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Exact renormalization group for wave functionals

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Abstract

Motivated by the construction of continuum tensor networks for interacting field theories, which are relevant in understanding the emergence of space-time in the gauge/gravity correspondence, we derive a non-perturbative functional differential equation for wave functionals in scalar field theories from the exact renormalization group equation. We check the validity of the equation using the perturbation theory. We calculate the wave functional up to the first-order perturbation and verify that it satisfies the equation.

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1 Introduction

It is recognized that the emergence of space-time is an essential feature of quantum gravity. Tensor network models such as the MERA [1,2], the HaPPY code [3] and the random tensor network [4] give networks that can be interpreted as discrete bulk space emerging from a boundary theory through quantum entanglement [5]. It should be crucial to construct continuum tensor networks from field theories to obtain continuum emergent space-time.

Continuum tensor networks for free-field theories have been successfully constructed based on the variational method. This is called the cMERA [6,7], which is a continuum counterpart of the MERA. Interestingly, the relationship between the information metric and geometry was studied in Ref. [7]. It is, however, important to construct continuum tensor networks non-perturbatively for interacting theories from the point of view of the gauge/gravity correspondence [8], because the strong coupling regime in boundary theories corresponds to classical geometry with quantum fluctuations. It seems non-trivial to construct trial functions for the cMERA in interacting theories (see Refs. [9–11] for an approach based on non-linear canonical transformations). We should remark here that there is a non-perturbative construction of the cMERA based on the Weyl transformation, which is called the path-integral optimization [12].

In the MERA, which represents a wave function of a quantum many-body system, the layers of the network are interpreted as representing the energy scale, which corresponds to the bulk direction in the gauge/gravity correspondence. Thus, constructing continuum tensor networks is considered to be equivalent to obtaining the scale dependence of a wave functional in a quantum field theory, which should be determined by the renormalization group[1]. In this paper, to obtain the scale dependence of the wave functional non-perturbatively, we consider an approach based on the exact renormalization group. Note here that there are works on a perturbative construction of continuum tensor networks for interacting theories based on the renormalization group [16] and the exact renormalization group approach to the continuum tensor networks for O(N) vector model at free fixed point [17].

The exact renormalization group (ERG) [18, 19] (for a review of the ERG, see [20–29]) serves as a powerful non-perturbative method for quantum field theories along with lattice field theories. The ERG gives a functional differential equation that describes the scale-dependence of the effective action. In this paper, we derive a functional differential equation obeyed by wave functionals of the ground states in scalar field theories from the ERG equation (the Polchinski equation [30]). While our equation is a non-perturbative one, we check the validity of our equation using the perturbation theory. We calculate the wave functional up to the first-order perturbation and verify that it satisfies our equation. While our motivation is

[1]For early work on renormalization of quantum field theory from a wave functional approach, see Refs. [13–15]

to study a tensor network description of emergent space, we expect our findings to contribute to developments in the non-perturbative studies of quantum field theories.

This paper is organized as follows. In Sect. 2, we derive an ERG equation for the wave functionals of the ground states in scalar field theories from the ERG equation (the Polchinski equation). We solve the equation to obtain a solution taking the Gaussian form and derive an ERG equation for the interaction part of the wave functionals. In Sect. 3, we perform the perturbative expansion of the wave functional up to the first order. We develop a systematic method for the perturbative expansion of the wave functional based on the path integral. In Sect. 4, we check the validity of the ERG equation for the wave functional derived in Sect. 2 by using the results obtained in Sect. 3. Section 5 is devoted to the conclusion and discussion. In the appendix, the flow equation for the mass counterterm is derived.

2 The ERG equation for wave functionals

In this section, we derive the ERG equation for the wave functionals of the ground states in scalar field theories from the Polchinski equation. Throughout this paper, we work in $(d+1)$-dimensional Euclidean space-time, where the time direction is parameterized by τ, and use the following compact notations for the integrals:

$$\int_p \equiv \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \ , \quad \int_{\vec{p}} \equiv \int \frac{d^d p}{(2\pi)^d} \ , \quad \int_\tau \equiv \int d\tau \ , \tag{2.1}$$

where p stands for the momentum in $d+1$ dimensions, while $\vec{p}$ stands for the d-dimensional spatial part of p. We also introduce a shorthand notation

$$\tilde{\delta}(\vec{p}) = (2\pi)^d \delta(\vec{p}) \tag{2.2}$$

and define V by

$$V = \tilde{\delta}(0) \tag{2.3}$$

which is the volume of space. In what follows, we frequently use $\phi(p)$ and $\phi(\tau,\vec{p})$, which transform each other by

$$\phi(p) = \phi(E,\vec{p}) = \int d\tau \phi(\tau,\vec{p}) e^{-iE\tau} \ . \tag{2.4}$$

2.1 The Polchinski equation

First, we briefly review the Polchinski equation for scalar field theories. The ERG equations for scalar field theories have the following general structure [31–34]

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta\phi(p)} \left[G_\Lambda[\phi](p) e^{-S_\Lambda[\phi]} \right] , \tag{2.5}$$

where Λ is the effective cutoff and S_Λ is the effective action at the scale Λ. The functional $G_\Lambda[\phi](p)$, which also depends on p, is required to correspond to a continuum blocking procedure and to ensure the UV regularization of the equation. The structure in Eq.(2.5) ensures the physical requirement that the partition function is unchanged under the infinitesimal change of the effective cutoff Λ:

$$-\Lambda\partial_\Lambda Z = -\Lambda\partial_\Lambda \int \mathcal{D}\phi e^{-S_\Lambda[\phi]} = \int_p \mathcal{D}\phi \frac{\delta}{\delta\phi(p)} \left[G_\Lambda[\phi](p)e^{-S_\Lambda[\phi]}\right] = 0 \ . \tag{2.6}$$

Typically $G_\Lambda[\phi](p)$ takes the following form

$$G_\Lambda[\phi](p) = \frac{1}{2}\dot{C}_\Lambda(p)\frac{\delta}{\delta\phi(-p)}(S_\Lambda - 2\hat{S}) \ , \tag{2.7}$$

where $\dot{C}_\Lambda \equiv -\Lambda\partial_\Lambda C_\Lambda$ is an ERG integration kernel that incorporates the UV regularization and specifies the coarse-graining procedure with $\hat{S}$, which is called the seed action. The Polchinski equation [30] corresponds to setting the seed action $\hat{S}$ to S_0, i.e., the free part of the effective action S_Λ taking the form

$$S_0 = \int_p \frac{1}{2}\phi(p)C_\Lambda^{-1}(p)\phi(-p) \ . \tag{2.8}$$

It is easily checked that S_0 satisfies Eq.(2.5) with Eq.(2.7). Now Eq.(2.5) reduces to

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{-S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta\phi(p)}\left[\frac{1}{2}\dot{C}_\Lambda(p)\left\{\frac{\delta}{\delta\phi(-p)}(S_\Lambda - 2S_0)\right\}e^{-S_\Lambda[\phi]}\right] \ . \tag{2.9}$$

By decomposing the effective action into the free part and the interaction part as

$$S_\Lambda = S_0 + S_{\rm int} \ , \tag{2.10}$$

we obtain from Eq.(2.9) a conventional form of the Polchinski equation for $S_{\rm int}$:

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{-S_{\rm int}} = -\frac{1}{2}\int_p \dot{C}_\Lambda(p)\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}e^{-S_{\rm int}} \ . \tag{2.11}$$

2.2 The ERG equation for wave functionals

Next, we derive the ERG equation for the wave functional of the ground state from the Polchinski equation (2.9).

Note first that the ground-state wave functional $\Psi_\Lambda[\varphi]$ has the following path-integral representation

$$\Psi_\Lambda[\varphi] = \int_{\phi(0,\vec{p})=\varphi(\vec{p})} \mathcal{D}\phi e^{-\int_{-\infty}^0 d\tau L_\Lambda[\phi]} \ , \tag{2.12}$$

where L_Λ is the effective Lagrangian and the boundary condition that the field $\phi(\tau,\vec{p})$ at $\tau=0$ is fixed to $\varphi(\vec{p})$ is imposed:

$$\phi(0,\vec{p})=\varphi(\vec{p}) \ . \tag{2.13}$$

We assume that L_Λ is real so that $\Psi_\Lambda[\varphi]$ is also real. This implies that $\Psi_\Lambda[\varphi]$ is also represented as

$$\Psi_\Lambda[\varphi]=\int_{\phi(0,\vec{p})=\varphi(\vec{p})}\mathcal{D}\phi e^{-\int_0^\infty d\tau L_\Lambda[\phi]} \ . \tag{2.14}$$

We see from Eqs.(2.12) and (2.14) that the square of the wave functional $\Psi_\Lambda[\varphi]$ is represented in terms of the effective action $S_\Lambda=\int_{-\infty}^{\infty}d\tau L_\Lambda$ as

$$\Psi_\Lambda^2[\varphi]=\int\mathcal{D}\phi\prod_{\vec{p}}\delta[\phi(0,\vec{p})-\varphi(\vec{p})]e^{-S_\Lambda[\phi]} \ . \tag{2.15}$$

We act $-\Lambda\frac{\partial}{\partial\Lambda}$ on both sides of (2.15). The left-hand side reduces to

$$2\Psi_\Lambda\left(-\Lambda\frac{\partial}{\partial\Lambda}\Psi_\Lambda\right) \ . \tag{2.16}$$

On the right-hand side, we use Eq.(2.9) to obtain

$$\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]\int_p\frac{\delta}{\delta\phi(p)}\left[\frac{1}{2}\dot{C}_\Lambda(p)\left\{\frac{\delta}{\delta\phi(-p)}(S_\Lambda-2S_0)\right\}e^{-S_\Lambda}\right] \ . \tag{2.17}$$

Substituting Eq.(2.8) into Eq.(2.17) and switching to the coordinate representation in the time direction yields

$$\begin{aligned}&\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]\int_{\tau,\tau',\vec{p}}\left[-\frac{1}{2}\dot{C}_\Lambda(\tau-\tau',\vec{p})\frac{\delta^2}{\delta\phi(\tau,\vec{p})\delta\phi(\tau',-\vec{p})}e^{-S_\Lambda[\phi]}\right]\\&-\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]\int_{\tau,\tau',\tau'',\vec{p}}\frac{\delta}{\delta\phi(\tau,\vec{p})}\left[\dot{C}_\Lambda(\tau-\tau',\vec{p})C_\Lambda^{-1}(\tau'-\tau'',\vec{p})\phi(\tau'',\vec{p})e^{-S_\Lambda[\phi]}\right] \ .\end{aligned} \tag{2.18}$$

By noting that the first line in Eq.(2.18) is the integral of the total derivative except at $\tau=\tau'=0$, as is the second line except at $\tau=0$, we obtain

$$\begin{aligned}&\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]\int_{\tau,\tau',\vec{p}}\left[-\frac{1}{2}\dot{C}_\Lambda(0,\vec{p})\frac{\delta^2}{\delta\phi(0,\vec{p})\delta\phi(0,-\vec{p})}e^{-S_\Lambda[\phi]}\right]\\&-\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]\int_{\tau',\tau'',\vec{p}}\frac{\delta}{\delta\phi(0,\vec{p})}\left[\dot{C}_\Lambda(-\tau',\vec{p})C_\Lambda^{-1}(\tau'-\tau'',\vec{p})\phi(\tau'',\vec{p})e^{-S_\Lambda[\phi]}\right] \ .\end{aligned} \tag{2.19}$$

Here we assume that $C_\Lambda(\tau,\vec{p})$ is factorized as

$$C_\Lambda(\tau,\vec{p}) = f(\tau,\vec{p})g_\Lambda(\vec{p}) \ , \tag{2.20}$$

where f is independent of Λ. Namely, only the spatial components of the momentum have the UV cutoff. This implies that

$$\int_{\tau''} \dot{C}_\Lambda(\tau-\tau'',\vec{p})C_\Lambda^{-1}(\tau''-\tau',\vec{p}) = \delta(\tau-\tau')\frac{\dot{g}_\Lambda(\vec{p})}{g_\Lambda(\vec{p})} = \delta(\tau-\tau')\frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})} \ . \tag{2.21}$$

We substitute Eq.(2.21) into Eq.(2.19), perform the partial integration with respect to $\phi(0,\vec{p})$ and use the relation

$$\frac{\delta}{\delta\phi(0,\vec{p})}\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})] = -\frac{\delta}{\delta\varphi(\vec{p})}\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})] \tag{2.22}$$

to gain

$$\begin{aligned} &-\frac{1}{2}\int_{\vec{p}}\dot{C}_\Lambda(0,\vec{p})\frac{\delta^2}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})}\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]e^{-S_\Lambda[\phi]} \\ &-\int_{\vec{p}}\frac{\delta}{\delta\varphi(\vec{p})}\left[\frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})}\varphi(\vec{p})\int\mathcal{D}\phi\prod_{\vec{k}}\delta[\phi(0,\vec{k})-\varphi(\vec{k})]e^{-S_\Lambda[\phi]}\right] \ . \end{aligned} \tag{2.23}$$

Furthermore, by using Eq.(2.15), we rewrite Eq.(2.23) as

$$\begin{aligned} &-\Psi_\Lambda\int_{\vec{p}}\dot{C}_\Lambda(0,\vec{p})\left\{\frac{\delta^2\Psi_\Lambda}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})}+\frac{1}{\Psi_\Lambda}\frac{\delta\Psi_\Lambda}{\delta\varphi(\vec{p})}\frac{\delta\Psi_\Lambda}{\delta\varphi(-\vec{p})}\right\} \\ &-2\Psi_\Lambda\int_{\vec{p}}\frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})}\varphi(\vec{p})\frac{\delta\Psi_\Lambda}{\delta\varphi(\vec{p})}-\Psi_\Lambda^2 V\int_{\vec{p}}\frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})} \ . \end{aligned} \tag{2.24}$$

Finally, combining Eqs.(2.16) and (2.24) leads us to the ERG equation for the ground-state wave functional $\Psi_\Lambda[\varphi]$:

$$\begin{aligned} -\Lambda\frac{\partial}{\partial\Lambda}\Psi_\Lambda = &-\frac{1}{2}\int_{\vec{p}}\dot{C}_\Lambda(0,\vec{p})\left\{\frac{\delta^2\Psi_\Lambda}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})}+\frac{1}{\Psi_\Lambda}\frac{\delta\Psi_\Lambda}{\delta\varphi(\vec{p})}\frac{\delta\Psi_\Lambda}{\delta\varphi(-\vec{p})}\right\} \\ &-\int_{\vec{p}}\frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})}\varphi(\vec{p})\frac{\delta\Psi_\Lambda}{\delta\varphi(\vec{p})}-\frac{V}{2}\Psi_\Lambda\int_{\vec{p}}\frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})} \ . \end{aligned} \tag{2.25}$$

2.3 A Gaussian solution

As a demonstration of solving Eq.(2.25), we search for a solution that has the Gaussian form:

$$\Psi_0[\varphi] = \mathcal{N}_\Lambda\exp\left[-\frac{1}{2}\int_{\vec{p}}\varphi(\vec{p})\mathcal{M}_\Lambda(\vec{p})\varphi(-\vec{p})\right] \ . \tag{2.26}$$

This solution would correspond to the case in which $S_\Lambda = S_0$. By substituting Eq.(2.26) into Eq.(2.25), we obtain flow equations for $\mathcal{N}_\Lambda$ and $\mathcal{M}_\Lambda(\vec{p})$:

$$-\Lambda \frac{\partial}{\partial \Lambda} \ln \mathcal{N} = \frac{V}{2} \int_{\vec{p}} \dot{C}_\Lambda(0,\vec{p})(\mathcal{M}_\Lambda(\vec{p}) - C^{-1}(0,\vec{p})) , \tag{2.27}$$

$$-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{M}_\Lambda(\vec{p}) = 2\dot{C}_\Lambda(0,\vec{p})\mathcal{M}_\Lambda(\vec{p})(\mathcal{M}_\Lambda(\vec{p}) - C^{-1}(0,\vec{p})) . \tag{2.28}$$

We can find the general solution to Eq.(2.28) as

$$\mathcal{M}_\Lambda(\vec{p}) = \frac{1}{2C_\Lambda(0,\vec{p}) + \alpha(\vec{p})C_\Lambda^2(0,\vec{p})} , \tag{2.29}$$

where $\alpha(\vec{p})$ is an arbitrary function of $\vec{p}$. Here we assume that S_0 reduces to the standard free action in the $\Lambda \to \infty$ limit:

$$S_0|_{\Lambda\to\infty} = \frac{1}{2} \int_p \phi(p)(p^2 + m^2)\phi(-p) . \tag{2.30}$$

namely $C_\infty(p) = 1/(p^2 + m^2)$, which implies $C_\infty(0,\vec{p}) = 1/(2\omega_{\vec{p}})$, where $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. On the other hand, when the action is given by Eq.(2.30), the ground-state wave functional is given by Eq.(2.26) with $\mathcal{M}_\Lambda(\vec{p}) = \omega_{\vec{p}}$ [35]. Thus, we imposed the boundary condition $\mathcal{M}_\infty(\vec{p}) = \omega_{\vec{p}}$, which fixes $\alpha(\vec{p}) = 0$ in Eq.(2.29). Then, we can solve Eq.(2.27) as

$$\mathcal{N}_\Lambda = N_0 \exp\left[-\frac{V}{4} \int_{\vec{p}} \ln C_\Lambda(0,\vec{p})\right] . \tag{2.31}$$

with N_0 being an arbitrary constant. We impose the boundary condition for $\mathcal{N}_\Lambda$ by requiring the normalization condition $1 = \int \mathcal{D}\varphi |\Psi_0[\varphi]|^2$ in the $\Lambda \to \infty$ limit. This fixes N_0 to 1. We therefore obtain a Gaussian solution to Eq.(2.25):

$$\Psi_0[\varphi] = \exp\left[-\frac{1}{2}\int_{\vec{p}} \varphi(\vec{p}) \frac{1}{2C_\Lambda(0,\vec{p})} \varphi(-\vec{p}) - \frac{V}{4}\int_{\vec{p}} \ln C_\Lambda(0,\vec{p})\right]. \tag{2.32}$$

In Sect. 3, we indeed obtain the above result by calculating the path integral directly in the free case in which $S_\Lambda = S_0$.

2.4 The ERG equation for the interaction part of wave functionals

We parameterize the ground-state wave functional as

$$\Psi_\Lambda[\varphi] = e^{I[\varphi]}\Psi_0[\varphi] . \tag{2.33}$$

$I[\varphi]$ is interpreted as representing the contribution of the interaction to the wave functional. By substituting Eqs.(2.32) and (2.33) into Eq.(2.25), we obtain the ERG equation for I:

$$-\Lambda \frac{\partial}{\partial \Lambda} I = -\frac{1}{2} \int_{\vec{p}} \dot{C}(0,\vec{p}) \left[\frac{\delta^2 I}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} + \frac{\delta I}{\delta\varphi(\vec{p})}\frac{\delta I}{\delta\varphi(-\vec{p})}\right] . \tag{2.34}$$

This equation is a non-perturbative functional differential equation for the interaction part of the ground-state wave functionals, which is a counterpart of Eq.(2.11).

In the remaining part of this paper, we use the following C_Λ:

$$C_\Lambda(p) = \frac{K(\vec{p}^2/\Lambda^2)}{p^2+m^2} \ , \quad C_\Lambda(0,\vec{p}) = \frac{K(\vec{p}^2/\Lambda^2)}{2\omega_{\vec{p}}} \tag{2.35}$$

which has the property shown in (2.20). $K(x)$ is assumed to have the following properties: $K(0)=1$, $K(x)\sim 1$ for $x<1$, and $K(x)$ damps rapidly for $x>1$. In this case, $\dot{C}_\Lambda(0,\vec{p})$ is given by

$$\dot{C}_\Lambda(0,\vec{p}) = \frac{\dot{K}(\vec{p}^2/\Lambda^2)}{2\omega_{\vec{p}}} \ , \tag{2.36}$$

where $\dot{K}(\vec{p}^2/\Lambda^2) = -\Lambda\partial_\Lambda K(\vec{p}^2/\Lambda^2)$.

3 The perturbative wave functional

In this section, we calculate the ground-state wave functional given by Eq.(2.12) to the first-order perturbation. For technical convenience, we introduce an infrared cutoff T in the time direction and eventually take the $T\to\infty$ limit:

$$\Psi_\Lambda[\varphi] = \lim_{T\to\infty}\int_{\phi(0,\vec{p})=\varphi(\vec{p})} \mathcal{D}\phi\ e^{-\int_{-T}^{0} d\tau L_\Lambda} \ . \tag{3.1}$$

We can impose an arbitrary boundary condition at $\tau=-T$ in addition to Eq.(2.13). Here, for convenience, we impose the condition

$$\phi(-T,\vec{p}) = 0 \ . \tag{3.2}$$

The effective Lagrangian L_Λ consists of the free part L_0 and the interaction part $L_{\rm int}$. The free part L_0 is read off from S_0 in Eq.(2.8) with Eq.(2.35) as

$$L_0 = \int_{\vec{p}} \frac{1}{2}K_{\vec{p}}^{-1}\left[\partial_\tau\phi(\tau,\vec{p})\partial_\tau\phi(\tau,-\vec{p}) + \omega_{\vec{p}}^2\phi(\tau,\vec{p})\phi(\tau,-\vec{p})\right] \ , \tag{3.3}$$

where we have introduced a shorthand notation

$$K_{\vec{p}} = K(\vec{p}^2/\Lambda^2) \ . \tag{3.4}$$

We assume that the interaction part $L_{\rm int}$ consists of the mass counterterm and the ϕ^4 interaction term as

$$\begin{aligned} L_{\rm int} = {}& \frac{\delta m^2}{2}\int_{\vec{p}}\phi(\tau,\vec{p})\phi(\tau,-\vec{p}) \\ & + \frac{\lambda}{4!}\int_{\vec{p}_1\cdots\vec{p}_4} \phi(\tau,\vec{p}_1)\phi(\tau,\vec{p}_2)\phi(\tau,\vec{p}_3)\phi(\tau,\vec{p}_4)\tilde{\delta}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4) \ . \end{aligned} \tag{3.5}$$

In order to perform the perturbative expansion, we expand $\phi(\tau,\vec{p})$ around a classical solution $\phi_c(\tau,\vec{p})$ in the free part L_0 as

$$\phi(\tau,\vec{p}) = \phi_c(\tau,\vec{p}) + \chi(\tau,\vec{p}) \ . \tag{3.6}$$

Namely, $\phi_c(\tau,\vec{p})$ satisfies the equation of motion

$$\partial_\tau^2 \phi_c(\tau,\vec{p}) = \omega_{\vec{p}}^2 \phi_c(\tau,\vec{p}) \ . \tag{3.7}$$

Here we set

$$\phi_c(0,\vec{p}) = \varphi(\vec{p}) \ , \quad \phi_c(-T,\vec{p}) = 0 \ , \quad \chi(0,\vec{p}) = 0 \ , \quad \chi(-T,\vec{p}) = 0 \tag{3.8}$$

such that the boundary conditions (2.13) and (3.2) are satisfied. We see from Eqs.(3.7) and (3.8) that $\phi(\tau,\vec{p})$ is explicitly given by

$$\begin{aligned}\phi_c(\tau,\vec{p}) &= \frac{e^{\omega_{\vec{p}}(\tau+T)} - e^{-\omega_{\vec{p}}(\tau+T)}}{e^{\omega_{\vec{p}}T} - e^{-\omega_{\vec{p}}T}}\varphi(\vec{p}) \\ &\xrightarrow{T\to\infty} \ e^{\omega_{\vec{p}}\tau}\varphi(\vec{p})\end{aligned} \tag{3.9}$$

and $\chi(\tau,\vec{p})$ has the following Fourier expansion

$$\chi(\tau,\vec{p}) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{T}\tau\right)\chi_n(\vec{p}) \ . \tag{3.10}$$

Substituting Eq.(3.6) into L_0 and using Eqs.(3.7) and (3.8) gives rise to

$$\int_{-T}^{0} d\tau L_0 = \frac{1}{2}\int_{\vec{p}} K_{\vec{p}}^{-1}\phi_c(\tau,\vec{p})\partial_\tau\phi_c(\tau,\vec{p})|_{\tau=0} + \frac{1}{2}\int_{-T}^{0} d\tau \int_{\vec{p}} K_{\vec{p}}^{-1}\chi(\tau,-\vec{p})(-\partial_\tau^2+\omega_{\vec{p}}^2)\chi(\tau,\vec{p}) \ . \tag{3.11}$$

We parameterize the ground-state wave functional as Eq.(2.33). First, noting that the lowest-order of the wave functional is given by

$$\Psi_0[\varphi] = \lim_{T\to\infty}\int \mathcal{D}\chi e^{-\int_{-T}^{0} d\tau L_0} \ , \tag{3.12}$$

we see from (3.9) and the first term in (3.11) that

$$\Psi_0[\varphi] = N_0 \exp\left[-\int_{\vec{p}} \frac{1}{2} K_{\vec{p}}^{-1}\omega_{\vec{p}}\varphi(\vec{p})\varphi(-\vec{p})\right] \ , \tag{3.13}$$

where N_0 is the normalization constant which is fixed by the condition $1 = \int \mathcal{D}\varphi |\Psi_0[\varphi]|^2$ as

$$N_0 = \exp\left[\frac{V}{4}\int_{\vec{p}} \log(2K_{\vec{p}}^{-1}\omega_{\vec{p}})\right] \ . \tag{3.14}$$

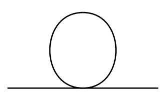
Equation (3.13) with Eq.(3.14) indeed agrees with Eq.(2.32).

Next, we calculate the contribution of the interaction to the wave functional, $I[\varphi]$, perturbatively. For that, we can use the Feynman diagrams in which the external and internal lines represent φ and the propagator for χ, respectively. We see from Eqs.(3.10) and (3.11) that the propagator for χ is given by

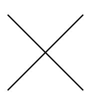
$$\begin{aligned}\langle\chi(\tau,\vec{p})\chi(\tau',\vec{p}')\rangle &= \sum_n \sin\left(\frac{n\pi}{T}\tau\right)\sin\left(\frac{n\pi}{T}\tau'\right)\frac{2}{T}\frac{K_{\vec{p}}\tilde{\delta}(\vec{p}+\vec{p}')}{\omega_{\vec{p}}^2+(\frac{n\pi}{T})^2} \\ &\xrightarrow{T\to\infty} \quad \frac{K_{\vec{p}}}{2\omega_{\vec{p}}}\left(e^{-\omega_{\vec{p}}|\tau-\tau'|}-e^{\omega_{\vec{p}}(\tau+\tau')}\right)\tilde{\delta}(\vec{p}+\vec{p}')\ . \end{aligned} \tag{3.15}$$

By substituting Eqs.(3.6) and (3.9) into L_{int}, we can read off the interaction vertices for φ and χ. Here we represent the interaction with δm^2 by the black dot vertex and the interaction with λ by the plain vertex. $I[\varphi]$ consists of the sum of the connected diagrams except the bubble diagrams and a constant that is fixed by the normalization condition $1=\int\mathcal{D}\varphi|\Psi[\varphi]|^2$. Here we concentrate on the first-order perturbation. The connected diagrams that we need are given by

$$= -\frac{\delta m^2}{2}\int_{\vec{p}}\varphi(\vec{p})\varphi(-\vec{p})\frac{1}{2\omega_{\vec{p}}}\ , \tag{3.16}$$



$$= -\frac{\lambda}{4!}\int_{\vec{p}_1\vec{p}_2}\varphi(\vec{p}_1)\varphi(-\vec{p}_1)\ \frac{3K_2}{2\omega_1(\omega_1+\omega_2)}\ , \tag{3.17}$$



$$= -\frac{\lambda}{4!}\int_{\vec{p}_1\cdots\vec{p}_4}\varphi_1\cdots\varphi_4\ \frac{\tilde{\delta}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)}{\omega_1+\omega_2+\omega_3+\omega_4}\ , \tag{3.18}$$

where we have introduced shorthand notations: $\varphi(\vec{p}_i)=\varphi_i$, $K(\vec{p}_i^{\,2}/\Lambda^2)=K_i$, and $\omega_{\vec{p}_i}=\omega_i$.

After fixing the constant in $I[\varphi]$ using the normalization condition, we obtain the final result for the ground-state wave functional to the first-order perturbation as

$$\Psi_\Lambda[\varphi] = e^{I[\varphi]}\Psi_0[\varphi]\ , \tag{3.19}$$

$$\begin{aligned} I[\varphi] = &-\frac{\delta m^2}{2}\int_{\vec{p}}\varphi(\vec{p})\varphi(-\vec{p})\frac{1}{2\omega_{\vec{p}}} \\ &-\frac{\lambda}{4!}\int_{\vec{p}_1\vec{p}_2}\varphi(\vec{p}_1)\varphi(-\vec{p}_1)\ \frac{3K_2}{2\omega_1(\omega_1+\omega_2)} \\ &-\frac{\lambda}{4!}\int_{\vec{p}_1\cdots\vec{p}_4}\varphi_1\cdots\varphi_4\ \frac{\tilde{\delta}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)}{\omega_1+\omega_2+\omega_3+\omega_4}+\mathcal{C} \end{aligned} \tag{3.20}$$

with

$$\mathcal{C} = \left\{\frac{\delta m^2}{2}+\frac{\lambda}{4!}\int_{\vec{p}}\frac{6K_{\vec{p}}}{2\omega_{\vec{p}}}\right\}\int_{\vec{k}}\frac{K_{\vec{k}}V}{4\omega_{\vec{k}}^2}-\frac{\lambda}{4!}\int_{\vec{p}_1,\vec{p}_2}\frac{3K_1K_2V}{\omega_1\omega_2(\omega_1+\omega_2)}\ . \tag{3.21}$$

The above result agrees with the one obtained using the canonical formalism [35]. The formulation of the perturbative expansion that we have developed here based on the path integral can be applied straightforwardly and systematically to the higher-order perturbations.

4 A perturbative check

In this section, as a check of the validity of the ERG equation (2.25) for the ground-state wave functional, we show that the ground-state wave functional (3.19) satisfies the ERG equation (2.25). Since the analysis in Sect. 2.3 ensures that Ψ_0 in Eq.(3.19) satisfies the ERG equation (2.25) in Sect. 2.3, our task is to check that $I[\varphi]$ in Eq.(3.19) satisfies Eq.(2.34).

Equation (2.34) to the first order perturbation reads

$$-\Lambda\frac{\partial}{\partial\Lambda}I = -\int_{\vec{p}}\frac{\dot{K}_{\vec{p}}}{4\omega_{\vec{p}}}\frac{\delta^2 I}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} \ . \tag{4.1}$$

Substituting I in Eq.(3.19) into the left-hand side of Eq.(4.1) yields

$$\begin{aligned}-\Lambda\frac{\partial}{\partial\Lambda}I = &-\left\{\frac{\dot{\delta m^2}}{2}+\frac{\lambda}{4!}\int_{\vec{p}}\frac{6\dot{K}_{\vec{p}}}{2\omega_{\vec{p}}}\right\}\int_{\vec{k}}\left[\frac{1}{2\omega_{\vec{k}}}\varphi(\vec{k})\varphi(-\vec{k})-\frac{K_{\vec{k}}V}{4\omega_{\vec{k}}^2}\right]\\ &+\left\{\frac{\delta m^2}{2}+\frac{\lambda}{4!}\int_{\vec{p}}\frac{6K_{\vec{p}}}{2\omega_{\vec{p}}}\right\}\int_{\vec{k}}\frac{\dot{K}_{\vec{k}}V}{4\omega_{\vec{k}}^2}\\ &+\frac{\lambda}{4!}\int_{\vec{k}_i}\frac{3}{\omega_1+\omega_2}\frac{\dot{K}_2}{2\omega_2}\phi(\vec{k}_1)\phi(-\vec{k}_1)\\ &+\frac{\lambda}{4!}\int_{\vec{k}_i}\frac{3}{\omega_1+\omega_2}\frac{\dot{K}_2}{2\omega_2}\left(-\frac{\dot{K}_1 V}{2\omega_1}\right) \ ,\end{aligned} \tag{4.2}$$

while the right-hand side gives

$$\begin{aligned}-\int_{\vec{p}}\frac{\dot{K}_{\vec{p}}}{4\omega_{\vec{p}}}\frac{\delta^2 I}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} = &\left\{\frac{\delta m^2}{2}+\frac{\lambda}{4!}\int_{\vec{p}}\frac{6K_{\vec{p}}}{2\omega_{\vec{p}}}\right\}\int_{\vec{k}}\frac{\dot{K}_{\vec{k}}V}{4\omega_{\vec{k}}^2}\\ &+\frac{\lambda}{4!}\int_{\vec{k}_i}\frac{3}{\omega_1+\omega_2}\frac{\dot{K}_2}{2\omega_2}\phi(\vec{k}_1)\phi(-\vec{k}_1)\\ &+\frac{\lambda}{4!}\int_{\vec{k}_i}\frac{3}{\omega_1+\omega_2}\frac{\dot{K}_2}{2\omega_2}\left(-\frac{\dot{K}_1 V}{2\omega_1}\right) \ .\end{aligned} \tag{4.3}$$

By using the flow equation for δm^2

$$\frac{\dot{\delta m^2}}{2} = -\frac{\lambda}{4!}\int_{\vec{p}}\frac{6\dot{K}_{\vec{p}}}{2\omega_{\vec{p}}} \tag{4.4}$$

shown in the appendix, we can see that Eq.(4.2) agrees with Eq.(4.3). Therefore $\Psi_\Lambda = e^I\Psi_0$ in Eq.(3.19) satisfies the ERG equation Eq.(2.25).

5 Conclusion and discussion

In this paper, we derived the ERG equation for the ground state wave functionals in scalar field theories from the ERG equation (the Polchinski equation). We checked the validity of the equation by performing a perturbative expansion up to the first order. Here we developed a systematic method for the perturbative expansion of the wave functionals.

While we considered the Polchinski equation in this paper, we could have started with other ERG equations. For instance, we can set the seed action $\hat{S}$ in (2.7) to S_Λ. Then, the ERG equation for the wave functional reads

$$-\Lambda \frac{\partial}{\partial \Lambda} \Psi_\Lambda = \frac{1}{2} \int_{\vec{p}} \dot{C}_\Lambda(0, \vec{p}) \left\{ \frac{\delta^2 \Psi_\Lambda}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} + \frac{1}{\Psi_\Lambda} \frac{\delta \Psi_\Lambda}{\delta\varphi(\vec{p})} \frac{\delta \Psi_\Lambda}{\delta\varphi(-\vec{p})} \right\} . \tag{5.1}$$

As discussed in the introduction, the wave functionals that are solutions to Eqs.(2.25) or (2.34) with Eq.(2.33) are considered to represent continuum tensor networks. By using the solutions, we can calculate the Fisher information metric which measures the distance between the states at different energy scales. As discussed in Refs. [7, 11], we expect to extract dual bulk geometry from the Fisher information metric. It is also desirable to obtain the dependence of entanglement entropy on the energy scale by using the solutions[2].

We would like to develop a method to solve Eqs.(2.25) or (2.34) based on an approximation such as the derivative expansion or a numerical method, for instance, a wave functional analog of a method proposed in Ref. [48].

In this paper, we restricted ourselves to scalar field theories. Extension to gauge field theories is of course important (for a recent development in the ERG equation for the effective action in gauge field theories, see Ref. [49]).

We hope to report progress in the above-mentioned issues in the near future.

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[2]For the calculation of entanglement entropy in interacting field theories, see [36–47]

Appendix: Flow equation for δm^2

In this appendix, we derive the flow equation for δm^2 from Eq.(2.11) up to the first-order perturbation. By using Eq.(2.35), Eq.(2.11) is rewritten as

$$-\Lambda \frac{\partial S_{\text{int}}}{\partial \Lambda} = -\frac{1}{2} \int_p \frac{\dot{K}_{\vec{p}}}{2\omega_{\vec{p}}} \left(\frac{\delta^2 S_{\text{int}}}{\delta\phi(p)\delta\phi(-p)} - \frac{\delta S_{\text{int}}}{\delta\phi(p)} \frac{\delta S_{\text{int}}}{\delta\phi(-p)} \right) . \tag{A.1}$$

S_{int} can be read off from Eq.(3.5) as

$$S_{\text{int}} = \frac{\delta m^2}{2} \int_k \phi(k)\phi(-k) + \frac{\lambda}{4!} \int_{k_i} (2\pi)^{d+1} \delta(\Sigma_{i=1}^4 k_i) \prod_{i=1}^{4} \phi(k_i) . \tag{A.2}$$

We substitute Eq.(A.2) into Eq.(A.1) and focus on the ϕ^2 terms. The left-hand side of Eq.(A.1) gives

$$\frac{\delta \dot{m}^2}{2} \int_k \phi(k)\phi(-k) \tag{A.3}$$

The first term on the right-hand side of Eq.(A.1) gives

$$\begin{aligned} & -\frac{1}{2} \int_p \frac{\dot{K}_{\vec{p}}}{2\omega_{\vec{p}}} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \left\{ \frac{\lambda}{4!} \int_{k_i} (2\pi)^{d+1} \delta(\Sigma_{i=1}^4 k_i) \prod_{i=1}^{4} \phi(k_i) \right\} \\ & = -\frac{\lambda}{4} \int_k \frac{\dot{K}_{\vec{p}}}{2\omega_{\vec{k}}} \phi(k)\phi(-k) \end{aligned} \tag{A.4}$$

The second-term of Eq.(A.1) gives a term that is higher order in λ. Thus, from Eqs.(A.3) and (A.4), we obtain the flow equation for δm^2 up to the first order in λ as

$$\frac{\dot{\delta m^2}}{2} = -\frac{\lambda}{4!} \int_{\vec{p}} \frac{6\dot{K}_{\vec{p}}}{2\omega_{\vec{p}}} . \tag{A.5}$$

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