PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actual counting):

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of -

(a) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is $m + n$ (known as addition principle).

Example : There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example : There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE &AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

***Illustration* 1 :** A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

***Solution* :** The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4ways.Hence the total number of ways $6 \times 4 = 24$. **Ans.(A)**

***Illustration* 2 :** A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

***Solution* :** The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways. Hence the total number of ways $6 + 4 = 10$. **Ans. (C)**

Do yourself - 1 :

(i) There are 3 ways to go from A to B, 2 ways to go from B to C and 1 way to go from A to C. In how many ways can a person travel from A to C ?

(ii) There are 2 red balls and 3 green balls. All balls are identical except colour. In how many ways can a person select two balls ?

2. FACTORIAL NOTATION :

(i) A Useful Notation : n! (factorial n) = n.(n – 1).(n – 2).........3. 2. 1; n! = n. (n – 1)! where $n \in N$

(ii) 0! = 1! = 1

(iii) Factorials of negative integers are not defined.

(iv) n! is also denoted by $\underline{|n}$

(v) $(2n)! = 2^n.n!\,[1.\,3.\,5.\,7........(2n-1)]$

(vi) Prime factorisation of n! : Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

where, $p^k \le n < p^{k+1}$ and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number n, then n can be written as $n = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}$ .... , where α_i are whole numbers.

***Illustration* 3 :** Find the exponent of 6 in 50!

***Solution* :**

$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right] \text{(where [ ] denotes integral part)}$$

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

$\Rightarrow$ 50! can be written as $50! = 2^{47}.\,3^{22}$.........

Therefore exponent of 6 in 50! = 22 **Ans.**

3. PERMUTATION & COMBINATION :

(a) **Permutation :** Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained. Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

^nP_r denotes the number of permutations of n **different** things, taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$$^nP_r = n(n-1)(n-2) .... (n-r+1) = \frac{n!}{(n-r)!}$$

Note : (i) $^nP_n = n!$, $^nP_0 = 1$, $^nP_1 = n$

(ii) Number of arrangements of n **distinct** things taken all at a time = n!

(iii) ^nP_r is also denoted by A_r^n or P(n,r).

1 2 3 4 5

5 4 ~~x~~ mans $^5P_2$ i.e, permutation of 5 distinct object taken two at a time which will be equal to

5 4 3 2 1 → 5! but we don't want 3, 2, 1 so $\frac{5!}{3\times2\times1}$ i.e, $\frac{5!}{(5-2)!}$ i.e, $\frac{n!}{(n-r)!}$ for two object at a time.

(b) **Combination :** Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION. Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

^nC_r denotes the number of combinations of n different things taken r at a time ($n \in N$, $r \in W$, $r \le n$)

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Note : (i) ^nC_r is also denoted by $\binom{n}{r}$ or C (n, r). **(ii)** $^nP_r = {^nC_r}.\ r!$

Illustration 4 : If a denotes the number of permutations of (x + 2) things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of(x – 11) things taken all at a time such that a = 182 bc, then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

Solution : $^{x+2}P_{x+2} = a \quad \Rightarrow a = (x+2)!$

$^xP_{11} = b \quad \Rightarrow b = \frac{x!}{(x-11)!}$

and $^{x-11}P_{x-11} = c \quad \Rightarrow c = (x-11)!$

$\because\ a = 182bc$

$(x+2)! = 182\frac{x!}{(x-11)!}(x-11)! \Rightarrow (x+2)(x+1) = 182 = 14 \times 13$

$\therefore\ x + 1 = 13 \Rightarrow x = 12$ **Ans. (B)**

Illustration 5 : A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?

Solution : The selections of 6 balls, consisting of atleast two balls of each colour from 5 red and 6white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	$^5C_2 \times {^6C_4} = 150$
3	3	$^5C_3 \times {^6C_3} = 200$
4	2	$^5C_4 \times {^6C_2} = 75$

Therefore total number of ways = 425 **Ans.**

Illustration 6 : How many 4 letter words can be formed from the letters of the word 'ANSWER'? How many of these words start with a vowel ?

Solution : Number of ways of arranging 4 different letters from 6 different letters are

$^6C_4 4! = \frac{6!}{2!} = 360.$

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ $= {}^5C_3 3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ $= {}^5C_3 3! = \frac{5!}{2!} = 60.$

$\therefore$ Total number of 4 letter words starting with a vowel = 60 + 60 = 120. **Ans.**

Illustration 7 : If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution : First of all, arrange all letters of given word alphabetically : 'ADIPR'

Total number of words starting with A _ _ _ _ = 4! = 24

Total number of words starting with D _ _ _ _ = 4! = 24

Total number of words starting with I _ _ _ _ = 4! = 24

Total number of words starting with P _ _ _ _ = 4! = 24

Total number of words starting with RAD _ _ = 2! = 2

Total number of words starting with RAI _ _ = 2! = 2

Total number of words starting with RAPD _ = 1

Total number of words starting with RAPI _ = 1

$\therefore$ Rank of the word RAPID = 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102 **Ans.**

Do yourself -2 :

(i) Find the exponent of 10 in $^{75}C_{25}$.

(ii) If $^{10}P_r = 5040$, then find the value of r.

(iii) Find the number of ways of selecting 4 even numbers from the set of first 100 natural numbers.

(iv) If all letters of the word 'RANK' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RANK'.

(v) How many words can be formed using all letters of the word 'LEARN' ? In how many of these words vowels are together ?

4. PROPERTIES OF ^nP_r and ^nC_r:

(a) The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.\ ^{n-p}C_{r-p} (p \le r \le n)$

(b) The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.

(c) Following properties of ^nC_r should be remembered :

(i) $^nC_r = {}^nC_{n-r}$; $^nC_0 = {}^nC_n = 1$

(ii) $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) $^nC_0 + {}^nC_1 + {}^nC_2 + \dots\dots\dots\dots + {}^nC_n = 2^n$

(v) $^nC_r = \dfrac{n}{r}\ {}^{n-1}C_{r-1}$

(vi) ^nC_r is maximum when $r = \dfrac{n}{2}$ if n is even and $r = \dfrac{n-1}{2}$ or $r = \dfrac{n+1}{2}$, if n is odd.

(d) The number of combinations of n different things taking r at a time,

(i) when p particular things are always to be included $= {}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded $= {}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$

Illustration 8 : There are 6 pockets in the coat of a person. In how many ways can he put 4pens in these pockets?

(A) 360 (B) 1296 (C) 4096 (D) none of these

Solution : First pen can be put in 6 ways.

Similarly each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways = $6 \times 6 \times 6 \times 6 = 1296$ **Ans.(B)**

Illustration 9 : A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if-

(a) all the students are equally willing ?

(b) two particular students have to be included in the delegation ?

(c) two particular students do not wish to be together in the delegation ?

(d) two particular students wish to be included together only ?

(e) two particular students refuse to be together and two other particular students wish to be together only in the delegation ?

Solution : (a) Formation of delegation means selection of 4 out of 12.

Hence the number of ways = $^{12}C_4 = 495$.

(b) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways = $^{10}C_2 = 45$.

(c) The number of ways in which both are selected = 45. Hence the number of ways in which the two are not included together = 495 – 45 = 450

(d) There are two possible cases

(i) Either both are selected. In this case, the number of ways in which the selection can be made = 45.

(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence the total number of ways of selection = 45 + 210 = 255

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i)	(A, B, C) selected,	(D) not selected
(ii)	(A, B, D) selected,	(C) not selected
(iii)	(A, B) selected,	(C, D) not selected
(iv)	(C) selected,	(A, B, D) not selected
(v)	(D) selected,	(A, B, C) not selected
(vi)	A, B, C, D not selected	

For (i) the number of ways of selection = ${}^8C_1 = 8$

For (ii) the number of ways of selection = ${}^8C_1 = 8$

For (iii) the number of ways of selection = ${}^8C_2 = 28$

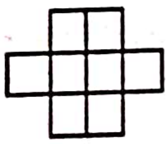
For (iv) the number of ways of selection = ${}^8C_3 = 56$

For (v) the number of ways of selection = ${}^8C_3 = 56$

For (vi) the number of ways of selection = ${}^8C_4 = 70$

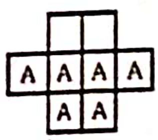
Hence total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226. **Ans.**

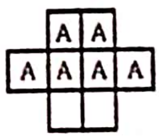
***Illustration* 10 :** In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how many number of ways is it possible ?



(A) 24 (B) 25 (C) 26 (D) 27

***Solution* :** There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^8C_6$ number of ways.

(I) 

(II) 

According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are = $({}^8C_6 - 2) = 28 - 2 = 26$. **Ans.(C)**

Illustration 11 : There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is :

(A) $3p^2(p-1)+1$ (B) $3p^2(p-1)$ (C) $p^2(4p-3)$ (D) none of these

Solution : The number of triangles with vertices on different lines $= {}^{p}C_1 \times {}^{p}C_1 \times {}^{p}C_1 = p^3$ Thenumber of triangles with two vertices on one line and the third vertex on any one of the other two lines $= {}^{3}C_1\{{}^{p}C_2 \times {}^{2p}C_1\} = 6p.\frac{p(p-1)}{2}$

So, the required number of triangles $= p^3 + 3p^2(p-1) = p^2(4p-3)$ **Ans. (C)**

Illustration 12 : There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Solution : Total number of remaining non-selected points = 6

• • • • • •

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x • • x • x • • x •

Total number of ways of selecting 4 gaps out of 7 gaps $= {}^{7}C_4$ **Ans.**

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive : $^{n-r+1}C_r$

Do yourself-3 :

(i) Find the number of ways of selecting 5 members from a committee of 5 men & 2 women such that all women are always included.

(ii) Out of first 20 natural numbers, 3 numbers are selected such that there is exactly one even number. How many different selections can be made ?

(iii) How many four letter words can be made from the letters of the word 'PROBLEM'. How many of these start as well as end with a vowel ?

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5. FORMATION OF GROUPS :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\frac{(m+n)!}{m!n!} (m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\frac{(2n)!}{n!n!2!}$ As in any one way it is possible to interchange the two groups with out obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways : $\frac{(2n)!}{n!n!2!} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\frac{(m+n+p)!}{m!n!p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups $= \frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each and m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ell!m!}$

Here $\ell p + mq = n$

***Illustration* 13 :** In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together ? Also find the number of ways if these groups are to be sent to three different colleges.

***Solution* :** Here first we separate those two particular students and make 3 groups of 5,5 and 3 of the remaining 13 so that these two particular students always go with the group of 3 students.

$\therefore$ Number of ways $= \frac{13!}{5!5!3!}\cdot\frac{1}{2!}$.

Now if these groups are to be sent to three different colleges, total number of ways $= \frac{13!}{5!5!3!}\cdot\frac{1}{2!}.3!$ **Ans.**

***Illustration* 14 :** Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

***Solution* :** Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups

$= \frac{48!}{(12!)^4 4!}$

Now, distribute exactly one Ace to each group of 12 cards. Total number of ways

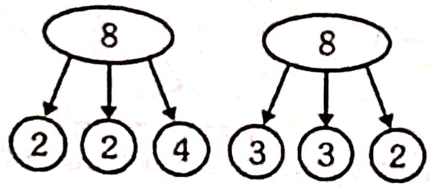
$= \frac{48!}{(12!)^4 4!} \times 4!$

Now, distribute these groups of cards among four players $= \frac{48!}{(12!)^4 4!} \times 4!\ 4!$

$= \frac{48!}{(12!)^4} \times 4!$ **Ans.**

***Illustration* 15 :** In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

***Solution* :** If each receives at least two books, then the division trees would be as shown below :



(i) (ii)

The number of ways of division for tree in figure (i) is $\left[\dfrac{8!}{(2!)^2 4!2!}\right]$

The number of ways of division for tree in figure (ii) is $\left[\dfrac{8!}{(3!)^2 2!2!}\right]$

The total number of ways of distribution of these groups among 3 students is

$\left[\dfrac{8!}{(2i)^2 4!2!}+\dfrac{8!}{(3!)^2 2!2!}\right]\times 3!.$ **Ans.**

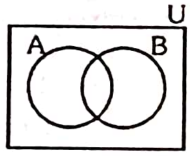
Do yourself-4 :

(i) Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.

(ii) In how many ways can 6 different books be distributed among 3 students such that none gets equal number of books ?

(iii) *n* different toys are to be distributed among n children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.

6. PRINCIPLE OF INCLUSION AND EXCLUSION :

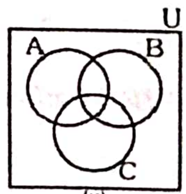


(i)

In the Venn's diagram (i), we get

$n(A\cup B) = n(A) + n(B) - n(A\cap B)$

$n(A'\cap B') = n(U) - n(A\cup B)$



(ii)

In the Venn's diagram (ii), we get

$n(A\cup B\cup C)$

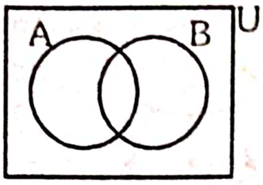
$= n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$

$n(A'\cap B'\cap C') = n(U) - n(A\cup B\cup C)$

In general, we have $n(A_1\cup A_2\cup........\cup A_n)$

$= \sum n(A_i) - \sum\limits_{i\neq j} n(A_i\cap A_j) + \sum\limits_{i\neq j\neq k} n(A_i\cap A_j\cap A_k) + ...+ (-1)^n \sum n(A_1\cap A_2\cap .....\cap A_n)$

***Illustration* 16 :** Find the number of permutations of letters a, b, c, d, e, f, g taken all at a time if neither 'beg' nor 'cad' pattern appear.



***Solution* :** The total number of permutations without any restrictions; n(U) = 7!

$\boxed{\text{b e g}}$ a c d f

Let A be the set of all possible permutations in which 'beg' pattern always appears : n(A)= 5!

$\boxed{\text{c a d}}$ b e f g

Let B be the set of all possible permutations in which 'cad' pattern always appears : n(B)= 5!

$\boxed{\text{c a db e g}}$ f

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear. $n(A \cap B) = 3!$.

Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear $n(A' \cap B') = n(U) - n(A \cap B) = n(U) - n(A) - n(B) + n(A \cap B)$

$= 7! - 5! - 5! + 3!$. **Ans.**

Do yourself-5 :

(i) Find the number of n digit numbers formed using first 5 natural numbers, which contain the digits 2 & 4 essentially.

7. PERMUTATIONS OF ALIKE OBJECTS :

Case-I : Taken all at a time -

The number of permutations of n things taken all at a time : when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n - (p + q + r)$ are all different is : $\dfrac{n!}{p!\ q!\ r!}$.

***Illustration* 17 :** In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels & consonants.

***Solution* :** The consonants in their positions can be arranged in $\dfrac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\dfrac{3!}{2!} = 3$ ways

$\therefore$ Total number of arrangements = $12 \times 3 = 36$ **Ans.**

***Illustration* 18 :** How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 (B) 18 (C) 19 (D) 20

***Solution* :** There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in $\frac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\frac{3!}{2!} = 3$ ways

∴ The required number of numbers = $6 \times 3 = 18$. **Ans. (B)**

***Illustration* 19:**
(a) How many permutations can be made by using all the letters of the word HINDUSTAN ?
(b) How many of these permutations begin and end with a vowel ?
(c) In how many of these permutations, all the vowels come together ?
(d) In how many of these permutations, none of the vowels come together ?
(e) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN ?

***Solution* :**
(a) The total number of permutations = Arrangements of nine letters taken all at a time = $\frac{9!}{2!} = 181440$.

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\frac{7!}{2!}$ ways.

Hence the total number of permutations = $3 \times 2 \times \frac{7!}{2!} = 15120$.

(c) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in $3! = 6$ ways.

Hence the total number of permutations = $\frac{7!}{2!} \times 6 = 15120$.

(d) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways.

× C × C × C × C × C × C × (Here C stands for a consonant and × stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in $^7C_3.3! = 210$ ways.

Hence the total number of permutations = $\frac{6!}{2!} \times 210 = 75600$.

(e) In this case, the vowels can be arranged among themselves in 3! = 6 ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence the total number of permutations $=\frac{6!}{2!}\times 6 = 2160$. **Ans.**

***Illustration* 20 :** If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER'.

***Solution* :** First of all, arrange all letters of given word alphabetically : EOPPRR Total number of words starting with-

E _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

O _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

PE _ _ _ _ $= \frac{4!}{2!} = 12$

PO _ _ _ _ $= \frac{4!}{2!} = 12$

PP _ _ _ _ $= \frac{4!}{2!} = 12$

PRE _ _ _ = 3! = 6

PROE _ _ = 2! = 2

PROPER = 1 = 1

Rank of the word PROPER = 105 **Ans.**

Case-II : Taken some at a time

***Illustration* 21 :** Find the total number of 4 letter words formed using four letters from the word" PARALLELOPIPED'.

***Solution* :** Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No. of ways of selection	No. of ways of arrangements	Total
All distinct	${}^8C_4$	${}^8C_4 \times 4!$	1680
2 alike, 2 distinct	${}^4C_1 \times {}^7C_2$	${}^4C_1 \times {}^7C_2 \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	${}^4C_2$	${}^4C_2 \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	${}^2C_1 \times {}^7C_1$	${}^2C_1 \times {}^7C_1 \times \frac{4!}{3!}$	56
		Total	2780

Ans.

Illustration 22 : Find the number of all 6 digit numbers such that all the digits of each number are selected from the set {1,2,3,4,5} and any digit that appears in the number appears at least twice.

Solution :

Cases	No. of ways of selection	No. of ways of arrangements	Total
All a like	${}^5C_1$	${}^5C_1 \times 1$	5
4alike + 2other alike	${}^5C_2 \times 2!$	${}^5C_2 \times 2 \times \frac{6!}{2!4!}$	300
3alike + 3other alike	${}^5C_2$	${}^5C_2 \frac{6!}{3!3!}$	200
2alike + 2other alike +2other alike	${}^5C_3$	${}^5C_3 \times \frac{6!}{2!2!2!}$	900
		Total	1405

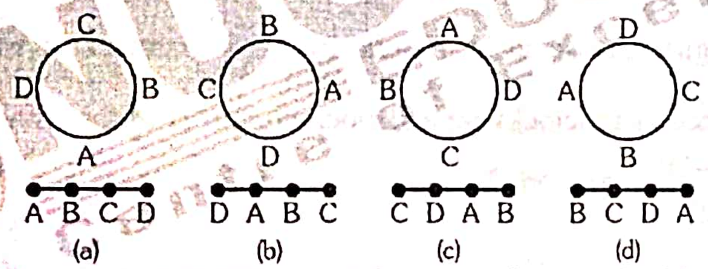
Ans.

Do yourself-6 :

(i) In how many ways can the letters of the word 'INDIA' be arranged ? Also find its rank if all these words are arranged as they are in dictionary.

(ii) How many numbers greater than 1000 can be formed from the digits 1, 1, 2, 2, 3 ?

8. CIRCULAR PERMUTATION :



Let us consider that persons A,B,C,D are sitting around a round table. If all of them (A,B,C,D) are shifted by one place in anticlockwise order, then we will get Fig.(b) from Fig.(a). Now, if we shift A,B,C,D in anticlockwise order, we will get Fig.(c). Again, if we shift them, we will get Fig.(d) and in the next time, Fig.(a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A,B,C,D does not change.

But if A,B,C,D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4. Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n × (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is

$$1/n \times (\text{number of linear arrangements of n different things}) = \frac{n!}{n} = (n-1)!$$

Therefore note that :

(i) The number of circular permutations of n different things taken all at a time is : $(n-1)!$. If clockwise & anti-clockwise circular permutations are considered to be same, then it is: $\frac{(n-1)!}{2}$

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise & anticlockwise arrangements is : $\frac{{}^nP_r}{r}$

***Illustration* 23 :** In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) $5! \times 5!$ (B) $5! \times 4!$ (C) $\frac{1}{2}(5!)^2$ (D) $\frac{1}{2}(5! \times 4!)$

***Solution* :** Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways = $4! \times 5!$

Ans. (B)

***Illustration* 24 :** The number of ways in which 7 girls can stand in a circle so that they do not have same neighbors in any two arrangements ?

(A) 720 (B) 380 (C) 360 (D) none of these

***Solution* :** Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \frac{(7-1)!}{2!} = 360$ **Ans. (C)**

***Illustration* 25 :** The number of ways in which 20 different pearls of two colours can be set alternately on a necklace, there being 10 pearls of each colour, is

(A) $9! \times 10!$ (B) $5(9!)^2$ (C) $(9!)^2$ (D) none of these

***Solution* :** Ten pearls of one colour can be arranged in $\frac{1}{2}.(10-1)!$ways. The number of arrangements of 10 pearls of the other colour in 10 places between the pearls of the first colour = 10!

$\therefore$ The required number of ways $= \frac{1}{2} \times 9! \times 10! = 5(9!)^2$ **Ans. (B)**

***Illustration* 26 :** A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Solution : (i) The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.

The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! = \frac{10!4!4!}{5!5!} = \frac{10!}{25}$

(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$.

The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!

Therefore, total number of arrangements is : ${}^{10}C_6 5! \times 3! = \frac{(10)!}{6!4!} 5!3! = \frac{(10)!}{24}$

Ans.(B)

Do yourself-7 :

(i) In how many ways can 3 men and 3 women be seated around a round table such that all men are always together ?

(ii) Find the number of ways in which 10 different diamonds can be arranged to make a necklace.

(iii) Find the number of ways in which 6 persons out of 5 men & 5 women can be seated at a round table such that 2 men are never together.

(iv) In how many ways can 8 persons be seated on two round tables of capacity 5 & 3.

9. TOTAL NUMBER OF COMBINATIONS :

(a) Given n different objects, the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(b) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by :

$(p + 1)(q + 1)(r + 1)\ldots\ldots\ldots -1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by :

$(p + 1)(q + 1)(r + 1)2^n - 1$.

Illustration 27 : A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-

(A) $2^{2n} - {}^{2n}C_n$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Solution : Let $A = \{a_1, a_2, a_3, \ldots\ldots a_n\}$. For $a_i \in A$, we have the following choices :

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \ldots. a_n$ belong to $P \cap Q$ is 3^n. **Ans. (D)**

Illustration 28 : There are 3 books of mathematics, 4 of science and 5 of English. How many different collections can be made such that each collection consists of-

(i) one book of each subject ? (ii) at least one book of each subject ?

(iii) at least one book of English ?

Solution :

(i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 60$

(ii) $(2^3 - 1)(2^4 - 1)(2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1)(2^3)(2^4) = 31 \times 128 = 3968$ **Ans.**

Illustration 29 : Find the number of groups that can be made from 5 red balls, 3 green balls and 4blackballs, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution : After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red, 2 green and 3 black balls. These will be $(4 + 1)(2 + 1)(3 + 1) = 60$ **Ans.**

Do yourself-8 :

(i) There are p copies each of n different books. Find the number of ways in which atleast one book can be selected ?

(ii) There are 10 questions in an examination. In how many ways can a candidate answer the questions, if he attempts atleast one question.

10. DIVISORS :

Let $N = p^a . q^b . r^c \ldots\ldots$ where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is $= (a + 1)(b + 1)(c + 1)\ldots\ldots$

(b) The sum of these divisors is

$= (p^0 + p^1 + p^2 + \ldots + p^a)(q^0 + q^1 + q^2 + \ldots + q^b)(r^0 + r^1 + r^2 + \ldots + r^c)\ldots$

(c) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a + 1)(b + 1)(c + 1)\ldots\ldots$ if N is not a perfect square

$\frac{1}{2}[(a + 1)(b + 1)(c + 1)\ldots\ldots + 1]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Note :

(i) Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.

(ii) A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.

(iii) Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.

(iv) 1 is neither prime nor composite however it is co-prime with every other natural number.

(v) Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g.5 & 7, 19 & 17 etc).

(vi) All divisors except 1 and the number itself are called proper divisors.

***Illustration* 30 :** Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution :

(i) The number $38808 = 2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$= (3 + 1) (2 + 1) (2 + 1) (1 + 1) - 2 = 70$

(ii) The sum of these divisors

$= (2^0 + 2^1 + 2^2 + 2^3) (3^0 + 3^1 + 3^2) (7^0 + 7^1 + 7^2) (11^0 + 11^1) - 1 - 38808$

$= (15) (13) (57) (12) - 1 - 38808 = 133380 - 1 - 38808 = 94571.$ **Ans.**

***Illustration* 31 :** In how many ways the number 18900 can be split in two factors which are relative prime(or coprime) ?

Solution : Here $N = 18900 = 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in $18900 = n = 4$

Hence number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) $= 2^{4-1} = 2^3 = 8.$ **Ans.**

***Illustration* 32 :** Find the total number of proper factors of the number 35700. Also find

(i) sum of all these factors,

(ii) sum of the odd proper divisors,

(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Solution :

$35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5,5), (2,2), (3),(7) and (17), which is given by $3 \times 3 \times 2 \times 2 \times 2 = 72$.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and35700) is $72 - 2 = 70$

(i) Sum of all these factors (proper) is :

$(5^0 + 5^1 + 5^2)(2^0 + 2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1 - 35700$

$= 31 \times 7 \times 4 \times 8 \times 18 - 1 - 35700 = 89291$

(ii) The sum of odd proper divisors is :

$(5^0 + 5^1 + 5^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 1$

$= 31 \times 4 \times 8 \times 18 - 1 = 17856 - 1 = 17855$

(iii) The number of proper divisors divisible by 10 is equal to number of selections from(5,5), (2,2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is :

$(5^1 + 5^2)(2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 35700$

$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$ **Ans.**

Do yourself-9 :

(i) Find the number of ways in which the number 94864 can be resolved as a product of two factors.

(ii) Find the number of different sets of solution of $xy = 1440$.

11. TOTAL DISTRIBUTION :

(a) Distribution of distinct objects : Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

(b) Distribution of alike objects : Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Illustration 33 : In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Solution : 5 different mangoes can be distributed by following ways among 3 children such that each gets atleast 1 :

3 1 1

2 2 1

Total number of ways : $\left(\dfrac{5!}{3!1!1!2!} + \dfrac{5!}{2!2!2!}\right) \times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) among 3 children = 3^7(as each fruit has 3 options).

$\therefore$ Total number of ways $= \left(\dfrac{5!}{3!2!} + \dfrac{5!}{(2!)^3}\right) \times 3! \times 3^7$ **Ans.**

Illustration 34 : In how many ways can 12 identical apples be distributed among four children if each gets at least 1 apple and not more than 4 apples.

Solution : Let x,y,z & w be the number of apples given to the children.

$\Rightarrow x + y + z + w = 12$

Giving one-one apple to each

Now, $x + y + z + w = 8$ ....(i)

Here, $0 \le x \le 3, 0 \le y \le 3, 0 \le z \le 3, 0 \le w \le 3$

$x = 3 - t_1, y = 3 - t_2, z = 3 - t_3, w = 3 - t_4$.

Putting value of x, y, z, w in equation (i)

Put $12 - 8 = t_1 + t_2 + t_3 + t_4$

$\Rightarrow t_1 + t_2 + t_3 + t_4 = 4$

(Here max. value that t_1, t_2, t_3& $t_4$can attain is 3, so we have to remove those cases when any of t_i getting value 4)

$= {}^7C_3 -$ (all cases when atleast one is 4)

$= {}^7C_3 - 4 = 35 - 4 = 31$

Ans.

***Illustration* 35 :** Find the number of non negative integral solutions of the in equation $x + y + z \le 20$.

Solution : Let w be any number ($0 \le w \le 20$), then we can write the equation as :

$x + y + z + w = 20$ (here x, y, z, w ≥ 0)

Total ways $= {}^{23}C_3$

Ans.

***Illustration* 36:** Find the number of integral solutions of $x + y + z + w < 25$, where $x > -2, y > 1, z \ge 2, w \ge 0$.

Solution : Given $x + y + z + w < 25$

$x + y + z + w + v = 25$ ....(i)

Let $x = -1 + t_1, y = 2 + t_2, z = 2 + t_3, w = t_4, v = 1 + t_5$ where ($t_1, t_2, t_3, t_4 \ge 0$)

Putting value of x, y, z, w, v in equation (i)

$\Rightarrow t_1 + t_2 + t_3 + t_4 + t_5 = 21$.

Number of solutions $= {}^{25}C_4$

Ans.

***Illustration* 37 :** Find the number of positive integral solutions of the in equation $x + y + z \ge 150$, where $0 < x \le 60, 0 < y \le 60, 0 < z \le 60$.

Solution : Let $x = 60 - t_1, y = 60 - t_2, z = 60 - t_3$ (where $0 \le t_1 \le 59, 0 \le t_2 \le 59, 0 \le t_3 \le 59$)

Given $x + y + z \ge 150$

or $x + y + z - w = 150$ (where $0 \le w \le 147$) ....(i)

Putting values of x, y, z in equation (i)

$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$

$30 = t_1 + t_2 + t_3 + w$

Total solutions $= {}^{33}C_3$

Ans.

Illustration 38 : Find the number of positive integral solutions of $xy = 12$

Solution :

$xy = 12$

$xy = 2^2 \times 3^1$

(i) 3 has 2 ways either 3 can go to x or y

(ii) 2^2 can be distributed between x & y as distributing 2 identical things between 2 persons (where each person can get 0, 1 or 2 things). Let two person be ℓ_1 & ℓ_2

$\Rightarrow \quad \ell_1 + \ell_2 = 2$

$\Rightarrow \quad {}^{2+1}C_1 = {}^3C_1 = 3$

So total ways $= 2 \times 3 = 6$.

Alternatively :

$xy = 12 = 2^2 \times 3^1$

$x = 2^{a_1}3^{a_2} \qquad 0 \le a_1 \le 2$

$\qquad\qquad\qquad 0 \le a_2 \le 1$

$y = 2^{b_1}3^{b_2} \qquad 0 \le b_1 \le 2$

$\qquad\qquad\qquad 0 \le b_2 \le 1$

$2^{a_1+b_1}3^{a_2+b_2} = 2^2 3^1$

$\Rightarrow a_1 + b_1 = 2 \rightarrow {}^3C_1$ ways

$a_2 + b_2 = 1 \rightarrow {}^2C_1$ ways

Number of solutions $= {}^3C_1 \times {}^2C_1 = 3 \times 2 = 6$ **Ans.**

Illustration 39 : Find the number of solutions of the equation $xyz = 360$ when

(i) $x,y,z \in N$

(ii) $x,y,z \in I$

Solution :

(i) $xyz = 360 = 2^3 \times 3^2 \times 5$ $(x,y,z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3, 0 \le a_2 \le 2, 0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3, 0 \le b_2 \le 2, 0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3, 0 \le c_2 \le 2, 0 \le c_3 \le 1$)

$\Rightarrow \quad 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow \quad 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow \quad a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions $= 10 \times 6 \times 3 = 180$.

(ii) If $x,y,z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways $= 180 + {}^3C_2 \times 180 = 720$ **Ans.**

Do yourself -10:

(i) In how many ways can 12 identical apples be distributed among 4 boys. (a) If each boy receives any number of apples. (b) If each boy receives atleast 2 apples.

(ii) Find the number of non-negative integral solutions of the equation $x + y + z = 10$.

(iii) Find the number of integral solutions of $x + y + z = 20$, if $x \geq -4$, $y \geq 1$, $z \geq 2$

12. DEARRANGEMENT :

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is $n!\left[1-\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{(-1)^n}{n!}\right]$

Proof : n letters are denoted by 1,2,3,........,n. Let Ai denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then, $n(A_i) = 1 \times (n-1)!$ [since the remaining n–1 letters can be placed in n –1 envelops in (n – 1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$(A_i \cap A_j) = 1 \times 1 \times (n-2)!$

Also $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$n(A_1' \cup A_2' \cup \ldots \cup A_n') = n! - n(A_1 \cup A_2 \cup \ldots \cup A_n)$

$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + (-1)^n \sum n(A_i \cap A_2 \ldots \cap A_n)\right]$

$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + \ldots + (-1)^{n-1} \times {}^nC_n 1]$

$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + \ldots + (-1)^{n-1}\right] = n!\left[1-\frac{1}{11}+\frac{1}{2!}+\ldots+\frac{(-1)^n}{n!}\right]$

***Illustration* 40 :** A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that

(i) all the letters are in the wrong envelopes.

(ii) at least two of them are in the wrong envelopes.

***Solution* :** (i) The number of ways is which all letters be placed in wrong envelopes

$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right) = 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$

$= 360 - 120 + 30 - 6 + 1 = 265.$

(ii) The number of ways in which at least two of them in the wrong envelopes

$= {}^6C_4.2!\left(1-\frac{1}{1!}+\frac{1}{2!}\right) + {}^6C_3.3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) + {}^6C_2.\,4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$

$+{}^6C_1.\,5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) + {}^6C_0\,6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$

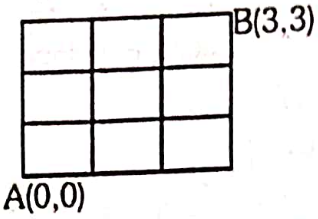
$= 15 + 40 + 135 + 264 + 265 = 719.$ **Ans.**

Do yourself - 11 :

(i) There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.

Miscellaneous Illustrations :

***Illustration* 41 :** In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem) ?



***Solution* :** To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips. Therefore, we have to arrange 3H and 3V in a row. Total number of ways $= \frac{6!}{3!3!} = 20$ ways **Ans.**

***Illustration* 42 :** Find sum of all numbers formed using the digits 2,4,6,8 taken all at a time and no digit being repeated.

***Solution* :** All possible numbers $= 4! = 24$

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occurring at unit's place

$= 6 \times (2 + 4 + 6 + 8)$

Same summation will occur for ten's, hundred's & thousand's place. Hence required sum

$= 6 \times (2 + 4 + 6 + 8) \times (1 + 10 + 100 + 1000) = 133320$ **Ans.**

***Illustration* 43 :** Find the sum of all the numbers greater than 1000 using the digits 0,1,2,2.

***Solution* :** (i) When 1 is at thousand's place, total numbers formed will be $= \frac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be $= 3! = 6$

(iii) When 1 is at hundred's, ten's or unit's place then total numbers formed will be-
Thousand's place is fixed i.e. only the digit 2 will come here, remaining two places can be filled in 2! ways.
So total numbers = 2!

(iv) When 2 is at hundred's, ten's or unit's place then total numbers formed will be-
Thousand's place has 2 options and other two places can be filled in 2 ways.

So total numbers $= 2 \times 2 = 4$

Sum $= 10^3(1 \times 3 + 2 \times 6) + 10^2(1 \times 2 + 2 \times 4) + 10^1(1 \times 2 + 2 \times 4) + (1 \times 2 + 2 \times 4)$

$= 15 \times 10^3 + 10^3 + 10^2 + 10$

$= 16110$ **Ans.**

Illustration 44 : Find the number of positive integral solutions of $x + y + z = 20$, if $x \neq y \neq z$.

Solution : $x \geq 1$

$y = x + t_1 \qquad t_1 \geq 1$

$z = y + t_2 \qquad t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$3x + 2t_1 + t_2 = 20$

(i) $x = 1 \quad 2t_1 + t_2 = 17$

$t_1 = 1, 2 \ldots\ldots\ldots 8 \qquad \Rightarrow 8$ ways

(ii) $x = 2 \quad 2t_1 + t_2 = 14$

$t_1 = 1, 2 \ldots\ldots\ldots 6 \qquad \Rightarrow 6$ ways

(iii) $x = 3 \quad 2t_1 + t_2 = 11$

$t_1 = 1, 2 \ldots\ldots\ldots 5 \qquad \Rightarrow 5$ ways

(vi) $x = 4 \quad 2t_1 + t_2 = 8$

$t_1 = 1, 2, 3 \qquad \Rightarrow 3$ ways

(v) $x = 5 \quad 2t_1 + t_2 = 5$

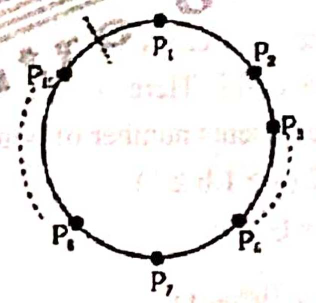
$t_1 = 1, 2 \qquad \Rightarrow 2$ ways

Total $= 8 + 6 + 5 + 3 + 2 = 24$

But each solution can be arranged by $3!$ ways.

So total solutions $= 24 \times 3! = 144$. **Ans.**

Illustration 45 : A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon ?



Solution : Select one point out of 15 point, therefore total number of ways $= {}^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point which are not consecutive.

since we cannot select P_2 & P_{15}.

Total points left are 12.

Now we have to select 2 points out of 12 points which are not consecutive

Total ways $= {}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

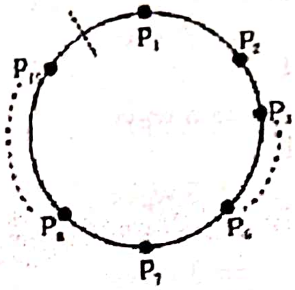
So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative :

First of all let us cut the polygon between points P_1 & P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.

x O y O z O w

Here bubbles represents the selected points,



x represents the number of points before first selected point,
y represents the number of points between Ist & IInd selected point,
z represents the number of points between IInd & IIIrd selected point
and w represents the number of points after IIIrd selected point.

$x + y + z + w = 15 - 3 = 12$

here $x \geq 0,\ y \geq 1,\ z \geq 1,\ w \geq 0$

Put $y = 1 + y'$ & $z = 1 + z'$ $(y' \geq 0, z' \geq 0)$

$\Rightarrow x + y' + z' + w = 10$

Total number of ways $= {}^{13}C_3$

These selections include the cases when both the points P_1& $P_{15}$are selected. We have to remove those cases. Here a represents number of points between P_1& 3^{rd}selected point& b represents number of points between 3rd selected point and P_{15}

$\Rightarrow a + b = 15 - 3 = 12\ (a \geq 1, b \geq 1)$

put $a = 1 + t_1$ & $b = 1 + t_2$

$t_1 + t_2 = 10$

Total number of ways $= {}^{11}C_1 = 11$

Therefore required number of ways $= {}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ **Ans.**

Illustration 46 : Find the number of ways in which three numbers can be selected from the set $\{5^1, 5^2, 5^3, \ldots\ldots 5^{11}\}$ so that they form a G.P.

Solution : Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is $= \{1,2,\ldots\ldots\ldots 6,7,\ldots\ldots 11\}$

By selecting any two numbers from $\{1,3,5,7,9,11\}$, the middle number is automatically fixed. Total number of ways $= {}^6C_2$

Now select any two numbers from $\{2,4,6,8,10\}$ and again middle number is automatically fixed. Total number of ways $= {}^5C_2$

$\therefore$ Total number of ways are $= {}^6C_2 + {}^5C_2 = 15 + 10 = 25$ **Ans.**

ANSWERS FOR DO YOURSELF

1 :	(i)	7	(ii)	3						
2 :	(i)	0	(ii)	r = 4	(iii)	$^{50}C_4$	(iv)	20	(v)	120, 48
3 :	(i)	10	(ii)	450	(iii)	840, 40				
4 :	(i)	$\frac{16!}{(2!)^8 8!} \times 8!$	(ii)	360	(iii)	$^nC_2.n!$				
5 :	(i)	$5^n - 4^n - 4^n + 3^n$								
6 :	(i)	60	(ii)	60						
7 :	(i)	36	(ii)	$\frac{9!}{2} = 181440$	(iii)	5400	(iv)	2688		
8 :	(i)	$(p+1)^n - 1$	(ii)	$2^{10} - 1$						
9 :	(i)	23	(ii)	36						
10 :	(i)	(a) $^{15}C_3$ (b) $^7C_3$	(ii)	$^{12}C_2$	(iii)	$^{23}C_2$				
11 :	(i)	9								