARTESIAN FORM

PROCEDURE OF FINDING ALL ASYMPTOTES OF A POLYNOMIAL CURVE IN CARTESIAN FORM

Step - 1: Find all asymptotes parallel to coordinate axes using the process defined earlier.

Step - 2: For finding all oblique asymptotes of a curve of degree n, we put $x = 1$ and $y = m$ in terms of degree n and n - 1 to find $\phi_n(m)$ and $\phi_{n-1}(m)$.

Step - 3: We put $\phi_n(m) = 0$ to find n values of m, If these n values of m are real and different i.e. $m = m_1, m_2, \ldots, m_n$, then we calculate c using the following formula:

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

Then by substituting $m = m_1, m_2, \ldots, m_n$ in above expression, we get $c = c_1, c_2, \ldots, c_n$. Combining every set of values of m and c, asymptotes are given as $\boxed{y = m_i x + c_i}$

Step - 4: If n values of m are real such that two values are identical and remaining n - 2 values are different i.e. $m = m_1, m_1, m_3, \ldots\ldots, m_n$, then we calculate c for $m = m_1, m_1$ using the following formula:

$$\frac{c^2}{\lfloor 2}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0,$$

where $\phi_{n-2}(m)$ can be obtained by substituting x = 1 and y = m in terms of degree n - 2. Also, for finding asymptotes for remaining different values, use step - 2.

Step - 5: If three values of m are identical, then c is calculated as per the formula

$$\frac{c^3}{\lfloor 3}\phi_n'''(m) + \frac{c^2}{\lfloor 2}\phi_{n-1}'''(m) c\,\phi_{n-2}'(m) + \phi_{n-3}(m) = 0 \quad \text{etc.}$$

Exa. *Find all asymptotes of the following curve:*

$$4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 = 7$$

Sol. $4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 = 7$

As coefficients of highest power of x and y are constant, there is no asymptote parallel to axes. Now, for further asymptotes (oblique) of the given curve, by substituting $x = 1$ and $y = m$ in terms of degree 3 and 2, we get

$$\phi_3(m) = (4 - m - 4m^2 + m^3) \text{ and } \phi_2(m) = (3 + 2m - m^2)$$

Now, $\phi_3(m) = 0$

$\Rightarrow \quad 4 - m - 4m^2 + m^3 = 0$

$\Rightarrow \quad (m - 4)(m^2 - 1) = 0$

$\Rightarrow \quad m = 4, \pm 1$

and $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(3+2m-m^2)}{(3m^2-8m-1)} = \frac{(m^2-2m-3)}{(3m^2-8m-1)}$$

Also, $$m = 1 \Rightarrow c = \frac{(1-2-3)}{(3-8-1)} = \frac{2}{3}$$

$$m = -1 \Rightarrow c = \frac{(1+2-3)}{(3+8-1)} = 0$$

$$m = 4 \Rightarrow c = \frac{(16-8-3)}{(48-32-1)} = \frac{5}{15} = \frac{1}{3}$$

So, asymptotes are $y = x + \frac{2}{3}, y = -x, y = 4x + \frac{1}{3}$

Exa. *Find all asymptotes of the following curve:*

$$(x-y)(x+y)(x+2y+2)=(x+9y-1)$$

Sol. $(x-y)(x+y)(x+2y+2)=x+9y-2$

As coefficients of highest powers of x and y are constant, there is no asymptote parallel to axes.

Now, for further asymptotes of the given curve, by substituting $x=1$ and $y=m$ in terms of degree 3 and 2, we get

$$\phi_3(m)=(1-m)(1+m)(1+2m)$$

and $$\phi_2(m)=2(1-m)(1+m)$$

Now, $$\phi_3(m)=0$$

$$\Rightarrow \quad (1-m)(1+m)(1+2m)=0$$

$$\Rightarrow \quad m=1,-1,-\frac{1}{2}$$

and
$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-2(1-m)(1+m)}{2(1-m^2)-2m(1+2m)}$$

Also,
$$m = 1 \Rightarrow c = \frac{-2(1-1)(1+1)}{2(1-1)-2(1+2)} = 0$$

$$m = -1 \Rightarrow c = \frac{-2(1+1)(1-1)}{2(1-1)+2(1-2)} = 0$$

$$m = -\frac{1}{2} \Rightarrow c = \frac{-2\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)}{2\left(1-\frac{1}{4}\right)+(1-1)} = -1$$

So, asymptotes are

$$y = x, y = -x, y = -\frac{x}{2} - 1$$

Exa. ***Find all asymptotes of the following curve :***

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$$

Sol. $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$

As coefficients of highest powers of x and y are constant, there is no asymptote parallel to axes. Now, for further asymptotes of the given curve, by substituting $x = 1$ and $y = m$ in terms of degree 3 and 2, we get

$$\phi_3(m) = 1 - m - m^2 + m^3 \text{ and } \phi_2(m) = 2 - 4m^2 + 2m$$

Now, $\phi_3(m) = 0$

$\Rightarrow \quad 1 - m - m^2 + m^3 = 0$

$\Rightarrow \quad (1 - m)(1 - m^2) = 0$

$\Rightarrow \quad m = 1, 1, -1$

and
$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(2 - 4m^2 + 2m)}{(-1 - 2m + 3m^2)}$$

Also, $\quad m = -1 \Rightarrow \quad c = -\dfrac{(2-4-2)}{(-1+2+3)} = 1$

So, corresponding asymptote is $y = -x + 1$

and for $m = 1, 1$

$$\frac{c^2}{\lfloor 2}\,\phi_3''(m) + \frac{c}{\lfloor 1}\,\phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{\lfloor 2}(-2+6m) + \frac{c}{\lfloor 1}(-8m+2) + (1+m) = 0 \qquad [\because \quad \phi_1(m) = 1 + m]$$

$$\Rightarrow \quad \frac{c^2}{2}(-2+6) + c(-8+2) + (1+1) = 0$$

$$\Rightarrow \quad 2c^2 - 6c + 2 = 0$$

$$\Rightarrow \quad c = \frac{3 \pm \sqrt{5}}{2}$$

Hence, corresponding to $m = 1$, two mutually parallel asymptotes are given by

$$y = x + \left(\frac{3 \pm \sqrt{5}}{2}\right).$$

Exa. *Find all asymptotes of the following curve:*

$$y^3 - 6xy^2 + 12x^2y - 8x^3 - 3y^2 + 12xy - 12x^2 + 2y - 4x = 0$$

Sol. $y^3 - 6xy^2 + 12x^2y - 8x^3 - 3y^2 + 12xy - 12x^2 + 2y - 4x = 0$

As coefficients of highest powers of x and y are constant, there is no asymptote parallel to axes. Now, for further asymptotes of the given curve, by putting $x = 1$ and $y = m$ in terms of degree 3 and 2, we get

$\phi_3(m) = m^3 - 6m^2 + 12m - 8$ and $\phi_2(m) = (-3m^2 + 12m - 12)$

Now, $\phi_3(m) = 0$

$\Rightarrow \quad m^3 - 6m^2 + 12m - 8 = 0$

$\Rightarrow \quad (m-2)(m^2 - 4m + 4) = 0$

$\Rightarrow \quad (m-2)^3 = 0 \quad \Rightarrow \quad m = 2, 2, 2.$

For $m = 2$ (repeated thrice)

$$\frac{c^3}{\lfloor 3}\phi_3'''(m) + \frac{c^2}{\lfloor 2}\phi_2''(m) + \frac{c}{\lfloor 1}\phi_1'(m) + \phi_0(m) = 0$$

$\Rightarrow \quad \dfrac{c^3}{6}(6) + \dfrac{c^2}{2}(-6) + c(2) = 0 \qquad [\because \quad \phi_1(m) = 2m - 4 \text{ and } \phi_0(m) = 0]$

$\Rightarrow \quad c^3 - 3c^2 + 2c = 0$

$\Rightarrow \quad c(c^2 - 3c + 2) = 0$

$\Rightarrow \quad c(c-1)(c-2) = 0$

$\Rightarrow \quad c = 0, 1, 2$

Thus asymptotes corresponding to $m = 2$ are given by

$$y = 2x,\ y = 2x + 1,\ y = 2x + 2$$

EXERCISE

Find all asymptotes of the following curves:

1. $x^3 + y^3 = 3axy$
2. $2x(y-3)^2 - 3y(x-1)^2 = 0$
3. $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$
4. $y^3 - 3xy^2 - x^2y + 3x^3 - 3x^2 + 10xy - 3y^2 - 10x - 10y + 7 = 0$
5. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$
6. $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 + 2y^2 - 3xy - 1 = 0$
7. $(x+y)^2(x+2y+2) = x + 9y - 2$
8. $x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$

ANSWERS

1. $x + y + a = 0$
2. $x = 0, y = 0, 2y = 3x + 6$
3. $y = x - 1, y = -x - 2, y = 2x$
4. $y = x + 1, y + x = 2, y = 3x$
5. $y = x, 2y + x - 1 = 0, 2y + x + 1 = 0$
6. $y = x, 2y = x, 2y = x + 1$
7. $x + 2y + 2 = 0, x + y = \pm 2\sqrt{2}$
8. $y = x + 1, x + y = 2, x + y + 1 = 0$

METHOD OF INSPECTION (SHORT CUT METHED) FOR FINDING ALL ASYMPTOTES OF A POLYNOMIAL CURVE IN CARTESIAN FORM

METHOD OF INSPECTION

If the equation of the curve is expressible in the form $\boxed{F_n + P = 0,}$ where F_n contains product of n linear and different factors of the type $(ax + by + c)$ and P contains terms of degree $(n-2)$ and lower, then all asymptotes of the given curve are obtained by

$$\boxed{F_n = 0}$$

Use method of inspection (if possible) to find all asymptotes of the following curves:

(i) $y(x-y)^3 = y(x-y) + 2$

(ii) $(y-3x)^2(4x+5y) + 3(y-3x)(4x+5y) = 0$

(iii) $xy(x^2-y^2)(x^2-16y^2) + 3xy(x^2-y^2) + x^2 + y^2 - 7 = 0$

(iv) $(x^2-y^2)(x+2y+3) + x + y + 1 = 0$

(i) $y(x-y)^3 = y(x-y) + 2$

We have $F_4 + P = 0$ from in the given problem with two degree difference in F_4 and P, but F_4 is not represented as product of four linear and different factors. So, method of inspection can not be applied here.

(ii) $(y-3x)^2(4x+5y) + 3(y-3x)(4x+5y) = 0$

We have $F_3 + P = 0$ from in the given problem with only one degree difference in F_3 and P. So, method of inspection can not be applied here.

(iii) $xy(x^2 - y^2)(x^2 - 16y^2) + 3xy(x^2 - y^2) + x^2 + y^2 - 7 = 0$

Above curve is expressible in the form $F_6 + P = 0$

where $F_6 = xy(x^2 - y^2)(x^2 - 16y^2)$ contains terms of degree six and

$$P = 3xy(x^2 - y^2) + x^2 + y^2 - 7$$

contains terms of degree four or lower and F_6 can be factorised into six linear and distinct factors.

Thus, above curve satisfies requirements of method of inspection, using which asymptotes are given by

$$F_6 = 0$$

$\Rightarrow \quad xy(x^2 - y^2)(x^2 - 16y^2) = 0$

$\Rightarrow \quad x = 0, y = 0, x + y = 0, x - y = 0, x + 4y = 0, x - 4y = 0$ are the required asymptoes.

(iv) $(x^2 - y^2)(x + 2y + 3) + x + y + 1 = 0$

Above curve is expressible in the form $F_3 + P = 0$

where $F_3 = (x^2 - y^2)(x + 2y + 3)$ contains terms of degree three and lower and $P = x + y + 1$ contains terms of degree one or lower and F_3 can be factorised into three linear and distinct factors.

Thus, above curve satisfies requirements of method of inspection, using which asymptotes are given by

$$F_3 = 0$$

$\Rightarrow \quad (x^2 - y^2)(x + 2y + 3) = 0$

$\Rightarrow \quad x + y = 0, x - y = 0, x + 2y + 3 = 0$ are the required asymptotes.

INTERSECTION OF THE CURVE AND ITS ASYMPTOTES

INTERSECTION OF THE CURVE AND ITS ASYMPTOTES

If $S_n = 0$ is equation of given curve of degree 'n' and $A_n = 0$ is combined equation of its all asymptotes, then curve which contains $n(n-2)$ points of intersection of given curve and its all asymptotes is given as :

$$\boxed{S_n + \lambda A_n = 0\ ;\ \lambda \text{ is any real number.}}$$

Exa. *Find all the asymptotes of the curve* $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5 = 0$ *and Show that the asymptotes meet the curve again in three points which lie on a straight line and find the equation of this line.*

Sol. $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5 = 0$

There is no parallel asymptote for this curve. To find oblique asymptotes, $\phi_3(m) = 3 + 2m - 7m^2 + 2m^3$ and $\phi_2(m) = -14m + 7m^2$.

Now, $\phi_3(m) = 0$

$\Rightarrow \quad 2m^3 - 7m^2 + 2m + 3 = 0$

$\Rightarrow \quad 2m^3 - 2m^2 - 5m^2 + 5m - 3m + 3 = 0$

$\Rightarrow (m-1)(2m^2 - 5m - 3) = 0$

$\Rightarrow \quad (m-1)(m-3)(2m+1) = 0 \quad \Rightarrow \quad m = 1, 3, -\dfrac{1}{2}$

and $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(7m^2 - 14m)}{(6m^2 - 14m + 2)}$$

Also, $$m = 1 \Rightarrow c = -\frac{(7 - 14)}{(6 - 14 + 2)} = -\frac{7}{6}$$

$$m = -\frac{1}{2} \Rightarrow c = -\frac{\left(\frac{7}{4} + \frac{14}{2}\right)}{\left(\frac{6}{4} + \frac{14}{2} + 2\right)} = -\frac{5}{6}$$

$$m = 3 \Rightarrow c = -\frac{(63 - 42)}{(54 - 42 + 2)} = -\frac{3}{2}$$

Thus, for given curve, asymptotes are given by

$$y = x - \frac{7}{6}, y = -\frac{x}{2} - \frac{5}{6}, y = 3x - \frac{3}{2}$$

So, combined equation of all asymptotes of given curve is given as

$$F_3 = \left(y - x + \frac{7}{6}\right)\left(y - 3x + \frac{3}{2}\right)\left(y + \frac{x}{2} + \frac{5}{6}\right) = 0$$

whereas given curve can be written as

$$S_3 = 3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5 = 0$$

Thus three points of intersection of S_3 and F_3 lie on

$$S_3 + \lambda F_3 = 0$$

$$\Rightarrow \quad (3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5) + \lambda\left[\left(y - x + \frac{7}{6}\right)\left(y - 3x + \frac{3}{2}\right)\left(y + \frac{x}{2} + \frac{5}{6}\right)\right] = 0$$

$$\Rightarrow \quad (3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5) + \lambda\left[\frac{1}{72}(72y^3 + 108x^3 - 252xy^2 + 72x^2y + 252y^2 - 504xy + 286y - 237x + 105)\right] = 0$$

Now by selecting $\lambda = -2$, we get

$$(3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5) - \left(2y^3 + 3x^3 - 7xy^2 + 2x^2y + 7y^2 - 14xy + \frac{286}{36}y - \frac{237}{36}x + \frac{105}{36}\right) = 0$$

$$\Rightarrow \qquad 4x + 5 - \frac{286}{36}y + \frac{237}{36}x - \frac{105}{36} = 0$$

$$\Rightarrow \qquad 381x - 286y + 75 = 0$$

which is required equation of straight line.

Exa. *Find the equation of cubic curve whose asymptotes are $x + a = 0$, $y - a = 0$, $x + y + a = 0$ and which touches the x-axis at origin and passes through the point $(-2a, -2a)$*

Sol. Combined equation of asymptotes is given as :

$(x + a)(y - a)(x + y + a) = 0$ or $F_3 = 0$

Thus, we can assume the cubic curve as $F_3 + P = 0$

where P is an expression of degree one or lower, which can be assumed as

$$P = (Ax + By + C)$$

Thus, required cubic curve is expressed as

$(x + a)(y - a)(x + y + a) + (Ax + By + C) = 0$

which passes through origin.

So, $-a^3 + C = 0 \Rightarrow C = a^3$

Hence, cubic curve is modified as :

$(x + a)(y - a)(x + y + a) + Ax + By + a^3 = 0$ ...(1)

To find equation of tangents at origin, equating to zero the lowest degree terms in the equation of curve, we get

$$(A - 2a^2)\,x + By = 0 \qquad \ldots(2)$$

But, we are given that tangent of cubic curve at origin is x-axis *i.e.*

$$y = 0 \qquad \ldots(3)$$

(2) and (3) $\Rightarrow$ $A - 2a^2 = 0$ or $A = 2a^2$

Hence, the equation of cubic curve can be further rewritten as

$$(x + a)\,(y - a)\,(x + y + a) + 2a^2x + By + a^3 = 0 \qquad \ldots(4)$$

Eqn. (4) passes through the point $(-2a, -2a)$

So, $(-2a + a)\,(-2a - a)\,(-2a - 2a + a) + 2a^2\,(-2a) + B\,(-2a) + a^3 = 0$

$\Rightarrow$ $B = -6a^2$

Thus, the cubic curve is given as

$$(x + a)\,(y - a)\,(x + y + a) + 2a^2x - 6a^2y + a^3 = 0$$

$\Rightarrow$ $xy\,(x + y) + a(y^2 - x^2 + xy) - 6a^2y = 0$

EXERCISE

1. Show that the asymptotes of the curve $x^3 - 2y^3 + 2x^2y - xy^2 + y(x-y) + 1 = 0$ cut the curve in three points which lie on the line $x - y + 1 = 0$
2. Show that the eight points of intersection of the curve $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$ and its asymptotes lie on a rectangular hyperbola
3. Find the equation of the curve on which the points of intersection of the curve $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0$ and its asymptotes lie
4. Find the equation of cubic curve, which has asymptotes on the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which passes through the points $(0, 0)$, $(1, 0)$ and $(0, 1)$
5. Show that the four asymptotes of the curve $(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2v - 3xv^2 + 2y^3 - x^2 + 3xy - 1 = 0$ cut the curve again in eight points which lie on the circle $x^2 + y^2 = 1$

ANSWERS

3. There doesn't exist any curve which contains points of intersection of given curve and its asymptotes.

4. $(y-x)\left(y-\dfrac{x}{2}\right)\left(y-\dfrac{x}{3}\right)+\dfrac{x}{6}-y=0$

THANKS

ENGINEERING MATHEMATICS - I
(I SEM)

CURVATURE

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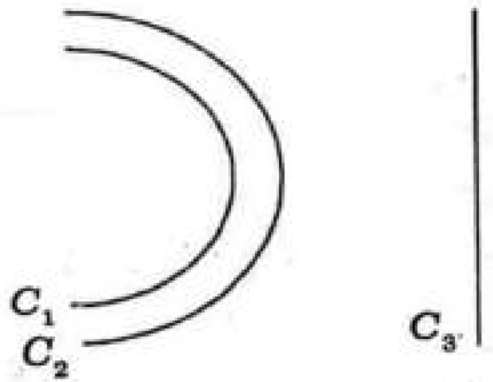
CONCEPT OF THE CURVATURE

Let us look at the following curves.



The curve C_1 bends more sharply than C_2 and bending tendency is zero in the straight line C_3.

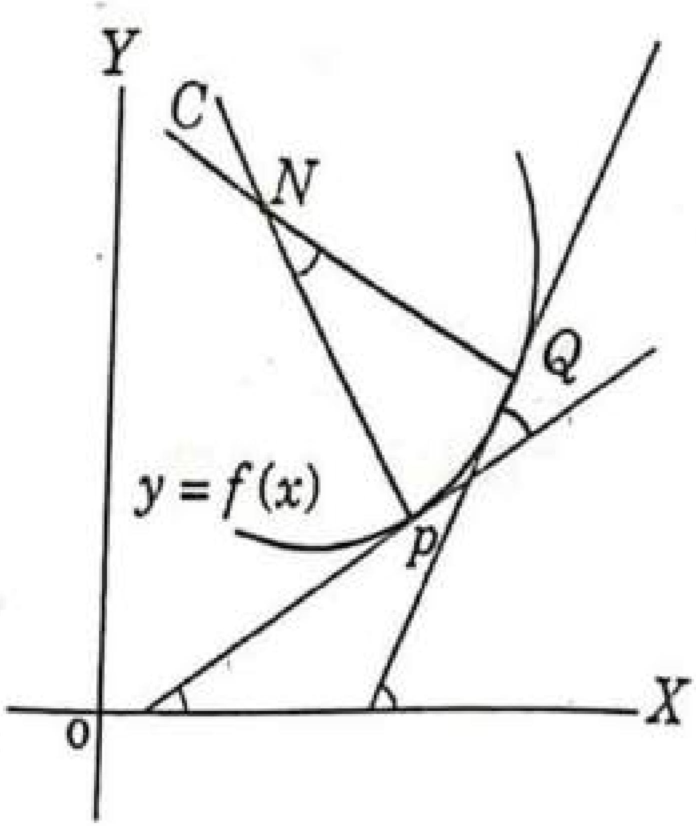
CURVATURE

Definition:

'Curvature' is used for the quantitative measure of the bending of the curve at a point or in other words, curvature is a mathematical tool which measures the bending of the curve at every point.

GEOMETRY OF CURVATURE

Let the normals at P and Q intersect at N. As Q approaches P, the point N takes up a definite position C, on the normal at P. The length CP is called the radius of curvature, and the point C is called centre of curvature at the point P, whereas the circle with centre C and radius CP is called the circle of curvature which touches the curve $y = f(x)$ at P.



The radius of curvature is donated by ρ and curvature is donated by $\boxed{\kappa = \dfrac{1}{\rho}}$

RADIUS OF CURVATURE IN CARTESIAN COORDINATE SYSTEM

CARTESIAN FORMULA FOR RADIUS OF CURVATURE

Let the equation of the curve be $y = f(x)$.

Then $$\boxed{\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

Remark: The above formula does not hold good if the tangent is parallel to the axis of y *i.e.* when $\frac{dy}{dx}$ is infinite. In such cases we interchange the axis of x and y because the value of ρ depends only on the curve and not on the axes.

Hence, we have

$$\boxed{\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}}$$

Exa. For the curve $y = ae^{x/a}$, prove that $\rho = a \sec^2 \theta \operatorname{cosec} \theta$

where $\theta = \tan^{-1}\left(\frac{y}{a}\right)$

Sol.

$$y = ae^{x/a}$$

diff. wrt x

$$\frac{dy}{dx} = e^{x/a} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{a} e^{x/a}$$

$\Rightarrow$

$$\frac{dy}{dx} = \frac{y}{a} \text{ and } \frac{d^2y}{dx^2} = \frac{y}{a^2}$$

Hence,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{y^2}{a^2}\right)^{3/2}}{\frac{y}{a^2}}$$

$\Rightarrow$

$$\rho = \frac{(1 + \tan^2 \theta)^{3/2}}{\frac{1}{a} \tan \theta} = \frac{a \sec^3 \theta}{\tan \theta} \qquad \left[\because \ \tan \theta = \frac{y}{a}\right]$$

$\Rightarrow$

$$\rho = a \sec^2 \theta \operatorname{cosec} \theta \qquad \text{...H.P.}$$

Exa. *For the curve $x^2y = a(x^2 + y^2)$, prove that radius of curvature is a constant at point $(-2a, 2a)$*

Sol.
$$x^2y = a(x^2+y^2)$$

diff. wrt x
$$2xy + x^2\frac{dy}{dx} = a\left(2x + 2y\frac{dy}{dx}\right)$$

$\Rightarrow$
$$\frac{dy}{dx} = \left(\frac{2ax - 2xy}{x^2 - 2ay}\right)$$

$\Rightarrow$
$$\left(\frac{dy}{dx}\right)_{(-2a,\,2a)} = \infty \Rightarrow \left(\frac{dx}{dy}\right)_{(-2a,\,2a)} = 0$$

Now,
$$\frac{dx}{dy} = \left(\frac{x^2 - 2ay}{2ax - 2xy}\right)$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{(2ax-2xy)\left(2x\frac{dx}{dy}-2a\right)-(x^2-2ay)\left(2a\frac{dx}{dy}-2x-2y\frac{dx}{dy}\right)}{(2ax-2xy)^2}$$

$$\Rightarrow \quad \left(\frac{d^2x}{dy^2}\right)_{(-2a,\,2a)} = \frac{(4a^2)(-2a)-(0)(4a)}{16a^4} = -\frac{1}{2a}$$

Hence, at the point $(-2a, 2a)$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{(1+0)^{3/2}}{-\frac{1}{2a}}$$

$\Rightarrow$ $\rho = -2a$, which is a constant.

EXERCISE

1. Show that the radius of curvature at any point $P(x, y)$ on the parabola $y^2 = 4ax$ is $\frac{2(SP)^{3/2}}{\sqrt{a}}$ where S is the focus of the parabola.

2. Find the radius of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

3. Find the radius of curvature at any point (x, y) on the following curves :
   (i) $a^2y = x^3 - a^3$
   (ii) $ay^2 = x^3$
   (iii) $y = c\cosh\left(\frac{x}{c}\right)$
   (iv) $y = a\log\left[\sec\left(\frac{x}{a}\right)\right]$

4. Find the radius of curvature of the curve $xy^2 = 4a^2(2a - x)$ at the point where the curve meets the x-axis.

ANSWERS

2. $\dfrac{3\sqrt{2}a}{16}$

3. (i) $\dfrac{(a^4+9x^4)^{3/2}}{6xa^4}$ (ii) $\dfrac{\sqrt{x}}{6a}(9x+4a)^{3/2}$ (iii) $\dfrac{y^2}{c}$ (iv) $a\sec\left(\dfrac{x}{a}\right)$

4. a

RADIUS OF CURVATURE FOR CURVE IN PARAMETRIC FORM

FORMULA FOR RADIUS OF CURVATURE FOR CURVE IN PARAMETRIC FORM

If $x = f(t)$, $y = g(t)$ is the parametric form of the given curve, then radius of curvature is given as:

$$\boxed{\rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - x''y')}}$$

where x', y', x'', y'' are respective first and second order derivatives of x and y with respect to 't'

Exa. *For an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *prove that* $\rho = \frac{a^2 b^2}{p^3}$, *where* p *is the perpendicular drawn from the centre of ellipse upon the tangent at any point* (x, y)

Sol. Equation of an ellipse is given in cartesian form, but to solve this problem we will choose point (x, y) as $x = a \cos \theta, y = b \sin \theta$ (Parametric Form).

$$\text{Hence, } \rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - x''y')} = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{(-a \sin \theta)(-b \sin \theta) - (-a \cos \theta)(b \cos \theta)}$$

$$\Rightarrow \quad \rho = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \qquad ...(1)$$

Now, equation of tangent of given ellipse at point ($a \cos \theta, b \sin 0$) is given as

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \qquad ...(2)$$

Thus, length of perpendicular from centre of ellipse (0, 0) to tangent (2) is given as

$$p = \frac{|-1|}{\sqrt{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}\right)}} = \frac{ab}{\sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}} \qquad ...(3)$$

(1) and (3) $\Rightarrow \quad \rho = \frac{a^2 b^2}{p^3}$

Exa. *If CP and CD be a pair of conjugate semi-diameter of an ellipse, prove that the radius of curvature at P is $\frac{(CD)^3}{ab}$, where a and b are the lengths of the semiaxes of an ellipse.*

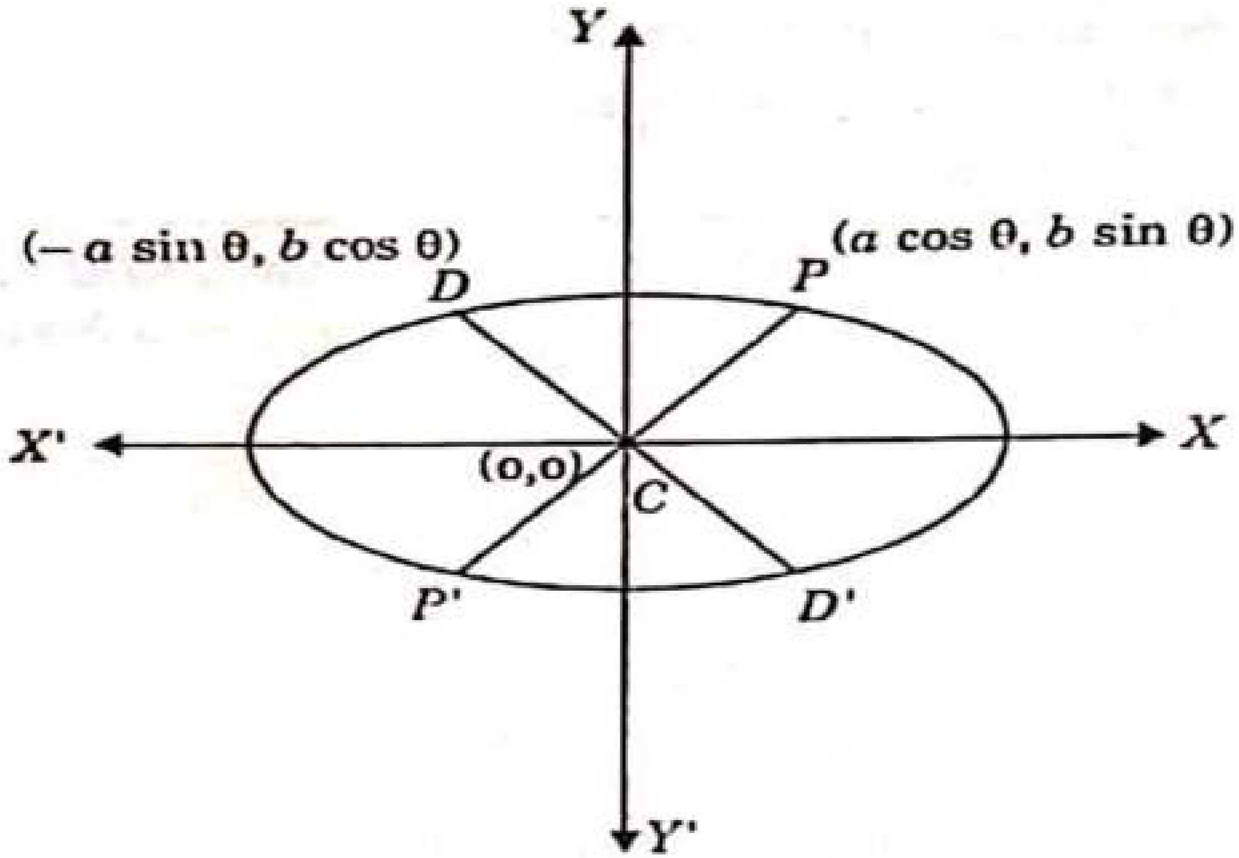
Sol. The equation of an ellipse with a and b as semiaxes in parametric form is given by

$$x = a \cos \theta, y = b \sin \theta$$

Hence,
$$\rho = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \quad ...(1)$$

[using solution of part (i) of Ex. 4]

Here, PCP' and DCD' are two conjugate diameters of the ellipse *i.e.* each diameter bisects chords parallel to the other. This concludes an important result that the difference in eccentric angles (θ) of vertices P, D, P' and D' (consecutive vertices of pair of conjugate diameters) is $\frac{\pi}{2}$. Hence, if point P is given as ($a \cos \theta, b \sin \theta$), then D will be given as ($-a \sin \theta, b \cos \theta$).



Thus, $$(CD)^2 = (a^2 \sin^2 \theta + b^2 \cos^2 \theta) \qquad \ldots(2)$$

(1) and (2) $\Rightarrow$ $$\rho = \frac{(CD)^3}{ab}$$

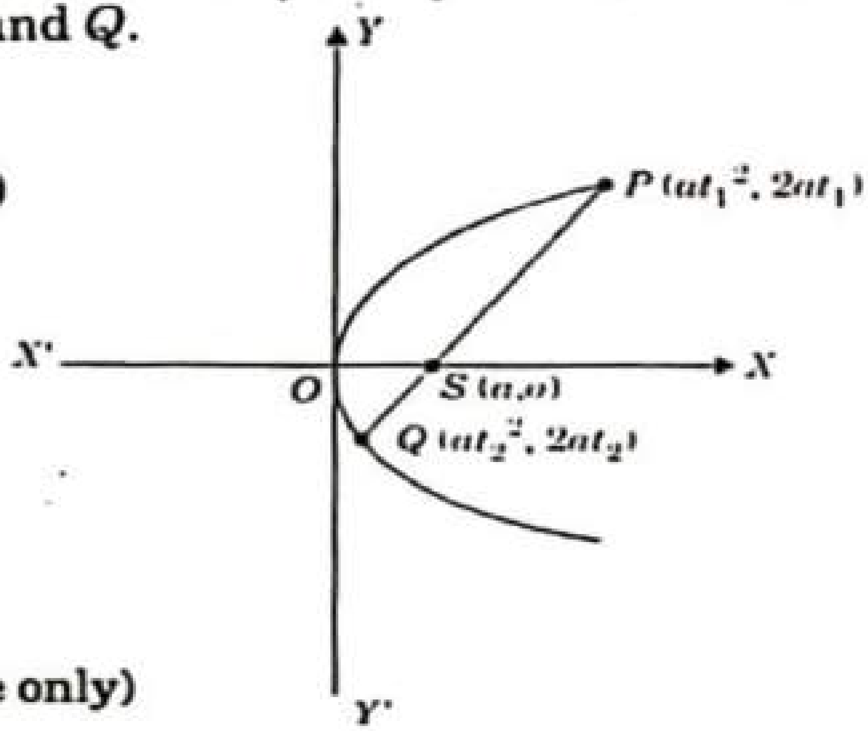
Exa. *If ρ_1, ρ_2 be the radii of curvature at the extremities of a focal chord of a parabola $y^2 = 4ax$, then prove that*

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}$$

Sol. The equation of parabola is given in cartesian form, but to solve this problem conveniently, we will choose point P as $(at_1^2, 2at_1)$ and point Q as $(at_2^2, 2at_2)$ *i.e.* parametric form of points P and Q.

We have from property of focal chord of parabola

$$t_1t_2 = -1 \qquad ...(1)$$



Now, $x = at^2$ and $y = 2at$

$\Rightarrow$ $x' = 2at$ and $y' = 2a$

$\Rightarrow$ $x'' = 2a$ and $y'' = 0$

Hence, $$\rho = \frac{(x'^2 + y'^2)^{3/2}}{(x'y'' - x''y')}$$

$$= \frac{(4a^2t^2 + 4a^2)^{3/2}}{(2at)(0) - (2a)(2a)}$$

$= 2a\,(1 + t^2)^{3/2}$ (Numerical value only)

Thus, at point $P\ (at_1^2, 2at_1)$, radius of curvature is given as

$$\rho_1 = 2a\,(1 + t_1^2)^{3/2}$$

and at point $Q\ (at_2^2, 2at_2)$, radius of curvature is given as

$$\rho_2 = 2a\,(1 + t_2^2)^{3/2}$$

So, $(\rho_1)^{-2/3} + (\rho_2)^{-2/3}$

$$= (2a)^{-2/3}\left[\frac{1}{(1+t_1^2)} + \frac{1}{(1+t_2^2)}\right]$$

$$= (2a)^{-2/3}\left[\frac{1}{(1+t_1^2)} + \frac{t_1^2}{(1+t_1^2)}\right] \quad \text{[using (1)]}$$

$$= (2a)^{-2/3}\left(\frac{1+t_1^2}{1+t_1^2}\right)$$

$$= (2a)^{-2/3}$$

EXERCISE

1. If ρ_1 and ρ_2 be the radii of curvature at the extremities of two conjugate diameters of an ellipse, then prove that
$(ab)^{2/3} (\rho_1^{2/3} + \rho_2^{2/3}) = (a^2 + b^2)$, where a and b are the semiaxes of the ellipse.

2. Prove that the radius of curvature at any point 't' on a cycloid
$x = a(t + \sin t), y = a(1 - \cos t)$ is $4a \cos \dfrac{t}{2}$

3. Prove that the radius of curvature at a point $(a \cos^3 \theta, a \sin^3 \theta)$ on an astroid
$x^{2/3} + y^{2/3} = a^{2/3}$ is $\dfrac{3a}{2} \sin 2\theta$

RADIUS OF CURVATURE AT ORIGIN

RADIUS OF CURVATURE AT ORIGIN

Step - 1 Equate the lowest degree terms of the given curve with zero to find all tangents of the curve at origin.

Step - 2 For any tangent except x - axis and y - axis, find the slope and assume that as p

Step - 3 Substitute $y = \left(px + \dfrac{qx^2}{\underline{|2}} + \dots \right)$ in the equation of the curve and by equating coefficient of like power of x, find the value of q

Step - 4 For finding value of radius of curvature concerned with that particular tangent at origin, use the formula

$$\boxed{\rho = \frac{(1 + p^2)^{3/2}}{q}}$$

[Maclaurine's expansion method]

Step - 5 If for the given curve, x - axis $(\text{line } y = 0)$ is obtained as tangent at origin, then for finding the concerned radius of curvature, use the formula

$$\boxed{(\rho)_{(0,0)} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2}{2y}\right)}$$

[Newton's Method]

Step - 6 If for the given curve, y - axis $(\text{line } x = 0)$ is obtained as tangent at origin, then for finding the concerned radius of curvature, use the formula

$$\boxed{(\rho)_{(0,0)} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y^2}{2x}\right)}$$

[Newton's Method]

Exa. *Find the radius of curvature at the origin for the curve* $y^2 - 3xy - 4x^2 + x^3 + x^4y + y^5 = 0$

Sol. $y^2 - 3xy - 4x^2 + x^3 + x^4y + y^5 = 0$ ...(1)

Equation of tangents at origin are obtained by equating to zero the lowest degree terms

i.e. $y^2 - 3xy - 4x^2 = 0$

$\Rightarrow$ $(y - 4x)(y + x) = 0$

$\Rightarrow$ $y = 4x, y = -x$

Now, radius of curvature (ρ_1) corresponding to tangent $y = 4x$ is given by

$$\rho_1 = \frac{(1 + p^2)^{3/2}}{q}$$

where p = slope of tangent = 4 and to find value of q, using maclaurine's expansion

$$y = px + \frac{qx^2}{2} + \ldots \text{ i.e. } y = 4x + \frac{qx^2}{2} + \ldots \text{ into (1), we have}$$

$$\left(4x + \frac{qx^2}{2} + \ldots\right)^2 - 3x\left(4x + \frac{qx^2}{2} + \ldots\right) - 4x^2 + x^3$$

$$+ x^4\left(4x + \frac{qx^2}{2} + \ldots\right) + \left(4x + \frac{qx^2}{2} + \ldots\right)^5 = 0$$

Thus, by comparing coefficients of x^3, we get

$$4q - \frac{3q}{2} + 1 = 0 \Rightarrow q = -\frac{2}{5}$$

$$\Rightarrow \qquad \rho_1 = \frac{[1 + (4)^2]^{3/2}}{-\frac{2}{5}} = \frac{85\sqrt{17}}{2} \text{ (Numerical value only)}$$

and radius of curvature (ρ_2) corresponding to tangent $y = -x$ is given by

$$\rho_2 = \frac{(1+p^2)^{3/2}}{q}$$

where p = slope of the tangent $= -1$

and to find value of q, using maclaurine's expansion

$$y = px + \frac{qx^2}{2} + \dots \text{ i.e. } y = -x + \frac{q}{2}x^2 + \dots \text{ into (1), we have}$$

$$\left(-x + \frac{qx^2}{2} + \dots\right)^2 - 3x\left(-x + \frac{qx^2}{2} + \dots\right) - 4x^2 + x^3$$

$$+ x^4\left(-x + \frac{qx^2}{2} + \dots\right) + \left(-x + \frac{qx^2}{2} + \dots\right)^5 = 0$$

Then, by comparing coefficients of x^3, we get

$$-q - \frac{3q}{2} + 1 = 0$$

$$\Rightarrow \qquad q = \frac{2}{5}$$

$$\Rightarrow \qquad \rho_2 = \frac{[1+(-1)^2]^{3/2}}{\frac{2}{5}} = 5\sqrt{2}$$

Exa. *Show that the radii of curvature at the origin for the curve $x^3 + y^3 = 3axy$ are each equal to $\frac{3a}{2}$*

Sol. $x^3 + y^3 = 3axy$ ...(1)

Equation of tangents at origin are obtained by equating to zero the lowest degree terms *i.e.*

$$xy = 0 \quad \Rightarrow \quad x = 0, \; y = 0$$

Now, radius of curvature (ρ_1) corresponding to tangent $x = 0$ (y – axis) is given by Newton's method as

$$\rho_1 = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}$$

dividing (1) by $2xy$, we get

$$\frac{x^2}{2y} + \frac{y^2}{2x} = \frac{3a}{2}$$

$$\Rightarrow \quad \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2}{2y}\right) + \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y^2}{2x}\right) = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{3a}{2}\right)$$

$$\Rightarrow \quad 0 + \rho_1 = \frac{3a}{2} \quad \Rightarrow \quad \rho_1 = \frac{3a}{2} \qquad \left[\because \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{4}\left(\frac{1}{\frac{y^2}{2x}}\right) = 0\right]$$

Also, radius of curvature (ρ_2) corresponding to tangent $y = 0$ (x-axis) is given by Newton's method as

$$\rho_2 = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}$$

So, dividing equation (1) by $2xy$ and by applying $\lim\limits_{\substack{x \to 0 \\ y \to 0}}$, we get

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} + \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{3a}{2}$$

$$\Rightarrow \qquad \rho_2 + 0 = \frac{3a}{2} \qquad \left[\because \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{4} \left(\frac{1}{\frac{x^2}{2y}} \right) = 0 \right]$$

$$\Rightarrow \qquad \rho_2 = \frac{3a}{2}$$

EXERCISE

Find the radius of curvature of the following curves at origin :

1. $y = x^4 - 4x^3 - 18x^2$
2. $y^2 = x^2\left(\dfrac{a+x}{a-x}\right)$
3. $3x^3 + y^3 + 5y^2 + 3x^2y + 2x = 0$
4. $a(y^2 - x^2) = x^3$

ANSWERS

1. $\dfrac{1}{36}$
2. $\pm a\sqrt{2}$
3. $\dfrac{1}{5}$
4. $\pm 2a\sqrt{2}$

CENTRE AND CIRCLE OF CURVATURE

CENTRE AND CIRCLE OF CURVATURE

For a curve $f(x, y) = c$, centre of curvature $(\bar{x}, \bar{y})$ is given as:

$$\boxed{\bar{x} = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}}$$

and

$$\boxed{\bar{y} = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}}$$

Using $(\bar{x}, \bar{y})$ as centre of curvature and ρ as radius of curvature, the equation of the circle of curvature is given as:

$$\boxed{(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2}$$

Exa. *If the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at one end of minor axis lies at the other end, then prove that eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$*

Sol.
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(1)}$$

Here, we are given that the centre of curvature of equation (1) at the point $(0, b)$ lies at the point $(0, -b)$. Differentiating (1) wrt x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)=\frac{-b^2}{a^2y^2}\left(y+\frac{b^2x^2}{a^2y}\right)$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(0,b)}=0 \text{ and } \left(\frac{d^2y}{dx^2}\right)_{(0,b)}=-\frac{b}{a^2}$$

Hence, $$(\rho)_{(0,b)} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2_{(0,b)}\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)_{(0,b)}} = \frac{a^2}{b} \text{ (Numerical value only)} \quad ...(2)$$

Now, at point $(0, b)$, centre of curvature lies at point $(0, -b)$

$\Rightarrow$ Radius of curvature at point$(0, b)$ is distance between points $(0, b)$ and $(0, -b)$ *i.e.* $(\rho)_{(0,b)} = 2b$

Thus, $$\frac{a^2}{b} = 2b \quad \Rightarrow \quad a^2 = 2b^2 \quad ...(3)$$

Now, eccentricity (e) of the ellipse (1) is given by

$$\frac{b^2}{a^2} = 1 - e^2 \quad \Rightarrow \quad \frac{1}{2} = 1 - e^2$$

$$\Rightarrow \quad e^2 = \frac{1}{2} \quad \Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

Exa. *Find the equation of circle of curvature of the parabola $y^2 = 4ax$ at the point $(am^2, 2am)$*

Sol. $$y^2 = 4ax \quad \text{...(1)}$$

Differentiating (1) wrt x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{4a^2}{y^3}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(am^2,\,2am)} = \frac{2a}{2am} = \frac{1}{m}$$

and $$\left(\frac{d^2y}{dx^2}\right)_{(am^2,\,2am)} = -\frac{4a^2}{8a^3m^3} = -\frac{1}{2am^3}$$

If centre of curvature is given as point $(\bar{x}, \bar{y})$ at point $(am^2, 2am)$, then

$$\bar{x} = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \Rightarrow \bar{x} = am^2 - \frac{\frac{1}{m}\left(1+\frac{1}{m^2}\right)}{-\frac{1}{2am^3}}$$

$$\Rightarrow \bar{x} = am^2 + 2a(m^2+1) \Rightarrow \bar{x} = 3am^2 + 2a$$

and $$\bar{y} = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 2am + \frac{\left(1+\frac{1}{m^2}\right)}{-\frac{1}{2am^3}}$$

$$\Rightarrow \quad \bar{y} = 2am - 2am\,(m^2+1) = -2am^3$$

Thus, $(3am^2+2a, -2am^3)$ is centre of curvature for given equation at point $(am^2, 2am)$

Also, $$(\rho)_{(am^2, 2am)} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\frac{1}{m^2}\right)^{3/2}}{-\frac{1}{2am^3}} = 2a\,(m^2+1)^{3/2} \text{ (Numerical value only)}$$

Hence, circle of curvature for equation (1) at point $(am^2, 2am)$ is given by

$$(x-3am^2-2a)^2 + (y+2am^3)^2 = 4a^2(m^2+1)^3$$

$$\Rightarrow \quad x^2+y^2-6am^2x-4ax+4am^3y-3a^2m^4 = 0$$

EXERCISE

Find the co-ordinates of centre of curvature at indicated points on each of the following curves :

1. $y = x^4 - x^2$ at point $(0, 0)$
2. $x^3 + y^3 = 3axy$ at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Find the circle of curvature at indicated points on each of the following curves :

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $(0, b)$
4. $y = mx + x^2$ at point $(0, 0)$

ANSWERS

1. $\left(0, -\frac{1}{2}\right)$

2. $\left(\frac{21a}{16}, \frac{21a}{16}\right)$

3. $x^2 + y^2 - \frac{2(b^2 - a^2)}{b}\, y + (b^2 - 2a^2) = 0$

4. $x^2 + y^2 = (1 + m^2)(y - mx)$

CHORDS OF CURVATURE

CHORDS OF CURVATURE

Length of Chord of Curvature Parallel to x-axis

$$= 2\,\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]\Bigg/\frac{d^2y}{dx^2}$$

Length of Chord of Curvature Parallel to y-axis

$$= \frac{2\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Exa. *For the curve $y = a \log \sec\left(\frac{x}{a}\right)$, prove that the chord of curvature parallel to y-axis is of constant length.*

Sol.

$$y = a \log \sec\left(\frac{x}{a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan\left(\frac{x}{a}\right) \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\left(\frac{x}{a}\right)$$

Hence, chord of curvature parallel to y-axis

$$= \frac{2\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = \frac{2\left[1+\tan^2\left(\frac{x}{a}\right)\right]}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)}$$

$= 2a$, which is constant

Thus, *the chord of curvature parallel to y-axis is of constant length.*

EXERCISE

1. *For catenary* $y = a\cosh\left(\frac{x}{a}\right)$, prove that the chord of curvature parallel to the axis of x is of length $a\sinh\left(\frac{2x}{a}\right)$ and the chord of curvature parallel to the axis of y is double the ordinate.

2. If C_x and C_y be the chords of curvature parallel to the co-ordinate axes at any point on the curve $y = ae^{x/a}$, then prove that

$$\frac{1}{C_x^{\,2}} + \frac{1}{C_y^{\,2}} = \frac{1}{2aC_x}$$

THANKS

ENGINEERING MATHEMATICS - I
(I SEM)

CONCAVITY, CONVEXITY AND POINT OF INFLEXION

By Dr. Anil Maheshwari

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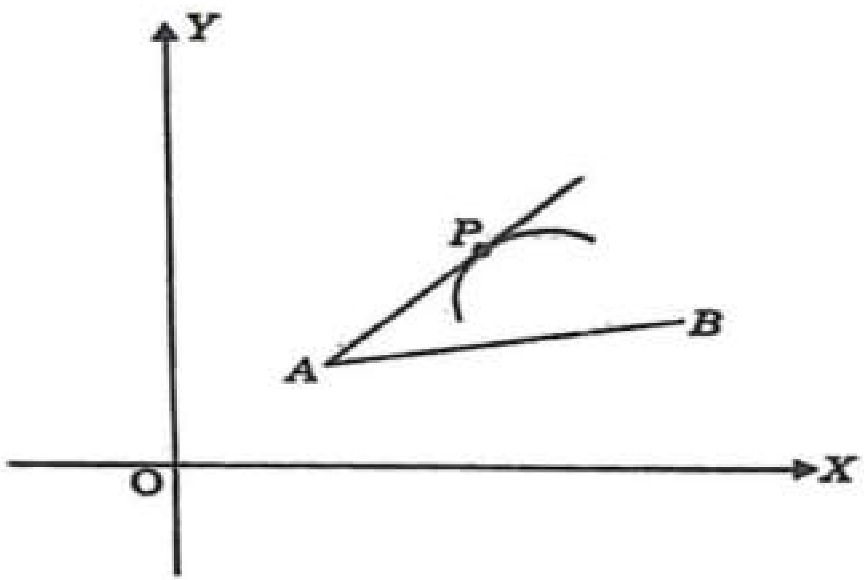
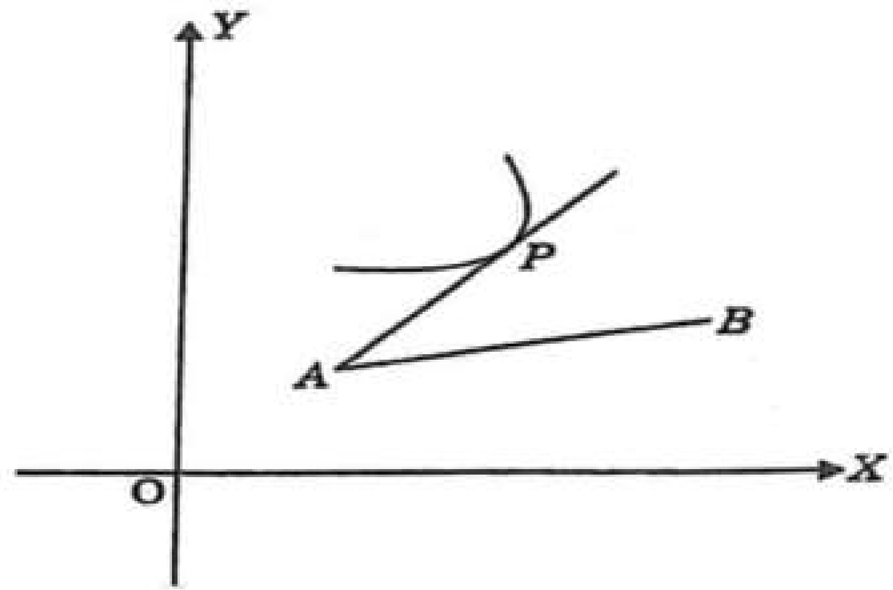
CONCAVITY AND CONVEXITY

CONCAVITY AND CONVEXITY OF A CURVE WITH RESPECT TO A GIVEN LINE

Let P be a given point on a curve and let AB be a given fixed straight line not passing through P. The curve is said to be concave or convex at P with respect to the line AB according as a sufficiently small arc containing P lies entirely within or without the acute angle formed by the tangent at P and the given line AB.



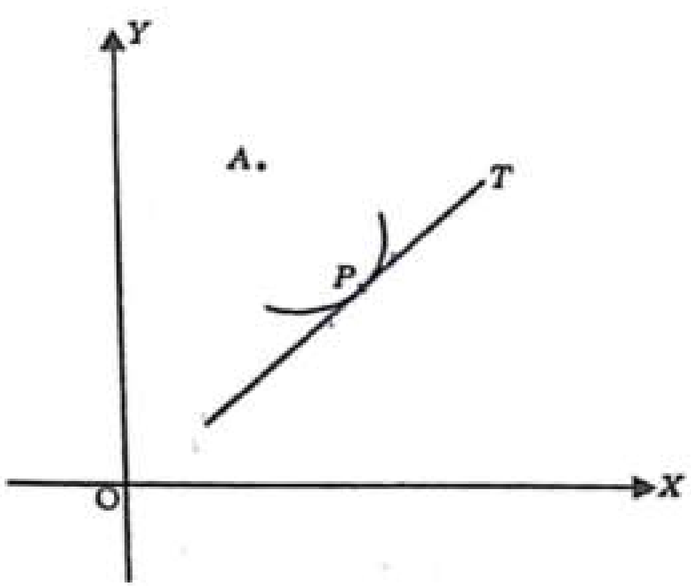
(a) Curve at P is concave w.r.t. AB



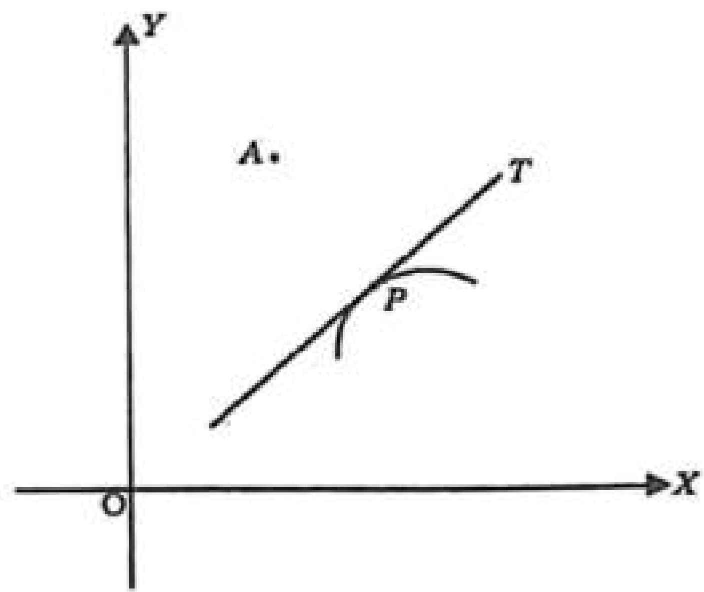
(b) Curve at P is convex w.r.t. AB

CONCAVITY AND CONVEXITY OF A CURVE WITH RESPECT TO A POINT

Let P be a given point on a curve. The curve is said to be concave or convex at P with respect to a given fixed point A, according as the curve in the immediate neighbourhood of P and the point A lie on the same side or opposite sides of the tangent PT to the curve at P.



(a) Curve at P is concave w.r.t A.



(b) Curve at P is convex w.r.t. A.

A TEST FOR CONCAVITY AND CONVEXITY

1. A curve is convex or concave at point P wrt x-axis according as $y\dfrac{d^2y}{dx^2}$ is positive or negative at point P.

2. A curve is convex or concave at point P wrt y-axis according as $x\dfrac{d^2x}{dy^2}$ is positive or negative at point P.

3. A curve is convex or concave at point P wrt the foot of the ordinate at point P according as $y\dfrac{d^2y}{dx^2}$ is positive or negative at point P.

4. A curve is convex in interval $[a, b]$, if $\dfrac{d^2y}{dx^2} > 0,\ \forall\ x \in [a, b]$ and concave in interval $[a, b]$, if $\dfrac{d^2y}{dx^2} < 0\ \forall\ x \in [a, b]$.

Exa. *Show that the curve $y = e^x$ is convex at every point with respect to the foot of the corresponding ordinate*

Sol. $y = e^x \quad \Rightarrow \quad \frac{dy}{dx} = e^x$ and $\frac{d^2y}{dx^2} = e^x$

Now, for a curve to be convex at a point with respect to the foot of the corresponding ordinate $y \frac{d^2y}{dx^2}$ should be positive at that point.

Here, $\quad y\frac{d^2y}{dx^2} = e^x \cdot e^x = e^{2x} > 0$ for all real values of x

Thus, given curve $y = e^x$ is convex at every point with respect to the foot of the corresponding ordinate.

Exa. *Find the intervals in which the curve* $y = \dfrac{2x}{(x+3)^2}$ *is concave or convex*

Sol.

$$y = \frac{2x}{(x+3)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(x+3)^2 - 2x[2(x+3)]}{(x+3)^4}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(3-x)}{(x+3)^3}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{(x+3)^6}\left[-(x+3)^3 - 3(3-x)(x+3)^2\right] = \frac{4(x-6)}{(x+3)^4}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} > 0 \ \forall \ x \in (6, \infty) \text{ and } \frac{d^2y}{dx^2} < 0 \ \forall \ x \in (-\infty, 6) - \{-3\}$$

Hence, given curve is convex in $(6, \infty)$ and concave in $(-\infty, 6) - \{-3\}$

EXERCISE

1. Show that the curve $y^2 = 4x$ is concave wrt x-axis and convex wrt y-axis at the point (1, 2)
2. Show that the curve $y = \log x$ is concave everywhere wrt the foot of the corresponding ordinate.
3. Find the intervals in which the curve $y = (\cos x + \sin x)e^x$ is convex or concave.

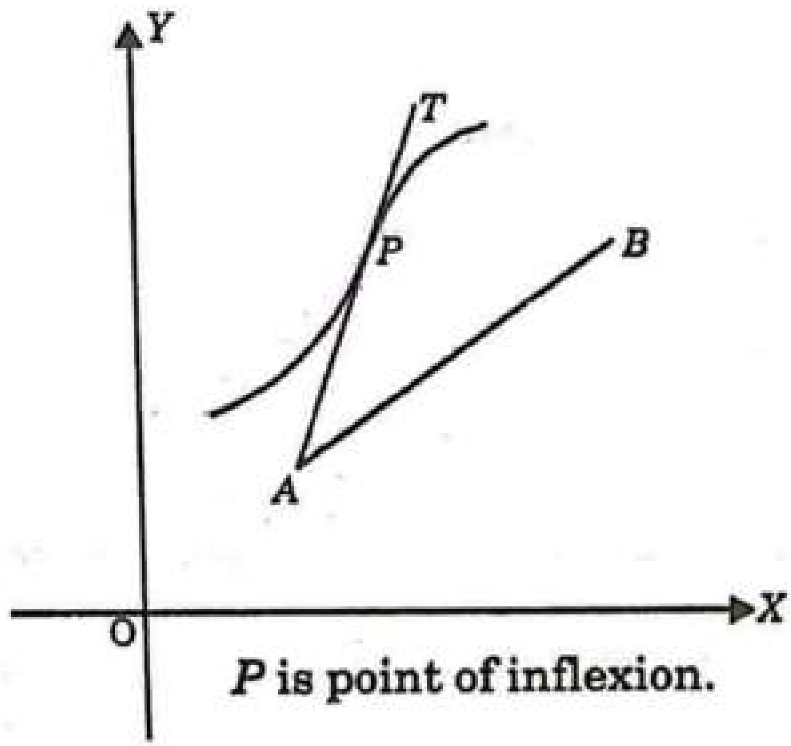
ANSWERS

3. Convex in $\left(0, \frac{\pi}{4}\right)$ and concave in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

POINT OF INFLEXION

POINT OF INFLEXION

A point on a curve, at which the curve changes from cancavity to convexity or from convexity to concavity, is called a point of inflexion *e.g.* for the following curve, point P is that point.



P is point of inflexion.

A TEST FOR POINT OF INFLEXION

For any curve $f(x, y) = 0$, all points where we obtain

$$\frac{d^2y}{dx^2} = 0 \text{ but } \frac{d^3y}{dx^3} \neq 0,$$

are called points of inflexion.

Exa. *Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the straight line $3x + a = 4b$*

Sol.

$$y^2 = (x-a)^2(x-b) \Rightarrow y = (x-a)\sqrt{(x-b)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{(x-b)} + \frac{(x-a)}{2\sqrt{(x-b)}} = \frac{(3x-2b-a)}{2\sqrt{(x-b)}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{2(x-b)}\left[3\sqrt{(x-b)} - \frac{(3x-2b-a)}{2\sqrt{(x-b)}}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(3x+a-4b)}{4(x-b)^{3/2}}$$

$$\Rightarrow \qquad \frac{d^3y}{dx^3} = \frac{1}{4(x-b)^3}\left[3(x-b)^{3/2} - (3x+a-4b)\cdot\frac{3}{2}(x-b)^{1/2}\right]$$

Now, $$\frac{d^2y}{dx^2} = 0 \Rightarrow 3x + a - 4b = 0$$

$$\Rightarrow \qquad x = \left(\frac{4b-a}{3}\right)$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x=\frac{4b-a}{3}} = \left[\frac{1}{4}\cdot\frac{3}{(x-b)^{3/2}}\right]_{x=\frac{4b-a}{3}} \neq 0$$

Hence, for the given curve, points of inflexion lie on the straight line

$3x + a = 4b$

EXERCISE

1. Prove that the curve $x^3 + y^3 = a^3$ has points of inflexion where it crosses the co-ordinate axes.

2. Prove that every point at which the curve $y = c \sin\left(\frac{x}{a}\right)$ meets the axis of x is a point of inflexion.

3. Prove that the locus of all points of inflexion of the curve $y(a^2 + x^2) = 2x^2$ for varying values of a is the straight line $2y = 1$

THANKS

ENGINEERING MATHEMATICS - I (I SEM)

CURVE TRACING

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CURVE TRACING IN CARTESIAN COORDINATE SYSTEM

CURVE TRACING IN CARTESIAN COORDINATE SYSTEM

INTRODUCTION

The main objective to study curve tracing is to find the approximate shape of a curve with the knowledge of asymptotes, tangents, points of inflexion and multiple points etc. without plotting a large number of points.

IMPORTANT DEFINITIONS AND CONCEPTS INVOLVED IN CURVE TRACING

(*a*) **Singular Point.** A point on a curve at which the curve shows an extra-ordinary or abnormal behaviour is called singular point. Point of inflexion, maxima, minima and multiple points are the examples of singular points.

(*b*) **Multiple Point.** A point on a curve 'through which more than one branch(es) of the curve pass' is called a multiple point.

(*c*) **Double Point.** A point on a curve is called a double point if two branches of the curve pass through it.

(*d*) **Triple Point.** A point on a curve is called a triple point if three branches of the curve pass through it.

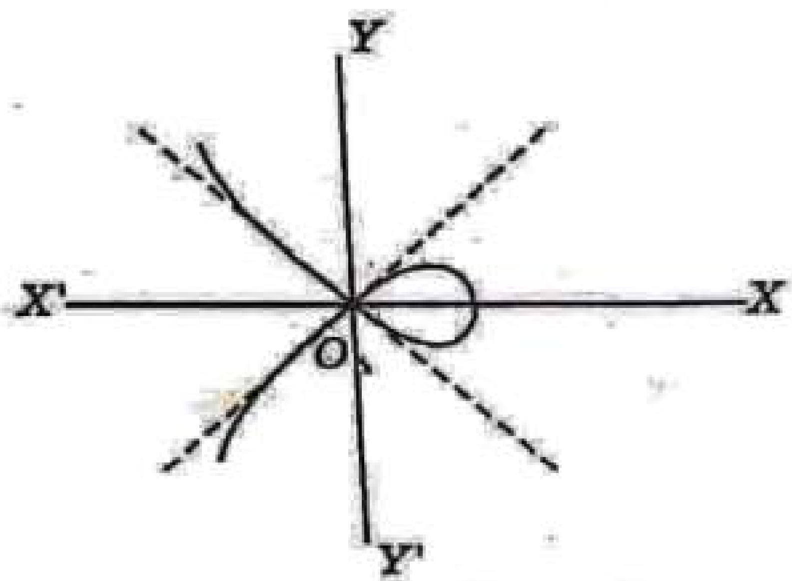
(*e*) **Tangents at the Origin/other Point.** To investigate the nature of a multiple point, the equation(s) of the tangent(s) at the point is/are required.

(f) Classification of the Double Point. There are three categories of double points.

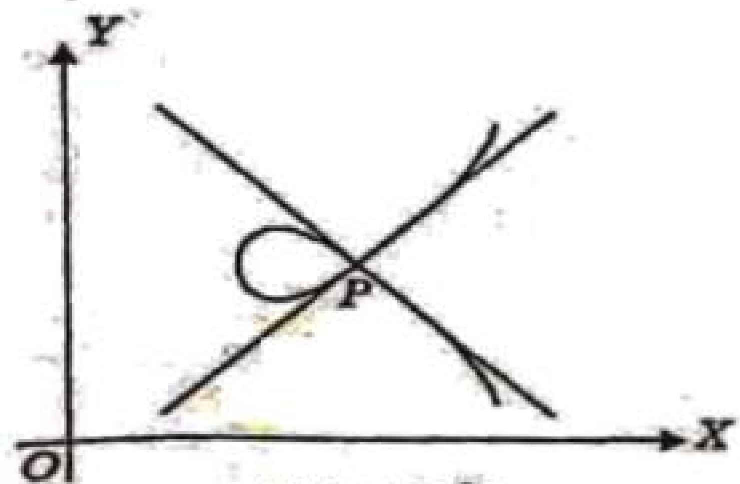
(i) Node. If the two branches of the curve passing through a point P are real and the tangents to them are also real and different, then P is called a node.

e.g. for $y^2(a+x) = x^2(3a-x)$,

$y = \pm x\sqrt{3}$ are real and different tangents at the origin

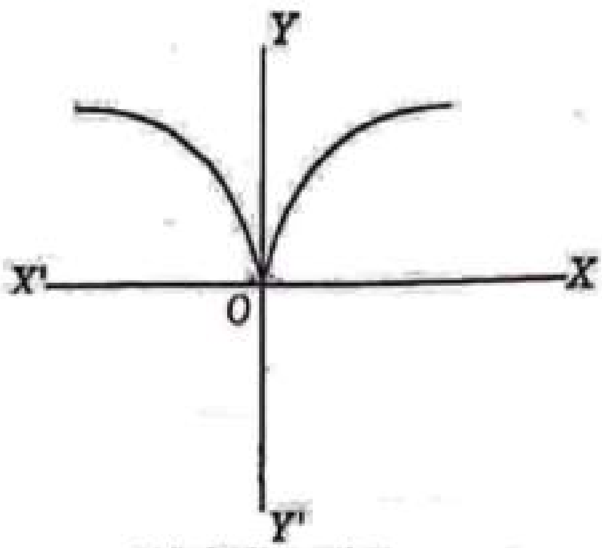


Origin is a node

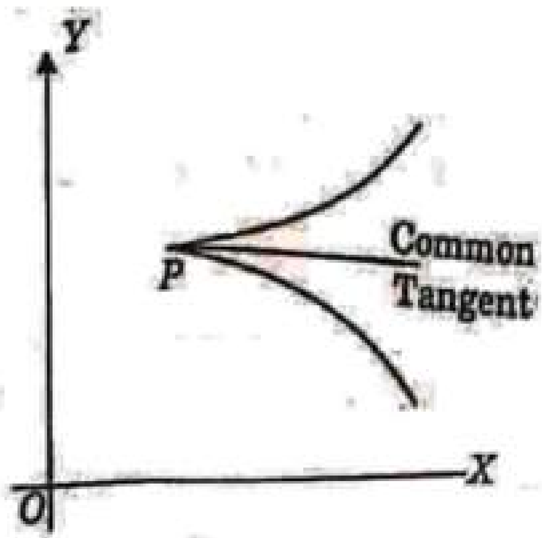


P is a node

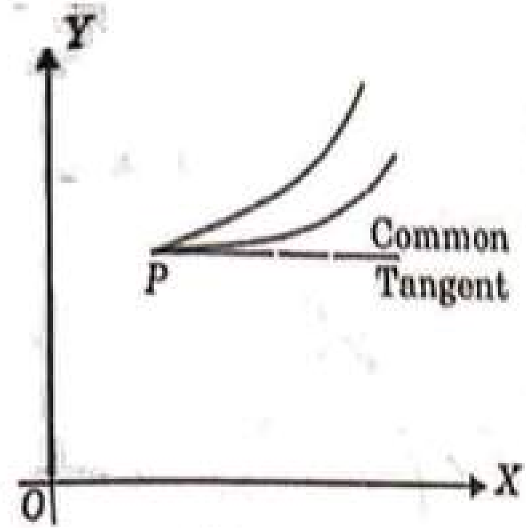
(*ii*) **Cusp.** If the two branches of the curve passing through a point P are real and the tangents to them are also real and coincident, then P is called a cusp e.g. for $x^2 = ay^3$, $x = 0$, $x = 0$ are real and coincident tangents at the origin.



Origin is a cusp

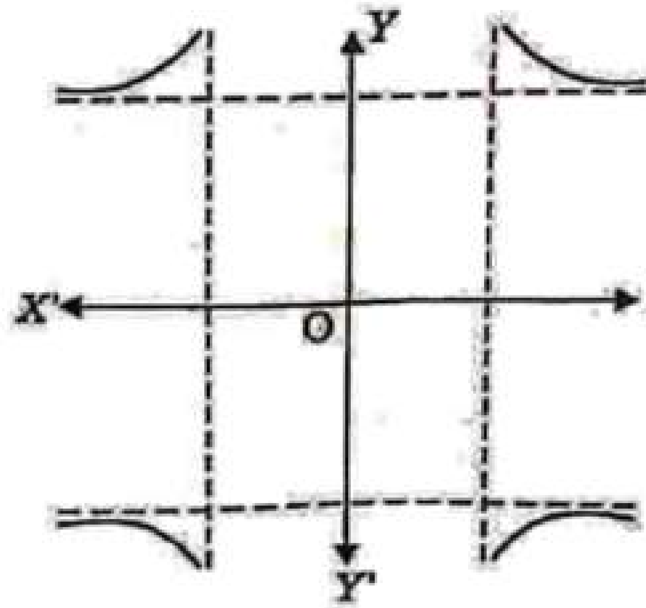


P is a cusp



P is a cusp

(*iii*) **Conjugate or Isolated Point.** If the two branches of a curve passing through a point P be imaginary *i.e.* if there are no real points of the curve in the neighbourhood of P, then P is called a conjugate or an isolated point of the curve. The co-ordinates of this point satisfy the equation of the curve and generally at such point there exist two imaginary tangents.



Origin is an isolated point

e.g. for $(x^2 - a^2)(y^2 - b^2) = a^2b^2$,

$y = \pm \dfrac{ibx}{a}$ are two imaginary tangents at the origin.

PROCEDURE FOR TRACING CARTESIAN CURVES

Apply following procedure for tracing of a curve whose equation is given in the cartesian form $f(x, y) = 0$

1. Symmetry. The following rules help in determining the symmetry of the curve :

(*i*) If $f(x, y) = f(x, -y)$ *i.e.* if the equation of the curve contains even powers of y only, then the curve is symmetrical about x-axis. *e.g.* $y^2 = 4ax$

(*ii*) If $f(x, y) = f(-x, y)$ *i.e.* if the equation of the curve contains even powers of x only, then the curve is symmetrical about y-axis. *e.g.*

$$x^2(y - a) = y^2(y + a)$$

(*iii*) If $f(x, y) = f(-x, -y)$ *i.e.* if the equation of the curve contains even powers of both x and y, then the curve is symmetrical about x and y-axis.

e.g. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(iv) If $f(x, y) = f(-x, -y)$ *i.e.* if x and y are replaced by $-x$ and $-y$ respectively and the equation of the curve remains unaltered without fulfilling the condition of even powers of x and y, then the curve is symmetrical in opposite quadrants *e.g.* $y = x(x^2 - 1)$

(v) If $f(x, y) = f(y, x)$ *i.e.* if x and y are replaced by y and x respectively and the equation of the curve remains unaltered, then the curve is symmetrical about the line $y = x$, *e.g.* $x^3 + y^3 = 3axy$

(vi) If $f(x, y) = f(-y, -x)$ *i.e.* if x and y are replaced by $-y$ and $-x$ respectively and equation of the curve remains unaltered, then the curve is symmetrical about the line $y = -x$, *e.g.* $x^3 - y^3 = 3axy$

2. Nature of Origin. Now, we have to find out whether the curve passes through the origin or not. If the equation of the curve does not contain any constant term, then the curve clearly passes through the origin or if on putting $x = 0$ in the equation, y also becomes zero, then the curve passes through the origin. *e.g.* $y^2 = 4ax$

If the curve passes through the origin, then find the equation(s) of the tangent(s) of the curve at the origin by equating to zero the lowest degree terms in the equation of the curve. In case origin is a double point, find whether it is a node, a cusp or a conjugate point.

3. Intersection with the Co-ordinate Axes. To find the points on co-ordinate axes, we first put $x = 0$ and calculate the values of y, then put $y = 0$ and calculate the values of x. After finding the points of intersection with co-ordinate axes, we find the tangents at those points by shifting the origin to those points. For example, if the origin is shifted to a point $(a, 0)$, then the given curve is transformed from $f(x, y) = 0$ to $f(x + a, y) = 0$

4. Asymptotes. Determine all the real asymptotes of the curve *i.e.*

(*i*) Asymptotes parallel to the co-ordinate axes.

(*ii*) Oblique asymptotes with the help of the methods already discussed.

5. Region of Existence. Determine the region in which the curve lies. To do this, solve the given equation of the curve for y (or x) and determine the values of x (or y) for which the values of y (or x) are imaginary. If y is seen to be imaginary in $a < x < b$, the curve does not exist in the region bounded by the lines $x = a$ and $x = b$. Above process can be simplified as :

(*i*) If any of x (or y) is given as pure quadratic variable, then solve given equation for x (or y) in terms of y (or x) by virtue of a square root to decide the certain region *e.g.* In $y^2(a - x) = x^2(a + x)$, y is a pure quadratic variable.

Thus, $$y = \pm x\sqrt{\left(\frac{a+x}{a-x}\right)}$$

$\Rightarrow$ $$x \in [-a, a)$$

So, curve exist only if $-a \leq x < a$

(*ii*) If neither x nor y is given as pure quadratic variable, then check the existence of the curve in all four quadrants one by one using sign combinations *i.e.* in first quadrant (+, +), in second quadrant (–, +), in third quadrant (–, –), in fourth quadrant (+, –) *e.g.* curve $x^3 + y^3 = 3axy$ doesn't exist in third quadrant because in third quadrant, $x^3 + y^3$ is negative and $3axy$ is positive. (Here, we consider 'a' as a positive constant).

6. Points of Maxima and Minima.

To find these points, use equation of curve to obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. By substituting $\frac{dy}{dx}=0$, we obtain points as probable maxima or minima and then by evaluating $\frac{d^2y}{dx^2}$ at these points, we can state as

$\frac{d^2y}{dx^2}>0 \Rightarrow$ point is maxima, $\frac{d^2y}{dx^2}<0 \Rightarrow$ point is minima,

$\frac{d^2y}{dx^2}=0 \Rightarrow$ calculate higher order derivatives at that point

and if $$\frac{d^3y}{dx^3}=\frac{d^4y}{dx^4}=\ldots\ldots=\frac{d^{n-1}y}{dx^{n-1}}=0\,;\ \frac{d^ny}{dx^n}\neq 0,$$

then there exist a maxima $\left(\text{if } n \text{ is even and } \frac{d^ny}{dx^n}<0\right)$

and minima $\left(\text{if } n \text{ is even and } \frac{d^ny}{dx^n}>0\right)$

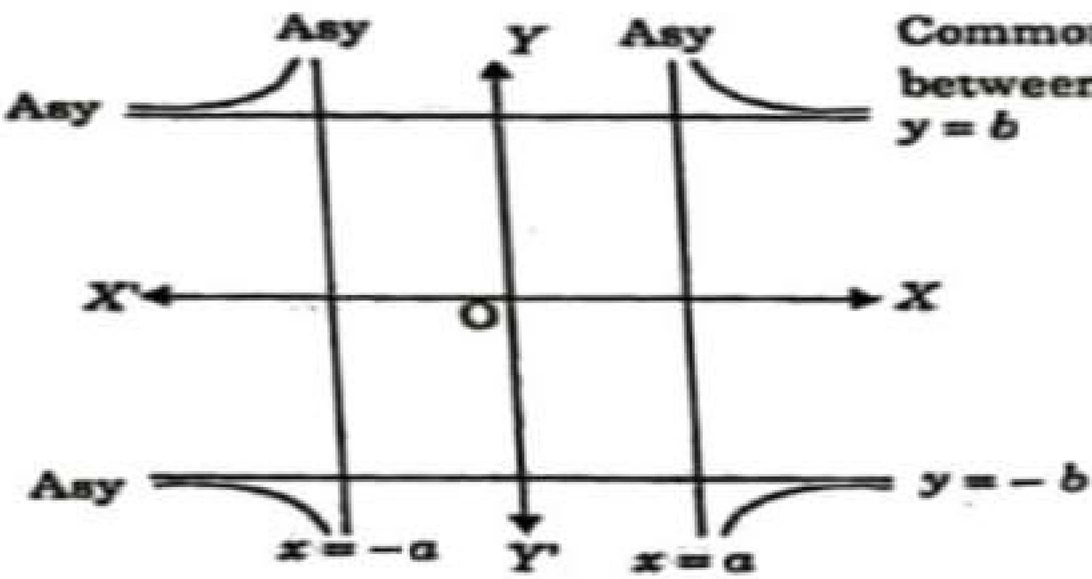
and there does not exist any maxima or minima (if n is odd).

7. Plotting of Points. For the correct shape of the curve, we can plot some points on the curve. Notice also how y varies as x varies continuously from $-\infty$ to $+\infty$.

Remarks : (1) For tracing a curve, we have to draw asymptotes, points of intersection of curve with the co-ordinate axes and tangents and at last apply symmetricity of the curve with the region of existence.

(2) For any curve if two asymptotes intersect, then there exist a common infinite branch of the curve *e.g.* For the curve $(x^2 - a^2)(y^2 - b^2) = a^2b^2$,

$x = \pm a$ and $y = \pm b$ are the asymptotes and its tracing is given as



(3) For any curve, if origin or any other point exist as a node, then there exists a loop at that node in absence of asymptote.

Exa. ***Trace the following curve:***

$$y^2(a+x) = x^2(a-x)$$

Sol. $$y^2(a+x) = x^2(a-x) \qquad \ldots(1)$$

(1) Symmetry. The curve is symmetrical about x-axis as all powers of y are even.

(2) Nature of Origin. The curve passes through the origin and the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the curve *i.e.* $(y^2 - x^2)\,a = 0 \Rightarrow y = \pm x$ which are two real and different tangents at the origin. Hence origin is a node.

(3) Points of Intersection of Curve with Co-ordinate Axes. On putting $x = 0$ into eqn. (1), we get $y = 0$

and on putting $y = 0$ into eqn. (1), we get $x = 0, a$

Hence, given curve cuts the co-ordinate axes at $(0, 0)$ and $(a, 0)$

To find tangent at the point $(a, 0)$, we have to shift origin to point $(a, 0)$ *i.e.* on replacing x by $x + a$ in equation (1), we get

$$y^2 (2a + x) = (x + a)^2 (-x)$$

and equating to zero the lowest degree terms, we get $x = 0$ as tangent *i.e.* a vertical line as a tangent at point $(a, 0)$

(4) Asymptotes. As coefficient of highest power of x is constant, there doesn't exist any asymptote parallel to x-axis and coefficient of highest power of y is $(a + x)$, so $x = -a$ is an asymptote parallel to y-axis and to find oblique asymptotes, $\phi_3(m) = m^2 + 1 = 0 \Rightarrow m = \pm i$.

So, there doesn't exist any oblique real asymptote.

(5) Region of Existence

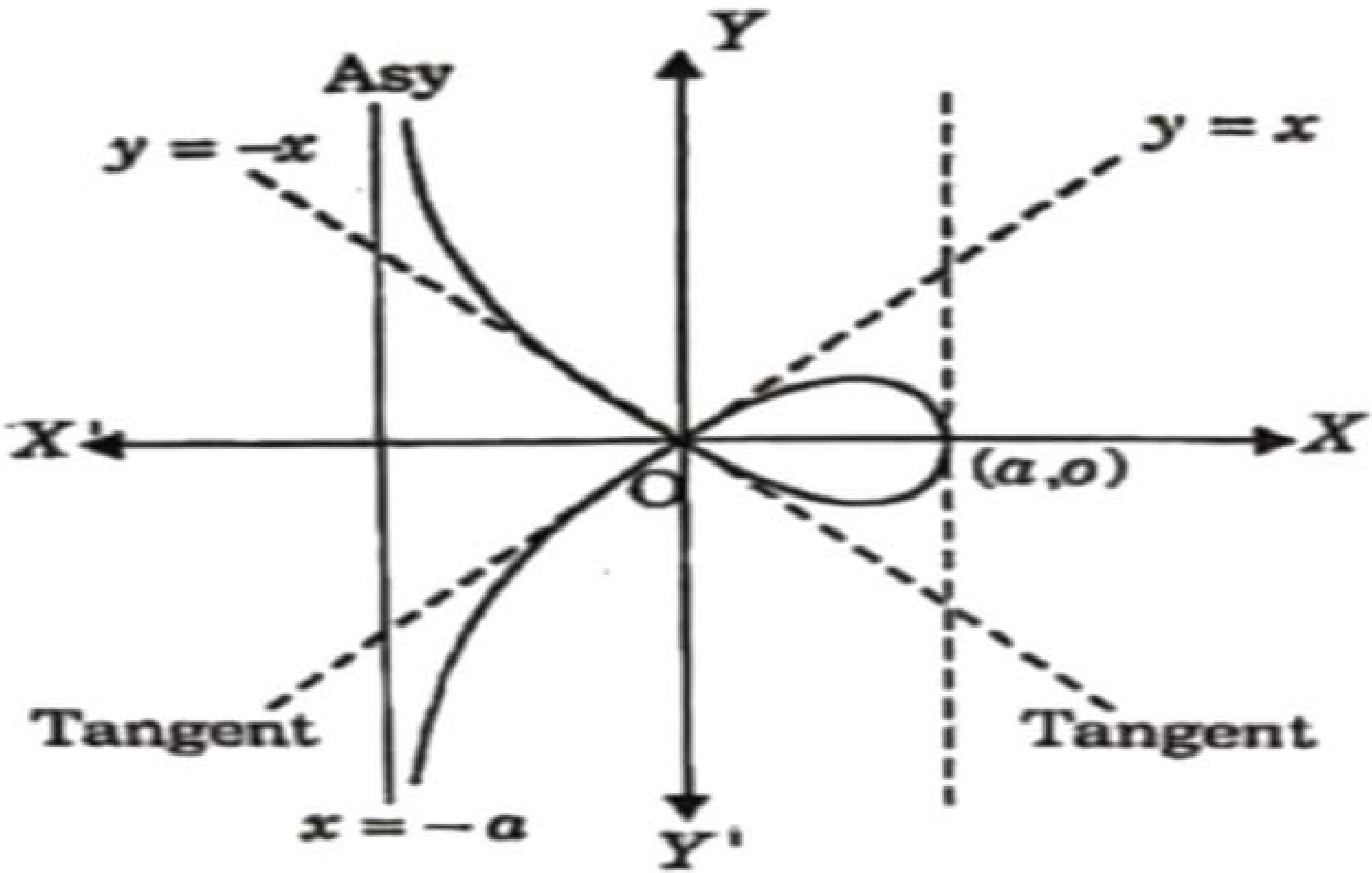
$$y^2 (a + x) = x^2 (a - x)$$

$$\Rightarrow \qquad y = \pm x \sqrt{\left(\frac{a - x}{a + x}\right)}$$

$$\Rightarrow \qquad \left(\frac{a - x}{a + x}\right) \geq 0 \quad \Rightarrow \quad x \in (-a, a]$$

Thus, curve lies in the region where $x \in (-a, a]$

Using above facts, the shape of the curve is given as


Y
Asy
y = −x
y = x
X'
X
O
(a,o)
Tangent
Tangent
x = −a
Y'

Exa. ***Trace the following curve:***

$$(x^2 - a^2)(y^2 - b^2) = a^2b^2$$

Sol. $$(x^2 - a^2)(y^2 - b^2) = a^2b^2 \quad ...(1)$$

(1) Symmetry. The curve is symmetrical about x-axis and y-axis as all the powers of x and y are even.

(2) Nature of Origin. The curve passes through origin and the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the curve

i.e. $$-a^2y^2 - b^2x^2 = 0$$

$\Rightarrow \quad y = \pm \dfrac{ibx}{a}$ which are two imaginary tangents at the origin. Hence, origin is a conjugate point

(3) Points of Intersection of Curve with Co-ordinate Axes

On putting $x = 0$ into eqn. (1), we get $y = 0$

and on putting $y = 0$ into eqn. (1), we get $x = 0$

Hence, only origin exist as point of intersection of curve with the co-ordinate axes.

(4) Asymptotes. As. coefficients of highest powers of x and y are $(y^2 - b^2)$ and $(x^2 - a^2)$ respectively.

So, asymptotes parallel to x-axis are

$$y^2 - b^2 = 0$$

$$\Rightarrow \quad y = \pm b$$

and asymptotes parallel to y-axis are

$$x^2 - a^2 = 0$$

$$\Rightarrow \quad x = \pm a$$

(5) Region of Existence

$$(x^2 - a^2)(y^2 - b^2) = a^2b^2$$

$$\Rightarrow \quad x^2y^2 - a^2y^2 - b^2x^2 = 0$$

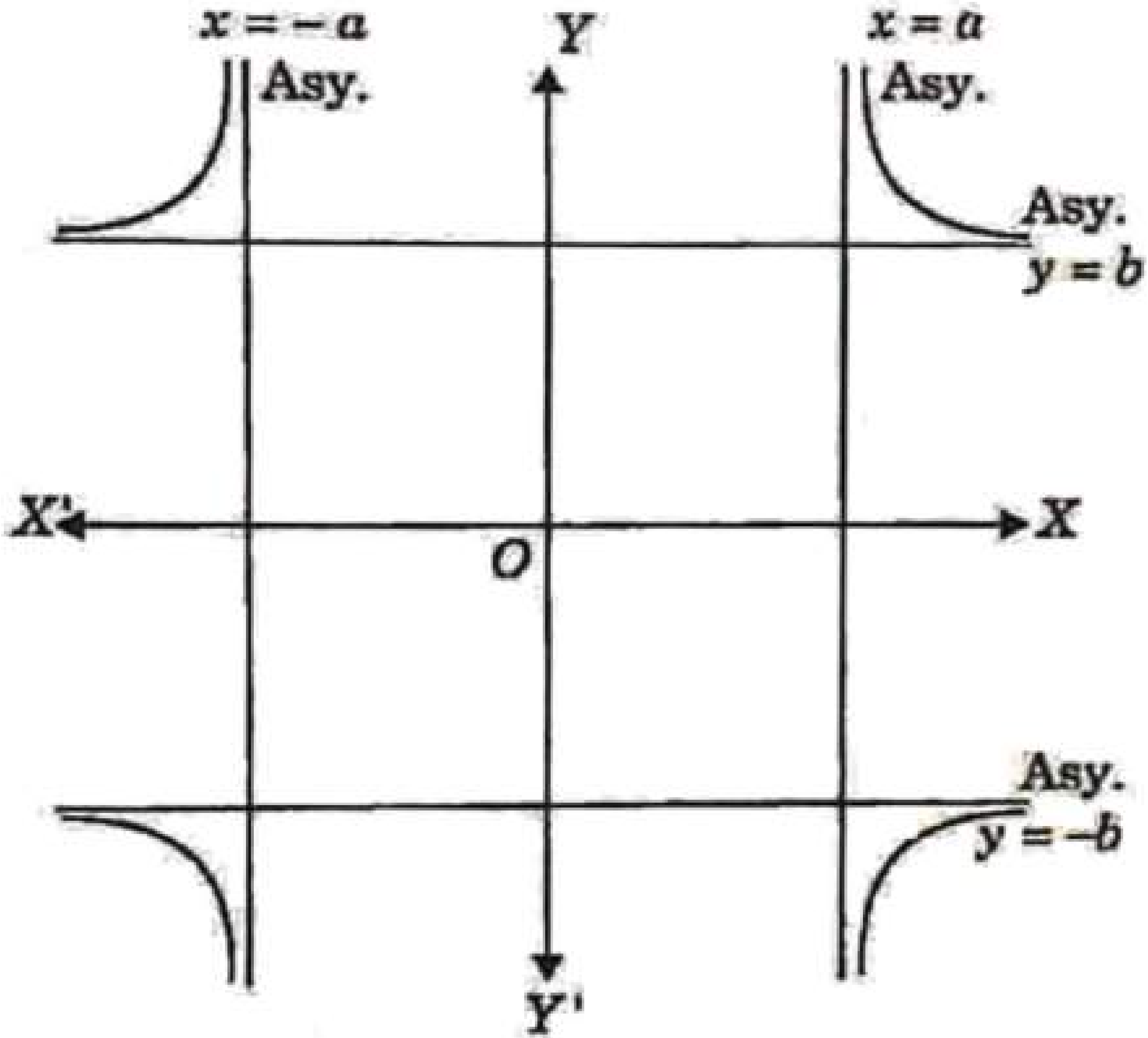
$$\Rightarrow \quad x = \pm \frac{ay}{\sqrt{(y^2 - b^2)}} \text{ and } y = \pm \frac{bx}{\sqrt{(x^2 - a^2)}}$$

$$\Rightarrow \quad y^2 - b^2 > 0 \text{ and } x^2 - a^2 > 0$$

$\Rightarrow$ $y \in (-\infty, -b) \cup (b, \infty)$ and $x \in (-\infty, -a) \cup (a, \infty)$

Thus, curve lies in the region where $x \in (-\infty, -a) \cup (a, \infty)$ and $y \in (-\infty, -b) \cup (b, \infty)$

Using above facts, the shape of the curve is given as



x = – a
Asy.
Y
x = a
Asy.
Asy.
y = b
X'
X
O
Asy.
y = –b
Y'

Exa. ***Trace the following curve:***

$$x^3+y^3=a^2x$$

Sol. $$x^3+y^3=a^2x \quad ...(1)$$

(1) Symmetry. On replacing x and y by $-x$ and $-y$ respectively, given curve remains unaltered. SO, the curve is symmetrical in opposite quadrants.

(2) Nature of Origin. The curve passes through the origin and the tangents at the origin are given as $xa^2=0 \Rightarrow x=0$, which is one real tangent at the origin.

(3) Points of Intersection of Curve with Co-ordinate axes.
On putting $x=0$ into eqn. (1), we get

$$y=0$$

and on putting $y=0$ into eqn. (1), we get

$$x^3=a^2x$$

$$\Rightarrow x\left(x^2-a^2\right)=0 \Rightarrow x=0,\pm a$$

Hence, given curve cuts the co-ordinate axis at $(0, 0)$, $(-a, 0)$ and $(a, 0)$.

To find tangent at the point $(-a, 0)$, we have to shift origin to $(-a, 0)$ *i.e.* on replacing x by $x-a$ in equation (1), we get

$$(x-a)^3 + y^3 = a^2(x-a)$$

and equating to zero the lowest degree terms, we get $x = 0$ *i.e.* a vertical line as a tangent at point $(-a, 0)$. Similarly, we can find a vertical line as a tangent at point $(a, 0)$.

(4) **Asymptotes.** As coefficients of highest power of x and y are constants, there does not exist any asymptote parallel to co-ordinate axes and to find oblique asymptotes,

$$\phi_3(m) = 1 + m^3, \ \phi_2(m) = 0$$

Now, $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0$

$\Rightarrow m = -1$ is the only real value of m

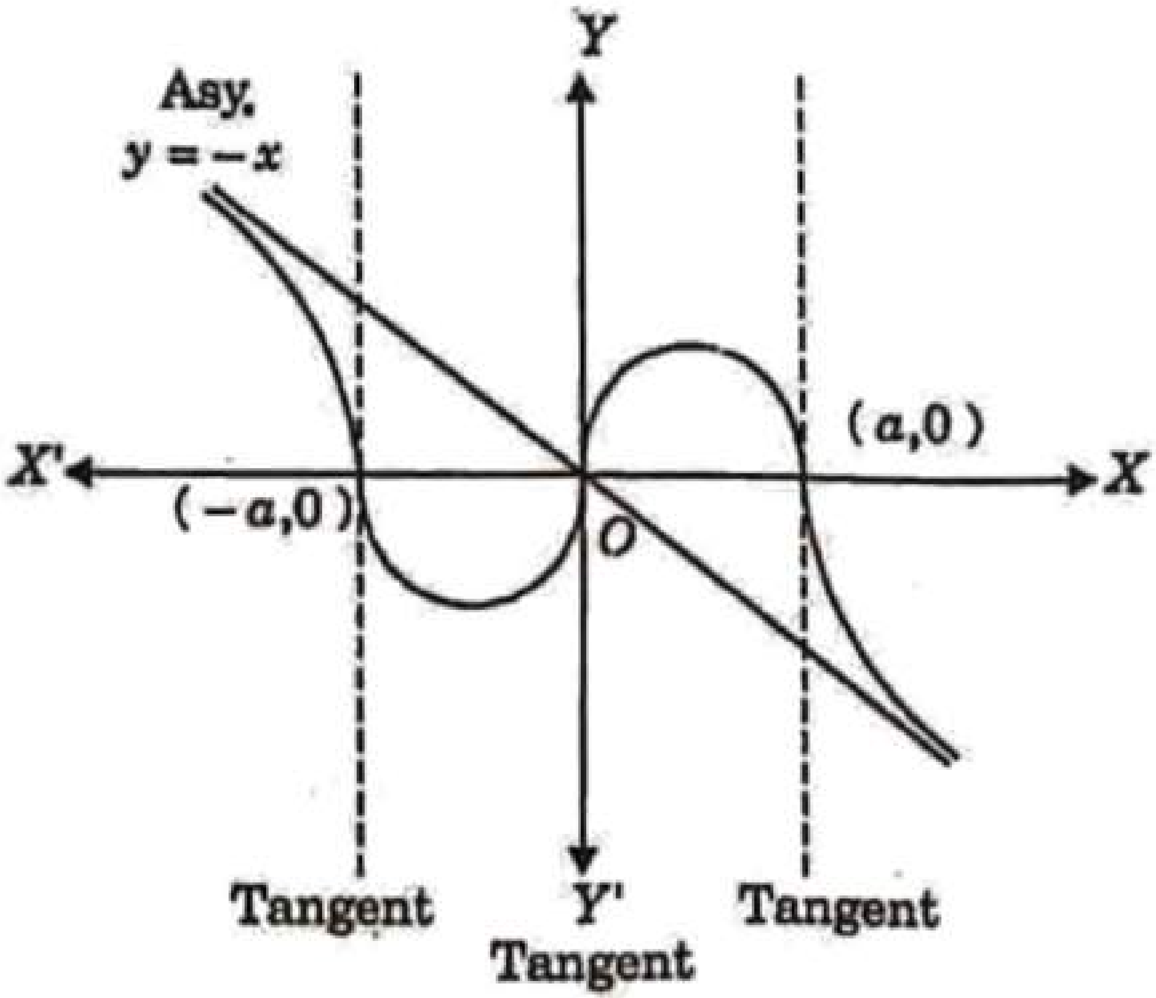
and $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = 0$ at $m = -1$

Thus, $y = -x$ is the only real asymptote of the given curve.

(5) Point of Inflexion. For the given curve, $\left(\dfrac{d^2y}{dx^2}\right)_{(0,\,0)} = 0$

and $\left(\dfrac{d^3y}{dx^3}\right)_{(0,\,0)} \neq 0 \Rightarrow$ origin is a point of inflexion.

(6) Region of Existence. Given curve exist in all four quadrants. Using above facts, the shape of the curve is given as



Y
Asy.
$y = -x$
$(a,0)$
X'
X
$(-a,0)$
O
Tangent
Y'
Tangent
Tangent

Exa. ***Trace the following curve:***

$$x^3 + y^3 = 3axy$$

Sol. $x^3 + y^3 = 3axy$ **(Follium of Descartes)** ...(1)

(1) Symmetry. On replacing x and y by y and x respectively, given curve remains unaltered. So, the curve is symmetrical with respect to $y = x$ line

(2) Nature of Origin. The curve passes through the origin and the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the curve *i.e.* $3axy = 0$

$\Rightarrow$ $x = 0, y = 0$ which are two real and different tangents at the origin. Hence, origin is a node.

(3) Points of Intersection of Curve with Co-ordinate Axes

On putting $x = 0$ into eqn. (1), we get $y = 0$

On putting $y = 0$ into eqn. (1), we get $x = 0$

Hence, only origin exist as point of intersection of curve with the co-ordinate axes.

(4) Asymptotes. As coefficients of highest powers of x and y are constants, there doesn't exist any asymptote parallel to co-ordinate axes and to find oblique asymptotes,

$$\phi_3(m) = (1 + m^3),\ \phi_2(m) = -3am$$

Now, $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0 \Rightarrow m = -1$ is the only real value of m

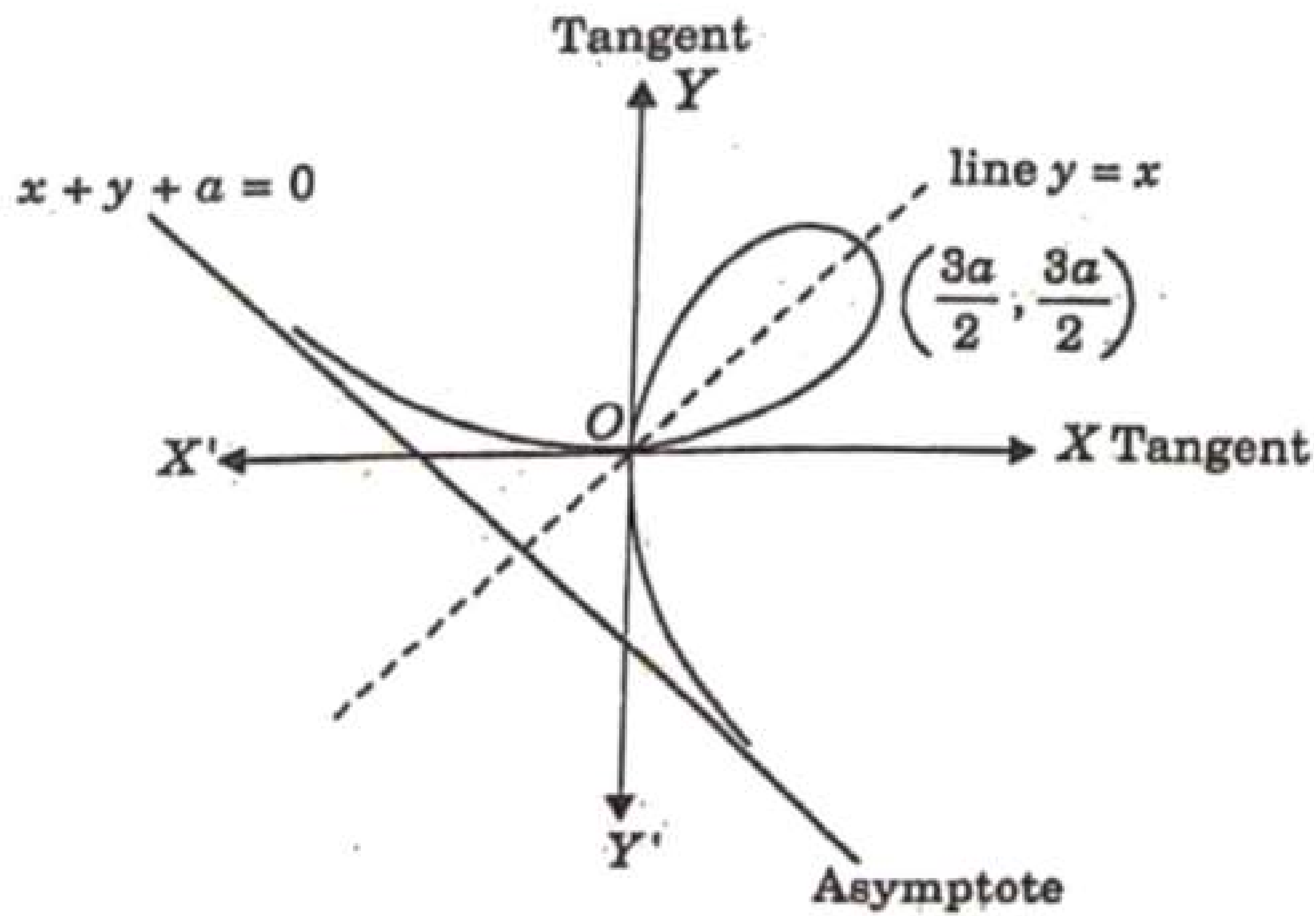
and
$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{3am}{3m^2} = -a \text{ at } m = -1$$

Thus, $y = -x - a$ or $x + y + a = 0$ is the only real asymptote of the given curve.

(5) Region of Existence. Curve $x^3 + y^3 = 3axy$ does not exist in III quadrant because in III quadrant $x^3 + y^3$ is negative and $3axy$ is positive. (Here, we consider 'a' as a positive constant).

(6) Other Points. For the given curve, point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ exists as a point of intersection of curve with line of symmetricity $y = x$.

Using above facts, the shape of the curve is given as


Tangent
Y
line y = x
x + y + a = 0
(3a/2, 3a/2)
O
X'
X Tangent
Y'
Asymptote

Exa. ***Trace the following curve:***

$$y^2(2a - x) = x^3$$

Sol. $$y^2(2a - x) = x^3 \qquad ...(1)$$

(1) **Symmetry.** The curve is symmetrical about x-axis as all powers of y are even.

(2) **Nature of Origin.** The curve passes through the origin and the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the curve *i.e.* $2ay^2 = 0 \Rightarrow y = 0, y = 0$ which are the two real and same tangents at the origin. Hence, origin is a cusp.

(3) **Points of Intersection of Curve with Co-ordinate Axes**

On putting $x = 0$ into eqn. (1), we get $y = 0$

On putting $y = 0$ into eqn. (1), we get $x = 0$

Hence, only origin exist as point of intersection of curve with the co-ordinate axes.

(4) Asymptotes. As coefficient of highest power of x is constant, there does not exist any asymptote parallel to x-axis and coefficient of highest power of y is $(2a - x)$,

So $x = 2a$ is an asymptote parallel to y-axis and to find oblique asymptotes,

$$\phi_3(m) = (m^2 + 1)$$

Now, $\phi_3(m) = 0 \Rightarrow m = \pm i$

So, there does not exist any oblique real asymptote.

(5) Region of Existence

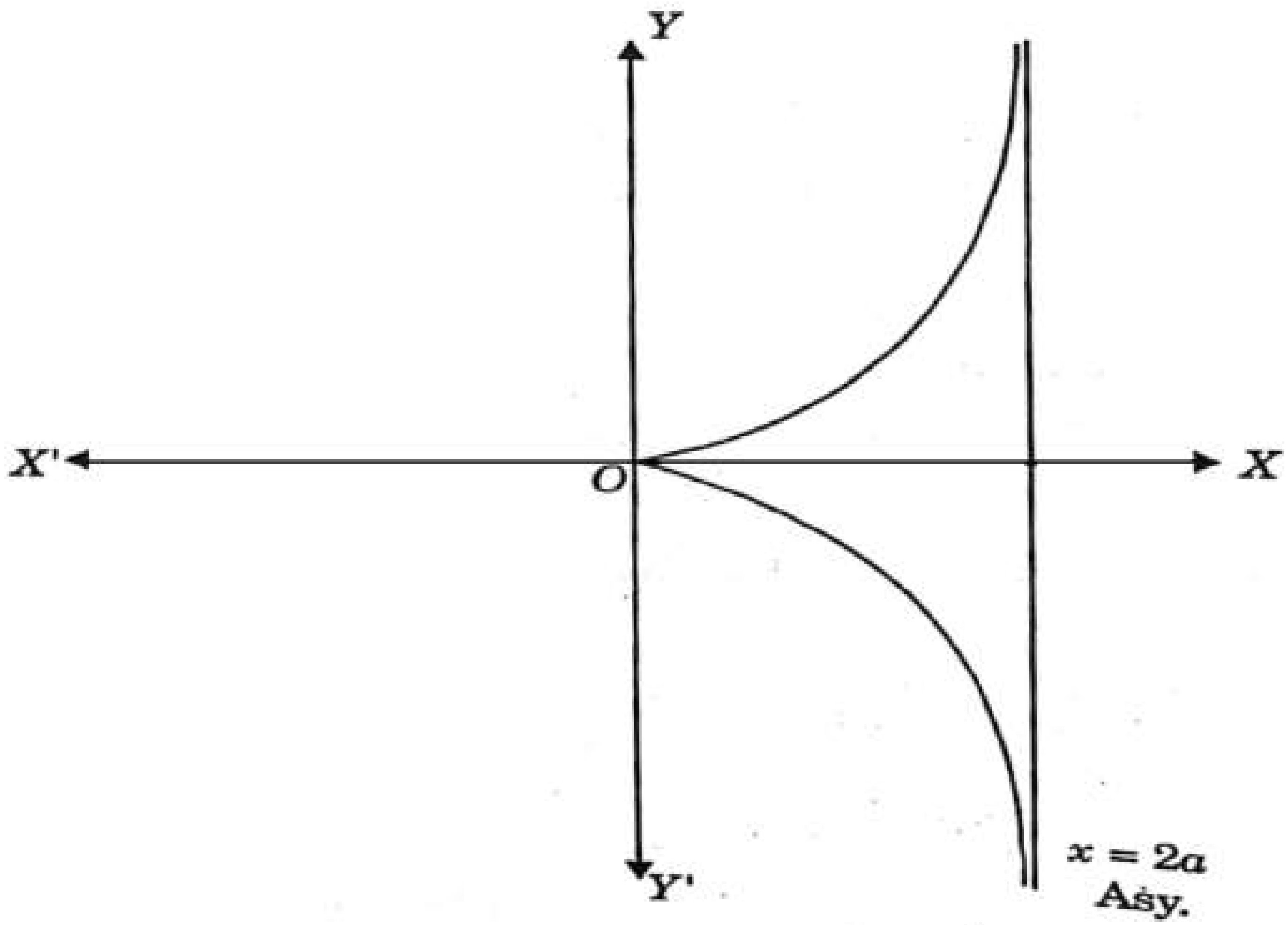
$$y^2(2a - x) = x^3$$

$$\Rightarrow \quad y = \pm x\sqrt{\frac{x}{(2a - x)}}$$

$$\Rightarrow \quad \frac{x}{(2a - x)} \geq 0 \Rightarrow x \in [0, 2a)$$

Thus, curve lies in the region where $x \in [0, 2a)$

Using above facts, the shape of the curve is given as


Y
X'
X
O
Y'
x = 2a
Asy.

Exa. ***Trace the following curve:***

$$x(x-2a)y^2 = a^2(x-a)\,(x-3a)$$

Sol. $$x(x-2a)y^2 = a^2(x-a)\,(x-3a) \qquad ...(1)$$

(1) **Symmetry.** The curve is symmetrical about x-axis as all the powers of y are even.

(2) **Nature of Origin.** The curve doesn't passes through the origin.

(3) **Points of Intersection of Curve with Co-ordinate Axes**

On putting $x = 0$ into eqn. (1), we get $y = \infty$

On putting $y = 0$ into eqn. (1), we get $x = a, 3a$

Hence, given curve cuts the co-ordinate axes at $(a, 0)$ and $(3a, 0)$

To find tangent at the point $(a, 0)$, we have to shift origin to point $(a, 0)$ *i.e.* on replacing x by $(x + a)$ in eqn. (1), we get

$$(x+a)\,(x-a)y^2 = a^2x(x-2a)$$

and equating to zero the lowest degree terms, we get $x = 0$ as tangent *i.e.* a vertical line as a tangent at point $(a, 0)$. In similar fashion, we can find a vertical line as a tangent at point $(3a, 0)$.

(4) Asymptotes. As coefficients of highest powers of x and y are $(y^2 - a^2)$ and $x(x - 2a)$ respectively.

So, asymptotes parallel to x-axis are

$$y^2 - a^2 = 0$$

$$\Rightarrow \qquad y = \pm a$$

and asymptotes parallel to y-axis are

$$x(x - 2a) = 0$$

$$\Rightarrow \qquad x = 0, 2a$$

(5) Region of Existence

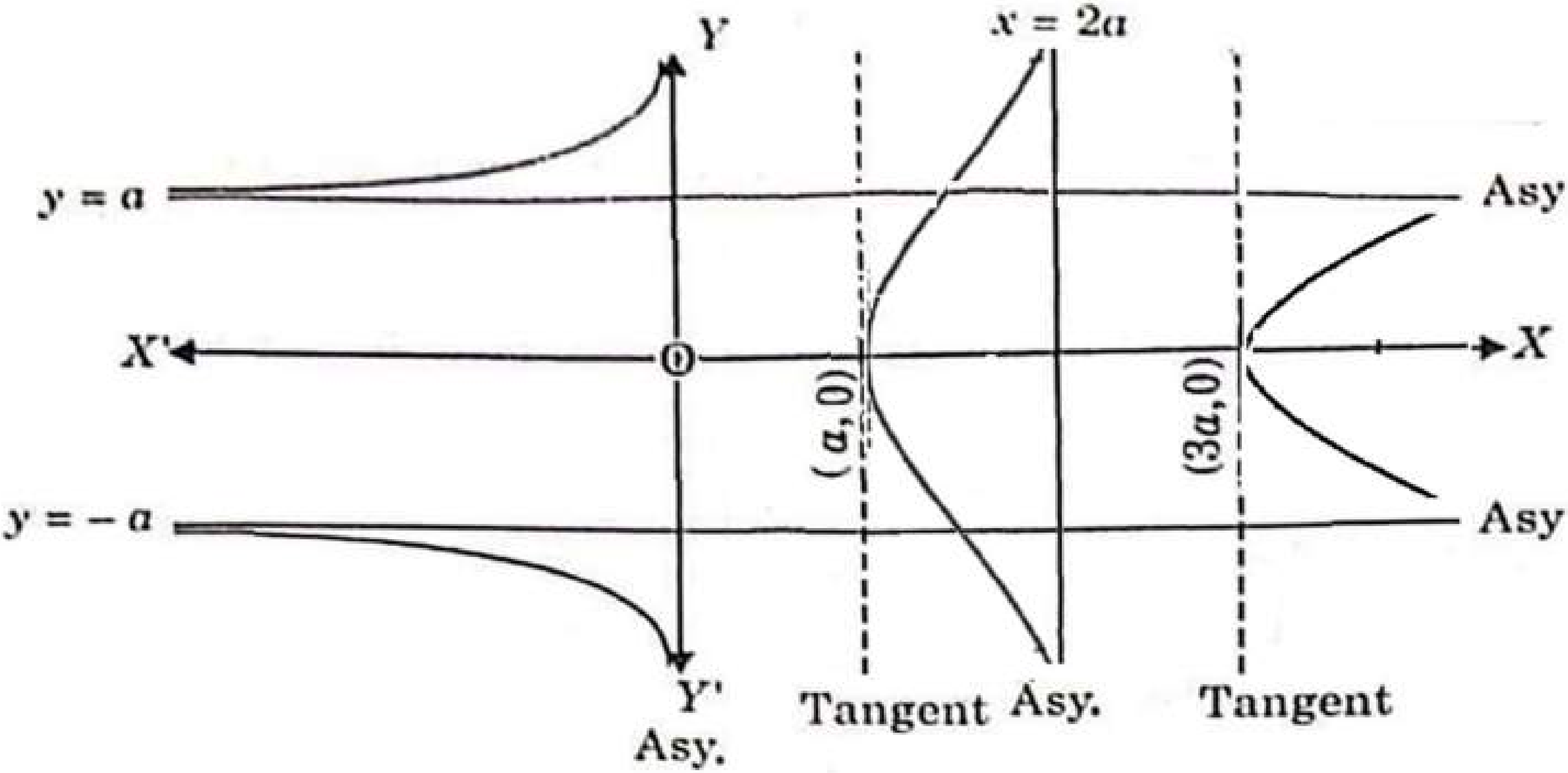
$$x(x - 2a)y^2 = a^2(x - a)(x - 3a)$$

$$\Rightarrow \qquad y = \pm a\sqrt{\frac{(x-a)(x-3a)}{x(x-2a)}}$$

$$\Rightarrow \qquad \frac{(x-a)(x-3a)}{x(x-2a)} \geq 0$$

$$\Rightarrow \quad x \in (-\infty, 0) \cup [a, 2a) \cup [3a, \infty).$$

Thus, curve lies in the region where $x \in (-\infty, 0) \cup [a, 2a) \cup [3a, \infty)$
Using above facts, the shape of the curve is given as



Y
x = 2a
y = a
Asy
X'
O
X
(a, 0)
(3a, 0)
y = – a
Asy
Y'
Asy.
Tangent
Asy.
Tangent

Exa. ***Trace the following curve:***

$$x = (y-1)(y-2)(y-3)$$

Sol. $x = (y-1)(y-2)(y-3)$ ...(1)

(1) **Symmetry.** There is no symmetry for the given curve.

(2) **Nature of Origin.** The curve doesn't passes through the origin.

(3) **Points of Intersection of Curve with Co-ordinate Axes**

On putting $x = 0$ into eqn. (1), we get $y = 1, 2, 3$

On putting $y = 0$ into eqn. (1), we get $x = -6$

Hence, given curve cuts the co-ordinate axes at (0, 1), (0, 2), (0, 3) and (−6, 0)

To find tangent at the point (0, 1), we have to shift origin to point (0, 1) *i.e.* on replacing y by $(y + 1)$ into eqn. (1), we get

$$x = y(y-1)(y-2)$$

and equating to zero the lowest degree terms, we get $x = 2y$ as a tangent at point (0, 1)

In similar fashion, we can find $x = -y$ as a tangent at point (0, 2), $x = 2y$ as a tangent at point (0, 3) and $x = 11y$ as a tangent at point (− 6, 0)

(4) Asymptotes. As coefficients of highest powers of x and y are constants, there doesn't exist any asymptote parallel to co-ordinate axes. Also, there is no oblique asymptote.

(5) Region of Existence

$$x = (y-1)(y-2)(y-3)$$

$$\Rightarrow \qquad x < 0 < \text{if } y < 1,$$

$$x \geq 0 \text{ if } 1 \leq y \leq 2,$$

$$x \leq 0 \text{ if } 2 \leq y \leq 3,$$

$$x > 0 \text{ if } y > 3$$

(6) Maxima and Minima. Here, highest power of y exceeds than highest power of x. Thus, to find maxima and minima, we obtain $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$ from eqn. (1)

as $$\frac{dx}{dy} = (3y^2 - 12y + 11) \text{ and } \frac{d^2x}{dy^2} = (6y - 12)$$

Now, $$\frac{dx}{dy} = 0 \Rightarrow 3y^2 - 12y + 11 = 0 \Rightarrow y = 2 \pm \frac{1}{\sqrt{3}}$$

and $$\left(\frac{d^2x}{dy^2}\right)_{y=\left(2+\frac{1}{\sqrt{3}}\right)} = 6\left(2 + \frac{1}{\sqrt{3}}\right) - 12 = 3\sqrt{3} > 0$$

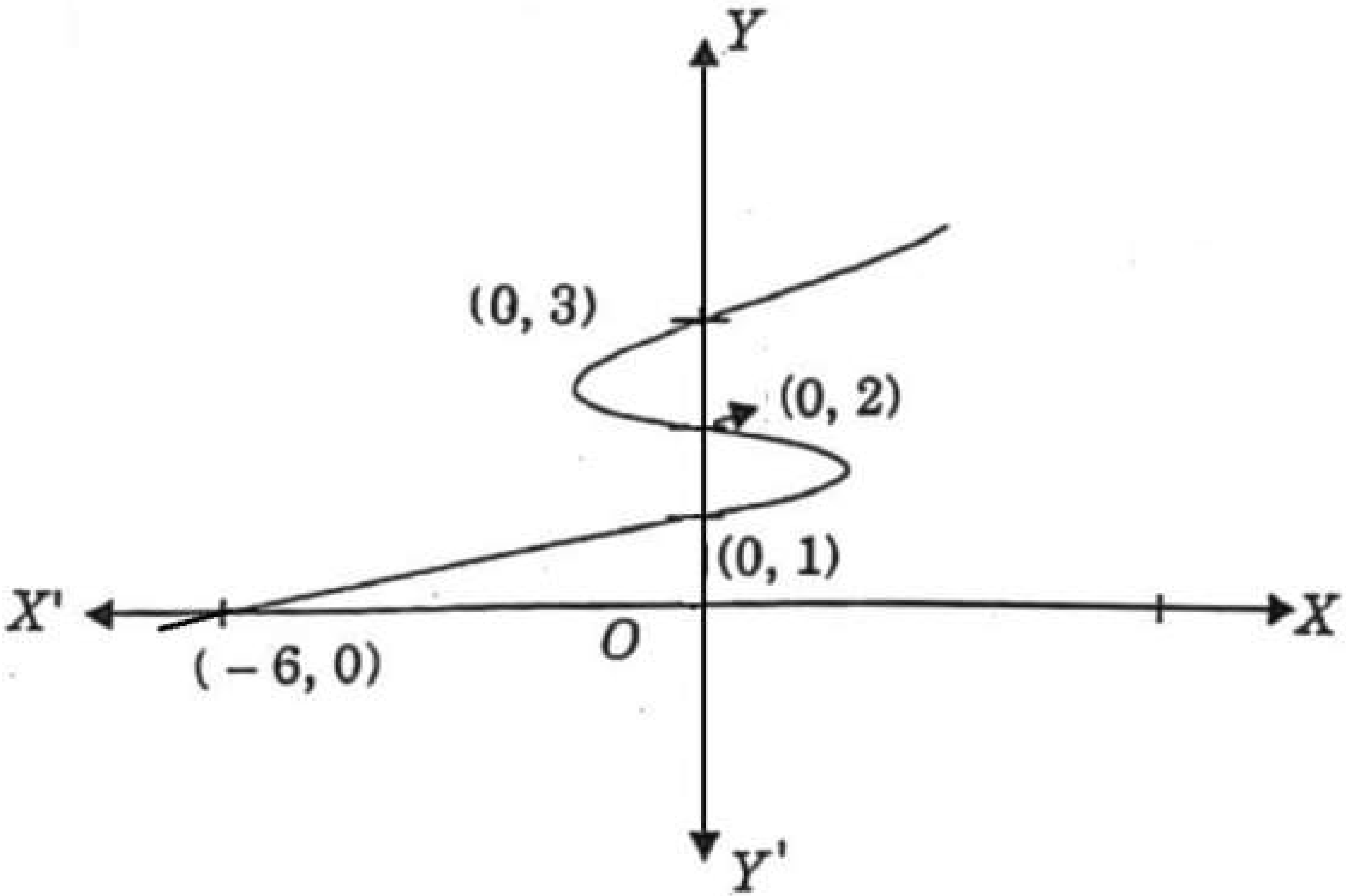
So, $$y = \left(2 + \frac{1}{\sqrt{3}}\right) \text{ is a point of minima where } x = -\frac{2}{3\sqrt{3}}$$

Also, $$\left(\frac{d^2x}{dy^2}\right)_{y=\left(2-\frac{1}{\sqrt{3}}\right)} = 6\left(2 - \frac{1}{\sqrt{3}}\right) - 12$$

$$= -3\sqrt{3} < 0$$

So, $y = \left(2 - \frac{1}{\sqrt{3}}\right)$ is a point of maxima where $x = \frac{2}{3\sqrt{3}}$

Using above facts, the shape of the curve is given as


Y
(0, 3)
(0, 2)
(0, 1)
X'
X
(– 6, 0)
O
Y'

Exa. ***Trace the following curve:***

$$x^5 + y^5 = 5a^2x^2y\,(y-3)$$

Sol. $$x^5 + y^5 = 5a^2x^2y \quad ...(1)$$

(1) Symmetry. Equation of the given curve remains unaltered, if we replace x and y by $-x$ and $-y$ respectively. Thus, curve is symmetrical in opposite quadrants.

(2) Nature of Origin. The curve passes through the origin and the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the curve *i.e.* $5a^2x^2y = 0 \Rightarrow x = 0, x = 0, y = 0$ which are three real tangents at the origin.

Hence, origin is a node. (considering $x = 0, y = 0$ as two real and different tangents)

as well as a cusp. (considering $x = 0, x = 0$ as two real and coincident tangents)

(3) Points of Intersection of Curve with Co-ordinate Axes

On putting $x = 0$ into eqn. (1), we get $y = 0$

On putting $y = 0$ into eqn. (1), we get $x = 0$

Hence, only origin exist as point of intersection of curve with the co-ordinate axes.

(4) Asymptotes. As coefficients of highest powers of x and y are constants, there doesn't exist any asymptote parallel to co-ordinate axes and to find oblique asymptotes,

$$\phi_5(m) = (1 + m^5),\ \phi_4(m) = 0$$

Now, $\phi_5(m) = 0$

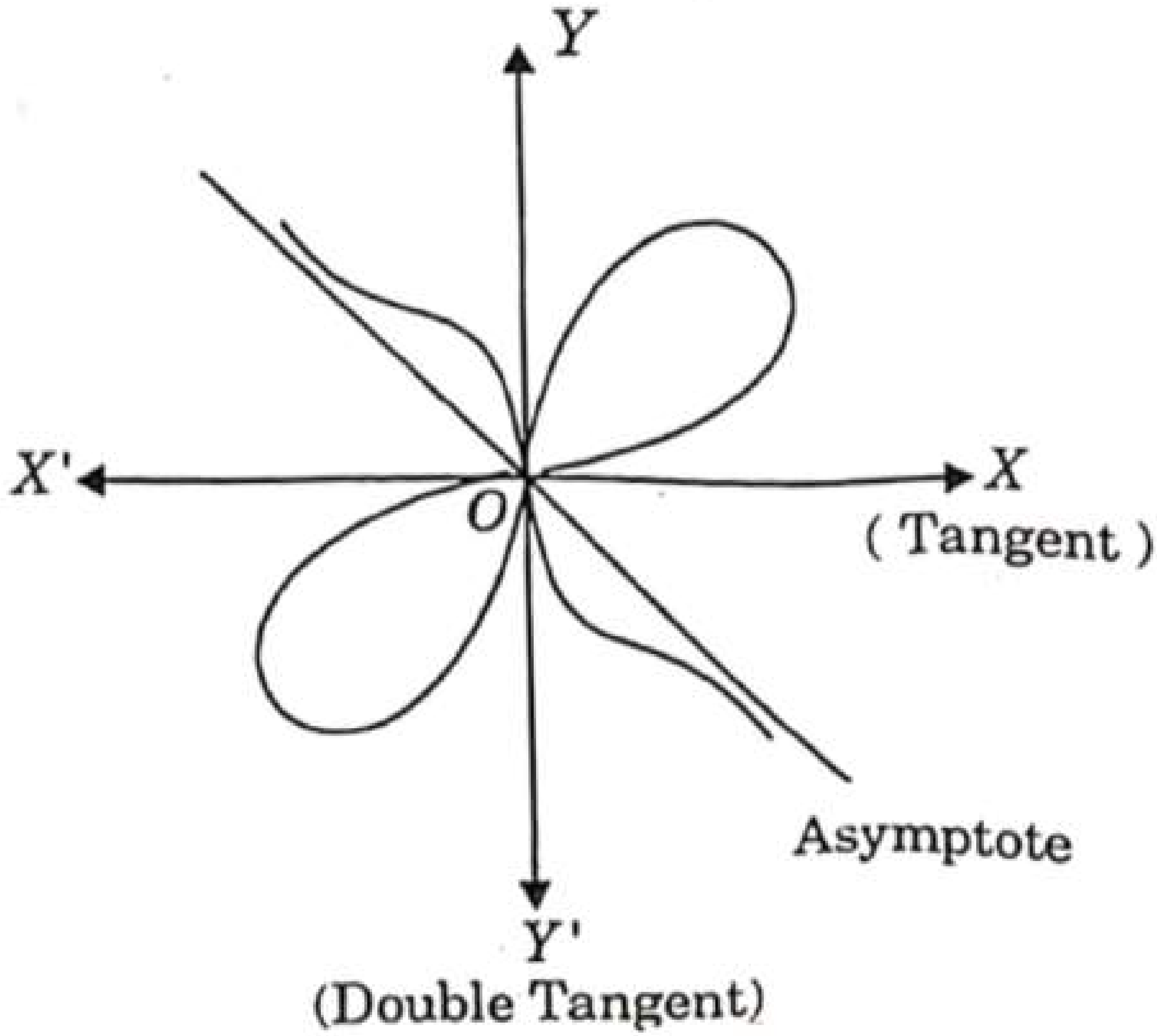
$\Rightarrow$ $1 + m^5 = 0 \Rightarrow m = -1$ is the only real value of m

and
$$c = -\frac{\phi_4(m)}{\phi_5'(m)} = 0$$

Thus $y = -x$ is the only real asymptote of the given curve.

(5) Region of Existence. Curve $x^5 + y^5 = 5a^2x^2y$ exists in all four quadrants.

Using above facts, the shape of the curve is given as


Y
X'
X
O
(Tangent)
Asymptote
Y'
(Double Tangent)

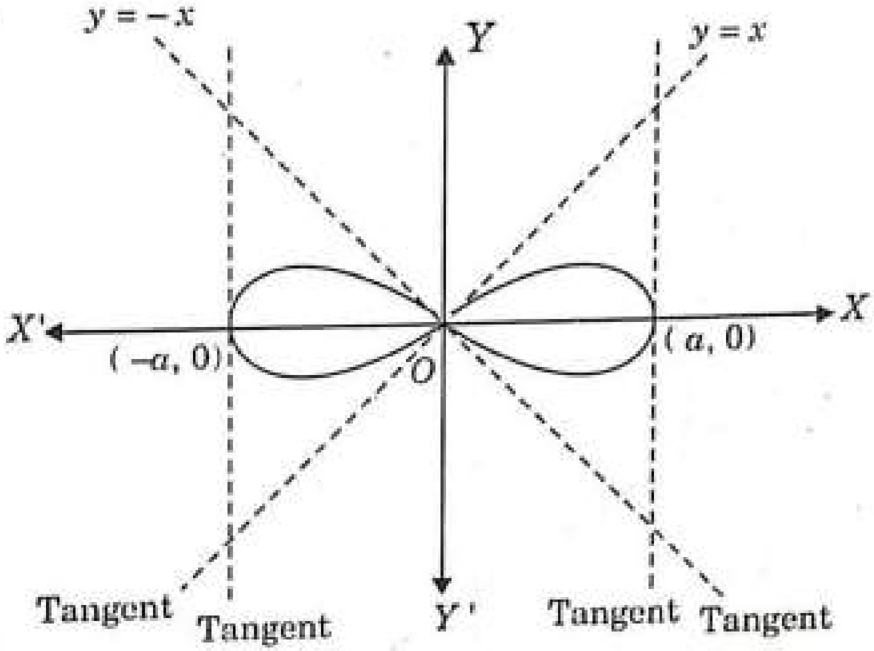
EXERCISE

Trace the following curves :

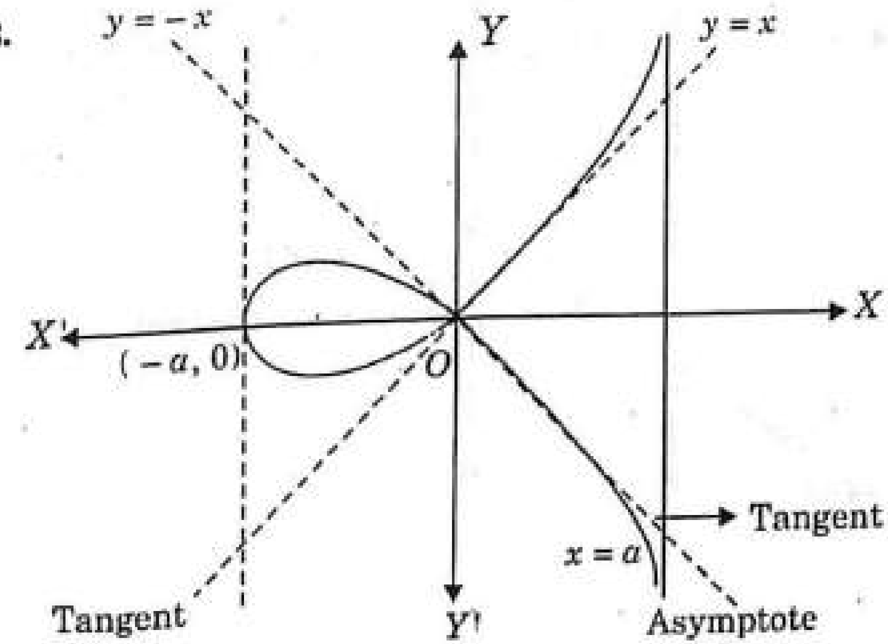
1. $a^2y^2 = x^2(a^2 - x^2)$
2. $y^2(a - x) = x^2(a + x)$
3. $y^2(a + x) = x^2(3a - x)$
4. $y^2(x + 3a) = x(x - a)(x - 2a)$
5. $x^2y^2 = (1 + y)^2(4 - y^2)$
6. $xy^2 + (x + a)^2(x + 2a) = 0$
7. $y^3 - x^3 = ax^2$
8. $y = x(x^2 + 1)$
9. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$
10. $xy^2 = 4a^2(2a - x)$

ANSWERS

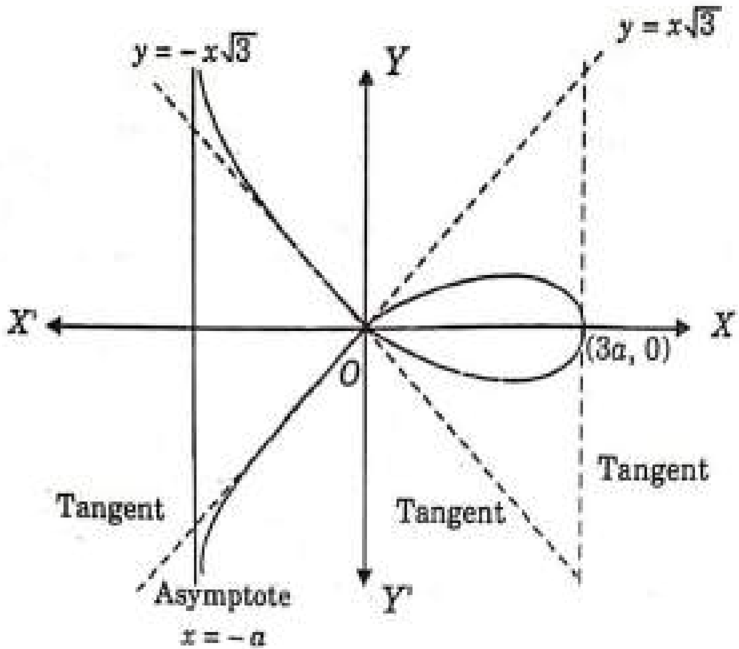
1.


y = − x
Y
y = x
X'
X
(−a, 0)
(a, 0)
O
Tangent
Tangent
Y'
Tangent
Tangent

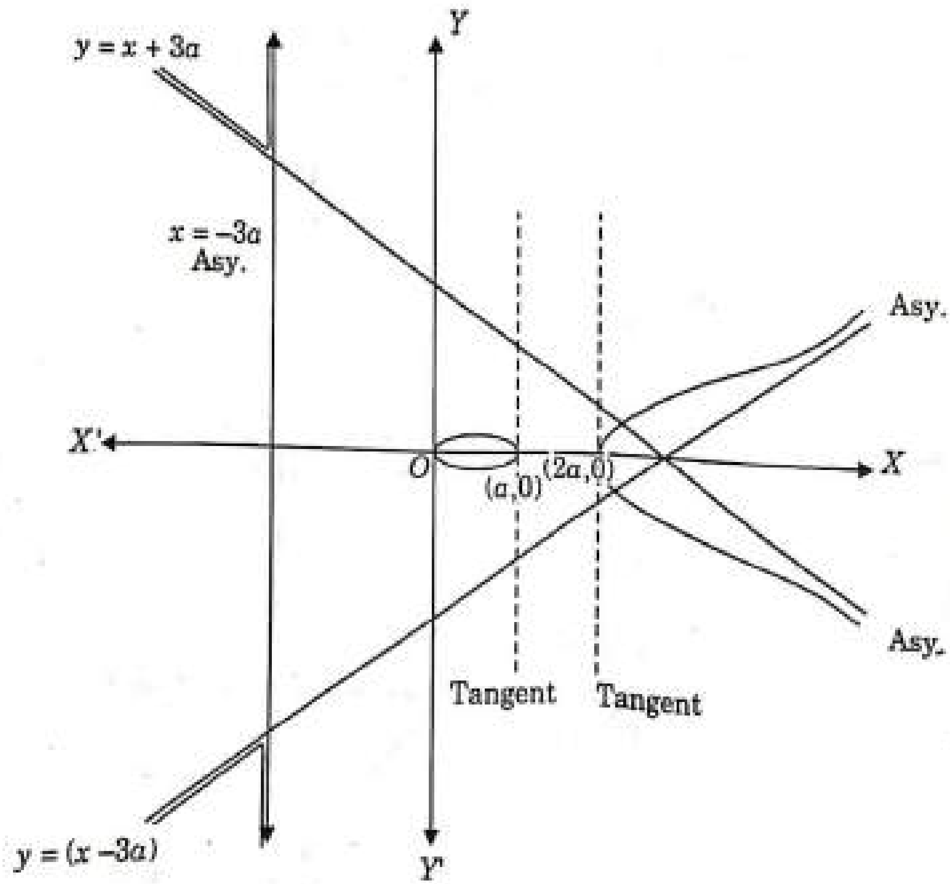
2.


y = − x
Y
y = x
X'
X
(−a, 0)
O
Tangent
x = a
Tangent
Y'
Asymptote

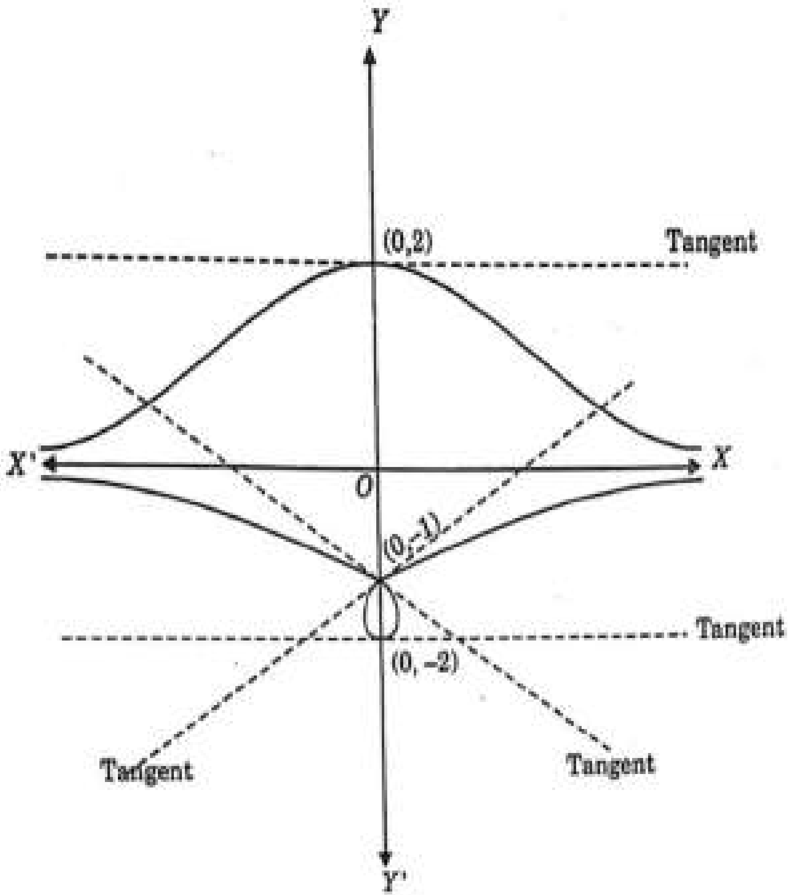
3.


y = – x√3
y = x√3
Y
X'
X
O
(3a, 0)
Tangent
Tangent
Tangent
Asymptote
x = – a
Y'

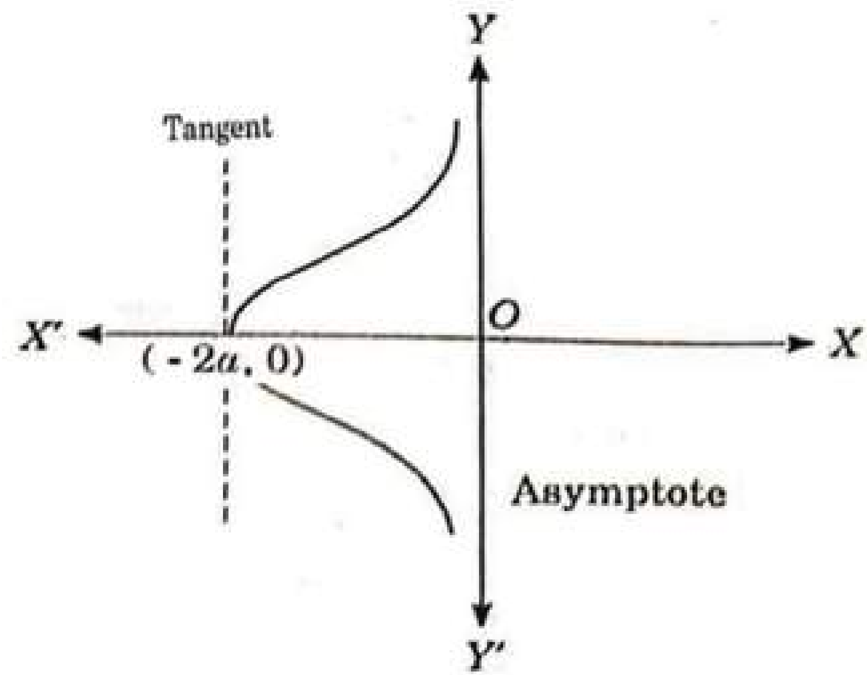
4.


y = x + 3a
Y
x = –3a
Asy.
Asy.
X'
O
(a,0)
(2a,0)
X
Asy.
Tangent
Tangent
y = (x –3a)
Y'

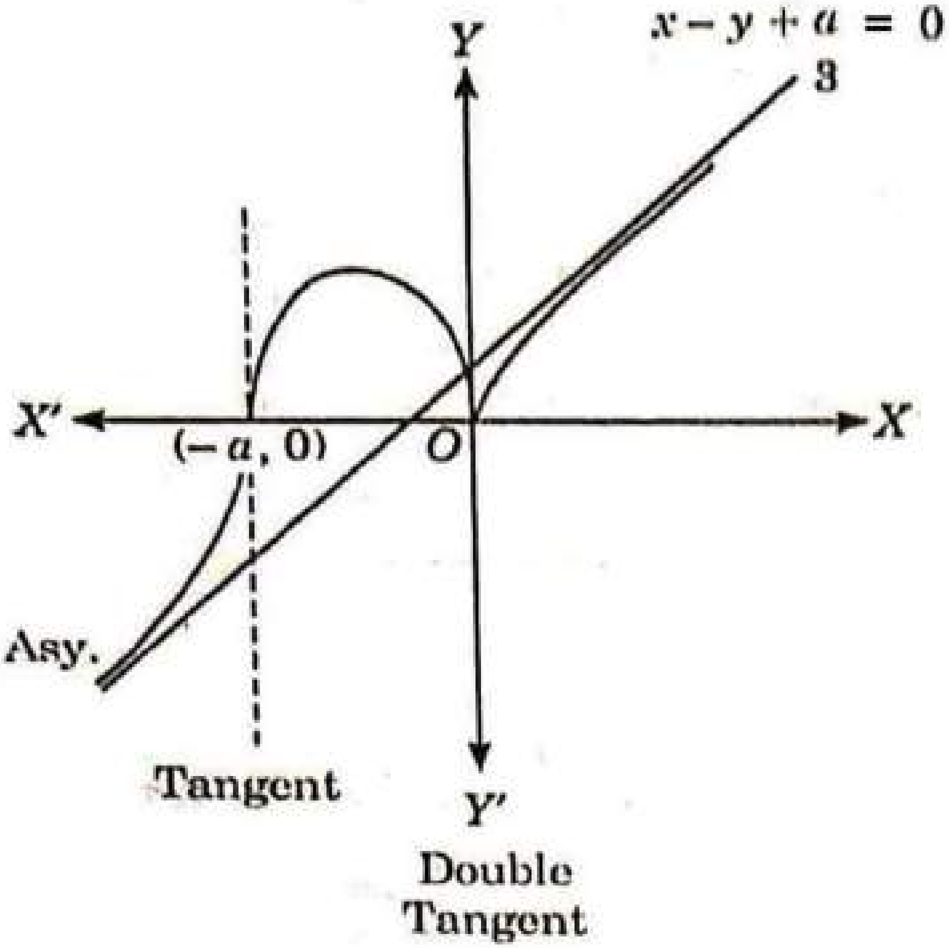
5.


Y
(0,2)
Tangent
X'
X
O
(0,-1)
Tangent
(0, -2)
Tangent
Tangent
Y'

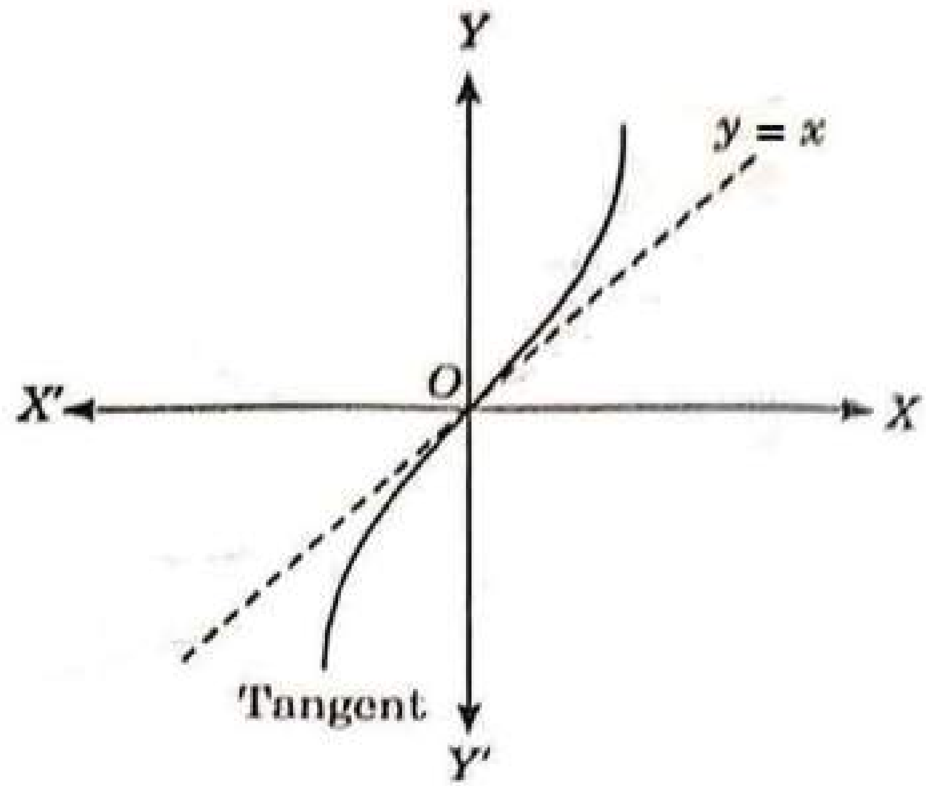
6.


Y
Tangent
X'
O
X
(- 2a, 0)
Asymptote
Y'

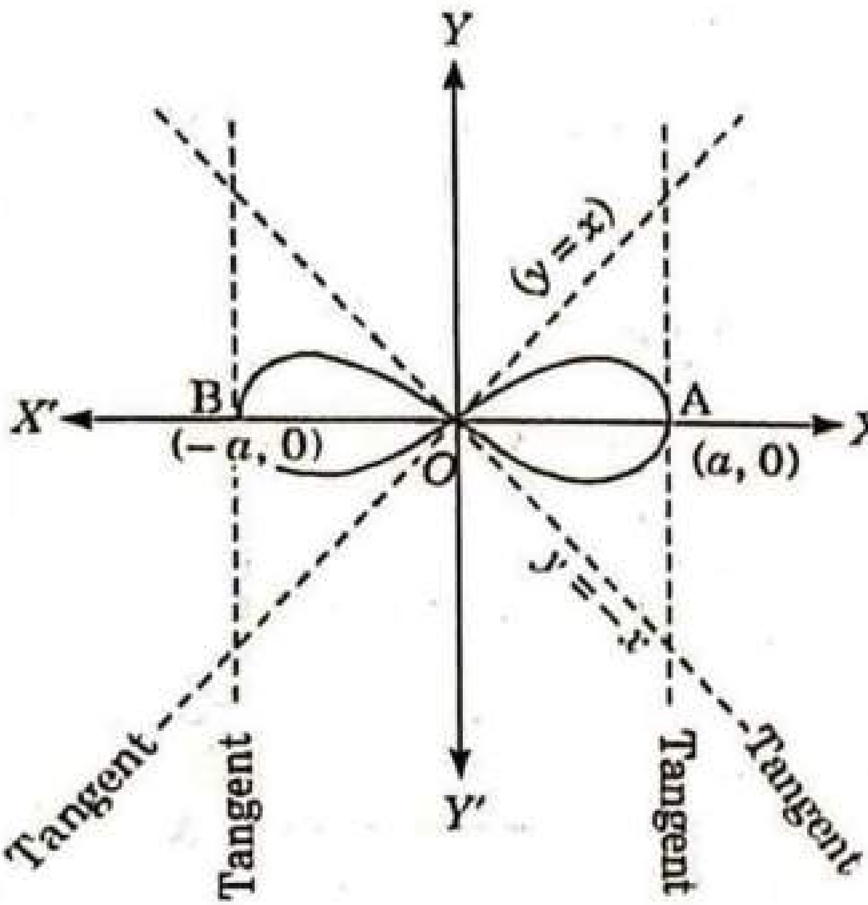
7.



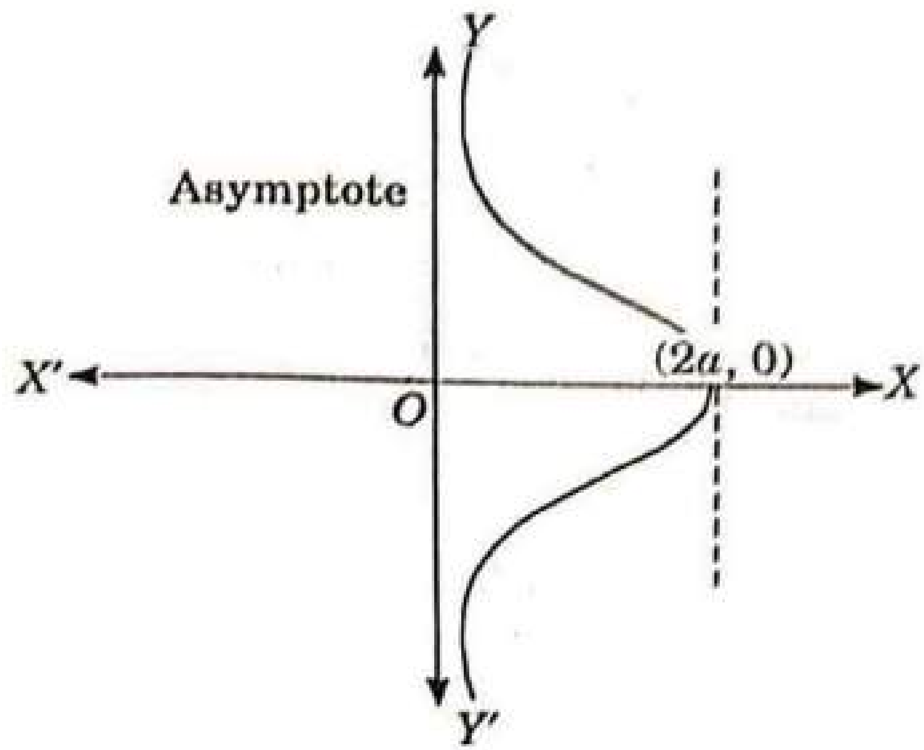
8.



9.


Y
(y = x)
B
A
X'
X
(−a, 0)
O
(a, 0)
y = −x
Tangent
Tangent
Y'
Tangent
Tangent

10.


Y
Asymptote
(2a, 0)
X'
X
O
Y'

CURVE TRACING IN POLAR COORDINATE SYSTEM

PROCEDURE FOR TRACING POLAR CURVES

Apply following procedure for tracing of a curve whose equation is given in the polar form $f(r, \theta) = 0$

(1) Symmetry. (*a*) If $f(r, \theta) = f(r, -\theta)$ *i.e.* if θ is replaced by $-\theta$ and the equation of the curve remains unaltered, then the curve is symmetrical about the initial line $\theta = 0$ (*i.e.* x-axis).

e.g. $\quad r = a(1 + \cos\theta)$

(*b*) If $f(r, \theta) = f(r, \pi - \theta)$ *i.e.* if θ is replaced by $(\pi - \theta)$ and the equation of the curve remains unaltered, then the curve is symmetrical about the line $\theta = \dfrac{\pi}{2}$ (*i.e.* y-axis).

e.g. $\quad r = a(1 + \sin\theta)$

(c) If $f(r, \theta) = f\left(r, \frac{\pi}{2} - \theta\right)$ *i.e.* if θ is replaced by $\left(\frac{\pi}{2} - \theta\right)$ and the equation of the curve remains unaltered, then the curve is symmetrical about the line $\theta = \frac{\pi}{4}$ (*i.e.* line $y = x$).

e.g.
$$r = \frac{3a \sin\theta \cos\theta}{(\cos^3\theta + \sin^3\theta)}$$

(d) If $f(r, \theta) = f\left(r, \frac{3\pi}{2} - \theta\right)$ *i.e.* if θ is replaced by $\left(\frac{3\pi}{2} - \theta\right)$ and the equation of the curve remains unaltered, then the curve is symmetrical about the line $\theta = \frac{3\pi}{4}$ (*i.e.* line $y = -x$).

e.g.
$$r = \frac{3a \sin\theta \cos\theta}{(\cos^3\theta - \sin^3\theta)}$$

(e) If $f(r, \theta) = f(-r, \theta)$ *i.e.* if r is replaced by $-r$ and the equation of the curve remains unaltered, then the curve is symmetrical about the pole.

e.g.
$$r^2 = a^2 \cos 2\theta.$$

(2) Nature of Pole. Now, we have to find out whether the curve passes through the pole or not. If for some finite value of θ (say) $\theta = \alpha$, $r = 0$ then curve passes through the pole and line $\theta = \alpha$ is considered as a tangent of the curve at the pole.

(3) Intersection with the Line $\theta = 0$ and $\theta = \frac{\pi}{2}$. If the curve intersects with the initial line $\theta = 0$, then the points of intersection can be obtained by putting $\theta = 0$ in the eqn. of the curve and if the curve intersects with the line $\theta = \frac{\pi}{2}$, then the points of intersection can be obtained by putting $\theta = \frac{\pi}{2}$ in the equation of the curve.

(4) Asymptotes. For a polar curve, generally asymptote exists if $r \to \infty$, when $\theta \to a$ finite angle, which can be obtained by the following process :

(*i*) Represent the given curve as $\dfrac{1}{r} = g(\theta)$

(*ii*) Solve the equation $g(\theta) = 0$

(*iii*) If $g(\theta) = 0$ provides a finite solution *i.e.* $\theta = \beta$, then calculate

$$g'(\beta) = [g'(\theta)]_{\theta=\beta}$$

(*iv*) Asymptote is given by $r \sin(\theta - \beta) = \dfrac{1}{g'(\beta)}$

Using above process, we can conclude that for any polar curve, asymptote doesn't exist in any of three conditions :

(*i*) If curve is not expressible as $\dfrac{1}{r} = g(\theta)$

(*ii*) If $g(\theta) = 0$ does not provide a finite solution

(*iii*) If $g'(\beta) = 0$

(5) Region of Existence. (*i*) If r^2 is given in the equation of the curve, then solve the equation for r in terms of θ by virtue of a square root to decide the certain region. *e.g.* In $r^2 = a^2 \cos 2\theta$,

$$r = \pm a\sqrt{\cos 2\theta} \quad \Rightarrow \quad \cos 2\theta \geq 0$$

$\Rightarrow$ Curve exist only if $\cos 2\theta \geq 0$

(*ii*) If $\sin\theta$ or $\cos\theta$ is given in the equation of the curve, then use $-1 \leq \sin\theta \leq 1$ or $-1 \leq \cos\theta \leq 1$ to select the region of existence.

(6) Periodicity. In polar equations generally periodic functions like $\sin\theta$ or $\cos\theta$ occur and so values of θ are considered from 0 to 2π. The remaining vlaues of θ will not provide any new branch of the curve.

(7) Table. Solve the given equation for r and see how r varies as θ increases from 0 to $+\infty$ and θ decreases from 0 to $-\infty$. Thus, construct a table between corresponding values of θ and r as

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	......
r	r_1	r_2	r_3	r_4	r_5	......

(8) Transformation to Cartesian Co-ordinate System. Sometimes it is useful to transform $f(r, \theta) = 0$ into $F(x, y) = 0$ by using the relations

$$r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Remarks : (1) While observing nature of pole for a particular polar curve, we can obtain $r = 0$ for several values of θ, which can create a confusion about pole for that curve as a multiple point. But it is not true and it happens because of periodicity of the given curve. *e.g.* for the curve $r = a(1 + \cos\theta)$, $r = 0$ for $\theta = \pm\pi, \pm 3\pi, \ldots$, but we consider pole as a single point and one of the tangents as $\theta = \pi$.

(2) For most of the curves, while calculating the values of θ and r, periodicity helps to consider range of θ from 0 to 2π and using symmetricity this range can be further minimized.

e.g. For curve $r = a(1 - \cos\theta)$, using periodicity, range of θ is considered from 0 to 2π and it is minimized from 0 to π only because of symmetricity of the curve with respect to the line $\theta = 0$

Exa. *Trace the following curve :*

$r = a + b\cos\theta; a < b$ *(Limacon)*

Sol. $r = a + b\cos\theta, a < b$

(1) **Symmetry.** On replacing θ by $-\theta$, the eqn. of curve remains unaltered, so given curve is symmetrical w.r.t. initial line $\theta = 0$

(2) **Nature of Pole**

$$r = 0 \quad \Rightarrow \quad a + b\cos\theta = 0 \quad \Rightarrow \quad \cos\theta = -\frac{a}{b}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{a}{b}\right)$$

Since $a < b$, curve passes through pole and $\theta = \cos^{-1}\left(-\frac{a}{b}\right)$ is tangent at pole.

(3) Intersection with the Line $\theta = 0$ and $\theta = \frac{\pi}{2}$

$$r = a + b\cos\theta \text{ and } \theta = 0 \quad \Rightarrow \quad r = a + b$$

and $$r = a + b\cos\theta \text{ and } \theta = \frac{\pi}{2} \quad \Rightarrow \quad r = a$$

Thus, curve intersects the line $\theta = 0$ at $r = a + b$ and line $\theta = \frac{\pi}{2}$ at $r = a$

(4) Asymptotes. Given curve can be rewritten as

$$\frac{1}{r} = \frac{1}{a + b\cos\theta}$$

and $$\frac{1}{a + b\cos\theta} = 0 \quad \not\Rightarrow \quad a \text{ finite value of } \theta$$

Thus, there doesn't exist any asymptote for the given curve.

(5) Region of Existence

Since $$-1 \le \cos\theta \le 1$$

or $$-b \le b\cos\theta \le b$$

or $$a - b \le a + b\cos\theta \le a + b$$

or $$a - b \le r \le a + b$$

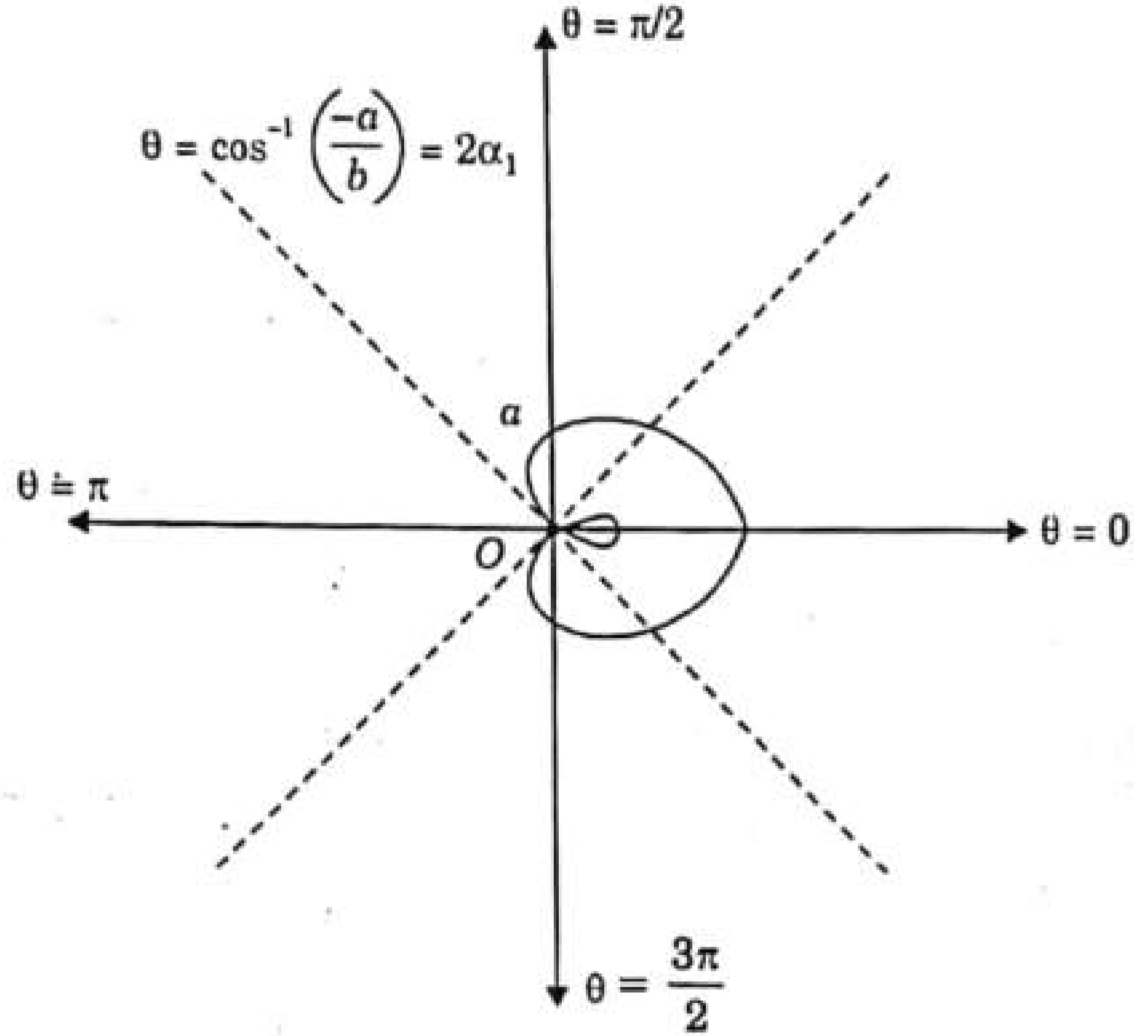
(6) Periodicity. In $r = a + b\cos\theta$, $\cos\theta$ is a periodic function with period 2π

(7) Points. On giving various values of θ into given curve, we can obtain values of r as

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\alpha_1 = \cos^{-1}\left(-\frac{a}{b}\right)$	π
r	$a+b$	$a+\frac{b}{\sqrt{2}}$	$a+\frac{b}{2}$	a	$a-\frac{b}{2}$	0	$a-b$

It is obvious to say that r decreases as θ increases and r is positive $(r = a)$ at $\theta = \frac{\pi}{2}$ and negative $(r = a - b)$ at $\theta = \pi$ so that $r = 0$ exist between $\theta = \frac{\pi}{2}$ and π.

Using above facts, the shape of the curve is given as



θ = π/2
θ = cos⁻¹(−a/b) = 2α₁
a
θ = π
O
θ = 0
θ = 3π/2

Exa. *Trace the following curve :*

$r^2 = a^2 \cos 2\theta$ *(Lemniscate of Bernoulli)*

Sol. $r^2 = a^2 \cos 2\theta$

(1) Symmetry. Given curve is symmetrical w.r.t. initial line $\theta = 0$ and line $\theta = \dfrac{\pi}{2}$ as on replacing θ by $-\theta$ and θ by $(\pi - \theta)$ respectively, the eqn. of curve remains unaltered.

Also the curve is symmetrical w.r.t. pole, as on replacing r by $-r$ the eqn. of curve remains unaltered.

(2) Nature of Pole

$$r = 0 \quad \Rightarrow \quad a^2 \cos 2\theta = 0$$

$$\Rightarrow \qquad \cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\Rightarrow \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$$

Since $\cos 2\theta$ is a periodic function with period 2π so values of θ which lie in between 0 to 2π are considered.

i.e.
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Thus, curve passes through pole and $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$ are two tangents at the pole.

(3) Intersection with the Line with $\theta = 0$ and $\theta = \dfrac{\pi}{2}$

$$r^2 = a^2 \cos 2\theta \text{ and } \theta = 0 \quad \Rightarrow \quad r = \pm a$$

and $$r^2 = a^2 \cos 2\theta \text{ and } \theta = \frac{\pi}{2} \quad \Rightarrow \quad r = \pm ia \text{ (imaginary values)}$$

Thus, curve intersects the line $\theta = 0$ at $r = \pm a$.

(4) Asymptotes. Given curve can be rewritten as

$$\frac{1}{r} = \pm \frac{1}{a\sqrt{\cos 2\theta}}$$

and $$\frac{1}{a\sqrt{\cos 2\theta}} = 0 \quad \not\Rightarrow \quad a \text{ finite value of } \theta$$

Thus, there doesn't exist any asymptote for the given curve.

(5) Region of Existence

Since $-1 \le \cos 2\theta \le 1$

or $-a^2 \le a^2 \cos 2\theta \le a^2$

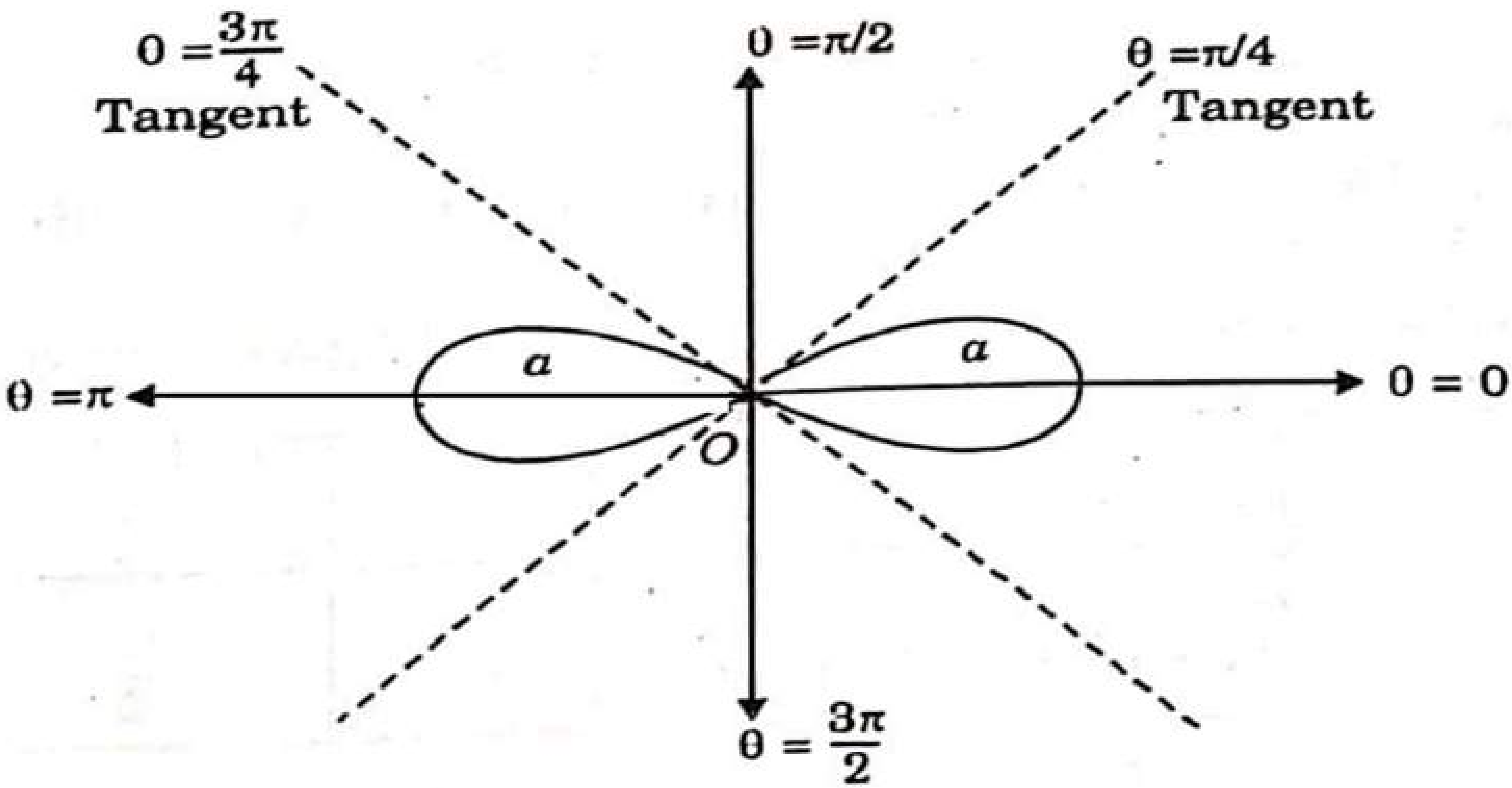
or $-a^2 \le r^2 \le a^2$.

(6) Points. On giving various values of θ into given curve, we can obtain values of r as

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	$\pm a$	$\pm \frac{a}{\sqrt{2}}$	0	*imag.*	*imag.*

It is obvious to say that r decreases as θ increases.

Using above facts, the shape of the curve is given as


θ = 3π/4
Tangent
θ = π/2
θ = π/4
Tangent
a
a
θ = π
θ = 0
O
θ = 3π/2

Exa. *Trace the following curve :*

$r = a(1 - \cos\theta)$ *(Cardioid)*

Sol. $r = a(1 - \cos\theta)$

(1) **Symmetry.** On replacing θ by $-\theta$, the eqn. of curve remains unaltered, so given curve is symmetrical w.r.t. initial line $\theta = 0$

(2) **Nature of Pole**

$$r = 0 \quad \Rightarrow \quad a(1 - \cos\theta) = 0$$

$$\Rightarrow \quad 1 - \cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = 1$$

$$\Rightarrow \quad \theta = 0$$

Thus, the curve passes through pole and $\theta = 0$ is the tangent at the pole.

(3) Intersection with the Line $\theta = 0$ and $\theta = \frac{\pi}{2}$

$$r = a(1 - \cos\theta) \text{ and } \theta = 0 \quad \Rightarrow \quad r = 0$$

and
$$r = a(1 - \cos\theta) \text{ and } \theta = \frac{\pi}{2} \quad \Rightarrow \quad r = a$$

Thus, curve intersects the line $\theta = 0$ at $r = 0$ and line $\theta = \frac{\pi}{2}$ at $r = a$

(4) Asymptotes. Given curve can be rewritten as

$$\frac{1}{r} = \frac{1}{a(1 - \cos\theta)}$$

and
$$\frac{1}{a(1 - \cos\theta)} = 0 \quad \not\Rightarrow \quad a \text{ finite value of } \theta$$

Thus, there doesn't exist any asymptote for the given curve.

(5) Region of Existence. Since $-1 \le \cos\theta \le 1$

or $\quad 1 \ge -\cos\theta \ge -1$

or $\quad 1+1 \ge 1-\cos\theta \ge 1-1$

or $\quad 2 \ge 1-\cos\theta \ge 0$

or $\quad 0 \le 1-\cos\theta \le 2$

or $\quad 0 \le a(1-\cos\theta) \le 2a$

or $\quad 0 \le r \le 2a.$

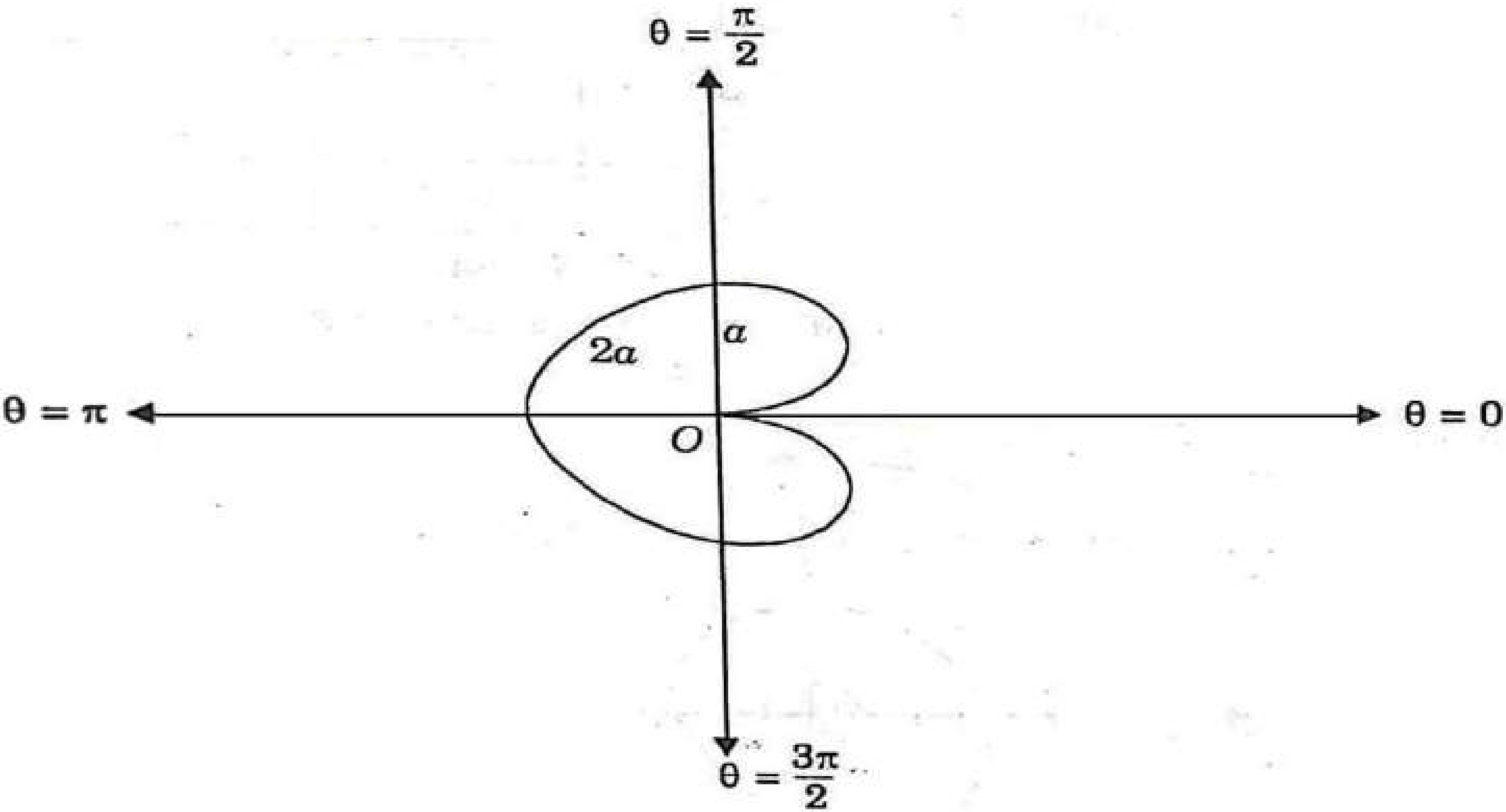
(6) Periodicity. In $r = a(1-\cos\theta)$, $\cos\theta$ is a periodic function with period 2π

(7) Points. On giving various values of θ into given curve, we can obtain values of r as

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a\left(1-\frac{1}{\sqrt{2}}\right)$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

It is obvious to say that r increases as θ increases.

Using above facts, the shape of the curve is given as



$\theta = \frac{\pi}{2}$
$\theta = \pi$
$\theta = 0$
$\theta = \frac{3\pi}{2}$
$2a$
a
O

Exa. *Trace the following curve :*

$r = ae^{m\theta}$ *(Equiangular spiral)*

Sol. $r = ae^{m\theta}$

(1) Symmetry. Curve is no where symmetrical.

(2) Nature of Pole. $r = 0 \quad \Rightarrow \quad ae^{m\theta} = 0$

$$\Rightarrow \quad e^{m\theta} = 0 \quad \Rightarrow \quad \theta = -\infty$$

Thus, curve doesn't passes through the pole.

(3) Intersection with the Line $\theta = 0$ and $\theta = \frac{\pi}{2}$

$$r = ae^{m\theta} \text{ and } \theta = 0 \quad \Rightarrow \quad r = a$$

and

$$r = ae^{m\theta} \text{ and } \theta = \frac{\pi}{2} \quad \Rightarrow \quad r = ae^{m\pi/2}$$

Thus, curve intersects the line $\theta = 0$ at $r = a$ and line $\theta = \frac{\pi}{2}$ at $r = ae^{m\pi/2}$

(4) **Asymptotes.** Given curve can be rewritten as

$$\frac{1}{r} = \frac{1}{a} e^{-m\theta}$$

and $$\frac{1}{a} e^{-m\theta} = 0 \not\Rightarrow \text{ a finite value of } \theta$$

Thus, there doesn't exist any asymptote for the given curve.

(5) **Region of Existence.** $r = ae^{m\theta}$ and $-\infty < \theta < \infty$

$\Rightarrow$ $$0 < r < \infty$$

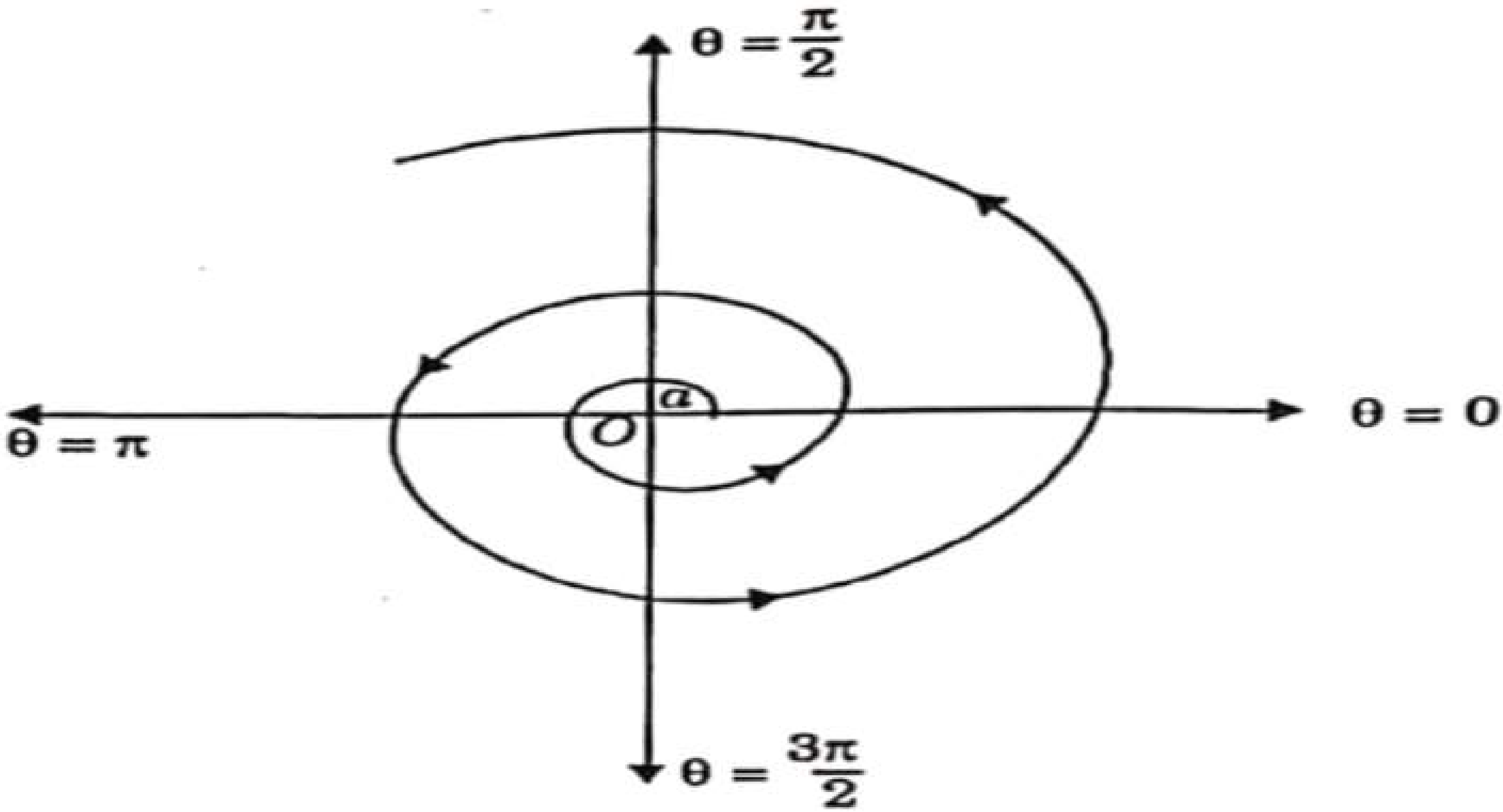
(6) **Periodicity.** There doesn't exist any periodic function in the given curve.

(7) Points. On giving various values of θ into given curve, we can obtain values of r as

θ	$-\infty$	0	$\frac{\pi}{2}$	π	2π	∞
r	0	a	$ae^{m\pi/2}$	$ae^{m\pi}$	$ae^{2m\pi}$	∞

It is obvious to say that r increases as θ increases.

Using above facts, the shape of the curve is given as



$\theta = \frac{\pi}{2}$
$\theta = 0$
$\theta = \pi$
$\theta = \frac{3\pi}{2}$
a
O

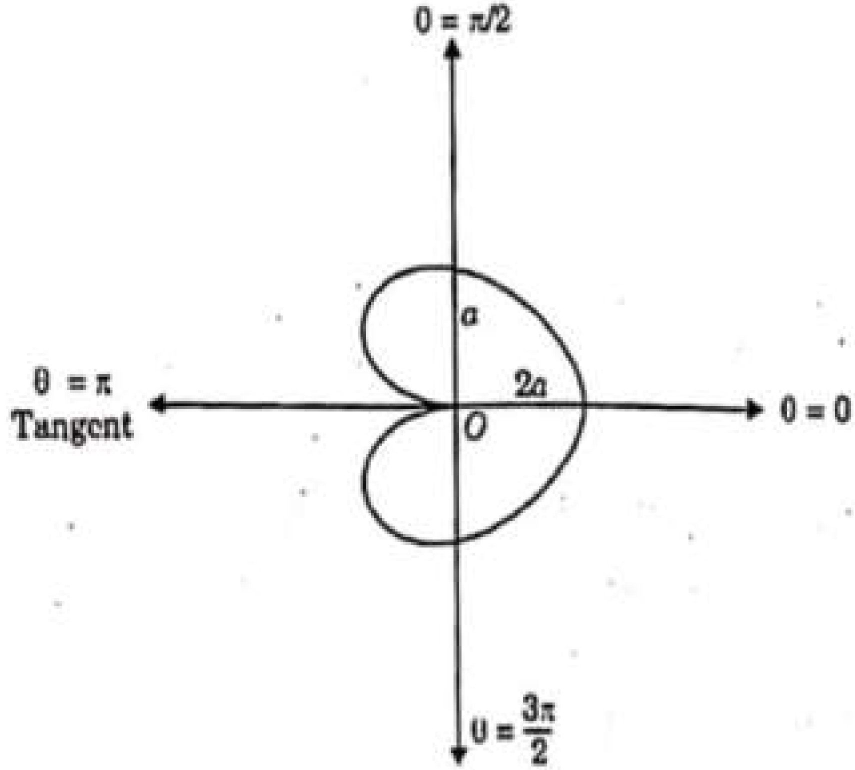
EXERCISE

Trace the following polar curves :

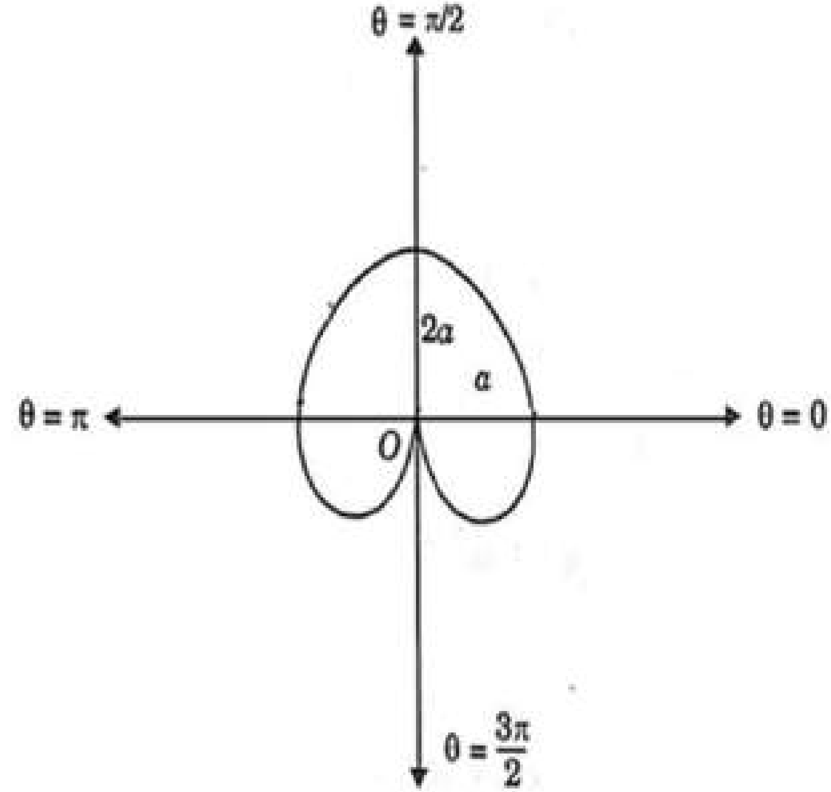
1. $r = a(1 + \cos\theta)$ (cardioid)
2. $r = a(1 + \sin\theta)$ (cardioid)
3. $r = a(1 - \sin\theta)$ (cardioid)
4. $r = a + b\cos\theta$; $a > b$ (limacon)

ANSWERS

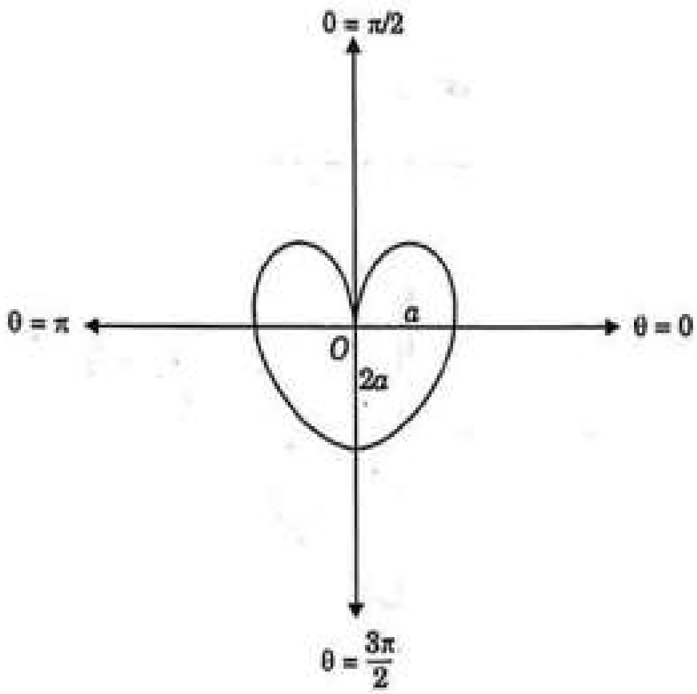
1.



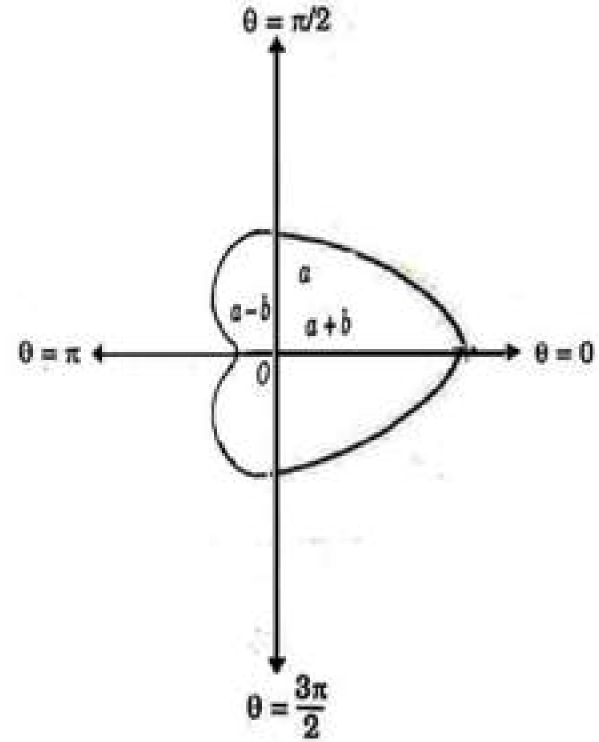
2.



3.



4.



THANKS