Largest Permutation Codes With the Kendall τ-Metric in S_5 and S_6

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***Abstract*—An important problem in the theory of permutation codes is finding the value of $P(n,d)$, the size of the largest subset of the set of all permutations S_n with minimum Kendall τ-distance d. Using an integer programming approach, we find the values of $P(5,d)$ for $d \geq 3$ and $P(6,d)$ for $d \geq 4$. We give instances of codes which achieve these values. We also show that $P(6,3) \geq 102$ by giving a code of cardinality 102 in S_6, which has minimum Kendall τ-distance 3.**

***Index Terms*—Rank modulation, flash memory, Kendall τ-metric, permutation codes.**

I. INTRODUCTION

RANK modulation was proposed as a solution to the challenges posed by flash memory [1]. In the rank modulation framework, codes are subsets of S_n, the set of all permutations of n elements. The Kendall τ-distance between permutations σ and π in S_n is defined as the minimum number of adjacent transpositions[1] needed to change σ into π. A subset of S_n in which any two distinct permutations are at a Kendall τ-distance of at least d can correct upto $\lfloor \frac{d-1}{2} \rfloor$ errors caused by charge-constrained errors [2]. Several researchers have presented bounds on $P(n,d)$, the size of the largest code in S_n with minimum Kendall τ-distance d [3]–[5]. The best known upper and lower bounds on $P(n,d)$ for $n \in \{5,6,7\}$ are presented in [5, Table II]. For convenience, we reproduce the bounds corresponding to $n=5$ and $n=6$ in Table I.

Previous work which uses optimization techniques to bound the size of permutation codes under various distance metrics (Hamming, Kendall, Ulam) can be found in [6]–[8]. These papers formulate optimization problems where the optimal value of the objective function only gives an upper bound on the maximum size of a permutation code. In this letter, we formulate the calculation of $P(n,d)$ as a binary integer program [9] with linear/quadratic constraints and find its value for $n \in \{5,6\}$. While we were able to find $P(5,d)$ for all $d \geq 3$, we could find $P(6,d)$ only for $d \geq 4$. These values are listed in Table II. In the appendix, we present specific codes which achieve $P(n,d)$ for each case except $P(6,3)$. We present a code of cardinality 102 in S_6 having minimum Kendall τ-distance 3 improving the lower bound on $P(6,3)$ from the previous value of 90 reported in [5]. For $n \geq 7$, the binary integer programs we formulate turn out to be difficult to solve using a standard solver [10] on a desktop computer. Specialized solution techniques for binary integer programs will be needed to find $P(n,d)$ for large n using our formulation.

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[1]Given a permutation $\sigma = [\sigma(1), \ldots, \sigma(i-1), \sigma(i), \sigma(i+1), \sigma(i+2) \ldots, \sigma(n)] \in S_n$, an adjacent transposition τ_i applied to σ will result in the permutation $[\sigma(1), \ldots, \sigma(i-1), \sigma(i+1), \sigma(i), \sigma(i+2) \ldots, \sigma(n)]$ for $1 \leq i \leq n-1$.

TABLE I

UPPER AND LOWER BOUNDS ON $P(5,d)$ AND $P(6,d)$ FROM [5]

n \ d	3	4	5	6	7	8	9
5	≥ 20 ≤ 23	≥ 10 ≤ 15	≥ 6 ≤ 8	≥ 4 ≤ 6			
6	≥ 90 ≤ 119	≥ 45 ≤ 72	≥ 23 ≤ 36	≥ 12 ≤ 24	≥ 10 ≤ 14	≥ 5 ≤ 10	≥ 4 ≤ 7

TABLE II

VALUES OF OR LOWER BOUNDS ON $P(5,d)$ AND $P(6,d)$

n \ d	3	4	5	6	7	8	9	10
5	20	12	6	5				
6	≥ 102	64	26	20	11	7	4	4

II. PROBLEM FORMULATION

For permutations σ and π in S_n, let $d_K(\sigma,\pi)$ denote the Kendall τ-distance between them. For $\sigma \in S_n$ and $r \in \mathbb{N}$, the ball of radius r around σ is given by

$$B(\sigma,r) = \left\{ \pi \in S_n \;\middle|\; d_K(\sigma,\pi) \leq r \right\}. \tag{1}$$

Let $\sigma_1, \sigma_2, \ldots, \sigma_{n!}$ be a fixed ordering of the elements of S_n. Given a subset C of S_n, let $x = (x_1, x_2, \ldots, x_{n!})$ denote the indicator vector of C where

$$x_i = \begin{cases} 1 & \text{if } \sigma_i \in C, \\ 0 & \text{if } \sigma_i \notin C. \end{cases} \tag{2}$$

The cardinality of C is given by $|C| = \sum_{i=1}^{n!} x_i$.

For a subset C of S_n containing at least two elements, let $d_{\min}(C)$ denote the minimum Kendall τ-distance between distinct elements in C, i.e.

$$d_{\min}(C) = \min \left\{ d_K(\sigma,\pi) \;\middle|\; \sigma,\pi \in C, \sigma \neq \pi \right\}. \tag{3}$$

The maximum Kendall τ-distance between permutations in S_n is $\binom{n}{2}$. For $d \leq \binom{n}{2}$, the cardinality of the largest subset of S_n with minimum Kendall τ-distance d is given by

$$P(n,d) = \max \left\{ |C| \;\middle|\; C \subseteq S_n, |C| \geq 2, d_{\min}(C) \geq d \right\}. \tag{4}$$

The restriction on C to contain at least two elements ensures that $d_{\min}(C)$ is well defined.

Our approach to finding $P(n,d)$ is through binary integer programming [9]. We present two different formulations; the first one is applicable only when d is odd while the second is applicable irrespective of whether d is even or odd. For odd d, solving an integer program with linear constraints is sufficient to find $P(n,d)$. But for even d, we need to solve an integer program with quadratic constraints. While the second

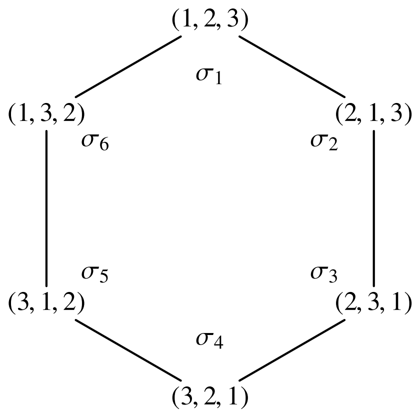

Fig. 1. A graph whose vertex set is S_3 with edges between permutations at Kendall τ-distance of one from each other.

formulation can be used to find $P(n, d)$ for odd d as well, we chose to include the first formulation on account of its simplicity and to illuminate the possibility of using relaxations or heuristics for linearly constrained binary integer programs to find $P(n, d)$ for $n \geq 7$ and odd d.

A. Odd d

Let $d = 2t + 1$ for some $t \in \mathbb{N}$. Suppose a set $C \subseteq S_n$ satisfies $d_{\min}(C) \geq 2t + 1$. Then a ball of radius t around any $\sigma \in S_n$ will contain at most one element of C, i.e.

$$|B(\sigma, t) \cap C| \leq 1 \text{ for all } \sigma \in S_n. \tag{5}$$

This is because the maximum Kendall τ-distance between elements in $B(\sigma, t)$ is $2t$ and the presence of two distinct elements from C in $B(\sigma, t)$ will violate the condition $d_{\min}(C) \geq 2t + 1$.

The converse is also true in the sense that if the condition in (5) holds for some $C \subseteq S_n$ which satisfies $|C| \geq 2$, then $d_{\min}(C) \geq 2t + 1$. To see this, suppose $d_{\min}(C) \leq 2t$. Then there exist distinct permutations $\sigma, \pi \in C$ such that $d_K(\sigma, \pi) \leq 2t$. So σ can be changed into π by applying $d_K(\sigma, \pi)$ adjacent transpositions. Consider the permutation $\gamma \in S_n$ which is obtained by applying the first $\lfloor d_K(\sigma, \pi)/2 \rfloor$ of these adjacent transpositions to σ. Applying the remaining $\lceil d_K(\sigma, \pi)/2 \rceil$ adjacent transpositions to γ will change it to π. As $d_K(\sigma, \pi) \leq 2t$, both $\lfloor d_K(\sigma, \pi)/2 \rfloor$ and $\lceil d_K(\sigma, \pi)/2 \rceil$ are less than or equal to t. Thus both σ and π belong to $B(\gamma, t)$. This contradicts the condition in equation (5).

The above discussion has given us the following set equality.

$$\left\{ C \subseteq S_n \;\middle|\; |C| \geq 2, d_{\min}(C) \geq 2t + 1 \right\} = \left\{ C \subseteq S_n \;\middle|\; |C| \geq 2, |B(\sigma, t) \cap C| \leq 1 \quad \text{for all } \sigma \in S_n \right\}. \tag{6}$$

Let $I(\sigma, r)$ be the set of indices of the permutations in $B(\sigma, r)$, i.e.

$$I(\sigma, r) = \{ i \mid \sigma_i \in B(\sigma, r) \}. \tag{7}$$

Then the condition in (5) can be expressed in terms of the indicator variables as

$$\sum_{i \in I(\sigma, t)} x_i \leq 1 \text{ for all } \sigma \in S_n. \tag{8}$$

From (4), (6), and (8), we can see that $P(n, d)$ is the solution to the following binary integer program (BIP) [9] where $d = 2t + 1$.

$$\begin{aligned} &\text{maximize} \sum_{i=1}^{n!} x_i \\ &\text{subject to} \sum_{i \in I(\sigma, t)} x_i \leq 1 \quad \text{for all } \sigma \in S_n, \\ &\qquad\qquad x_i \in \{0, 1\} \quad \text{for } i = 1, 2, \ldots, n!. \end{aligned} \tag{9}$$

Note that the constraint $|C| = \sum_{i=1}^{n!} x_i \geq 2$ does not appear explicitly in the above BIP. This is because when $d \leq \binom{n}{2}$, the set $\{(1, 2, \ldots, n), (n, n-1, \ldots, 1)\}$ consisting of the identity permutation and its reverse satisfies the constraints of the above BIP[2]. This ensures that $\sum_{i=1}^{n!} x_i \geq 2$.

Example 1: In this example, we consider the BIP for $n = d = 3$. The ordering of the permutations in S_3 is fixed as shown in Fig. 1. The ball of radius one around a permutation σ consists of σ itself and the two permutations adjacent to σ in the graph. For instance, $B(\sigma_1, 1) = \{\sigma_1, \sigma_2, \sigma_6\} \implies I(\sigma_1, 1) = \{1, 2, 6\}$. The BIP in (9) specializes to the following.

$$\begin{aligned} &\text{maximize } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ &\text{subject to } x_1 + x_2 + x_6 \leq 1, \quad x_2 + x_1 + x_3 \leq 1, \\ &\qquad x_3 + x_2 + x_4 \leq 1, \quad x_4 + x_3 + x_5 \leq 1, \\ &\qquad x_5 + x_4 + x_6 \leq 1, \quad x_6 + x_1 + x_5 \leq 1, \\ &\qquad x_i \in \{0, 1\} \quad \text{for } i = 1, 2, \ldots, 6. \end{aligned}$$

B. Even d

Let $d = 2t$ for some $t \in \mathbb{N}$. Suppose a set $C \subseteq S_n$ satisfies $d_{\min}(C) \geq 2t$. Since $d_{\min}(C) \geq 2t - 1$, by the argument in the previous subsection we have

$$|B(\sigma, t - 1) \cap C| \leq 1 \text{ for all } \sigma \in S_n, \tag{10}$$

which is just (5) with t replaced by $t - 1$. If the condition in (10) holds for some $C \subseteq S_n$ satisfying $|C| \geq 2$, then it only guarantees that $d_{\min}(C) \geq 2t - 1$. We need more constraints to formulate the calculation of $P(n, d)$ in terms of the indicator variables.

If a permutation σ is in a set $C \subseteq S_n$ satisfying $d_{\min}(C) \geq d$, then a ball of radius $d - 1$ around σ will contain exactly one element (σ) of C, i.e.

$$|B(\sigma, d - 1) \cap C| = 1 \text{ for all } \sigma \in C. \tag{11}$$

This condition holds irrespective of whether d is even or odd. We did not need it for the case when d was odd. The constraint in (11) cannot be directly used to formulate an optimization problem since we do not know in advance which permutations σ belong to C.

Let $I_e(\sigma, r)$ be the set of indices[3] of the permutations in $B(\sigma, r) \setminus \{\sigma\}$, i.e.

$$I_e(\sigma, r) = \{ i \mid \sigma_i \in B(\sigma, r), \sigma_i \neq \sigma \}. \tag{12}$$

[2] Any permutation in S_n and its reverse are at a Kendall τ-distance of $\binom{n}{2}$ from each other [5].

[3] The subscript e indicates that $I_e(\sigma, r)$ is obtained from $I(\sigma, r)$ by *excluding* the index of σ.

Then the following condition on the indicator variables of C holds.

$$x_i \left(\sum_{j \in I_e(\sigma_i, d-1)} x_j \right) = 0 \text{ for all } \sigma_i \in S_n. \tag{13}$$

If $\sigma_i \notin C$, then $x_i = 0$ and the product in (13) is zero. If $\sigma_i \in C$, then $x_i = 1$. But the ball of radius $d-1$ around σ_i contains only σ_i, i.e. $x_j = 0$ for all $j \in I_e(\sigma_i, d-1)$. So once again, the product in (13) is zero. Note that (13) is equivalent to (11) for $\sigma_i \in C$.

If the condition (13) holds for some $C \subseteq S_n$ satisfying $|C| \geq 2$, then $d_{\min}(C) \geq d$. This gives us the following set equality (which holds irrespective of whether d is even or odd).

$$\left\{ C \subseteq S_n \;\middle|\; |C| \geq 2, d_{\min}(C) \geq d \right\} = \left\{ C \subseteq S_n \;\middle|\; |C| \geq 2, x_i\text{'s of } C \text{ satisfy (13)} \right\}. \tag{14}$$

Hence for even d, $P(n,d)$ is the solution of the following quadratically constrained binary integer program (QCBIP).

$$\begin{aligned} &\text{maximize } \sum_{i=1}^{n!} x_i \\ &\text{subject to } x_i \left(\sum_{j \in I_e(\sigma_i, d-1)} x_j \right) = 0 \quad \text{for } i = 1, 2, \ldots, n!, \\ &\qquad x_i \in \{0, 1\} \text{ for } i = 1, 2, \ldots, n!. \end{aligned} \tag{15}$$

Example 2: In this example, we consider the QCBIP for $n = 3$ and $d = 2$. The choice of $d = 2$ is only for illustration. Once again, we use the ordering of the permutations in S_3 as shown in Fig. 1. The QCBIP in (15) specializes to the following.

$$\begin{aligned} &\text{maximize } x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ &\text{subject to } x_1(x_2 + x_6) = 0, \quad x_2(x_1 + x_3) = 0, \\ &\qquad x_3(x_2 + x_4) = 0, \quad x_4(x_3 + x_5) = 0, \\ &\qquad x_5(x_4 + x_6) = 0, \quad x_6(x_1 + x_5) = 0, \\ &\qquad x_i \in \{0, 1\} \quad \text{for } i = 1, 2, \ldots, 6. \end{aligned}$$

III. Largest Codes in S_5 and S_6

We used software from Gurobi Optimization Inc [10] to solve the optimization problems formulated in the previous section. The values of $P(5,d)$ for $d \in \{3,4,5,6\}$ and $P(6,d)$ for $d \in \{4,5,\ldots,10\}$ were successfully found[4]. The values are listed in Table II and the corresponding codes are listed in the appendix. For ease of comparison, the upper and lower bounds on $P(n,d)$ for $n = 5$ and $n = 6$ from [5] are listed in Table I.

The solver could not find the optimal value of $P(6,3)$ even after several weeks of computation. The solver did find a heuristic solution of cardinality 102 improving the lower bound on $P(6,3)$ from the previous value of 90 reported in [5]. For $n = 7$, the solver was not able to solve the optimization problems.

[4]By [5, Th. 10], $P(n,d) = 2$ for $d > \frac{2}{3}\binom{n}{2}$.

IV. Conclusion

The problem of finding $P(n,d)$ is now completely resolved for $n = 5$ and partially resolved for $n = 6$. The approach proposed in this letter does not work for larger values of n, at least when a commercial off-the-shelf solver for integer programs is used. It would be interesting to investigate the algebraic structure (if any) of the optimal codes found and give constructions of provably optimal codes for larger values of n.

Appendix

This appendix contains instances of codes of size $P(5,d)$ in S_5 for $d \in \{3,4,5,6\}$ and codes of size $P(6,d)$ in S_6 for $d \in \{4,5,\ldots,10\}$. We also give a code of cardinality 102 in S_6 with minimum Kendall τ-distance 3 which shows that $P(6,3)$ is at least 102.

A. Code of Size $P(5,3) = 20$

(1, 2, 3, 4, 5)	(1, 4, 2, 5, 3)	(1, 5, 2, 3, 4)
(1, 5, 4, 3, 2)	(2, 1, 5, 4, 3)	(2, 3, 1, 5, 4)
(2, 3, 4, 5, 1)	(2, 4, 1, 3, 5)	(2, 5, 4, 3, 1)
(3, 1, 4, 2, 5)	(3, 1, 5, 2, 4)	(3, 4, 5, 1, 2)
(3, 5, 2, 4, 1)	(4, 1, 3, 2, 5)	(4, 2, 5, 1, 3)
(4, 3, 2, 5, 1)	(4, 5, 1, 3, 2)	(5, 2, 3, 1, 4)
(5, 3, 1, 4, 2)	(5, 4, 3, 2, 1)	

B. Code of Size $P(5,4) = 12$

(1, 2, 3, 4, 5)	(1, 4, 3, 5, 2)	(1, 5, 3, 2, 4)
(2, 1, 5, 4, 3)	(2, 4, 3, 1, 5)	(2, 5, 3, 4, 1)
(3, 2, 1, 5, 4)	(3, 4, 1, 2, 5)	(3, 5, 1, 4, 2)
(4, 1, 2, 5, 3)	(4, 3, 5, 2, 1)	(5, 1, 4, 2, 3)

C. Code of Size $P(5,5) = 6$

(1, 2, 3, 4, 5)	(1, 4, 5, 3, 2)	(2, 5, 4, 1, 3)
(3, 4, 5, 1, 2)	(4, 2, 3, 1, 5)	(5, 3, 2, 1, 4)

D. Code of Size $P(5,6) = 5$

(1, 2, 3, 4, 5)	(3, 5, 1, 4, 2)	(4, 1, 5, 3, 2)
(4, 2, 3, 5, 1)	(5, 2, 1, 4, 3)	

E. Code Which Shows That $P(6,3) \geq 102$

(1, 2, 3, 4, 5, 6)	(1, 2, 4, 6, 5, 3)	(1, 2, 6, 3, 4, 5)
(1, 3, 2, 5, 6, 4)	(1, 3, 4, 2, 6, 5)	(1, 3, 4, 5, 6, 2)
(1, 3, 6, 2, 4, 5)	(1, 4, 2, 5, 3, 6)	(1, 4, 5, 6, 3, 2)
(1, 4, 6, 3, 2, 5)	(1, 5, 2, 3, 6, 4)	(1, 5, 2, 4, 6, 3)
(1, 5, 3, 4, 2, 6)	(1, 5, 6, 3, 2, 4)	(1, 5, 6, 4, 2, 3)
(1, 6, 2, 5, 3, 4)	(1, 6, 3, 4, 5, 2)	(1, 6, 4, 2, 5, 3)
(2, 1, 3, 6, 5, 4)	(2, 1, 4, 3, 6, 5)	(2, 1, 5, 4, 3, 6)
(2, 1, 5, 6, 3, 4)	(2, 3, 1, 5, 4, 6)	(2, 3, 4, 6, 1, 5)
(2, 3, 5, 4, 6, 1)	(2, 3, 6, 5, 1, 4)	(2, 4, 1, 5, 6, 3)
(2, 4, 3, 5, 1, 6)	(2, 4, 5, 6, 3, 1)	(2, 4, 6, 1, 3, 5)
(2, 5, 6, 3, 1, 4)	(2, 5, 6, 4, 1, 3)	(2, 6, 1, 4, 5, 3)
(2, 6, 3, 1, 4, 5)	(2, 6, 4, 3, 5, 1)	(3, 1, 5, 2, 4, 6)
(3, 1, 6, 5, 4, 2)	(3, 2, 1, 6, 4, 5)	(3, 2, 4, 1, 5, 6)
(3, 2, 5, 1, 6, 4)	(3, 2, 6, 4, 5, 1)	(3, 4, 1, 5, 2, 6)

(3, 4, 1, 6, 2, 5) (3, 4, 2, 5, 6, 1) (3, 4, 6, 5, 2, 1)
(3, 5, 1, 4, 6, 2) (3, 5, 2, 4, 1, 6) (3, 5, 6, 1, 2, 4)
(3, 6, 1, 2, 5, 4) (3, 6, 4, 1, 5, 2) (3, 6, 5, 2, 4, 1)
(4, 1, 2, 6, 3, 5) (4, 1, 3, 2, 5, 6) (4, 1, 3, 6, 5, 2)
(4, 1, 5, 2, 6, 3) (4, 2, 3, 6, 5, 1) (4, 2, 5, 1, 3, 6)
(4, 2, 6, 5, 1, 3) (4, 3, 2, 1, 6, 5) (4, 3, 5, 2, 1, 6)
(4, 3, 5, 6, 1, 2) (4, 5, 1, 3, 2, 6) (4, 5, 2, 3, 6, 1)
(4, 5, 6, 1, 3, 2) (4, 6, 1, 5, 2, 3) (4, 6, 2, 3, 1, 5)
(4, 6, 3, 1, 5, 2) (4, 6, 5, 2, 3, 1) (5, 1, 3, 6, 4, 2)
(5, 1, 4, 2, 3, 6) (5, 2, 1, 3, 4, 6) (5, 2, 1, 6, 4, 3)
(5, 2, 3, 6, 4, 1) (5, 2, 4, 3, 1, 6) (5, 3, 1, 2, 6, 4)
(5, 3, 4, 2, 6, 1) (5, 3, 6, 4, 1, 2) (5, 4, 1, 6, 2, 3)
(5, 4, 2, 6, 1, 3) (5, 4, 3, 1, 6, 2) (5, 4, 6, 3, 2, 1)
(5, 6, 1, 2, 3, 4) (5, 6, 1, 4, 3, 2) (5, 6, 2, 4, 3, 1)
(5, 6, 3, 2, 1, 4) (6, 1, 2, 4, 3, 5) (6, 1, 3, 5, 2, 4)
(6, 1, 4, 5, 3, 2) (6, 1, 5, 2, 4, 3) (6, 2, 1, 3, 5, 4)
(6, 2, 5, 1, 4, 3) (6, 2, 5, 3, 4, 1) (6, 3, 1, 4, 2, 5)
(6, 3, 2, 4, 1, 5) (6, 3, 2, 5, 1, 4) (6, 3, 5, 1, 4, 2)
(6, 4, 1, 3, 2, 5) (6, 4, 2, 1, 5, 3) (6, 4, 3, 2, 5, 1)
(6, 4, 5, 3, 1, 2) (6, 5, 3, 4, 2, 1) (6, 5, 4, 1, 2, 3)

F. Code of Size $P(6,4) = 64$

(1, 2, 3, 4, 5, 6) (1, 2, 5, 6, 3, 4) (1, 3, 5, 6, 4, 2)
(1, 4, 2, 5, 6, 3) (1, 4, 3, 2, 6, 5) (1, 4, 6, 5, 3, 2)
(1, 5, 4, 3, 2, 6) (1, 5, 6, 4, 2, 3) (1, 6, 2, 3, 4, 5)
(1, 6, 5, 3, 2, 4) (2, 1, 3, 6, 5, 4) (2, 1, 5, 4, 3, 6)
(2, 1, 6, 4, 5, 3) (2, 3, 4, 5, 1, 6) (2, 3, 6, 4, 1, 5)
(2, 4, 1, 3, 6, 5) (2, 4, 5, 1, 6, 3) (2, 4, 6, 3, 5, 1)
(2, 5, 3, 6, 4, 1) (2, 6, 3, 5, 1, 4) (2, 6, 5, 4, 1, 3)
(3, 1, 4, 6, 5, 2) (3, 1, 5, 2, 4, 6) (3, 1, 6, 2, 5, 4)
(3, 2, 1, 4, 6, 5) (3, 2, 5, 1, 6, 4) (3, 4, 2, 6, 5, 1)
(3, 4, 5, 1, 2, 6) (3, 5, 2, 4, 6, 1) (3, 5, 6, 1, 2, 4)
(3, 6, 2, 5, 4, 1) (3, 6, 4, 1, 2, 5) (4, 1, 3, 5, 6, 2)
(4, 1, 6, 2, 3, 5) (4, 2, 5, 3, 6, 1) (4, 2, 6, 1, 5, 3)
(4, 3, 2, 1, 5, 6) (4, 3, 6, 5, 1, 2) (4, 5, 1, 2, 3, 6)
(4, 5, 6, 3, 2, 1) (4, 6, 3, 2, 1, 5) (4, 6, 5, 1, 2, 3)
(5, 1, 2, 4, 6, 3) (5, 1, 3, 2, 6, 4) (5, 2, 3, 1, 4, 6)
(5, 2, 6, 1, 3, 4) (5, 3, 1, 4, 6, 2) (5, 3, 6, 4, 2, 1)
(5, 4, 1, 6, 3, 2) (5, 4, 2, 6, 1, 3) (5, 4, 3, 2, 1, 6)
(5, 6, 1, 3, 4, 2) (5, 6, 2, 4, 3, 1) (6, 1, 4, 2, 5, 3)
(6, 2, 1, 5, 3, 4) (6, 2, 4, 1, 3, 5) (6, 3, 1, 5, 4, 2)
(6, 3, 2, 1, 4, 5) (6, 3, 4, 5, 2, 1) (6, 4, 1, 3, 5, 2)
(6, 4, 2, 5, 3, 1) (6, 5, 1, 2, 4, 3) (6, 5, 3, 2, 1, 4)
(6, 5, 4, 3, 1, 2)

G. Code of Size $P(6,5) = 26$

(1, 2, 3, 4, 5, 6) (1, 3, 5, 6, 4, 2) (1, 4, 5, 3, 2, 6)
(1, 6, 5, 2, 3, 4) (2, 1, 4, 6, 5, 3) (2, 3, 5, 4, 6, 1)
(2, 5, 1, 3, 6, 4) (2, 6, 4, 3, 1, 5) (3, 1, 4, 6, 2, 5)
(3, 2, 1, 6, 5, 4) (3, 4, 5, 6, 1, 2) (3, 5, 1, 2, 4, 6)
(3, 6, 2, 4, 5, 1) (4, 1, 6, 3, 5, 2) (4, 2, 3, 1, 5, 6)
(4, 2, 5, 6, 1, 3) (5, 1, 2, 4, 6, 3) (5, 2, 6, 4, 3, 1)
(5, 3, 6, 2, 1, 4) (5, 4, 3, 2, 1, 6) (5, 4, 6, 1, 3, 2)
(6, 1, 3, 2, 4, 5) (6, 2, 5, 1, 4, 3) (6, 3, 5, 1, 4, 2)
(6, 4, 1, 2, 5, 3) (6, 4, 5, 3, 2, 1)

H. Code of Size $P(6,6) = 20$

(1, 2, 3, 4, 5, 6) (1, 2, 6, 5, 4, 3) (1, 5, 3, 6, 2, 4)
(1, 6, 4, 3, 5, 2) (2, 5, 4, 1, 3, 6) (2, 5, 6, 3, 1, 4)
(3, 2, 1, 6, 5, 4) (3, 2, 4, 5, 6, 1) (3, 4, 6, 1, 5, 2)
(3, 5, 1, 4, 2, 6) (4, 1, 5, 3, 2, 6) (4, 2, 3, 1, 6, 5)
(4, 5, 2, 6, 3, 1) (4, 6, 1, 2, 5, 3) (5, 1, 4, 6, 2, 3)
(5, 3, 6, 4, 2, 1) (6, 2, 1, 3, 4, 5) (6, 3, 5, 1, 2, 4)
(6, 4, 3, 2, 5, 1) (6, 5, 2, 4, 1, 3)

I. Code of Size $P(6,7) = 11$

(1, 2, 3, 4, 5, 6) (1, 5, 4, 3, 6, 2) (2, 1, 6, 5, 4, 3)
(2, 5, 3, 4, 6, 1) (3, 4, 5, 6, 1, 2) (3, 6, 1, 2, 5, 4)
(4, 3, 2, 1, 6, 5) (4, 5, 2, 1, 6, 3) (5, 6, 1, 2, 3, 4)
(6, 4, 1, 2, 3, 5) (6, 5, 4, 3, 2, 1)

J. Code of Size $P(6,8) = 7$

(1, 2, 3, 4, 5, 6) (1, 4, 6, 5, 3, 2) (3, 2, 6, 5, 1, 4)
(3, 5, 4, 1, 6, 2) (4, 2, 5, 3, 6, 1) (5, 2, 1, 6, 4, 3)
(6, 2, 4, 1, 3, 5)

K. Code of Size $P(6,9) = 4$

(1, 2, 3, 4, 5, 6) (1, 5, 6, 4, 3, 2) (2, 4, 6, 5, 3, 1)
(3, 5, 4, 6, 2, 1)

L. Code of Size $P(6,10) = 4$

(1, 2, 3, 4, 5, 6) (1, 6, 5, 4, 3, 2) (3, 6, 4, 2, 5, 1)
(5, 2, 4, 6, 3, 1)

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