Acceptance-Rejection Technique

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The rejection method

- Suppose we wish to generate Y from a distribution with PDF $f(y)$.
- Assume that we are able to generate X from a distribution with PDF $g(x)$ and that there is a constant c such that
  $\frac{f(x)}{g(x)} \leq c$, for all y.
- According to the rejection method, we can generate Y using the following steps:
  step 1: generate X from distribution with density g.
  step 2: generate a random number U.
  step 3: if $U \leq \frac{f(X)}{cg(X)}$, set $Y = X$.
  step 4: else return to step 1.

Important results

$X \sim U(a,b)\ , \quad a < b$

Use inverse transform method

$F(x) = \int_a^x \frac{1}{b-a} dz = \frac{x-a}{b-a}$

- Generate a random number U
- $\frac{x-a}{b-a} = U \Longrightarrow X = a + U(b-a)$.

  $$\begin{aligned} \textit{Examples}: \quad & X \sim U(0, 2\pi) \Longrightarrow X = 2\pi U \\ & X \sim U(-1,1) \Longrightarrow X = 2U - 1. \end{aligned}$$

–continued

Theorem :

- The random variable Y generated by the rejection method has density f .
- The number of iterations required for the rejection algorithm has a geometric distribution with mean c.

proof : to be discussed in class.

To verify the validity of the acceptance rejection method, let Y be the sample returned by the algorithm and observe that Y has the distribution of X conditional on $U \leq \frac{f(x)}{cg(x)}$.
Thus for any $A \subseteq \chi$,

$$\begin{aligned}
P[Y \in A] &= P\left[X \in A | U \leq \frac{f(x)}{cg(x)}\right] \\
&= \frac{P\left[X \in A, U \leq \frac{f(x)}{cg(x)}\right]}{P\left[U \leq \frac{f(x)}{cg(x)}\right]} \\
&= \frac{\int_A P[U \leq \frac{f(x)}{cg(x)} | X = x] g(x) dx}{P\left[U \leq \frac{f(x)}{cg(x)}\right]} \\
&= \int_A f(x) dx
\end{aligned}$$

- **Beta Distribution :**
  The Beta distribution on $[0, 1]$ with parameters $\alpha_1, \alpha_2 > 0$ is given by

$$f(x) = \frac{1}{B(\alpha_1, \alpha_2)} x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1} \ ; \ \ 0 \leq x \leq 1.$$

  with

$$B(\alpha_1, \alpha_2) = \int_0^1 x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1} dx = \frac{\Gamma\alpha_1 \Gamma\alpha_2}{\Gamma\alpha_1 + \alpha_2}$$

- Let c be the value of the density f at this point.
- Then $f(x) \leq c, \ \ \forall x$.

Algorithm :

1. Generate $U_1, U_2 \in U[0,1]$ until $cU_2 \leq f(U_1)$.
2. Return U_1

Normal-from Double exponential

- A sample from the double exponential can be generated easily.
- The double exponential density on $(-\infty, \infty)$ is $g(x) = \frac{1}{2}\exp(-|x|)$.
- The normal density is $g(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$.
- The ratio is $\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}}e^{(-\frac{|x|}{2}+|x|)} \leq \sqrt{\frac{2e}{\pi}} = const$
- The rejection test $u > \frac{f(x)}{cg(x)}$ can be implemented as :

$$u > \frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{1}{ce^{-|x|}/2} = e^{-\frac{1}{2}(|x|-1)^2}$$

- In light of symmetry of both f and g it is sufficient to generate a positive sample and determine the sign only if sample is accepted.

The acceptance-rejection technique

- Suppose we wish to simulate from a discrete distribution with mass function $\{p_j, j \geq 0\}$
- Suppose we have an efficient method to simulate from $\{q_j, j \geq 0\}$ where
  $\frac{p_j}{q_j} \leq c$, for all j such that $p_j > 0$,
  where c is a fixed positive constant.
- The acceptance-rejection method is as follows :
  step 1 : Simulate Y with mass function $\{q_j\}$.
  step 2 : Generate a random number U.
  step 3 : If $U < \frac{p_Y}{cq_Y}$, set $X = Y$ and STOP.
  step 4 : Else return to step 1.
- We must prove that the random variable generated comes from the distribution $\{p_j\}$. We will prove in class.

Illustrative examples

- Suppose we want to simulate from the discrete distribution

j	1	2	3	4	5
p_j	0.05	0.25	0.45	0.15	0.10

- We can use the acceptance-rejection method by choosing the discrete uniform for q and then the constant $c = \max \frac{p_j}{q_j} = 2.25$.
- The algorithm can then be summarized as follows :

  step 1 : generate a random number $U_1 \sim U(0, 1)$ and set $Y = int(5 \cdot U_1) + 1$

  step 2 : generate a second random number $U_2 \sim U(0, 1)$.

  step 3 : if $U_2 < \frac{p_Y}{0.45}$, set $X = Y$, and STOP.

  step 4 : else return to step 1.