THE SUCCESS NOTES

PARTIAL DIFFERENTIATION

NOTES BY

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TOPIC INDEX

FORMATION OF PDE AND SOLUTION OF FIRST ORDER PDE

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INTRODUCTION

Introduction: In this chapter, we will form PDE by eliminating arbitrary constants or arbitrary function. Then we will solve various first order PDE.

FORMATION OF PDE BY ELIMINATING ARBITRARY CONSTANTS

FORMATION OF PARTIAL DIFFERENTIAL EQUATION BY THE ELIMINATION OF ARBITRARY CONSTANTS

Consider an equation

$$f(x, y, z, a, b) = 0 \qquad \text{...(1)}$$

where a and b are arbitrary constants.

Let z be a function of two independent variables x and y. Differentiating equation (1) partially with respect to x and y, we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \qquad \text{...(2)}$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \qquad \text{...(3)}$$

Eliminating two constants a and b from these equations (1), (2) and (3), we obtain a partial differential equation of the form

$$g(x, y, z, p, q) = 0 \qquad \text{...(4)}$$

Q. 1. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol. Differentiating the given equation partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2x + a^2z\frac{\partial z}{\partial x} = 0 \qquad \text{...(1)}$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2y + b^2z\frac{\partial z}{\partial y} = 0 \qquad \text{...(2)}$$

Now, differentiating (1) with respect to x, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \qquad \text{...(3)}$$

From equation (1), we have .

$$\Rightarrow \quad c^2 = -\frac{a^2z}{x}\frac{\partial z}{\partial x} \qquad \text{...(4)}$$

Putting this value of c^2 in (3) and dividing by a^2, we get

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

or $$zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \qquad \text{...(5)}$$

Q. 2. Find the differential equation of spheres, whose centres lie on the z-axis.

Sol. Equation of spheres whose centres lie on the z-axis will be

$$(x-0)^2+(y-0)^2+(z-c)^2=c^2$$

$$x^2+y^2+z^2-2cz=0$$

Differentiating the given equation partially with respect to x and y, we have

$$2x+2z\frac{\partial z}{\partial x}-2c\frac{\partial z}{\partial x}=0$$

or
$$2x+2zp-2cp=0 \qquad ...(1)$$

and
$$2y+2z\frac{\partial z}{\partial y}-2c\frac{\partial z}{\partial y}=0$$

or
$$2y+2zq-2cq=0 \qquad ...(2)$$

Equation (1) $\times q$ $\Rightarrow$ $2xq+2zpq-2cpq=0$ ...(3)

Equation (2) $\times p$ $\Rightarrow$ $2yp+2zpq-2cpq=0$ ...(4)

On subtracting the equations (3) and (4), we have

$$2(qx-py)=0$$

$$py-qx=0$$

FORMATION OF PDE BY ELIMINATING ARBITRARY FUNCTION

FORMATION OF PARTIAL DIFFERENTIAL EQUATION BY THE ELIMINATION OF ARBITRARY FUNCTION

Let u and v are two functions of x, y and z which are connected by the relation

$$\phi(u, v) = 0$$

or

$$u = \phi(v) \qquad \ldots(5)$$

Differentiating equation (5) partially with respect to x and y, we obtain

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(6)$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(7)$$

Here, z is considered as a function of x and y.

Now, eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between equations (6) and (7) by the determinant method, we get

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial x} + p\dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial y} + q\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial y} + q\dfrac{\partial v}{\partial z} \end{vmatrix} = 0$$

or
$$p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right) + q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)$$

or
$$\frac{\partial(u,v)}{\partial(y,z)}\,p + \frac{\partial(u,v)}{\partial(z,x)}\,q = \frac{\partial(u,v)}{\partial(x,y)} \qquad \ldots(8)$$

where $\dfrac{\partial(u,v)}{\partial(x,y)} = \left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}\right)$

Above equation (8) is generally denoted by

$$Pp + Qq = R \qquad \ldots(9)$$

where P, Q and R are functions of x, y and z.

Above PDE (9) is known as Lagrange's PDE

Q. 1. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Sol. $\frac{z - y^2}{2} = f\left(\frac{1}{x} + \log y\right)$

Let $u = \frac{z - y^2}{2}$

and $v = \frac{1}{x} + \log y$

Now, to eliminate the arbitrary function $u = f(v)$, we calculate P, Q and R as

$$P = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} = \frac{1}{2y} - 0 = \frac{1}{2y}$$

$$Q = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial z} = 0 + \frac{1}{2x^2} = \frac{1}{2x^2}$$

and $$R = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} = \frac{y}{x^2} - 0 = \frac{y}{x^2}$$

Hence, the required partial differential equation $pP + qQ = R$ is written as

$$p\left(\frac{1}{2y}\right) + q\left(\frac{1}{2x^2}\right) = \frac{y}{x^2}$$

or $$\frac{px^2 + qy}{2x^2 y} = \frac{y}{x^2}$$

$$px^2 + qy = 2y^2.$$

EXERCISE

Eliminate the arbitrary constants from the following equations and form partial differential equations

1. $(x-a)^2+(y-b)^2+z^2=c^2$ (eliminate only a and b)
2. $z=(x^2+a)(y^2+b)$

Eliminate the arbitrary function fro.n the following equations and form partial differential equations

3. $f(x+y+z, x^2+y^2-z^2)=0$
4. $lx+my+nz=\phi(x^2+y^2+z^2)$

ANSWERS

1. $z^2(p^2 + q^2 + 1) = c^2$

2. $4xyz = pq$

3. $(y + z)p - (z + x)q = x - y$

4. $(ny - mz)p + (lz - nx)q = (mx - ly)$

DIFFERENT SOLUTIONS OF FIRST ORDER PDE

Different types of solutions of a first order PDE

First order PDE: $f(x, y, z, p, q) = 0$; $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$

Complete solution: $g(x, y, z, a, b) = 0$

Using a given condition

Using both given conditions

Using complete solution with $\dfrac{\partial g}{\partial a} = 0$ and $\dfrac{\partial g}{\partial b} = 0$

General solution: $g[x, y, z, a, \phi(a)] = 0$

Particular solution: $u(x, y, z) = 0$

Singular solution: $v(x, y, z) = 0$

LAGRANGE’S PDE

WORKING PROCEDURE FOR SOLVING $Pp + Qq = R$

(1) Put the given linear partial differential equation of the first order in the standard form $Pp + Qq = R$.

(2) Form the auxiliary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(3) Solve above auxiliary equations by the method of multipliers or by the method of grouping or both to get two independent solutions $u = a$ and $v = b$, where a and b are arbitrary constants.

(4) The general solution of equation written in (1) is then written in one of the following three equivalent forms :

$$\phi(u, v) = 0 \quad \text{or } u = \phi(v) \quad \text{or } v = \phi(u)$$

Q. 1. **Solve:** $px - qy = 2x - z$

Sol. Here, $P = x$, $Q = -y$ and $R = 2x - z$

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{(2x - z)}$$

Taking the first two members, we get

$$\frac{dx}{x} = -\frac{dy}{y}$$

Integrating it, we have

$$\log x = -\log y + \log c_1$$

or $$xy = c_1$$

Again taking first and third members, we get

$$\frac{dx}{x} = \frac{dz}{2x - z}$$

or $$(2x - z)\,dx = x\,dz$$

or $$2\,x dx = (z\,dx + x\,dz)$$

or $$2\,x\,dx = d\,(xz)$$

Integrating it, we have

$$x^2 = xz + c_2$$

or $$x^2 - xz = c_2$$

Therefore the general integral is

$$f\,(xy,\, x^2 - xz) = 0.$$

Q. 2. **Solve:** $(y+z)p+(z+x)q=(x+y)$

Sol. *Here, $P = y + z$, $Q = z + x$ and $R = x + y$*

The subsidiary equations are

$$\frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)}$$

Taking 1, 1, 1 as multipliers, each fraction is equal to

$$= \frac{1.dx + 1.dy + 1.dz}{(y+z)+(z+x)+(x+y)}$$

$$= \frac{(dx+dy+dz)}{2(x+y+z)}$$

Taking 1, – 1, 0 as multipliers, each fraction equals

$$= \frac{dx-dy}{(y+z)-(z+x)} = \frac{(dx-dy)}{(y-x)}$$

Again taking 0, 1, – 1 as multipliers each fraction

$$= \frac{dy-dz}{(z+x)-(x+y)} = \frac{(dy-dz)}{(z-y)}$$

Combining all these three fractions, we get

$$\frac{(dx+dy+dz)}{2(x+y+z)} = \frac{(dx-dy)}{-(x-y)} = \frac{(dy-dz)}{-(y-z)}$$

Now taking the first two members, we have

$$\frac{(dx+dy+dz)}{(x+y+z)} + \frac{2(dx-dy)}{(x-y)} = 0$$

Integrating it, we get

$$\log(x+y+z) + 2\log(x-y) = \log c_1$$

or
$$(x+y+z)(x-y)^2 = c_1$$

Again taking the last two members, we have

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$\therefore$
$$\log(x-y) = \log(y-z) + \log c_2$$

or
$$\frac{x-y}{y-z} = c_2$$

Therefore the general integral is

$$f\left[(x-y)^2(x+y+z), \frac{x-y}{y-z}\right] = 0$$

Q. 3. **Solve:** $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol. *Here,* $P = x^2(y-z)$, $Q = y^2(z-x)$ *and* $R = z^2(x-y)$

The subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Taking $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, each fraction

$$= \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{(y-z)+(z-x)+(x-y)}$$

$$= \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$$

So, $$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating it, we get

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c$$

or $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$$

Again taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers each fraction

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{x(y-z) + y(z-x) + z(x-y)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{xy - xz + yz - yx + zx - zy}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

So, $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating it, we get

$$\log x + \log y + \log z = \log c_2$$

or $$xyz = c_2$$

Therefore the general integral is

$$f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$$

Q. 4. **Solve:** $(xz+y)p-(x+yz)q=(x^2-y^2)$

Sol. Here, $P = xz+y$, $Q = -(x+yz)$ and $R = x^2-y^2$

The subsidiary equations are

$$\frac{dx}{(xz+y)} = \frac{dy}{-(x+yz)} = \frac{dz}{(x^2-y^2)}$$

Taking $y, x, 0$ as multipliers each fraction

$$= \frac{y\,dx + x\,dy}{y(xz+y) - x(x+yz)} = \frac{y\,dx + x\,dy}{(y^2-x^2)}$$

So,
$$\frac{dx}{(xz+y)} = \frac{dy}{-(x+yz)} = \frac{dz}{(x^2-y^2)} = \frac{y\,dx + x\,dy}{(y^2-x^2)}$$

Taking last two members, we get

$$\frac{dz}{(x^2-y^2)} = \frac{-(y\,dx + x\,dy)}{(x^2-y^2)}$$

or
$$dz = -(y\,dx + x\,dy)$$

or
$$dz = -d(xy)$$

or
$$dz + d(xy) = 0$$

which on integration gives

$$z + xy = c_1$$

Taking $x, y, -z$ as multipliers each fraction

$$= \frac{x\,dx + y\,dy - z\,dz}{x\,(xz + y) + y\,(-x - yz) - z\,(x^2 - y^2)}$$

$$= \frac{x\,dx + y\,dy - z\,dz}{x^2z + xy - xy - y^2z - zx^2 + zy^2}$$

$$= \frac{x\,dx + y\,dy - z\,dz}{0}$$

$$\Rightarrow \qquad x\,dx + y\,dy - zdz = 0$$

Integrating it, we get

$$x^2 + y^2 - z^2 = c_2$$

Therefore the general integral is

$$f\,(xy + z,\, x^2 + y^2 - z^2) = 0$$

Q. 5. ***Solve:*** $z - px - qy = a\sqrt{(x^2 + y^2 + z^2)}$

Sol. $z - px - qy = a\sqrt{(x^2 + y^2 + z^2)}$

or $px + qy = z - a\sqrt{(x^2 + y^2 + z^2)}$

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{(x^2 + y^2 + z^2)}}$$

Taking x, y, z as mutlipliers, each fraction

$$= \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}$$

Putting $x^2 + y^2 + z^2 = u^2$

$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 2u\,du$

or $xdx + ydy + zdz = udu$

$$\Rightarrow \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}} = \frac{u\,du}{u^2 - auz} = \frac{du}{(u - az)}$$

So, $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{(z - au)} = \frac{du}{(u - az)} = \frac{(du + dz)}{(z + u)(1 - a)}$$

Taking first two members, we get

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating it, $\log y = \log x + \log c_1$

or $y = c_1 x$

Taking first and last terms, we have

$$\frac{dx}{x} = \frac{(du + dz)}{(u + z)(1 - a)} \Rightarrow (1-a)\frac{dx}{x} = \frac{(du+dz)}{(u+z)}$$

Integrating it $(1 - a) \log x = \log (u + z) + \log c_2$

or $x^{1-a} = (u + z)\, c_2$

or $x^{1-a} = \left[z + \sqrt{(x^2 + y^2 + z^2)}\right] c_2$

Therefore the general integral is

$$f\left(\frac{y}{x}, \frac{x^{1-a}}{z + \sqrt{(x^2 + y^2 + z^2)}}\right) = 0$$

Q.6. **Solve:** $p+3q=5z+\tan(y-3x)$

Sol. Here, $P=1$, $Q=3$ and $R=5z+\tan(y-3x)$

The subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}$$

Taking first two members, we get

$$\frac{dx}{1}=\frac{dy}{3}$$

or $$3dx=dy,$$

which on integration gives

$$y-3x=c_1$$

Using the above equation, first and third members of subsidiary equations are written as:

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1},$$

which on integration gives

$$e^{-5x}\left(5z + \tan c_1\right) = c_2$$

Using value of c_1, we get

$$e^{-5x}\left[5z + \tan\left(y - 3x\right)\right] = c_2$$

Therefore the general integral is

$$f\left[y - 3x, e^{-5x}\left\{5z + \tan\left(y - 3x\right)\right\}\right] = 0$$

EXERCISE

Solve the following differential equations :

1. $zp + yq = x$
2. $(mz - ny)\, p + (nx - lz)\, q = (ly - mx)$
3. $(x^2 - yz)\, p + (y^2 - zx)\, q = (z^2 - xy)$
4. $(x^3 + 3xy^2)\, p + (y^3 + 3x^2y)\, q = 2\,(x^2 + y^2)z$
5. $z\,(xp - yq) = y^2 - x^2$
6. $(x + 2z)\, p + (4\,zx - y)\, q = 2x^2 + y$

ANSWERS

1. $f\left(x^2 - z^2, \dfrac{x+z}{y}\right) = 0$

2. $f(x^2 + y^2 + z^2, lx + my + nz) = 0$

3. $f\left(\dfrac{x-y}{y-z}, \dfrac{z-x}{y-z}\right) = 0$

4. $\dfrac{xy}{z^2} = f\left[\dfrac{1}{(x+y)^2} - \dfrac{1}{(x-y)^2}\right]$

5. $f(xy, x^2 + y^2 + z^2) = 0$

6. $f(xy - z^2, x^2 - y - z) = 0$

NON-LINEAR PDE OF FIRST ORDER

NON-LINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FIRST ORDER

General form of any non-linear partial differential equation of first order can be given as

$$f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

Obviously it is a partial differential equation which contains p and q with degree higher than one and terms involving product of p and q.

As there are two independent variables (usually x and y) is partial differential equation (1), its complete solution involves two arbitrary constants.

There exist a general method of solution for partial differential equation (1). Before discussing that, we discuss some standard forms of equation (1) with different combinations of variables x, y, z, p and q which can be solved by specific methods.

STANDARD FORM - I

STANDARD FORM I

There are some partial differential equations in which x, y and z do not occur, so that those can be written as

$$\boxed{f(p, q) = 0} \quad \ldots (1)$$

Complete integral of equation (1) is given by

$$z = ax + by + c \quad \ldots (2)$$

where a, b and c are arbitrary constants and a, b are connected by the relation

$$f(a, b) = 0 \quad \ldots (3)$$

If $f(a, b) = 0$ reduces to $b = \phi(a)$, the complete integral of the given partial differential equation will be of the form

$$z = ax + \phi(a)\, y + c \quad \ldots (4)$$

Remark : Differentiating equation (4) partially with respect to a and c, we get

$$0 = x + \phi'(a)\, y$$

and $$0 = 1$$

As $0 = 1$ is absurd, hence singular solution does not exist for standard form I.

Q.1. *Solve:* $p = e^q$

Sol. $p = e^q$

The given equation is of the form $f(p, q) = 0$, hence its solution is given by $z = ax + by + c$,

where $f(a, b) = 0$

$f(a, b) = 0 \quad \Rightarrow \quad a = e^b$

i.e. $\quad b = \log a$

Hence, the complete integral is

$$z = ax + (\log a)y + c$$

Q. 2. *Solve*: $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$

Sol. $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ ...(1)

Substituting $x + y = X^2$

and $x - y = Y^2$, we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \left(\frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y}\right)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \left(\frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}\right)$$

then $p + q = \dfrac{1}{X}\dfrac{\partial z}{\partial X}$ and $p - q = \dfrac{1}{Y}\dfrac{\partial z}{\partial Y}$

Using these substitutions in equation (1), we get

$$X^2 \frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2 \frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

or
$$\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

or
$$P^2 + Q^2 = 1$$

where
$$P = \frac{\partial z}{\partial X} \quad \text{and} \quad Q = \frac{\partial z}{\partial Y}$$

which is a partial differential equation of standard form I.

Hence its complete solution is given by

$$z = aX + bY + c \,;\, a^2 + b^2 = 1$$

or
$$z = a\sqrt{(x+y)} + \sqrt{(1-a^2)}\sqrt{(x-y)} + c$$

Q. 3. *Solve :* $(y-x)(qy-px)=(p-q)^2$

Sol. $(y-x)(qy-px)=(p-q)^2$ ...(1)

Substituting $x+y=X$ and $xy=Y$, we get

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}=\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)$$

and
$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}=\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}\right)$$

Using these substitutions in equation (1), we get

$$(y-x)\left[\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}\right)y-\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)x\right]$$
$$=\left[\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)-\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}\right)\right]^2$$

or
$$(y-x)(y-x)\frac{\partial z}{\partial X}=(y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$$

or
$$\frac{\partial z}{\partial X}=\left(\frac{\partial z}{\partial Y}\right)^2$$

or
$$P=Q^2$$

where
$$P=\frac{\partial z}{\partial X} \text{ and } Q=\frac{\partial z}{\partial Y}$$

which is a partial differential equation of standard form I.

Hence, its complete solution is given by

$$z=aX+bY+c\ ;\ a=b^2$$

or
$$z=a(x+y)\pm\sqrt{a}\ xy+c$$

EXERCISE

Solve the following partial differential equations :

1. $\sqrt{p} + \sqrt{q} = 1$
2. $3p^2 = 2q^2 + 4pq$
3. $q = e^{-p/a}$

ANSWERS

1. $z = ax + (1 - \sqrt{a})^2 y + c$
2. $z = a[x + (\pm\sqrt{10} - 1)y] + c$
3. $z = bx + e^{-b/a} y + d$

STANDARD FORM - II

STANDARD FORM II

There are some partial differential equations in which x, y, z, p and q are combined specifically as

$$\boxed{z = px + qy + f(p, q)} \qquad \ldots(1)$$

Complete integral of equation (1) is given by

$$z = ax + by + f(a, b) \qquad \ldots(2)$$

where a and b are arbitrary constants.

Remark: For above standard form II, singular solution is to be provided in any problem.

Q. 1. *Solve*: $(p-q)(z-px-qy)=1$

Sol. $(p-q)(z-px-qy)=1$ ...(1)

or $z-px-qy=\dfrac{1}{(p-q)}$

or $z=px+qy+\dfrac{1}{(p-q)}$

The given equation is of the form

$$z=px+qy+f(p,q)$$

Hence its solution is given by

$$z=ax+by+f(a,b)$$

Therefore the complete integral of partial differential equation (1) is

$$z=ax+by+\frac{1}{(a-b)} \qquad \text{...(2)}$$

Now, differentiating (2) partially with respect to a and b **treating** x, y and z **as constants, we get**

$$0 = x - \frac{1}{(a-b)^2} \qquad \ldots(3)$$

and

$$0 = y + \frac{1}{(a-b)^2} \qquad \ldots(4)$$

Using (3) and (4), we get $x + y = 0$ i.e. $y = -x$, **which converts (2) in the following form:** $z = (a-b)x + \frac{1}{(a-b)}$, **in which we put value of** $(a-b)$ **from (3) and we get the singular solution as:**

$$z = \pm 2\sqrt{x}$$

Q. 2. Solve: $4\,xyz = pq + 2px^2y + 2qxy^2$

Sol. $4xyz = pq + 2p\,x^2y + 2q\,xy^2$ ...(1)

Substituting $x^2 = X$

and $y^2 = Y$, we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = 2x\frac{\partial z}{\partial X}$$

and

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = 2y\frac{\partial z}{\partial Y}$$

Using these substitutions in equation (1), we get

$$4xyz = 4xy\frac{\partial z}{\partial X}\frac{\partial z}{\partial Y} + 4x^3y\frac{\partial z}{\partial X} + 4xy^3\frac{\partial z}{\partial Y}$$

or

$$z = \frac{\partial z}{\partial X}\frac{\partial z}{\partial Y} + x^2\frac{\partial z}{\partial X} + y^2\frac{\partial z}{\partial Y}$$

or

$$z = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\frac{\partial z}{\partial Y}$$

or

$$z = PX + QY + PQ$$

where $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$

which is a partial differential equation of standard form II.

Hence its complete solution is given by

$$z = aX + bY + ab$$

or

$$z = ax^2 + by^2 + ab$$

Note: Also, find the singular solution from complete solution like previous problem.

EXERCISE

Solve the following partial differential equations :

1. $z = px + qy + pq$

2. $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$

3. $z = px + qy - 2\sqrt{pq}$

ANSWER

1. $z = ax + by + ab\,;\, z = xy$

2. $z = ax + by + c\sqrt{(1 + a^2 + b^2)}\,;\, x^2 + y^2 + z^2 = c^2$

3. $z = ax + by - 2\sqrt{ab}\,;\, xy = 1$

STANDARD FORM - III

STANDARD FORM III

There are some differential equations which do not contain x and y, so that those can be written as

$$\boxed{f(z, p, q) = 0} \qquad \ldots(1)$$

Remark : Singular solution may exist for equation (1)

WORKING PROCEDURE FOR SOLVING $f(z, p, q) = 0$

1. We have to put the values of p and q in the partial differential equation as $\frac{dz}{dX}$ and $a\frac{dz}{dX}$ respectively.
2. Solve the ordinary differential equation of first order in z and X.
3. Substitute $X = (x + ay)$ in the solution of above ordinary differential equation to find the complete integral of given partial differential equation.

Q. 1. *Solve:* $z^2(p^2+q^2+1)=c$

Sol. $z^2(p^2+q^2+1)=c$

The given equation is of the form $f(z,p,q)=0$

Taking $p=\dfrac{dz}{dX}$, $q=a\dfrac{dz}{dX}$,

where $X=(x+ay)$ and a is an arbitrary constant.

Therefore the given equation becomes

$$z^2\left[\left(\frac{dz}{dX}\right)^2+a^2\left(\frac{dz}{dX}\right)^2+1\right]=c$$

$$\Rightarrow \quad z^2\left(\frac{dz}{dX}\right)^2(1+a^2)+z^2=c \quad \Rightarrow \quad \left(z\frac{dz}{dX}\right)^2=\frac{(c-z^2)}{(1+a^2)}$$

$$\Rightarrow \quad z\frac{dz}{dX}=\frac{\sqrt{(c-z^2)}}{\sqrt{(1+a^2)}} \quad \Rightarrow \quad \frac{z}{\sqrt{(c-z^2)}}\,dz=\frac{1}{\sqrt{(1+a^2)}}\,dX$$

$$\Rightarrow \quad dX=\frac{\sqrt{(1+a^2)}\,z}{\sqrt{(c-z^2)}}\,dz,$$ which on integration gives

$$-\sqrt{(1+a^2)}\sqrt{(c-z^2)}=X+b$$

$$\Rightarrow \quad -\sqrt{(1+a^2)(c-z^2)}=(x+ay+b)$$

$$\Rightarrow \quad (1+a^2)(c-z^2)=(x+ay+b)^2$$

Q. 2. *Solve:* $9(p^2z + q^2) = 4$

Sol. $9(p^2z + q^2) = 4$

The given equation is of the form $f(z, p, q) = 0$

Taking $p = \dfrac{dz}{dX}$, $q = a\dfrac{dz}{dX}$,

where $X = (x + ay)$ and a is an arbitrary constant.

Therefore the given equation becomes

$$9\left[z\left(\frac{dz}{dX}\right)^2 + \left(a\frac{dz}{dX}\right)^2\right] = 4$$

$$\Rightarrow \quad 9\left(\frac{dz}{dX}\right)^2 (z + a^2) = 4 \quad \Rightarrow \quad \left(\frac{dz}{dX}\right)^2 = \frac{4}{9(z + a^2)}$$

$$\Rightarrow \quad \frac{dz}{dX} = \pm\frac{2}{3\sqrt{(z + a^2)}} \quad \Rightarrow \quad \pm\frac{3}{2}\sqrt{(z + a^2)}\, dz = dX,$$

which on integration gives

$$\pm\frac{3}{2}\frac{(z + a^2)^{3/2}}{\frac{3}{2}} = X + b$$

$$\Rightarrow \quad \pm(z + a^2)^{3/2} = x + ay + b$$

$$\Rightarrow \quad (z + a^2)^3 = (x + ay + b)^2$$

EXERCISE

Solve the following partial differential equations :

1. $c(p+q)=z$
2. $zpq = p + q$
3. $p(1+q) = qz$
4. $z^2p^2 + q^2 = p^2q$
5. $p^3 + q^3 = 27z$

ANSWERS

1. $x + ay + b = c\,(1 + a) \log z$
2. $x + ay + b = \dfrac{4\,az}{1 - a}$
3. $\log\,(az - 1) = (x + ay + b)$
4. $z = a \tan\,(x + ay + b)$
5. $(1 + a^3)\,z^2 = 8\,(x + ay + b)^3$

STANDARD FORM - IV

STANDARD FORM IV

Equations of the form $\boxed{f_1(x, p) = f_2(y, q)}$

i.e. partial differential equations in which the variable z does not appear as well as the terms containing p and x are separated from those containing q and y.

To solve such partial differential equations, we put each side equal to an arbitrary constant a

i.e. $\quad f_1(x, p) = f_2(y, q) = a$

Thus, we get $\quad p = F_1(x, a)$ and $q = F_2(y, a)$

Since $\quad dz = p\,dx + q\,dy$

We have, $\quad dz = F_1(x, a)\,dx + F_2(y, a)\,dy$

Integrating it, we get

$$z = \int F_1(x, a)\,dx + \int F_2(y, a)\,dy + b$$

which is the required complete integral of the given partial differential equation.

Remarks : (1) Partial differential equations in which functions of x, p and y, q are separated as well as z is contained but distributed in the same ratio with p and q are also solvable by the above method.

e.g. $z^2(p+q) = x + y$, $x^2y^3p^2q = z^3$ etc.

As $z^2(p+q) = x + y$ can be written as

$$pz^2 - x = y - qz^2$$

i.e. z is multiplied with p as well as with q

and $x^2y^3p^2q = z^3$ can be written as

$$\frac{x^2p^2}{z^2} = \frac{z}{y^3q}$$

i.e., $\frac{1}{z^2}$ is multiplied with p^2 as well as $\frac{1}{z}$ is multiplied with q.

(2) While solving a partial differential equation like

$$yzp^2 = q \quad \text{or } z^{2/3}p^2 = \frac{q}{yz^{1/3}},$$

z can not be distributed in the same ratio with p and q

i.e. $z^{2/3}$ is multiplied with p^2 but $z^{1/3}$ is not multilied with q.

(3) Partial differential equations of the form

$$y - p = f(x - q)$$

or $$x - q = \_\,(y - p)$$

can also be solved by the above method.

Assuming $x - q = a,\quad y - p = k\quad$ and $k = f(a)$

i.e. $p = y - k\quad$ and $q = x - a$

and $dz = pdx + q\,dy$ gives

$$dz = (y-k)\,dx + (x-a)\,dy$$

$$= (ydx + xdy) - kdx - ady$$

or $$\int dz = \int d(xy) - k\int dx - a\int dy$$

or $$z = xy - kx - ay + b$$

or $$z = xy - f(a)\,x - ay + b \qquad [\because k = f(a)]$$

(4) Singular solution may exist for above standard form.

Q. 1. *Solve :* $q(p - \cos x) = \cos y$

Sol. $q(p - \cos x) = \cos y$

Separating p and x from q and y, we get

$$p - \cos x = \frac{\cos y}{q}$$

Let $p - \cos x = a$ and $\dfrac{\cos y}{q} = a$

so that $p = (a + \cos x)$ and $q = \dfrac{\cos y}{a}$

Putting these in $dz = p\,dx + q\,dy$, we get

$$dz = (a + \cos x)\,dx + \frac{\cos y}{a}\,dy$$

which on integration gives

$$z = ax + \sin x + \frac{1}{a}\sin y + b$$

Q. 2. *Solve :* $q = xyp^2$

Sol. $q = xyp^2$

Separating p and x from q and y, we get $p^2x = \dfrac{q}{y}$

Let $p^2x = a$ and $\dfrac{q}{y} = a$

so that $p = \sqrt{\dfrac{a}{x}}$ and $q = ay$

Putting these in $dz = pdx + qdy$, we get

$$dz = \frac{\sqrt{a}}{\sqrt{x}}\,dx + ay\,dy$$

which on integration gives $z = 2\sqrt{ax} + \dfrac{ay^2}{2} + b$

Q. 3. *Solve :* $z^2(p^2+q^2)=x^2+y^2$

Sol. $z^2(p^2+q^2)=x^2+y^2$

Separating p and x from q and y, we get $z^2p^2-x^2=y^2-z^2q^2$

Let $z^2p^2-x^2=a^2=y^2-z^2q^2$

so that $p=\dfrac{\sqrt{(a^2+x^2)}}{z}$ and $q=\dfrac{\sqrt{(y^2-a^2)}}{z}$

Putting these in $dz=pdx+qdy$, we get

$$dz=\frac{\sqrt{(a^2+x^2)}}{z}\,dx+\frac{\sqrt{(y^2-a^2)}}{z}\,dy$$

or $$zdz=\sqrt{(a^2+x^2)}\,dx+\sqrt{(y^2-a^2)}\,dy$$

which on integration gives

$$\frac{z^2}{2}=\frac{x}{2}\sqrt{(a^2+x^2)}+\frac{a^2}{2}\log\left\{x+\sqrt{(x^2+a^2)}\right\}$$
$$+\frac{y}{2}\sqrt{(y^2-a^2)}-\frac{a^2}{2}\log\left\{y+\sqrt{(y^2-a^2)}\right\}+b$$

or $$z^2=x\sqrt{(a^2+x^2)}+y\sqrt{(y^2-a^2)}$$
$$+a^2\log\left\{\frac{x+\sqrt{(x^2+a^2)}}{y+\sqrt{(y^2-a^2)}}\right\}+b_1 \text{ where } b_1=2b$$

Q. 4. *Solve:* $(y-p) = (x-q)^2$

Sol. $(y-p) = (x-q)^2$

Taking $x - q = a$ and $y - p = b$, we get

$$p = y - b, \quad q = x - a \quad \text{and} \quad b = a^2$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (y-b)\,dx + (x-a)\,dy$$

or $$dz = (ydx + xdy) - bdx - ady$$

or $$dz = d\,(xy) - bdx - ady$$

which on integration gives, $z = xy - bx - ay + c$

or $$z = xy - a^2x - ay + c$$

Q. 5. *Solve :* $(x^2 + y^2)(p^2 + q^2) = 1$

Sol. $(x^2 + y^2)(p^2 + q^2) = 1$ ...(1)

Substituting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right)$$

Using these substitutions in (1), we get

$$r^2\left[\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right)^2 + \left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right)^2\right] = 1$$

or $$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2\right] = 1$$

or $$r^2\left(\frac{\partial z}{\partial r}\right)^2 = 1 - \left(\frac{\partial z}{\partial \theta}\right)^2$$

or $$r^2P^2 = 1 - Q^2$$

where $P = \dfrac{\partial z}{\partial r}$ and $Q = \dfrac{\partial z}{\partial \theta}$

which is a partial differential equation of standard form IV.

Hence, to find its complete solution, we assume

$$r^2P^2 = 1 - Q^2 = a$$

$$\Rightarrow \qquad P = \frac{\sqrt{a}}{r} \quad \text{and} \quad Q = \sqrt{(1-a)}$$

Thus, complete solution is given by

$$dz = Pdr + Qd\theta = \frac{\sqrt{a}}{r}\,dr + \sqrt{(1-a)}\,d\theta$$

which on integration gives,

$$z = \sqrt{a}\,\log r + \sqrt{(1-a)}\,\theta + b$$

or $$z = \sqrt{a}\,\log\sqrt{(x^2+y^2)} + \sqrt{(1-a)}\,\tan^{-1}\left(\frac{y}{x}\right) + b$$

EXERCISE

Solve the following partial differential equations :

1. $x(1+y)p = y(1+x)q$
2. $p^2 + q^2 = x + y$
3. $p^2 + q^2 = z^2(x+y)$
4. $z^2(p^2x^2 + q^2) = 1$
5. $x^2y^3p^2q = z^3$
6. $(x-q) = (y-p)^{1/3}$

ANSWERS

1. $z = a \log x + ax + a \log y + ay + b$

2. $z = \dfrac{2}{3}(x+a)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + b$

3. $\log z = \dfrac{2}{3}(x+a)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + b$

4. $\sqrt{(1+a^2)}\, z^2 = \pm 2(\log x + ay) - b$

5. $\log z = a \log x - \dfrac{1}{2a^2y^2} + b$

6. $z = xy - ax - a^{1/3} y + b$

GENERAL METHOD (CHARPIT'S METHOD)

GENERAL METHOD FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE BUT OF ANY DEGREE (CHARPIT'S METHOD)

This is a general method for solving partial differential equations of order one with two independent variables. Let the given equation be

$$\boxed{f(x, y, z, p, q) = 0} \quad \text{...(1)}$$

Now, if we are able to find another relation

$$g(x, y, z, p, q) = 0 \quad \text{...(2)}$$

containing x, y, z, p and q, then we can solve equations (1) and (2) for p and q and substitute in

$$dz = p\,dx + q\,dy \quad \text{...(3)}$$

Solution of equation (3), (if it exists) is the complete solution of given partial differential equation (1).

Remarks : (1) Since the solution by this method is generally more complicated, this method is applied to solve equations which can not be reduced to any of the standard forms as discussed before.

(2) This method is also applicable to solve Lagrange's linear partial differential equations of first order.

WORKING PROCEDURE WHILE USING CHARPIT'S METHOD

(1) Shift all the terms of the given partial differential equation to L.H.S. and consider the given partial differential equation as

$$f(x, y, z, p, q) = 0$$

(2) Write down the Charpit's auxilliary equation

$$\frac{dp}{\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} = \frac{dz}{\left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

(3) Substitute the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}$ and $\frac{\partial f}{\partial q}$ in the auxilliary equation written in step (2).

(4) Select two proper fractions from the auxilliary equation so that the resulting integral may come out to be the simplest relation containing atleast one of p and q.

(5) Solve the relation obtained in step (4) with given partial differential equation to find values of p and q in terms of x, y, and z.

(6) Substitute values of p and q in

$$dz = p\,dx + q\,dy$$

which on integration yields the complete solution of given partial differential equation.

Q. 1. *Solve:* $(p^2 + q^2)\, x = pz$

Sol. $(p^2 + q^2)\, x = pz$

Here $f = (p^2 + q^2)\, x - pz = 0$ ...(1)

Charpit's Auxiliary equations are

$$\frac{dp}{\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} = \frac{dz}{\left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

$$\Rightarrow \quad \frac{dp}{p^2 + q^2 - p^2} = \frac{dq}{-pq} = \frac{dz}{-2p^2x + pz - 2q^2x} = \frac{dx}{-2px + z} = \frac{dy}{-2qx} = \frac{df}{0}$$

From first two members, we have

$$\frac{dp}{q^2} = \frac{dq}{-pq} \quad \Rightarrow \quad p\,dp + q\,dq = 0$$

which on integrating gives

$$\frac{p^2}{2} + \frac{a^2}{2} = a \quad \Rightarrow \quad p^2 + q^2 = 2a \qquad \text{...(2)}$$

Using (2) in (1), we get $2ax = pz \quad \Rightarrow \quad p = \dfrac{2ax}{z}$ ...(3)

Putting the value of p in (2), we have $\dfrac{4a^2x^2}{z^2} + q^2 = 2a$

$$\Rightarrow \quad q = \sqrt{\left(2a - \frac{4a^2x^2}{z^2}\right)}$$

Putting these values of p and q in

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \quad dz = \frac{2ax}{z}\,dx + \sqrt{\left(2a - \frac{4a^2x^2}{z^2}\right)}\,dy$$

$$\Rightarrow \quad dz = \frac{2ax}{z}\,dx + \frac{\sqrt{(2az^2 - 4a^2x^2)}}{z}\,dy$$

$$\Rightarrow \quad z\,dz - 2ax\,dx = \sqrt{2a\,(z^2 - 2ax^2)}\,dy$$

$$\Rightarrow \quad \frac{(z\,dz - 2ax\,dx)}{\sqrt{(z^2 - 2ax^2)}} = \sqrt{2a}\,dy$$

$$\left[\begin{array}{l} \text{Put } (z^2 - 2ax^2) = t^2 \\ \Rightarrow \ (2z\,dz - 4ax\,dx) = 2t\,dt \\ \Rightarrow \ (z\,dz - 2ax\,dx) = t\,dt \end{array}\right]$$

$$\Rightarrow \quad \int \frac{t}{t}\,dt = \sqrt{2a}\int dy$$

$$\Rightarrow \quad t = y\sqrt{2a} + b$$

$$\Rightarrow \quad \sqrt{(z^2 - 2ax^2)} = (y\sqrt{2a} + b)$$

$$\Rightarrow \quad z^2 - 2ax^2 = (y\sqrt{2a} + b)^2$$

Q. 2. *Solve:* $p = (qy + z)^2$

Sol. $p = (qy + z)^2$

Here, $f = -p + (qy + z)^2 = 0$ ...(1)

Charpit's auxilliary equations are

$$\frac{dp}{\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} = \frac{dz}{\left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

$$\Rightarrow \quad \frac{dp}{2p(qy+z)} = \frac{dq}{4q(qy+z)} = \frac{dz}{(-p)(-1) - q \cdot 2y(qy+z)}$$

$$= \frac{dx}{1} = \frac{dy}{-2y(qy+z)} = \frac{df}{0}$$

$$\Rightarrow \quad \underset{(i)}{\frac{dp}{2p(qy+z)}} = \underset{(ii)}{\frac{dq}{4q(qy+z)}} = \underset{(iii)}{\frac{dz}{p - 2qy(qy+z)}} = \underset{(iv)}{\frac{dx}{1}} = \underset{(v)}{\frac{dy}{-2y(qy+z)}} = \frac{df}{0}$$

(i) and $(v) \Rightarrow \quad \frac{dp}{2p(qy+z)} = \frac{dy}{-2y(qy+z)}$

$$\Rightarrow \qquad \frac{dp}{p} + \frac{dy}{y} = 0$$

which on integration gives

$$\log p + \log y = \log a \qquad \Rightarrow \qquad py = a$$

$$\Rightarrow \qquad p = \frac{a}{y} \qquad \ldots(2)$$

Using (2) in (1), we get $\frac{a}{y} = (qy + z)^2$

$$\Rightarrow \qquad \sqrt{\frac{a}{y}} = (qy + z) \qquad \Rightarrow \qquad q = \left(\frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}\right) \qquad \ldots(3)$$

Putting these values of p and q in

$$dz = pdx + qdy \qquad \Rightarrow \qquad dz = \frac{a}{y}dx + \left(\frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}\right)dy$$

$$\Rightarrow \qquad ydz + zdy = adx + \frac{\sqrt{a}}{\sqrt{y}}dy \Rightarrow \qquad d(yz) = adx + \sqrt{\frac{a}{y}}\,dy$$

Integrating it, $yz = ax + 2\sqrt{ay} + b$

Q. 3. *Solve:* $yzp^2 = q$

Sol. $yzp^2 = q$

Here $f = yzp^2 - q = 0$ ...(1)

Charpit's auxiliary equations are

$$\frac{dp}{\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)} = \frac{dz}{\left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{df}{0}$$

$$\Rightarrow \frac{dp}{p^3y} = \frac{dq}{zp^2 + yp^2q} = \frac{dz}{(-2p^2yz + q)} = \frac{dx}{-2pyz} = \frac{dy}{1} = \frac{df}{0}$$

$$\Rightarrow \frac{dp}{p^3y} = \frac{dq}{p^2(z + qy)} = \frac{dz}{(q - yzp^2) - yzp^2} = \frac{dx}{-2pyz} = \frac{dy}{1} = \frac{df}{0}$$

$$\Rightarrow \underset{(i)}{\frac{dp}{p^3y}} = \underset{(ii)}{\frac{dq}{p^2(z + qy)}} = \underset{(iii)}{\frac{dz}{-yzp^2}} = \underset{(iv)}{\frac{dx}{-2pyz}} = \underset{(v)}{\frac{dy}{1}} = \frac{df}{0}$$

From (i) and (iii), we get $\dfrac{dp}{p^3y} = \dfrac{dz}{-yzp^2} \quad \Rightarrow \quad \dfrac{dp}{p} = -\dfrac{dz}{z}$

On integrating, we get $\quad \log p = -\log z + \log a$

$$\Rightarrow \qquad p = \frac{a}{z} \qquad \text{...(2)}$$

Equations (1) and (2) altogether provide $\quad q = yz\dfrac{a^2}{z^2}$

$$\Rightarrow \qquad q = \frac{a^2y}{z} \qquad \text{...(3)}$$

Substituting the values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{a}{z}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow \qquad \int zdz = a\int dx + a^2\int ydy$$

$$\Rightarrow \qquad \frac{z^2}{2} = ax + \frac{a^2y^2}{2} + b$$

$$\Rightarrow \qquad z^2 = 2ax + a^2y^2 + c \,;\; c = 2b$$

EXERCISE

Solve the following partial differential equations by Charpit's method:

1. $q = (px + z)^2$
2. $(p^2 + q^2)y = qz$
3. $px + qy = pq$
4. $(p + q)(px + qy) - 1 = 0$
5. $2zx - px^2 - 2qxy + pq = 0$
6. $p^2 + q^2 - 2px - 2qy + 2xy = 0$
7. $2(z + px + qy) = p^2y$
8. $pxy + pq + qy - yz = 0$

ANSWERS

1. $xz = ay + 2\sqrt{ax} + b$
2. $z^2 = a^2y^2 + (ax + b)^2$
3. $az = \dfrac{1}{2}(y + ax)^2 + b$
4. $z = \dfrac{2}{\sqrt{(1 + a)}}\sqrt{(ax + y)} + b$
5. $z = ay + b\,(x^2 - a)$
6. $$2z = x^2 + y^2 + ax + ay + \frac{1}{\sqrt{2}}\left[(x - y)\sqrt{(x - y)^2 - \frac{a^2}{2}} - \frac{a^2}{2}\log\left\{(x - y) + \sqrt{(x - y)^2 - \frac{a^2}{2}}\right\} + b\right]$$
7. $4y^3z = 4axy + 4\,by^2 - a^2$
8. $(z - ax)\,(y + a)^a = be^y$

THANKS