MA 224 (Real Analysis)

Hints/Solutions of Quiz-1 Problems

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1. Let $\mathbf{u} = (1,0)$, $\mathbf{v} = (0,1)$ and let $f(\mathbf{x}) = \mathbf{v} + \frac{3}{2}(\mathbf{x} - \mathbf{u}) = (\frac{3x}{2} - \frac{3}{2}, 1 + \frac{3y}{2})$ for all $\mathbf{x} = (x,y) \in \Omega_1$. If $\mathbf{x} \in \Omega_1$, then $\|f(\mathbf{x}) - \mathbf{v}\|_2 = \frac{3}{2}\|\mathbf{x} - \mathbf{u}\|_2 < 3$ and so $f(\mathbf{x}) \in \Omega_2$. Thus f maps Ω_1 to Ω_2 and clearly f is continuous (since both the component functions of f are continuous). Again, if $\mathbf{y} \in \Omega_2$, then $\mathbf{x} = \mathbf{u} + \frac{2}{3}(\mathbf{y} - \mathbf{v}) \in \mathbb{R}^2$ and $\|\mathbf{x} - \mathbf{u}\|_2 = \frac{2}{3}\|\mathbf{y} - \mathbf{v}\|_2 < 2$, *i.e.* $\mathbf{x} \in \Omega_1$, and also $f(\mathbf{x}) = \mathbf{y}$. Thus $f : \Omega_1 \to \Omega_2$ is onto. Therefore there exists a continuous function from Ω_1 onto Ω_2.

2. Let $f(x,y) = \frac{x^3+y^2}{x^2+y}$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 + y \neq 0$. We have $(\frac{1}{n}, 0) \to (0,0)$, $(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}) \to (0,0)$ as $n \to \infty$ and $f(\frac{1}{n}, 0) = \frac{1}{n} \to 0$, $f(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}) = 1 + \frac{1}{n}(\frac{1}{n} - 1)^2 \to 1$ as $n \to \infty$. Consequently $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

3. Since L is linear, $L(\mathbf{0}) = \mathbf{0}$. Let $\mathbf{x}(\neq \mathbf{0}) \in \mathbb{R}^n$. Then $\mathbf{x}_0 + \frac{r\mathbf{x}}{2\|\mathbf{x}\|_2} \in \mathbb{R}^n$ and $\|\mathbf{x}_0 + \frac{r\mathbf{x}}{2\|\mathbf{x}\|_2} - \mathbf{x}_0\|_2 = \frac{r}{2} < r$. By hypothesis, $L(\mathbf{x}_0 + \frac{r\mathbf{x}}{2\|\mathbf{x}\|_2}) = \mathbf{0} \Rightarrow L(\mathbf{x}_0) + \frac{r}{2\|\mathbf{x}\|_2}L(\mathbf{x}) = \mathbf{0} \Rightarrow L(\mathbf{x}) = \mathbf{0}$ (since $L(\mathbf{x}_0) = \mathbf{0}$). Therefore $L(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$.

4. Let $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ and $\Omega_2 = \{(x,y) \in \mathbb{R}^2 : xy < 0\}$. Then $f(x,y) = xy$ for all $(x,y) \in \Omega_1$ and $f(x,y) = -xy$ for all $(x,y) \in \Omega_2$. Since $f_x(x,y) = y$ and $f_y(x,y) = x$ for all $(x,y) \in \Omega_1$, we find that both $f_x : \Omega_1 \to \mathbb{R}$ and $f_y : \Omega_1 \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of Ω_1. By a similar argument, we can show that f is differentiable at every point of Ω_2. If $\alpha(\neq 0) \in \mathbb{R}$, then $f_y(\alpha, 0) = \lim\limits_{t\to 0} \frac{f(\alpha,t) - f(\alpha,0)}{t} = \lim\limits_{t\to 0} \frac{|\alpha||t|}{t}$ does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = 0$, $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = 0$ and $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|h||k|}{\sqrt{h^2+k^2}} = 0$ (since $|h||k| \leq h^2 + k^2$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\} \cup \{(0,0)\}$.