Assignment 2

AI1110: Probability and Random Variables
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12.13.6.18: Question. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Answer: 0.5

Solution: Using Markov Chain Approach,
Let the states **S** be,

1) State 0 : The initial state.
2) State 1 : 3 comes on die throw.
3) State 2 : 6 comes on die throw.
4) State 3 : non-multiple of 3 comes on die throw.
5) State 4 : Heads comes up on coin toss.
6) State 5 : Tails comes up on coin toss.

- The Markov Chain for the given states is as follows,

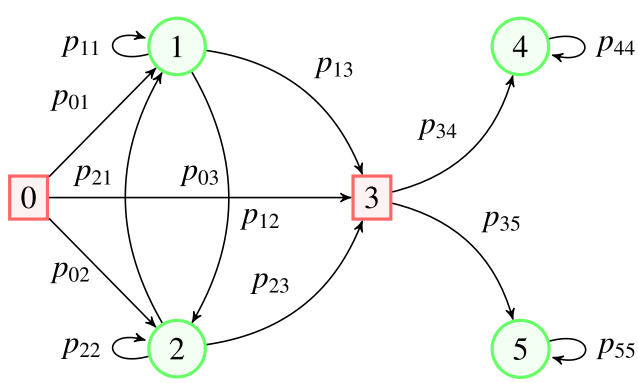

Figure: Markov Chain

- Here, **States 4 and 5** are **Absorbing States** indicating that once they are reached, They cannot be left.While **States 0, 1, 2, 3** are **Transient States** whose probability will eventually become 0.

- The State Transition Matrix is as follows,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{4}{6} & \frac{4}{6} & \frac{4}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \tag{1}$$

Now, Let the Probability of state i at time t be $P_i^{(n)}$.Then the state vector will be:

$$\mathbf{B_n} = \begin{pmatrix} P_0^{(n)} \\ P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \\ P_4^{(n)} \\ P_5^{(n)} \end{pmatrix} \tag{2}$$

From here we can write probability of states one step in time after t as,

$$P_0^{n+1} = 0 \tag{3}$$

$$P_1^{n+1} = \frac{1}{6} \times P_0^n + \frac{1}{6} \times P_1^n + \frac{1}{6} \times P_2^n \tag{4}$$

$$P_2^{n+1} = \frac{1}{6} \times P_0^n + \frac{1}{6} \times P_1^n + \frac{1}{6} \times P_2^n \tag{5}$$

$$P_3^{n+1} = \frac{4}{6} \times P_0^n + \frac{4}{6} \times P_1^n + \frac{4}{6} \times P_2^n \tag{6}$$

$$P_4^{n+1} = \frac{1}{2} \times P_3^n + 1 \times P_4^n \tag{7}$$

$$P_5^{n+1} = \frac{1}{2} \times P_3^n + 1 \times P_5^n \tag{8}$$

- These equations can be written directly as:

$$\mathbf{B_{n+1}} = \mathbf{A} \times \mathbf{B_n} \tag{9}$$

To Find,

$$\lim_{n\to\infty} P_5^{(n)} \tag{10}$$

From (9),

$$\mathbf{B_n} = \mathbf{B_{n-1}} \times \mathbf{A} = \mathbf{B_{n-2}} \times \mathbf{A^2} \cdots = \mathbf{B_0} \times \mathbf{A^n} \tag{11}$$

The Initial State $\mathbf{B_0}$ can be denoted by given that 3 occurs atleast once,

$$\mathbf{B_0} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

The last column of $\mathbf{A^n}$ when $n \to \infty$ can be written as,

$$\mathbf{A^n} = \begin{pmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix} \tag{13}$$

And,

$$\lim_{n\to\infty} \mathbf{B_n} = \begin{pmatrix} . \\ . \\ . \\ . \\ . \\ 0.5 \end{pmatrix} \tag{14}$$

Hence From (2),

$$\implies \lim_{n\to\infty} P_5^{(n)} = 0.5 \tag{15}$$