Indefinite Integration

Integration is known as the reverse operation of diffirentiation

Integration

eg:- $\frac{d}{dx}(F(x)+C) = f(x)$

diffirentiation

$$\int f(x)\, dx = F(x) + C$$

Integrator

Anti derivative or integral

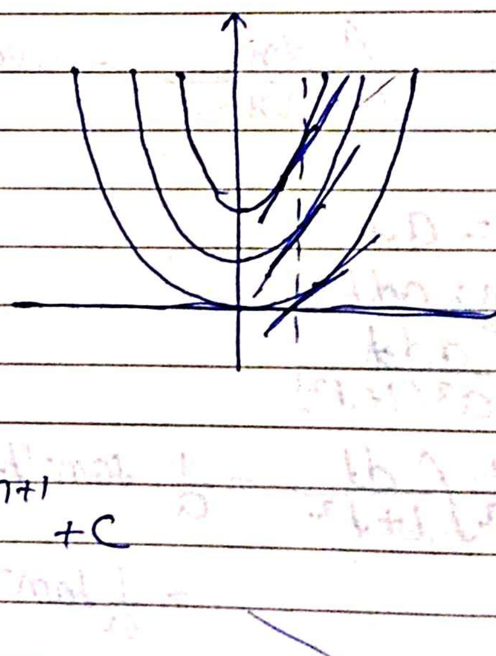
$$\int 2x\, dx = x^2 + C$$

Integration

Integrand

It geomotrically represent a family of curve having some common property i.e, slope of tangent at $x = x_1$ is same for each member of the family

for example:-



1)* $\int x^n dx = \frac{x^{n+1}}{n+1}$

* $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$

2)* $\int \frac{1}{x} dx = \ln|x| + C$

* $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + C$

3) $\int e^x dx = e^x + C$

Indefinite Integration

Integration is known as the reverse operation of diffirentiation

$$\text{eg:- } \frac{d}{dx}(F(x)+C) = f(x)$$

integration

diffirentiation

$$\int f(x)\,dx = F(x)+C$$

Integrator

Integrand

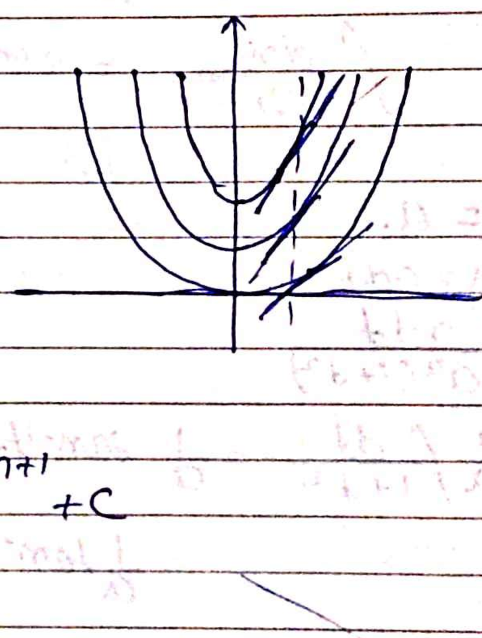
Anti derivative or integral

$$\int 2x\,dx = x^2+C$$

Integration

It geomatrically represents a family of curve having some common property i.e slope of tangent at $x=x_1$ is same for each member of the family

for example:-



1)* $\int x^n dx = \frac{x^{n+1}}{n+1}$

* $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$

2)* $\int \frac{1}{x} dx = \ln|x| + c$

* $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

3) $\int e^x dx = e^x + c$

* $\int e^{(ax+b)}\,dx = \frac{1}{a}e^{(ax+b)} + c$

* $\int a^x\,dx = \frac{a^x}{\ln a} + c$

* $\int \sin x\,dx = -\cos x$
* $\int \cos x\,dx = \sin x + c$
* $\int \sec x \tan x\,dx = \sec x + c$
* $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
* $\int \sec^2 dx = \tan x + c$
* $\int \operatorname{cosec}^2 dx = \cot x + c$
* $\int \frac{dx}{1+x^2} = \tan^{-1} x \Rightarrow \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
* $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Rightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$
* $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c \Rightarrow \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

Put $x = at$

$dx = a\,dt$

$I = \int \frac{a\,dt}{a^2(1+t^2)}$

$= \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\tan^{-1} t + c$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Q) i) $\int e^{\ln\sqrt{x}}\,dx$

$$\int \sqrt{x}\,dx = \frac{x^{1/2+1}}{3/2}+c = \frac{2}{3}x^{3/2}+c$$

ii) $\int \frac{dx}{2\sqrt{x}}$

$$\frac{1}{2}\int\frac{dx}{\sqrt{x}} = \frac{1}{2}\int\frac{(x)^{-1/2+1}}{1/2} = \sqrt{x}$$

iii) $\int 2^{\ln x}\,dx$

$$\int x^{\ln 2}\,dx$$

$$\frac{x^{\ln 2+1}}{\ln 2+1}+c$$

iv) $\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}\,dx$

$$\int\frac{(x^2+x^{-2})}{x^3}\,dx = \int\left(\frac{1}{x}+x^{-5}\right)dx$$

$$= \ln x + \left(\frac{x^{-4}}{-4}\right)+c$$

v) $\int\frac{dx}{3-2x}$

$$= -\frac{1}{2}\ln(3-2x)+c$$

vi) $\int\frac{x}{a+bx}\,dx$

$$= \frac{1}{b}\int\frac{(bx+a)-a}{a+bx}\,dx$$

$$= \frac{1}{b}\int\left(1-\frac{a}{a+bx}\right)dx = \frac{1}{b}\left[x-\frac{a}{b}\ln(a+bx)\right]$$

vii) $\int \frac{2x+3}{x^2+3x-10}\,dx$

$= \int \frac{2x+3}{(x+5)(x-2)}\,dx$

$= \int \frac{(x+5)+(x-2)}{(x+5)(x-2)}\,dx$

$= \int \frac{dx}{x-2} + \frac{dx}{x+5}$

$= \ln(x-2) + \ln(x+5) + c$

viii) $\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,dx$

$= \cancel{\int \frac{2^{x+1}-5^{x-1}}{2^{x+1}\,5^{x+1}\,10}\,dx}$ $\quad \int \frac{2 \times 2^x - 5^x/5}{2^x 5^x}\,dx$

$= \cancel{\int \frac{dx}{5^{x-1}} - \frac{dx}{2^{x+1}}}$ $\quad = \int \left(2\left(\frac{1}{5}\right)^x - \frac{1}{5}\left(\frac{1}{2}\right)^x\right)dx$

$= \frac{2\,(1/5)^x}{\ln\frac{1}{5}} - \frac{1}{5}\,\frac{(1/2)^x}{\ln 1/2} + c$

ix) $\int \frac{e^{3x}+e^{5x}}{e^x+e^{-x}}\,dx$

$= \int \frac{e^{3x}(\cancel{1+e^{2x}})}{\cancel{(1+e^{2x})}}\,e^x\,dx$

$\Rightarrow \int e^{3x}e^{x}\,dx$

$= \int e^{4x}\,dx$

$= \frac{e^{4x}}{4} + c$

~~vi) $\int \sqrt{1+\sin x}\, dx$~~

~~vii) $\int \frac{\cos 7x - \cos 2x}{1-\cos x}\, dx$~~

viii) $\int \sin^2 dx$

$$= \int \frac{1-\cos 2x}{2}\, dx$$

$$= \frac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$$

ix) $\int \cos^3 x\, dx$

$$\because \cos 3x = 4\cos^3 x - 3\cos x$$

$$\Rightarrow \frac{\cos 3x + 3\cos x}{4} = \cos^3 x$$

$$\int \frac{\cos 3x + 3\cos x}{4} = \frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x + C\right)$$

x) $\int \sin^4 x\, dx$

$$= \int (\sin^2 x)^2\, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1+\cos^2 2x - 2\cos 2x)\, dx$$

$$= \frac{1}{4}\int \left(1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right) dx$$

$$= \frac{1}{8}\int (2+1+\cos 4x - 4\cos 2x)\, dx$$

$$= \frac{1}{8}\int (3+\cos 4x - 4\cos 2x)\, dx$$

$$= \frac{1}{8}\left(3x + \frac{\sin 4x}{4} - 2\sin 2x\right) + c$$

xi) $\int \frac{\cos x - \cos 2x}{1-\cos x}\,dx$

$$= \int \frac{\cos x - (2\cos^2 x - 1)}{1-\cos x}\,dx$$

$$= \int \frac{\cos x - 2\cos^2 x + 1}{1-\cos x}\,dx$$

$$= -\int \frac{2\cos^2 x - \cos x - 1}{1-\cos x}\,dx$$

$$= \int \frac{2\cos^2 x - 2\cos x + \cos x - 1}{\cos x - 1}\,dx$$

$$= \int \frac{2\cos x(\cos x - 1) + 1(\cos x - 1)}{\cos x - 1}\,dx$$

$$= \int \frac{\cancel{(\cos x - 1)}(2\cos x + 1)}{\cancel{(\cos x - 1)}}\,dx$$

$$= \int 2\cos x + 1\,dx$$

$$= 2\sin x + x + c.$$

xii) $\int \frac{dx}{1+\cos x}$

$$= \int \frac{(1-\cos x)}{(1-\cos x)(1+\cos x)}\,dx$$

$$= \int \frac{(1-\cos x)}{\sin^2 x}\,dx$$

$$= \int \frac{\cancel{2}\sin^2 x/2}{2\cdot\cancel{4}\sin^2 x/2\cos^2 x/2}\,dx = \int \frac{1}{2\cos^2 x/2}\,dx$$

$$= \int \frac{\sec^2 x/2}{2}\,dx = \frac{\cancel{2}\tan x/2}{\cancel{2}} + c$$

$$= \tan\frac{x}{2} + c$$

xiii) $\int \frac{1-\cos x}{1+\cos x}\,dx$ (xi) $\int \frac{a\sin^3 x + b\cos^3 x}{\sin^2 x\cos^2 x}\,dx$

ix) $\int \cot^2 x\,dx$ (xii) $\int a^x e^x\,dx$ Pg-148 } Remm to do

x) $\int \tan^2 x \sin^2 x\,dx$ (xiii) $\int \sqrt{1+\sin x}\,dx$ } must do

Pg-148, 149

xiv) $\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}\,dx$

$$\Rightarrow \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)}{1-2\left(2\cos^2\frac{3x}{2}-1\right)}\,dx$$

$$= \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)}{3-4\cos^2\frac{3x}{2}}\,dx$$

$$= \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)}{3\cos\frac{3x}{2}-4\cos^3\frac{3x}{2}}\,dx$$

$$= -\int \frac{2\cancel{\cos\left(\frac{9x}{2}\right)}\cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)}{\cancel{\cos\left(\frac{9x}{2}\right)}}\,dx$$

$$= -\int (\cos 2x + \cos x)\,dx$$

$$= -\left(\frac{\sin 2x}{2} + \sin x + C\right)$$

xiii) $\int \frac{2\sin^2 x/2}{2\cos^2 x/2}\,dx = \int \tan^2 x/2\,dx$

$$= \int (\sec^2 x/2 - 1)\,dx$$

ix) $\int \cot^2 x$

$= \int (\operatorname{cosec}^2 x - 1)\, dx$

$= \int -\cot x - x + c$

x) $\int \tan^2 x \sin^2 x\, dx$

$= \int \tan^2 x (1 - \cos^2 x)\, dx$

$= \int (\tan^2 x - \sin^2 x)\, dx$

$= \int \left[(\sec^2 x - 1 - \left(\frac{1 - \cos 2x}{2}\right)\right] dx$

$= \int \cancel{2\sec^2 x \, 1} \int \tan x - x - \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$

xi) $\int \frac{a \sin^3 x + b \cos^3 x}{\sin^2 x \cos^2 x}\, dx$

$= a \int \sec x \tan x + b \int \operatorname{cosec} x \cot x\, dx$

$= a \sec x - b \operatorname{cosec} x + c$

xii) $\int a^x e^x\, dx$

$\int e^{x \ln a} e^x$

$= \int e^{x(1+\ln a)}$

$= \frac{e^{x(1+\ln a)}}{(1+\ln a)}$

xii) $\int \sqrt{1+\sin x}\, dx$

$\int \frac{\cos x}{\sqrt{1-\sin x}}\, dx$

Let $1 - \sin x = t^2$

$-\cos x\, dx = 2t\, dt$

$\therefore -\int \frac{2\cancel{t}\, dt}{\cancel{t}}$

$= -2t + c$

$= -2\sqrt{1-\sin x} + c$

H/W

29/05/18

1) $\int \frac{x^2 + \cos^2 x}{x^2+1} \operatorname{cosec}^2 x\, dx$

$= \int \frac{x^2 + 1 - \sin^2 x}{x^2+1} \operatorname{cosec}^2 x\, dx$

$= \int \left(1 - \frac{\sin^2 x}{x^2+1}\right) \operatorname{cosec}^2 x\, dx$

$= \int \operatorname{cosec}^2 x\, dx - \int \frac{1}{x^2+1}\, dx$

$= -\cot x - \tan^{-1} x + C$

2) $\int \frac{x^2}{1+x^2}\,dx$

$= \int \frac{x^2+1-1}{1+x^2}\,dx$

$= \int dx - \frac{1}{1+x^2}\,dx$

$= x - \tan^{-1} x + c$

xii) $\sqrt{1+\sin x}\,dx$

$\Rightarrow \int \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}\,dx$

$= \int \cos\frac{x}{2} + \sin\frac{x}{2}\,dx$

$= \frac{2}{1}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) + c$

3) $\int \frac{x^4}{1+x^2}\,dx$

$= \int \frac{x^4-1+1}{1+x^2}\,dx$

$= \int \frac{(x^2-1)(x^2+1)+1}{1+x^2}\,dx$

$= \int (x^2-1)\,dx + \int \frac{1}{1+x^2}\,dx$

$= \frac{x^3}{3} - x + \tan^{-1} x + C$

4) $\int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}}$

$= \int \cancel{\frac{(x-3)-(x-4)}{(2x-7)\sqrt{(x-3)(x-4)}}}\,dx = \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}}$

$= \cancel{\int \sqrt{x-3}}$

$= \int \frac{2\,dx}{(2x-7)\sqrt{4x^2-28x+48}}$

$= \int \frac{2\,dx}{(2x-7)\sqrt{(2x-7)^2-1}}$

Let $2x-7=t$ $\qquad 2dx=dt$

$$\int \frac{2dx}{t\sqrt{t^2-1}} = \int \frac{dt}{t\sqrt{t^2-1}}$$

$= \sec^{-1} t + c$

$= \sec^{-1}(2x-7)+c$

$= \sec^{-1}(2x-7)+c$

Q) Find a function f satisfying the given condition

i) $f'(x^2) = \frac{1}{x}$ for $x>0$, $f(1)=1$

ii) $f'(\sin^2 x) = \cos^2 x$ $\forall x$, $f(1)=1$

i) Let $x^2=t$

$f'(t) = \frac{1}{\sqrt{t}}$

Int. w.r.t t

$f(t) = 2\sqrt{t}+c$

$= 2\sqrt{t}+C$

$\Rightarrow f(1)=1$ (Given)

$\Rightarrow 2+C=1$

$\Rightarrow C=-1$

$\therefore f^n$ is $2\sqrt{t}-1$

$= 2x-1$

ii) Let $\sin^2 x=t$ $\Rightarrow 1-\cos^2 x=t$

$t \in [0,1]$

$f'(t) = \cos^2 x$

$f'(t) = (1-t)$

Int. w.r.t

$f(t) = t-\frac{t^2}{2}+c$

$f(1)=1$

$\Rightarrow 1-\frac{1}{2}+c=1$

$\Rightarrow c=\frac{1}{2}$

$\therefore f^n$ is $f(t) = t - \frac{t^2}{2} + \frac{1}{2} \quad \forall t \in [0, 1]$

* Techniques of Integration

1) Substitution Method:-

Let $I = \int f(x)dx$ Let $x = \phi(t)$

$dx = \phi'(t)dt$

$\Rightarrow \boxed{I = \int f(\phi(t))\, \phi'(t)dt}$

* $\int (f(x))^n f'(x)dx$

or

* $\frac{f'(x)}{(f(x))^n} dx$

Let $f(x) = t$

~~** $\int f(x)dx =$~~

1. $\int \tan x\, dx$

$= \int \frac{\sin x}{\cos x} dx$ Let $\cos x = t$

$dx(-\sin x) = dt$

$\sin x\, dx = -dt$

$= \int \frac{-dt}{t}$

$= -\ln t + c$

V.V.I

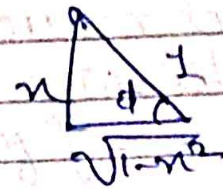
$\therefore \boxed{\int \tan x\, dx = -\ln \cos x + c \text{ or } \ln \sec x + c}$

Q) $\int \frac{\tan(\sin^{-1}x)}{\sqrt{1-x^2}}\,dx$

Let $\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}\,dx = dt$

$\sin t = x$ $\quad \cancel{x = \sin t}$

$\cancel{dx = \cos t\, dt}$



$= \cancel{\int \frac{\tan t}{\sqrt{1-\sin^2 t}}\,dx}$ $\quad$ Now, $\int \frac{\tan t}{\sqrt{1-x^2}}\,dx$

$= \cancel{\int \frac{\tan t}{\cos t}\,dx}$ $\quad = \int \tan t\, dt$

$= \cancel{\int \frac{\sin t}{\cos^2 t}\,dx}$ $\quad = \ln \sec t + c$

$= \ln \frac{1}{\sqrt{1-x^2}} + c$

$= -\ln\sqrt{1-x^2} + c.$

Q) $\int \frac{\cos x}{\cos(x-a)}\,dx$

$= \cancel{\int \frac{\cos x}{\cos x\cos a + \sin x \sin a}\,dx}$ $\quad$ Let $x-a = t$

$x = t+a$

$dx = dt$

$= \int \frac{\cos(a+t)}{\cos(t)}\,dt$

$= \int \frac{\cos a\cos t - \sin a\sin t}{\cos t}\,dt$

$= \int (\cos a - \sin a\tan t)\,dt$

$= \cos a\int dt - \int \sin a\tan t\,dt$

$= \cos a\, t - \sin a \ln \sec t + c$

$\Rightarrow (x-a)\cos a - \sin a \ln \sec(x-a) + c$

Q) $\int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$

Let $x\sin x+\cos x = t$

$dx(x\cos x+\sin x-\sin x) = dt$

$\Rightarrow x\cos x\,dx = dt$

Now $\int \frac{dt}{t^2}$

$= -\frac{1}{t}+c$

$= \frac{-1}{x\sin x+\cos x}+c$

Q) $\int \frac{dx}{\cos(x-a)\cos(x-b)}$ Nice

$= \int \frac{2dx}{2\cos(x-a)\cos(x-b)}$

$= \int \frac{2dx}{\cos(b-a)+\cos(2x-a-b)}$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)\,dx}{\cos(x-a)\cos(x-b)}$

$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]\,dx}{\cos(x-a)\cos(x-b)}$

$= \frac{1}{\sin(a+b)}\int \left(\tan(x-b)-\tan(x-a)\right)dx$

$= \frac{1}{\sin(a+b)}\left\{\ln\sec(x-b)-\ln\sec(x-a)\right\}+c$

Q) $\int \frac{\sin 2x}{\sin 3x \sin 5x} dx$

$= \int \frac{\sin(5x-3x)}{\sin 3x \sin 5x} dx$

$= \int (\cot 3x - \cot 5x)\, dx$

$= \frac{\ln \sin 3x}{3} - \frac{\ln \sin 5x}{5} + C$

Q) $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$

Let $\tan^{-1} x^3 = t$ $\quad \frac{1}{1+x^6} 3x^2 dx = dt$

$\cancel{x^3 = \tan t}$

$\cancel{3x^2 dx =}$ $\quad \cancel{= \int \frac{dt}{3}} \cancel{= \frac{x^2 dx}{1+x^6}}$

$\frac{1}{3} \int t\, dt$

$= \frac{t^2}{6} = \frac{(\tan^{-1} x^3)^2}{6}$

Q) $\int \frac{x^5}{1+x^{12}} dx$

Let $x^6 = t$ $\quad$ Now, $\int \frac{dt}{6(1+t^2)}$

$6x^5 dx = dt$

$x^5 dx = \frac{dt}{6}$ $\quad = \frac{1}{6} \tan^{-1} t$

$= \frac{1}{6} \tan^{-1} x^6 + C$

Q) $\int \sec x \ln(\sec x + \tan x)\, dx$

Let $\ln \sec x + \tan x = t$

$$\frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\, dx = dt$$

$$\Rightarrow \sec x\, dx = dt$$

$$\Rightarrow dx = \frac{dt}{\sec x}$$

$$\int \sec x \ln(\sec x + \tan x)\, dx$$

$$= \int t\, dt$$

$$= \frac{t^2}{2} + C = \frac{\left(\ln(\sec x + \tan x)\right)^2}{2} + C$$

Q) $\int \frac{\sqrt{\tan x}}{\sin 2x}\, dx$

Let $\cancel{\int \frac{\sqrt{\sin x}}{\sqrt{\cos x} \times 2 \sin x \cos x}\, dx}$ $\because \sin 2x = \frac{2\tan x}{1 + \tan^2 x}$

$\cancel{= \int \frac{1}{2\sqrt{\sin x}\cos x \cos x}\, dx}$

$$= \int \frac{1 + \tan^2 x}{2\sqrt{\tan x}}\, dx$$

$$= \frac{1}{2} \int \frac{1 + \tan^2 x}{\sqrt{\tan x}}\, dx \quad \text{Let } \tan x = t^2$$

$$\sec^2 x\, dx = 2t\, dt$$

$$\Rightarrow dx = \frac{2t\, dt}{\sec^2 x}$$

$\cancel{= \frac{1}{2}\int \frac{2t\,dt}{t} = t + C} = \sqrt{\tan x} + C.$

$$\frac{1}{2} \int \frac{\sec^2 x}{t}\, dx$$

$$= \frac{1}{2} \int \frac{\sec^2 x}{t} \times \frac{2t\, dt}{\sec^2 x} = t + C$$

$$= \sqrt{\tan x} + C$$

Q) $\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x} dx$

Let $a\sin^2 x + b\cos^2 x = t$

$(a\,2\sin x\cos x - b\,2\cos x\sin x)\,dx = dt$

$\Rightarrow \sin 2x(a-b)\,dx = dt$

$\int \frac{\sin 2x}{a\sin^2 x + b\cos^2 x} dx$

$= \int \frac{1}{(a-b)} \frac{dt}{t}$ $\quad \frac{1}{(a-b)} \ln t + c \Rightarrow \frac{1}{a-b} \ln(a\sin^2 x + b\cos^2 x) + c$

$= \cancel{\frac{1}{(a-b)t} + c}$

$= \cancel{\frac{-1}{(a-b)(a\sin^2 x + b\cos^2 x)} + c}$

Q) $\int \frac{\ln\left(\frac{x}{x+1}\right)^2}{x(x+1)} dx$

$= \int \frac{\ln\left(\frac{x}{x+1}\right)^2}{x(x+1)} dx$ $\quad$ Let $\ln\left(\frac{x}{x+1}\right) = t$

$dx\left(\frac{(x+1)}{x} \times \frac{(x+1-x)}{(x+1)^2}\right) = dt$

$= \frac{1}{2}\int t\,dt$ $\quad \frac{dx}{x(x+1)} = dt$

$= \frac{1}{2} t^2 + c$

$= \frac{1}{2}\left[\ln\left(\frac{x}{x+1}\right)\right]^2 + c$

* $\boxed{\int \sec x \, dx}$

$= \int \left[\frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} \right] dx$

Let $\sec x + \tan x = t$

$(\sec x \tan x + \sec^2 x) dx = dt$

$I = \int \frac{dt}{t} = \ln t + C.$

$= \ln(\sec x + \tan x) + C$

$= \ln\left(\frac{1 + \sin x}{\cos x}\right) + C$

$= \ln\left(\frac{(\sin x/2 + \cos x/2)^2}{(\cos^2 x/2 - \sin^2 x/2)}\right) + C$

$= \ln\left(\frac{\sin x/2 + \cos x/2}{\cos x/2 - \sin x/2}\right) + C$

$= \ln\left(\frac{1 + \tan x/2}{1 - \tan x/2}\right) + C$

$= \boxed{\ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C}$

* $\int \operatorname{cosec} x \, dx = \ln(\operatorname{cosec} x - \cot x) + C$

$= \ln\left(\frac{1 - \cos x}{\sin x}\right) + C$

$= \ln\left(\frac{2\sin^2 x/2}{2\sin x/2 \cos x/2}\right) + C$

$= \boxed{\ln\left(\tan \frac{x}{2}\right) + C}$

Questions

1) $\int e^{x^2+\ln x}\,dx$ (3) $\int \frac{e^x-1}{e^x+1}\,dx$

2) $\int \frac{dx}{e^x+1}$ (4) $\int \frac{e^x(1+x)}{\sin^2(xe^x)}\,dx$

1) $\int e^{x^2+\ln x}\,dx$

Let $e^{x^2}=t$

$(e^{x^2}2x)\,dx = dt$

$dx = \frac{dt}{e^{x^2}2x}$

$= \int \frac{1}{2}\,dt$

$= \frac{t}{2}+c$

$= \frac{e^{x^2}}{2}+c$

2) Multiply Nr and Dr by e^{-x}

$\int \frac{e^{-x}dx}{e^{-x}(e^x+1)}$

$= \int \frac{e^{-x}dx}{1+e^{-x}}$

Let $1+e^{-x} = t$

$-e^{-x}\,dx = dt$

$dx = -\frac{dt}{e^{-x}}$

$= -\int \frac{dt}{t}$

$= -\ln t + c$

$= -\ln(1+e^{-x})+c.$

3) $\int \frac{(e^x+1)-2}{(e^x+1)}\,dx$

$= \int \left[1 - \frac{2}{e^x+1}\right]dx$

$= \int dx - 2\int \frac{dx}{e^x+1} = x + 2\ln(1+e^{-x})+c$

previous prob.

4) $\int \frac{e^x(x+1)}{\sin^2(xe^x)}\,dx$

$xe^x = t$

$(xe^x + e^x)\,dx = dt$

$\Rightarrow e^x(x+1)\,dx = dt$

Now, $I = \int \frac{e^x(x+1)}{\sin^2(xe^x)}\,dx$

$I = \int \frac{dt}{\sin^2 t}$

$= \int \operatorname{cosec}^2 t\,dt$

$= -\cot(xe^x) + c$

Practice Problem

1) $\int \frac{\sec x\,dx}{\sqrt{\cos(2x+\alpha)\cos x}} \rightarrow$ cos C + cos D

2) $\int \frac{x\,dx}{\sqrt{(1+x^2)} + \sqrt{(1+x^2)^3}}$

1) $\int \frac{\sec x\,dx}{\sqrt{2\cos(\alpha+x)\cos x}}$

$= \int \frac{\sec x\,dx}{\sqrt{2[(\cos x\cos\alpha - \sin x\sin\alpha)\cos x]}}$

$= \int \frac{\sec^2 x\,dx}{\sqrt{2(\cos\alpha - \sin\alpha\tan x)}}$

Let $2(\cos\alpha - \sin\alpha\tan x) = t^2$

~~$-2\sin\alpha\sec^2 x\,dx = 2t\,dt$~~

~~$\sec^2 x\,dx = \frac{-t}{\sin\alpha}\,dt$~~

$-2\sin\alpha\sec^2 x = 2t\,dt$

$\sec^2 x\,dx = \frac{-t\,dt}{\sin\alpha}$

$I = \frac{-1}{\sin\alpha}\int \frac{t\,dt}{t} = \frac{-t}{\sin\alpha} + c$

put value of t.

2> $\int \frac{x\,dx}{\sqrt{(1+x^2)} + \sqrt{(1+x^2)^3}}$

Let $1+x^2 = t^4$

$2x\,dx = 4t^3\,dt$

$x\,dx = 2t^3\,dt$

$= 2\int \frac{t^3\,dt}{\sqrt{t^4+t^6}}$

$= 2\int \frac{t\,dt}{\sqrt{1+t^2}}$

$1+t^2 = u^2$

$t\,dt = u\,du$

$= 2\int \frac{\cancel{u}\,du}{\cancel{u}}$

$= 2u + C$

$= 2\sqrt{1+t^2} + C$

$= 2\sqrt{1+\sqrt{1+x^2}} + C$

* Profile — Sugested sub.

1> $\sqrt{a^2-x^2}$ — Put $x = a\sin\theta$ or $a\cos\theta$

2> $\sqrt{x^2-a^2}$ — Put $x = a\sec\theta$

3> $\sqrt{a^2+x^2}$ — $x = a\tan\theta$

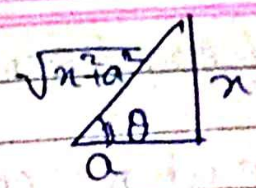
4> $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ — $x^2 = a^2\cos 2\theta$

Q> * $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\{x+\sqrt{x^2+a^2}\} + C$

Put $x = a\tan\theta \longrightarrow \tan\theta = \frac{x}{a}$

$dx = a\sec^2\theta\,d\theta$

$$I = \int \frac{a \sec^2\theta \, d\theta}{\sqrt{a^2(1+\tan^2\theta)}}$$



$$\int \sec\theta \, d\theta$$

$$= \ln(\sec\theta + \tan\theta) + c$$

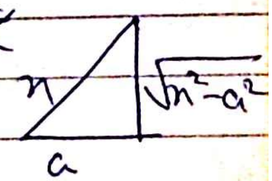
$$= \ln\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) + c$$

$$= \ln\{x + \sqrt{x^2+a^2}\} + \underbrace{(c - \ln a)}_{\lambda}$$

$$= \ln(x + \sqrt{x^2+a^2}) + \lambda$$

* $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\{x + \sqrt{x^2-a^2}\} + c$$

Proof:- $x = a\sec\theta$ $\Rightarrow \sec\theta = \frac{x}{a}$



$\cancel{dx = a(\ln(\sec\theta + \tan\theta))d\theta}$

$\cancel{\int \frac{a(\ln(\sec\theta + \tan\theta))d\theta}{a\tan\theta}}$

$$\Rightarrow dx = a\sec\theta\tan\theta\, d\theta$$ $\sec\theta = \frac{x}{a} \Rightarrow \cos\theta = \frac{a}{x}$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{\cancel{a}\sec\theta\cancel{\tan\theta}\, d\theta}{\cancel{a\tan\theta}}$$

$$= \int \sec\theta\, d\theta$$

$$= \ln(\sec\theta + \tan\theta) + c$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + c$$

$$= \ln\{x + \sqrt{x^2-a^2}\} + \underbrace{(c - \ln a)}_{\lambda}$$

$$= \ln\{x + \sqrt{x^2-a^2}\} + \lambda$$

Q) $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right) dx$

Put $x = \cos 2\theta$

$dx = -2\sin 2\theta \, d\theta$

$I = -2\int \cos\left(2\cot^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right) \sin 2\theta \, d\theta$

$= -2\int \cos\left(2\cot^{-1}(\tan\theta)\right) \sin 2\theta \, d\theta$

$= -2\int \cos\left(2\left(\frac{\pi}{2} - \tan^{-1}(\tan\theta)\right)\right) \sin 2\theta \, d\theta$

$= -2\int \cos(\pi - 2\theta) \sin 2\theta \, d\theta$

$= -\int 2\sin 2\theta \cos 2\theta \, d\theta$

$= -\int \sin 4\theta \, d\theta = -\frac{\cos 4\theta}{4} + C$

$= -\frac{1}{4}(2\cos^2 2\theta - 1) + C$

$= -\frac{1}{2}x^2 + \underbrace{\frac{1}{4} + C}_{A} = A - \frac{1}{2}x^2$

~~Q) $\int \sqrt{\frac{x}{a-x}} dx$~~ M–II

Let $\cot^{-1}\sqrt{\frac{1-x}{1+x}} = \theta$

$\cot\theta = \sqrt{\frac{1-x}{1+x}}$

$I = \int \cos 2\theta \, dx$

$= \int \frac{1-\tan^2\theta}{1+\tan^2\theta} = \int \frac{1 - \frac{1+x}{1-x}}{1 + \frac{1+x}{1-x}} = \int \frac{-2x}{2} = -\frac{x^2}{2} + C$

Q) $\int \sqrt{\frac{x}{a-x}}\,dx$

$\cos^2\theta = \frac{x}{a}$ $\quad \frac{1+\cos 2\theta}{2} = \frac{x}{a}$

$\cos 2\theta = \frac{2x}{a} - 1$

$\cos 2\theta = \frac{2x-a}{a}$

$x = a\cos^2\theta \Rightarrow dx = -2a\cos\theta\sin\theta\,d\theta$

$\int \sqrt{\frac{a\cos^2\theta}{a-a\cos^2\theta}}\,dx$

$\int \sqrt{\frac{\cos^2\theta}{\sin^2\theta}}\,dx$

$= \int \frac{\cos\theta}{\cancel{\sin\theta}} \times -2a\cos\theta\,\cancel{\sin\theta}\,d\theta$

$= -2\int a\cos^2\theta\,d\theta$

$= -a\int (1+\cos 2\theta)\,d\theta$

$= -a\left[\theta + \frac{\sin 2\theta}{2}\right] + c$

$a^2 - (2x-a)^2$

$a^2 - 4x^2 - a^2 + 4ax$

$4ax - 4x^2$

$2\sqrt{ax - x^2}$

$\sin 2\theta = \frac{2\sqrt{ax-x^2}}{a}$

$= -a\left[\cos^{-1}\frac{2x-a}{a}\right] = -a\left[\frac{\cos^{-1}\left(\frac{2x-a}{a}\right)}{2} + \frac{\cancel{2}\sqrt{ax-x^2}}{\cancel{2}a}\right] + c$

Q) $\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$

Put $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\int \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta+4)\sqrt{16\tan^2\theta+1}}$

$= \int \frac{\sec^2\theta\,d\theta}{2(1+\tan^2\theta)\sqrt{1+(4\tan\theta)^2}}$

$\frac{1}{2}\int \frac{d\theta}{\sqrt{16\tan^2\theta+1}} = \frac{1}{2}\int \frac{\cos\theta\,d\theta}{\sqrt{16\sin^2\theta+\cos^2\theta}} = \frac{1}{2}\int \frac{\cos\theta\,d\theta}{\sqrt{15\sin^2\theta+1}}$

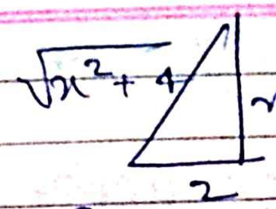
let $\sin\theta = t$

$\cos\theta\,d\theta = dt$

$= \frac{1}{2}\int \frac{dt}{\sqrt{15t^2+1}} = \frac{1}{2\sqrt{15}}\int \frac{dt}{\sqrt{t^2+\left(\frac{1}{\sqrt{15}}\right)^2}}$

$$= \frac{1}{2\sqrt{15}} \ln\left\{ t + \sqrt{t^2 + \frac{1}{15}} \right\} + C$$

$$= \frac{1}{2\sqrt{15}} \ln\left\{ \frac{x}{\sqrt{x^2+4}} + \sqrt{\frac{x^2}{x^2+4} + \frac{1}{15}} \right\} + C$$



$$\sin\theta = \frac{x}{\sqrt{x^2+4}} = t$$

Q) $\displaystyle\int \frac{x^2}{\sqrt{a^6 - x^6}}\, dx$

$x^3 = a^3 \sin\theta \Rightarrow 3x^2\,dx = a^3 \cos\theta\, d\theta$

$\Rightarrow dx = \dfrac{a^3 \cos\theta\, d\theta}{3x^2}$

$$\int \frac{x^2}{\sqrt{a^6 - a^6 \sin^2\theta}}$$

$$= \int \frac{x^2}{a^3\sqrt{(1-\sin^2\theta)}}\, dx = \int \frac{\cancel{x^2}}{\cancel{a^3} \times \cancel{\cos\theta}} \frac{\cancel{a^3}\,\cancel{\cos\theta}\, d\theta}{3\cancel{x^2}} = \frac{1}{3}\int d\theta$$

$$= \frac{1}{3}\theta$$

$$= \frac{1}{3} \sin^{-1} \frac{x^3}{a^3}$$

Q) $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-1}}\, dx$

Let $e^x = t$

$e^x\, dx = dt$

Now, $I = \displaystyle\int \frac{dt}{\sqrt{t^2-1}} = \ln\{t + \sqrt{t^2-1}\} + c$

Q) $\displaystyle\int \frac{\sin 2x}{\sqrt{9 - \sin^4 x}}\, dx$

Let $\sin^2 x = t$

$2\sin x \cos x\, dx = dt$

$I = \dfrac{1}{3}\displaystyle\int \frac{dt}{\sqrt{9-t^2}} = \frac{1}{3} \sin^{-1} \frac{t}{3} + C$

* $\int \frac{dx}{\text{Quadratic}}$ or $\int \frac{dx}{\sqrt{\text{Quadratic}}}$

make a perfect square and then integrate.

* $\int \frac{\text{Linear}}{\text{Quadratic}}$ or $\int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}$

Let $\text{Linear} = A \frac{d}{dx}(\text{Quadratic}) + B$

Q) $\int \frac{dx}{\sqrt{2ax - x^2}}$

$\int \frac{dx}{\sqrt{-(x^2 - 2ax)}} = \int \frac{dx}{\sqrt{-(x^2 + a^2 - 2ax) + a^2}}$

$= \int \frac{dx}{\sqrt{-(x-a)^2 + a^2}} = \int \frac{dx}{\sqrt{a^2 - (x-a)^2}}$

~~Let $x - a = a\sin\theta$~~

$= \frac{1}{a} \sin^{-1} \frac{(x-a)}{a} + C$

Q) $\int \frac{dx}{4x^2 + 4x + 5}$

$= \int \frac{dx}{(4x^2 + 4x + 1) + 4} = \int \frac{dx}{(2x+1)^2 + 4} = \frac{1}{4} \int \frac{dx}{(x+\frac{1}{2})^2 + 1}$

$= \frac{1}{4} \tan^{-1}(x + \frac{1}{2})$

Q) $\int \frac{4x+3}{3x^2+3x+1} dx$

derivative of

$\int \frac{\frac{2}{3}(6x+3)+1}{3x^2+3x+1} dx = \frac{2}{3} \int \frac{(6x+3)}{3x^2+3x+1} dx + \int \frac{dx}{3x^2+3x+1}$

$= \frac{2}{3} \ln(3x^2+3x+1) + \frac{1}{3} \int \frac{dx}{(x^2 + x + \frac{1}{4}) + \frac{1}{12}}$

$$= \frac{2}{3}\ln(3x^2+3x+1) + \frac{1}{3}\int \frac{dx}{\left(\frac{1}{\sqrt{12}}\right)^2 + \left(x+\frac{1}{2}\right)^2}$$

$$\quad + \frac{1}{3} \times \sqrt{12} \tan^{-1}\left(\frac{x+1/2}{1/\sqrt{12}}\right)$$

Q) $\int \frac{2x+1}{\sqrt{4x^2+4x+2}}\,dx$

$$= \int \frac{\frac{1}{4}(8x+4)-2}{\sqrt{4x^2+4x+1+1}} = \int \frac{\frac{1}{4}(8x+4)-2}{\sqrt{(2x+1)^2+1}}\,dx$$

$$= \frac{1}{4}\int \frac{(8x+4)-2}{\sqrt{(2x+1)^2+1}}\,dx$$

$$\cancel{= \frac{1}{4}\int \frac{(8x+4)}{\sqrt{(2x+1)^2+1}}\,dx + 2\int \frac{1}{\sqrt{(2x+1)^2+1}}\,dx}$$

$$\frac{1}{4}\int \frac{(8x+4)}{\sqrt{4x^2+4x+2}}\,dx - 2\int \frac{dx}{\sqrt{4x^2+4x+2}}$$

Put $4x^2+4x+2 = t^2$

$$\frac{1}{4}\int \frac{2t}{t}\,dt - \frac{2}{2}\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$= \frac{1}{2}\sqrt{4x^2+4x+2} - \ln\left\{\left(x+\frac{1}{2}\right) + \sqrt{\left(\frac{x+1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right\} + C$$

Integration by parts (V.V.I)

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f(x)\,g'(x) + g(x)\,f'(x)$$

Int. w.r.t x

$$\Rightarrow f(x)\,g(x) = \int f(x)\,g'(x)\,dx + \int g(x)\,f'(x)\,dx$$

$$\Rightarrow \boxed{\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx}$$

* Here the choice of first and second function is done by ILATE

I : Inverse
L : Logarithmic
A : Algebric
T : Trignometric
E :- exponential.

Q) $\int x \tan^{-1} x \, dx$

$= I = \int x \tan^{-1} x \, dx$

$\Rightarrow I = \tan^{-1} x \frac{x^2}{2} - \int \frac{1}{1+x^2} \times \frac{x^2}{2}$

$= \tan^{-1} x \frac{x^2}{2} - \int \frac{x^2}{2(1+x^2)} dx$

$= \tan^{-1} x \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2+1-1}{(1+x^2)}$

$= \tan^{-1} x \frac{x^2}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \tan^{-1} \frac{x^2}{2} - \frac{1}{2} (x - \tan^{-1} x) + C$

Q) $\int \operatorname{cosec}^2 x \ln(\sec x) \, dx$

$I = \ln \sec x (-\cot x) - \int \frac{1}{\sec x} \sec x \tan x (-\cot x \, dx)$

$- \cot x \ln \sec x + \int 1 \, dx$

$= -\cot x \ln \sec x + x + C$

Q) $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\, dx$

$= \int \sin^{-1}x\,(1-x^2)^{-3/2}\, dx$

$\cancel{= \int (1-x^2)^{-3/2} \sin^{-1}x\, dx}$

$\cancel{= (1-x^2)^{-3/2} \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{-3}{2}(1-x^2)^{-5/2} \int \frac{dx}{\sqrt{1-x^2}}}\, dx$

Let $\sin^{-1}x = t$ $\quad$ $\frac{1}{\sqrt{1-x^2}}\, dx = dt$

$x = \sin t$

$I = \int \frac{t\, dt}{(1-\sin^2 t)} = \int \frac{t\, dt}{\cos^2 t} = \int \sec^2 t\, t\, dt$

$I = \int t \cdot \sec^2 t\, dt$

$= t \cdot \tan t - \int 1 \cdot \tan t$

$\quad '' \quad - \int \tan t \cdot 1 \cdot dt$

$t\tan t - \int \left[\tan t \cdot t - \int \sec^2 t\, t \right.$

$I = t\tan t - \cancel{t\tan t} - I$

$\Rightarrow 2I = 0 \quad \Rightarrow I = 0$

* Integration of alone inverse function and logarithmic function is to be done by parts by taking unity as the second function.

$\int \ln x\, dx = \int \ln x \cdot 1 \cdot dx$

$= \ln x \cdot x - \int \frac{1}{x} \times x\, dx$

$= x\ln x - x + C$

* $\int \cos^{-1}\frac{1}{x}\,dx$

$\int \cos^{-1}\frac{1}{x}\cdot 1\cdot dx$

$= \cos^{-1}\frac{1}{x}\times x - \int \frac{-1}{\sqrt{1-x^2}}\times\frac{-1}{x^2}\,x\,dx$

$= x\cos^{-1}\frac{1}{x} - \int\frac{dx}{\sqrt{x^2-1}}$

$= x\cos^{-1}\frac{1}{x} - \ln\left[x+\sqrt{x^2-1}\right]+c$

Formulae

* $\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]+c$

* $\int\sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2}-a^2\ln\left\{x+\sqrt{x^2-a^2}\right\}\right]+c$

* $\int\sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2}+a^2\ln\left\{x+\sqrt{x^2+a^2}\right\}\right]+c$

First one

$I = \int\sqrt{a^2-x^2}\,dx = \int\underset{I}{\sqrt{a^2-x^2}}\cdot\underset{II}{1}\cdot dx$

$= \sqrt{a^2-x^2}\,x + \int\frac{1\times 2x}{2\sqrt{a^2-x^2}}\,x\,dx$

$= x\sqrt{a^2-x^2} + \int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$

$I = x\sqrt{a^2-x^2} + a^2\int\frac{dx}{\sqrt{a^2-x^2}} - \underbrace{\int\sqrt{a^2-x^2}\,dx}_{I}$

$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}$

$\boxed{I = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]+c}$

$I = \int \sec^3 x \, dx$

$= \int \sec x \sec^2 x \, dx$

$I = \sec x \tan x - \int \sec x \tan x \tan x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$2I = \sec x \tan x + \ln(\sec x + \tan x + c)$

Q) $I = \int x^2 e^{3x} \, dx$

$= x^2 \frac{e^{3x}}{3} - \int 2x \frac{e^{3x}}{3} \, dx$

$= x^2 \frac{e^{3x}}{3} - \frac{2}{3} \int x e^{3x} \, dx$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{3} \left[x \frac{e^{3x}}{3} - \int 1 \times \frac{e^{3x}}{3} \, dx \right]$

$= \frac{x^2 e^{3x}}{3} - \frac{2}{9} \left(x e^{3x} - \frac{1}{3} e^{3x} \, dx \right)$

$= x^2 \frac{e^{3x}}{3} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + c$

Q) $I = \int x^2 e^{3x} \, dx = e^{3x} (Ax^2 + Bx + c) + \lambda$

diff. w.r.t x

$x^2 \cancel{e^{3x}} = \cancel{e^{3x}} (2Ax + B) + (Ax^2 + Bx + c) \times 3 \cancel{e^{3x}}$

$x^2 = 3Ax^2 + x(2A + 3B) + (B + 3c) \quad \forall x \in R$

on comparing

$A = 3 \qquad 3B + 2A = 0 \qquad B + 3c = 0$

$B = -\frac{2}{9} \qquad c = \frac{2}{27}.$

M - II

$$\begin{array}{ccc} x^2 & \oplus & e^{3x} \\ 2x & \ominus & \frac{e^{3x}}{3} \\ 2 & \oplus & \frac{e^{3x}}{9} \\ 0 & & \frac{e^{3x}}{27} \end{array}$$

Q) $\int \sin x \ln(\sec x + \tan x)\,dx$

$= \int \ln(\sec x + \tan x)\,dx \sin x$

$= \ln(\sec x + \tan x) \times (-\cos x) - \int \frac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x)(-\cos x)\,dx$

$= \ln(\sec x + \tan x)(-\cos x) + x + C.$

Q) $\int \ln(x + \sqrt{x^2+a^2}) \cdot 1 \cdot dx$

$= \ln(x+\sqrt{x^2+a^2})\, x - \int \frac{1}{x+\sqrt{x^2+a^2}} \times \left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right) x\,dx$

$= x\ln(x+\sqrt{x^2+a^2}) - \int \frac{x}{\sqrt{x^2+a^2}}\,dx$

Let $x^2+a^2 = t^2$

$\cancel{2}x\,dx = \cancel{2}t\,dt$

$= x\ln(x+\sqrt{x^2+a^2}) - \int \frac{\cancel{t}\,dt}{\cancel{t}}$

$= x\ln(x+\sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + C.$

Q) $\int \frac{x}{1+\sin x}\,dx$

~~Let $\sin x = t$, $\cos x\,dx = dt$~~

$\int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$

$= \int \frac{x(1-\sin x)}{\cos^2 x}\,dx$

$= \int x\sec^2 x(1-\sin x)\,dx$

$= \int x(\sec^2 x - \tan x \sec x)\,dx$

$= x(\tan x - \sec x) - \int 1(\tan x - \sec x)\,dx$

$= x(\tan x - \sec x) - \{\ln\sec x + \ln(\sec x + \tan x)\} + C$

Q) $\int \underset{I}{x}\,\underbrace{\sin x \cos^2 x}_{II}\,dx$

~~$\int \sin x \cos^2 x\,dx$, $\cos x = t$, $\sin x\,dx = -dt$~~

$x\left(-\frac{\cos^3 x}{3}\right) + \int 1\frac{\cos^3 x}{3}\,dx$

$= -\frac{x\cos^3 x}{3} + \frac{1}{3}\int \cos x(1-\sin^2 x)\,dx$

$\sin x = t$

$\cos x\,dx = dt$

$= -\frac{x\cos^3 x}{3} + \frac{1}{3}\int (1-t^2)\,dt$

$= -\frac{x\cos^3 x}{3} + \frac{1}{3}\left[\sin x - \frac{\sin^3 x}{3}\right] + C.$

Q) $\int \underset{I}{\ln x} \underset{II}{\frac{1}{(1+x)^2}} dx$

$= \ln x\left(\frac{-1}{1+x}\right) + \int \frac{1}{x} \frac{1}{x+1} dx$

$= -\frac{\ln x}{x+1} + \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \frac{-\ln x}{x+1} + \ln x - \ln(x+1) + c$

Q) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

Let, $x = a\tan^2\theta$ $\Rightarrow \theta = \tan^{-1}\sqrt{\frac{x}{a}}$

$dx = 2a\tan\theta\sec^2\theta d\theta$

$\int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\; 2a\tan\theta\sec^2\theta d\theta$

$= \int \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}$ "

$= 2a\int \sin^{-1}\sin\theta \; \tan\theta\sec^2\theta d\theta$

$= 2a\int \underset{I}{\theta} \underset{II}{\tan\theta\sec^2\theta} d\theta$

$= 2a\left[\frac{\theta\tan^2\theta}{2} - \int 1 \frac{\tan^2\theta}{2} d\theta\right]$

$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1) d\theta$

$= a\theta\tan^2\theta - a\int \sec^2\theta d\theta \int d\theta$

$= a\theta\tan^2\theta - a\tan\theta - a\theta$

$= a\tan^{-1}\sqrt{\frac{x}{a}} \times \frac{x}{a} - a\sqrt{\frac{x}{a}} - a\tan^{-1}\sqrt{\frac{x}{a}}$

Two classic integrals

1) $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+c$

proof:-

$$\int e^x f(x)\,dx + \int e^x f'(x)\,dx$$

$$= f(x)e^x - \cancel{\int f'(x)e^x dx} + \cancel{\int e^x f'(x) dx}$$

$$= e^x f(x) + c$$

2) $\int \left(f(x)+xf'(x)\right)dx = xf(x)+c$

Proof:-

$$\int 1\cdot f(x)\,dx + \int x f'(x)\,dx$$

$$= f(x)\,x - \cancel{\int f'(x)x\,dx} + \cancel{\int x f'(x)\,dx}$$

$$= xf(x)+c.$$

Q) $\int e^x(\underbrace{\tan x}_{f'} - \underbrace{\ln\cos x}_{f})\,dx$

$$= e^x(-\ln\cos x)+c$$

Q) $\int \frac{xe^x}{(x+1)^2}dx$

~~x = x+1~~

$$\int \left(\frac{(x+1-1)e^x}{(x+1)^2}\right)$$

$$= \int e^x\left(\underbrace{\frac{1}{x+1}}_{f} - \underbrace{\frac{1}{(x+1)^2}}_{f'}\right)dx$$

$$= e^x\left(\frac{1}{x+1}\right)+c$$

Q) $\int e^{\tan^{-1}x} \frac{(1+x+x^2)}{1+x^2} dx$

Let $\tan^{-1}x = t$ $\quad x = \tan t$

$\frac{1}{1+x^2} dx = dt$

$I = \int e^t (1+\tan t + \tan^2 t) dt$

$= \int e^t (\underbrace{\tan t}_{f} + \underbrace{\sec^2 t}_{f'}) dt$

$= e^t \tan t + c$

$= e^{\tan^{-1}x} \tan \tan^{-1}x + c$

$= x e^{\tan^{-1}x} + c$

M - II

$\int e^{\tan^{-1}x} (1 + \frac{x}{1+x^2}) dx$

$= \int (\underbrace{e^{\tan^{-1}x}}_{f} + \underbrace{\frac{x e^{\tan^{-1}x}}{1+x^2}}_{xf'}) dx$

$= x e^{\tan^{-1}x} + c$

Q) $\int e^{2x} \frac{(\sin 4x - 2)}{(1-\cos 4x)} dx$

$\int e^{2x} \frac{\cancel{2}(\sin 2x \cos 2x - 1)}{\cancel{2}\sin^2 2x}$

$= e^{2x} \int (\cot 2x - \csc^2 2x) dx$

$2x = t$

$dx = \frac{1}{2} dt$

$= \frac{1}{2}\int e^t (\underbrace{\cot t}_{f} \underbrace{- \csc^2 t}_{f'})$

$= \frac{1}{2} e^t \cot t + C$

Q) $\int \frac{\ln x}{(1+\ln x)^2} dx$

$\cancel{\int \frac{(\ln x + 1) - 1}{(1+\ln x)^2} dx = \int \frac{1}{(1+\ln x)} - \frac{1}{(\ln x + C)^2} dx}$

$\ln x = t$

$x = e^t$

$dx = e^t dt$

$I = \int \frac{e^t t}{(1+t)^2} dx \rightarrow$ Previous

M. II

$\int \frac{(\ln x + 1) - 1}{(\ln x + 1)^2} dx$

$\int \left(\underbrace{\frac{1}{(1+\ln x)}}_{fx} - \underbrace{\frac{1}{(1+\ln x)^2}}_{x f' x} \right) dx$

$= x \frac{1}{(1+\ln x)} + c.$

Q) $\int \left(\underbrace{\sin(\ln x)}_{f} + \underbrace{\cos(\ln x)}_{x f'} \right) dx$

$= x \sin(\ln x) + c$

Q) $\int \frac{e^x (1+x+x^3)}{(1+x^2)^{3/2}} dx$

$= \int e^x \left(\frac{x(1+x^2) + 1}{(1+x^2)^{3/2}} \right) dx$

$= \int e^x \left(\underbrace{\frac{x}{\sqrt{1+x^2}}}_{f} + \underbrace{\frac{1}{(1+x^2)^{3/2}}}_{f' x} \right) dx$

$= e^x \left(\frac{x}{\sqrt{1+x^2}} \right)$

Q) $\int \left(\ln(\ln x) + \frac{1}{\ln^2 x} \right) dx$

Let $\ln x = t \quad x = e^t$

$\int e^t \left(\ln t + \frac{1}{t^2} \right) dt$

$\int e^t \left(\underbrace{\left(\ln t + \frac{1}{t}\right)}_{f} + \underbrace{\left(\frac{1}{t} + \frac{1}{t^2}\right)}_{f'} \right) dt$

$= e^t \left(\ln t - \frac{1}{t} \right) + c$

$= e^t \left(\ln \ln x + \frac{1}{\ln x} \right) + c.$

Q) $\int \frac{x - \sin x}{1 - \cos x} \, dx$

$= \int \frac{x - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$

$= \int \left(\underbrace{-\cot\left(\frac{x}{2}\right)}_{f} + \underbrace{\frac{1}{2} x \operatorname{cosec}^2\frac{x}{2}}_{xf'} \right) dx$

$= x\left(-\cot\frac{x}{2}\right) + c$

Q) $\int e^x \frac{(x^2+5x+7)}{(x+3)^2} \, dx$

$\int e^x \left[\frac{(x+3)^2 - (2+x)}{(x+3)^2} \right] dx \quad \int e^x \left(\frac{(x+2)(x+3) - 1}{(x+3)^2} \right) dx$

$= \int e^x \left(\underbrace{\frac{x+2}{x+3}}_{f(x)} - \underbrace{\frac{1}{(x+3)^2}}_{f'(x)} \right) dx = e^x \left(\frac{x+2}{x+3} \right) + c$

Q) $\int \frac{x^2 e^x}{(x+2)^2}\,dx$

$= \int e^x \left(\frac{(x^2-4)+4}{(x+2)^2}\right) dx$

$= \int e^x \left(\underbrace{\frac{x-2}{x+2}}_{f} + \underbrace{\frac{4}{(x+2)^2}}_{f'}\right) dx$

$= e^x\left(\frac{x-2}{x+2}\right) + c$

\# $\cancel{I = \int e^{ax} \sin bx\, dx}$

$\cancel{= e^{ax} \int \underset{I}{\sin bx}\, \underset{II}{e^{ax}}}$

$\cancel{= \sin bx \frac{e^{ax}}{a} - \int \underset{II}{\cos bx} \underset{II}{\frac{e^{ax}}{a}}}$

$\cancel{= \frac{e^{ax}}{a}\sin bx - \left[-\frac{e^{ax}}{a^2}\cos bx + \int \frac{e^{ax}}{a^2}\sin bx\right.}$

\# $I = \int \underset{II}{e^{ax}}\, \underset{I}{\sin bx}\, dx$

$\sin bx \frac{e^{ax}}{a} - \int b \cos bx \frac{e^{ax}}{a}\, dx$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int \underset{I}{\cos bx}\, \underset{II}{e^{ax}}\, dx$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\cos bx \frac{e^{ax}}{a} - \int -b \sin bx \frac{e^{ax}}{a}\, dx\right]$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a^2} e^{ax}\cos bx - \frac{b^2}{a^2}\underbrace{\int e^{ax}\sin bx\, dx}_{I}$

$$\left(1+\frac{b^2}{a^2}\right) I = \frac{e^{ax}}{a^2}(a\sin bx - b\cos bx) + c$$

$$\boxed{I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c}$$

* $\int \underbrace{e^{ax}}_{②} \underbrace{\cos bx}_{①} dx$

$$I = \cos bx \frac{e^{ax}}{a} - \int -b\sin bx \frac{e^{ax}}{a} dx$$

$$I = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int \sin bx\, e^{ax} dx$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left[\frac{\sin bx\, e^{ax}}{a} - \int \frac{\cos bx \times b\, e^{ax}}{a}\right] dx$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{b}{a^2}\sin bx\, e^{ax} - \frac{b^2}{a^2}\underbrace{e^{ax}\cos bx\, dx}_{I}$$

$$I\left(\frac{a^2+b^2}{a^2}\right) = \frac{e^{ax}\cos bx}{a} + \frac{b}{a^2} e^{ax}\sin bx$$

$$I = \frac{e^{ax}}{(a^2+b^2)}(b\sin bx + a\cos bx) + c$$

* Manipulating Integrands (Force concept).

$$\int \frac{dx}{x(x^n+1)}$$

out

$$= \int \frac{dx}{x^{n+1}(1+x^{-n})} \qquad 1+x^{-n} = t$$

$$= \int \frac{x^{-(n+1)}}{1+x^{-n}}\,dx \qquad = -n\,x^{-n-1}\,dx = dt$$

$$x^{-(n+1)}\,dx = -\frac{1}{n}\,dt$$

$$= -\frac{1}{n}\int \frac{dt}{t}$$

Q) $\int \frac{x^7}{(1-x^2)^5}\,dx$

$$= \int \frac{x^7}{x^{10}(x^{-2}-1)^5} \qquad \text{Let } x^{-2}-1 = t$$

$$-2x^{-3}\,dx = dt$$

$$= \int \frac{x^{-3}}{(x^{-2}-1)^5} \qquad x^{-3}\,dx = -\frac{1}{2}\,dt$$

$$= -\frac{1}{2}\frac{dt}{t^5}$$

Q) $\int \frac{x}{(1-x^4)^{3/2}}\,dx$

$$= \int \frac{x}{x^6(x^{-4}-1)^{3/2}}\,dx$$

$$= \int \frac{x^{-5}}{(x^{-4}-1)}\,dx \qquad \text{Let } x^{-4}-1 = t^2$$

$$dx(-4x^{-5}) = dt \times 2t$$

$$= -\frac{1}{2}\int \frac{dt}{t^2} \qquad dx\,x^{-5} = -\frac{dt}{4} \times 2t$$

$$= \cancel{\frac{1}{4}\ln t + c = \frac{1}{4}\ln(x^{-4}-1)+c}$$

$$= +\frac{1}{2}\,\frac{1}{t} = \frac{1}{2\sqrt{x^{-4}-1}}$$

Q) $\int \frac{dx}{x^2(1+x^n)^{\frac{n-1}{n}}}$

$= \int \frac{dx}{x^{2+n-1}(x^{-n}+1)}$ Let $x^{-n}+1 = t^n$

$= \int \frac{x^{-n-1}}{(x^{-n}+1)} dx$ $(-nx^{-n-1})dx = dt\, n \times t^{n-1}$

$x^{-n-1} dx = \frac{dt \times n \times t^{n-1}}{-n}$

$= \cancel{-\frac{1}{n}\int \frac{dt}{t}}$

$= \cancel{-\frac{1}{n} \ln t + C}$ $= \cancel{-\frac{1}{n} \ln (x^{-n}+1) + C}$

$= \int -\frac{t^{n-1}}{t^{n-1}} dt$

$= -t + C$

$= -(\sqrt[n]{x^{-n}+1}) + C$

Q) $\int \frac{dx}{x^2(x+\sqrt{1+x^2})}$

$= \int \frac{dx}{x^3(1+\sqrt{x^{-2}+1})}$

$= \int \frac{x^{-3}dx}{1+\sqrt{x^{-2}+1}}$ Let $x^{-2}+1 = t^2$

$-2x^{-3}dx = 2t\,dt$

$x^{-3}dx = -t\,dt$

$= \int \frac{-t\,dt}{1+t}$

$= -\int \frac{(1+t)-1}{(1+t)} dt = \int \left(\frac{1}{1+t} - 1\right) dt$

$= \ln(1+t) - t + C$

Q) $\int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

$= \cancel{\int \frac{x^2(a - bx^{-2})}{x^3\sqrt{c^2x^{-4} - (a+bx^{-2})^2}}}$

$= \cancel{\int \frac{x^{-3}(a-bx^{-2})}{\sqrt{c^2x^{-4}-(a+bx^{-2})^2}}}$

$$= \int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$$

$$= \int \frac{(ax^2-b)\,dx}{x^2\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$$

$$= \int \frac{\left(a-\frac{b}{x^2}\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$$

$$ax+\frac{b}{x}=t$$

$$\left(a-\frac{b}{x^2}\right)dx = dt$$

$$\int \frac{dt}{\sqrt{c^2-t^2}}$$

$$= \sin^{-1}\frac{t}{c}+\lambda$$

Q) $\displaystyle\int \frac{x^2-x^{-2}}{x\sqrt{x^2+x^{-2}+1}}\,dx$

$$= \int \frac{x-x^{-3}}{\sqrt{x^2+x^{-2}+1}}\,dx$$

$= \int$ Let $x^2+x^{-2}+1 = t^2$

$(2x-2x^{-3})\,dx = 2t\,dt \Rightarrow (x-x^{-3})\,dx = t\,dt$

$$\int \frac{t}{t}\,dt = t+C.$$

Integration by partial fraction

* $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + c$

Proof:-

$$I = \int \frac{dx}{(a-x)(a+x)} = \frac{1}{2a}\int \frac{(a-x)+(a+x)}{(a-x)(a+x)}$$

$$= \frac{1}{2a}\int \left(\frac{1}{a-x} + \frac{1}{a+x}\right) dx$$

$$= \frac{1}{2a}\left(\ln(a+x) - \ln(a-x)\right) + c$$

$$= \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + c$$

* $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) + c.$

$$= \int \frac{dx}{(x-a)(x+a)}$$

$$= \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a}\left(\ln(x-a) - \ln(x+a)\right) + c$$

$$= \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) + c.$$

Rule:- 1

When Denominator contains linear non repeated factors.

Q) $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} dx$

Let $\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x+3)} + \frac{C}{(x-4)}$

$\Rightarrow 2x^2+41x-91 = (x+3)(x-4)A + B(x-4)(x-1) + C(x-1)(x+3)$

For A

$\frac{A}{x-1}$ $\Rightarrow$ Put, $x-1=0 \Rightarrow x=1$

as in $x=1$ B and C becomes 0.

$2+41-91 = -12 \times A$

$\Rightarrow A = 4$

For B

Put $x+3=0 \Rightarrow x=-3$

$18-123-91 = 28 \times B$

$\Rightarrow 18-214 = -4 \times -7 \times B$

$\Rightarrow B = -7$

$\therefore I = 4\int \frac{dx}{x-1} - 7\int \frac{dx}{x+3} + 5\int \frac{dx}{x-4}$

$= 4\ln(x-1) - 7\ln(x+3) + 5\ln(x-4)$

For C

$x-4=0 \Rightarrow x=4$

$32+164-91 = C(3)(7)$

$\Rightarrow 196-91 = 21C$

$C = 5.$

$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{4}{(x-1)} + \frac{7}{(x-3)} + \frac{5}{(x-4)}$

Rule 2:-

When Denomenator contains repeated linear factor

$\int \frac{x^3-2x^2+4}{x^2(x-2)^2}$

by rule 1

Let, $\frac{x^3-2x^2+4}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$

For B

$x^2=0$

$x=0$

$B = \left.\frac{x^3-2x^2+4}{(x-2)^2}\right]_{x=0} = 1.$

For D

$(x-2)^2=0$

$x=2$

$\left.\frac{x^3-2x^2+4}{x^2}\right]_{x=2}$ $\frac{8-8+4}{4} = 1$

Now, $x^3 - 2x^2 + 4 = Ax(x-2)^2 + (x-2)^2 + Cx^2(x-2) + x^2$

Put $x = 1$

$3 = A - C + 2$

$\Rightarrow A - C = 1$ — (i)

Put $x = -1$

$-1 - 2 + 4 = -9A + 9 + 3C + 1$

$\Rightarrow 9A - 3C = 9$

$\Rightarrow 3A + C = 3$ — (ii)

(i) + (ii)

$4A = 4$ $\therefore C = 0$

$A = 1$

$$\frac{x^3 - 2x^2 + 4}{x^2(x-2)^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{0}{x-2} + \frac{1}{(x-2)^2} \quad \text{—} \ (1)$$

$$\therefore I = \int \frac{dx}{x} + \int \frac{dx}{x^2} + \int \frac{dx}{(x-2)^2}$$

Rule :- 3

When dinomenator contain non-repeated quadratic eqn.

$$\int \frac{x}{(x-1)(x^2+4)} dx$$

Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+4)}$ — (1)

$$A = \left.\frac{x}{x^2+4}\right]_{x=1} = \frac{1}{5}$$

By eqn 1

$$x = \frac{1}{5}(x^2+4) + (Bx+C)(x-1)$$

It's an identity hence true for all x.

put $x = 0$ $C = 4/5$

put $x = 2$

$$2 = \frac{8}{5} + 2B + \frac{4}{5}$$

$$B = -\frac{1}{5}$$

By eqn I

$$I = \int \left(\frac{1/5}{x-1} - \frac{1/5(x-4)}{x^2+4} \right) dx$$

linear / quad

$$I = \frac{1}{5} \ln(x-1) - \frac{1}{10} \int \frac{2x}{x^2+4} dx + \int \frac{4}{5(x^2+4)} dx$$

$$I = \frac{1}{5} \ln(x-1) - \frac{1}{10} \ln(x^2+4) + \frac{4}{5} \frac{1}{2} \tan^{-1}\frac{x}{2} + c$$

Q) $\int \frac{x^3}{x^4+3x^2+2} dx$

$$= \int \frac{x \cdot x^2}{x^4+3x^2+2} dx$$

Let $x^2 = t$

$x dx = \frac{1}{2} dt$

$$= \frac{1}{2} \int \frac{t}{t^2+3t+2} dt$$

$$= \frac{1}{2} \int \frac{t}{(t+1)(t+2)} dt$$

$$= \cancel{\frac{1}{2} \int \frac{t}{(t+1)(t+2)}} \quad \frac{1}{2} \int \left(\frac{-1}{t+1} + \frac{2}{t+2} \right) dt$$

$$= \frac{1}{2} \left(-\ln(t+1) + 2\ln(t+2) \right) + c$$

Q) $\int \frac{x}{(x^2+2)(x^2+1)} dx$

Let $x^2 = t \Rightarrow x\,dx = \frac{1}{2} dt$

$\frac{1}{2}\int \frac{dt}{(t+2)(t+1)}$

$= \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt$

$= \frac{1}{2}\left(\ln(t+1) - \ln(t+2)\right) + C$

Q) $\int \frac{dx}{x^3+1} dx$

$\int \frac{x^2 - (x^2-1)}{x^3+1} dx$

$= \frac{1}{3}\int \frac{3x^2}{x^3+1} dx - \int \frac{x-1}{x^2-x+1} dx$

$= \frac{1}{3}\ln(x^3+1) - \int \frac{\frac{1}{2}(2x-1) - \frac{1}{2}}{x^2-x+1} dx$

$= \frac{1}{3}\ln(x^3+1) - \frac{1}{2}\int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$

$= \frac{1}{3}\ln(x^3+1) - \frac{1}{2}\ln(x^2-x+1) + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$

$= \frac{1}{3}\ln(x^3+1) - \frac{1}{2}\ln(x^2-x+1) + \frac{1}{2}\int \frac{dx}{1 + \left(\frac{x-1/2}{\sqrt{3}/2}\right)^2}$

$= \frac{1}{3}\ln(x^3+1) - \frac{1}{2}\ln(x^2-x+1) + \frac{1}{2}\left[\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)\right]\frac{\sqrt{3}}{2} + C.$

Q) $\int \frac{dx}{\sin 2x - \sin x}$

$= \int \frac{dx}{2\sin x\cos x - \sin x} = \int \frac{dx}{\sin x(2\cos x - 1)}$

$$= \int \frac{\sin x \, dx}{\sin^2 x(2\cos x - 1)}$$

$$= \int \frac{\sin x \, dx}{(1-\cos^2 x)(2\cos x - 1)}$$

Let $\cos x = t$

$\sin x \, dx = -dt$

$$= -\int \frac{dt}{(1-t^2)(2t-1)}$$

$$= -\int \frac{dt}{(1+t)(1-t)(2t-1)}$$

$$= \int \frac{dt}{(t+1)(t-1)(2t-1)}$$

$$= \int \left(\frac{dt}{(t+1)} + \frac{dt}{(t-1)} + \frac{dt}{(2t-1)} \right)$$

$= \ln(1+t) + \ln(t-1) + \ln(2t-1)$

$= \ln(1+\cos x) + \ln(\cos x - 1) + \ln(2\cos x - 1)$

Q) $\int \frac{\sin x}{\sin 4x} dx$

$$= \int \frac{\sin x}{\sin(2x+2x)} dx$$

$$= \int \frac{\sin x}{2\sin 2x \cos 2x} dx$$

$$= \int \frac{\sin x}{4\sin x \cos x \cos 2x} dx$$

$$= \frac{1}{4} \int \frac{dx}{\cos x \cos 2x}$$

$$= \frac{1}{4} \int \frac{dx}{(\cos x)\cos 2x} = \frac{1}{4} \int \frac{\cos x \, dx}{\cos^2 x \cos 2x}$$

$$= \frac{1}{4} \int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)} dx$$

Let $\sin x = t$

$\cos x\, dx = dt$

$$\frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}$$

$$= \frac{1}{4}\int\left(\frac{-1}{(1-t^2)} + \frac{2}{(1-2t^2)}\right)dt$$

$$= -\frac{1}{4}\left[\int \frac{dt}{(1-t^2)} - \frac{1}{2}\int \frac{2\,dt}{1-2t^2}\right]$$

$$= -\frac{1}{4}\cdot\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) + \frac{1}{4}\int \frac{dt}{\frac{1}{2}-t^2}$$

$$= -\frac{1}{8}\ln\left(\frac{1+t}{1-t}\right) + \frac{1}{4}\cdot\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\ln\left(\frac{\frac{1}{\sqrt{2}}+t}{\frac{1}{\sqrt{2}}-t}\right) + C$$

Integration of trigonometric function

* $\int \frac{dx}{a+b\sin^2 x}$ or $\int \frac{dx}{a+b\cos^2 x}$ or $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Key point :- divide N^r and D^r by $\cos^2 x$ and put $\tan x = t$

* $\int \frac{dx}{a+b\cos x}$ or $\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a\sin x + b\cos x + c}$

Key point :- Put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

and put $\tan\frac{x}{2} = t$ $\quad \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

Put $\tan x/2 = t$.

Q) $\int \frac{dx}{4-5\sin^2 x}$

$= \cancel{\int \frac{\cos^2 x\,dx}{4\cos^2 x - 5\sin^2 x \cos^2 x}} = \int \frac{dx}{4-4\sin^2 x - \sin^2 x}$

$= \int \frac{dx}{4\cos^2 x - \sin^2 x}$

$= \int \frac{\frac{dx}{\cos^2 x}}{4 - \tan^2 x}$

Let $\tan x = t$

$\sec^2 x\,dx = dt$

$= \int \frac{\sec^2 x\,dx}{4-\tan^2 x}$

$= \int \frac{dt}{4-t^2}$

$= \cancel{\int \frac{dt}{(2-t)(2+t)}}$

$= \frac{1}{2 \times 2} \ln\left(\frac{2+t}{2-t}\right) + C$

Q) $\int \frac{dx}{(3\sin x - 4\cos x)^2}$

Divide Nr and Dr by $\cos^2 x$

$\int \frac{\sec^2 x\,dx}{(3\tan x - 4)^2}$

Put $\tan x = t$

$\sec^2 x\,dx = dt$

$\int \frac{dt}{(3t-4)^2}$

$= -\frac{1}{3}\left(\frac{1}{3t-4}\right) + C$

Q) $\int \frac{dx}{5+4\cos x} = \int \frac{dx}{5+4\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$

$= \int \frac{\sec^2 \frac{x}{2}\,dx}{9+\tan^2 x/2}$

Let $\tan \frac{x}{2} = t$

$\frac{1}{2}\sec^2 \frac{x}{2}\,dx = dt$

$\sec^2 \frac{x}{2}\,dx = 2dt$

$$= \int \frac{2\,dt}{3^2+t^2}$$

$$= \frac{2}{9} \tan^{-1}\frac{t}{3} + C$$

Q) $\int \frac{dx}{3+2\sin x+\cos x}$

$$= \int \frac{dx}{3+\frac{\cancel{4}\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \int \frac{\sec^2 x/2\,dx}{3+3\tan^2 x/2+4\tan x/2+1-\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2\,dx}{4+2\tan^2 x/2+4\tan x/2}$$

$$= \int \frac{2\,dt}{4+2t^2+4t}$$

Let $\tan\frac{x}{2} = t$

$\frac{\sec^2 x/2}{2} = dt$

$\sec^2 x/2 = 2dt$

$$= \int \frac{dt}{\cancel{\ }t^2+2t+2}$$

$$= \int \frac{dt}{1+(t+1)^2}$$

$$= \tan^{-1}(t+1) + C.$$

H/W

Q) 1) $\int \frac{dx}{\cos x(5+3\cos x)}$

2) $\int \frac{\cos x}{5-3\cos x}\,dx$

3) $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

1) $\int \frac{dx}{\cos x(5+3\cos x)}$

$= \frac{1}{5}\int \frac{(5+3\cos x) - 3\cos x}{\cos x(5+3\cos x)} dx$

$= \frac{1}{5}\int \frac{dx}{\cos x} - \frac{3}{5}\int \frac{dx}{5+3\cos x}$ Let $\tan\frac{x}{2}$ t

$\sec^2 x/_2\, dx = dt$

$= \frac{1}{5}\int \frac{dx\,(1+\tan^2 x/_2)}{(1-\tan^2 x/_2)} - \frac{3}{5}\int \frac{dx}{5+\frac{3(1-\tan^2 x/_2)}{(1+\tan^2 x/_2)}}$

$= \frac{1}{5}\int \frac{\sec^2 x\, dx}{1-t^2} - \frac{3}{10}\int \frac{\sec^2 x/_2\, dx}{4 + \tan^2 x/_2}$

$= \boxed{\frac{1}{5}\int \frac{t\,dt}{1-t^2} - \frac{3}{10}\int \frac{dt}{4+t^2}}$

2) $\int \frac{\cos x}{5-3\cos x} dx$

$= \frac{1}{3}\int \frac{5-(5-3\cos x)}{5-3\cos x} dx$

$= \frac{5}{3}\int \frac{dx}{5-3\cos x} - \frac{1}{3}\int dx$

$= \frac{5}{3}\int \frac{dx}{5-\frac{3(1-\tan^2 x/_2)}{1+\tan^2 x/_2}} - \frac{x}{3}$ Let $\tan x/_2 = t$

$\sec^2 x/_2\, dx = dt$

$= \frac{5}{3}\int \frac{\sec^2 x/_2\, dx}{2+8\tan^2 x/_2} + \frac{x}{3} \Rightarrow \frac{5}{6}\int \frac{dt}{1+4t^2} + \frac{x}{3}$

3) $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

Divide N^r and D^r by $\cos^4 x$

$I = \int \frac{\frac{2\sin x \cos x}{\cos^4 x}}{1+\tan^4 x}$

$\Rightarrow I = \int \frac{2\tan x \sec^2 x\, dx}{1+\tan^4 x}$

Let $\tan^2 x = t$

$2\tan x \sec^2 x\, dx = dt$

$I = \int \frac{dt}{1+t^2}$

$I = \tan^{-1} t + c$

$I = \tan^{-1} \tan^2 x + c$

* $\int \frac{a \sin x + b\cos x + c}{l\sin x + m\cos x + n} dx$

Let $Nr = A\, Dr + B \frac{d}{dx} Dr + c$

Q) $\int \frac{6+3\sin x + 14\cos x}{3+4\sin x + 5\cos x} dx$

$6+3\sin x + 14\cos x = A(3+4\sin x + 5\cos x) + B(4\cos x - 5\sin x) + c$

On comparing

$3A = 6$ — ① $4A - 5B = 3$ — ②

$A = 2$ $5A + 4B = 14$

$\therefore A = 2$ $c = 0$,

$B = 1$

$I = \int \frac{2(3+4\sin x + 5\cos x) + (4\cos x - \sin x)}{3+4\sin x + 5\cos x} dx$

$I = 2\int dx + \int \frac{4\cos x - \sin x}{3+4\sin x + 5\cos x} dx$

$= 2x + \ln(3+4\sin x + 5\cos x) + c.$

Q) $\int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}} dx$

Let $3e^x + 5e^{-x} = A(4e^x - 5e^{-x}) + B(4e^x + 5e^{-x}) = 0$

$4A + 4B = 3$ and $-5A + 5B = 5$

$A + B = 3/4$

$B - A = \cancel{5}$

$A + B = 4/3$

$2B = \frac{7}{4} \Rightarrow B = \frac{7}{8}$

$\therefore A = -\frac{1}{8}$

Now, $I = \int \frac{-\frac{1}{8}(4e^x - 5e^{-x}) + \frac{7}{8}(4e^x + 5e^{-x})}{4e^x - 5e^{-x}}$

$I = -\frac{1}{8}\int dx + \frac{7}{8}\int \frac{(4e^x + 5e^{-x})}{4e^x - 5e^{-x}} dx$

$I = -\frac{x}{8} + \frac{7}{8} \ln(4e^x - 5e^{-x}) + c$

Q) $\int \frac{\sin x}{e^x - \sin x - \cos x} dx$

~~$\Rightarrow$ Let $\sin x = A(e^x - \sin x - \cos x) + B($~~

$= -\frac{1}{2}\int \frac{(e^x - \sin x - \cos x) - (e^x - \cos x + \sin x)}{e^x - \sin x - \cos x}$

$= -\frac{1}{2}\int dx + \frac{1}{2}\int \frac{(e^x - \cos x + \sin x)}{(e^x - \sin x - \cos x)}$

$= -\frac{x}{2} + \ln(e^x - \sin x - \cos x) + c.$

* Algebric twins

$\int \frac{x^2+1}{x^4 + Kx^2 + 1}$ or $\int \frac{x^2-1}{x^4 + Kx^2 + 1} dx$

Key point :- Divide N^r and D^r by x^2 and put $x - \frac{1}{x} = t$ or $x + \frac{1}{x} = t$

Q) $\int \frac{x^2+1}{x^4+7x^2+1}\,dx$

divide N^r and D^r by x^2

$\int \frac{1+1/x^2}{x^2+7+1/x^2}\,dx$

let $x-\frac{1}{x}=t \Rightarrow x^2+\frac{1}{x^2}=t^2+2$

$dx\left(1+\frac{1}{x^2}\right)=dt$

$= \int \frac{dt}{t^2+2+7} = \int \frac{dt}{t^2+9}$

$= \frac{1}{3}\tan^{-1}\frac{x}{3}$

Q) $\int \frac{x^2+1}{x^4-23x^2+1}\,dx$

divide N^r and D^r by x^2

$\int \frac{1+1/x^2}{x^2-23+1/x^2}\,dx$ let $x+\frac{1}{x}=t \Rightarrow x^2+\frac{1}{x^2}=t^2-2$

diff w.r.t x

$\left(1-\frac{1}{x^2}\right)dx=dt$

$= \int \frac{dt}{t^2-23+2}$

$= \int \frac{dt}{t^2-25} = \frac{1}{10}\ln\left(\frac{t-5}{t+5}\right)+C.$

Q) $\int \frac{x^2+1}{x^4+1}\,dx$

$= \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+1}\,dx$

$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}dx + \frac{1}{2}\int \frac{(x^2-1)dx}{x^4+1}$

$= \frac{1}{2}\int \frac{1+1/x^2}{x^2+1/x^2}dx + \frac{1}{2}\int \frac{1-1/x^2}{x^2+1/x^2}\,dx$

Let $x - \frac{1}{x} = t$ and $x + \frac{1}{x} = t$

$x^2 + \frac{1}{x^2} = t^2 + 2$ $\qquad$ $x^2 + \frac{1}{x^2} = t^2 - 2$

diff w.r.t x $\qquad$ diff w.r.t x

$dx\left(1 + \frac{1}{x^2}\right) = dt$ $\qquad$ $dx\left(1 - \frac{1}{x^2}\right) = dt$

Now $= \frac{1}{2}\int \frac{dt}{t^2+2} + \frac{1}{2}\int \frac{dt}{t^2-2}$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\frac{t}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln\left(\frac{t-\sqrt{2}}{t+\sqrt{2}}\right) + c$

Put value of t

Q) $\int \frac{dx}{x^4+1}$

$= \frac{1}{2}\int \frac{(x^2+1) - (x^2-1)}{x^4+1}$

$= \frac{1}{2}\int \frac{x^2+1}{x^4+1} - \frac{1}{2}\int \frac{(x^2-1)}{x^4+1}$ $\qquad$ Let

$= \frac{1}{2}\int \frac{1 + 1/x^2}{x^2 + \frac{1}{x^2}} - \frac{1}{2}\int \frac{1 - 1/x^2}{x^2 + 1/x^2}$ $\qquad$ $x - \frac{1}{x} = t$ $\qquad$ $x + \frac{1}{x} = t$

diff. w.r.t x $\qquad$ $1 - \frac{1}{x^2} = dt$

$1 + \frac{1}{x^2} = dt$

$= \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{dt}{t^2-2}$ $\qquad$ and $x^2 + \frac{1}{x^2} = t^2 + 2$ $\qquad$ $x^2 + \frac{1}{x^2} = t^2 - 2$

$= \frac{1}{4} \tan^{-1}\frac{t}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \ln\left(\frac{t-\sqrt{2}}{t+\sqrt{2}}\right) + c.$

Put value of t

Q) $\int \frac{x^{17}}{1+x^{24}} dx$

Let $\int \frac{x^{12} x^5}{1+x^{24}} dx$

Let $x^6 = t \Rightarrow t^2 = x^{12}$

$(6x^5) dx = dt$ Let $t - \frac{1}{t} = u$

$= \frac{1}{6} \int \frac{t^2 dt}{1+t^4}$ Previous question.

$= \frac{1}{12} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \ln\left(\frac{u-\sqrt{2}}{u+\sqrt{2}}\right) + c.$

Put value of u and then t

Q) $\int \frac{x^2+3}{x^4+8x^2+9}$

divide N^r and D^r by x^2

$I = \int \frac{1 + 3/x^2}{x^2 + 8 + 9/x^2} dx$

$I =$ Let $x - \frac{3}{x} = t \Rightarrow x^2 + \frac{9}{x^2} = t^2 + 6$

$dx\left(1 + \frac{3}{x^2}\right) = dt$

$I = \int \frac{dt}{t^2 + 14}$

$I = \frac{1}{\sqrt{14}} \tan^{-1} \frac{t}{\sqrt{14}} + c.$

Q) $\int \sqrt{\tan x}\, dx$

Let $\tan x = t^2$

$\sec^2 x\, dx = 2t\, dt$

$dx = \frac{2t\, dx}{1 + \tan^2 x}$

$I = 2\int \frac{t^2 dx}{1+t^4}$ Previous prob. $\Rightarrow I = 2\int \frac{dx}{1 + 1/t^2}$

$\cancel{\int \frac{(t^2-1)+(t^2+1)}{(1+t^2)} dx = \int dx + \int \frac{(t^2-1)}{(t^2+1)} dx}$

Trigonometric twins

If $(\cos x+\sin x)$ or $(\cos x-\sin x)$ present in N^r along with $b+\sin x$ or $\sqrt{b+\sin 2x}$ in D^r then put

$$\sin x+\cos x = t$$

or

$$\sin x - \cos x = t$$

Q) $\int \frac{\cos x+\sin x}{\sqrt{10+\sin 2x}}\,dx$

Let $\sin x - \cos x = t$

$dx(\cos x+\sin x) = dt$

Let $1-\sin 2x = t^2$

$\sin 2x = 1-t^2$

$I = \int \frac{dt}{\sqrt{11-t^2}} = \frac{1}{\sqrt{11}}\sin^{-1}\frac{t}{\sqrt{11}}+c$

Q) $\int \frac{\cos x-\sin x}{\sqrt{8+\sin 2x}}\,dx$

Let ~~cos x - sin x~~

$\sin x+\cos x = t$

$dx(\cos x-\sin x) = dt$

$I = \int \frac{dt}{\sqrt{9-t^2}}$ ~~dx~~ Let $1-\sin 2x = t^2$

$\sin 2x = 1-t^2$

$= \int \frac{dt}{\sqrt{3^2-t^2}}$

Q) $\int (\cos x+\sin x)\sqrt{\sin 2x}\,dx$

Let $\sin x-\cos x = t$

$= \int dt\sqrt{1-t^2}$

$dx(\cos x+\sin x) = dt$

$1-\sin 2x = t^2$

$\sin 2x = 1-t^2$

$= \frac{1}{2}\left[t\sqrt{1+t^2}+\sin^{-1}t\right]$

Q) $\int \frac{\cos x}{\sqrt{10+\sin 2x}}\,dx$

$= \frac{1}{2}\int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{\sqrt{10+\sin 2x}}\,dx$

$= \frac{1}{2}\int \frac{\cos x+\sin x}{\sqrt{10+\sin 2x}}\,dx + \frac{1}{2}\int \frac{(\cos x-\sin x)}{\sqrt{10+\sin 2x}}\,dx$

$= \frac{1}{2}\int \frac{dt}{\sqrt{11-t^2}} + \frac{1}{2}\int \frac{dt}{\sqrt{3+t^2}}$

$= \frac{1}{2}\sin^{-1}\frac{t}{\sqrt{11}} + \frac{1}{2}\ln\{t+\sqrt{3+t^2}\}+C$

Let $\sin x-\cos x=t$
$(\cos x+\sin x)dx=dt$
$\Rightarrow 1-\sin 2x=t^2$
$\Rightarrow \sin 2x=1-t^2$
and
$\sin x+\cos x=t$
$(\cos x-\sin x)dx=dt$
$1+\sin 2x=t^2$
$\sin 2x=t^2-1$

Put value of t.

NOTE :- Put value of t as taken above.

Q) $\int \frac{dx}{\operatorname{cosec} x+\cos x}$

$= \int \frac{\sin x}{1+\frac{\sin 2x}{2}}\,dx$ Let t =

$= \int \frac{2\sin x}{2+\sin 2x}\,dx$

$= \int \frac{(\cos x+\sin x)-(\cos x-\sin x)}{2+\sin 2x}\,dx$

$= \int \frac{\cos x+\sin x}{2+\sin 2x} - \int \frac{\cos x-\sin x}{2+\sin 2x}\,dx$

$= \int \frac{dt}{3-t^2} - \int \frac{dt}{1+t^2}$

$= \frac{1}{2\sqrt{3}}\ln\left(\frac{\sqrt{3}+t}{\sqrt{3}-t}\right) - \tan^{-1}(t) \to \sin x+\cos x$

Put value of t

$\sin x-\cos x$

Q) $\int \frac{dx}{\operatorname{cosec} x - \sec x} dx$

$\int \frac{\sin x \cos x}{\cos x - \sin x} dx$ Let ~~$\sin x + \cos x = t$~~ ~~$(\cos x - \sin x) dx = dt$~~

$= \frac{1}{2}\int \frac{1-(\cos x - \sin x)^2}{(\cos x - \sin x)} dx$

$= \frac{1}{2}\int \frac{dx}{\cos x - \sin x} - \frac{1}{2}\int (\cos x - \sin x) dx$

$= \frac{1}{2}\int \frac{dx}{\frac{1-\tan^2 x/2}{1+\tan^2 x/2} - \frac{2\tan x/2}{1+\tan^2 x/2}} - \frac{1}{2}\int (\sin x + \cos x)$

$= \frac{1}{2}\int \frac{\sec^2 x\, dx}{1-\tan^2 x/2 - 2\tan x/2} - \frac{(\sin x + \cos x)}{2}$

* Integral of the form $\int \sin^m x \cos^n x\, dx$

i) if m = odd, Put $\cos x = t$

ii) if n = odd, Put $\sin x = t$

iii) If m, n both odd Put $\frac{\sin x}{\cos x} = t$

iv) If m, n both even then then use multiple angle formulae.

v) if $m+n$ = (–)ve even Integer, Put $\tan x = t$

Q) $\int \sin^3 x \cos^2 x\, dx$

$\downarrow$

$\sin^2 x \sin x$

$\Rightarrow I = \int (1-\cos^2 x)\cos^2 x \sin x\, dx$

Let $\cos x = t$

$\sin x\, dx = dt$

$= -\int t^2(1-t^2)dt$

$= \int (t^4 - t^2) dt$

$= \frac{t^5}{5} - \frac{t^3}{3}$

Q) $\int \sin^4 x \cos^2 x \, dx$

$$I = \frac{1}{8}\int (4\sin^2 x \cos^2 x)(2\sin^2 x)\,dx$$

$$I = \frac{1}{8}\int \sin^2 2x \,(1-\cos^2 2x)\,dx$$

$$= \frac{1}{8}\int \frac{(1-\cos 4x)}{2}(1-\cos^2 2x)\,dx$$

$$\cancel{= \frac{1}{16}\int 1-\cos^2 2x-\cos 4x+\cos 4x\cos^2 2x}$$

$$\cancel{= \frac{1}{16}\int 2\sin^2 2x+\cos^2 2x\,(1-\cos 4x)\,dx}$$

$$\cancel{= \frac{1}{16}\int 2\sin^2 2x-\cos^2 2x\times 2\sin^2 2x\,dx}$$

$$\cancel{= \frac{1}{16}\int 2\sin^2 2x\,(1-\cos^2 2x)\,dx}$$

$$\cancel{= \frac{1}{16}\int 2\sin^2 2x\times\sin^2 2x\,dx}$$

$$\cancel{= \frac{1}{8}\int \sin^4 2x\,dx}$$

$$\frac{1}{16}\int 1-\cos 4x-\cos 2x+\cos 4x\cos 2x$$

$$= \frac{1}{16}\int 1-\cos 4x-\cos 2x+\frac{\cos 6x}{2}+\frac{\cos 2x}{2}$$

$$= \frac{1}{16}\int 1-\frac{\cos 4x}{2}+\frac{\cos 6x}{2}-\cos 2x$$

$$= \frac{1}{16}\left(x-\frac{\sin 4x}{2}+\frac{\sin 6x}{2}-\frac{\sin 2x}{2}\right)$$

$$\cancel{I = \frac{1}{16}\int (1-\cos 4x)\left[1-\left(\frac{1+\cos 4x}{2}\right)\right]dx}$$

$$\cancel{= \frac{1}{32}\int (1-\cos 4x)(1-\cos 4x)\,dx}$$

$$\cancel{= \frac{1}{32}\int 1-\cos 4x-\cos}$$

* **Forcing Integration by parts :-**

$$\int \frac{dx}{(x^4-1)^2}$$

$$I = \int \underset{II}{\frac{4x^3}{(x^4-1)^2}} \times \underset{I}{\frac{1}{4x^3}}\, dx$$

$$\frac{1}{4x^3}\left(-\frac{x^4}{x^4-1}\right) - \int \frac{1}{4}\cdot\left(\frac{3}{x^4}\right)\frac{1}{x^4-1}\, dx$$

$$= \frac{-x^4}{4x^3(x^4-1)} - \frac{3}{4}\int \frac{dx}{x^4(x^4-1)}$$

$$= \frac{-x^4}{4x^3(x^4-1)} - \frac{3}{4}\int\left(\frac{1}{x^4-1} - \frac{1}{x^4}\right)dx$$

$$= \cancel{\frac{-1}{4x^3(x^4-1)} - \frac{3}{4}\int\left(\frac{1}{x^4-1} - \frac{1}{x^4}\right)}dx$$

$$= \frac{-x}{4(x^4-1)} - \frac{3}{4}\int\frac{dx}{(x^2-1)(x^2+1)} + \frac{3}{4}\int x^{-4}dx$$

$$= \frac{-x}{4(x^4-1)} - \frac{3}{8}\int dx\left(\frac{1}{(x^2-1)} - \frac{1}{x^2+1}\right) + \frac{3}{34x^3}$$

$$= \frac{-x}{4(x^4-1)} - \frac{1}{4x^3} + \frac{3}{8}\tan^{-1}x - \frac{3}{8}\int\frac{dx}{(x-1)(x+1)}$$

$$\quad '' \qquad '' \qquad '' \qquad - \frac{3}{16}\int\left(\frac{1}{(x-1)} - \frac{1}{x+1}\right)dx$$

Q) $\displaystyle\int \frac{x^2}{(x\sin x + \cos x)^2}\, dx$

$$\Rightarrow I = \int \frac{x\cos x}{(x\sin x+\cos x)^2}\,\frac{x}{\cos x}\, dx$$

$$= \frac{x}{\cos x}\left(-\frac{1}{x\sin x+\cos x}\right) + \int \frac{\cos x + x\sin x}{\cos^2 x}\left(\frac{dx}{\cos x + x\sin x}\right)$$

$$= \frac{-x}{\cos x(x\sin x+\cos x)} + \tan x + c$$

$$\boxed{\int \frac{dx}{(ax+b)^m (cx+d)^n} \text{ if } m+n=2}$$

Let $\frac{ax+b}{cx+d} = t$ Put $\frac{ax+b}{cx+d} = t$

i) $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$ ii) $\int e^{\sin x}\left(\frac{x^2\cos^3 x - \sin x}{\cos^2 x}\right) dx$

i) $\int \frac{dx}{(x-1)^{3/4}}$

$\Downarrow$

Q) $\int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$

$$I = \int \frac{dx}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}$$

Let

$\frac{x-1}{x+2} = t \Rightarrow \frac{(x+2)-3}{(x+2)} = t$

$1 - \frac{3}{(x+2)} = t$

$\left(\frac{3}{(x+2)^2}\right) dx = dt$

$\frac{dx}{(x+2)^2} = \frac{dt}{3}$

$I = \frac{1}{3}\int \frac{dt}{t^{3/4}}$

$I = \frac{4}{3}(t)^{1/4}$

$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}$

ii) $\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$

$= \int \underbrace{x}_{I}\, \underbrace{e^{\sin x}\cos x}_{II}\, dx - \int \underbrace{e^{\sin x}}_{I}\, \underbrace{\sec x \tan x}_{II}\, dx$

$= \left(x e^{\sin x} + \int 1 \cdot e^{\sin x} dx\right) - \left(e^{\sin x}\sec x - e^{\sin x}\cos x \sec x dx\right)$

$= x e^{\sin x} - \cancel{\int e^{\sin x} dx} - e^{\sin x}\sec x + \cancel{\int e^{\sin x} dx}$

$e^{\sin x}(x - \sec x) + c$

M–II Using classic Integral.

$\int e^{\sin x}\left(x - \frac{\sin x}{\cos^3 x}\right)\cos x\, dx$

Let $\sin x = t$

$\cos x\, dx = dt$

$I = \int e^t\left(x - \frac{t}{(1-t^2)^{3/2}}\right)dt$

$I = \int e^t\left(\underbrace{\left(\sin^{-1}t - \frac{1}{\sqrt{1-t^2}}\right)}_{f} + \underbrace{\left(\frac{1}{\sqrt{1-t^2}} - \frac{t}{(1-t^2)^{3/2}}\right)}_{f'}\right)dt$

$= e^t\left(\sin^{-1}t - \frac{1}{\sqrt{1-t^2}}\right) + c$

$= e^{\sin x}(x - \sec x) + c.$

Integration of Irrational Algebraic function :-

Type	Subtitution	Example
1) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$	Put $px+q = t^2$	$\int \frac{dx}{(2x+1)\sqrt{4x+3}}$
2) $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$	Put $px+q = t^2$	$\int \frac{dx}{(2x+2)\sqrt{4x^2+5x-2}}$
3) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$	Put $ax+b = \frac{1}{t}$	$\int \frac{dx}{(x+1)\sqrt{1+x+x^2}}$ $x+1 = \frac{1}{t}$
4) $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$	Put $x = \frac{1}{t}$ or any other ~~substitution~~ Suitable trigonometric substitution	$\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$ Put $x = 2\tan\theta$

$$* \int \sqrt{(x-\alpha)(\beta-x)}\,dx$$

or

$$\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx \qquad \text{Put } x = \alpha\cos^2\theta + \beta\sin^2\theta$$

or

$$\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$$

$$* \int \sqrt{(x-\alpha)(x-\beta)}\,dx$$

or $\qquad$ Put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx \quad \text{or} \quad \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$$

2000
4000
6000

* Reduction formula * $\sin^n x\,dx$

$$I = \int \sin^n x\,dx = \int \underset{I}{\sin^{n-1}x}\ \underset{II}{\sin x}\,dx$$

$$= -\sin^{n-1}x\cos x + \int (n-1)(\sin x)^{n-2}\underset{1-\sin^2 x}{\cos^2 x}$$

$$= -\sin^{n-1}x\cos(x) + (n-1)\int \sin^{n-2}x\,dx - (n-1)\underbrace{\int \sin^n x\,dx}_{I}$$

$$(1+n-1)I = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\Rightarrow \boxed{\int \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx}$$

Q) $\int \sin^8 x\,dx =$

$$= -\frac{\sin^7 x\cos x}{8} + \frac{7}{8}\int \sin^6 x\,dx$$

$$= -\frac{\sin^7 x\cos x}{8} + \frac{7}{8}\left\{-\frac{\sin^5 x\cos x}{6} + \frac{5}{6}\int \sin^4 x\,dx\right\}$$

* $\int \tan^n x\,dx$

$$I = \int \tan^{n-2}x\ \underset{(\sec^2 x - 1)}{\tan^2 x}\,dx$$

$$= \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

$\tan x = t$

$\sec^2 x\,dx = dt$

$$= \int t^{n-2}\,dt - \int \tan^{n-2}x\,dx$$

$$\Rightarrow \boxed{\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx}$$

* $\boxed{I_n = \int \frac{dx}{(x^2+1)^n}}$

$\int \underset{I}{\frac{1}{(x^2+1)^n}} \cdot \underset{II}{1}\, dx$

$= \frac{1}{(x^2+1)^n} x + \int \frac{n}{(x^2+1)^{n+1}} \times 2x^2\, dx$

$= \frac{x}{(x^2+1)^n} + 2n \int \frac{(x^2+1)-1}{(x^2+1)^{n+1}}\, dx$

$I_n = \frac{x}{(x^2+1)^n} + 2n \underbrace{\int \frac{dx}{(x^2+1)^n}}_{I_n} - 2n \underbrace{\int \frac{dx}{(x^2+1)^{n+1}}}_{I_{n+1}}$

$\Rightarrow \boxed{2n\, I_{n+1} = \frac{x}{(x^2+1)^n} + (2n-1) I_n}$

* $\int \frac{\text{Quad}}{(\text{Quad})^2}\, dx$

Let Quad in N^r = A quad in D^r + B $\frac{d}{dx}$(quad in D^r) + C

Sample Prob.

Q) $\int \frac{5x^2-12}{(x^2-6x+13)^2}\, dx$

$5x^2-12 = A(x^2-6x+13) + B(2x-6) + C$ — (1)

comparing cofficions

$A = 5$

$-6A + 2B = 0$

$B = 15$

$13A - 6B + C = -12$

$\Rightarrow 65 - 90 + C = -12$

$\Rightarrow C = 13$

$\therefore$ Putting value

$5x^2-12 = 5(x^2-6x+13)+15(2x-6)+13$

$$I = 5\int\frac{dx}{x^2-6x+13} + 15\int\frac{2x-6}{(x^2-6x+13)^2} + 13\int\frac{dx}{(x^2-6x+13)^2}$$

$\downarrow$ I_1 $\qquad$ $\downarrow$ I_2 $\qquad$ $\downarrow$ I_3

For I_3

$$13\int\frac{dx}{(x^2-6x+13)^2}$$

$$= 13\int\frac{dx}{((x-3)^2+2^2)^2}$$

Let $x-3 = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\cos^2\theta\, dx = 2d\theta$

$$= 13\int\frac{dx}{\{4(\tan^2\theta+1)\}^2}$$

$$= \frac{13}{16}\int\cos^2\theta\, dx$$

$\tan^{-1}\left(\frac{x-3}{2}\right) = \theta$

$$= \frac{13}{16}\int 2d\theta$$

$$= \frac{13}{16}\, 2\theta = \frac{13}{8}\tan^{-1}\left(\frac{x-3}{2}\right)$$

$I = \frac{5}{4}\tan^{-1}\left(\frac{x-3}{4}\right) - \frac{15}{(x^2-6x+13)} + \frac{13}{8}\tan^{-1}\left(\frac{x-3}{2}\right)$

9) $$\int\frac{dx}{(\sin x+2\sec x)^2}$$

Multiply N^r and D^r by $\sec^2 x$

$$= \int\frac{\sec^2 x\, dx}{(\tan x+2\sec^2 x)^2}$$

$$= \int\frac{\sec^2 x\, dx}{(2\tan^2 x+\tan x+2)^2}$$

$\cancel{\tan x + 2\sec^2 x = t}$

$\sec^2 x$

Let $\tan x = t$

$\tan x = t$

$(dx\, \sec^2 x) = dt$

$$\int\frac{dt}{(2t^2+t+2)^2}$$

$$= \frac{1}{4}\int \frac{dt}{\left(t^2+\frac{t}{2}+1\right)^2} = \frac{1}{4}\int \frac{dt}{\left[\left(t+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2\right]^2}$$

Put $t+\frac{1}{4} = \frac{\sqrt{15}}{4}\tan\theta$

$$dt = \frac{\sqrt{15}}{4}\sec^2\theta\, dx$$

$$= \frac{1}{4}\int \frac{dt}{\left[\frac{15^2}{16^2}(\tan^2\theta+1)^2\right]}$$

$$= \frac{16^2}{4\times 15^2}\int \cos^2 x\, dx$$

$$= \frac{16\times 4}{15^2\times 2}\int 1+\cos 2x\, dx$$

$$\frac{32}{225}(x+\sin 2x)$$

Q) If $\int \frac{x^n}{1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{x^n}{n!}}dx = n!\,(x-f(x))+c$ where $f(0)=0$

$$g(x) = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$$

$$g'(x) = 1+\frac{2x}{2!}+\frac{3x^2}{3!}+\cdots+\frac{n(x)^{n-1}}{n!}$$

$$= 1+x+\frac{x^2}{2!}+\cdots+\frac{(x)^{n-1}}{(n-1)!}$$

$$g(x)-g'(x) = \frac{x^n}{n!} \Rightarrow x^n = n!\,(g(x)-g'(x))$$

Now, $\int \frac{n!\,(g(x)-g'(x))}{g(x)}dx$

$$n!\int \frac{g(x)-g'(x)}{g(x)}dx$$

$$n!\int\left(1-\frac{g'(x)}{g(x)}\right)dx$$

$$= n!\,(x-\underbrace{\ln(g(x))}) + c$$

On comparing

$\ln g(x) = f(x)$

$f(0) = \ln g(0)$ and $g(0) = 1$

$f(0) = 0$

$$\lim_{n\to\infty} f(1) = \lim_{n\to\infty} \ln\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{n!}\right)$$

$= \ln e = 1.$