**Nature, Definition & Characteristics of Operations research**

The term Operation Research (OR) related to military operations during the Second World War. Scientists used various techniques to deal with strategic and tactical problems during the war. After the war military OR group scientists tried to apply OR techniques to civilian problems relating to business, industry and research development. During the 1950s educational institutions introduced OR in their curricula. Today service organizations such as airlines, railways, hospitals, libraries and banks employ OR to improve their efficiency.

1. **According to H.M. Wagner**

O.R. is a scientific approach to problem solving for executive management.

1. **According to Operations Research Society of America**

O.R. is an experimental and applied science devoted to observing, understanding and predicting the behavior of purposeful man-machine systems and operations research workers are actively engaged in applying this knowledge to practical problems in business, government and society.

1. **According to E.L. Arnoff and M.J. Netzorg**

O.R. is the systematic, method oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making.

1. **According to C.W. Churchman**

Operation Research is the application of scientific methods, techniques and tools to problems involving the operation of a system so as to provide those in control of the system with optimum solution to the problem.

1. **According to C. Kittel**

O.R. is an aid for the executive in making his decision by providing him with the needed quantitative information based on the scientific method of analysis.

**It can be used for solving different types of problems, such as:**

1. Problems dealing with the waiting line, the arrival of units or persons requiring ser­vice.
2. Problems dealing with the allocation of material or activities among limited facilities.
3. Equipment replacement problems.
4. Problems dealing with production processing i.e., production control and material ship­ment.

But it may be remembered that operation research never replaces a manager as decision maker. The ultimate and full responsibility for analysing all factors and making decision will be of the manager.

In the more wide sense, operation research does not deal with the everyday problems such as output by the one worker or machine capacity; instead it is concerned with the overall aspect of business operation such as something as the relationship between inventory, sales, production and scheduling. It may also deal with the overall flow of goods and services from plants to consumers.

The team doing operation research may have statisticians, psychologists, labour specialists, mathematicians and others depending upon the requirement for the problems.

**Nature of Operation Research:**

In its recent years of organised development, O.R. has solved successfully many cases of research for military, the government and industry. The basic problem in most of the develop­ing countries in Asia and Africa is to remove poverty and hunger as quickly as possible. So there is a great scope for economist, statisticians, administrators, politicians and technicians working in a team to solve this problem by an O.R. approach.

On the other hand, with the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land for various crops in accordance with climatic conditions and available facilities. The problem of optimal distribution of water from a resource like a canal for irrigation purposes is faced by developing country. Hence a good amount of scientific work can be done in this direction.

In the field of Industrial Engineering, there is a claim of problems, starting from the pro­curement of material to the despatch of finished products. Management is always interested in optimizing profits.

Hence in order to provide decision on scientific basis, O.R. study team con­siders various alternative methods and their effects on existing system. The O.R. approach is equally useful for the economists, administrators, planners, irrigation or agricultural experts and statisticians etc.

Operation research approach helps in operation management. Operation management can be defined as the management of systems for providing goods or services, and is concerned with the design and operation of systems for the manufacture, transport, supply or service. The operating systems convert the inputs to the satisfaction of customers need.

Thus the operation management is concerned with the optimum utilisation of resources i.e. effective utilisation of resources with minimum loss, under utilisation or waste. In other words, it is concerned with the satisfactory customer service and optimum resource utilisation. Inputs for an operating system may be material, machine and human resource.

O.R. study is complete only when we also consider human factors to the alternatives made available. Operation Research is done by a team of scientists or experts from different related disciplines.

For example, for solving a problem related to the inventory management, O.R. team must include an engineer who knows about stores and material management, a cost ac­countant a mathematician-cum-statistician. For large and complicated problems, the team must include a mathematician, a statistician, one or two engineers, an economist, computer program­mer, psychologist etc.

1. **Finance, Budgeting and Investment:**

* Cash flow analysis, long range capital requirement, investment portfolios, divi­dend policies,
* Claim procedure, and
* Credit policies.

**2. Marketing:**

* Product selection, competitive actions
* Number of salesmen, frequencies of calling on
* Advertising strategies with respect to cost and time

**3. Purchasing:**

* Buying policies, varying prices
* Determination of quantities and timing of purchases
* Bidding policies
* Replacement policies
* Exploitation of new material resources

**4. Production Management:**

* Physical distribution: Location and size of warehouses, distribution centres and retail outlets, distribution policies.
* Facilities Planning: Number and location of factories, warehouses etc. Loading and unloading facilities.
* Manufacturing: Production scheduling and sequencing stabilisation of produc­tion, employment, layoffs, and optimum product mix.
* Maintenance policies, crew size.
* Project scheduling and allocation of resources.

**5. Personnel Management:**

* Mixes of age and skills
* Recruiting policies
* Job assignments

**6. Research and Development:**

* Areas of concentration for R&D.
* Reliability and alternate decisions.
* Determination of time-cost trade off and control of development projects.

**Characteristics of Operations research**

1. **Decision making**

OR is a decision science which helps management to make better decisions.

1. **Use of Information Technology (IT)**

O.R. often requires a computer to solve the complex mathematical model or to perform a large number of computations that ae involved. Use of digital computer has become an integral part of the operations research approach to decision making.

1. **Quantitative solution**

Operations research provides the managers with a quantitative basis for decision making. OR attempts to provide a systematic and rational approach for quantitative solution to the various managerial problems.

1. **Human factors**

In deriving quantitative solution we do not consider human factors, which doubtlessly plays a great role in the problems. So study of the OR is incomplete without a study of human factors.

1. **System orientation**

O.R. study the situation or problem as a whole. This means that an activity by any part of an organization has some effect on the activity of every other part. The optimum result of one part of a system may not be the optimum for some other part. Therefore, to evaluate a decision, one must identify all possible interactions and determine their impact on the organization as a whole.

1. **Scientific approach**

O.R. uses scientific methods to solve the problems. Most of the scientific studies such as chemistry, physics, biology etc. can be carried out in the laboratories, without much interference form the outside world. Bust same is not true in the systems under study by OR teams. So, OR is an formalized process of reasoning. Under OR the problem is to be analysed ad defined clearly. Observations are made under different conditions to  study the behavior of the system. On the basis of these observations a hypothesis describing how the various factors involved are believed to interact and the best solution to the problem is formulated. To test the hypothesis experiment is designed and executed. Observations are made and measurement s are recorded. Finally results of the experiments are studied and the hypothesis is accepted or rejected. So, OR is the use of scientific method to solve the problem under study.

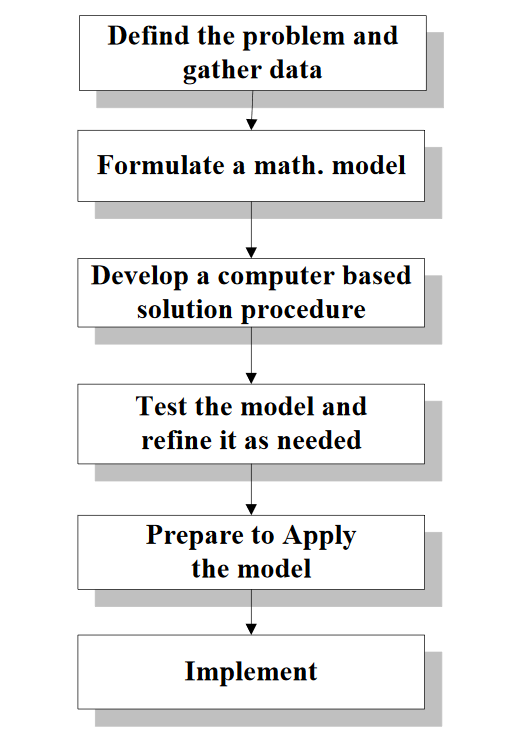
1. **Inter-disciplinary team approach**

O.R. is performed by a team of scientists whose individual members have been drawn from different scientific and engineering disciplines. For example, one may find a mathematician, statistician, physicist, psychologist, economist and an engineer working together on an OR problem.

1. **Uncovering new problems**

Solution of an OR problem may uncover a number of new problems. In order to derive the maximum benefit each one of them must be solved. OR is not effectively used if It is restricted to one shot problems only. In order to derive full benefits, continuity of research must be maintained

* **Overview of the Operations Research Modelling Approach**



‰**Modelling approach for problem solving**

* Defining the problem and gathering data
* This procedure is crucial. It is difficult to extract a “right” answer from the “wrong” problem.
* Most practical problems encountered by OR teams are initially described in a vague and imprecise way.
* Who are the decision makers?
* What are the objectives?
* What are the constraints (relationships)?
* How to collect relevant data?

‰**Formulating a mathematical model**

* Construct a mathematical model that represents the essence of the problem.
* Define decision variables.
* Define the objective function.
* Define the constraints (relations among decision variables).
* Usually, there is more than one way to formulate a problem.
  + Some degree of simplifications and approximations are inevitable.
  + Different formulations may require different solution techniques.

‰**Deriving solutions from the model**

* Need an algorithm (systematic solution procedures).
* Solve to optimality or pseudo-optimal.
  + Exact algorithm vs. heuristics algorithms.
* Conduct postoptimality (what-if) analysis.
  + What would happen to the optimal solution if different assumptions are made?

‰**Testing (validating) the model**

* Make sure there are no serious flaws.
* Make changes if necessary.
* Repeat the above procedures until satisfied.

‰**Preparing to apply the model and implementation**

* Install a well-documented system for applying the model.
  + Include the model, solution procedure, and operating procedures for implementation. Even include personnel changes.
  + This system is usually computer-based. A considerable number of computer programs often need to be used and integrated.
  + Databases and management information systems may provide up-to-date input for the model.
  + Usually take several months.

‰**Implement**

* The step yields the real benefits.
* Need support from both top management and operating management.
* It’s better to keep management well informed and encourage management’s active guidance throughout the course of the study.
* Require good communication skills.
* Continue to obtain feedback on how well the system is working and whether the assumptions of the model continue to be satisfied.
  + Need to revise or re-build models when significant deviations occur.
* Linear Programming Problem

A large number of military programming and planning problem could be formulated as maximizing/minimizing a linear form of profit/cost function whose variables were restricted to values satisfying a system of linear constraints.

A linear form is meant a mathematical expression of the type a1x1+a2x2+……..anxn where a1,a2……….an are constants and x1,x2,………xn are variables.

The term program refers to the process of determining a particular programme or plan of action.

So Linear Programming(LP) is one of the most important optimization (maximization/minimization) techniques developed in the field of Operation Research.

* Definition

The general LPP calls for optimizing (maximizing/minimizing) a linear function of variables called the OBJECTIVE FUNCTION subject to a set of linear equations and/or inequalities called the CONSTRAINTS or RESTRICTIONS.

* Formulation of LP Problem

1. Production Allocation Problem

A firm manufactures two types of products A and B and sells them at a profit of Rs 2 on type A & Rs 3 on type B. Each product is processed on two machines G & H. Type A requires one minute of processing time on G and two minutes on H; type B requires one minute on G & one minute on H. The machine G is available for not more than 6 hour 40 minutes while machine H is available for 10 hours during any working day.

Formulate the problem as a linear programming problem.

x1 A

x2 B

|  |  |  |  |
| --- | --- | --- | --- |
|  | A | B | Avail Time min |
| G | 1 | 1 | 400 |
| H | 2 | 1 | 600 |
| Profit per unit | 2 | 3 |  |

1. A company produces two types of hats. Each hat of first type requires twice as much labour time as the second type. If all hats are of second type only the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profit per hat are Rs 8 for type A and Rs 5 for type B. Formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

Max Z=8x1+5x2

Subject to constraints

2x1+x2<=500

x1<=150

x2<=250

x1,x2>=0

1. A firm manufactures 3 products A, B and C. The profits are Rs 3, Rs 2 and Rs 4respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product

Machine G and H have 2000 & 2500 machine minutes, respectively. The firm must manufacturers 100 A’s, 200 B’s and 50 C’s but not more than 150 A’s.

Setup LP problem to maximize profit.

PRODUCT

|  |  |  |  |
| --- | --- | --- | --- |
| MACHINE | A | B | C |
| G | 4 | 3 | 5 |
| H | 2 | 2 | 4 |
|  |  |  |  |

X1🡪A x2 🡪 B x3🡪C

Max Z=3x1+2x2+4x3

Subject to

4x1+3x2+5x3<=2000

2x1+2x2+4x3<=2500

100<=x1<=150

200<=x2

50<=x3

1. Diet Problem

Dieticians tells us that a balanced diet must contain quantities of nutrients such as calories, minerals, vitamins, etc. Suppose that we are asked to find out the food that should be recommended from large number of alternative sources of these nutrients so that the total cost of food satisfying minimum requirements of balanced diet is the lowest.

The medical experts and dieticians tell us that it is necessary for an adult to consume at least 75 gm of protein, 85 gm of fats and 300 gm of carbohydrates daily. The following table gives the food item (which are readily available in the market) , analysis and their respective cost.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| FOOD TYPE | FOOD VALUE(GMS) PER 100 GMS | | | COST PER KG(Rs) |
| PROTEIN | FATS | CARBOHYDRATES |
| 1 | 8.0 | 1.5 | 35.0 | 1.00 |
| 2 | 18.0 | 15.0 | - | 3.00 |
| 3 | 16.0 | 4.0 | 7.0 | 4.00 |
| 4 | 4.0 | 20.0 | 2.5 | 2.00 |
| 5 | 5.0 | 8.0 | 40.0 | 1.50 |
| 6 | 2.5 | - | 25.0 | 3.00 |
| MINIMUM DAILY REQUIREMENT | 75 | 85 | 300 |  |

X1🡪1,x2 2,x3 3,x4 4,x5 5,x6 6

Min Z= x1+3x2+4x3+2x4+1.5x5+3x6

Subject to constraint

8x1+18x2+16x3+4x4+5x5+2.5x6>=75

1.5x1+15x2+4x3+20x4+8x5>=85

35x1+7x3+2.5x4+40x5+25x6>=300

X1,x2,x3,x4,x5,x6>=0

**GRAPHICAL METHOD**

Graphical Solution of Two Variable Problems

* Graphical Procedure

Simple Linear programming problem of two decision variables can be easily solved by graphical method. The outline of graphical procedure are as follows.

**Step1:**

Consider each inequality constraint as equation.

**Step2:**

Plot each equation on the graph as each one will geometrically represent a straight line.

**Step3:**

Shade the feasible region. Every point on the line will satisfy the equation of the line.

If the inequality constraint corresponding to that line is <= then the region below the line lying in the first quadrant(due to nonnegativity of variables) is shaded.

For the inequality constraint with >= sign, the region above the line in the first quadrant is shaded.

The points lying in common region will satisfy all the constraints simultaneously.

The common region thus obtained is called feasible region.

**Step4:**

Choose the convenient value of z (say its 0) and plot the objective function line.

**Step 5:**

Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop furthest from the origin & passing through at least one corner of the feasible region.

**Step 6:**

Read the coordinates of the extreme points selected in Step 5 and find the maximum or minimum (as the case may be) value of z.

Example:

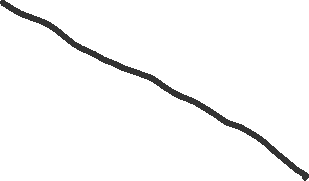
* 1. Find a geometrical interpretation and solution as well for the following LP problem.

Maximize z=3x1+5x2

subject to constraint

Consider x2=0 then x1=2000 A(2000,0)

Now x1=0 then x2=1000 B(0,1000)



* 1. A company manufactures two products X & Y using three machines A, B & C. Machine A has 4 hrs of capacity, machine B has 24 hrs of capacity & machine C has 35 hrs of capacity. One unit of product of X requires 1 hr on A, 3 hrs on B and 10 hrs on machine C. Product Y requires 1 hr on machine A, 8 hrs on machine B and 7 hrs on machine C. When 1 unit of X sold to the market it gives a profit of Rs 5 and one unit of Y gives a profit of Rs 7. Solve the problem using graphical method.



|  |  |  |  |
| --- | --- | --- | --- |
| Product 🡪  Machine | X | Y | Available  Capacity |
| A | 1 | 1 | **4** |
| B | 3 | 8 | **24** |
| C | 10 | 7 | **35** |
| Profit | **5** | **7** |  |

Max z=5x+7y

X+y<=4

3x+8y<=24

10x+7y<=35

x, y>=0

* 1. X+y<=4

X+y=4

A(4,0) B(0,4)

* 1. 3x+8y<=24

C(8,0) D(0,3)

* 1. 10x+7y<=35

E(3.5,0) F(0,5)

OEGHD

O(0,0) z=o

E(3.5,0) z=17.5

D(0,3) 21

G(2.4,1.6) 23.2

H(1.6,2.4) 24.8

* 1. A machine component requires a drill operation followed by welding and assembling. Two versions of product are produced: ordinary service and heavy duty. A single unit of ordinary design requires 10 minutes of drilling, 5 minutes of welding and 15 mins of assembling. A single unit of heavy duty component require 5 mins of drilling, 15 mins of welding and 5 mins of assembling. The profit for ordinary design is Rs 100 per unit & for heavy duty it is Rs 150 per unit. The total capacity drilling is 1500 mins, 1000 mins for welding and 2000 mins for assembling. What is the optimum mins so as to achieve maximum profit.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Ordinary Service | Heavy Duty | Available Capacity |
| Drilling | 10 | 5 | 1500 |
| Welding | 5 | 15 | 1000 |
| Assembling | 15 | 5 | 2000 |
| Profit | 100 | 150 |  |

x🡪ordinary y🡪heavyduty

Max z=100x+150y

10x+5y<=1500

5x+15y<=1000

15x+5y<=2000

X,y>=0

* 1. A(150,0) B(0,300)
  2. C(200,0) D(0,0.67)
  3. E(133.3,0) F(0,400)

OEGD

O(0,0) z=0

E(133.33,0) z=13333

D(0,66.67) z=10000.5

**G(125,25) z=16250**

* 1. A manufacturer of furniture makes two products chair and tables. Processing of these products is done on 2 machines A & B.

A chair requires 2 hrs on machine A & 6 hrs on machine B. A table requires 5 hrs on machine A

There is 16 hrs of time per day available on machine A 30 hrs on machine B.

Profit gained from a chair is Rs 2 & for table is Rs 10. What should be daily production of both the products.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Chair | Table | Available Time |
| A | 2 | 5 | 16 |
| B | 6 | - | 30 |
| Profit | 2 | 10 |  |
|  |  |  |  |

Max z=2x+10y

2x+5y<=16

6x<=30

2x+5y=16 A(8,0) B(0,3.2)

6x=30 x=5 c(5,0)

D(5,1.2)

O(0,0) z=0

C(5,0) z=10

B(0,3.2) z=32

D(5,1.2) z=22

* 1. A dietician wants to decide a breakfast menu for certain patients. The menu is to include 2 items of A & B. Suppose each spoon of A divides 2 unit of Vitamin C and 2 units of iron & each spoon of B provide 1 unit of vitamin C and 2 units of iron. The cost of A is Rs 4 per spoon and cost of B is Rs 3 per spoon. The breakfast menu must provide at least8 units of Vitamin C & 10 units of iron.

How many units of A & B should a person intake? How much will the breakfast cost?

|  |  |  |  |
| --- | --- | --- | --- |
|  | A🡪x | B🡪y | Requirement |
| Vitamin C | 2 | 1 | 8 |
| Iron | 2 | 2 | 10 |
| Cost | 4 | 3 |  |

Min z=4x+3y

2x+y>=8

2x+2y>=10

X,y>=0

2x+y = 8

A(4,0) B(0,8)

2x+2y=10

C(5,0) D(0,5)

Min z=4x+3y

BCE

B(0,8) z=24

C(5,0) z=20

E(3,2) z=18

* 1. A small business makes gear and non-gear bikes at two factories. Factory A produces 16 geared & 20 non geared bikes daily. While factory B produces 12 geared & 20 non geared bikes daily. It costs Rs 1000 per day to operate factory A and Rs 800 per day to operate factory B. An order for 96 geared and 140 non geared bikes has just arrived. How many days should each factory be operated in order to fulfil the order at minimum cost.

|  |  |  |  |
| --- | --- | --- | --- |
|  | A🡪x | B🡪y | Requirement |
| Geared | 16 | 12 | 96 |
| Nongeared | 20 | 20 | 140 |
| Cost | 1000 | 800 |  |

Min z=1000x+800y

16x+12y>=96

20x+20y>=140

X,y>=0

16x+12y=96

A(6,0)

B(0,8)

20x+20y>=140

C(7,0)

D(0,7)

Min z=1000x+800y

B(0,8) z=6400

C(7,0) z=7000

E(3,4) z=6200

**Special cases in Graphical method**

* 1. Solve the following LPP using graphical method

Max Z=3x+2y

Subject to

x + y <= 1

2x + 2y >= 6

x, y >= 0

x + y = 1

A(1,0)

B(0,1)

2x + 2y >= 6

C(3,0)

D(0,3)

Definition:

**Infeasibility:**

Infeasibility is a case where there is no solution which satisfies all the constraints simultaneously.

* 1. Max Z = 3x+5y

Subject to

2x + y >= 7

x + y >= 6

x + 3y >= 9

x, y >= 0

2x + y >= 7

A(3.5,0)

B(0,7)

x + y >= 6

C(6,0)

D(0,6)

x + 3y >= 9

E(9,0)

F(0,3)

Definition:

**Unbounded:**

An LPP can fail to have an optimum solution if the objective is infinitely large without violating any of the constraints. An unbounded solution occurs in minimization problem if all the constraints are less than (<) type and it will occur in maximization problem if all the constraints are greater than (>) type.

* 1. Solve the LPP using graphical method.

Max Z=6x+2y

Subject to constraints

x + y <= 4

4x + 3y <= 12

-x + y >=1

x + y <= 6

x,y>= 0

x + y <= 4

A(4,0)

B(0,4)

4x + 3y <= 12

C(3,0)

D(0,4)

-x + y >=1

E(-1,0)

F(0,1)

x + y <= 6

G(6,0)

H(0,6)

F(0,1) z=2

B(0,4) z=8

I(1.28,2.28) z=12.24

Definition:

**Redundancy:**

A constraint which does not affect the feasible region is called as redundant constraints. Therefore, it can be admitted, and the problem can still be solved.

* 1. Max Z=18x +6y

Subject to constraints,

3x + y <= 120

X + 2y <= 160

x <= 35

x, y >= 0

3x + y <= 120

A(40,0)

B(0,120)

X + 2y <= 160

C(160,0)

D(0,80)

x <= 35

E(35,0)

O(0,0) z=0

E(35,0) z=630

D(0,80) z=480

F(35,15) z=720

G(16,72) z=720

Definition:

**Alternate optima:**

When there is more than one solution with the same optimal value then we say that we have alternative optima.

**SIMPLEX METHOD**

It has not been possible to obtain the graphical solution to the LP problem of more than two variables.

The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of prescribed operation) which is based on fundamental theorem of linear programming.

The simplex method is iterative (step by step) procedure for solving LP problems. It consists of :-

* + 1. Having a trial basic feasible solution to constraint equation
    2. Testing whether it is optimal solution
    3. Improving first trial solution by a set of rules and repeating the process till optimal solution is obtained.

The computational procedure requires at most m (equal to the number of equations) nonzero variables in the solution at any step.

In case of less than m nonzero variables at any stage of computations the degeneracy arises in LP problem.

A feasible solution at any iteration is related to the feasible solution at the successive iteration in the following way:

One of the non-basic variables (which are zero now) at one iteration becomes basic(nonzero) at the following iteration and is called an entering variable. To compensate, one of the basic variables (which are nonzero now) at one iteration becomes non-basic (zero) at the following iteration and is called a departing variable.

The other non-basic variable remain zero and other basic variable remain non zero(though their values may change)

Consider LP problem in standard form:

Max z=c1x1+c2x2+-------------+cnxn+0xn+1+0xn+2+------------0xn+m

subject to constraints

a11x1+a12x2+-----------a1nxn+xn+1+ =b1

a21x1+a22x2+-----------a2nxn+xn+2+ =b2

…

….

am1x1+am2x2+-----------amnxn+xm+n =bm

x1,x2,……………..xn,xn+1,…………xn+m>=0

The starting basicfeasible solution of m equations is usually taken as

x1=x2=,……………..=xn=0,

xn+1=b2,…………..…xn+m=bm

For this solution value of objective function is zero.

Here x1,x2,……………..xn(each equal to zero) are non-basic variables and remaining variables (xn+1,…………xn+m) are basic variables (some of them may also have value zero)

**Definitions:**

* 1. **Simplex method**

It is based on the property that the optimal solution to the LPP will be found in one of the basic feasible solution.

* 1. **Slack variable**

A variable which is used to convert less than equal to constraints into equality constraints.

It is added to the left-hand side (LHS) of the constraint.

* 1. **Surplus variable**

A variable which is used to convert greater than or equal to constraint into equality constraint. It is subtracted from LHS of the constraint.

* 1. **Artificial variable**

A variable which is added to convert equal to and greater than equal to constraint into equality constraint.

* 1. **Canonical form of LPP**

A LPP is said to be canonical form if

* + 1. The objective function is to maximize or minimize.
    2. All constraints must be less than equal to or greater than equal to
    3. All variable should be nonnegative
  1. **Standard form of LPP**

A LPP is said to be standard form if

* + 1. The objective function is to maximize or minimize.
    2. All conditions except the nonnegativity condition should be equal to type
    3. All variables should be nonnegative.
  1. **Solution Of LPP**

Any set of variables which satisfy the given constraint is called solution of LPP.

* 1. **Basic Solution**

By setting any m variables equal to zero and solving the remaining n variables, we get a basic solution.

The m variables are called as basic variables and n variables are called as non basic variables.

* 1. **Basic feasible solution**

A basic solution that is feasible (all basic variables are non negative) is called as basic feasible solution. There are 2 types of basic feasible solutions

* + 1. **Degenerate basic feasible solution**

If any of the basic variable of a basic feasible solution are zero. Then it is called as degenerate basic feasible solution.

* + 1. **Nondegenerate basic feasible solution**

In this solution all basic variables should be positive and the remaining variables should be zero.

* 1. **Optimum basic feasible solution**

A basic feasible solution is said to be optimum if it optimizes (maximize/minimize) the objective function.

* Computational Procedure of Simplex Method

1. Consider the linear programming problem

Max z= 3x1 + 2x2

Subject to constraints

x1 + x2<= 4

x1- x2<= 2

x1 , x2>=0

**Step 1:** First observe whether all the right-side constants of the constraints are non-negative. If not it can be changed into positive value on multiplying both side of the constraints by -1

In this example, all the bi’s(right side of the constraint) are already positive.

**Step 2:** Next covert the inequality constraints to equations by introducing the non negative slack or surplus variables.

The coefficient of slack or surplus variables are always taken zero in the objective function.

In this example all inequality constraints being <= , only slack variables x3 and x4 are needed.

Therefore given problem now becomes

Max z= 3x1 + 2x2 +0x3 + 0x4

Subject to constraints

x1 + x2+ x3 + 0x4= 4

x1- x2+ 0x3 + x4= 2

x1 , x2,x3,x4>= 0

**Step 3**: Now present the constraint equations in matrix form:

**Step 4:** Construct the starting simplex table as follows:

It should be remembered that the values of non basic variables are always zero at each iteration.

So x1= x2= 0 here.

The complete starting basic feasible solution can be immediately read from following table as

x1 = 0, x2=0,x3=4, x4=2

Note:

In this step the variables x3 and x4 are corresponding to the columns of basis matrix (identity matrix), so will be called basic variables.

Other variables, x1 , x2 are non basic variables which always have the value zero.

Starting simplex table:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | Cj🡪 | 3 | 2 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** | 0 | 4 | 1 | 1 | 1 | 0 | 4 |
| **X4** | 0 | 2 | 1\* | -1 | 0 | 1 | 2 |
|  |  | ∆j=**CB XB -** Cj | -3 | -2 | 0 | 0 |  |

Incoming outgoing

Step 5: Now proceed to test basic feasible solution for optimality by the rules given below. This is done by computing the net evaluation ∆j for each variable xj (column vector xj) by the formula

∆j=**Zj -Cj =CBXj–**Cj

Thus

∆1=**CB X1–**C1

= (0\*1 + 0\*1) – 3

= -3

∆2=**CB X1–**C1

= -2

∆3=**CB X1–**C1

= 0

∆4=**CB X1–**C1

= 0

Optimality Test:

* 1. If all ∆j >=0 then the solution under test will be optimal. Alternative optimal solutions will exist, if any non basic ∆j is also zero.
  2. If atleast one ∆j is negative then the solution under test is not optimal then proceed to improve the solution in the next step.
  3. If corresponding to any negative ∆j all elements of the column Xj are negative or zero (<= 0) then the solution under test will be unbounded.

Applying these rules for testing the optimality of starting basic feasible solution , it is observed that ∆1 & ∆2 both are negative. Hence we have to proceed to improve this solution in step 6.

**Step 6:**In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as “incoming vector” and “outgoing vector” respectively.

**Incoming vector:**

The incoming vector Xk is always selected corresponding to the most negative value of ∆j (say ∆k)

Here, ∆k = min [∆1, ∆2]= min [ , ] =

Therefore, k=1 and hence column vector x1 must enter the basis matrix, The column x1 is marked by an upward arrow (↑)

**Outgoing vector:**

The outgoing vector Br is selected corresponding to the minimum ratio elements of XB by the corresponding positive elements of predefined incoming vector Xk. This rule is called **Minimum ratio rule.**

Here,

XB / Xk= min [XB1/X11, XB2/X21]

The X4 marked with downward arrow ( ↓) should be removed from basis matrix.

Now table 1 will be modified to table 2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | Cj🡪 | 3 | 2 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** | 0 | 4 | 1 | 1 | 1 | 0 | 4 |
| **X4** | 0 | 2 | 1\* | -1 | 0 | 1 | 2 |
|  |  | ∆j=**CB XB -** Cj | -3 | -2 | 0 | 0 |  |

R1🡪 R1-R2

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **Min Ratio** |
| X3 | 0 | 2 | 0 | 2\* | 1 | 1 |
| X1 | 3 | 2 | 1 | -1 | 0 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | -5 | 0 |  |

R1 🡪 R1/2

R2 🡪 R2 + R1

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **Min Ratio** |
| X2 | 2 | 1 | 0 | 1 |  |
| X1 | 3 | 3 | 1 | 0 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 |  |

X1= 3, x2=1

Max z= 3x1 + 2x2

**Max z=11**

Step 7: The element at intersection of minimum ratio arrow and incoming vector arrow is called key element or **pivot element.**

In order to bring pivot element equal to 1, unity must occupy in the marked \* position and 0 at all other places of xi

If the number in the marked \* position is other than unity then divide all elements of that row by the key element.

Then substract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column incoming vector(X1)

Thus process can be fortified by simple matrix transformation as follows:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  |  |  |  |  |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** |  |  |  |  |  |  |  |
| **X4** |  |  |  |  |  |  |  |
|  |  | ∆j=**CB XB -** Cj |  |  |  |  |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  |  |  |  |  |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** |  |  |  |  |  |  |  |
| **X4** |  |  |  |  |  |  |  |
|  |  | ∆j=**CB XB -** Cj |  |  |  |  |  |

As all ∆j values are greater than zero. So by definition of optimality test, As all ∆j>= 0 therefor the solution under test will be optimal

At

Max Z =

1. Max z= 3x1+ 2x2

Subject to

x1 + x2<= 4

x1 - x2<= 2

X1, x2>= 0

1. Min z= x1-3x2+2x3

Subject to

3x1 -x2 +3x3<= 7

-2x1 + 4x2<= 12

-4x1 + 3x2 + 8x3<= 10

X1, x2, x3>= 0

Max(z’)=min z=-x1+3x2-2x3+0x4+0x5+0x6

X4,x5 and x6 slack variables

3x1 -x2 +3x3+x4+0x5+0x6 = 7

-2x1 + 4x2+0x3+0x4+x5+0x6= 12

-4x1 + 3x2 + 8x3+0x4+0x5+x6= 10

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -1 | 3 | -2 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **Min Ratio** |
| X4 | 0 | 7 | 3 | -1 | 3 | 1 | 0 | 0 | - |
| X5 | 0 | 12 | -2 | 4\* | 0 | 0 | 1 | 0 | 3 |
| X6 | 0 | 10 | -4 | 3 | 8 | 0 | 0 | 1 | 3.3 |
|  |  | ∆j=**CB XB–**Cj | 1 | -3 | 2 | 0 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -1 | 3 | -2 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X6** | **Min Ratio** |
| **X4** | 0 | 10 | 5/2\* | 0 | 3 | 1 | 0 | 4 |
| **X2** | 3 | 3 | -1/2 | 1 | 0 | 0 | 0 | - |
| **X6** | 0 | 1 | -5/2 | 0 | 8 | 0 | 1 | - |
|  |  | ∆j=**CB XB–**Cj | -1/2 | 0 | 2 | 0 | 0 |  |

R2🡪 R2/4 R1🡪R1+R2 R3🡪 R3-3R2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -1 | 3 | -2 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X6** | **Min Ratio** |
| **X1** | -1 | 4 | 1 | 0 | 6/5 | 0 |  |
| **X2** | 3 | 5 | 0 | 1 | 3/5 | 0 |  |
| **X6** | 0 | 11 | 0 | 0 | 11 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 13/5 | 0 |  |

R1🡪R1\*2/5 R2🡪 R2+1/2R1 R3🡪R3+5/2R1

X1=4 x2=5

Max(z’)=min z=-x1+3x2-2x3+0x4+0x5+0x6=11

Min z=-11

1. Max z= 3x1 + 2x2+5x3

Subject to

x1 + 2x2 +x3<= 430

3x1 + 2x3<= 460

x1 + 4x2<= 420

X1, x2, x3>= 0

Slack variables x4,x5,x6

Max z= 3x1 + 2x2+5x3+0x4+0x5+0x6

Subject to

x1 + 2x2 +x3+x4+0x5+0x6= 430

3x1 +0x2+2x3+0x4+x5+0x6= 460

x1 + 4x2+0x3+0x4+0x5+x6 = 420

X1, x2, x3>= 0

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 5 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **Min Ratio** |
| **X4** | 0 | 430 | 1 | 2 | 1 | 1 | 0 | 0 | 430 |
| **X5** | 0 | 460 | 3 | 0 | 2\* | 0 | 1 | 0 | 230 |
| **X6** | 0 | 420 | 1 | 4 | 0 | 0 | 0 | 1 | - |
|  |  | ∆j=**CB XB–**Cj | -3 | -2 | -5 | 0 | 0 | 0 |  |

R2🡪 R2/2 R1🡪 R1-R2

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 5 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X6** | **Min Ratio** |
| **X4** | 0 | 200 | -1/2 | 2\* | 0 | 1 | 0 | 100 |
| **X3** | 5 | 230 | 3/2 | 0 | 1 | 0 | 0 | - |
| **X6** | 0 | 420 | 1 | 4 | 0 | 0 | 1 | 105 |
|  |  | ∆j=**CB XB–**Cj | 9/2 | -2 | 0 | 0 | 0 |  |

R1🡪 R1/2 R3🡪R3-4R1

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 5 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X6** | **Min Ratio** |
| **X2** | 2 | 100 | -1/4 | 1 | 0 | 0 |  |
| **X3** | 5 | 230 | 3/2 | 0 | 1 | 0 |  |
| **X6** | 0 | 20 | 2 | 0 | 0 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 4 | 0 | 0 | 0 |  |

Optimal solution

Max z= 3x1 + 2x2+5x3

x2=100 x3=230 Max z==1350

1. Max z= 5x1+ 3x2

Subject to

3x1 + 5x2<= 15

5x1+2x2<= 10

X1, x2>= 0

X3, x4

Max z= 5x1+ 3x2+0x3+0x4

Subject to

3x1 + 5x2 +x3+0x4= 15

5x1 + 2x2 +0x3+x4= 10

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 5 | 3 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** | 0 | 15 | 3 | 5 | 1 | 0 | 5 |
| **X4** | 0 | 10 | 5\* | 2 | 0 | 1 | 2 |
|  |  | ∆j=**CB XB -** Cj | -5 | -3 | 0 | 0 |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 5 | 3 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **Min Ratio** |
| **X3** | 0 | 9 | 0 | 19/5\* | 1 | 45/19=2.3 |
| **X1** | 5 | 2 | 1 | 2/5 | 0 | 5 |
|  |  | ∆j=**CB XB -** Cj | 0 | -1 | 0 |  |

R2🡪 R2/5

R1 🡪 R1-3R2

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 5 | 3 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **Min Ratio** |
| **X2** | 3 | 45/19 | 0 | 1 |  |
| **X1** | 5 | 20/19 | 1 | 0 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 |  |

R1🡪 R1\*5/19 R2🡪 R2-(2/5R1)

X1=20/19 x2=45/19 Max z=235/19= 12.36

1. Max z= 2x1+3x2+ x3

Subject to

3x1 + 6x2 + x3<= 6

4x1 + 2x2 + x3<= 4

x1 - x2+ x3<= 3

X1, x2, x3>= 0

X4,x5,x6

Max z= 2x1+3x2+ x3 + 0x4+0x5+0x6

Subject to

3x1 + 6x2 + x3 + x4+0x5+0x6 = 6

4x1 + 2x2 + x3 +0x4 +x5+0x6 = 4

x1 - x2+ x3 +0x4+0x5+x6 = 3

X1, x2, x3>= 0

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | 3 | 1 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **Min Ratio** |
| **X4** | 0 | 6 | 3 | 6\* | 1 | 1 | 0 | 0 | 1 |
| **X5** | 0 | 4 | 4 | 2 | 1 | 0 | 1 | 0 | 2 |
| **X6** | 0 | 3 | 1 | -1 | 1 | 0 | 0 | 1 | - |
|  |  | ∆j=**CB XB–**Cj | -2 | -3 | -1 | 0 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | 3 | 1 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X5** | **X6** | **Min Ratio** |
| **X2** | 3 | 1 | 1/2 | 1 | 1/6 | 0 | 0 | 2 |
| **X5** | 0 | 2 | 3\* | 0 | 2/3 | 1 | 0 | 2/3 |
| **X6** | 0 | 4 | 3/2 | 0 | 7/6 | 0 | 1 | 8/3 |
|  |  | ∆j=**CB XB–**Cj | -1/2 | 0 | -1/2 | 0 | 0 |  |

R1🡪R1/6 R2🡪R2-2R1 R3🡪R3+R1

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | 3 | 1 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X6** | **Min Ratio** |
| **X2** | 3 | 2/3 | 0 | 1 | 1/18 | 0 |  |
| **X1** | 2 | 2/3 | 1 | 0 | 2/9 | 0 |  |
| **X6** | 0 | 3 | 0 | 0 | 5/6 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 10/18 | 0 |  |

R2🡪 R2/3 R1🡪 R1-(1/2R2) R3🡪 R3-(3/2R2)

X1=2/3 x2=2/3 Max z= 2x1+3x2+ x3 =10/3

1. Min z= x1-3x2+ 2x3

Subject to

3x1 - x2 + 3x3<= 7

-2x1 + 4x2<= 12

-4x1 + 3x2+ 8x3<= 10

X1, x2, x3>= 0

**Big M method**

Computational steps of Big – M method are as stated below:

**Step 1:**

Express the problem in standard format.

**Step 2:**

Add nonnegative artificial variables to the left side of each of the equations corresponding to constraints of type >= or =.

Whhen artificial variable are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificiall variable will be zero in the final solution (provided solution of problem exists.)

On the other hand, if the problem does not have a solution at least one of the artificial variable will appear in final solution with positive value. This is achieved by assigning a very large price (per unit penalty) to these variables in the objective function. Such large price will be designated by -M for maximization problems and +M for minimization problems where M>0.

**Step 3:**

In the last, use the artificial variable for the starting solution and proceed with usual simplex routine until the optimal solution is obtained.

Example

Solve by using big – M method the following linear programming problems

1. Max z= - 2x1-x2

Subject to

3x1 + x2 = 3

4x1 + 3x2>= 6

x1 + 2x2<= 4

X1, x2, x3>= 0

= artificial variable 🡪 a1

>= surplus variable ,artificial variable 🡪 x3,a2

<= slack variable 🡪 x4

Max z= - 2x1-x2+0x3+0x4 -Ma1 -Ma2

Subject to

3x1 + x2 +0x3+0x4+a1+0a2 = 3

4x1 + 3x2 -x3 +0x4 +0a1+a2 = 6

x1 + 2x2 +0x3+x4+0a1+0a2 = 4

3 1 0 0 1 0

4 3 -1 0 0 1

1 2 0 1 0 0

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -2 | -1 | 0 | 0 | -M | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **a1** | **a2** | **Min Ratio** |
| a1 | -M | 3 | 3\* | 1 | 0 | 0 | 1 | 0 | 1 |
| a2 | -M | 6 | 4 | 3 | -1 | 0 | 0 | 1 | 1.5 |
| X4 | 0 | 4 | 1 | 2 | 0 | 1 | 0 | 0 | 4 |
|  |  | ∆j=**CB XB -** Cj | -7M+2 | -4M+1 | M | 0 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -2 | -1 | 0 | 0 | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **a2** | **Min Ratio** |
| **x1** | -2 | 1 | 1 | 1/3 | 0 | 0 | 0 | 3 |
| **a2** | -M | 2 | 0 | 5/3\* | -1 | 0 | 1 | 1.2 |
| **X4** | 0 | 3 | 0 | 5/3 | 0 | 1 | 0 | 1.8 |
|  |  | ∆j=**CB XB -** Cj | 0 | -5/3M+1/3 | M | 0 | 0 |  |

R1🡪R1/3 R2🡪R2-4R1 R3🡪R3-R1

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -2 | -1 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X1** | -2 | 3/5 | 1 | 0 | 1/5 | 0 |  |
| **X2** | -1 | 6/5 | 0 | 1 | -3/5 | 0 |  |
| **X4** | 0 | 1 | 0 | 0 | 1 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 1/5 | 0 |  |

R2🡪 R2\*3/5 R1🡪 R1-1/3R2 R3🡪R3-5/3R2

X1=3/5 x2=6/5

Max Z= -2(3/5)-6/5 = **-12/5**

1. Max z= 3x1 + 2x2

Subject to

2x1 + x2<= 1

3x1 + 4x2>= 4

X1, x2, x3>= 0

X3 as slack variable and x4 as surplus variable and a1 artificial variable

Max z= 3x1 + 2x2+0x3+0x4 -Ma1

Subject to

2x1 + x2+x3 + 0x4 +0a1= 1

3x1 + 4x2+0x3 -x4 + a1= 4

X1, x2, x3>= 0

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 0 | 0 | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **Min Ratio** |
| X3 | 0 | 1 | 2 | 1 | 1 | 0 | 0 | 1 |
| a1 | -M | 4 | 3 | 4\* | 0 | -1 | 1 | 1 |
|  |  | ∆j=**CB XB -** Cj | -3M-3 | -4M-2 | 0 | M | 0 |  |

R2🡪R2/4 R1🡪R1-R2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** | 0 | 0 | 5/4\* | 0 | 1 | 1/4 | 0 |
| **X2** | 2 | 1 | 3/4 | 1 | 0 | -1/4 | 4/3 |
|  |  | ∆j=**CB XB -** Cj | -6/4 | 0 | 0 | -1/2 |  |

R1🡪 R1\*4/5 R2🡪R2-3/4R1

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X4** | **Min Ratio** |
| **X1** | 3 | 0 | 1 | 0 | 1/5\* | 0 |
| **X2** | 2 | 1 | 0 | 1 | -2/5 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | -1/5 |  |

R1🡪R1\*5 R2🡪R2+2/5R1

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 2 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X4** | **Min Ratio** |
| **X4** | 0 | 0 | 5 | 0 | 1 |  |
| **X2** | 2 | 1 | 2 | 1 | 0 |  |
|  |  | ∆j=**CB XB–**Cj | 1 | 0 | 0 |  |

Optimal x2=1

Max z= 3x1 + 2x2= 2

1. Max z= 3x1-x2

Subject to

2x1 + x2>=2

x1 + 3x2<= 3

x2<= 4

X1, x2>= 0

X3 surplsus, a1 artificial, x4,x5 slack

Max z= 3x1-x2+0x3 -Ma1+0x4+0x5

Subject to

2x1 + x2 -x3+a1+0x4+0x5 =2

x1 + 3x2 +0x3+0a1 +x4+0x5 = 3

0x1+x2 +0x3+ 0a1+ 0x4+ x5= 4

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | -1 | 0 | -M | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **A1** | **X4** | **X5** | **Min Ratio** |
| A1 | -M | 2 | 2\* | 1 | -1 | 1 | 0 | 0 | 1 |
| **X4** | 0 | 3 | 1 | 3 | 0 | 0 | 1 | 0 | 3 |
| **X5** | 0 | 4 | 0 | 1 | 0 | 0 | 0 | 1 | - |
|  |  | ∆j=**CB XB -** Cj | -2M-3 | -M+1 | M | 0 | 0 | 0 |  |

R1🡪R1/2 R2🡪R2-R1

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | -1 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **Min Ratio** |
| **X1** | 3 | 1 | 1 | 1/2 | -1/2 | 0 | 0 | - |
| **X4** | 0 | 2 | 0 | 5/2 | ½\* | 1 | 0 | 4 |
| **X5** | 0 | 4 | 0 | 1 | 0 | 0 | 1 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | 5/2 | -3/2 | 0 | 0 |  |

R2🡪R2\*2 R1🡪 R1+1/2R2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | -1 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X5** | **Min Ratio** |
| **X1** | 3 | 3 | 1 | 3 | 0 | 0 |  |
| **X3** | 0 | 4 | 0 | 5 | 1 | 0 |  |
| **X5** | 0 | 4 | 0 | 1 | 0 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 10 | 0 | 0 |  |

X1=3 Max z=9

1. Min z= 4x1 + x2

Subject to

3x1 + 4x2>= 20

x1 + 5x2>= 15

x1, x2>= 0

x3,x4 surplus a1,a2 artificial

Max( z’)= min z= -4x1-x2+0x3+0x4-Ma1 -Ma2

Subject to

3x1 + 4x2 -x3 +0x4 +a1+0a2 = 20

x1 + 5x2 +0x3 -x4 +0a1+a2 = 15

x1, x2>= 0

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -4 | -1 | 0 | 0 | -M | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **A2** | **Min Ratio** |
| A1 | -M | 20 | 3 | 4 | -1 | 0 | 1 | 0 | 5 |
| A2 | -M | 15 | 1 | 5\* | 0 | -1 | 0 | 1 | 3 |
|  |  | ∆j=**CB XB -** Cj | -4M+4 | -9M+1 | M | M | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -4 | -1 | 0 | 0 | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **Min Ratio** |
| **A1** | -M | 8 | 11/5\* | 0 | -1 | 4/5 | 1 | 40/11 |
| **X2** | -1 | 3 | 1/5 | 1 | 0 | -1/5 | 0 | 15 |
|  |  | ∆j=**CB XB–**Cj | -11M/5+19/5 | 0 | M | -4M/5+1/5 | 0 |  |

R2🡪R2/5 R1🡪R1-4R2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -4 | -1 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| X1 | -4 | 40/11 | 1 | 0 | -5/11 | 4/11\* | 10 |
| **X2** | -1 | 25/11 | 0 | 1 | 1/11 | -3/11 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 19/11 | -13/11 |  |

R1🡪 R1\*5/11 R2🡪 R2-1/5R1 (-1/5) \*(-1/5)(4/11)=-15/55=-3/11

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -4 | -1 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X4** | 0 | 10 | 11/4 | 0 | -5/4 | 1 |  |
| **X2** | -1 | 5 | ¾ | 1 | -1/4 | 0 |  |
|  |  | ∆j=**CB XB -** Cj | 13/4 | 0 | 1/4 | 0 |  |

R1🡪R1\*11/4 R2🡪R2+3/11R1

Min z= 4x1 + x2=5

1. Min z= 2x1 + 3x2

Subject to

3x1 + 5x2>= 30

5x1 + 3x2>= 60

x1, x2>= 0

3 a1🡪x1

5/3 a2🡪x3

X1=12 x2=0 Minz=24

**Two phase simplex method**

Comparing the Big M simplex method and the Two-Phase simplex method, weobserve the following:

•The basic approach to both methods is the same.Both add the artificialvariables to get the initial canonical system and then derive them to zero assoon as possible.

•The sequence of tableaus and the basis changes are identical.

•The number of iterations are the same.

•The Big M simplex method solves the linear problem in one pass while theTwo-Phase simplex method solves it in two stages as two linear program.

Algorithm:

***Phase 1:***

Step 1:

Form a new objective function by assigning zero to every original variable including slack and surplus variables

e.g. Max z= -a1 – a2

Step 2:

Using simplex method, try to eliminate the artificial variables from the basis.

Step 3:

The solution at the end of the phase 1 is the initial basic feasible solution for phase 2.

***Phase 2:***

Step 1:

The original objective function is used, and coefficient of artificial variable is 0. (So artificial variable is removed from the calculation process)

Step 2:

Then use simplex method as usual way to find optimal solution.

Example:

* 1. Find optimal solution using two phase method

Min z = x1+x2

2x1+ x2>= 4

x1+ 7x2>= 7

x1,x2>=0

Maxz’=Min(- z) = -x1-x2 +0x3+0x4 -a1 -a2

X3,x4 are surplus a1,a2 are artificial

2x1+ x2-x3 +0x4 + a1 + 0a2 = 4

x1+ 7x2+0x3 – x4 +0a1 +a2 = 7

x1,x2>=0

Phase 1:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | -1 | -1 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **A2** | **Min Ratio** |
| A1 | -1 | 4 | 2 | 1 | -1 | 0 | 1 | 0 | 4 |
| A2 | -1 | 7 | 1 | 7\* | 0 | -1 | 0 | 1 | 1 |
|  |  | ∆j=**CB XB -** Cj | -3 | -8 | 1 | 1 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | -1 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **Min Ratio** |
| A1 | -1 | 3 | 13/7\* | 0 | -1 | 1/7 | 1 | 21/13 |
| X2 | 0 | 1 | 1/7 | 1 | 0 | -1/7 | 0 | 7 |
|  |  | ∆j=**CB XB -** Cj | -13/7 | 0 | 1 | -1/7 | 0 |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| x1 | 0 | 21/13 | 1 | 0 | -7/13 | 1/13 |  |
| X2 | 0 | 10/13 | 0 | 1 | 1/13 | -14/91 |  |
|  |  |  | 0 | 0 | 0 | 0 |  |

R1🡪 R1\*7/13 R2🡪R2-1/7R1

Phase 2:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -1 | -1 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| x1 | -1 | 21/13 | 1 | 0 | -7/13 | 1/13 |  |
| X2 | -1 | 10/13 | 0 | 1 | 1/13 | -14/91 |  |
|  |  |  | 0 | 0 | 6/13 | 1/13 |  |

X1=21/13 x2=10/13

Minz=31/13

* 1. Use two phase simplex method to maximize

Z=2x1 + 3x2- 5x3

Subject to

x1 + x2+ x3 = 7

2x1 - 5x2>= 10

x1,x2,x3>= 0

Max Z=2x1 + 3x2- 5x3 + 0x4 – a1 -a2

Subject to

x1 + x2+ x3 +0x4 +a1+0a2 = 7

2x1 - 5x2+0x3- x4 + 0a1 +a2 = 10

Phase 1:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | -1 | -1 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **A2** | **Min Ratio** |
| A1 | -1 | 7 | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| A2 | -1 | 10 | 2\* | -5 | 0 | -1 | 0 | 1 | 5 |
|  |  |  | -3 | 4 | -1 | 1 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | -1 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **A1** | **Min Ratio** |
| A1 | -1 | 2 | 0 | 7/2\* | 1 | 1/2 | 1 | 4/7 |
| X1 | 0 | 5 | 1 | -5/2 | 0 | -1/2 | 0 | - |
|  |  |  | 0 | -7/2 | -1 | -1/2 | 0 |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| X2 | 0 | 4/7 | 0 | 1 | 2/7 | 1/7 |  |
| X1 | 0 | 45/7 | 1 | 0 | 5/7 | -1/7 |  |
|  |  |  | 0 | 0 | 0 | 0 |  |

Phase 2:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | 3 | -5 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| X2 | 3 | 4/7 | 0 | 1 | 2/7 | 1/7 |  |
| X1 | 2 | 45/7 | 1 | 0 | 5/7 | -1/7 |  |
|  |  |  | 0 | 0 | 51/7 | 1/7 |  |
|  |  |  |  |  |  |  |  |

X1=45/7 x2=4/7 Max z=102/7

R2🡪 R2+5/2R1

* 1. Use two phase method to maximize

Z= 3x1 – x2

Subject to

2x1 + x2>= 2

x1 + 3x2<= 2

x1, x2>= 0

* 1. Use two phase method

Min z = 5x1 - 2x2 + 5x3

Subject to

2x1 + 2x2 - x3>= 2

3x1 - 4x2 + x3<= 3

x2 + 3x3<= 5

x1, x2 ,x3>= 0

x4 surplus a1 artificial x5,x6 slack

Min z=Max z’ = -5x1 + 2x2 - 5x3

Subject to

2x1 + 2x2 - x3 -x4 + 0x5 +0x6 + a1 = 2

3x1 - 4x2 + x3 +0x4 +x5 +0x6 +0a1 = 3

0x1+ x2 + 3x3 + 0x4 +0x5 +x6 +0a1= 5

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 00 | 0 | 0 | 0 | -1 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **A1** | **Min Ratio** |
| A1 | -1 | 2 | 2\* | 2 | -1 | -1 | 0 | 0 | 1 | 1 |
| X5 | 0 | 3 | 3 | -4 | 1 | 0 | 1 | 0 | 0 | 1 |
| X6 | 0 | 5 | 0 | 1 | 3 | 0 | 0 | 1 | 0 | - |
|  |  |  | -2 | -2 | 1 | 1 | 0 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 00 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **Min Ratio** |
| X1 | 0 | 1 | 1 | 1 | -1/2 | -1/2 | 0 | 0 |  |
| X5 | 0 | 0 | 0 | -7 | 5/2 | 3/2 | 1 | 0 |  |
| X6 | 0 | 5 | 0 | 1 | 3 | 0 | 0 | 1 |  |
|  |  |  | 0 | 0 | 0 | 0 | 00 | 0 |  |

R1🡪 R1/2 R2🡪2-3R1

Phase 2

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -5 | 2 | -5 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **Min Ratio** |
| X1 | -5 | 1 | 1 | 1\* | -1/2 | -1/2 | 0 | 0 | 1 |
| X5 | 0 | 0 | 0 | -7 | 5/2 | 3/2 | 1 | 0 | - |
| X6 | 0 | 5 | 0 | 1 | 3 | 0 | 0 | 1 | 5 |
|  |  |  | 0 | -7 | 15/2 | 5/2 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -5 | 2 | -5 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **X6** | **Min Ratio** |
| X2 | 2 | 1 | 1 | 1 | -1/2 | -1/2 | 0 | 0 | - |
| X5 | 0 | 7 | 7 | 0 | -1 | -2 | 1 | 0 | - |
| X6 | 0 | 4 | -1 | 0 | 7/2 | ½\* | 0 | 1 | 8 |
|  |  |  | 7 | 0 | 4 | -1 | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -5 | 2 | -5 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **Min Ratio** |
| X2 | 2 | 5 | 0 | 1 | 3 | 0 | 0 |  |
| X5 | 0 | 23 | 3 | 0 | 13 | 0 | 1 |  |
| X4 | 0 | 8 | -2 | 0 | 7 | 1 | 0 |  |
|  |  |  | 5 | 0 | 11 | 0 | 0 |  |

R2🡪R2+2R3 R1🡪R1+1/2R3

X2=5

Min z = 5x1 - 2x2 + 5x3 =-10

***Special cases in Simplex Method***

* 1. Degeneracy in Simplex Method

In some casesthere may be ambiguity in selecting the variable that should be introduced in the basis, i.e. there is a tie between the replacement ratio of two variables.

To resolve degeneracy in simplex method, we select one of them arbitrarily.

Example:

Max z= 3x1 + 9x2

subject to

x1 + 4x2<= 8

x1 + 2x2<= 4

x1, x2>= 0

Max z= 3x1 + 9x2 + 0x3 +0x4

subject to

x1 + 4x2+x3 +0x4 = 8

x1 + 2x2 + 0x3 + x4 = 4

x1, x2>= 0

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 9 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| X3 | 0 | 8 | 1 | 4 | 1 | 0 | 2 |
| X4 | 0 | 4 | 1 | 2 | 0 | 1 | 2 |
|  |  |  | -3 | -9 | 0 | 0 |  |

Here degeneracy arise

Case 1)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 9 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| X3 | 0 | 8 | 1 | 4 | 1 | 0 | 2 |
| X4 | 0 | 4 | 1 | 2 | 0 | 1 | 2 |
|  |  |  | -3 | -9 | 0 | 0 |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 9 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X4** | **Min Ratio** |
| X2 | 9 | 2 | 1/4 | 1 | 0 | 8 |
| X4 | 0 | 0 | 1/2 | 0 | 1 | 0 |
|  |  |  | -3/4 | 0 | 0 |  |

R2🡪 R2-2R1

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 9 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **Min Ratio** |
| X2 | 9 | 2 | 0 | 1 |  |
| X1 | 3 | 0 | 1 | 0 |  |
|  |  |  | 0 | 0 |  |

R1🡪R1-1/4R2

X1=0 x2=2 Max z= 18

Case 2)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 9 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| X3 | 0 | 8 | 1 | 4 | 1 | 0 | 2 |
| X4 | 0 | 4 | 1 | 2 | 0 | 1 | 2 |
|  |  |  | -3 | -9 | 0 | 0 |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 3 | 9 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **Min Ratio** |
| X3 | 0 | 0 | -1 | 0 | 1 |  |
| X2 | 9 | 2 | 1/2 | 1 | 0 |  |
|  |  |  | 3/2 | 0 | 0 |  |

R1🡪 R1-4R2

X2=2 x1=0 Maxz= 18

* 1. No feasible Solution

If in course of Simplex method computation, one or more artificial variables remain in the basis at positive level at the end of the phase 1computation, the problem has no feasible solution or it is called as Infeasible solution

Example

Max Z=200x1 - 300x2

2x1 + 3x2>= 1200

x1 + x2<=400

2x1 + 3/2x2>=900

x1, x2>=0

x3 surplus a1 artificial x4 slack x5 surplus a2 artificial

Max Z=200x1 - 300x2+0x3 +0x4 +0x5 -Ma1 -M a2

2x1 + 3x2-x3 +0x4 + 0x5 +a1 +0a2 = 1200

x1 + x2+ 0x3 +x4 +0x5 +0a1+0a2 =400

2x1 + 3/2x2+0x3 +0x4 -x5 +0a1 +a2=900

x1, x2>=0

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 200 | -300 | 0 | 0 | 0 | -M | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **A1** | **A2** | **Min Ratio** |
| a1 | -M | 1200 | 2 | 3\* | -1 | 0 | 0 | 1 | 0 | 400 |
| X4 | 0 | 400 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 400 |
| **a2** | -M | 900 | 2 | 3/2 | 0 | 0 | -1 | 0 | 1 | 600 |
|  |  | ∆j=**CB XB -** Cj | -4M-200 | -9/2M+300 | M | 0 | M | 0 | 0 |  |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 200 | -300 | 0 | 0 | 0 | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **X5** | **A2** | **Min Ratio** |
| X2 | -300 | 400 | 2/3 | 1 | -1/3 | 0 | 0 | 0 | 600 |
| X4 | 0 | 0 | 1/3\* | 0 | 1/3 | 1 | 0 | 0 | 0 |
| **a2** | -M | 300 | 1 | 0 | 1/2 | 0 | -1 | 1 | 300 |
|  |  | ∆j=**CB XB -** Cj | -M-400 | 0 | -1/2M+100 | 0 | M | 0 |  |

R1🡪R1/3 R2🡪R2-R1 R3🡪R3-3/2R1

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 200 | -300 | 0 | 0 | -M |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X5** | **A2** | **Min Ratio** |
| X2 | -300 | 400 | 0 | 1 | -1 | 0 | 0 |  |
| X1 | 200 | 0 | 1 | 0 | 1 | 0 | 0 |  |
| **a2** | -M | 300 | 0 | 0 | -1/2 | -1 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 1/2M+500 | M | 0 |  |

R2🡪R2\*3 R1🡪 R1-2/3R2 R3🡪R3-R2

In the above table the optimal solution still contains the artificial variable a2 in the basis at positive level, this indicates that the linear program problem has no feasible solution or infeasible solution.

* 1. Unbounded Solution

If in course of simplex computation zj-cj< 0 but minimum ratio value is <= 0 then problem has unbounded solution

Max z= 5x1 + 4x2

x1<=7

x1- x2<=8

x1, x2>=0

x3,x4 are slack variables

Max z= 5x1 + 4x2

x1 +0x2 +x3+0x4 =7

x1 - x2 + 0x3 + x4 =8

x1, x2>=0

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 5 | 4 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** | 0 | 7 | 1\* | 0 | 1 | 0 | 7 |
| **X4** | 0 | 8 | 1 | -1 | 0 | 1 | 8 |
|  |  | ∆j=**CB XB–**Cj | -5 | -4 | 0 | 0 |  |

R2🡪R2-R1

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 5 | 4 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X4** | **Min Ratio** |
| **X1** | 5 | 7 | 1 | 0 | 0 | - |
| **X4** | 0 | 1 | 0 | -1 | 1 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | -4 | 0 |  |

Since minimum ratio value is <= 0, it is not possible to proceed with the simplex computation any further, so it leads to unbounded solution.

* 1. Unrestricted/ Unconstrained Variables

Sometimes decision variables are unrestricted in sign(positive, negative or zero). In all such cases, the decision variables can be expressed as the difference between two non negative variables. For example x1is unrestricted in sign then put x1 = x1’-x1’’

e.g.

Max z=2x1 + 3x2

-x1 + 2x2<=4

x1 + x2<= 6

x1 + 3x2<=9

where x1,x2 are unrestricted variables

x3,x4,x5 slack variables

Max z=2x1’-2x1’’ + 3x2’-3x2’’ +0x3+0x4+0x5

-x1’ + x1’’ + 2x2’- 2x2’’+x3+0x4+0x5=4

x1’-x1’’+ x2’-x2’’+0x3+x4+0x5= 6

x1’-x1’’+ 3x2’-3x2’’+ 0x3+0x4+x5=9

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | -2 | 3 | -3 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1’** | **X1’’** | **X2’** | **X2’’** | **X3** | **X4** | **X5** | **Min Ratio** |
| **X3** | 0 | 4 | -1 | 1 | 2\* | -2 | 1 | 0 | 0 | 2 |
| **X4** | 0 | 6 | 1 | -1 | 1 | -1 | 0 | 1 | 0 | 6 |
| **X5** | 0 | 9 | 1 | -1 | 3 | -3 | 0 | 0 | 1 | 3 |
|  |  | ∆j=**CB XB -** Cj | -2 | 2 | -3 | 3 | 0 | 0 | 0 |  |

R1🡪R1/2 R2🡪R2-R1 R3🡪R3-3R1

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | -2 | 3 | -3 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1’** | **X1’’** | **X2’** | **X2’’** | **X4** | **X5** | **Min Ratio** |
| **X2’** | 3 | 2 | -1/2 | 1/2 | 1 | -1 | 0 | 0 | - |
| **X4** | 0 | 4 | 3/2 | -3/2 | 0 | 0 | 1 | 0 | 8/3 |
| **X5** | 0 | 3 | 5/2\* | -5/2 | 0 | 0 | 0 | 1 | 6/5 |
|  |  | ∆j=**CB XB -** Cj | -7/2 | 7/2 | 0 | 0 | 0 | 0 |  |

R3🡪R3\*2/5 R1🡪R1+1/2R3 R2🡪R2-3/2R3

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2 | -2 | 3 | -3 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1’** | **X1’’** | **X2’** | **X2’’** | **X4** | **Min Ratio** |
| **X2’** | 3 | 13/5 | 0 | 0 | 1 | -1 | 0 |  |
| **X4** | 0 | 11/5 | 0 | 0 | 0 | 0 | 1 |  |
| **X1’** | 2 | 6/5 | 1 | -1 | 0 | 0 | 0 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 0 | 0 | 0 |  |

X1’=6/5

X2’=13/5

x1=x1’-x1’’=6/5

x2=x2’-x2’’=13/5

Max z=2x1 + 3x2= 12/5+39/5=51/5

* 1. Multiple optimal solutions

The optimal solution may not be unique, if the non-basic variables have a zero coefficient in the index row(zj-cj)

This implies that bringing the non-basic variable into the basis will neither increase nor decrease the value of objective function

Thus the linear program problem has multiple optimal solutions or alternative solutions

Max z=2000x1 + 3000x2

6x1 + 9x2<=100

2x1 + x2<= 20

x1,x2>=0

Max z=2000x1 + 3000x2 +0x3+0x4

6x1 + 9x2 +x3+0x4=100

2x1 + x2+0x3+x4 = 20

x1,x2>=0

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2000 | 3000 | 0 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** | 0 | 100 | 6 | 9\* | 1 | 0 | 11. |
| **X4** | 0 | 20 | 2 | 1 | 0 | 1 | 20 |
|  |  | ∆j=**CB XB -** Cj | -2000 | -3000 | 0 | 0 |  |

R1🡪R1/9 R2🡪R2-R1

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2000 | 3000 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X4** | **Min Ratio** |
| **X2** | 3000 | 100/9 | 2/3 | 1 | 0 |  |
| **X4** | 0 | 80/9 | 4/3 | 0 | 1 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 0 |  |

R2🡪R2/9 R1🡪R1-9R2

X2=100/9 **Max z=100000/3**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2000 | 3000 | 0 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X4** | **Min Ratio** |
| **X2** | 3000 | 100/9 | 2/3 | 1 | 0 | 50/3 |
| **X4** | 0 | 80/9 | 4/3\* | 0 | 1 | 20/3 |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 0 |  |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 2000 | 3000 |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **Min Ratio** |
| **X2** | 3000 | 20/3 | 0 | 1 |  |
| **X1** | 2000 | 20/3 | 1 | 0 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 |  |

R2🡪R2\*3/4 R1🡪R1-2/3R2

X1=20/3 x2=20/3

Max z=2000x1 + 3000x2 = 2000(20/3)+3000(20/3)=100000/3

Hence solution has not improved only x2=100/9 is shifted to x1=20/3 and x2=20/3

DUALITY IN LINEAR PROGRAMMING

One of the most important discoveries in the early development of linear programming was the concept of duality and its division into important branches.

The discovery closed the fact that every linear programming problem has associated with its another linear programming problem.

The original problem is called the “primal” while another is called “dual”.

The optimal solution of either problem reveals information concerning the optimal solution of the other.

If the optimal solution to one is known, then the optimal solution of other is readily available.

This fact is important because the situation can arise where the dual is easier to solve than primal.

**🡪General Rules for Converting Any Primal into its Dual**

If the system of constraints in a given LPP consists of a mixture of equations, inequalities (<= or >=), nonnegative variables, or unrestricted variables, then the dual of the given problem can be obtained by reducing it to the standard primal form by adopting the following algorithm

Step 1:

First convert the objective function to maximization form if not.

Step 2:

If a constraint has inequality sign >=, then multiply both sides by -1 and make the inequality sign <=

Step 3:

If a constraint has an equality sign(=), then it is replaced by two constraintsinvolving the inequalities going in opposite directions, simultaneously.

For example, an equation x1+2x2=4 is replaced by two opposite inequalities (<= and >=) constraints:

x1+2x2<=4 and x1+2x2>=4

The second inequality with >= sign can further write as -x1- 2x2<= -4

Step 4:

Every unrestricted variable is replaced by the difference of two non negative variables.

Step 5:

We get the standard primal form, of given LPP in which –

* + 1. All the constraints have <= sign where the objective function is of maximization form, or
    2. All the constraints have >= sign where the objective function is of minimization form

Step 6:

Finally, the dual of the given problem is obtained by,

1. Transpose the rows and columns of constraint coefficients.
2. Transposing the coefficients (c1,c2,…cn) of the objective function and the right side constraints (b1,b2,…bm).
3. Changing the inequalities from <= to >= sign.
4. Minimizing the objective function instead of maximizing it.

Example:

1. Find the dual of the following primal problem

Min z= 2x2 + 5x3

Subject to

x1 + x2 >= 2

2x1 + x2 + 6x3 <= 6

x1 – x2 + 3x3 = 4

x1,x2,x3 >= 0

Min z= 0x1+2x2 + 5x3

Subject to

x1 + x2 +0x3>= 2

-2x1 - x2 - 6x3 >= -6

-x1 + x2 - 3x3 >= -4

x1 – x2 + 3x3 >= 4

x1,x2,x3 >= 0

Consider w1,w2,w3,w4 are dual variables

Max z’= 2w1 -6w2 -4w3 +4w4

W1-2w2-w3+w4<=0

W1-w2+w3-w4<=2

0w1-6w2-3w3+3w4<=5

W1,w2,w3,w4>=0

W3-w4=w5

Max z’= 2w1 -6w2 -4(w3 -w4)

W1-2w2-(w3-w4)<=0

W1-w2+(w3-w4)<=2

0w1-6w2-3(w3-w4)<=5

W1,w2,w3,w4>=0

Max z’= 2w1 -6w2 -4w5

W1-2w2-w5<=0

W1-w2+w5<=2

0w1-6w2-3w5<=5

W1,w2,w3,w4>=0

1. Obtain the dual of the following LP problem

Min z=6x1+3x2+0x3

Sub to

6x1-3x2+2x3>=2

3x1+4x2+4x3>=5

X1,x2,x3>=0

W1,w2 are dual variables where w1,w2>=0

Max z’=2w1+5w2

6w1+3w2<=6

-3w1+4w2<=3

2w1+4w2<=0

W1,w2>=0

1. Min z=2x2+5x3

X1+x2>=2

2x1+x2+6x3<=6

X1-x2+3x3=4

X1,x2,x3>=0

Miz z= 0x1+2x2+5x3

X1+x2+0x3>=2

-2x1-x2-6x3>=-6

X1-x2+3x3>=4

-X1+x2-3x3>=-4

W1,w2,w3,w4 dual variables

Max z’= 2w1-6w2+4w3-4w4

W1-2w2+w3-w4<=0

W1-w2-w3+w4<=2

0w1-6w2+3w3-3w4<=5

Max z’= 2w1-6w2+4(w3-w4)

W1-2w2+1(w3-w4)<=0

W1-w2-1(w3-w4)<=2

0w1-6w2+3(w3-w4)<=5

W3-w4=w5

Maxz’=2w1-6w2+4w5

W1-2w2+w5<=0

W1-w2-w5<=2

-6w2+3w5<=5

1. Find the dual of

Min z=x1+2x2+3x3

Sub to

2x1-x2+x3>=4

x1+x2+2x3<=8

x2-x3>=2

x1, x3>=0, x2 is unrestricted

x2=x2’-x2’’

Min z=x1+2x2’-2x2’’+3x3

Sub to

2x1-x2’+x2’’+x3>=4

x1+x2’-x2’’+2x3<=8

x2’-x2’’-x3>=2

Min z=x1+2x2’-2x2’’+3x3

Sub to

2x1-x2’+x2’’+x3>=4

-x1-x2’+ x2’’-2x3>=-8

0x1+x2’-x2’’-x3>=2

W1,w2,w3 dual variables

Max z’= 4w1-8w2+2w3

2w1-w2+0w3<=1

-w1-w2+w3<=2

W1+w2-w3<=-2

W1-2w2-w3<=3

Max z’= 4w1-8w2+2w3

2w1-w2+0w3<=1

-w1-w2+w3<=2

-W1-w2+w3>=2

W1-2w2-w3<=3

Max z’= 4w1-8w2+2w3

2w1-w2+0w3<=1

-w1-w2+w3=2

W1-2w2-w3<=3

1. Find the dual of

Max z=8x1+2x2+5x3

Sub to

3x1-2x2+5x3<=40

X1+7x2-4x3<=20

5x1-2x3>=12

X1,x2,x3>=0

Max z=8x1+2x2+5x3

Sub to

3x1-2x2+5x3 <=40

X1+7x2-4x3<=20

-5x1+0x2+2x3<=-12

W1,w2,w3 are dual variables

Min z=40w1+20w2-12w3

3w1+w2-5w3>=8

-2w1+7w2+0w3>=2

5w1-4w2+2w3>=5

W1,w2,w3>=0

1. Write the LP dual of the following LP problem.

Min z = 3x1 – 2x2 + 4x3

3x1 + 5x2 + 4x3 >= 7

6x1 + x2 + 3x3 >= 4

7x1 – 2x2 - x3 <= 10

X1 – 2x2 + 5x3 >= 3

4x1 + 7x2 -2x3 >= 2

X1, x2, x3 >=0

Min z = 3x1 – 2x2 + 4x3

3x1 + 5x2 + 4x3 >= 7

6x1 + x2 + 3x3 >= 4

-7x1 + 2x2 + x3 >= -10

X1 – 2x2 + 5x3 >= 3

4x1 + 7x2 -2x3 >= 2

W1,w2,w3,w4,w5 are dual variables

Max z’=7w1+4w2-10w3+3w4+2w5

3w1+6w2-7w3+w4+4w5<=3

5w1+w2+2w3-2w4+7w5<=-2

4w1+3w2+w3+5w4-2w5<=4

1. Obtain the dual of the following LP problem

Max z=2x1+3x2+x3

Sub to

4x1 + 3x2 + x3 = 6

X1 + 2x2+5x3 =4

x1,x2,x3>=0

Max z=2x1+3x2+x3

Sub to

4x1 + 3x2 + x3 <= 6

-4x1 - 3x2 - x3 <= -6

X1 + 2x2+5x3 <=4

-X1 - 2x2- 5x3 <= -4

W1,w2,w3,w4 dual variables

Min z’=6w1-6w2+4w3-4w4

4w1-4w2+w3-w4>=2

3w1-3w2+2w3-2w4>=3

W1-w2+5w3-5w4>=1

Min z’=6(w1-w2)+4(w3-w4)

4(w1-w2)+1(w3-w4)>=2

3(w1-w2)+2(w3-w4)>=3

1(W1-w2)+5(w3-w4)>=1

W1-w2=w5

W3-w4=w6

Minz’=6w5+4w6

4w5+w6>=2

3w5+2w6>=3

W5+5w6>=1

1. Obtain the dual of the following LP problem

Min z=2x1+3x2+4x3

Sub to

2x1+3x2+5x3>=2

3x1+x2+7x3=3

X1+4x2+6x3<=5

X1,x2>=0, x3 is unrestricted

**DUAL SIMPLEX METHOD**

* 1. Max z=40x1+50x2

Subject to

2x1+3x2<=3

8x1+4x2<=5

X1,x2>=0

Consider w1 w2 are dual variables

Min z’= 3w1+5w2

2w1+8w2>=40

3w1+4w2>=50

Max z’’=-3w1-5w2

W3,w4 surplus variables

A1,a2 are artificial variables

Max z’’=-3w1-5w2+0w3+0w4-a1-a2

2w1+8w2-w3+0w4+a1+0a2=40

3w1+4w2+0w3-w4+0a1+a2=50

Phase 1

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | -1 | -1 |  |
| **Basic Variables** | **CB** | **WB** | **w1** | **w2** | **w3** | **w4** | **a1** | **A2** | **Min Ratio** |
| **A1** | -1 | 40 | 2 | 8\* | -1 | 0 | 1 | 0 | 5 |
| A2 | -1 | 50 | 3 | 4 | 0 | -1 | 0 | 1 | 12. |
|  |  | ∆j=**CB XB -** Cj | -5 | -12 | 1 | 1 | 0 | 0 |  |

R1🡪R1/8 R2🡪R2-4R1

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | -1 |  |
| **Basic Variables** | **CB** | **WB** | **w1** | **w2** | **w3** | **w4** | **A2** | **Min Ratio** |
| W2 | 0 | 5 | 1/4 | 1 | -1/8 | 0 | 0 | 20 |
| A2 | -1 | 30 | 2\* | 0 | 1/2 | -1 | 1 | 15 |
|  |  | ∆j=**CB XB -** Cj | -2 | 0 | -1/2 | 1 | 0 |  |

R2🡪R2/2 R1🡪R1-1/4R2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 |  |
| **Basic Variables** | **CB** | **WB** | **w1** | **w2** | **w3** | **w4** | **Min Ratio** |
| W2 | 0 | 5/4 | 0 | 1 | -3/16 | 1/8 |  |
| W1 | 0 | 15 | 1 | 0 | 1/4 | -1/2 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 0 | 0 |  |

Phase2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | -3 | -5 | 0 | 0 |  |
| **Basic Variables** | **CB** | **WB** | **w1** | **w2** | **w3** | **w4** | **Min Ratio** |
| W2 | -5 | 5/4 | 0 | 1 | -3/16 | 1/8 |  |
| W1 | -3 | 15 | 1 | 0 | 1/4 | -1/2 |  |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 3/16 | 7/8 |  |

W2=5/4 w1=15

Min z’= 3w1+5w2

=3(15)+5(5/4) = 45+25/4=205/4

* 1. Solve following problem by dual simplex method.

Max z=5x1+12x2+4x3

Subject to

X1+2x2+x3<=5

2x1-x2+3x3=2

X1,x2,x3>=0

Max z=5x1+12x2+4x3

Subject to

X1+2x2+x3<=5

2x1-x2+3x3<=2

-2x1+x2-3x3<=-2

W1,w2,w3

Min z’=5w1+2w2-2w3

W1+2w2-2w3>=5

2w1-w2+w3>=12

W1+3w2-3w3>=4

W4,w5,w6 surplus

A1,a2,a3 artificial

Max z’’=-5w1-2w2+2w3+0w4+0w5+0w6-a1-a2-a3

W1+2w2-2w3 -w4+0w5+0w6+a1+0a2+0a3=5

2w1-w2+w3 +0w4 -w5+0w6+0a1+a2+0a3=12

W1+3w2-3w3 +0w4+0w5-w6+0a1+0a2+a3=4

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | -1 |  |
| **Basic Variables** | **CB** | **WB** | **W1** | **w2** | **w3** | **w4** | **W5** | **W6** | **A1** | **A2** | **A3** | **Min Ratio** |
| A1 | -1 | 5 | 1 | 2 | -2 | -1 | 0 | 0 | 1 | 0 | 0 | 5 |
| **A2** | -1 | 12 | 2 | -1 | 1 | 0 | -1 | 0 | 0 | 1 | 0 | 6 |
| A3 | -1 | 4 | 1\* | 3 | -3 | 0 | 0 | -1 | 0 | 0 | 1 | 4 |
|  |  | ∆j=**CB XB -** Cj | -4 | -4 | 4 | 1 | 1 | 1 | 0 | 0 | 0 |  |

R1🡪r1-R3 R2🡪R2-2R3

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 |  |
| **Basic Variables** | **CB** | **WB** | **W1** | **w2** | **w3** | **w4** | **W5** | **W6** | **A1** | **A2** | **Min Ratio** |
| A1 | -1 | 1 | 0 | -1 | 1 | -1 | 0 | 1 | 1 | 0 | 1 |
| **A2** | -1 | 4 | 0 | -7 | 7\* | 0 | -1 | 2 | 0 | 1 | 4/7 |
| W1 | 0 | 4 | 1 | 3 | -3 | 0 | 0 | -1 | 0 | 0 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | 8 | -8 | 1 | 1 | -3 | o | 0 |  |

R2🡪R2/7 R1🡪R1-R2 R3🡪R3+3R2

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  | 0 | 0 | 0 | 0 | 0 | 0 | -1 |  |
| **Basic Variables** | **CB** | **WB** | **W1** | **w2** | **w3** | **w4** | **W5** | **W6** | **A1** | **Min Ratio** |
| A1 | -1 | 3/7 | 0 | 0 | 0 | -1 | 1/7 | 5/7\* | 1 | 3/5 |
| **W3** | 0 | 4/7 | 0 | -1 | 1 | 0 | -1/7 | 2/7 | 0 | 2 |
| W1 | 0 | 40/7 | 1 | 0 | 0 | 0 | -3/7 | -1/7 | 0 | - |
|  |  | ∆j=**CB XB -** Cj | 0 | 0 | 0 | 0 | -1/7 | -5/7 | 0 |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  |  |  |  |  |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** |  |  |  |  |  |  |  |
| **X4** |  |  |  |  |  |  |  |
|  |  | ∆j=**CB XB -** Cj |  |  |  |  |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  |  |  |  |  |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** |  |  |  |  |  |  |  |
| **X4** |  |  |  |  |  |  |  |
|  |  | ∆j=**CB XB -** Cj |  |  |  |  |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  |  |  |  |  |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** | **Min Ratio** |
| **X3** |  |  |  |  |  |  |  |
| **X4** |  |  |  |  |  |  |  |
|  |  | ∆j=**CB XB -** Cj |  |  |  |  |  |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Cj🡪 |  |  |  |  |  |  |  |
| **Basic Variables** | **CB** | **XB** | **X1** | **X2** | **X3** | **X4** |  | **Min Ratio** |
| **X3** |  |  |  |  |  |  |  |  |
| **X4** |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  | ∆j=**CB XB -** Cj |  |  |  |  |  |  |

**TRANSPORTAION PROBLEM**

If there are more than one centres, called origins from where goods need to be shipped to more than one places called “destinations” and the cost of shipping from each of the origins to each of the destinations being different and known, the problem is to ships the goods from various origins to different destinations in a such manner that the cost of shipping or transportation is minimum.

Definition:

The transportation problem is to transport various amounts of single homogenous commodity, that are initially stored at various origins to different destinations in such a way that the total transportation cost is minimum.

Example:-

A tyre manufacturing concern has m factories located at m different cities. The total supply potential of manufactured product is absorbed by n retail dealers in n different cities of the country. Then the transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

There are n factories and n godowns. Each factory manufactures or supplies number of items.

Cost of transporting one unit of the item from factory to godown is given in the question.

Our aim is to decide how many items should be transported from factory to the godown so that total cost of transportation get minimized.

* 1. Demand and supply conditions are fulfilled.
  2. Total cost of transportation is minimized.
  3. If total demand=total supply then it is called as balanced transportation problem.

Only balanced transportation problem can be solved solution for transportation problem.

Step 1:-

Find initial basic feasible solution using any of the following method

* 1. North West Corner Rule
  2. Least Cost Method
  3. Vogel’s Transportation method

Step 2:-

Find optimum solution using modified distribution method (MODI method) It is also called as UV method.

1. **North West Corner Method**

Step1:

Select the north west corner.

Step2:

Allocate maximum possible units to this cell.

Step3:

Subtract the number allocated from corresponding demand and supply.

Step4:

Delete the row/column fulfilling demand supply condition.

Step5:

Go to step 1.

Example:

* 1. Find an initial basic feasible solution using north west corner rule.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Destinations | | | | |  |
| Plant |  | A | B | C | D | Supply |
|  | 6 |  |  |  |  |
| 1 | 2 | 3 | 11 | 7 | 6 -0 |
|  | 1 |  |  |  |  |
| 2 | 1 | 0 | 6 | 1 | 1-0 |
|  |  | 5 | 3 | 2 |  |
| 3 | 5 | 8 | 15 | 9 | 10-5-2-0 |
|  | DEMAND | 7-1-0 | 5-0 | 3-0 | 2-0 | 17 |

Total cost=(6\*2)+(1\*1)+(5\*8)+(3\*15)+(2\*9)

= 12+1+40+45+18 =116 units

* 1. Find an initial basic feasible solution using north west corner rule.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Warehouse | | | | |  |
| Factory |  | W1 | W2 | W3 | W4 | Factory Capacity |
|  | 5 | 2 |  |  |  |
| F1 | 90 | 30 | 50 | 10 | 7-2-0 |
|  |  | 6 | 3 |  |  |
| F2 | 70 | 30 | 40 | 60 | 9-3-0 |
|  |  |  | 4 | 14 |  |
| F3 | 40 | 8 | 70 | 20 | 18-14-0 |
|  | Warehouse Requirement | 5-0 | 8-6-0 | 7-4 | 14-0 |  |

Total cost=(5\*90)+(2\*30)+(6\*30)+(3\*40)+(4\*70)+(14\*20)

= 450+60+180+120+280+280

=1370

* 1. Find an initial basic feasible solution using north west corner rule.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  | | | | |  |
|  |  | D1 | D2 | D3 | D4 | Supply |
|  | 6 | 8 |  |  |  |
| O1 | 6 | 4 | 1 | 5 | 14-8-0 |
|  |  | 2 | 14 |  |  |
| O2 | 8 | 9 | 2 | 7 | 16-14-0 |
|  |  |  | 1 | 4 |  |
| O3 | 4 | 3 | 6 | 2 | 5-4 |
|  | DEMAND | 6-0 | 10-2-0 | 15-1-0 | 4 | 35 |

Total cost=128 units

1. LOWEST COST METHOD

Step 1:

Determine the smallest cost in the cost matrix of the transportation table.

Let it be Cij

Allocate xij=min(ai, bj) in the cell (i, j)

Step 2:

* 1. If xij=ai, cross out the ith row of the transportation table, and decreased bj by ai. Go to step 3.
  2. If xij=bj, cross out jth column of the transportation table and decreased ai by bj. Go to step 3
  3. If xij=ai=bj, cross out either the ith row or jth column but not both.

Step 3:

Repeat steps 1 & 2 for the resulting reduced transportation table until all requirements are satisfied.

Whenever minimum cost is not unique, make an arbitrary choice among the minima.

Example:

* 1. Find an initial basic feasible solution using lowest cost method.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Destinations | | | | |  |
| Plant |  | A | B | C | D | Supply |
|  | 6 |  |  |  |  |
| 1 | 2 | 3 | 11 | 7 | 6-0 |
|  |  | 1 |  |  |  |
| 2 | 1 | 0 | 6 | 1 | 1-0 |
|  | 1 | 4 | 3 | 2 |  |
| 3 | 5 | 8 | 15 | 9 | 10-9-5-3-0 |
|  | DEMAND | 7-1-0 | 5-4-0 | 3-0 | 2-0 | 17 |

Total cost=112

* 1. Find an initial basic feasible solution using lowest cost method.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Warehouse | | | | |  |
| Factory |  | W1 | W2 | W3 | W4 | Factory Capacity |
|  |  |  |  | 7 |  |
| F1 | 90 | 30 | 50 | 10 | 7-0 |
|  | 2 |  | 7 |  |  |
| F2 | 70 | 30 | 40 | 60 | 9-2-0 |
|  | 3 | 8 |  | 7 |  |
| F3 | 40 | 8 | 70 | 20 | 18-10-3-0 |
|  | Warehouse Requirement | 5-2-0 | 8-0 | 7-0 | 14-7-0 | 34 |

Total cost=814

* 1. Find an initial basic feasible solution using lowest cost method.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | | | | |  |  |
|  |  | D1 | D2 | D3 | D4 | Supply |  |
|  |  |  |  |  |  |  |
| O1 | 6 | 4 | 1 | 5 | 14 | (3) |
|  |  |  |  |  |  |  |
| O2 | 8 | 9 | 2 | 7 | 16 | (5) |
|  |  |  |  |  |  |  |
| O3 | 4 | 3 | 6 | 2 | 5 | (1) |
|  | DEMAND | 6 | 10 | 15 | 4 |  |  |
|  |  | (2) | (1) | (1) | (3) |  |  |

1. Vogel’s Approximation Method(VAM)

Step1:

For each row of the transportation table identify smallest and next to smallest cost. Determine the difference between them for each row. This are called penalties.

Put them along side the transportation table by enclosing them in the parentheses against the respective rows. Similarly compute these penalties for each column.

Step 2:

Identify the row or column with largest penalty among all the rows and columns.

If a tie occurs, use any arbitrary tie breaking choice. Let the largest penalty corresponds to ith row and let cijbe the smallest cost in the ith row

Allocate the largest possible amount xij=min(ai,bj) in the cell (i,j) and cross out the ith row and jth column in the usual manner.

Step 3:

Again compute the column and row penalties for the reduced transportation table and then go to step2. Repeat the procedure until all the requirements are satisfied.

Example:

* 1. Find an initial basic feasible solution using VAM method.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Destinations | | | | |  |  |
| Plant |  | A | B | C | D | Supply |  |
|  | 1 | 5 |  |  |  |  |
| 1 | 2 | 3 | 11 | 7 | 6 1 0 | (1)(1)(5) |
|  |  |  |  | 1 |  |  |
| 2 | 1 | 0 | 6 | 1 | 1 0 | (1) |
|  | 6 |  | 3 | 1 |  |  |
| 3 | 5 | 8 | 15 | 9 | 10 0 | (3)(3)(4) |
|  | DEMAND | 7 6 0 | 5 0 | 3 0 | 2 1 0 |  |  |
|  |  | (1)(3)(3) | (3)(5) | (5)(4)(4) | (6)(2)(2) |  |  |

Total cost= (1\*2)+(5\*3)+(1\*1)+(6\*5)+(3\*15)+(1\*9)=102

* 1. Find an initial basic feasible solution using VAM method.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | Warehouse | | | | |  |
| Factory |  | W1 | W2 | W3 | W4 | Factory Capacity |
|  |  |  |  |  |  |
| F1 | 19 | 30 | 50 | 10 | 7 |
|  |  |  |  |  |  |
| F2 | 70 | 30 | 40 | 60 | 9 |
|  |  |  |  |  |  |
| F3 | 40 | 8 | 70 | 20 | 18 |
|  | Warehouse Requirement | 5 | 8 | 7 | 14 |  |

* 1. Find an initial basic feasible solution using VAM method.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | | | | |  |  |
|  |  | D1 | D2 | D3 | D4 | Supply |  |
|  | 4 | 10 |  |  |  |  |
| O1 | 6 | 4 | 1 | 5 | 14 4 | (3)(1)(2)(2) |
|  | 1 |  | 15 |  |  |  |
| O2 | 8 | 9 | 2 | 7 | 16 1 | (5)(1)(1)(1) |
|  | 1 |  |  | 4 |  |  |
| O3 | 4 | 3 | 6 | 2 | 5 1 0 | (1)(1)(1) |
|  | DEMAND | 6 5 | 10 0 | 15 0 | 4 0 |  |  |
|  |  | (2)(2)(2)(2) | (1)(1)(1)(5) | (1) | (3)(3) |  |  |

Total Cost=(4\*6)+(10\*4)+(1\*8)+(15\*2)+(1\*4)+(4\*2)=114

* OPTIMALITY TEST FOR TRANSPORTION PROBLEM

There are basically two methods

1. Modified Distribution Method
2. Stepping Stone Method
   1. MODIFIED DISTRIBUTION METHOD(MODI Method)

The modified distribution method also known as MODI or u-v method provides minimum cost solution to the transportation problem.

In the stepping stone method we have to draw as many closed paths as equal to the unoccupied cells for their evaluation.

To the contrary, in MODI method, only closed path with unoccupied cell with highest opportunity cost is drawn.

Step 1:

Determine an initial basic feasible solution using any of the three methods given below

NWCR

LCM

VAM

Solution should have m+n-1 allocations in independent positions.

Step2:

Determine the values of the dual variables ui and vj using ui+vj=cij for occupied cells

Step3:

Compute opportunity cost using dij= Cij – (ui + vj) for unoccupied cells.

Step4:

Check the sign of each opportunity cost

1. If opportunity cost of all unoccupied cells are either positive or zero, then given solution is optimal solution.
2. If one or more unoccupied cells have negative opportunity cost, the given solution is not optimum solution and further saving in transportation cost are possible.

Step5:

Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in next solution.

Step 6:

Draw a closed path or loop for the unoccupied cell selected in previous step.

Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

A close up of a whiteboard

Description automatically generated

Step 7:

Assign alternate plus and minus sign at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.

Step 8:

Determine maximum number of units that should be shipped to this unoccupied cell.

The smallest value with negative position on closed path indicates the number of units that can be shipped to the entering cell.

Now add this quantity to all the cells on the corner points of the closed path marked with plus sign and subtract it from those cells marked with minus sign.

In this way , an unoccupied cell becomes occupied cell

Step 9:

Repeat the whole procedure until an optimum solution is obtained.

Example:

* 1. Obtain an initial basic feasible solution to the transportation problem

If a company is spending Rs 1000 of transportation of its units to four warehouses from these factories. What can be the maximum saving by optimal scheduling?

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | Warehouse | | | | |  |
|  |  |  | V1 | V2 | V3 | V4 |  |
| Factory |  |  | W1 | W2 | W3 | W4 | Factory Capacity |
|  |  | 5 |  |  | 2 |  |
| U1 | F1 | 19 | 30 | 50 | 10 | 7 |
|  |  |  |  | 7 | 2 |  |
| U2 | F2 | 70 | 30 | 40 | 60 | 9 |
|  |  |  | 8 |  | 10 |  |
| U3 | F3 | 40 | 8 | 70 | 20 | 18 |
|  |  | Warehouse Requirement | 5 | 8 | 7 | 14 |  |

For occupied cell

U1+v1=19 U1+v4=10 U2+v3=40 U2+v4=60 U3+v2=8 U3+v4=20

U1=-10 U2=40 U3=0 V1=29 V2=8 V3=0 V4=20

Unoccupied cells

dij= Cij – (ui + vj)

d12=30-(u1+v2) =30-(-10+8)=32

d13=50-(u1+v3) =50-(-10+0)=60

d21=70-(u2+v1) =70-(40+29)= 1

d22=30-(u2+v2) =30-(40+8) =-18

d31=40-(u3+v1) =40-(0+29) =11

d33=70-(u3+v3) =70-(0+0) =70

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | Warehouse | | | | |  |
|  |  |  | V1 | V2 | V3 | V4 |  |
| Factory |  |  | W1 | W2 | W3 | W4 | Factory Capacity |
|  |  | 5 |  |  | 2 |  |
| U1 | F1 | 19 | 30 | 50 | 10 | 7 |
|  |  |  | \*+ | 7 | 2- |  |
| U2 | F2 | 70 | 30 | 40 | 60 | 9 |
|  |  |  | 8- |  | 10+ |  |
| U3 | F3 | 40 | 8 | 70 | 20 | 18 |
|  |  | Warehouse Requirement | 5 | 8 | 7 | 14 |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | Warehouse | | | | |  |
|  |  |  | V1 | V2 | V3 | V4 |  |
| Factory |  |  | W1 | W2 | W3 | W4 | Factory Capacity |
|  |  | 5 |  |  | 2 |  |
| U1 | F1 | 19 | 30 | 50 | 10 | 7 |
|  |  |  | 2 | 7 |  |  |
| U2 | F2 | 70 | 30 | 40 | 60 | 9 |
|  |  |  | 6 |  | 12 |  |
| U3 | F3 | 40 | 8 | 70 | 20 | 18 |
|  |  | Warehouse Requirement | 5 | 8 | 7 | 14 |  |

For all allocated cells

U1v1 u1+v1=19

U1v4 u1+v4=10

U2v2 u2+v2=30

U2v3 u2+v3=40

U3v2 u3+v2=8

U3v4 u3+v4=20

U1=-10 u2=22 u3=0 v1=29 v2=8 v3=18 v4=20

For unallocated cells

Dij=cij-(ui+vj)

U1v2=30-(-10+8)= 32

U1v3=50-(-10+18)= 42

U2v1=70-(22+29)= 19

U2v4=60-(22+20)= 18

U3v1=40-(0+29)= 11

U3v3=70-(0+18)= 52

Total cost=(5\*19)+(2\*10)+(2\*30)+(7\*40)+(6\*8)+(12\*20)

=95+20+60+280+48+240=743

Saving=1000-743=257/-

* 1. Solve following transportation problem in which cell entries represent unit costs

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | To | | | | |
| From |  |  |  |  | Available |
| +3\* | 2 | 3-3 |  |  |
| 2 | 7 | 4 |  | 5 |
|  |  | 8 |  |  |
| 3 | 3 | 1 |  | 8 |
|  | 7 |  |  |  |
| 5 | 4 | 7 |  | 7 |
| 3-7 |  | 7+3 |  |  |
| 1 | 6 | 2 |  | 14 |
|  |  |  |  |  |
| Requirement | 7 | 9 | 18 |  |  |

M+n-1=6

For allocated cells

Ui+vj=cij

U1+v2=7

U1+v3=4

U2+v3=1

U3+v2=4

U4+v1=1

U4+v3=2

U1=2 u2=-1 u3=-1 u4=0

V1=1 v2=5 v3=2

For non allocated cells

Dij=cij-(ui+vj)

D11=2-(2+1)=-1

D21=3-(-1+1)=3

D22=3-(-1+5)=-1

D31=5-(-1+1)=5

D33=7-(-1+2)=6

D42=6-(0+5)=1

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | To | | | | |
| From |  |  |  |  | Available |
| +3 | 2- |  |  |  |
| 2 | 7 | 4 |  | 5 |
|  | \*+ | 8- |  |  |
| 3 | 3 | 1 |  | 8 |
|  | 7 |  |  |  |
| 5 | 4 | 7 |  | 7 |
| -4 |  | 10+ |  |  |
| 1 | 6 | 2 |  | 14 |
|  |  |  |  |  |
| Requirement | 7 | 9 | 18 |  |  |

For allocate cells

U1+v1=2

U1+v2=7

U2+v3=1

U3+v2=4

U4+v1=1

U4+v3=2

U1=1 u2=-1 u3=-2 u4=0

V1=1 v2=6 v3=2

D13=4-(1+2)=1

D21=3-(-1+1)=3

D22=3-(-1+6)=-2

D31=5-(-2+1)=6

D33=7-(-2+2)=7

D42=6-(0+6)=0

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | To | | | | |
| From |  |  |  |  | Available |
| 5 |  |  |  |  |
| 2 | 7 | 4 |  | 5 |
|  | 2 | 6 |  |  |
| 3 | 3 | 1 |  | 8 |
|  | 7 |  |  |  |
| 5 | 4 | 7 |  | 7 |
| 2 |  | 12 |  |  |
| 1 | 6 | 2 |  | 14 |
|  |  |  |  |  |
| Requirement | 7 | 9 | 18 |  |  |

For allocated cells

U1+v1=2

U2+v2=3

U2+v3=1

U3+v2=4

U4+v1=1

U4+v3=2

U1=1 u2=-1 u3=0 u4=0

V1=1 v2=4 v3=2

For nonallocated cells

D12=7-(1+4)=2

D13=4-(1+2)=1

D21=3-(-1+1)=3

D31=5-(0+1)=4

D33=7-(0+2)5

D42=6-(0+4)=2

Total cost=(5\*2)+(2\*3)+(6\*1)+(7\*4)+(2\*1)+(12\*2)=76

* 1. Solve the transportation problem

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  | To | | | | |
| From |  | D1 | D2 | D3 | D4 | Available |
|  | 20 |  | 10 |  |  |
| O1 | 1 | 2 | 1 | 4 | 30 10 0 |
|  |  | 20 | 20 | 10 |  |
| O2 | 3 | 3 | 2 | 1 | 50 40 20 0 |
|  |  | 20 |  |  |  |
| O3 | 4 | 2 | 5 | 9 | 20 0 |
|  |  |  |  |  |  |
|  | Requirement | 20 0 | 40 20 0 | 30 20 0 | 10 0 |  |

Allocations=6 =m+n-1

* Special case in transportation problem

Unbalanced transportation problem:

When the total supply of all the sources is not equal to total demand of all destinations, the problem is unbalanced transportation problem.

Total Demand ≠ TotalSupply

These unbalanced problems can be easily solved by introducing dummy sources and dummy destinations. If the total supply is greater than total demand, a dummy destination (dummy column) with demand equal to the supply surplus is added.

If total demand is greater than total supply, dummy source (dummy row) with supply equal to the demand surplus is added.

The unit transportation cost for the dummy column and dummy row are assigned zero values, because no shipment is actually made in case of dummy source and dummy destinations.

* 1. Solve the transportation problem when the unit transportation cost, demand and supply are as given below:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | To | | | | |  |
| From |  | D1 | D2 | D3 | D4 | Supply |  |
|  | 25-65 | 5+25 |  |  |  |  |
| O1 | 6 | 1 | 9 | 3 | 70 5 0 | 2 2 |
|  |  | 30-25 | 25+25 |  |  |  |
| O2 | 11 | 5 | 2 | 8 | 55 25 0 | 3 3 6 |
|  | +25\* |  | 25-25 | 45 |  |  |
| O3 | 10 | 12 | 4 | 7 | 70 0 | 3 3 3 |
|  | 20 |  |  |  |  |  |
|  | Dummy | 0 | 0 | 0 | 0 | 20 0 | 0 |
|  | Demand | 85 65 0 | 35 30 | 50 25 0 | 45 0 | 215>195 |  |
|  |  | 6 4 | 1 4 4 7 | 2 2 2 2 2 | 3 4 4 1 1 |  |  |

215-195=20

For allocated cells

Ui+vj=cij

U1+v1=6

U1+v2=1

U2+v2=5

U2+v3=2

U3+v3=4

U3+v4=7

U4+v1=0

U1=-6 u2=-2 u3=0 u4=-12

V1=12 v2=7 v3=4 v4=7

For nonallocated cells

Dij=cij-(ui+vj)

D13=9-(-6+4)=11

D14=3-(-6+7)=2

D21=11-(-2+12)=1

D24=8-(-2+7)=3

D31=10-(0+12)=-2

D32=12-(0+7)=5

D42=0-(-12+7)=5

D43=0-(-12+4)=8

D44=0-(-12+7)=5

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  | To | | | | |  |
| From |  | D1 | D2 | D3 | D4 | Supply |  |
|  | 40 | 30 |  |  |  |  |
| O1 | 6 | 1 | 9 | 3 | 70 5 0 | 2 2 |
|  |  | 5 | 50 |  |  |  |
| O2 | 11 | 5 | 2 | 8 | 55 25 0 | 3 3 6 |
|  | 25 |  |  | 45 |  |  |
| O3 | 10 | 12 | 4 | 7 | 70 0 | 3 3 3 |
|  | 20 |  |  |  |  |  |
|  | Dummy | 0 | 0 | 0 | 0 | 20 0 | 0 |
|  | Demand | 85 65 0 | 35 30 | 50 25 0 | 45 0 | 215>195 |  |
|  |  | 6 4 | 1 4 4 7 | 2 2 2 2 2 | 3 4 4 1 1 |  |  |

Total cost=(40\*6)+(30\*1)+(5\*5)+(50\*2)+(25\*10)+(45\*7)+(20\*0)=960

Ui+vj=cij

U1+v1=6

U1+v2=1

U2+v2=5

U2+v3=2

U3+v1=10

U3+v4=7

U4+v1=0

U1=-4 u2=0 u3=0 u4=-10

V1=10 v2=5 v3=2 v4=7

D13=9-(-4+2)=11

D14=3-(-4+7)=0

D21=11-(0+10)=1

D24=8-(0+7)=1

D32=12-(0+5)=7

D33=4-(0+2)=2

D42=0-(-10+5)=5

D43=0-(-10+2)=8

D44=0-(-10+7)=3

* 1. A product is produced by four factories F1, F2, F3, F4. Their unit production cost are Rs 2, 3, 1 and 5 respectively. Production capacity of factories are 50, 70,30,50 units respectively. The product is supplied to 4 stores S1, S2, S3, S4, the requirement of which are 25, 35, 105 and 20 respectively. Unit cost of transportation are given below.

Find the transportation plan such that the total production transportation cost is minimum.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Stores🡪  Factories | S1 | S2 | S3 | S4 |
| F1 | 2 | 4 | 6 | 11 |
| F2 | 10 | 8 | 7 | 5 |
| F3 | 13 | 3 | 9 | 12 |
| F4 | 4 | 6 | 8 | 3 |

Total cost=unit cost of production+transportation cost

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Stores🡪  Factories | S1 | S2 | S3 | S4 | dummy | capacity |
| F1 | 4 | 6 | 8 | 13 | 0 | 50 |
| F2 | 13 | 11 | 10 | 8 | 0 | 70 |
| F3 | 14 | 4 | 10 | 13 | 0 | 30 |
| F4 | 9 | 11 | 13 | 8 | 0 | 50 |
| Requirement | 25 | 35 | 105 | 20 | 15 | 185\200 |

**ASSIGNMENT PROBLEM**

Assignment problem is special type of linear programming problem which deals with the allocation of various resources to the various activities on one to one basis. It does it in such a way that the cost or time involved in this process is minimum and profit or sale is maximum

Example:

In a factory, a supervisor may have a six workers available and six jobs to fire. He will have to take decision regarding which job should be given to which worker, Problem forms one to one basis. This is assignment problem.

Suppose there are n facilities and n jobs it is clear that in this case there will be n assignments. Each facility or say worker can perform each job one at a time. But their should be certain procedure by which assignment should be made so that profit is maximized or the cost or time is minimized.

Job of Work

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Worker | 1 | 2 | 3 | 4 | ……… | n |
| 1 | C11 | C12 | C13 | C14 | ……… | C1n |
| 2 | C21 | C22 | C23 | C24 | ………. | C2n |
| 3 | C31 | C32 | C33 | C34 | ……….. | C3n |
| 4 | C41 | C42 | C43 | C44 | ………. | C4n |
| … | ……….. | ,……… | ………. | ……… | ………. | ………. |
| N | Cn1 | Cn2 | Cn3 | Cn4 | ………. | Cnn |

Here Cij is defined as the cost when jth job is assigned to ith worker

* HUNGARIAN METHOD FOR ASSIGNMENT PROBLEM

Various steps of the computational procedure for obtaining an optimal assignment may be summarised as follows:

STEP 1:

Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows.

STEP 2:

Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus obtain the first modified matrix.

STEP 3:

Then draw the minimum number of horizontal and vertical lines to cover all zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities:

* 1. If N=n, then number of rows of given matrix, then an optimal assignment can be made. So make the zero assignment to get required solution.
  2. If N<n, then proceed to step 4.

RULES TO DRAW MINIMUM NUMBER OF LINES

A very convenient rule of drawing minimum number of lines to cover all the 0’s of the reduced matrix is given in the following steps:

Step 1:

Tick (✓) rows that do not have any marked (▢ ) zero.

Step 2:

Tick (✓) columns having any marked (▢ ) zeros or otherwise in ticked rows.

Step3:

Tick (✓) rows having marked (▢ ) zeros in ticked columns.

Step 4:

Repeat step 2 and 3 until the chain of ticking is complete.

Step 5:

Draw lines through all unticked rows and ticked columns.

This will give us the minimal system of lines.

STEP 4:

Determine the smallest element in the matrix not covered by the N lines. Subtract this minimum element from all uncovered elements & add the same element at the intersection of horizontal and vertical lines. Thus second modified matrix is obtained.

STEP 5:

Again repeat step 3 and 4 until minimum number of lines become equal to the number of rows(columns) of the given matrix i.e. N=n

Step 6:

(To make zero assignment). Examine the rows successively until a row wise exactly single zero is found, mark this zero by ‘’ ‘ to make the assignment. Then mark cross (×) over all zeros if laying in the column of the marked ‘ ‘ zero, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for column also.

Step 7:

Repeat the step 6 successively until one of the following situations arise.

1. If no unmarked zero is left, then the process ends or
2. If their lie more than one of the unmarked zeros in any column or row then mark ‘ ‘ one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its rows and column. Repeat the process until no unmarked zeros is left in the matrix.

Step 8:

Thus exactlyone markedzero in eachcolumn of the matrix is obtained. The assignment corresponding to these marked zeros will give the optimal assignment.

Example:

1. A department head has four subordinates and four task to be performed. Subordinates differ in efficiency and task differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allotted to each person so as to minimize the total man hours.



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Subordinates | | | |
|  |  | I | II | III | IV |
| Tasks | A | 8 | 26 | 17 | 11 |
| B | 13 | 28 | 4 | 26 |
| C | 38 | 19 | 18 | 15 |
| D | 19 | 26 | 24 | 10 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Subordinates | | | |
|  |  | I | II | III | IV |
| Tasks | A | 0 | 18 | 9 | 3 |
| B | 9 | 24 | 0 | 22 |
| C | 23 | 4 | 3 | 0 |
| D | 9 | 16 | 14 | 0 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Subordinates | | | |
|  |  | I | II | III | IV |
| Tasks | A | 0 | 14 | 9 | 3 |
| B | 9 | 20 | 0 | 22 |
| C | 23 | 0 | 3 | 0 |
| D | 9 | 12 | 14 | 0 |

min time=A🡪I,B🡪III,C🡪II,D🡪IV



=8+4+19+10=41 man hours

1. A car hire company has one car at each of the five depots a, b, c, d & e. A customer requires a car in each town, namely A, B, C, D & E. Distance (in kms) between depots(origins) & towns(destinations) are given in following distance matrix.



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | a | b | c | d | e |
| A | 160 | 130 | 175 | 190 | 200 |
| B | 135 | 120 | 130 | 160 | 175 |
| C | 140 | 110 | 155 | 170 | 185 |
| D | 50 | 50 | 80 | 80 | 110 |
| E | 55 | 35 | 70 | 80 | 105 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | a | b | c | d | e |
| A | 30 | 0 | 45 | 60 | 70 |
| B | 15 | 0 | 10 | 40 | 55 |
| C | 30 | 0 | 45 | 60 | 75 |
| D | 0 | 0 | 30 | 30 | 60 |
| E | 20 | 0 | 35 | 45 | 70 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | A | b | c | d | e |
| A | 30 | 0 | 35 | 30 | 15 |
| B | 15 | 0 | 0 | 10 | 0 |
| C | 30 | 0 | 35 | 30 | 20 |
| D | 0 | 0 | 20 | 0 | 5 |
| E | 20 | 0 | 25 | 15 | 15 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | A | b | c | d | E |
| A | 15 | 0 | 20 | 15 | 0 |
| B | 15 | 15 | 0 | 10 | 0 |
| C | 15 | 0 | 20 | 15 | 5 |
| D | 0 | 15 | 20 | 0 | 5 |
| E | 5 | 0 | 10 | 0 | 0 |

A🡪e B🡪c C🡪b D🡪a E🡪d



200+130+110+50+80=570kms

1. Solve the following assignment problem

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Man🡪  Work | 1 | 2 | 3 | 4 |
| I | 12 | 30 | 21 | 15 |
| II | 18 | 33 | 9 | 31 |
| III | 44 | 25 | 24 | 21 |
| IV | 23 | 30 | 28 | 14 |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Man🡪  Work | 1 | 2 | 3 | 4 |
| I | 0 | 18 | 9 | 3 |
| II | 9 | 24 | 0 | 22 |
| III | 23 | 4 | 3 | 0 |
| IV | 9 | 16 | 14 | 0 |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Man🡪  Work | 1 | 2 | 3 | 4 |
| I | 0 | 14 | 9 | 3 |
| II | 9 | 20 | 0 | 22 |
| III | 23 | 0 | 3 | 0 |
| IV | 9 | 12 | 14 | 0 |

I🡪1, II🡪3,III🡪2,IV🡪4

Total cost=12+9+25+14=60



1. Solve the following assignment problem

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 11 | 10 | 18 | 5 | 9 |
| B | 14 | 13 | 12 | 19 | 6 |
| C | 5 | 3 | 4 | 2 | 4 |
| D | 15 | 18 | 17 | 9 | 12 |
| E | 10 | 11 | 19 | 6 | 14 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 6 | 5 | 13 | 0 | 4 |
| B | 8 | 7 | 6 | 13 | 0 |
| C | 3 | 1 | 2 | 0 | 2 |
| D | 6 | 9 | 8 | 0 | 3 |
| E | 4 | 5 | 13 | 0 | 8 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 3 | 4 | 11 | 0 | 4 |
| B | 5 | 6 | 4 | 13 | 0 |
| C | 0 | 0 | 0 | 0 | 2 |
| D | 3 | 8 | 6 | 0 | 3 |
| E | 1 | 4 | 11 | 0 | 8 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 2 | 3 | 10 | 0 | 3 |
| B | 5 | 6 | 4 | 14 | 0 |
| C | 0 | 0 | 0 | 1 | 2 |
| D | 2 | 7 | 5 | 0 | 2 |
| E | 0 | 3 | 10 | 0 | 7 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 0 | 1 | 8 | 0 | 1 |
| B | 5 | 6 | 4 | 16 | 0 |
| C | 0 | 0 | 0 | 3 | 2 |
| D | 0 | 5 | 3 | 0 | 0 |
| E | 0 | 3 | 10 | 2 | 7 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 0 | 0 | 7 | 0 | 1 |
| B | 5 | 5 | 3 | 16 | 0 |
| C | 1 | 0 | 0 | 4 | 3 |
| D | 0 | 4 | 2 | 0 | 0 |
| E | 0 | 2 | 9 | 2 | 7 |



A🡪II, B🡪V, C🡪III, D🡪IV, E🡪I

10+6+4+9+10=39

* ALTERNATIVE SOLUTION TO ASSIGNMENT PROBLEM

1. Solve the following assignment problem

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 45 | 30 | 65 | 40 | 55 |
| B | 50 | 30 | 25 | 60 | 30 |
| C | 25 | 20 | 15 | 20 | 40 |
| D | 35 | 25 | 30 | 30 | 20 |
| E | 80 | 60 | 60 | 70 | 50 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 15 | 0 | 35 | 10 | 25 |
| B | 25 | 5 | 0 | 35 | 5 |
| C | 10 | 5 | 0 | 5 | 25 |
| D | 15 | 5 | 10 | 10 | 0 |
| E | 30 | 10 | 10 | 20 | 0 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 5 | 0 | 35 | 5 | 25 |
| B | 15 | 5 | 0 | 30 | 5 |
| C | 0 | 5 | 0 | 0 | 25 |
| D | 5 | 5 | 10 | 5 | 0 |
| E | 20 | 10 | 10 | 15 | 0 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 5 | 0 | 35 | 5 | 30 |
| B | 15 | 5 | 0 | 30 | 10 |
| C | 0 | 5 | 0 | 0 | 30 |
| D | 0 | 0 | 5 | 0 | 0 |
| E | 15 | 5 | 5 | 10 | 0 |

A🡪II B🡪III C🡪I D🡪IV E🡪V 30+25+25+30+50=160

A🡪II B🡪III C🡪IV D🡪I E🡪V 30+25+20+35+50=160

1. Solve the following assignment problem

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 2 | 9 | 2 | 7 | 1 |
| B | 6 | 8 | 7 | 6 | 1 |
| C | 4 | 6 | 5 | 3 | 1 |
| D | 4 | 2 | 7 | 3 | 1 |
| E | 5 | 3 | 9 | 5 | 1 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 1 | 8 | 1 | 6 | 0 |
| B | 5 | 7 | 6 | 5 | 0 |
| C | 3 | 5 | 4 | 2 | 0 |
| D | 3 | 1 | 6 | 2 | 0 |
| E | 4 | 2 | 8 | 4 | 0 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 0 | 7 | 0 | 4 | 0 |
| B | 4 | 6 | 5 | 3 | 0 |
| C | 2 | 4 | 3 | 0 | 0 |
| D | 2 | 0 | 5 | 0 | 0 |
| E | 3 | 1 | 7 | 2 | 0 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 0 | 7 | 0 | 4 | 1 |
| B | 3 | 5 | 4 | 2 | 0 |
| C | 2 | 4 | 3 | 0 | 1 |
| D | 2 | 0 | 5 | 0 | 1 |
| E | 2 | 0 | 6 | 1 | 0 |



|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs 🡪  Machine | I | II | III | IV | V |
| A | 0 | 9 | 0 | 6 | 3 |
| B | 1 | 5 | 2 | 2 | 0 |
| C | 0 | 4 | 1 | 0 | 1 |
| D | 0 | 0 | 3 | 0 | 1 |
| E | 0 | 0 | 4 | 1 | 0 |



A🡪III B🡪V C🡪I D🡪IV E🡪II

2+1+4+3+3=13

* Unbalanced Assignment Problem

If the cost of matrix of an assignment problem is not a square matrix (number of sources are not equal to number of destinations), the assignment problem is called as Unbalanced Assignment problem.

In such cases, dummy row or column with zero costs are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem.

* 1. A method engineer wants to assign four new methods to three work centres. The assignment of the new methods will increase production and they are given below. If only one method can be assigned to a work centre, determine the optimal assignment

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | Increase in production(unit) | | |
| METHODS |  | A | B | C |
| 1 | 10 | 7 | 8 |
| 2 | 8 | 9 | 7 |
| 3 | 7 | 12 | 6 |
| 4 | 10 | 10 | 8 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Increase in production(unit) | | |  |
| METHODS |  | A | B | C | Dummy |
| 1 | 10 | 7 | 8 | 0 |
| 2 | 8 | 9 | 7 | 0 |
| 3 | 7 | 12 | 6 | 0 |
| 4 | 10 | 10 | 8 | 0 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Increase in production(unit) | | |  |
| METHODS |  | A | B | C | Dummy |
| 1 | 2 | 5 | 4 | 0 |
| 2 | 4 | 3 | 5 | 0 |
| 3 | 5 | 0 | 6 | 0 |
| 4 | 2 | 2 | 4 | 0 |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Increase in production(unit) | | |  |
| METHODS |  | A | B | C | Dummy |
| 1 | 0 | 5 | 0 | 0 |
| 2 | 2 | 3 | 1 | 0 |
| 3 | 3 | 0 | 2 | 0 |
| 4 | 0 | 2 | 0 | 0 |

1🡪A 2🡪Dummy 3🡪B 4🡪C 10+0+12+8=30units



* 1. A city corporation has decided to carry out road repairs on main four arteries of the city. The government has agreed to make a special grant of Rs 50 lakhs towards the cost with a condition that the repairs must be done at lowest cost and quickest time. If conditions warrant, then a supplementary token grant will also be considered favourably. The corporation has floated tenders and 5 contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | Cost of Repair (Rs Lakhs) | | | |
| Contractors/Roads | Road🡪 | R1 | R2 | R3 | R4 |
| C1 | 9 | 14 | 19 | 15 |
| C2 | 7 | 17 | 20 | 19 |
| C3 | 9 | 18 | 21 | 18 |
| C4 | 10 | 12 | 18 | 19 |
| C5 | 10 | 15 | 21 | 16 |

* 1. Find the best way of assigning the repair work to the contractor and the cost
  2. If it is necessary to seek supplementary grants then what should be amount sought?
  3. Which out of the five contractors will be unsuccessful in his bid?

**SOLUTION OF A SEQUENCING PROBLEM**

**15.1  Introduction**

When a number of jobs are given to be done and they require processing on two or more machines, the main concern of a manager is to find the order or sequence to perform these jobs. We shall consider the sequencing problems in respect of the jobs to be performed in a factory and study the method of their solution. Such sequencing problems can be broadly divided in two groups. In the first one, there are n jobs to be done, each of which requires processing on some or all of the k different machines. We can determine the effectiveness of each of the sequences that the technologically feasible (that is to say, those satisfying the restrictions on the order in which each job must be processed through the machines) and choose a sequence which optimizes the effectiveness. To illustrate, the timings of processing of each of the n jobs on each of the **k** machines, in a certain given order, may be given and the time for performing the jobs may be the measure of effectiveness. We shall select the sequences for which the total time taken in processing all the jobs on the machines would be the minimum.

In this unit we will look into solution of a sequencing problem. In this lesson the solutions of following cases will be discussed:

a)       n jobs and two machines A and B, all jobs processed in the order AB.

b)       n jobs and three machines A, B and C all jobs processed in the order ABC

c)       Problems with n jobs and m machines.

**15.1.1  Processing of n jobs through two machines**

The simplest possible sequencing problem is that of n job two machine sequencing problem in which we want to determine the sequence in which **n**-job should be processed through two machines so as to minimize the total elapsed time T. The problem can be described as:

a)   Only two machines A and B are involved;

b)   Each job is processed in the order AB.

c)   The exact or expected processing times A1,A2,A3, --- , An ; B1,B2,B3, --- , Bn are known and are provided in the following table

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Machine | Job(s) | | | | | | | | |
| 1 | 2 | 3 | -- | - | I | - - | - | n |
| A | A1 | A2 | A3 | -- | - | Ai | -- | - | An |
| B | B1 | B2 | B3 | -- | - | Bi | -- | - | Bn |

The problem is to find the sequence (or order) of jobs so as to minimize the total elapsed time T. The solution of the above problem is also known as Johnsons procedure which involves the following steps:

Step 1.  Select the smallest processing time occurring in the list A1,A2,A3, --- , An ; B1,B2,B3, --- , Bn if there is a tie, either of the smallest processing times can be selected.

Step 2.If the least processing time is Ar, select the rth job first. If it is Bs, do the sth job last  as the given order is AB

Step 3.There are now (n-1) jobs left to be ordered. Repeat steps I and II for the remaining set of processing times obtained by deleting the processing time for both the machines corresponding to the job already assigned.

Step 4.Continue in the same manner till the entire jobs have been ordered. The resulting ordering will minimize the total elapsed time T and is called the optimal sequence.

Step 5.After finding the optimal sequence as stated above find the total elapsed time and idle times on machines A and B as under:

|  |  |
| --- | --- |
| Total elapsed time = | The time between starting the first job in the optimal sequence on machine A and completing the last job in the optimal machine B. |
| Idle time on machine A = | (Time when the last job in the optimal sequence on sequences is completed on machine B)- (Time when the last job in the optimal sequences is completed on machine A) |
| Idle time on machine B = | (Time when the first job in the optimal sequences is completed on machine A)+ |

The Johnsons procedure can be illustrated by following examples:

**Sequencing Problem is a problem-solving technique used to find proper sequence or order of a particular project or jobs which help business organisation to have proper control over their production.**

**Objective:**

* **To select proper sequencing**
* **Which reduces the total time to complete the jobs.**
* **And which also reduces ideal time between the process of completing the jobs**

**Conditions:**

* **The time for each job must be constant(fixed)**
* **Job sequence does not have an impact on job times.**
* **All jobs must go through first work centre before go through the second work centre.**
* **Their must be no job priorities.**

**Example 1**

**Solve the following sequencing problem using Johnson’s Algorithm method and find out**

1. **Total run time/ elapse time**
2. **Total ideal time**

|  |  |  |
| --- | --- | --- |
| **Jobs** | **Machine 1**  **(time in hrs)** | **Machine 2**  **(time in hrs)** |
| **A** | **4** | **2** |
| **B** | **3** | **9** |
| **C** | **5** | **1** |
| **D** | **7** | **3** |
| **E** | **8** | **3** |

|  |  |  |
| --- | --- | --- |
| **Jobs** | **Machine 1**  **(time in hrs)** | **Machine 2**  **(time in hrs)** |
| **A** | **4** | **2** |
| **B** | **3** | **9** |
| **C** | **5** | **1** |
| **D** | **7** | **3** |
| **E** | **8** | **3** |



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **B** | **E** | **D** | **A** | **C** |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Jobs** | **Machine 1**  **(time in hrs)** | | **Machine 2**  **(time in hrs)** | |
|  | **IN TIME** | **OUT TIME** | **IN TIME** | **OUT TIME** |
| **B** | **0** | **3** | **3** | **12** |
| **E** | **3** | **11** | **12** | **15** |
| **D** | **11** | **18** | **18** | **21** |
| **A** | **18** | **22** | **22** | **24** |
| **C** | **22** | **27** | **27** | **28** |



1. **Total run time/ elapse time= 28 HRS**
2. **Total ideal time MACHINE 1=28-27=1 HRS**
3. **Machine2=3+(18-15)+(22-21)+(27-24)=3+3+1+3=10hrs**

**Example 2**

There are nine jobs, each of which must go through two machines P and Q in the order PQ, the processing times (in hours) are given below:

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Machine | Job(s) | | | | | | | | |
| A | B | C | D | E | F | G | H | I |
| P | 2 | 5 | 4 | 9 | 6 | 8 | 7 | 5 | 4 |
| Q | 6 | 8 | 7 | 4 | 3 | 9 | 3 | 8 | 11 |

Find the sequence that minimizes the total elapsed time T. Also calculate the total idle time for the machines in this period.

**Solution**



|  |  |  |
| --- | --- | --- |
|  | **P** | **Q** |
| **A** | **2** | **6** |
| **B** | **5** | **8** |
| **C** | **4** | **7** |
| **D** | **9** | **4** |
| **E** | **6** | **3** |
| **F** | **8** | **9** |
| **G** | **7** | **3** |
| **H** | **5** | **8** |
| **I** | **4** | **11** |

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **A** | **C** | **I** | **B** | **H** | **F** | **D** | **G** | **E** |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **MACHINE P** | | **MACHINE Q** | |
|  | **IN** | **OUT** | **IN** | **OUT** |
| **A** | **0** | **2** | **2** | **8** |
| **C** | **2** | **6** | **8** | **15** |
| **I** | **6** | **10** | **15** | **26** |
| **B** | **10** | **15** | **26** | **34** |
| **H** | **15** | **20** | **34** | **42** |
| **F** | **20** | **28** | **42** | **51** |
| **D** | **28** | **37** | **51** | **55** |
| **G** | **37** | **44** | **55** | **58** |
| **E** | **44** | **50** | **58** | **61** |

**Total run time=61 hrs**

**Machine p idle time=61-50=11hrs**

**Machine q idle time=2**

**Total elapsed time of machine P=50hrs**

**Machine Q=61-2=59hrs**

**15.2 Processing of n Jobs through Three Machines**

The type of sequencing problem can be described as follows:

a)     Only three machines A, B and C are involved;

b)     Each job is processed in the prescribed order ABC

c)      No passing of jobs is permitted i.e. the same order over each machine is maintained.

d)     The exact or expected processing times A1,A2,A3, --- , An ; B1,B2,B3, --- , Bn and C1,C2,C3, --- , Cn are known and are denoted by the following table

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Machine | Job(s) | | | | | | | | |
| 1 | 2 | 3 | -- | - | i | - - | - | N |
| A | A1 | A2 | A3 | -- | - | Ai | -- | - | An |
| B | B1 | B2 | B3 | -- | - | Bi | -- | - | Bn |
| C | C1 | C2 | C3 |  |  | Ci |  |  | Cn |

Our objective will be to find the optimal sequence of jobs which minimizes the total elapsed time. No general procedure is available so far for obtaining an optimal sequence in such case. However, the Johnson�s procedure can be extended to cover the special cases where either one or both of the following conditions hold:

 ����a)      The minimum processing time on machine A ≥ the maximum processing time on machine B.

b)      The minimum processing time on machine C ≥ the maximum processing time on machine B.

The method is to replace the problem by an equivalent problem involving n jobs and two machines. These two fictitious machines are denoted by G and H and the corresponding time Gi and Hi are defined by

�����������Gi = Ai + Bi� and Bi + Ci

Now this problem with prescribed ordering GH is solved by the method with n jobs through two machines, the resulting sequence will also be optimal for the original problem. The above methodology is illustrated by following example:

**Conditions for 3 machine sequencing**

1. **Min of A 3>= Max of B 5 OR**
2. **Min of C 5>= Max of B 5**

**Note: If any one condition is ful-filled we can use this method.**

1. **You are given the following regarding the processing time of some jobs of three machines A, B & C. The order of processing is A, B, and C. Determine the sequence that minimize the total elapsed time required to complete the jobs, also evaluate time of A, B and C as an idle time.**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Machine (in hours)** | | |
| **Jobs** | **A** | **B** | **C** |
| **I** | **3** | **4** | **6** |
| **II** | **8** | **3** | **7** |
| **III** | **7** | **2** | **5** |
| **IV** | **4** | **5** | **11** |
| **V** | **9** | **1** | **5** |
| **VI** | **8** | **4** | **6** |
| **VII** | **7** | **3** | **12** |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **Machine (in hours)** | | | **I** | **II** |
| **Jobs** | **A** | **B** | **C** | **A+B** | **B+C** |
| **I** | **3** | **4** | **6** | **7** | **10** |
| **II** | **8** | **3** | **7** | **11** | **10** |
| **III** | **7** | **2** | **5** | **9** | **7** |
| **IV** | **4** | **5** | **11** | **9** | **16** |
| **V** | **9** | **1** | **5** | **10** | **6** |
| **VI** | **8** | **4** | **6** | **12** | **10** |
| **VII** | **7** | **3** | **12** | **10** | **15** |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **I** | **IV** | **VII** | **VI** | **II** | **III** | **V** |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Jobs** | **A IN** | **AOUT** | **B IN** | **B OUT** | **C IN** | **C OUT** |
| **I** | **0** | **3** | **3** | **7** | **7** | **13** |
| **IV** | **3** | **7** | **7** | **12** | **13** | **24** |
| **VII** | **7** | **14** | **14** | **17** | **24** | **36** |
| **VI** | **14** | **22** | **22** | **26** | **36** | **42** |
| **II** | **22** | **30** | **30** | **33** | **42** | **49** |
| **III** | **30** | **37** | **37** | **39** | **49** | **54** |
| **V** | **37** | **46** | **46** | **47** | **54** | **59** |

Total elapsed time=59hrs

Total elapsed time for A=46

B=44

C=52

Idle timeA=59-46=13hrs

B=3+(14-12)+(22-17)+(30-26)+(37-33)+(46-39)+(59-47)

=3+2+5+4+4+7+12=37hrs

C=7hrs

**Example 2**

There are five jobs (namely 1,2,3,4 and 5), each of which must go through machines A, B and C in the order ABC.  Processing Time (in hours) are given below:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs | 1 | 2 | 3 | 4 | 5 |
| Machine A | 5 | 7 | 6 | 9 | 5 |
| Machine B | 2 | 1 | 4 | 5 | 3 |
| Machine C | 3 | 7 | 5 | 6 | 7 |

Find the sequence that minimum the total elapsed time required to complete the jobs.

1. Find the sequence that minimizes the total time required in performing the following jobs on three machines in order ABC. Processing time(in hours) are given in following table

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jobs | 1 | 2 | 3 | 4 | 5 |
| Machine A | 8 | 10 | 6 | 7 | 11 |
| Machine B | 5 | 6 | 2 | 3 | 4 |
| Machine C | 4 | 9 | 8 | 6 | 5 |

**Solution**

**15.3 Problems with n Jobs and m Machines**

Let there be n jobs, each of which is to be processed through m machines, say M1,M2, --- , Mm in the order M1,M2,M3, --- , Mm. Let T ij be the time taken by the ithmachine to complete the jth job.

The iterative procedure of obtaining an optimal sequence is as follows:

**Step I:** Find (i) minj(T1j)� ii) minj (Tmj)  iii) maxj (T2j,T3j,T4j, --- , T(m-1)j)for j=1,2,---, n

**Step II:** Check whether

a.  minj(T1j)  ≥maxj (Tij) for i=2,3,----,m-1

                                    Or

b.   minj(Tmj)  ≥ maxj (Tij) for i=2,3,---,m-1

**Step III:** If the inequalities in Step II are not satisfied, method fails, otherwise, go to next step.

**Step IV:** Convert the m machine problem into two machine problem by introducing two fictitious machines G and H, such that

**�����������������**TGj **=** T1j + T2j + --- +T(m-1)j� and�THj **=** T2j + T3j + --- +Tmj

Determine the optimal sequence of n jobs through 2 machines by using optimal sequence algorithm.

**Step V:** In addition to condition given in Step IV, if�Tij **=** T2j + T3j + --- +Tmj = C is a fixed positive constant for all i = 1, 2, 3, � , n then determine the optimal sequence of n jobs and two machines M1 and Mm in the order M1Mm by using the optimal sequence algorithm.

**Example 1**

**Four jobs 1,2,3 and 4 are to be processed on each of the 5 machines A, B, C, D and E in**

**the order ABCDE. Find the total minimum elapsed time. Also find the idle time for each**

**machine.**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Job🡪**  **Machine** | **1** | **2** | **3** | **4** |
| **A** | **7** | **6** | **5** | **8** |
| **B** | **5** | **6** | **4** | **3** |
| **C** | **2** | **4** | **5** | **3** |
| **D** | **3** | **5** | **6** | **2** |
| **E** | **9** | **10** | **8** | **6** |

**Min(Ai) >=max(Bi,Ci,Di)**

**5>=6**

**Or**

**min(Ei)>=max(Bi,Ci,Di )**

**6 = 6**

**Sequence**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Job** | **1** | **2** | **3** | **4** |
| **G(A+B+C+D)** | **17** | **21** | **20** | **16** |
| **H(B+C+D+E)** | **19** | **25** | **23** | **14** |

|  |  |  |  |
| --- | --- | --- | --- |
| **1** | **3** | **2** | **4** |

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **A** | | **B** | | **C** | | **D** | | **E** | |
|  | **In** | **Out** | **In** | **Out** | **In** | **Out** | **In** | **Out** | **In** | **Out** |
| **1** | **0** | **7** | **7** | **12** | **12** | **14** | **14** | **17** | **17** | **26** |
| **3** | **7** | **12** | **12** | **16** | **16** | **21** | **21** | **27** | **27** | **35** |
| **2** | **12** | **18** | **18** | **24** | **24** | **28** | **28** | **33** | **35** | **45** |
| **4** | **18** | **26** | **26** | **29** | **29** | **32** | **33** | **35** | **45** | **51** |

**Total elapse time =51 hrs**

**Idle time for A=51-26=25hrs**

**Idle time for B=7+2+2+22=33hrs**

**Idle time for C=37**

**Idle time for D=14+4+1+16=35**

**Idle time for E=17+1=18**

**INTEGER PROGRAMMING**

**THE BRANCH AND BOUND METHOD**

The general idea of the method is to solve the problem first as a continuous linear

programming problem and then the original problem is partitioned (branched) into two

subproblems by imposing the integer conditions on one of its integer variables that currently

has a fractional optimal value.

At the rth iteration we have available a lower bound (say zr) for the optimal value of the

objective function. For convenience we suppose that at the first iteration , z1 is either strictly

less than optimal value or equals the value of the objective function for a feasible solution

that we have noted. In case, if we have no information about the problem we let z1= -∞. In

addition to lower bound z1 we also have a master list of linear programming problems to be

solved differing only in the revisions of the bounds(10.10). At the first iteration, the master

list has only one problem consisting of (10.6), (10.7), (10.8) and (10.10)

The step by step procedure at this rth ( r=0,1,2,…..) iteration can be outlined as follows:

STEP 1:

Two possibilities may arise at rth iteration

1. If the master list does not contain any linear programming problem (i.e. empty), stop the computations.
2. Otherwise, go to step2 for removing a linear programming problem from the master list.

STEP 2:

Solve the chosen problem to obtain the optimal solution by using bounded variable technique. Again two possibilities may arise:

1. If it has no feasible solution or if the resulting optimal value of the objective function z is <= zr, then let zr+1 = zrand return back to step 1.
2. Otherwise go to step 3.

Step 3:

1. If the optimal solution to the linear programming problem thus obtain satisfies the integer condition, then record it , let zr+1  be associated optimal value of the objective function and return back to step 1.
2. Otherwise go to step 4.

Step 4:

Select any variable xj, for j=1,2,……k that does not have an integer value in the obtained optimal solution to the chosen linear programming problem. Let xj\* denote this value, and

[xj\*] stand for largest integer less than or equal to xj\*. Now include two linear programming problems in the master list. These two sub problems are :

**Sub-problem 1:**

Same as the problem chosen in step 1, except that the lower bound Lj on xj is replaced by [xj\*] + 1.

**Sub-problem 1:**

Same as the problem chosen in step 1, except that the upper bound Uj on xj is replaced by

[xj\*].

Let zr+1  =zr , and return back to step 1. At the termination of the process if we find a integer valued feasible solution giving zr, it will be optimal otherwise no integer valued feasible solution exists.

**GOMORY’S CUTTING PLANE METHOD FOR INTEGER PRAGRAMMING**

In this technique, we first find the optimum solution of the given IPP by regular simplex method, disregarding the integer condition of variables. Then we observe the following:

1. If all the variables in the optimum solution thus obtained have integer values, the current solution will be the desired optimum integer solution.
2. If not, the considered LPP requires modification by introducing secondary constraint ( also called as Gomory’s constraint) that reduces some of the non integer values of variables to integer one, but does not eliminate any feasible integer.
3. Then an optimum solution to this modified LPP is obtained by using any standard algorithm. If all the variables in this solution are integers then the optimum integer solution is obtained. Otherwise another secondary constraint is added to the LPP and entire procedure is repeated.

In this way, the optimal integer solution will be obtained eventually after introducing the sufficient number of new constraints.

**Gomory’s Cutting Plane Algorithm:**

The step by step procedure for the solution of all integer programming problem is as follows:

Step 1:

If the IPP is in minimization form, convert it into maximization form.

Step 2:

Then convert inequalities into equations by introducing slack and/or surplus variables( if necessary) and obtain the optimum solution of the LPP (after ignoring the integer conditions) by using simplex method.

Step 3:

Now test integrability of the optimum solution which is obtained in step2.

1. If the optimal solution contains all integer value, then an optimum integer basic feasible solution has been achieved.
2. If not go to next step

Step 4:

Examine the constraint equation corresponding to current optimal solution. Let this constraint be expressed by XBi = xi + xijxj ( i=1,2,….m)

Select the largest fraction of XBi i.e. find maxi[fBi] . Let it be fBk for i=k

Step 5:

Express the negetive fraction, if any in the kth row of the optimum simplex table, as the sum of a negative integer and a non negative fraction.

Step 6:

At this stage, construct the Gomorian constraint:

fBi - xijxj fijxj<=0

As described in the preceding section, and then introduce Gomorian equation

fBi - j fijxj + gi

to the current set of equation constrint.

Step 7:

Starting with this new set of constraint equations obtain the new optimum solution by using dual simplex method in order to clear infeasibility. The slack variable giwill be the initial leaving basic variable.

Step 8:

Now two possibilities may arise:

1. If this new optimum solution for the modified LPP is an all integer solution, it is also feasible and optimum for given LPP.
2. Otherwise we return to Step 4 and repeat the entire process until an optimum feasible integer solution is obtained.