Differential Equation II

1. Partial Differentiation

In engineering problems, many quantities depend on more than one variable. Partial differentiation is the process of finding the rate of change of a function with respect to one variable while keeping the other variables constant.

Definition: If $z = f(x, y)$, then $\partial z/\partial x$ and $\partial z/\partial y$ are called partial derivatives.

Notation:

$\partial z/\partial x \rightarrow$ Partial derivative of z with respect to x (y constant)
$\partial z/\partial y \rightarrow$ Partial derivative of z with respect to y (x constant)

Example:

Let $z = x^2y + 3xy^2$.
$\partial z/\partial x = 2xy + 3y^2$
$\partial z/\partial y = x^2 + 6xy$

Higher Order Partial Derivatives:

Second and higher order partial derivatives are obtained by differentiating the first order derivatives again with respect to x or y. Example: $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$, $\partial^2 z/\partial x \partial y$.

2. Euler's Theorem on Homogeneous Functions

A function is said to be homogeneous if all its terms are of the same degree. Euler's theorem provides a simple relation between a homogeneous function and its partial derivatives.

Homogeneous Function:

A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^{\blacksquare} f(x, y)$.

Euler's Theorem Statement:

If $z = f(x, y)$ is a homogeneous function of degree n, then:
$x(\partial z/\partial x) + y(\partial z/\partial y) = nz$

Illustrative Example:

Let $z = x^2y + xy^2$. This is homogeneous of degree 3. By calculating $\partial z/\partial x$ and $\partial z/\partial y$ and substituting in Euler's formula, the theorem is verified.

3. Maxima and Minima of Functions of Two Variables

In engineering applications, it is often required to find the maximum or minimum value of a function of two variables such as cost, efficiency, stress, or temperature.

Stationary Points:

Points where both first order partial derivatives vanish are called stationary or critical points.

Procedure:

1. Find $\partial z/\partial x$ and $\partial z/\partial y$.
2. Equate them to zero to find stationary points.
3. Find second order derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$ and $\partial^2 z/\partial x \partial y$.

Second Derivative Test:

Let $D = f_{xx} f_{yy} - (f_{xy})^2$.
If $D > 0$ and $f_{xx} > 0 \rightarrow$ Minimum
If $D > 0$ and $f_{xx} < 0 \rightarrow$ Maximum
If $D < 0 \rightarrow$ Saddle Point

If D = 0 → Test fails

4. Lagrange's Method of Multipliers

Lagrange's method is used when we need to optimize a function subject to a constraint. This method avoids direct substitution and uses an auxiliary variable called a Lagrange multiplier.

Given:

Objective function: $f(x, y)$
Constraint: $g(x, y) = 0$

Lagrange's Conditions:

$\partial f/\partial x = \lambda \, \partial g/\partial x$
$\partial f/\partial y = \lambda \, \partial g/\partial y$
$g(x, y) = 0$

Steps:

1. Form equations using Lagrange's condition.
2. Solve for x, y, and λ.
3. Substitute values in $f(x, y)$ to obtain maximum or minimum value.