Assignment 2

AI1110: Probability and Random Variables
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12.13.6.18: Question. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Answer: 0.5

Solution: Using Markov Chain Approach,
Let the states **S** be,

1) 0 : The initial state.
2) 1 : 3 comes on die throw.
3) 2 : 6 comes on die throw.
4) 3 : non-multiple of 3 comes on die throw.
5) 4 : Heads comes up on coin toss.
6) 5 : Tails comes up on coin toss.

The Markov Chain for the given states is as follows,

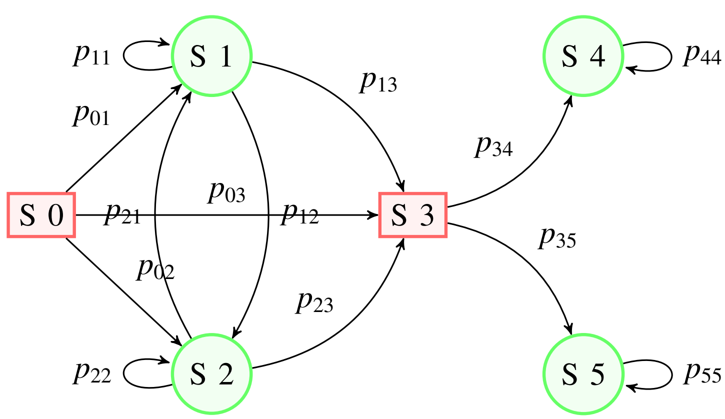

Figure: Markov Chain

- Here, **States 4 and 5** are **Absorbing States** indicating that once they are reached, They cannot be left.While **States 0, 1, 2, 3** are **Transient States** whose probability will eventually become 0.

The State Transition Matrix is as follows,

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Now, Let the Probability of state i at time t be $P_i^{(t)}$.Then the state vector will be:

$$\mathbf{B_t} = \begin{pmatrix} P_0^{(t)} & P_1^{(t)} & P_2^{(t)} & P_3^{(t)} & P_4^{(t)} & P_5^{(t)} \end{pmatrix} \tag{2}$$

From here we can write probability of states one step in time after t as,

$$P_0^{t+1} = 0 \tag{3}$$

$$P_1^{t+1} = \frac{1}{6} \times P_0^t + \frac{1}{6} \times P_1^t + \frac{1}{6} \times P_2^t \tag{4}$$

$$P_2^{t+1} = \frac{1}{6} \times P_0^t + \frac{1}{6} \times P_1^t + \frac{1}{6} \times P_2^t \tag{5}$$

$$P_3^{t+1} = \frac{4}{6} \times P_0^t + \frac{4}{6} \times P_1^t + \frac{4}{6} \times P_2^t \tag{6}$$

$$P_4^{t+1} = \frac{1}{2} \times P_3^t + 1 \times P_4^t \tag{7}$$

$$P_5^{t+1} = \frac{1}{2} \times P_3^t + 1 \times P_5^t \tag{8}$$

- These equations can be written directly as:

$$\mathbf{B_{t+1}} = \mathbf{B_t} \times \mathbf{A} \tag{9}$$

To Find,

$$\lim_{t \to \infty} P_5^{(t)} \tag{10}$$

From (9),

$$\mathbf{B_t} = \mathbf{B_{t-1}} \times \mathbf{A} = \mathbf{B_{t-2}} \times \mathbf{A^2} \cdots = \mathbf{B_0} \times \mathbf{A^t} \tag{11}$$

The Initial State $\mathbf{B_0}$ can be denoted by given that 3 occurs atleast once,

$$\mathbf{B_0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

The last column of $\mathbf{A^t}$ when $t \to \infty$ can be written as,

$$\mathbf{A^t} = \begin{pmatrix} . & . & . & . & . & \frac{1}{2} \\ . & . & . & . & . & \frac{1}{2} \\ . & . & . & . & . & \frac{1}{2} \\ . & . & . & . & . & \frac{1}{2} \\ . & . & . & . & . & 0 \\ . & . & . & . & . & 1 \end{pmatrix} \tag{13}$$

$$\lim_{t \to \infty} \mathbf{B_0} = \begin{pmatrix} . & . & . & . & . & 0.5 \end{pmatrix} \tag{14}$$

Hence From (2),

$$\implies \lim_{t \to \infty} P_5^{(t)} = 0.5 \tag{15}$$