

Assignment 2

AI1110: Probability and Random Variables
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12.13.6.18: Question. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Answer: 0

Solution: Using Markov Chain Approach,
Let the states **S** be,

- 1) State 1 : 3 or 6 comes on die throw.
- 2) State 2 : non-multiple of 3 comes on die throw.
- 3) State 3 : Heads comes up on coin toss.
- 4) State 4 : Tails comes up on coin toss.

- The Markov Chain for the given states is as follows,

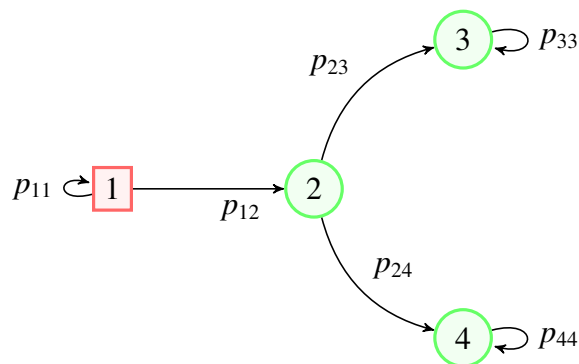


Fig.1 : MarkovChain

- Here, **States 4 and 5** are **Absorbing States** indicating that once they are reached, They cannot be left. While **States 0, 1, 2, 3** are **Transient States** whose probability will eventually become 0.

- The State Transition Matrix is as follows,

$$\mathbf{A} = \begin{pmatrix} \frac{2}{6} & 0 & 0 & 0 \\ \frac{4}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \quad (1)$$

Now, Let the Probability of state i at time t be $P_i^{(n)}$. Then the state vector will be:

$$\mathbf{B}_n = \begin{pmatrix} P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \\ P_4^{(n)} \end{pmatrix} \quad (2)$$

From here we can write probability of states one step in time after t as,

$$P_1^{(n+1)} = \frac{2}{6} \times P_1^{(n)} \quad (3)$$

$$P_2^{(n+1)} = \frac{4}{6} \times P_1^{(n)} \quad (4)$$

$$P_3^{(n+1)} = \frac{1}{2} \times P_2^{(n)} + 1 \times P_3^{(n)} \quad (5)$$

$$P_4^{(n+1)} = \frac{1}{2} \times P_2^{(n)} + 1 \times P_4^{(n)} \quad (6)$$

$$(7)$$

- These equations can be written directly as:

$$\mathbf{B}_{n+1} = \mathbf{A}\mathbf{B}_n \quad (8)$$

To Find,

$$\lim_{n \rightarrow \infty} P_4^{(n)} \quad (9)$$

From (8),

$$\mathbf{B}_n = \mathbf{B}_{n-1} \times \mathbf{A} = \mathbf{B}_{n-2} \times \mathbf{A}^2 \cdots = \mathbf{B}_0 \times \mathbf{A}^n \quad (10)$$

The Initial State \mathbf{B}_0 can be denoted by given that 3 occurs atleast once,

$$\mathbf{B}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (11)$$

Now using **Eigen Decomposition** to find the Conditional Probability, let λ be the eigen value for the transition matrix \mathbf{A}

$$\Rightarrow |P - \lambda I_4| = 0 \quad (12)$$

$$\Rightarrow \begin{pmatrix} 2/6 - \lambda & 0 & 0 & 0 \\ 4/6 & -\lambda & 0 & 0 \\ 0 & 1/2 & 1 - \lambda & 0 \\ 0 & 1/2 & 0 & 1 - \lambda \end{pmatrix} = 0 \quad (13)$$

$$\Rightarrow \left(\frac{2}{6} - \lambda\right)(-\lambda)(1 - \lambda^2) = 0 \quad (14)$$

$$\Rightarrow \lambda = \frac{2}{6}, 0, 1, 1 \quad (15)$$

Now Finding the eigen vectors, Lets assume it to be \mathbf{E}

$$\mathbf{E} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad (16)$$

$$\Rightarrow (A - \lambda I)\mathbf{E} = 0 \quad (17)$$

$$1) \lambda = \frac{2}{6}$$

$$\mathbf{E} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \\ 1 \end{pmatrix} \quad (18)$$

$$2) \lambda = 0$$

$$\mathbf{E} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \quad (19)$$

$$3) \lambda = 1$$

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (20)$$

Now, forming the eigenvector matrix from all the value of \mathbf{E} obtained, Lets assume it to be \mathbf{X}

$$\mathbf{X} = \begin{pmatrix} -2/3 & 0 & 0 & 0 \\ -4/3 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad (21)$$

Now, we can also write A as :

$$A = \mathbf{X}\mathbf{Z}\mathbf{X}^{-1} \quad (22)$$

Where \mathbf{Z} is eigenvalue matrix,

$$\mathbf{Z} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (23)$$

We also know,

$$B_n = A^n B_0 \quad (24)$$

and

$$A^n = (XZX^{-1})(XZX^{-1}) \dots (XZX^{-1}) \quad (25)$$

$$\Rightarrow A^n = XZ^n X^{-1} \quad (26)$$

$$\Rightarrow \lim_{n \rightarrow \infty} A^n = \lim_{n \rightarrow \infty} XZ^n X^{-1} \quad (27)$$

Now,

$$\lim_{n \rightarrow \infty} \mathbf{Z}^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (28)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \mathbf{XZ}^n \mathbf{X}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{A}^n \quad (29)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \mathbf{B}_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{B}_0 \quad (30)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \mathbf{B}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (31)$$

From here we can say that the Required Conditional Probability is,

$$\lim_{n \rightarrow \infty} P_4^{(n)} = 0 \quad (32)$$