TORSION HOMOLOGY GROWTH OF POLYNOMIALLY GROWING FREE-BY-CYCLIC GROUPS

NAOMI ANDREW, SAM HUGHES, AND MONIKA KUDLINSKA

ABSTRACT. We show that the homology torsion growth of a free-by-cyclic group with polynomially growing monodromy vanishes in every dimension independently of the choice of Farber chain. It follows that the integral torsion $\rho^{\mathbb{Z}}$ equals the ℓ^2-torsion $\rho^{(2)}$ verifying a conjecture of Lück for these groups.

1. INTRODUCTION

There are a number of notions of volume for a hyperbolic 3-manifold M; namely, its hyperbolic volume $\mathrm{Vol}(M)$, the minimal volume entropy $\mathcal{E}_{\min}(M)$, and the ℓ^2-torsion $\rho^{(2)}(M)$. Combining results of Lück–Schick [LS99] for $\rho^{(2)}$, and Gromov [Gro82], Soma [Som81], and Thurston [Thu78] for $\mathcal{E}_{min}$, we see that for finite volume hyperbolic 3-manifolds each of the invariants are proportional. Conjecturally, (see [BV13]) they are also proportional to a fourth invariant, the integral torsion

$$\rho^{\mathbb{Z}}(M) := \sum_{j \geqslant 0} t_j^{(2)}(M; M_n),$$

where

$$t_j^{(2)}(M; M_n) := \limsup_{n \to \infty} \frac{\log |H_j(M_n; \mathbb{Z})_{\mathrm{tors}}|}{|M : M_n|}$$

is the torsion homology gradient with respect to a *Farber sequence* $(M_n)_{n \in \mathbb{N}}$ of finite covers. See Section 3 for a definition of Farber sequence — the main example is a sequence of regular covers such that the intersection of the inclusions of the fundamental groups into $\pi_1 M$ is trivial.

Note that in particular, the conjecture states that the limit supremums are actually genuine limits and the limit is independent of the choice of Farber sequence.

Conjecture 1.1 (Lück). [Lüc13, Conjecture 1.11(3)] *Let G be an infinite residually finite ℓ^2-acyclic group of type* VF. *Then,* $\rho^{(2)}(G) = \rho^{\mathbb{Z}}(G)$.

Lück proved the conjecture for fundamental groups of closed aspherical manifolds which admit non-trivial S^1-action, or where the group contains a non-trivial elementary amenable normal subgroup [Lüc13, Corollary 1.13]. The conjecture also appears to be known for amenable groups (see [KKN17] for $\rho^{\mathbb{Z}}$ and [LT14] for $\rho^{(2)}$) and for groups acting on simplicial complexes with strict fundamental domain and with stabilisers containing normal abelian subgroups [OS21].

MATHEMATICAL INSTITUTE, ANDREW WILES BUILDING, OBSERVATORY QUARTER, UNIVERSITY OF OXFORD, OXFORD OX2 6GG, UK

E-mail addresses: naomi.andrew@maths.ox.ac.uk, sam.hughes@maths.ox.ac.uk, kudlinska@maths.ox.ac.uk.

Free-by-cyclic groups (understood to mean {finitely generated free}-by-{infinite cyclic}) are a natural algebraic generalisation of cusped 3-manifolds which fibre over the circle. However, due to their generality there is no notion of volume for them. Due to the proportionality principle alluded to above it is tempting to take one of the other invariants $\rho^{\mathbb{Z}}$, $\rho^{(2)}$, or $\mathcal{E}_{\mathrm{min}}$ as a definition of volume for free-by-cyclic groups. Unfortunately computing these invariants is extremely difficult. Indeed, there is a lower bound for $\mathcal{E}_{\mathrm{min}}$ [BC21] (using [BS20]) and for $\rho^{(2)}$ [Cla17] as well as some partial progress towards computing $\rho^{(2)}$ via a 'chain flaring' condition [Cla21]. There is an upper bound for $\rho^{\mathbb{Z}}$ for hyperbolic 3-manifolds [Lê18]. But for both hyperbolic 3-manifolds and all free-by-cyclic groups with infinite order monodromy there are no explicit computations of $\rho^{\mathbb{Z}}$.

Theorem A. *Let $\Gamma = F_m \rtimes_\varphi \mathbb{Z}$. If φ is polynomially growing, then $t_j^{(2)}(\Gamma; \Gamma_n) = 0$, independently of the choice of a Farber sequence $(\Gamma_n)_{n \in \mathbb{N}}$, for all $j \geqslant 0$.*

The theorem follows from establishing the *cheap rebuilding property* and deducing the vanishing from the breakthrough work in [ABFG21]. The cheap rebuilding property was introduced by Abert–Bergeron–Fraczyk–Gaboriau in loc. cit. and used to prove homology torsion growth vanishing for many families of groups. We will give the background relevant to us below.

The next corollary follows from Clay's upper bound on the ℓ^2-torsion of a free-by-cyclic group [Cla17, Theorem 5.1].

Corollary B. *Let $\Gamma = F_m \rtimes_\varphi \mathbb{Z}$. If φ is polynomially growing, then $\rho^{\mathbb{Z}}(\Gamma) = \rho^{(2)}(\Gamma) = 0$.*

Combining the theorem with a result of Bregman–Clay [BC21] where they compute the minimal volume entropy of free-by-cyclic groups we obtain the following corollary.

Corollary C. *Let $\Gamma = F_m \rtimes_\varphi \mathbb{Z}$. If φ is polynomially growing but Γ is not virtually tubular, then $\rho^{\mathbb{Z}}(\Gamma) \neq c \cdot \mathcal{E}_{\mathrm{min}}(\Gamma)$ for any non-zero real-number c.*

Here Γ being *virtually tubular* implies there is a finite index subgroup $\Lambda \leqslant \Gamma$ such that Λ splits as a graph of groups with vertex groups isomorphic to $\mathbb{Z}^2$ and edge groups is isomorphic to $\mathbb{Z}$. Note that such a splitting implies that φ is linearly growing.

It is tempting to conjecture a sharper relation between the degree of the monodromy and the homology torsion growth.

Conjecture 1.2. *Let $\Gamma = F_m \rtimes_\varphi \mathbb{Z}$. If φ is polynomially growing of degree d, then,*

$$\lim_{n \to \infty} \frac{|H_1(\Gamma_n; \mathbb{Z})_{\mathrm{tors}}|}{|\Gamma : \Gamma_n|^d} = c$$

for some $c \in \mathbb{R}_{\geqslant 0}$ and any Farber sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of Γ.

Structure of the paper. In Section 2 we recall the necessary background on free-by-cyclic groups. In Proposition 2.5 we give a self-contained proof of a splitting of a polynomially growing free-by-cyclic group with stabilisers of strictly lower polynomial growth (note that this result is well known to experts). In Section 3 we give the relevant background from [ABFG21] which we will need to compute the homology torsion growth. Finally, in Section 4 we prove the main theorem.

2. Background on free-by-cyclic groups

Let F be a finite rank free group; we write F_m if the rank m is relevant. Let φ be an element of $\mathrm{Aut}(F)$, representing an element Φ of $\mathrm{Out}(F)$. Throughout, we work with free-by-cyclic groups $F \rtimes_\varphi \mathbb{Z}$. Note that the free-by-cyclic groups defined by representatives of the same outer automorphism Φ are isomorphic, and so we are free to consider the group $F \rtimes_\Phi \mathbb{Z}$ without specifying a representative.

Fix a free set of generators of F. For any $g \in F$, we denote by $|g|$ the length of the reduced word representative of g. We write $|\bar{g}|$ to denote the minimal length of a cyclically reduced word representing a conjugate of g.

An outer automorphism $\Phi \in \mathrm{Out}(F)$ acts on the conjugacy classes of elements in F. Given a conjugacy class $\bar{g}$ of an element $g \in F$, we say that $\bar{g}$ grows polynomially of degree d under the iteration of Φ, if there exist constants $C_1, C_2 > 0$ such that for all $n \geqslant 1$,

$$C_1 n^d \leqslant |\Phi^n(\bar{g})| \leqslant C_2 n^d.$$

Note that if $H \leqslant F$ is a subgroup whose conjugacy class in F is preserved by Φ, then for any $g \in H$ the order of growth of the conjugacy class $\bar{g}$ in H is equal to the order of growth of the conjugacy class in F. We say Φ grows polynomially of degree d if every conjugacy class of elements of F grows polynomially of degree at most d under the iteration of Φ, and there exists a conjugacy class which grows polynomially of degree exactly d.

For a detailed discussion of the growth of elements or conjugacy classes under automorphisms or outer automorphisms see [Lev09]. Note that by [Lev09, Lemma 2.3] and the subsequent example, there are linearly growing outer automorphisms for which there are elements growing quadratically (with respect to reduced word length) under certain representative automorphisms, so the choice to use conjugacy growth is necessary for our arguments.

An element $\Phi \in \mathrm{Out}(F_m)$ is said to be *unipotent polynomially growing (UPG)*, if it is polynomially growing and it induces a unipotent element of $\mathrm{GL}(m, \mathbb{Z})$. It follows from [BFH00, Corollary 5.7.6] that if $\Phi \in \mathrm{Out}(F)$ is polynomially growing, then it has a power Φ^k that is UPG; in fact k can be taken to depend only on the rank of F. Note that the restriction of a UPG automorphism to an invariant free factor H is again UPG. The unipotence can be seen for instance by taking the restriction of the induced element of $\mathrm{GL}(m, \mathbb{Z})$ to the image of H.

We will consider certain topological representatives f of Φ on a graph G, and will need to understand growth of edges, paths and (cyclically reduced) loops γ in G under iteration of f: this is defined analogously to growth of conjugacy growth, but taking the lengths of reduced images $f^n(\gamma)$ in the path metric on G. (In fact, the definition of conjugacy growth can be recovered by representing Φ on a rose, and taking cyclically reduced loops representing each conjugacy class.)

By [BFH00, Theorem 5.1.8], any UPG outer automorphism Φ admits a relative train track representative $f\colon G \to G$ with the following additional properties. There exists an associated filtration $\varnothing = G_0 \subset G_1 \subset \ldots \subset G_n = G$, such that G_{i+1} is constructed from G_i by adding an oriented edge E_{i+1}. The map f fixes every vertex of G, and for each $i \leqslant n$ there exists a closed path ρ_i whose image lies in G_{i-1}, and such that $f(E_i) = E_i \cdot \rho_i$. In the language of [BFH00] this image is *split*: $f^k(E_i \cdot \rho_i) = f^k(E_i)f^k(\rho_i)$, and no cancellation occurs between these images at any iterate. In particular, this means that no cancellation can occur between $f^k(\rho_i)$ and $f^{k+1}(\rho_i)$. If ρ_i is non-constant, we may also assume that it is immersed, and we can assume that distinct E_i have distinct suffixes ρ_i ([BFH05, Remark 3.12]). We call any such relative train track representative an *improved relative train track*.

The following facts about improved relative train tracks representing UPG outer automorphisms are established by combining [BFH00, Lemmas 4.1.4 and 5.5.1] (see also [Lev09, Lemma 6.5]). A *Nielsen path* is a path that is fixed under iterating f (after *tightening* to remove backtracks as necessary). An *exceptional path* is a path of the form $E_i\tau^k\overline{E}_j$, where k is an integer, τ is a loop which is a Nielsen path (one can assume it is not a proper power), and ρ_i, ρ_j are both positive powers of τ. Exceptional paths do not split; under the assumption that distinct edges have distinct suffixes they are Nielsen paths if $i = j$ and have linear growth otherwise.

Lemma 2.1. *Let f be an improved relative train track representative for a UPG automorphism, and let u be a path or loop in G_k. Then there is a power M so that $f_{\#}^M(u)$ splits into E_k, $\overline{E}_k$, paths γ contained in G_{k-1}, and exceptional paths (which may be Nielsen paths). If E_k grows at least quadratically then the exceptional paths do not occur.*

The following result is true in the more general setting of *Kolchin maps*, however to avoid introducing extra terminology we will restrict to the case of improved relative train tracks, and indicate a proof using this technology.

Lemma 2.2. [Mac02, Lemma 2.16] *Let $f\colon G \to G$ be an improved relative train track representative of an UPG outer automorphism. Let E_i be an edge of G which grows polynomially of degree $d \geqslant 2$, and suppose that $f(E_i) = E_i \cdot \rho_i$. Then every edge in ρ_i grows polynomially of degree at most $d-1$, and there exists an edge which grows polynomially of degree $d-1$.*

Note that this lemma fails for linear growth: paths containing linearly growing edges can be fixed, and so E_i may be linearly growing despite mapping over another such edge.

Proof. That the bound is attained (including the "moreover" statement) follows since a finite difference argument gives that if u_n grows polynomially of degree d, then $f_{\#}^{\ell}(E_n) = E_n \cdot u_n \cdot f_{\#}(u_n) \cdot f_{\#}^2(u_n) \cdot \ldots \cdot f_{\#}^{\ell-1}(u_n)$ grows polynomially with degree $d + 1$, and the growth of u_n is bounded above by the growth of its edges, and the paths in its splitting(s).

The proof of the first claim is by induction on height in the filtration. The base case is the first i in the filtration for which E_i is at least quadratically growing; this cannot map over any other such edge by definition.

We need a fairly strong inductive hypothesis. Suppose that the claim is true for every edge E_i with height less than n, and note that this means that the filtration can be rearranged so that the filtration on G_{n-1} respects the degrees of growth (in the sense that the top edge may be taken to have the fastest growth). We now show that if E_n maps over an edge known to have polynomial growth of degree at least 2, then E_n has higher growth.

Consider E_n and suppose that E_k grows polynomially with degree $d \geqslant 2$ and is the topmost edge in u_n. From the inductive hypothesis, we can safely assume that E_k has the fastest growth of edges in u_n. Lemma 2.1 implies that for some (perhaps very large) M, $f_\#^M(u_n)$ splits as edges E_k or $\overline{E}_k$, and paths strictly lower in the filtration. On further iteration of f, the edges E_k and $\overline{E}_k$ grow polynomially with degree d; in particular so does any sufficiently high iterate of u_n. So E_n grows polynomially with degree $d + 1$. $\square$

We now tie together the two notions of growth: that of edges under an improved relative train track representing Φ, and that of conjugacy classes under Φ.

Lemma 2.3. *Any non-trivial UPG outer automorphism Φ of F_n enjoys the same degree of polynomial growth as the fastest growing edge in an improved relative train track representing Φ.*

This fact is certainly used implicitly in the literature, but we are unaware of an explicit proof and so we include one here for completeness.

Proof. Note that since the universal covers of G and the rose with n petals are quasi-isometric, the growth of a conjugacy class under Φ is equal to the growth of any cyclically reduced loop in G realising it.

The growth of such a loop is bounded above by the growth of its edges. We will see that (provided Φ is non-trivial) it is always possible to construct a loop growing polynomially with the same degree. By Lemma 2.2, we can assume that the top edge in the filtration on G has the highest growth (for the linear case, we might not have free choice as to which linearly growing edge is topmost, but one can certainly ensure that all linearly growing edges appear after all fixed edges).

There are two cases to consider, with some additional subtleties in the case of linear growth. First suppose the top edge is non-separating, so there is a loop $u = E_n\gamma$, where γ lives in G_{n-1} (this loop is cyclically reduced, and so are its iterates). By Lemma 2.1, eventually some iterate splits either as $E_n \cdot \gamma'$, or with an exceptional path $E_n\tau^k E_i \cdot \gamma'$ (here $i < n$, and γ' could be trivial). In the first case u grows at least as E_n does; in the second case both E_n and $E_n\tau^k\overline{E}_i$ are linearly growing and again this bounds the growth of u below.

Now suppose the top edge is separating, so we must consider loops of the form $E_n\gamma_1\overline{E}_n \cdot \gamma_2$, where γ_1, γ_2 are non-trivial loops. This loop is split as indicated (between $\overline{E}_n$ and γ_2), and as before all iterates remain cyclically reduced. We apply Lemma 2.1: if eventually some iterate of u splits either as $E_n \cdot \gamma_1' \cdot \overline{E}_n \cdot \gamma_2'$, or with one exceptional path and one edge (either as $E_n\tau^k\overline{E}_i \cdot \gamma_1' \cdot \overline{E}_n \cdot \gamma_2'$ or as $E_n \cdot \gamma_1' \cdot E_j\tau^\ell\overline{E}_n \cdot \gamma_2'$), then u grows as E_n does and we are done. Otherwise, E_n is linearly growing and both E_n and $\overline{E}_n$ form part of exceptional paths in the

splitting eventually provided by Lemma 2.1. If this splitting is of the form $E_n \tau^k \overline{E_i} \cdot \gamma_1' \cdot E_j \tau^\ell \overline{E_n} \cdot \gamma_2'$, then as before $E_n \tau^k \overline{E_i}$ is linearly growing and this gives a lower bound for the growth of u.

The remaining possibility is that $E_n \gamma_1 \overline{E_n}$ is already an exceptional path. Provided the connected component containing γ_1 is not a single cycle, one can choose some other γ_1 so that this does not occur. If this is not possible, E_n does not map over another linearly growing edge, so the filtration can be rearranged so E_n is below all other linearly growing edges. Provided there are linearly growing edges that do not separate a single cycle, such an edge can be assumed to be topmost, and one of the arguments above will go through. If all linearly growing edges are of this form, then the outer automorphism induced by this topological representative is trivial. □

Remark 2.4. The kind of pathological representative of the trivial automorphism described at the end of this proof does seem to be a valid improved relative train track, though it would be outlawed by various improvements on this technology – we chose to simplify our exposition by not taking these extra definitions and properties.

The following proposition is well known to the experts (see for instance [Mac02, Hag19], and also [BFH05, Theorem 4.22]). We include the proof here for completeness;

Proposition 2.5. *Let $\Phi \in \mathrm{Out}(F_m)$ be an UPG outer automorphism with polynomial growth of degree $d \geqslant 2$ and let $\Gamma = F_m \rtimes_\Phi \mathbb{Z}$. Then, Γ splits as a finite graph of groups with edge groups isomorphic to $\mathbb{Z}$, and such that every vertex group is either conjugate to an incident edge group, or is isomorphic to $F_{m_v} \rtimes_\Psi \mathbb{Z}$ where $\Psi \in \mathrm{Out}(F_{m_v})$ is an UPG outer automorphism with growth of degree at most $d - 1$, and $m_v < m$.*

Proof. Suppose $f \colon G \to G$ is an improved relative train track representative of the UPG outer automorphism $\Phi \in \mathrm{Out}(F_m)$ with growth of degree $d \geqslant 2$. We let $\mathcal{H}$ be the set of edges of G with growth of degree d, and G' be the subgraph of G obtained by removing the interiors of the edges in $\mathcal{H}$. Note that $f(G') \subseteq G'$, since every edge in G' grows polynomially with degree strictly less than d and thus by Lemma 2.2, the image of each edge in G' does not contain an edge which grows with order d. Let $\{G_i'\}_{i \in I}$ be the set of connected components of G'. Since f maps each edge E to an edge-path which traverses E, it follows that f also preserves each G_i'. Let $J \subseteq I$ be the indexing set of the non-simply-connected components of G'. Note that if G_k' is simply-connected then each edge of G_k' is fixed by f, since a non-trivial immersed loop does not map into G_k'.

Edges growing with degree d cannot appear in the image of any other edge, so we can assume that the top edge E_n in the filtration associated to f grows with degree d. Since f is an improved relative train track, G cannot have vertices of valence 1, and so if the edge E_n is separating the connected components of G_{n-1} cannot be simply connected, and must have non-trivial fundamental groups with ranks strictly less than m. (If it is non-separating then G_{n-1} consists of a single connected component whose rank is $m-1$.) Therefore no G_i' carries the whole fundamental group F_m.

Consider the graph of groups $\mathcal{G}$ obtained from G by collapsing each G_i' to a single vertex. The Bass–Serre covering tree $\mathcal{T}$ of $\mathcal{G}$ is an F_m-tree and thus gives rise to a splitting of F_m, where the vertex stabilisers correspond to the fundamental groups of the components G_i', and the edge stabilisers are trivial. Furthermore, since f fixes every vertex and edge of $\mathcal{G}$, the F_m-tree $\mathcal{T}$ is

preserved by Φ. Hence there is an action of $\Gamma = F_m \rtimes_\Phi \mathbb{Z}$ on $\mathcal{T}$. (This follows from [CM87, Theorem 3.7] or [AB87, Theorem 7.13(b)], since the pre-image of $\langle \Phi \rangle$ in $\mathrm{Aut}(F_m)$ preserves the translation length function of the F_m action on $\mathcal{T}$.)

The vertex stabilisers are given by conjugates of $\pi_1(G_i') \rtimes_{\varphi_i} \mathbb{Z}$, where φ_i is the restriction of an appropriate representative of Φ to the subgroup $\pi_1(G_i') < \pi_1(G) \simeq F_m$, for every $i \in I$. Note that if G_i' is not simply connected, then $\pi_1(G_i')$ is a proper free factor of $\pi_1(G)$ and thus φ_i represents a UPG outer automorphism Φ_i of $\pi_1(G_i')$, and $\mathrm{rank}(\pi_1(G_i')) < m$. Furthermore, since the edges of the components G_i' grow polynomially of degree at most $d-1$ under the action of f, it follows that the Φ_i have polynomial growth of degree at most $d-1$. The edge stabilisers are infinite cyclic groups generated by (conjugates of) the stable letter of the original presentation as a semidirect product. $\qquad\square$

We also record a similar decomposition theorem in the linearly growing case, obtained by the first author and Martino:

Proposition 2.6. [AM22, Proposition 5.2.2] *Let $\Lambda = F_m \rtimes_\Phi \mathbb{Z}$. Suppose that Φ is unipotent and linearly growing. Then, Λ splits as a finite graph of groups such that the vertex groups are isomorphic to $F_{m_v} \times \mathbb{Z}$ with $m_v \leqslant m$ and the edge groups are isomorphic to $\mathbb{Z}^2$.*

3. Affordable housing

Let following definition is due to Farber [Far98]. Let Γ be a countable group and let Sub_Γ denote the space of subgroups of Γ equipped with the induced topology from $\{0,1\}^\Gamma$ equipped with the topology of pointwise convergence. The subspace $\mathrm{Sub}_\Gamma^{\mathrm{fi}}$ consisting of finite index subgroups of Γ is countable if Γ is finitely generated. Note that Γ acts continuously on both spaces. For $\gamma \in \Gamma$ define the *fixed point ratio function* as

$$\mathrm{fx}_{\Gamma,\gamma} \colon \mathrm{Sub}_\Gamma^{\mathrm{fi}} \to [0,1] \quad \text{by} \quad \Gamma' \mapsto \frac{|\{g\Gamma' \mid \gamma g \Gamma' = g\Gamma'\}|}{|\Gamma : \Gamma'|}.$$

A sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of subgroups of Γ is a *Farber sequence* if it consists of finite index subgroups and for every $\gamma \in \Gamma \backslash \{1\}$ we have

$$\lim_{n \to \infty} \mathrm{fx}_{\Gamma,\gamma}(\Gamma_n) = 0.$$

To compute the homology gradients of Farber sequences we will use the *cheap α-rebuilding* property of Abert–Bergeon–Fraczyk–Gaboriau [ABFG21]. The actual definition of this property is technical and need not concern us — instead we will use a combination theorem from loc. cit.. The relevance of the α-rebuilding property for us is the following deep theorem.

Theorem 3.1 (Abert–Bergeron–Fraczyk–Gaboriau). [ABFG21, Theorem H] *Let Γ be a countable group of type $\mathsf{F}_{\alpha+1}$ that has the cheap α-rebuilding property for some $\alpha \in \mathbb{N}$. Then, for every Farber sequence (Γ_n) of Γ and $0 \leqslant j \leqslant n$ we have $t_j^{(2)}(G; \Gamma_n) = 0$.*

We will need the following lemmas.

Lemma 3.2 (Abert–Bergeron–Fraczyk–Gaboriau). [ABFG21, Corollary 10.13(1)] *Let $\alpha \in \mathbb{N}$. Let G be a residually finite group with finite index subgroup H. Then G has the cheap α-rebuilding property if only if H does.*

Lemma 3.3 (Abert–Bergeron–Fraczyk–Gaboriau). [ABFG21, Corollary 10.13(2)] *If a residually finite group G of type VF has a normal subgroup isomorphic to $\mathbb{Z}$, then G has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.*

At last, the promised combination theorem:

Theorem 3.4 (Abert–Bergeron–Fraczyk–Gaboriau). [ABFG21, Theorem F] *Let Γ be a residually finite group acting on a CW complex X such that any element stabilising a cell fixes it pointwise. Let $\alpha \in \mathbb{N}$ and assume that the following conditions hold:*

(1) X/Γ has finite α-skeleton;
(2) X is $(\alpha - 1)$-connected;
(3) the stabiliser of every cell of dimension $j \leqslant \alpha$ has the cheap $(\alpha - j)$-rebuilding property.

Then, Γ itself has the cheap α-rebuilding property.

4. Proof of the main theorem

Proposition 4.1. *Let $\Gamma = F_m \rtimes_\varphi \mathbb{Z}$ be a free-by-cyclic group with polynomially growing monodromy φ of degree d. Then, Γ has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.*

Proof. We proceed by induction on d. Note that for any $k \neq 0$, the proposition holds for φ if and only if it holds for φ^k. Indeed, the group $F_m \rtimes_{\varphi^k} \mathbb{Z}$ is a finite index subgroup of Γ, and by Lemma 3.2, the cheap α-rebuilding property is a commensurability invariant. Therefore for all d we begin by passing to a unipotent power (when $d = 0$, a unipotent power is inner), and carry out the induction assuming Φ is UPG.

Base cases: *periodic and linearly growing.*

If $d = 0$, then $F_m \rtimes_\varphi \mathbb{Z} \cong F_m \times \mathbb{Z}$. By Lemma 3.3 such a group has the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.

Suppose that $d = 1$. By Proposition 2.6, Γ splits as a finite graph of groups such that the vertex groups are isomorphic to $F_{k_v} \times \mathbb{Z}$ with $k_v \leqslant n$ and the edge groups are isomorphic to $\mathbb{Z}^2$. Note that this implies the stabilisers of Γ on the Bass-Serre tree $\mathcal{T}$ of this splitting are all isomorphic to $F_{k_v} \times \mathbb{Z}$ or $\mathbb{Z}^2$ with $k_v \leqslant m$. Observe that by Lemma 3.3 the groups $F_{k_v} \times \mathbb{Z}$ and $\mathbb{Z}^2$ have the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$. The result follows from applying Theorem 3.4 to $\mathcal{T}$.   ◆

Induction step:

We now assume that the growth d of the monodromy φ is at least quadratic and that any free-by-cyclic group with monodromy polynomially growing of degree at most $d-1$ has the cheap α-rebuilding property for all $\alpha \geqslant 0$. We continue to assume φ is unipotent.

It follows from Proposition 2.5 that Γ splits as a graph of groups such that the vertex groups are free-by-cyclic groups with polynomially growing monodromy of degree at most $d - 1$ and the edge groups are isomorphic to $\mathbb{Z}$. Now, Γ acts cocompactly on the Bass-Serre tree $\mathcal{T}$ of this graph of groups. By the induction hypothesis the stabilisers of this action admit the cheap α-rebuilding property for all $\alpha \in \mathbb{N}$.   ◆

This completes the proof of the proposition.   □

Proof of Theorem A. This follows immediately from Proposition 4.1 and Theorem 3.1.   □

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