Sandwich Theorem or Squeeze Theorem

If $\{a_n\}, \{b_n\}, \{c_n\}$ are three sequences such that

(i) $a_n \le b_n \le c_n \quad \forall n \in \mathbb{N}$

$\downarrow l \quad \downarrow l \quad \downarrow l$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$ then $\lim_{n\to\infty} b_n = l$

Proof:- Since $\{a_n\}$, $\{c_n\}$ converges to 'l'.

Let $\varepsilon > 0$, $\exists$ m_1, m_2 such that

$|a_n - l| < \varepsilon \quad \forall n > m_1$ $\qquad$ $|c_n - l| < \varepsilon \quad \forall n > m_2$

$l - \varepsilon < a_n < l + \varepsilon \longrightarrow (1)$ $\qquad$ $l - \varepsilon < c_n < l + \varepsilon \longrightarrow (2)$

let $m = \max(m_1, m_2)$ $\qquad$ and

$a_n \le b_n \le c_n$

let $\varepsilon > 0$ $\exists$ m such that

$$l - \varepsilon < a_n \le b_n \le c_n < l + \varepsilon$$

$$l - \varepsilon < b_n < l + \varepsilon$$

$$|b_n - l| < \varepsilon \quad \forall n > m$$

$$\therefore \lim_{n \to \infty} b_n = l.$$

ex: Given sequence converges or diverges

whether (or) diverges

$$\{b_n\} = \frac{\sin n}{n} \qquad n \in \mathbb{N}.$$

Proof: $\quad -1 \le \sin n \le 1 \quad \forall n \in \mathbb{N}$

divide by n

$$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$a_n \qquad b_n \qquad c_n$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{-1}{n} = 0 = \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} c_n$$

By Sandwich theorem

$$\lim_{n \to \infty} b_n = 0 \Rightarrow \lim_{n \to \infty} \frac{\sin n}{n} = 0$$

$\{b_n\} = \dfrac{\sin n}{n}$ converges to '0'

Monotonic Sequences

A sequence $\{S_n\}$ is said to be monotonic increasing if $S_{n+1} \geq S_n \ \forall n$ $\{S_1, S_2, S_3 \ldots\}$

ex :- $\{S_n\} = n^2$ $S_{n+1} \geq S_n$

and monotonic decreasing if $S_{n+1} \leq S_n \ \forall n$, $\longrightarrow$ Ex :- $S_n = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}\}$

It is said to be a monotonic sequence if it is either monotonic increasing or monotonic decreasing

n^2: $\{1, 4, 9, 16 \ldots\}$

Strictly Increasing and decreasing sequence

A sequence $\{S_n\}$ is said to be strictly increasing if $S_{n+1} > S_n \ \forall n$

and strictly decreasing if $S_{n+1} < S_n \ \forall n$

Theorem:

A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

Proof :- necessary condition :- [If monotonic sequence is convergent only then it is bounded]

★ Every convergent is bounded

Sufficient condition :- [Monotonic sequence is bounded then it is convergent]

Proof :- let $\{S_n\}$ → Monotonic increasing sequence is bounded

Range = S is also bounded $\{1, 4, 9, 16, \ldots\}$

★ [Every bounded above sequence has a supremum] = Completeness axiom

let 'M' is supremum. We want to show that

$$\lim_{n \to \infty} S_n = M$$

$1, \frac{1}{2}, \frac{1}{3}$ -0.5 0.5

let $\varepsilon > 0$

Since M is supremum $S_n \leq M < M + \varepsilon$ — (1)

M is supremum, $M - \varepsilon$ is not supremum. ∃ atleast $S_m > M - \varepsilon$

$M - \varepsilon < S_n < M + \varepsilon$ $\forall n$ $S_1, S_2 \ldots S_m, S_{m+1} \ldots$

$M-\varepsilon$ M $M+\varepsilon$

$|S_n - M| < \varepsilon$ $\forall n \geq m$ S_n converges

** Corollary:

1. A monotonic increasing bounded above sequence converges to its least upper bound (Supremum) and a monotonic decreasing bounded below to converges to greatest lower bound (Infimum)

2. Every monotonic increasing sequence which is not bounded above diverges to $+\infty$ $\{n^2\}$

3. Every monotonic decreasing sequence which is not bounded ~~above~~ below diverges to $-\infty$ $\{-n^2\}$

ex:- $\{S_n\} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n} \quad \forall n \in \mathbb{N}$

Converges or not ??

Monotonic increasing (or) decreasing

$S_{n+1} - S_n > 0$ Increasing $\Rightarrow$ bounded above

$S_{n+1} - S_n < 0$ decreasing $\Rightarrow$ bounded below

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n-2} + \frac{1}{n+n-1} + \frac{1}{n+n} \quad \frac{1}{2n}$$

$$S_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{n+1+n+1-2} + \frac{1}{2(n+1)-1} + \frac{1}{n+1+n+1}$$

$$S_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{(n+1)} = \frac{1}{2(n+1)(2n+1)}$$

$$= \frac{1}{2n+1} + \frac{1}{n+1}\left(\frac{1}{2} - 1\right) = \frac{1}{2n+1} - \frac{1}{2(n+1)} = \frac{2n+2-2n-1}{2(n+1)(2n+1)}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0$$

$S_{n+1} - S_n > 0$ Increasing function

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \cdots \frac{1}{n+n}$$

$$S_n < \frac{1}{n} + \frac{1}{n} + \cdots \cdots \frac{1}{n} = \frac{n}{n} = 1$$

$$\boxed{0 < S_n < 1}$$

Monotonic increasing function and it is bounded.

So S_n converges

Subsequences: If $\{S_n\} = \{S_1, S_2, S_3, \dots\}$ be a sequence then any infinte succession of its terms picked out in any way (but preserving the original order) is called subsequence of $\{S_n\}$ or in other words if $\{n_k\}$ be srictly monotonic increasing sequence of natural numbers. i.e $n_1 < n_2 < n_3 \dots$ then $\{S_{n_k}\}$ is a subsequence of $\{S_n\}$.

1. $\{S_2, S_4, S_6 \dots S_{2n} \dots\}$ ✓

$S_n = \{S_1, S_2, S_3, S_4 \dots\}$

2. $\{S_1, S_4, S_9, \dots S_{n^2}\}$ ✓

(1) $S_n = \{\underset{s_1}{1}, \underset{s_2}{5}, \underset{s_3}{9}, \underset{s_4}{10}, \underset{s_5}{2}, \underset{s_6}{4}, \underset{s_7}{6}, \dots\}$

3. $\{S_1, S_5, S_7, S_9 \dots\}$ ✓

(i) $S_{n_k} = \{\underset{s_1}{1}, \underset{s_3}{9}, \underset{s_6}{4}, \underset{s_7}{6} \dots\}$ Subsequence

4. $\{S_2, S_3, S_1, \dots \dots\}$

not subsequence

(ii) $S_{n_k} = \{\underset{s_2}{5}, \underset{s_5}{2}, \underset{s_7}{6}, \underset{s_3}{9}, \dots\}$

Not subsequence

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\}$ $\{\frac{1}{3}, \frac{1}{5}, \dots\}$

*1. A sequence $\{S_n\}$ converges to 'S' if only if its every subsequence converges to 'S'.

similarly $\lim S_n = \infty\ (-\infty)$ if and only if every subsequence of $\{S_n\}$ tends to $\infty\ (-\infty)$

2. If 'ξ' is a limit point of sequence $\{S_n\}$ then $\exists$ subsequence $\{S_{n_k}\}$ of S_n which converges to ξ i.e $\lim_{k\to\infty} S_{n_k} = \xi$

$S_{n_k} = \{-1, -1, -1, \dots\}$

$S_n = (-1)^n = \{-1, 1, -1, 1, \dots \dots\}$

limit points of $\{S_n\} = -1, 1$

$S_{n_1} = \{1, 1, 1, \dots\}$ limit 1

$S_{n_2} = \{-1, -1, -1, \dots\}$ -1 limit

(1) $S_{n_k} = \{-1, -1, -1, \dots\}$ limit $= -1$

(2) $S_{n_k} = \{1, 1, 1, \dots\}$ limit $= 1$

Sequence