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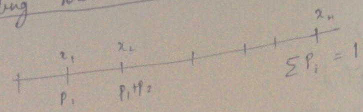
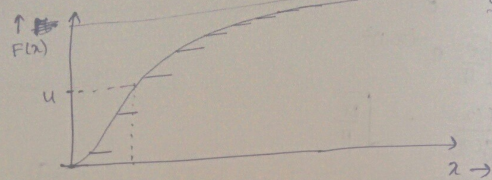
Simulating Discrete Distribution



$$P\{X = x_j\} = p_j = P\left(\sum_{i=1}^{j-1} p_i < U < \sum_{i=1}^{j} p_i\right)$$

$$\left(= \sum_{i=1}^{j-1} p_i - \sum_{i=1}^{j} p_i = p_j \right)$$



Step function

$$u = F_n(x) = \begin{cases} 0 & x < x_1 \\ 1/n & x_{(1)} < x < x_{(2)} \\ 2/n & x_{(2)} < x < x_{(3)} \\ \vdots & \\ 1 & x \geq x_{(n)} \end{cases}$$

(Empirical Dist^n)

Generate uniform u, invert $u = F_n(x)$

Take $F^{-1}(u) = \inf\{x : F(x) \geq u\}$

$$= \text{Greatest integer}\underbrace{(nU)}_{x_{(\ )}} + \underbrace{0}_{x_0}$$

Class missed: Box-Muller & Marsaglia-Bray

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Generating Geometric random variable (discrete)

no. of trials for first success

$P[X = i] = pq^{i-1}$, $i = 1, 2, \ldots$

$P(X \leq x) = \sum_{i=1}^{[x]} P[X = i] = 1 - q^{[x]}$

$x \in \mathbb{R}$

$$\boxed{X = \text{Int}\left(\frac{\log U}{\log q}\right) + 1}$$

Proof: Generate U

$P[X \leq j-1] \leq U \leq P[X \leq j]$

$\Rightarrow 1 - q^{j-1} \leq U \leq 1 - q^j$

$\Rightarrow q^j \leq 1 - U \leq q^{j-1}$ $\Rightarrow$ So min j ~~which~~ such that $q^j < 1-U$ is the reqd. X

$X = \min\{j \mid q^j < 1 - U\}$

Poisson Distribution (discrete)

$$\frac{p_{i+1}}{p_i} = \frac{\dfrac{e^{-\lambda}\lambda^{i+1}}{\lfloor i+1}}{\dfrac{e^{-\lambda}\lambda^i}{\lfloor i}} = \frac{\lambda}{i+1}$$

$$\Rightarrow p_{i+1} = p_i \times \frac{\lambda}{i+1}$$

Using this we recursively calculate all probabilities

Binomial Dist^n (discrete)

$$\frac{p_{i+1}}{p_i} = \frac{n-i}{i+1} \cdot \frac{p}{1-p}$$

$$\underbrace{P[X \leq j-1]}_{p_1 + \ldots + p_{j-1}} < U \leq \underbrace{P[X \leq j]}_{p_1 + \ldots + p_j}$$

Bernoulli r.v

$u \sim unif(0,1)$

$u < p \Rightarrow x = 1$

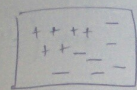
o/w $\Rightarrow x = 0$

Binomial is summation of Bernoulli

$X_{Bn} = X_1 + X_2 + \dots + X_n$

Composition of Distributions

X



2 types of population

$X = \alpha X_1 + (1-\alpha) X_2$

↳ This is a mixture of populations not a mixture distribution So it is incorrect.

$P(X = x_i) = \alpha p_j^{(1)} + (1-\alpha) p_j^{(2)}$

$X = [X_1, X_2] \rightarrow$ To generate

$u_1, u_2 \sim unif(0,1)$

$u_1 < \alpha$

$X_1 = \frac{1}{\theta_1}[\ln(1-u_2)]^{1/\beta_1}$

$u_1 > \alpha$

$X_2 = \frac{1}{\theta_2}[\ln(1-u_2)]^{1/\beta_2}$

$\Rightarrow X = [X_1, X_2]$

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Multivariate Normal

$(X_1, \dots, X_d) \sim N_d(\mu, \Sigma)$

$(z_1, \dots, z_n) \sim N_n(\underline{0}, I)$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & 0 \\ 0 & & \ddots & \\ 0 & & 0 & 1 \end{pmatrix}\right)$$

$z_i \overset{iid}{\sim} N(0,1)$ all z_i independent

$z_i \overset{iid}{\sim} N(0,1)$

$$f(\underline{z}) = \frac{1}{(2\pi)^{n/2}} \prod_{i=1}^{n} e^{-z_i^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{z'z}{2}}$$

$$\underset{\sim}{X}^d \sim N(\underline{\mu}^{d\times 1}, \Sigma^{d\times d}) \Rightarrow f(\underline{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})}$$

d-dimension of r.v

$\boxed{\Sigma = \text{Variance}}$

Σ^{-1} = inverse of var matrix

Σ - is the var. covariance matrix

$z_i \sim N(0,1)$

$x_i \sim N(\mu, \sigma^2)$

$\Rightarrow x_i = \mu + \sigma z_i$

$$\boxed{\underset{\sim}{X} = \Sigma^{1/2} \underline{z} + \underline{\mu}}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}, \quad \lambda^{1/2} =$$

Spectral decomposition, $\Sigma = P \Lambda P'$, $PP' = I$

$$= \underbrace{P \Lambda^{1/2} P'}_{} \underbrace{P \Lambda^{1/2} P'}_{\text{same}}$$

$\Rightarrow \Sigma^{1/2} = P \Lambda^{1/2} P'$

Jacobian $= \left|\frac{\partial(old)}{\partial(new)}\right| = |\Sigma^{-1/2}| = |\Sigma|^{1/2}$

$x = |\Sigma|^{1/2} z + \mu$

$\frac{x - \mu}{|\Sigma|^{1/2}} = z$

Algorithm:

$$(X_1, X_2) \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

$X_1 \sim N(\mu_1, \sigma_1^2)$

Moment generating fn

$$M_{\underset{\sim}{x}}(\underline{t}) = E(e^{\underline{t}'\underset{\sim}{x}}) = e^{\underline{t}'\underline{\mu} + \underline{t}'\Sigma\underline{t}}$$

Proof:

$$\underset{\sim}{x} = \Sigma^{1/2}\underline{z} + \underline{\mu}$$

$$E(e^{\underline{t}'\underline{z}}) = E(e^{\sum t_i z_i}) = E\left(\prod e^{t_i z_i}\right)$$

$$= \prod E(e^{t_i z_i}) = e^{\frac{1}{2}\underline{t}'\underline{t}}$$

$$M_{\underset{\sim}{x}}(\underline{t}) = E(e^{\underline{t}'\underline{x}}) = E(e^{\underline{t}'(\Sigma^{1/2}\underline{z} + \underline{\mu})})$$

For $\overset{m\times n}{A}\overset{n\times 1}{X} + \overset{m\times 1}{b}$

$$E(e^{\underline{t}'(A\underline{X} + b)}) = E(e^{\overset{t'A}{t^{*\prime}}\underline{X}})\, e^{\underline{t}'\underline{b}}$$

$$= e^{\underline{t}'A\underline{\mu} + \frac{1}{2}\underline{t}'A\Sigma A'\underline{t} + \underline{t}'\underline{b}}$$

$$= e^{\underline{t}'(A\underline{\mu} + b) + \frac{1}{2}\underline{t}'A\Sigma A'\underline{t}}$$

So $AX + b \sim N(A\underline{\mu} + b,\ A\Sigma A')$

$X = (X_1, X_2)$

We can find a a' s.t. $a'X = X_1 \sim (\mu_1, \sigma_1^2)$

★ $(X_2 \mid X_1) \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1),\ \sigma_2^2(1-\rho^2)\right)$

conditional distn

To generate a vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ on bivariate normal—

Generate x_1 from X_1

Generate x_2 from X_2 given X_1

(For third x_3, generate from $X_3 \mid \underbrace{X_1, X_2}_{\text{vector generated from above 2 steps}}$

$$X = \Sigma^{1/2}\underbrace{\underline{Z}}_{\text{d independent normal dist}^{\text{n}}\text{s}} + \mu$$

From Cholesky's Decomposition—

Σ can be decomposed into $\Sigma = AA'$, A is lower triangular, A' is upper triangular

$$\Rightarrow \underbrace{\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} & \\ & \end{pmatrix}}_{A'} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A_{11}^2 & A_{11}A_{12} \\ A_{21}A_{11} & A_{21}^2 + A_{22}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$A_{11} = \sigma_1, \quad A_{12} = \rho\sigma_2, \quad A_{22} = \sqrt{1-\rho^2}\,\sigma_2$$

$$X_1 = A_{11}Z_1 + \mu$$

$$\Rightarrow X_1 = \sigma_1 Z_1 + \mu$$

$$X_2 = \mu_2 + A_{12}Z_1 + A_{22}Z_2$$

$$X_2 = \mu_2 + \rho\sigma_2 Z_1 + \sqrt{1-\rho^2}\,\sigma_2 Z_2$$

$$\Rightarrow X_2 = \mu_2 + \rho\sigma_2 Z_1 +$$

$$\underset{\sim}{X} = \begin{pmatrix} \overset{m\times 1}{\underset{\sim}{X_1}} \\ \underset{\sim}{X_2}^{\,n-m\times 1} \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \overset{m\times m}{\Sigma_{11}} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$X_1 \sim N(\mu_1, \sigma_1^2)$$

Result:- If $\overset{\text{cov.}}{\Sigma_{12}} = 0 \iff \underline{X}_1, \underline{X}_2$ are independent
(for normal distn)

Proof: $t = (t_1^{m\times 1}\ t_2^{(n-m)\times 1})$

$$M_{\underset{\sim}{X}}(\underline{t}) = e^{\underline{t}_1'\underline{\mu}_1 + \underline{t}_2'\underline{\mu}_2 + \frac{1}{2}(t_1'\Sigma_{11}t_1 + t_1'\cancel{\Sigma_{12}}\,\underline{t}_2 + t_2'\cancel{\Sigma_{21}}\,\underline{t}_1 + \underline{t}_2'\Sigma_{22}\underline{t}_2)}$$

$$M_{\underset{\sim}{X}}(\underline{t}) = M_{\underset{\sim}{X_1}}(t_1)\, M_{\underset{\sim}{X_2}}(t_2)$$

Since $M_{\underset{\sim}{X}}$ can be splitted into mom. gen. funcs of X_1 & X_2, so X_1 & X_2 are independent.

* $(\underset{\sim}{X_2} \mid \underset{\sim}{X_1}) \sim N\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1),\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$

Proof: $W = X_2 - \underbrace{\Sigma_{21}\Sigma_{11}^{-1}X_1}$

subtracting from X_2 so that W ~~gets~~ becomes indep. of X_1 (to be proved)

$W \mid X_1 \equiv X_2 - \Sigma_{21}\Sigma_{11}^{-1}x_1 \mid \underset{\sim}{X_1}$

$\cancel{\begin{pmatrix} X_1 \\ W \end{pmatrix} = \begin{pmatrix} I_m & -\Sigma_{21}\Sigma_{11}^{-1} \\ 0 & I_p \end{pmatrix}\begin{pmatrix} X \\ X_1 \end{pmatrix}}$

$$\begin{pmatrix} X_1 \\ W \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_p \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$\begin{pmatrix} X \\ W \end{pmatrix} = AX \sim N(A\mu, A\Sigma A')$

↳ an A can be found

$$\begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \underbrace{\Sigma_{22} - \Sigma_{12}\Sigma_{11}^{-1}\Sigma_{12}}_{\Sigma^{**}} \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_p \end{pmatrix}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

cov are 0 ⇒ W & X_1 are indep

$W \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}\mu_1, \Sigma^{**})$

$W \mid X_1 \approx W \sim (\mu_2 \cancel{+} \Sigma_{21}\Sigma_{11}^{-1}\mu_1, \Sigma^{**})$

$\sim N\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1),\ \Sigma^{**}\right)$

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Variance Reduction Techniques

Application of MC: approximation of integrals

$$E(h(x)) = \int_{-\infty}^{\infty} h(x)\, p(x)\, dx \approx \frac{1}{n}\sum_{i=1}^{n} h(X_i) \rightarrow \text{Estimate}$$

from Weak Law of Large Nos. (WLLN)

Estimator: a function of samples, with no parameter involved

$E(h(x))$ is a function of some parameters θ of X but the estimator estimates the parameters

Confidence interval :-

$\{x_1, \dots, x_n\}$ - observations

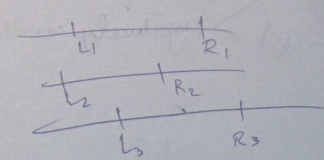
$100(1-\alpha)\%$ CI for $\theta \in (L, R)$

(L,R) includes θ with prob. $(1-\alpha)$

ex. $N(\mu, 2)$

↳ Conf. Int. for μ

Generate samples again & again



$\hat{\theta}_n = \frac{1}{n}\sum Y_j \rightarrow \theta$ (estimate of θ)

estimate's error $= E(\hat{\theta}_n - \theta)^2$ $\left(\overset{?}{=} Var(\hat{\theta}_n)\right)$

↳ function of θ

If $Y_j \to \theta$, $E(Y_j) = \theta \quad \forall j$

Then $\hat{\theta}_n \doteq \theta$ because $\left[E\left(\frac{1}{n}\sum Y_j\right) = \frac{1}{n}\sum E(Y_j) = \theta \right]$

is an estimator of

$\hat{\theta}$ be an estimator

$Var(\hat{\theta})$ = error for estimating θ

Some var. reduction techniques:—

1) Antithetic variables

$\bar{Y}_1 \xrightarrow{\text{is an estimator of}} \theta$

$\bar{Y}_2 \longrightarrow \theta$

Then $\frac{1}{2}(\bar{Y}_1 + \bar{Y}_2) \longrightarrow \theta$

Method ~~generally~~ $\bar{Y}_1 = \frac{1}{n}\sum \bar{Y}_i U_i$

$\bar{Y}_2 = \frac{1}{n}\sum \bar{Y}_i (1-U_i)$.

ex. $X \sim exp(\theta)$, $E(X) = \theta$

$x = -\frac{1}{\theta}\ln U_i$
$x = -\frac{1}{\theta}\ln(1-U_i)$ > Negatively correlated

Example

$\theta = E(e^U) = \int_0^1 e^U dU$

$U \sim unif(0,1)$

Antithetic estimator $T = \frac{1}{2}(e^U + e^{1-U})$

Calculate $\frac{V(T) - V(e^U)}{V(e^U)}$

$\int_0^\infty x \underbrace{\theta e^{-\theta x}}_{p(x)} dx = \theta$

$= \frac{1}{\theta}\int_0^\infty y e^{-y} dy$

2) Control Variates

Estimate $\theta = E(X)$

Suppose Y, $E(Y) = \mu_Y$

$W = X + c(Y - \mu_Y)$ will also be an estimator of θ

(choose c)

Amt. of reduction of var $= \frac{Var(W) - Var(X)}{Var(X)}$

ex. estimate $\theta = E(e^U)$

Take $W = e^U + c(\underbrace{U - \tfrac{1}{2}}_{Y})$

Antithetic variable is a special case of control variate.

3) Conditioning with another r.v.

$E_Y E(X|Y) = E(X) = \theta$

ex

To estimate $\theta = P(X > 1)$

$\theta = E(I(X>1))$

where $I(X > ~~2~~1) = \begin{cases} 1, & x > 1 \\ 0, & x \le 1 \end{cases}$

$E\left(\frac{1}{n}\sum I_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(I_i) = \theta$

$\Rightarrow \frac{1}{n}\sum I_i \xrightarrow{\text{estimator}} \theta$

$E(I) = E_Y E(I|Y)$

$E(I|Y) = P[X>1 | Y=y] = 1 - \Phi\left(\frac{1-y}{2}\right)$

exp. r.v. mean 1 ; CDF

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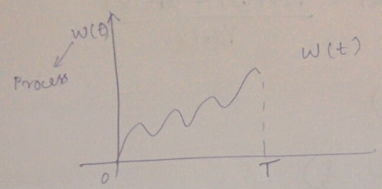
generating Stochastic Processes

$X_t : \Omega \times T \longrightarrow \mathbb{R}$

Brownian motion – a stochastic process



$W(t)$ is cont^s wrt t

$W(0) = 0$

$W(t) - W(s) \sim N(0, t-s)$

$\Rightarrow W(t) - W(0) \sim N(0,t)$

$\Rightarrow \boxed{W(t) \sim N(0,t)}$ → $W(t)$ is standard Brownian motion

$\Rightarrow \frac{1}{\sqrt{t}} W(t) \sim N(0,1)$ —

$X(t) = \mu t + \sigma W(t)$ — Brownian motion with drift μ & diffusion coeff. σ^2

mean of $X(t) = \mu t$

var of $X(t) = \sigma^2 t$

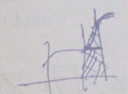
$\Rightarrow X(t) \sim N(\mu t, \sigma^2 t)$

$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \cdot Z_{i+1}$ ↳ std. norm

$\left(\frac{W(t_{i+1}) - W(t_i)}{\sqrt{t_{i+1} - t_i}} = Z_{i+1} \right)$

$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\, Z_{i+1}$

(from above)



13/4/16 modelling X_t using $W(t)$

$dX_t = \mu dt + \sigma dW(t)$

$X_0 = 0$

$X_t \sim N(\mu t, \sigma^2 t)$

Modelling price using $W(t)$ has a prob coz. $S(t)$ cannot be -ve

$r_t = \frac{\Delta X_t}{X_t} = \frac{X_{t+h} - X_t}{X_t} \sim N(\mu t, \sigma^2 t)$

return of stock

So we model $\frac{dX_t}{X_t}$ using BM (as return can be -ve)

(Here X_t is $\ln(S(t))$)

~~$H(X_t, t)$~~

$\frac{dX_t}{X_t} = \mu dt + \sigma dW(t)$

$\Rightarrow dX_t = \mu X_t dt + \sigma X_t dW(t)$

$$dH(X_t,t) = \frac{\partial H}{\partial X}dX + \frac{\partial H}{\partial t}dt + \frac{1}{2}\frac{\partial^2 H}{\partial X^2}(dX)^2 + \frac{\partial^2 H}{\partial X \partial t}dX\,dt + \frac{1}{2}\frac{\partial^2 H}{\partial t^2}(dt)^2$$

Approximations :-

$(dt)^2 \approx 0$, $dW(t)\,dt \approx 0$, $\underbrace{(dW(t))^2 = dt}_{\text{as in stocks } W(t+h)-W(t)=\sqrt{h}}$

$$\Rightarrow \underset{\nabla X_t}{\underset{\|}{X_{t+h} - X_t}} \approx \underbrace{\mu X_t}_{a_{x,t}} \nabla t + \underbrace{\sigma X_t}_{b_{x,t}} \sqrt{\nabla t}\, Z$$

$$dH(X_t,t) = \frac{\partial H}{\partial X}a_{x,t}dt + \frac{\partial H}{\partial t}dt + \frac{1}{2}\frac{\partial^2 H}{\partial X^2}(\nabla X_t)^2 + \frac{\partial H}{\partial X}b_{x,t}\sqrt{dt}\,Z$$

$$(\nabla X_t)^2 = \cancel{a_{x,t}^2(\nabla t)^2}^{\,0} + b_{x,t}^2 \nabla t Z^2 + \cancel{2a_{x,t}(\nabla t)^{3/2} Z b_{x,t}}^{\,0}$$

$E(Z^2) = 1$ as $Z \sim N(0,1)$

$$\Rightarrow (\nabla X_t)^2 = b_{x,t}^2 \nabla t$$

$$☆ \Rightarrow dH(X_t,t) = \frac{\partial H}{\partial X}a_{x,t}\nabla t + \frac{\partial H}{\partial t}\nabla t + \frac{1}{2}\frac{\partial^2 H}{\partial X^2}\nabla t\, b_{x,t}^2 + \frac{\partial H}{\partial X}b_{x,t}\sqrt{\nabla t}\,Z$$

Ito's Formula

$$\Rightarrow dH = \left(\frac{\partial H}{\partial X}a_{x,t} + \frac{\partial H}{\partial t} + \frac{1}{2}b_{x,t}^2\frac{\partial^2 H}{\partial X^2}\right)dt + b_{x,t}\frac{\partial H}{\partial X}dW(t)$$

$H(X_t, t) = \ln X_t$

$$d\log X_t = \left(\frac{1}{X_t}\mu X_t + 0 - \frac{1}{2}\sigma^2\frac{X_t^2}{X_t^2}\right)dt + \sigma X_t \frac{1}{X_t}d(W(t))$$

$$\Rightarrow \boxed{d\log X_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)}$$

$$\therefore \Updownarrow$$

$$\boxed{dX_t = \mu X_t dt + \sigma X_t dW(t)}$$

Generate Geometric Brownian path :-

$$\log X_{t_i} = \log X_{t_{i-1}} + \left(\mu - \frac{1}{2}\sigma^2\right)\nabla t + \sigma\sqrt{\nabla t}\,Z$$

where $\nabla t = t_i - t_{i-1}$

$$\therefore \log X_T \sim N\left(\log X_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T,\ \sigma^2 T\right)$$

$$X_T = X_0\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W(T)}$$

$\log X_T \sim N(\mu^*, \sigma^*)$, $\quad Z = \dfrac{\log X_T - \mu^*}{\sigma^*}$

$$\Rightarrow E(X_T) = e^{\mu_*}\underbrace{E(e^{\sigma^* Z})}_{\text{moment gen. func.}}$$

$$\Rightarrow \boxed{E(X_T) = e^{\mu^* + \frac{1}{2}\sigma^{*2}}}$$

$$V(X_T) = E(X_T^2) - E^2(X_T)$$
$$= e^{\mu^*}\left[e^{2\sigma^{*2}} - \mu^{*2}\right]$$

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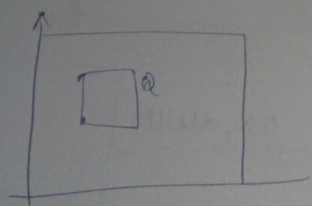
$$\int_0^{1/n} f(u)\,du \approx \frac{1}{n}\sum_{i=1}^{N} f(u_i)$$



$$D_N = \left| \frac{\#\text{ of } x_i \in Q}{N} - \text{Vol of } Q \right|$$

discrepancy

Halton sequence

in 2-D

Vander seq

$$(\phi_2(i),\ \phi_3(i))$$

First two primes

ex. 9

1001 →

Repres. base 2 :- $1\cdot\frac{1}{2^4} + 0\cdot\frac{1}{2^3} + 0\cdot\frac{1}{2^2} + 1\cdot\frac{1}{2^1} = \frac{9}{16}$

100

Repres. base 3 :- $\cancel{\frac{}{3^4}} + 1\cdot\frac{1}{3^3} + 0\cdot\frac{1}{3^2} + 0\cdot\frac{1}{3}$