An Alternate Definition of the Dimension of a Vector Space Using Isomorphisms

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Abstract

The standard definition of the dimension of a vector space, the algebraic dimension, states that dimension is equal to the cardinality of any basis of that vector space. This definition requires an understanding of the concepts of basis, spanning set, and linear independence. We pose a new definition for the dimension of a vector space using linear isomorphisms, called the isomorphic dimension. In the finite case, for a vector space over field F, we define the isomorphic dimension of the vector space to be equal n if and only if there exists a linear isomorphism between F^n and that vector space. There is a similar definition in the infinite case. For any vector space, the isomorphic dimension equals the dimension, but the definition of the isomorphic dimension doesn't require the concepts of basis, spanning set, and linear independence. This allows for some fundamental linear algebra results to be seen more easily or to be proven more similarly to other algebraic proofs involving isomorphisms and homomorphisms. We follow this definition of isomorphic dimension with alternate versions of basis, spanning set, and linear independence in terms of linear transformations. While this definition of dimension is equivalent to the algebraic dimension and the corresponding results are well-known, this article provides an alternate approach to teaching dimension.

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1 Introduction

In the history of the epistemology of abstract vector space theory, the definition of dimension has developed to represent an enumeration of a set of elements satisfying certain conditions. Those conditions have evolved from various presentations including a maximal linearly independent set or a minimal generating set to the standard approach employed today where these notions meet.

The following is a brief recounting of some of the history of basis and dimension. In 1844 in [6] and subsequently in 1862 in[7], Grassmann defines dimension in terms of the theory of extension and creates an exchange method from which the exchange theorem can be derived. In 1888 in [13], Peano provided an early axiomatic definition of what we now call a vector space. He defined dimension as the maximum number of independent elements in a system and without connecting it to the notion of a generating set. In 1893 in the fourth edition of [3], Dedekind contributed a supplement defining a basis in the

context of field extensions as an irreducible, or linearly independent, generating set, and he proves a result equivalent to the exchange theorem and presents the size of the basis as an invariant number. In 1909 in [2], Burali-Forti and Marcolongo continued Peano's work and defined dimension as the maximum number of independent elements as well. In 1910 in [15], Steinitz gave an explicit definition of linear dependence and defined the dimension of a finite field extension to be the maximum number of linearly independent elements. He considered a basis to be a generating set for which every element has a unique linear expression. In his paper, he proved that the dimension is the minimal size of a generating set, provided results equivalent to the exchange theorem, and included in his theorems the invariance of the size of a basis. In 1918 in [18], Weyl defined linear vector-manifolds in an axiomatic approach, which is similar to the definition of vector spaces, and he similarly defines dimension as the maximum number of linearly independent vectors. In 1930, see [16] and [17], van der Waerden published a text with a general approach to modern algebra. In his book, van der Waerden used the exchange theorem and proved results involving bases, the invariance of the number of elements in a basis, and dimension. He also introduced the concepts of linear dependence and basis for modules. His book was based on lectures by Emil Artin and Emmy Noether. In 1921 in [12], Emmy Noether defined modules and the basis of a module and considered vector spaces as special cases of modules. See [4], [5], and [9] for a more detailed overview of this history.

In the exposition of abstract vector space theory leading up to dimension, linear combinations and spans are first defined. The spanning set and linear independence are subsequently defined. A basis of a vector space is then defined as a linearly independent spanning set of that vector space, or it is equivalently defined as a set of vectors for which all vectors in the vector space can be represented as unique linear combinations of this basis. The size of a linearly independent set is less than or equal to the size of a spanning set, and this is demonstrated to show the number of elements in a basis is invariant. The dimension of a vector space is then the cardinality of every basis of the vector space. From here, it can be proven that any two vector spaces of equal dimension are isomorphic. The linear algebra texts [1], [8], [10], and [14] were used for reference, and they are examples of texts that follow the above approach to dimension.

In this article, we present a different approach with an alternate definition for dimension called the isomorphic dimension. In the finite case, for a vector space over field F, we define the isomorphic dimension of the vector space to be equal n if and only if there exists a linear isomorphism between F^n and that vector space. There is a similar definition in the infinite case. This produces a reordering of the fundamental results of abstract vector space theory involving dimension. While the standard approach relies on sets of vectors of the vector space and elemental arguments, this approach focuses on linear transformations between vector spaces. With this alternate definition, we prove statements with functional arguments. Some arguments still require an appeal to elemental work, but most of these arguments involve vectors as outputs of the standard basis of F^n.

The standard definition of dimension requires students to learn the definitions of linear combination, span, spanning set, linear independence, and basis before dimension is mentioned. These definitions are also relatively particular to the context of vector spaces (or similar constructions). This alternate definition requires students to use general abstract algebra concepts of function, homomorphism (linearity in the case of vector spaces), injective, surjective, bijective, and isomorphism along with some elementary matrix theory. In this paper, alternate definitions are constructed for basis, linear combination, span, spanning set, and linear independence after dimension is established.

With the isomorphic definition, F^n becomes a focal vector space. When defining dimension in terms of isomorphisms with F^n, the isomorphic dimension of the zero vector space is equal to zero using the linear isomorphism $f : F^0 = \{\mathbf{0}\} \rightarrow V = \{\mathbf{0}_V\}$ such that $f(\mathbf{0}) = \mathbf{0}_V$. This definition provides a constructive way of determining the dimension of the zero vector space and does not rely on the abstraction of using the empty set as a basis for the zero vector space.

In our approach, some fundamental linear algebra results are seen more easily or are

proven more similarly to other algebraic proofs involving isomorphisms and homomorphisms. We provide the details of the proofs to show the concepts used.

2 Preliminaries

We assume the reader is familiar with the definitions of vector space, subspace, linear transformation, kernel, image, and quotient vector space.

The following theorems are well-known fundamental theorems in linear algebra. These theorems are elementary for many algebraic structures, but we state them here in the context of vector spaces. We use these theorems as some of the main building blocks in our development of the concept of dimension and its results.

Theorem 2.1. *Let V and W be vector spaces over field F, and let $f : V \to W$ be a linear transformation. The following are true.*

1. *The kernel of f is a subspace of V.*

2. *The image of a f is a subspace of W*

3. *The linear transformation f is injective if and only if $\ker(f) = \{\mathbf{0}_V\}$.*

4. *The linear transformation f is an isomorphism with an inverse linear transformation, f^{-1}, if and only if f is bijective.*

Theorem 2.2. *The following are true for compositions of linear transformations.*

1. *The composition of linear transformations is a linear transformation.*

2. *The composition of injective linear transformations is an injective linear transformation.*

3. *The composition of surjective linear transformations is a surjective linear transformation.*

4. *The composition of bijective linear transformations is a bijective linear transformation.*

5. *The composition of linear isomorphisms is a linear isomorphism.*

The First Isomorphism Theorem for vector spaces is as follows.

Theorem 2.3. *Let V and W be vector spaces over field F, and let $f : V \to W$ be a linear transformation. Then $\bar{f} : V/\ker(f) \to \operatorname{im}(f)$ such that $\bar{f}(v + \ker(f)) = f(v)$ for any $v \in V$ is a linear isomorphism.*

Proof. $\bar{f}$ is a well-defined linear transformation that is injective and surjective. Thus it is a linear isomorphism. $\qquad\blacksquare$

Let $\{e_1, \ldots, e_n\} \subset F^n$ for field F be the standard basis of F^n. For every $i = 1, \ldots, n$, e_i has each jth coordinate equal to 0 for $j \neq i$ and the ith coordinate equal to 1. Here the name standard basis is used, but we will only use this as a label without any specific meaning attached to the term basis.

The following theorem is a known elementary theorem that is true by the definition of linear transformation and because for any $(x_1, \ldots, x_n) \in F^n$, $(x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n$.

Theorem 2.4. *Let V be a vector space over field F. Then for any set or multiset $\{v_1, \ldots, v_n\}$ for $v_1, \ldots, v_n \in V$, $f : F^n \to V$ such that*

$$f(x_1, \ldots, x_n) = x_1 v_1 + \cdots + x_n v_n$$

for all $x_1, \ldots, x_n \in F$ is the unique linear transformation such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$.

When $n = 0$, $F^0 = \{\mathbf{0}\}$, and the only linear transformation $f : \{\mathbf{0}\} \to V$ is defined by $f(\mathbf{0}) = \mathbf{0}_V$.

3   Isomorphic Dimension in the Finite Case

One of the difficulties in proving the dimension of a vector space is well-defined is in showing that any two bases of V contain the same number of vectors. This is made simpler with our alternate definition and using elementary matrix theory.

Lemma 3.1. *Let F be a field, and let n and m be nonnegative integers. If $f : F^n \to F^m$ is an injective linear transformation, then $n \leq m$.*

When $n = 0$, we mean $F^0 = \{\mathbf{0}\}$.

In the following proof, we will interchangeably write vectors of F^n as n-tuples and the corresponding $n \times 1$ columns.

Proof. If $n = 0$, then $0 \leq m$ is trivially true.

If $m = 0$, then $f : F^n \to F^0$ is the linear transformation with $f(x_1, \ldots, x_n) = \mathbf{0}$ for all $(x_1, \ldots, x_n) \in F^n$, and for positive n, $f(1, \ldots, 1) = \mathbf{0} = f(0, \ldots, 0)$. This contradicts f being injective. So $n = m = 0$ in this case.

Now consider $n, m > 0$, and assume $n > m$. For any $(x_1, \ldots, x_n) \in F^n$,

$$f(x_1, \ldots, x_n) = x_1 f(e_1) + \cdots + x_n f(e_n)$$

because f is a linear transformation. Therefore,

$$f(x_1, \ldots, x_n) = \begin{bmatrix} f(e_1) & \cdots & f(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Here $A = \begin{bmatrix} f(e_1) & \cdots & f(e_n) \end{bmatrix}$ is an $m \times n$ matrix in $M(m \times n, F)$.

The kernel of f is equal to $\{X \in F^n | AX = \mathbf{0}_{m \times 1}\}$. From elementary matrix theory, we know that $\ker(f) = \{X \in F^n | \mathrm{rref}(A)X = \mathbf{0}_{m \times 1}\}$, and since $n > m$, the number of independent variables in the solution set of $\mathrm{rref}(A)X = \mathbf{0}_{m \times 1}$ is at least one. Therefore, $\mathbf{0}_{n \times 1}$ and some nonzero vector in F^n both have output $\mathbf{0}_{m \times 1}$, and f is not injective. This is a contradiction, so we must have $n \leq m$. $\qquad\blacksquare$

Proposition 3.2. *Let V be a vector space over field F. If $f_m : F^m \to V$ is an injective linear transformation and $f_n : F^n \to V$ is a linear isomorphism, then $m \leq n$.*

Proof. $f_n^{-1} \circ f_m : F^m \to F^n$ is an injective linear transformation, and by Lemma 3.1, $m \leq n$. $\qquad\blacksquare$

Corollary 3.3. *Let V be a vector space over field F. If $f_m : F^m \to V$ is a linear isomorphsim and $f_n : F^n \to V$ is a linear isomorphism, then $m = n$.*

Proof. By Proposition 3.2, $m \leq n$ and $n \leq m$, so $m = n$. $\qquad\blacksquare$

If there exists a linear isomorphism between F^n and V, then n is unique.

Definition 3.4. *Let V be a vector space over field F. The **isomorphic dimension** of V is n, labeled $\dim_I(V) = n$, if and only if there exists a linear isomorphism $f : F^n \to V$.*

When $n = 0$, $F^0 = \{\mathbf{0}\}$, and $f : \{\mathbf{0}\} \to V$ is a linear isomorphism if and only if $f(\mathbf{0}) = \mathbf{0}_V$ and $V = \{\mathbf{0}_V\}$. Therefore $\dim_I(V) = 0$ if and only if $V = \{\mathbf{0}_V\}$.

For any nonnegative integer n, $\dim_I(F^n) = n$ using the identity function on F^n.

Let V be a vector space over field F. If $\dim_I(V) = n$ for some nonnegative integer n, then V is **finite-dimensional**, and if for any nonnegative integer n, there does not exists an isomorphism $f : F^n \to V$, then V is **infinite-dimensional**.

The isomorphic definition of dimension allows for a short proof of the following.

Theorem 3.5. *Let V and W be vector spaces over field F, and let V be finite-dimensional. There exists linear isomorphism $f : V \to W$ if and only if W is finite-dimensional and $\dim_I(V) = \dim_I(W)$.*

Proof. Let $\dim_I(V) = n$. There exists linear isomorphism $f_V : F^n \to V$.

If $f : V \to W$ is a linear isomorphism, then $f \circ f_V : F^n \to W$ is a linear isomorphism. Therefore W is finite-dimensional and $\dim_I(W) = n$.

If W is finite-dimensional and $\dim_I(W) = n$, then there exists linear isomorphism $f_W : F^n \to W$, and $f_W \circ f_V^{-1} : V \to W$ is a linear isomorphism. $\square$

We also say V and W are isomorphic if and only if there exists a linear isomorphism between them. If V is finite-dimensional, V and W are isomorphic if and only if $\dim_I(V) = \dim_I(W)$.

For every pair of nonnegative integers (i, j), define $p_i^j : F^i \to F^j$ such that for any $(x_1, \ldots, x_i) \in F^i$, $p_i^j(x_1, \ldots, x_i) = (x_1, \ldots, x_i, 0, \ldots, 0)$ if $i \leq j$ and $p_i^j(x_1, \ldots, x_i) = (x_1, \ldots, x_j)$ if $i \geq j$.

Lemma 3.6. *Let V be a vector space over field F. Let $f_n : F^n \to V$ be a linear transformation with $v_i = f_n(e_i)$ for all $i = 1, \ldots, n$. If f_n is injective but not surjective, then there exists $v_{n+1} \in V$ with $v_{n+1} \notin \operatorname{im}(f_n)$, and for any such $v_{n+1} \in V$, there exists an injective linear transformation $f_{n+1} : F^{n+1} \to V$ with $f_{n+1}(e_i) = v_i$ for all $i = 1, \ldots, n+1$, $f_{n+1} \circ p_n^{n+1} = f_n$, and $\operatorname{im}(f_n) \subsetneq \operatorname{im}(f_{n+1})$.*

In this statement, $\{e_1, \ldots, e_n\}$ is the standard basis of F^n when used in the context of $f_n : F^n \to V$, and $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of F^{n+1} when used in the context of $f_{n+1} : F^{n+1} \to V$.

Proof. If $f_n : F^n \to V$ is not surjective, there exists $v_{n+1} \in V$ such that $v_{n+1} \notin \operatorname{im}(f_n)$. Then for any $a \in F$, $av_{n+1} \in \operatorname{im}(f_n)$ if and only if $a = 0$ because $\operatorname{im}(f_n)$ is a subspace.

Define $f_{n+1} : F^{n+1} \to V$ to be the linear transformation such that $f_{n+1}(e_i) = v_i$ for all $i = 1, \ldots, n+1$. It follows that $f_{n+1} \circ p_n^{n+1} = f_n$ and $\operatorname{im}(f_n) \subset \operatorname{im}(f_{n+1})$ with $\operatorname{im}(f_n) \neq \operatorname{im}(f_{n+1})$ because $v_{n+1} \notin \operatorname{im}(f_n)$ but $v_{n+1} \in \operatorname{im}(f_{n+1})$.

For any $(x_1, \ldots, x_{n+1}) \in \ker(f_{n+1})$,

$$f_n(x_1, \ldots, x_n) + x_{n+1} v_{n+1} = \mathbf{0},$$

which implies $f_n(x_1, \ldots, x_n) = -x_{n+1} v_{n+1}$. Therefore, $f_n(x_1, \ldots, x_n) = \mathbf{0}_V$ and $x_{n+1} = 0$. Since f_n is injective, $(x_1, \ldots, x_n) = (0, \ldots, 0)$, and $(x_1, \ldots, x_n, x_{n+1}) = (0, \ldots, 0, 0)$. This proves f_{n+1} is injective. $\square$

Corollary 3.7. *Let V be a finite-dimensonal vector space over field F with $\dim_I(V) = n$. If $f : F^n \to V$ is an injective linear transformation, then f is a linear isomorphism.*

Proof. If f is not surjective, then by Lemma 3.6, there exists injective linear transformation $f_{n+1} : F^{n+1} \to V$. By Proposition 3.2, this is a contradiction, so f must be surjective and a linear isomorphism. $\square$

Theorem 3.8. *Let V be a vector space over field F.*

V is finite-dimensional with dimension n if and only if there exists a sequence of injective linear transformations:

$$f_0 : F^0 \to V, \, f_1 : F^1 \to V, \ldots, \, f_n : F^n \to V$$

such that $f_{k+1} \circ p_k^{k+1} = f_k$ and $\operatorname{im}(f_k) \subsetneq \operatorname{im}(f_{k+1})$ for any $k \leq n - 1$ and $\operatorname{im}(f_n) = V$.

V is infinite-dimensional if and only if there exists an infinite sequence of injective linear transformations:

$$f_0 : F^0 \to V, \, f_1 : F^1 \to V, \ldots, \, f_n : F^n \to V, \ldots$$

such that $f_{k+1} \circ p_k^{k+1} = f_k$ and $\operatorname{im}(f_k) \subsetneq \operatorname{im}(f_{k+1})$ for any $k \geq 0$.

Proof. For any vector space V, there exists the injective linear transformation $f_0 : F^0 \to V$ defined by $f(\mathbf{0}) = \mathbf{0}_V$. If $V = \{\mathbf{0}_V\}$, then f_0 satisfies the statement of the theorem. If V is nonzero, there exists nonzero $v_1 \in V$, and there exists the injective linear transformation $f_1 : F^1 \to V$ defined by $f(ae_1) = av_1$ for any $a \in F$. By Lemma 3.6, we can continue building injective linear transformations. The process terminates if we reach an isomorphism.

For any such sequence, using Proposition 3.2, V is finite-dimensional with dimension n if and only if this sequence ends with isomorphism $f_n : F^n \to V$, and V is infinite-dimensional if and only if there is no such isomorphism, and the sequence never ends. $\qquad\square$

This lemma creates a chain of subspaces of V in the form of the images of the injective linear transformations, and it is reminiscent of the length of a module, which is a generalization of dimension.

Theorem 3.9. *Let V be a finite-dimensional vector space over field F, and let U be a subspace of V. Then U is finite-dimensional and $\dim_I(U) \leq \dim_I(V)$.*

Proof. Let $\dim_I(V) = n$. There exists a linear isomorphism $f : F^n \to V$. Let $h : F^k \to U$ be an injective linear transformation for some nonnegative integer k. We know such an injective linear transformation exists for at least $k = 0$ defined by $h(\mathbf{0}) = \mathbf{0}_V$.

Let $\mathrm{inc} : U \to V$ be the injective linear transformation $\mathrm{inc}(u) = u$ for any $u \in U$. Then $f^{-1} \circ \mathrm{inc} \circ h : F^k \to F^n$ is an injective linear transformation. By Lemma 3.1, $k \leq n$. Therefore U is finite-dimensional by Theorem 3.8, and $\dim_I(U) \leq \dim_I(V)$. $\qquad\square$

Theorem 3.10. *Let V be a finite-dimensional vector space over field F, and let U be a subspace of V. $U = V$ if and only if $\dim_I(U) = \dim_I(V)$.*

Proof. Since U is a subspace of V, $U \subset V$.

If $U = V$, then $\dim_I(U) = \dim_I(V)$ trivially.

Let $\dim_I(U) = \dim_I(V) = n$. There exists a linear isomorphism $f : F^n \to U$ because $\dim_I(U) = n$. Let $\mathrm{inc} : U \to V$ be the injective linear transformation $\mathrm{inc}(u) = u$ for any $u \in U$. Then $\mathrm{inc} \circ f : F^n \to V$ is an injective linear transformation. By Corollary 3.7, $\mathrm{inc} \circ f$ is a linear isomorphism. Since f is also a linear isomorphism, $\mathrm{inc} = (\mathrm{inc} \circ f) \circ f^{-1}$ is a linear isomorphism, and inc is surjective. Thus $U = V$. $\qquad\square$

Lemma 3.11. *Let V be a finite-dimensional vector space over field F with dimension n, and let U be a subset of V. U is a subspace of V with dimension k if and only if there exists a sequence of injective linear transformations:*

$$f_0 : F^0 \to V, \, f_1 : F^1 \to V, \ldots, f_n : F^n \to V$$

such that $f_{m+1} \circ p_m^{m+1} = f_m$ and $\mathrm{im}(f_m) \subsetneq \mathrm{im}(f_{m+1})$ for any $m \leq n-1$, $\mathrm{im}(f_n) = V$, and $\mathrm{im}(f_k) = U$.

Proof. If $\mathrm{im}(f_k) = U$ for some nonnegative integer $k \leq n$, then U is a subspace of V because the image of a linear transformation is a subspace of the codomain. As a consequence of the First Isomorphism Theorem U is isomorphic to F^k and $\dim_I(U) = k$.

If U is a subspace of V with $\dim_I(U) = k$, then $k \leq n$. By Theorem 3.8, there exists a sequence of injective linear transformations:

$$h_0 : F^0 \to U, \, h_1 : F^1 \to U, \ldots, h_k : F^n \to U$$

such that $h_{m+1} \circ p_m^{m+1} = h_m$ and $\mathrm{im}(h_m) \subsetneq \mathrm{im}(h_{m+1})$ for any $m \leq k-1$ and $\mathrm{im}(h_k) = U$.

Let $\mathrm{inc} : U \to V$ be the injective linear transformation such that $\mathrm{inc}(u) = u$ for any $u \in U$, and define $f_i = \mathrm{inc} \circ h_i : F^i \to V$. Therefore, $\mathrm{im}(f_k) = U$. Using Lemma 3.6, we continue constructing injective linear transformations, until we reach the linear isomorphism $f_n : F^n \to V$. $\qquad\square$

Lemma 3.12. *Let V and W be vector spaces over field F, let $f : V \to W$ be a linear isomorphism, and let U be a subspace of V. Then $\bar{f} : V/U \to W/f(U)$ such that $\bar{f}(v+U) = f(v) + f(U)$ for any $v \in V$ is a linear isomorphism.*

Proof. $\bar{f}$ is a well-defined linear transformation that is injective and surjective. Thus it is a linear isomorphism. □

Theorem 3.13. *Let V be a finite-dimensional vector space over field F, and let U be a subspace of V. V/U is finite-dimensional with $\dim_I(V/U) = \dim_I(V) - \dim_I(U)$.*

Proof. Let $\dim_I(V) = n$. If U is a subspace of V, then U is finite-dimensional with $\dim_I(U) = k \leq n$. By Lemma 3.11, there exists a sequence of injective functions:

$$f_0 : F^0 \to V, \, f_1 : F^1 \to V, \ldots, \, f_n : F^n \to V$$

such that $f_{m+1} \circ p_m^{m+1} = f_m$ and $\operatorname{im}(f_m) \subsetneq \operatorname{im}(f_{m+1})$ for any $m \leq n - 1$, $\operatorname{im}(f_n) = V$, and $\operatorname{im}(f_k) = U$. Then $f_n(p_k^n(F^k)) = U$ for $p_k^n : F^k \to F^n$ such that $p_k^n(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$ for all $(x_1, \ldots, x_k) \in F^k$.

By Lemma 3.12, there exists an isomorphism between $F^n/p_k^n(F^k)$ and $V/f_n(p_k^n(F^k))$ with $f_n(p_k^n(F^k)) = U$. By Theorem 3.5, $\dim_I(V/U) = \dim_I(F^n/p_k^n(F^k))$, which equals $n - k$ using the linear isomorphism $h : F^{n-k} \to F^n/p_k^n(F^k)$, such that $h(x_{k+1}, \ldots, x_n) = (0, \ldots, 0, x_{k+1}, \ldots, x_n) + p_k^n(F^k)$ for all $(x_{k+1}, \ldots, x_n) \in F^{n-k}$. □

Corollary 3.14. *Let V be a finite-dimensional vector space over field F with $\dim_I(V) = n$. If $f : F^n \to V$ is a surjective linear transformation, then f is a linear isomorphism.*

Proof. By the First Isomorphism Theorem, $F^n/\ker(f)$ is isomorphic to V. By Theorem 3.5, $\dim_I(F^n/\ker(f)) = \dim(V) = n$. By Theorem 3.13, $\dim_I(F^n/\ker(f)) = \dim_I(F^n) - \dim_I(\ker(f)) = n - \dim_I(\ker(f))$, so $n - \dim_I(\ker(f)) = n$. Therefore $\dim_I(\ker(f)) = 0$ and $\ker(f) = \{\mathbf{0}\}$, so f is injective and a linear isomorphism. □

Theorem 3.15. *Let V and W be vector spaces over field F and such that W is finite-dimensional. There exists an injective linear transformation $f : V \to W$ if and only if V is finite-dimensional and $\dim_I(V) \leq \dim_I(W)$.*

Proof. Let $\dim_I(W) = m$. There exists linear isomorphism $f_W : F^m \to W$.

If V is finite-dimensional, then let $\dim_I(V) = n$ and there exists a linear isomorphism $f_V : F^n \to V$. If $n \leq m$, then $p_n^m : F^n \to F^m$ is the injective linear transformation $p_n^m(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$ for any $(x_1, \ldots, x_n) \in F^n$, and $f_W \circ p_n^m \circ f_V^{-1} : V \to W$ is an injective linear transformation.

If $f : V \to W$ is an injective linear transformation, then by the First Isomorphism Theorem, V is isomorphic to $\operatorname{im}(f)$. Since $\operatorname{im}(f)$ is a subspace of finite-dimensional W, it is finite-dimensional and its dimension is less than or equal to m by Theorem 3.9. By Theorem 3.5, V is finite-dimensional with $\dim_I(V) = \dim_I(\operatorname{im}(f))$. Therefore $\dim_I(V) \leq \dim_I(W)$. □

Theorem 3.16. *Let V and W be vector spaces over field F and such that V be finite-dimensional. There exists surjective linear transformation $f : V \to W$ if and only if W is finite-dimensional and $\dim_I(V) \geq \dim_I(W)$.*

Proof. Let $\dim_I(V) = n$. There exists linear isomorphism $f_V : F^n \to W$.

If W is finite-dimensional, then let $\dim_I(W) = m$ and there exists linear isomorphism $f_W : F^m \to W$. If $n \geq m$, then $p_n^m : F^n \to F^m$ is the surjective linear transformation $p_n^m(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_n) \in F^n$, and $f_W \circ p_n^m \circ f_V^{-1} : V \to W$ is a surjective linear transformation.

If $f : V \to W$ is a surjective linear transformation, then by the First Isomorphism Theorem, W is isomorphic to $V/\ker(f)$. Since $\ker(f)$ is a subspace of finite-dimensional V, $V/\ker(f)$ is finite-dimensional with $\dim_I(V/\ker(f)) = \dim_I(V) - \dim_I(\ker(f))$ by Theorem 3.13. By Theorem 3.5, W is finite-dimensional with $\dim_I(W) = \dim_I(V) - \dim_I(\ker(f))$. Therefore $\dim_I(V) \geq \dim_I(W)$. □

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The following theorem is the Rank Nullity Theorem using the definition of isomorphic dimension.

Theorem 3.17. *Let V and W be vector spaces over field F, let V be finite-dimensional, and let $f : V \to W$ be a linear transformation.*

$$\dim_I(\ker(f)) + \dim_I(\operatorname{im}(f)) = \dim_I(V)$$

Proof. By the first isomorphism theorem, $\bar{f} : V/\ker(f) \to \operatorname{im}(f)$ is a linear isomorphism, and by Theorem 3.5, $\dim_I(V/\ker(f)) = \dim_I(\operatorname{im}(f))$. By Theorem 3.13, $\dim_I(V) - \dim_I(\ker(f)) = \dim_I(\operatorname{im}(f))$. □

4 Isomorphic Basis in the Finite Case

In a similar fashion to our latest definition of dimension, we can also construct a new definition for basis that is equivalent to the algebraic definition of a basis.

Definition 4.1. *Let V be a finite-dimensional vector space over field F. The set (or multiset) $\{v_1, \ldots, v_n\}$ for $v_1, \ldots, v_n \in V$ is an **isomorphic basis** of V if and only if the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is an isomorphism.*

The standard basis of F^n, $\{e_1, \ldots, e_n\}$, is also an isomorphic basis of F^n according to this definition using the identity map.

Let V be a vector space over field F. Since a linear transformation with domain F^n is uniquely determined by the outputs of the standard basis, for $\dim_I(V) = n$, there is a one-to-one correspondence between the linear isomorphisms $f : F^n \to V$ and the ordered bases of V.

Theorem 4.2. *Let V be a vector space over field F. Then V is finite-dimensional with $\dim_I(V) = n$ if and only if there exists an isomorphic basis of V containing exactly $\dim_I(V)$ vectors.*

Proof. If $\dim_I(V) = n$, then there exists a linear isomorphism $f : F^n \to V$. Then $\{f(e_1), \ldots, f(e_n)\}$ is an isomorphic basis of V by definition.

If $\{v_1, \ldots, v_n\}$ is an isomorphic basis of V, then the linear transformation $f : F^n \to V$ defined by $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is an isomorphism. Since there exists a linear isomorphism between F^n and V, V is finite-dimensional and $\dim_I(V) = n$. □

If V is a finite-dimensional vector space, $\dim_I(V)$ is unique, so we get the following corollary.

Corollary 4.3. *Let V be a finite-dimensional vector space over field F. Every isomorphic basis of V contains exactly $\dim_I(V)$ vectors.*

When $\dim_I(V) = 0$, the only isomorphic basis of V is the empty set.

Definition 4.4. *Let V be a vector space over field F. For the set (or multiset) $\{v_1, \ldots, v_n\}$ with $v_1, \ldots, v_n \in V$, a **linear combination** of $\{v_1, \ldots, v_n\}$ is an output of the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$.*

This means that a linear combination is $f(x_1, \ldots, x_n) = x_1 v_1 + \cdots + x_n v_n$ for some $x_1, \ldots, x_n \in F$, which is the same as the usual definition.

Theorem 4.5. *Let V be a finite-dimensional vector space over field F, and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ for some $v_1, \ldots, v_n \in V$. Then $\mathcal{B}$ is an isomorphic basis of V if and only if every vector in V is a unique linear combination of $\mathcal{B}$.*

Proof. Let $f : F^n \to V$ be the linear transformation such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$.

If $\mathcal{B}$ is an isomorpic basis, then f is an isomorphism and every vector in V is a unique output for some input $(x_1, \ldots, x_n) \in F^n$, which means every vector in V is a unique linear combination $x_1 v_1 + \cdots + x_n v_n$ for that $(x_1, \ldots, x_n) \in F^n$.

If every vector in V is a unique linear combination of $\mathcal{B}$, then f is surjective because every vector in V is an output of f, and f is injective because $f(0, \ldots, 0) = 0v_1 + \cdots + 0v_n = \mathbf{0}_V$ is unique, which implies $\ker(f) = \{\mathbf{0}\}$. Thus f is an isomorphism, and $\mathcal{B}$ is an isomorphic basis. $\qquad\square$

5 Surjective Set in the Finite Case

Definition 5.1. *Let V be a vector space over field F. For the set (or multiset) $S = \{v_1, \ldots, v_n\}$ for $v_1, \ldots, v_n \in V$, the* **span** *of S is the image of the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$, or in other words, $\mathrm{span}(S) = \mathrm{im}(f)$.*

This means that $\mathrm{span}(v_1, \ldots, v_n) = \{x_1 v_1 + \cdots + x_n v_n | x_1, \ldots, x_n \in F\}$, which is the same as the usual definition.

Definition 5.2. *Let V be a vector space over field F. The set (or multiset) $\{v_1, \ldots, v_n\}$ for $v_1, \ldots, v_n \in V$ is a* **surjective set** *in V if and only if the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is surjective.*

Therefore, $S = \{v_1, \ldots, v_n\}$ is a surjective set in V if and only if $\mathrm{span}(S) = V$. The linear transformation $f : F^n \to V$ defined by $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is also surjective if and only if S is a spanning set of V. Therefore the definitions of surjective set in V and spanning set of V are equivalent (for finite-dimensional V as currently defined).

Theorem 5.3. *Let V be a vector space over field F, and let $S = \{v_1, \ldots, v_n\}$ be a set or multiset of vectors in V. Then S is a spanning set of V if and only if S is a surjective set in V.*

We can prove the results involving spanning sets for surjective sets using the definition of surjective sets.

Theorem 5.4. *Let V be a vector space over field F. V is finite-dimensional if and only if there exists a finite surjective set in V.*

Proof. If V is finite-dimensional with $\dim_I(V) = n$, then there exists a linear isomorphism $f : F^n \to V$, and $\{f(e_1), \ldots, f(e_n)\}$ is a surjective set in V.

If $\{v_1, \ldots, v_n\}$ is a surjective set in V, then there exists a surjective linear transformation $g : F^n \to V$ such that $g(e_i) = v_i$ for all $i = 1, \ldots, n$. By the First Isomorphism Theorem, there exists a linear isomorphism between $F^n / \ker(g)$ and V. Since $F^n / \ker(g)$ is finite-dimensional by Theorem 3.13, V is finite-dimensional by Theorem 3.5. $\qquad\square$

Theorem 5.5. *Let V be a finite-dimensional vector space over field F, and let $\{v_1, \ldots, v_m\}$ be a surjective set in V. Then $m \geq \dim_I(V)$.*

Proof. Let $\dim_I(V) = n$. There exists a linear isomorphism $f : F^n \to V$. Since $\{v_1, \ldots, v_m\}$ is a surjective set in V, there exists a surjective linear transformation $g : F^m \to V$ such that $g(e_i) = v_i$ for all $i = 1, \ldots, m$.

$f^{-1} \circ g : F^m \to F^n$ is a surjective linear transformation. By the First Isomorphism Theorem, there exists a linear isomorphism between $F^m / \ker(f^{-1} \circ g)$ and F^n. Since $F^m / \ker(f^{-1} \circ g)$ is finite-dimensional with dimension $m - \dim_I(\ker(f^{-1} \circ g)$ by Theorem 3.13, $m - \dim_I(\ker(f^{-1} \circ g) = n$ by Theorem 3.5 and $m \geq n$. $\qquad\square$

Theorem 5.6. *Let V be a finite-dimensional vector space over field F with $\dim_I(V) = n$, and let $\{v_1, \ldots, v_n\}$ be a surjective set in V, then $\{v_1, \ldots, v_n\}$ is an isomorphic basis of V.*

Proof. Since $\{v_1, \ldots, v_n\}$ is a surjective set in V, the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is surjective. By Corollary 3.14, f is a linear isomorphism, and thus $\{v_1, \ldots, v_n\}$ is an isomorphic basis. $\qquad\square$

Theorem 5.7. *Let V be a finite-dimensional vector space over field F with $\dim_I(V) = n$, and let $\{v_1, \ldots, v_m\}$ be a surjective set in V, then there exists some subset $\{v_{i_1}, \ldots, v_{i_n}\} \subset \{v_1, \ldots, v_m\}$ such that $\{v_{i_1}, \ldots, v_{i_n}\}$ is an isomorphic basis of V.*

Proof. Since $\{v_1, \ldots, v_m\}$ is a surjective set in V, the linear transformation $f : F^m \to V$ defined by $f(e_i) = v_i$ for all $i = 1, \ldots, m$ is surjective. By Theorem 5.5, $m \geq n$.

If $\ker(f) = \{\mathbf{0}\}$, then f is bijective and an isomorphism, and $m = n$ and $\{v_1, \ldots, v_m\}$ is an isomorphic basis. If $\ker(f) \neq \{\mathbf{0}\}$, then $m > n$ and there exists nonzero $(x_1, \ldots, x_m) \in \ker(f)$ with some nonzero coordinate. Without loss of generality, assume $x_m \neq 0$. Define $f_{m-1} : F^{m-1} \to V$ by $f_{m-1}(e_i) = v_i$ for all $i = 1, \ldots, m-1$. Since f is surjective, for any $v \in V$, there exists $(a_1, \ldots, a_m) \in F^m$ such that $f(a_1, \ldots, a_m) = a_1 v_1 + \cdots + a_m v_m = v$. If $a_m = 0$, then $f_{m-1}(a_1, \ldots, a_{m-1}) = a_1 v_1 + \cdots + a_{m-1} v_{m-1} = v$. If $a_m \neq 0$, define $(b_1, \ldots, b_m)$ by

$$(b_1, \ldots, b_m) = (a_1, \ldots, a_m) - \frac{a_m}{x_m}(x_1, \ldots, x_m) = (a_1 - \frac{a_m}{x_m}, \ldots, a_{m-1} - \frac{a_m}{x_m}, 0).$$

Thus $f(b_1, \ldots, b_m) = f(a_1, \ldots, a_m)$ because $(x_1, \ldots, x_m) \in \ker(f)$, and since $b_m = 0$, $f_{m-1}(b_1, \ldots, b_{m-1}) = v$. Therefore f_{m-1} is surjective.

Continue with this process a total of $m - n$ times until $f_n : F^n \to V$ is the linear transformation $f_n(e_i) = v_i$ for all $i = i_1, \ldots, i_n$ and f_n is surjective. By Corollary 3.14, f_n is a linear isomorphism, and $\{v_{i_1}, \ldots, v_{i_n}\}$ is a basis. $\square$

6   Injective Set in the Finite Case

Definition 6.1. *Let V be a vector space over field F. The set (or multiset) $\{v_1, \ldots, v_n\}$ for $v_1, \ldots, v_n \in V$ is an **injective set** in V if and only if the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is injective.*

If $n = 0$ and $\{v_1, \ldots, v_n\} = \emptyset$, then $F^0 = \{\mathbf{0}\}$ and the linear transformation $f : \{\mathbf{0}\} \to V$ is defined by $f(\mathbf{0}) = \mathbf{0}_V$, so f is injective. This implies the empty set is an injective set in V.

Theorem 6.2. *Let V be a vector space over a field F, and let $S = \{v_1, \ldots, v_n\}$ be a set or multiset of vectors in V. Then S is a linearly independent set in V if and only if S is an injective set in V.*

Proof. Let $f : F^n \to V$ be the linear transformation defined by $f(e_i) = v_i$ for all $i = 1, \ldots, n$. Then $f(x_1, \ldots, x_n) = x_1 v_1 + \cdots + x_n v_n$ for all $x_1, \ldots, x_n \in F$ and $(x_1, \ldots, x_n) \in \ker(f)$ if and only if $x_1 v_1 + \cdots + x_n v_n = \mathbf{0}_V$.

If S is a linearly independent set in V, then $x_1 v_1 + \cdots + x_n v_n = \mathbf{0}_V$ implies $(x_1, \ldots, x_n) = (0, \ldots, 0)$, $\ker(f) = \{(0, \ldots, 0)\}$, and f is injective, so S is an injective set in V. If S is an injective set in V, then f is injective and $\ker(f) = \{(0, \ldots, 0)\}$, and $x_1 v_1 + \cdots + x_n v_n = \mathbf{0}_V$ implies $(x_1, \ldots, x_n) = (0, \ldots, 0)$. $\square$

We can prove the results involving linearly independent set for injective sets using the definition of injective sets.

Theorem 6.3. *Let V be a finite-dimensional vector space over field F. If $\{v_1, \ldots, v_k\}$ is an injective set in V, then $k \leq \dim_I(V)$.*

Proof. Let $\dim_I(V) = n$. There exists a linear isomorphism $f : F^n \to V$. Since $\{v_1, \ldots, v_k\}$ is an injective set in V, there exists an injective linear transformation $g : F^k \to V$ such that $g(e_i) = v_i$ for all $i = 1, \ldots, m$.

$f^{-1} \circ g : F^k \to F^n$ is an injective linear transformation. By Lemma 3.1, $k \leq n$. $\square$

Corollary 6.4. *Let V be a finite-dimensional vector space over field F. If $\{v_1, \ldots, v_k\}$ is an injective set in V and $\{w_1, \ldots, w_m\}$ is a surjective set in V, then $k \leq m$.*

Proof. By Theorem 5.5, $m \geq \dim_I(V)$. By Theorem 6.3, $k \leq \dim_I(V)$. Therefore $k \leq m$. $\square$

Theorem 6.5. *Let V be a finite-dimensional vector space over field F with $\dim_I(V) = n$, and let $\{v_1, \ldots, v_n\}$ be an injective set in V, then $\{v_1, \ldots, v_n\}$ is an isomorphic basis of V.*

Proof. Since $\{v_1, \ldots, v_n\}$ is an injective set in V, the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is injective. By Corollary 3.7, f is a linear isomorphism, and $\{v_1, \ldots, v_n\}$ is an isomorphic basis of V. $\square$

Theorem 6.6. *Let V be a finite-dimensional vector space over field F with $\dim_I(V) = n$, and let $\{v_1, \ldots, v_m\}$ be an injective set in V. There exist $v_{m+1}, \ldots, v_n \in V$ such that $\{v_1, \ldots, v_n\}$ is an isomorphic basis of V.*

Proof. Since $\{v_1, \ldots, v_m\}$ is an injective set in V, the linear transformation $f : F^m \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, m$ is injective.

If $m = n$, then $\{v_1, \ldots, v_m\}$ is an isomorphic basis of V by the previous theorem. If $m < n$, then f is not an isomorphism, and there exists $v_{m+1} \in V$ such that $v_{m+1} \notin \text{im}(f)$. Define the linear transformation $f_{m+1} : F^{m+1} \to V$ by $f_{m+1}(e_i) = v_i$ for all $i = 1, \ldots, m+1$. By Lemma 3.6, f_{m+1} is injective. If $m + 1 = n$, then $\{v_1, \ldots, v_m, v_{m+1}\}$ is an isomorphic basis of V by the previous theorem. If $m + 1 < n$, continue with this process $n - m$ times. Then $\{v_1, \ldots, v_{m+n-m} = v_n\}$ is an isomorphic basis of V by the previous theorem. $\square$

Definition 6.7. *Let V be a vector space over field F. The set (or multiset) $\{v_1, \ldots, v_n\}$ for $v_1, \ldots, v_n \in V$ is a **noninjective set** in V if and only if the linear transformation $f : F^n \to V$ such that $f(e_i) = v_i$ for all $i = 1, \ldots, n$ is not injective.*

7 Preliminaries for the Infinite Case

For set A, we denote the cardinality of A by $|A|$. For sets A and B, $|A| \leq |B|$ if and only if there exists an injective function from A to B, and $|B| \geq |A|$ if and only if there exists a surjective function from B to A. Then $|A| \leq |B|$ if and only if $|B| \geq |A|$. Also $|A| = |B|$ if and only if there exists a bijective function from A to B.

We used [11] as a reference.

The following is the Axiom of Choice.

Axiom 7.1. *For any set A, there exists a function $f : A \to A$, such that for any nonempty subset $X \subset A$, $f(X) \subset X$.*

The following is the Well-Ordering Theorem, and it is equivalent to the Axiom of Choice.

Theorem 7.2. *Every set can be well-ordered.*

A well-ordered set has a total order such that every nonempty subset has a least element.

The following is Zorn's Lemma, and it is equivalent to the Axiom of Choice.

Lemma 7.3. *For any partially-ordered set A, if every chain in A has an upper bound in A, then A contains at least one maximal element.*

The following is called the Schröder-Bernstein Theorem or Cantor-Bernstein Theorem or Cantor-Schröder-Bernstein Theorem.

Theorem 7.4. *Let A and B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

If there exists an injective function $f : A \to B$ and an injective function $g : B \to A$, then there exists a bijective function $h : A \to B$.

Using the arithmetic developed for the cardinalities of infinite sets, we have the following results. For set A, let $\mathcal{P}(A)$ be the power set of A, and let $\mathcal{P}_0(A)$ be the set of finite subsets of A.

1) $|\mathcal{N}| = \aleph_0$.

2) If A is infinite, $\aleph_0 \leq |A|$.

3) If A is infinite, then $|\mathcal{P}_0(A)| = |A|$.

4) If A is infinite, then $|A \times \mathbb{N}| = |A| \cdot \aleph_0 = |A|$.

5) If A is infinite, then $|\mathcal{P}_0(A) \times \mathbb{N}| = |A|$.

Let F be a field, and let S be a set. Let $\{e_i | i \in S\}$ be the standard basis of F indexed by S, and define F_0^S to be the vector space of all linear combinations of finite subsets of the standard basis of F indexed by S. We can also think of F_0^S as the vector space of all functions with finite support from S to F, and here e_i is the function from S to F such that $e_i(s) = 0$ if $s \neq i$ and $e_i(s) = 1$ if $s = i$. If S is finite with $|S| = n$, then we can apply our results for F^n to F_0^S because these are isomorphic by sending the standard basis of F^n to the standard basis of F indexed by S.

Theorem 7.5. *Let V be a vector space over field F, and let S be a set. Then for any set (or multiset) $\{v_i | i \in S\}$ with $v_i \in V$ for all $i \in S$, $f : F_0^S \to V$ such that*

$$f(x_{i_1} e_{i_1} + \cdots + x_{i_n} e_{i_n}) = x_{i_1} v_{i_1} + \cdots + x_{i_n} v_{i_n}$$

for all nonnegative integers n and all $x_{i_1}, \ldots, x_{i_n} \in F$, is the unique linear transformation such that $f(e_i) = v_i$ for all $i \in S$.

When $S = \emptyset$ and $|S| = 0$, $F_0^S = \{\mathbf{0}\}$, and the only linear transformation $f : \{\mathbf{0}\} \to V$ is defined by $f(\mathbf{0}) = \mathbf{0}_V$.

8 Isomorphic Dimension with the Infinite Case

Lemma 8.1. *Let F be a field, let S be a set, and let n be a finite nonnegative integer. $|S| = n$ if and only if $\dim_I(F_0^S) = n$.*

Proof. If S is finite with $|S| = n$, then $S = \{s_1, \ldots, s_n\}$ for some distinct $s_1, \ldots, s_n \in S$. The linear transformation $f : F^n \to F_0^S$ such that $f(e_i) = e_{s_i}$ for all $i = 1, \ldots, n$ is bijective and thus a linear isomorphism. By definition, $\dim_I(F_0^S) = n$.

If S is infinite, then $\aleph_0 \leq |S|$, and there exists a subset of S that can be written as $\{s_i | i \in \mathbb{N}\}$ of distinct elements from S. For every nonnegative integer n, define $f_n : F^n \to F_0^S$ as the linear transformation such that $f(e_i) = e_{s_i}$ for all $i = 1, \ldots, n$. This creates a sequence of injective linear transformations.

$$f_0 : F^0 \to F_0^S, \ f_1 : F^1 \to F_0^S, \ldots, \ f_n : F^n \to F_0^S, \ldots$$

such that $f_{k+1} \circ p_k^{k+1} = f_k$ and $\operatorname{im}(f_k) \subsetneq \operatorname{im}(f_{k+1})$ for any $k \geq 0$. By Theorem 3.8, F_0^S is infinite-dimensional. $\square$

Lemma 8.2. *Let F be a field, and let S and T be sets. If $f : F_0^S \to F_0^T$ is an injective linear transformation, then $|S| \leq |T|$.*

Proof. If T is finite, then by Lemma 8.1 and by Theorem 3.15, S is finite and $|S| = \dim_I(F_0^S) \leq \dim_I(F_0^T) = |T|$.

Consider T to be infinite. For any $i \in S$, $f(e_i)$ is a linear combination of a finite subset of $\{e_j | j \in T\}$. Let A_i be the finite subset of T of the indexes of the basis elements in the linear combination of $f(e_i)$.

For any finite subset T' of T, let $S' = \{i \in S | A_i = T'\}$. Either $S' = \emptyset$ and $|S'| = 0$, or if $|S'| > 0$, then the linear transformation $f' : F_0^{S'} \to F_0^{T'}$ such that $f'(e_i) = f(e_i)$ for all $i \in S'$ is an injective function and $|S'|$ is finite with $|S'| \leq |T'|$.

Define $g : S \to \mathcal{P}_0(T) \times \mathbb{N}$ such that $g(i) = A_i \times j$ for every $i \in S$, and since for every $A \in \mathcal{P}_0(T)$, there will only be a finite number of i with $A_i = A$, we can order these using the positive integers j, $1 \leq j \leq |A|$ for those i with $A_i = A$. The function g is injective, and $|S| \leq |\mathcal{P}_0(T) \times \mathbb{N}| = |T|$ because T is infinite. $\qquad\square$

Lemma 8.3. *Let F be a field, and let S and T be sets. If $f : F_0^S \to F_0^T$ is a surjective linear transformation, then $|S| \geq |T|$.*

Proof. If S is finite, then by Lemma 8.1 and Theorem 3.16, $|S| = \dim_I(F_0^S) \geq \dim_I(F_0^T) = |T|$.

Consider S to be infinite. For any $i \in S$, $f(e_i)$ is a linear combination of a finite subset of $\{e_j | j \in T\}$. Let A_i be the finite subset of T of the indexes of the basis elements in the linear combination of $f(e_i)$. Since f is surjective, $\cup_{i \in S} A_i = T$. Therefefore $|T| = |\cup_{i \in S} A_i| \leq |S \times \mathbb{N}| = |S| \cdot \aleph_0 = |S|$ because S is infinite. $\qquad\square$

Proposition 8.4. *Let V be a vector space over field F, and let S and T be sets. Let $f_S : F_0^S \to V$ be a linear transformation and let $f_T : F_0^T \to V$ be a linear isomorphism. If f_S is injective, then $|S| \leq |T|$, and if f_S is surjective, then $|S| \geq |T|$.*

Proof. If f_S is injective, then $f_T^{-1} \circ f_S : F_0^S \to F_0^T$ is an injective linear transformation, and by Lemma 8.2, $|S| \leq |T|$.

If f_S is surjective, then $f_T^{-1} \circ f_S : F_0^S \to F_0^T$ is a surjective linear transformation, and by Lemma 8.3, $|S| \geq |T|$. $\qquad\square$

Corollary 8.5. *Let V be a vector space over field F, and let S and T be sets. If $f_S : F_0^S \to V$ is a linear isomorphism and $f_T : F_0^T \to V$ is a linear isomorphism, then $|S| = |T|$.*

Proof. By Proposition 8.4, $|S| \leq |T|$ and $|S| \geq |T|$, so $|S| = |T|$ by the Schröder-Bernstein Theorem. $\qquad\square$

Theorem 8.6. *Let V be a vector space over field F, and let S and T be sets. If $f_S : F_0^S \to V$ is a linear isomorphism and $|S| = |T|$, then there exists a linear isomorphism $f_T : F_0^T \to V$.*

Proof. If $|S| = |T|$, then there exists an isomorphism $g : T \to S$. Define the linear transformation $f_T : F_0^T \to V$ by $f_T(e_i) = f_S(e_{g(i)})$ for all $i \in T$. Since g is an isomorphism and f_S is a linear isomorphism, then f_T is a linear isomorphism. $\qquad\square$

If there exists a linear isomorphism between F_0^S and V, then $|S|$ is unique, and for any set T with the cardinality $|T| = |S|$, there exists a linear isomorphism between F_0^T and V. We can generalize our previous definition of isomorphic dimension to include dimensions of other cardinalities. This definition coincides with our previous definition because $F^n = F_0^S$ for $S = \{1, \ldots, n\}$, where $\{e_i | i \in S\} = \{e_1, \ldots, e_n\}$ is the standard basis of F^n.

Definition 8.7. *Let V be a vector space over field F, and let S be a set. The **isomorphic dimension** of V is $|S|$, labeled $\dim_I(V) = |S|$, if and only if there exists a linear isomorphism $f : F_0^S \to V$.*

For every pair of sets $S \subset T$, define $p_S^T : F_0^S \to F_0^T$ to be the linear transformation such that $p_S^T(e_i) = e_i$ for all if $i \in S$, and define $p_T^S : F_0^T \to F_0^S$ to be the linear transformation such that $p_T^S(e_i) = e_i$ for all $i \in S$ and $p_T^S(e_i) = \mathbf{0}$ for all $i \in T - S$.

Lemma 8.8. *Let V be a vector space over field F, and let S be a set. Let $f : F_0^S \to V$ be a linear transformation with $f(e_i) = v_i$ for all $i \in S$. If f is injective but not surjective, then for any $v \in V$ with $v \notin \mathrm{im}(f)$ and any $j \notin S$, there exists an injective linear transformation $f' : F_0^{S'} \to V$ for $S' = S \cup \{j\}$ such that $f'(e_i) = v_i$ for all $i \in S$ and $f'(e_j) = v$ with $f' \circ p_S^{S'} = f$ and $\mathrm{im}(f) \subsetneq \mathrm{im}(f')$.*

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Proof. If $f : F_0^S \to V$ is not surjective, there exists $v \in V$ such that $v \notin \text{im}(f)$. Then for any $a \in F$, $av \in \text{im}(f)$ if and only if $a = 0$ because $\text{im}(f)$ is a subspace.

For $S' = S \cup \{j\}$, define $f' : F_0^{S'} \to V$ to be the linear transformation such that $f'(e_i) = v_i$ for all $i \in S$ and $f'(e_j) = v$. It follows that $f' \circ p_S^{S'} = f$ and $\text{im}(f) \subset \text{im}(f')$ with $\text{im}(f) \neq \text{im}(f')$ because $v \notin \text{im}(f)$ but $v \in \text{im}(f')$.

For any $x_1 e_{i_1} + \cdots + x_n e_{i_n} + x e_j \in \text{ker}(f')$,

$$f(x_1 e_{i_1} + \cdots + x_n e_{i_n}) + xv = \mathbf{0},$$

which implies $f(x_1 e_{i_1} + \cdots + x_n e_{i_n}) = -xv$. Therefore, $f(x_1 e_{i_1} + \cdots + x_n e_{i_n}) = \mathbf{0}_V$ and $x = 0$. Since f is injective, $x_1 e_{i_1} + \cdots + x_n e_{i_n} = \mathbf{0}$, and $x_1 e_{i_1} + \cdots + x_n e_{i_n} + x e_j = \mathbf{0} + 0 e_j = \mathbf{0}$. This proves f' is injective. $\square$

Lemma 8.9. *Let V be a vector space over field F, and let S and T be sets such that $S \subset T$. If $f_T : F_0^T \to V$ is a surjective linear transformation such that the linear transformation $f_S : F_0^S \to V$ defined by $f_S(e_i) = f_T(e_i)$ for all $i \in S$ is injective, then there exists a set B, such that $S \subset B \subset T$ and the linear transformation $f_B : F_0^B \to V$ defined by $f_B(e_i) = f_T(e_i)$ for all $i \in B$ is an isomorphism.*

Proof. Let U be the set of all subsets A such that $S \subset A \subset T$ and the linear transformation $f_A : F_0^A \to V$ defined by $f_A(e_i) = f_T(e_i)$ for all $i \in A$ is injective. Since $S \in U$, U is nonempty.

Let $\{A_j | j \in K\}$ be a chain in U. Then $C = \cup_{j \in K} A_j$ is also in U because $S \subset C \subset T$ and the linear transformation $f_C : F_0^C \to V$ defined by $f_C(e_i) = f_T(e_i)$ for all $i \in C$ is injective. Therefore C is an upperbound for chain $\{A_j | j \in K\}$. Since every chain in U has an upper bound, U has a maximal element B by Zorn's Lemma.

Since B is in U, $S \subset B \subset T$ and the linear transformation $f_B : F_0^B \to V$ defined by $f_B(e_i) = f_T(e_i)$ for all $i \in B$ is injective. If $f_T(e_i) \in \text{im}(f_B)$ for all $i \in T$, then f_B is surjective because f_T is surjective. If $f_T(e_j) \notin \text{im}(f_B)$ for some $j \in T - B$, then for $B' = B \cup \{j\}$, $S \subset B' \subset T$ and the linear transformation $f_{B'} : F_0^{B'} \to V$ defined by $f_{B'}(e_i) = f_T(e_i)$ for all $i \in B'$ is injective by Lemma 8.8. This contradicts the maximality of B, so $f_T(e_i) \in \text{im}(f_B)$ for all $i \in T$ and f_B is surjective.

Therefore f_B is a linear isomorphism. $\square$

Theorem 8.10. *Let V be a vector space over field F. There exists a set B such that $\dim_I(V) = |B|$.*

Proof. Let $S = \emptyset$ and $T = V$. Then $S \subset T$. Define the linear transformation $f_T : F_0^T \to V$ by $f_T(e_i) = i$ for all $i \in T$. Then f_T is surjective. The linear transformation $f_S : F_0^S \to V$ defined by $f_S(e_i) = f_T(e_i)$ for all $i \in S$ is the same as the linear transformation $f_S : F_0^S \to V$ defined by $f_S(\mathbf{0}) = \mathbf{0}_V$, which is injective. By Lemma 8.9, there exists a set B such that $S \subset B \subset T$ and a linear isomorphism $f_B : F_0^B \to V$. By definition, $\dim_I(V) = |B|$. $\square$

The isomorphic definition of dimension allows for a short proof of the following.

Theorem 8.11. *Let V and W be vector spaces over field F. There exists linear isomorphism $f : V \to W$ if and only if $\dim_I(V) = \dim_I(W)$.*

Proof. Let $\dim_I(V) = |S|$. There exists linear isomorphism $f_V : F_0^S \to V$.

If $f : V \to W$ is a linear isomorphism, then $f \circ f_V : F_0^S \to W$ is a linear isomorphism. Therefore $\dim_I(W) = |S|$.

If $\dim_I(W) = |S|$, then there exists linear isomorphism $f_W : F_0^S \to W$, and $f_W \circ f_V^{-1} : V \to W$ is a linear isomorphism. $\square$

We also say V and W are isomorphic if and only if there exists a linear isomorphism between them, so V and W are isomorphic if and only if $\dim_I(V) = \dim_I(W)$.

Theorem 8.12. *Let V be a vector space over field F, and let U be a subspace of V. Then $\dim_I(U) \leq \dim_I(V)$.*

Proof. By Theorem 8.10, there exist a set S such that $\dim_I(U) = |S|$ and set T such that $\dim_I(V) = |T|$. By definition, there exist linear isomorphisms $f_S : F_0^S \to U$ and $f_T : F_0^T \to V$. Let $g : U \to V$ be the injective linear transformation defined by $g(u) = u$ for all $u \in U$. Then $f_T^{-1} \circ g \circ f_S : F_0^S \to F_0^T$ is injective. By Lemma 8.2, $\dim_I(U) = |S| \leq |T| = \dim_I(V)$. $\qquad\square$

Theorem 8.13. *Let V and W be vector spaces over field F. There exists an injective linear transformation $f : V \to W$ if and only if $\dim_I(V) \leq \dim_I(W)$.*

Proof. By Theorem 8.10, there exist a set S such that $\dim_I(V) = |S|$ and a set T such that $\dim_I(W) = |T|$. By definition, there exist linear isomorphisms $f_V : F_0^S \to V$ and $f_W : F_0^T \to W$.

If $f : V \to W$ is an injective linear transformation, then $f_W^{-1} \circ f \circ f_V : F_0^S \to F_0^T$ is an injective linear transformation, and by Lemma 8.2, $\dim_I(V) = |S| \leq |T| = \dim_I(W)$.

If $|S| = \dim_I(V) \leq \dim_I(W) = |T|$, then there exists an injective function $g : S \to T$. Define the linear transformation $g' : F_0^S \to F_0^T$ such that $g'(e_i) = e_{g(i)}$ for all $i \in S$. Since g is injective, g' is an injective linear transformation. The linear transformation $f_W \circ g' \circ f_V^{-1} : V \to W$ is injective. $\qquad\square$

Theorem 8.14. *Let V and W be vector spaces over field F. There exists a surjective linear transformation $f : V \to W$ if and only if $\dim_I(V) \geq \dim_I(W)$.*

Proof. By Theorem 8.10, there exist a set S such that $\dim_I(V) = |S|$ and a set T such that $\dim_I(W) = |T|$. By definition, there exist linear isomorphisms $f_V : F_0^S \to V$ and $f_W : F_0^T \to W$.

If $f : V \to W$ is a surjective linear transformation, then $f_W^{-1} \circ f \circ f_V : F_0^S \to F_0^T$ is an surjective linear transformation, and by Lemma 8.3, $\dim_I(V) = |S| \geq |T| = \dim_I(W)$.

If $|S| = \dim_I(V) \geq \dim_I(W) = |T|$, then there exists a surjective function $g : S \to T$. Define the linear transformation $g' : F_0^S \to F_0^T$ such that $g'(e_i) = e_{g(i)}$ for all $i \in S$. Since g is surjective, g' is a surjective linear transformation. The linear transformation $f_W \circ g' \circ f_V^{-1} : V \to W$ is surjective. $\qquad\square$

Theorem 8.15. *Let V be a vector space over field F, and let U be a subspace of V.*

$$\dim_I(U) + \dim_I(V/U) = \dim_I(V)$$

Proof. By Theorem 8.10, there exists a set S such that $\dim_I(U) = |S|$, and there exists a linear isomorphism $f : F_0^S \to U$. Let $\text{inc} : U \to V$ be the injective linear transformation such that $\text{inc}(u) = u$ for any $u \in U$, and let $f_S = \text{inc} \circ f : F_0^S \to V$. As a composition of injective linear transformations, f_S is injective.

Let $T = S \sqcup (V - \{f(e_i)|i \in S\})$. Then the linear transformation $f_T : F_0^T \to V$ defined as $f_T(e_i) = f_S(e_i)$ for all $i \in S$ and $f_T(e_i) = i$ for all $i \in T - S$ is surjective. By Lemma 8.9, there exist a set B such that $S \subset B \subset T$ and a linear isomorphism $g : F_0^B \to V$ such that $g(e_i) = f_S(e_i)$ for all $i \in S$ and thus $\text{im}(g|_{F_0^S}) = U$. Since g is a linear isomorphism, $\dim_I(V) = |B|$.

Let $q : V \to V/U$ be the surjective linear transformation such that $q(v) = v + U$ for all $v \in V$. Then $h = q \circ g|_{F_0^{B-S}} : F_0^{B-S} \to V/U$ is a linear transformation. Since g is a injective, $\text{im}(g|_{F_0^S}) \cap \text{im}(g|_{F_0^{B-S}}) = \{\mathbf{0}\}$. Thus $U \cap \text{im}(g|_{F_0^{B-S}}) = \{\mathbf{0}\}$ and h is injective. Since g is surjective, for any $v + U$ in V/U, v is a linear combination of the outputs from g of the standard basis of F_0^B. This linear combination can be split into a sum of a linear combination with standard basis vectors from F_0^S and and a linear combination from F_0^{B-S} because $B = S \sqcup (B - S)$. Any linear combination of outputs from g of standard basis vectors from F_0^S is in U. Thus $v + U = v' + U$ for some v' equal to a linear combination of outputs from g of standard basis vectors from F_0^{B-S}, and h is surjective.

As a bijective linear transformation, h is a linear isomorphism and $\dim_I(V/U) = |B-S|$. Therefore $|S| + |B - S| = |B|$ or $\dim_I(U) + \dim_I(V/U) = \dim_I(V)$. $\qquad\square$

The following theorem is the Rank Nullity Theorem using the definition of isomorphic dimension.

Theorem 8.16. *Let V and W be vector spaces over field F, and let $f : V \to W$ be a linear transformation.*

$$\dim_I(\ker(f)) + \dim_I(\operatorname{im}(f)) = \dim_I(V)$$

Proof. By the first isomorphism theorem, $\bar{f} : V/\ker(f) \to \operatorname{im}(f)$ is a linear isomorphism, and by Theorem 8.11, $\dim_I(V/\ker(f)) = \dim_I(\operatorname{im}(f))$. By Theorem 8.15, $\dim_I(\ker(f)) + \dim_I(\operatorname{im}(f)) = \dim_I(V)$. $\square$

9 Isomorphic Basis with the Infinite Case

In a similar fashion to our latest definition of dimension, we can also construct a new definition for basis that is equivalent to the algebraic definition of a basis.

Definition 9.1. *Let V be a vector space over field F, and let S be a set. The set (or multiset) $\{v_i | i \in S\}$ for $v_i \in V$ for all $i \in S$ is an* **isomorphic basis** *of V if and only if the linear transformation $f : F_0^S \to V$ such that $f(e_i) = v_i$ for all $i \in S$ is an isomorphism.*

The standard basis of F_0^S, $\{e_i | i \in S\}$, is also an isomorphic basis of F_0^S according to this definition using the identity map $I : F_0^S \to F_0^S$, which is the linear transformation with $I(e_i) = e_i$ for all $i \in S$ and is an isomorphism.

Let V be a vector space over field F. Since a linear transformation with domain F_0^S is uniquely determined by the outputs of the standard basis, for $\dim_I(V) = |S|$, there is a one-to-one correspondence between the linear isomorphisms $f : F_0^S \to V$ and the ordered bases of V (based on the ordering of S).

Theorem 9.2. *Let V be a vector space over field F, and let S be a set. Then $\dim_I(V) = |S|$ if and only if there exists an isomorphic basis of V containing $|S|$ vectors.*

Proof. If $\dim_I(V) = |S|$, then there exists a linear isomorphism $f : F_0^S \to V$. Then $\{f(e_i) | i \in S\}$ is an isomorphic basis of V by definition.

If $\{v_i | i \in S\}$ is an isomorphic basis of V, then the linear transformation $f : F_0^S \to V$ defined by $f(e_i) = v_i$ for all $i \in S$ is an isomorphism. Since there exists an isomorphism between F_0^S and V, $\dim_I(V) = |S|$. $\square$

For vector space V, $\dim_I(V)$ is unique, so we get the following corollary.

Corollary 9.3. *Let V be a vector space over field F. Every isomorphic basis of V has cardinality equal to $\dim_I(V)$.*

Definition 9.4. *Let V be a vector space over field F, and let S be a set. For the set (or multiset) $\{v_i | i \in S\}$ with $v_i \in V$ for all $i \in S$, a* **linear combination** *of $\{v_i | i \in S\}$ is an output of the linear transformation $f : F_0^{S'} \to V$ such that $f(e_i) = v_i$ for all $i \in S'$ for a finite subset $S' \subset S$.*

This means that a linear combination is $f(x_1 e_{i_1} + \cdots + x_n e_{i_n}) = x_1 v_{i_1} + \cdots + x_n v_{i_n}$ for some nonnegative integer n and some $x_1, \ldots, x_n \in F$, which is the same as the usual definition.

Theorem 9.5. *Let V be a vector space over field F, let S be a set, and let $\mathcal{B} = \{v_i | i \in S\}$ such that $v_i \in V$ for all $i \in S$. Then $\mathcal{B}$ is an isomorphic basis of V if and only if every vector in V is a unique linear combination of $\mathcal{B}$.*

Proof. Let $f : F_0^S \to V$ be the linear transformation such that $f(e_i) = v_i$ for all $i \in S$.

If $\mathcal{B}$ is an isomorphic basis, then f is an isomorphism and every vector in V is a unique output for some input $x_1 e_{i_1} + \cdots + x_n e_{i_n} \in F_0^S$, which means it is a unique linear combination $x_1 v_{i_1} + \cdots + x_n v_{i_n}$.

If every vector in V is a unique linear combination of $\mathcal{B}$, then f is surjective because every vector in V is an output of f, and f is injective because $f(\mathbf{0}) = \mathbf{0}_V$ is unique, which implies $\ker(f) = \{\mathbf{0}\}$. Thus f is an isomorphism, and $\mathcal{B}$ is an isomorphic basis. $\square$

10 Surjective Set with the Infinite Case

Definition 10.1. *Let V be a vector space over field F, and let S be a set. For the set (or multiset) $\{v_i | i \in S\}$ with $v_i \in V$ for all $i \in S$, the **span** of $\{v_i | i \in S\}$ is the image of the linear transformation $f : F_0^S \to V$ such that $f(e_i) = v_i$ for all $i \in S$, or in other words, $\mathrm{span}(\{v_i | i \in S\}) = \mathrm{im}(f)$.*

This means that $\mathrm{span}(\{v_i | i \in S\})$ is the set of all linear combinations of finite subsets of $\{v_i | i \in S\}$, which is the same as the usual definition.

Definition 10.2. *Let V be a vector space over field F, and let S be a set. The set (or multiset) $\{v_i | i \in S\}$ with $v_i \in V$ for all $i \in S$ is a **surjective set** in V if and only if the linear transformation $f : F_0^S \to V$ such that $f(e_i) = v_i$ for all $i \in S$ is surjective.*

Therefore, $\{v_i | i \in S\}$ is a surjective set in V if and only if $\mathrm{span}(\{v_i | i \in S\}) = V$. The linear transformation $f : F_0^S \to V$ defined by $f(e_i) = v_i$ for all $i \in S$ is also surjective if and only if $\{v_i | i \in S\}$ is a spanning set of V. Therefore the definitions of surjective set in V and spanning set of V are equivalent.

Theorem 10.3. *Let V be a vector space over field F, let S be a set, and let $\{v_i | i \in S\}$ be a set or multiset with $v_i \in V$ for all $i \in S$. Then $\{v_i | i \in S\}$ is a spanning set of V if and only if $\{v_i | i \in S\}$ is a surjective set in V.*

We can prove the results involving spanning sets for surjective sets using the definition of surjective sets.

Theorem 10.4. *Let V be a vector space over field F, let S be a set, and let $\{v_i | i \in S\}$ be a surjective set in V. Then $|S| \geq \dim_I(V)$.*

Proof. By definition, the linear transformation $f_S : F_0^S \to V$ such that $f_S(e_i) = v_i$ for all $i \in S$ is surjective. By Theorem 8.10, there exists a set B such that $\dim_I(V) = |B|$, and by definition, there exists a linear isomorphism $f_B : F_0^B \to V$. The linear transformation $f_B^{-1} \circ f_S : F_0^S \to F_0^B$ is surjective. By Lemma 8.3, $|S| \geq |B| = \dim_I(V)$. $\qquad\square$

Theorem 10.5. *Let V be a vector space over field F, and let S be a set. If $\{v_i | i \in S\}$ is a surjective set in V, then there exists some subset $\{v_i | i \in S'\}$ for some $S' \subset S$, such that $\{v_i | i \in S'\}$ is an isomorphic basis of V.*

Proof. By definition, the linear transformation $f_S : F_0^S \to V$ such that $f_S(e_i) = v_i$ for all $i \in S$ is surjective. Let $R = \emptyset$. Then $R \subset S$ and the linear transformation $f_R : F_0^R \to V$ defined by $f_R(e_i) = f_S(e_i)$ for all $i \in R$ is the same as the linear transformation $f_R : F_0^R \to V$ defined by $f_R(\mathbf{0}) = \mathbf{0}_V$, which is injective. By Lemma 8.9, there exists a set S' such that $R \subset S' \subset S$ and a linear isomorphism $f_{S'} : F_0^{S'} \to V$. By definition, $\{v_i | i \in S'\}$ is an isomorphic basis of V. $\qquad\square$

We don't have some of the same results for surjective sets of finite-dimensional vector spaces when including infinite-dimensional vector spaces. For example, let $V = P(F)$, the vector space of all polynomials over field F. Then $\dim_I(V) = \aleph_0 = |S|$ for $S = \mathbb{Z}_{\geq 0}$ because the linear transformation $f : F_0^S \to V$ defined by $f(e_i) = x^i$ for all $i \in S$ is an isomorphism. The linear transformation $g : F_0^S \to V$ defined by $g(e_0) = 1$ and $g(e_i) = x^{i-1}$ for all $i \in \mathbb{Z}_{>0}$ is surjective but not an isomorphism.

11 Injective Set with the Infinite Case

Definition 11.1. *Let V be a vector space over field F, and let S be a set. The set (or multiset) $\{v_i | i \in S\}$ with $v_i \in V$ for all $i \in S$ is an **injective set** in V if and only if the linear transformation $f : F_0^S \to V$ such that $f(e_i) = v_i$ for all $i \in S$ is injective.*

If $S = \emptyset$ and $\{v_i | i \in S\} = \emptyset$, then $F_0^S = \{\mathbf{0}\}$ and the linear transformation $f : \{\mathbf{0}\} \to V$ is defined by $f(\mathbf{0}) = \mathbf{0}_V$, so f is injective. This implies the empty set is an injective set in V.

Theorem 11.2. *Let V be a vector space over a field F, let S be a set, and let $\{v_i | i \in S\}$ be a set or multiset with $v_i \in V$ for all $i \in S$. Then $\{v_i | i \in S\}$ is a linearly independent set in V if and only if $\{v_i | i \in S\}$ is an injective set in V.*

Proof. Let $f : F_0^S \to V$ be the linear transformation defined by $f(e_i) = v_i$ for all $i \in S$. Then $f(x_1 e_{i_1} + \cdots + x_n e_{i_n}) = x_1 v_{i_1} + \cdots + x_n v_{i_n}$ for all $x_1 e_{i_1} + \cdots + x_n e_{i_n} \in F_0^S$ and $x_1 e_{i_1} + \cdots + x_n e_{i_n} \in \ker(f)$ if and only if $x_1 v_{i_1} + \cdots + x_n v_{i_n} = \mathbf{0}_V$.

If $\{v_i | i \in S\}$ is a linearly independent set in V, then $x_1 v_{i_1} + \cdots + x_n v_{i_n} = \mathbf{0}_V$ implies $x_1 e_{i_1} + \cdots + x_n e_{i_n} = \mathbf{0}$, $\ker(f) = \{\mathbf{0}\}$, and f is injective, so $\{v_i | i \in S\}$ is an injective set in V. If $\{v_i | i \in S\}$ is an injective set in V, then f is injective and $\ker(f) = \{\mathbf{0}\}$, and $x_1 v_{i_1} + \cdots + x_n v_{i_n} = \mathbf{0}_V$ implies $x_1 e_{i_1} + \cdots + x_n e_{i_n} = \mathbf{0}$. $\square$

We can prove the results involving linearly independent set for injective sets using the definition of injective sets.

Theorem 11.3. *Let V be a vector space over field F, and let S be a set. If $\{v_i | i \in S\}$ is an injective set in V, then $|S| \leq \dim_I(V)$.*

Proof. By definition, the linear transformation $f_S : F_0^S \to V$ such that $f_S(e_i) = v_i$ for all $i \in S$ is injective. By Theorem 8.10, there exists a set B such that $\dim_I(V) = |B|$, and by definition, there exists a linear isomorphism $f_B : F_0^B \to V$. The linear transformation $f_B^{-1} \circ f_S : F_0^S \to F_0^B$ is injective. By Lemma 8.2, $|S| \leq |B| = \dim_I(V)$. $\square$

Corollary 11.4. *Let V be a vector space over field F, and let S and T be sets. If $\{v_i | i \in S\}$ is an injective linearly independent set in V and $\{w_j | j \in T\}$ is a surjective set in V, then $|S| \leq |T|$.*

Proof. By Theorem 10.4, $|T| \geq \dim_I(V)$. By Theorem 11.3, $|S| \leq \dim_I(V)$. Therefore $|S| \leq |T|$. $\square$

Theorem 11.5. *Let V be a vector space over field F, and let S be a set. If $\{v_i | i \in S\}$ is an injective set in V, then there exist a set S' with $S \subset S'$ and a set $\{v_i | i \in S' - S\} \subset V$ such that $\{v_i | i \in S'\}$ is an isomorphic basis of V.*

Proof. By definition, the linear transformation $f_S : F_0^S \to V$ such that $f_S(e_i) = v_i$ for all $i \in S$ is injective. Let $T = S \sqcup (V - \{v_i | i \in S\})$, so that $S \subset T$. Define the linear transformation $f_T : F_0^S \to V$ by $f_T(e_i) = v_i$ for all $i \in S$ and $f_T(e_v) = v$ for all $v \in V - \{v_i | i \in S\}$, which is surjective. By Lemma 8.9, there exists a set S' such that $S \subset S' \subset T$ and $f_{S'} : F_0^{S'} \to V$ is a linear isomorphism. Then $\{f_T(e_i) = v_i | i \in S'\}$ is an isomorphic basis. $\square$

We don't have some of the same results for injective sets of finite-dimensional vector spaces when including infinite-dimensional vector spaces. For example, let $V = P(F)$, the vector space of all polynomials over field F. Then $\dim_I(V) = \aleph_0 = |S|$ for $S = \mathbb{Z}_{\geq 0}$ because the linear transformation $f : F_0^S \to V$ defined by $f(e_i) = x^i$ for all $i \in S$ is an isomorphism. The linear transformation $g : F_0^S \to V$ defined by $g(e_i) = x^{i+1}$ for all $i \in S$ is injective but not an isomorphism.

Definition 11.6. *Let V be a vector space over field F, and let S be a set. The set (or multiset) $\{v_i | i \in S\}$ with $v_i \in V$ for all $i \in S$ is a* **noninjective set** *in V if and only if the linear transformation $f : F_0^S \to V$ such that $f(e_i) = v_i$ for all $i \in S$ is not injective.*

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