Assignment 2

AI1110: Probability and Random Variables
Indian Institute of Techonology Hyderabad

Yasir Usmani
AI22BTECH11031

12.13.6.18: Question. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Answer: 0

Solution: Using Markov Chain Approach,
Let the states **S** be,

1) State 1 : 3 or 6 comes on die throw.

2) State 2 : non-multiple of 3 comes on die throw.

3) State 3 : Heads comes up on coin toss.

4) State 4 : Tails comes up on coin toss.

- The Markov Chain for the given states is as follows,

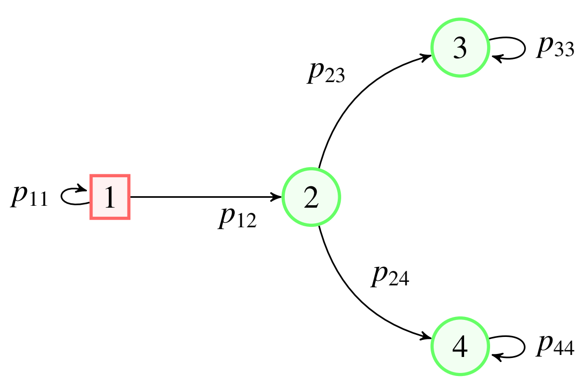

Fig. 4: Markov Chain

- Here, **States 4 and 5** are **Absorbing States** indicating that once they are reached, They cannot be left.While **States 0, 1, 2, 3** are **Transient States** whose probability will eventually become 0.

- The State Transition Matrix is as follows,

$$\mathbf{A} = \begin{pmatrix} \frac{2}{6} & 0 & 0 & 0 \\ \frac{4}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \tag{1}$$

Now, Let the Probability of state i at time t be $P_i^{(n)}$.Then the state vector will be:

$$\mathbf{B_n} = \begin{pmatrix} P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \\ P_4^{(n)} \end{pmatrix} \tag{2}$$

From here we can write probability of states one step in time after t as,

$$P_1^{(n+1)} = \frac{2}{6} \times P_1^{(n)} \tag{3}$$

$$P_2^{(n+1)} = \frac{4}{6} \times P_1^{(n)} \tag{4}$$

$$P_3^{(n+1)} = \frac{1}{2} \times P_2^{(n)} + 1 \times P_3^{(n)} \tag{5}$$

$$P_4^{(n+1)} = \frac{1}{2} \times P_2^{(n)} + 1 \times P_4^{(n)} \tag{6}$$

$$\tag{7}$$

- These equations can be written directly as:

$$\mathbf{B_{n+1}} = \mathbf{AB_n} \tag{8}$$

To Find,

$$\lim_{n\to\infty} P_4^{(n)} \tag{9}$$

From (8),

$$\mathbf{B_n} = \mathbf{B_{n-1}} \times \mathbf{A} = \mathbf{B_{n-2}} \times \mathbf{A^2} \cdots = \mathbf{B_0} \times \mathbf{A^n} \tag{10}$$

The Initial State $\mathbf{B_0}$ can be denoted by given that 3 occurs atleast once,

$$\mathbf{B_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

Now using **Eigen Decomposition** to find the Conditional Probability, let λ be the eigen value for the transition matrix **A**

$$\implies |\ P - \lambda I_4\ | = 0 \tag{12}$$

$$\implies \begin{pmatrix} 2/6-\lambda & 0 & 0 & 0 \\ 4/6 & -\lambda & 0 & 0 \\ 0 & 1/2 & 1-\lambda & 0 \\ 0 & 1/2 & 0 & 1-\lambda \end{pmatrix} = 0 \tag{13}$$

$$\implies (\frac{2}{6} - \lambda)(-\lambda)(1 - \lambda^2) = 0 \tag{14}$$

$$\implies \lambda = \frac{2}{6}, 0, 1, 1 \tag{15}$$

Now Finding the eigen vectors,Lets assume it to be E

$$\mathbf{E} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{16}$$

$$\implies (A - \lambda I)E = 0 \tag{17}$$

1) $\lambda = \frac{2}{6}$

$$\mathbf{E} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{4}{3} \\ 1 \\ 1 \end{pmatrix} \tag{18}$$

2) $\lambda = 0$

$$\mathbf{E} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \tag{19}$$

3) $\lambda = 1$

$$\mathbf{E} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

Now, forming the eigenvector matrix from all the value of **E** obtained,Lets assume it to be **X**

$$\mathbf{X} = \begin{pmatrix} -2/3 & 0 & 0 & 0 \\ -4/3 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{21}$$

Now, we can also write A as :

$$A = XZX^{-1} \tag{22}$$

Where Z is eigenvalue matrix,

$$\mathbf{Z} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

We also know,

$$B_n = A^n B_0 \tag{24}$$

and

$$A^n = (XZX^{-1})(XZX^{-1})\dots(XZX^{-1}) \tag{25}$$

$$\implies A^n = XZ^nX^{-1} \tag{26}$$

$$\implies \lim_{n\to\infty} A^n = \lim_{n\to\infty} XZ^nX^{-1} \tag{27}$$

Now,

$$\lim_{\mathbf{n}\to\infty} \mathbf{Z^n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

$$\implies \lim_{\mathbf{n}\to\infty} \mathbf{XZ^nX^{-1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{A^n} \tag{29}$$

$$\implies \lim_{\mathbf{n}\to\infty} \mathbf{B_n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{B_0} \tag{30}$$

$$\implies \lim_{\mathbf{n}\to\infty} \mathbf{B_n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

From here we can say that the Stationary Probability is,

$$\lim_{n\to\infty} P_4^{(n)} = 0 \tag{32}$$

To check whether the Required conditional probability is equal to the Stationary probability we will use concept of detailed balance, which is

$$z(i)p_{ij} = z(j)p_{ji} \tag{33}$$

- where **z(i)** represents the probability of being in **state i**
- If this condition satisfies for all states, then we can say that the obtained stationary probability is equal to the required condiitional probability.

From (31), We can say that as $n \to \infty$, $\mathbf{z(i)} = 0$ for all states i.

$$\implies z(i)p_{ij} = z(j)p_{ji} = 0 \tag{34}$$

$\therefore$ Obtained Stationary Probability = Required Conditional Probability

$$\implies \lim_{n\to\infty} P_4^{(n)} = 0 \tag{35}$$

Required Conditional probability