University of Hertfordshire UH

Dr S. Bunk, Dr M. Gordovskyy
School of Physics,
Astronomy and Mathematics
University of Hertfordshire

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Week 4 Tutorial/Lab

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(P1) Consider the following three vectors in $\mathbb{R}^3$:

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Is the set $\{b_1, b_2, b_3\}$ linearly independent?

(b) Does it form a basis of $\mathbb{R}^3$?

Hint: Recall that the linear independence is a condition on the solutions to the system of three linear equations given by

$$\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0\,.$$

Here, the variables are λ_1, λ_2, and λ_3.

(P2) Consider the basis of $\mathbb{R}^2$ given by the two vectors

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \in \mathbb{R}^2\,.$$

Let

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

be an arbitrary vector in $\mathbb{R}^2$. Compute the coefficients of v with respect to the basis (b_1, b_2), i.e. the (unique) numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$v = \lambda_1 b_1 + \lambda_2 b_2\,.$$

Hint: You can achieve this either by solving a system of two linear equations, or by inverting a certain (2×2)-matrix.

(P3) (a) Compute the determinants of the following matrices once using row or column operations, and once using the explicit formulas for determinants of (2×2)-matrices and (3×3)-matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ 1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix}.$$

(b) Decide whether these matrices are invertible and whether their columns for a basis of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Justify your answer.

(c) Bonus problem: use only a single row operation and $\det(B)$ from part (a) to compute the determinant of

$$B' = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix} .$$

(P4) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} .$$

(a) Let $\lambda \in \mathbb{R}$ be any real number, and I_2 the (2×2)-unit matrix. Compute the matrix

$$A - \lambda I_2 .$$

(b) Compute the determinant of $A - \lambda I_2$. This will depend on the parameter λ. For which values of λ is the matrix $A - \lambda I_2$ **not** invertible?

Hint: there are two such values; call them λ_1 and λ_2.

(c) For λ_1, find all solutions v to the system of linear equations

$$(A - \lambda_1 I_2)v = 0. \tag{1}$$

For λ_2, find all solutions v to the system of linear equations

$$(A - \lambda_2 I_2)v = 0. \tag{2}$$

Congratulations, you have just done your first matrix diagonalisation!

Bonus problem: Choose any non-zero solutions v_1 and v_2 to the equations (1) and (2), respectively. Write them as the two columns of a (2×2)-matrix S. Check that

$$A = S \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} S^{-1} .$$

FDS Week 4— Solutions

(P1) (a) The three vectors are linearly independent precisely when the *only* solution to

$$\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0$$

is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. The above equation of vectors in $\mathbb{R}^3$ amounts to the three linear equations

$$\begin{array}{llll} \lambda_1 & & +\lambda_3 & = 0 \\ \lambda_1 & +\lambda_2 & +2\lambda_3 & = 0 \\ \lambda_1 & +\lambda_2 & +\lambda_3 & = 0\,. \end{array}$$

We solve this, as always, using row operations:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \overset{\substack{r_2 \mapsto r_2 - r_1 \\ r_3 \mapsto r_3 - r_1}}{\longmapsto} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \overset{r_3 \mapsto r_3 - r_2}{\longmapsto} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$
$$\overset{r_3 \mapsto -r_3}{\longmapsto} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \overset{\substack{r_1 \mapsto r_1 - r_3 \\ r_2 \mapsto r_2 - r_3}}{\longmapsto} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

From this we can read off that the only(!) solution to the original system $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0$ is given by $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, the set $\{b_1, b_2, b_3\}$ is linearly independent.

(b) A tuple $(b_1, \dots, b_k)$ of vectors in $\mathbb{R}^n$ is a basis precisely if it is linearly independent *and* $k = n$. Thus, (b_1, b_2, b_3) above form a basis.

(P2) We have to find coefficients λ_1 and λ_2 such that

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

That is, we have to solve the system of linear equations

$$\left(\begin{array}{cc|c} 1 & 1 & v_1 \\ 1 & -1 & v_2 \end{array}\right) .$$

We compute

$$\left(\begin{array}{cc|c} 1 & 1 & v_1 \\ 1 & -1 & v_2 \end{array}\right) \overset{r_2 \mapsto r_2 - r_1}{\longmapsto} \left(\begin{array}{cc|c} 1 & 1 & v_1 \\ 0 & -2 & v_2 - v_1 \end{array}\right) \overset{r_2 \mapsto -\frac{1}{2} r_2}{\longmapsto} \left(\begin{array}{cc|c} 1 & 1 & v_1 \\ 0 & 1 & \frac{1}{2}(v_1 - v_2) \end{array}\right)$$
$$\overset{r_1 \mapsto r_1 - r_2}{\longmapsto} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2}(v_1 + v_2) \\ 0 & 1 & \frac{1}{2}(v_1 - v_2) \end{array}\right) .$$

We thus obtain

$$\lambda_1 = \frac{1}{2}(v_1 + v_2)\,, \qquad \text{and} \qquad \lambda_2 = \frac{1}{2}(v_1 - v_2)\,.$$

Alternatively, we denote the coefficient matrix by

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

This is the basis transformation matrix from B to the standard basis E_2.

To obtain the transformation matrix from E_2 to B, we compute its inverse; we obtain

$$S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then, we have that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = S^{-1}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

(P3) Using row operations,we compute:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \overset{r_2 \mapsto r_2 - r_1}{\longmapsto} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}.$$

Adding a multiple of one row of A to another row of A produces a new matrix A' with the same determinant, i.e. $\det(A) = \det(A')$. Therefore,

$$\det(A) = \det\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = 1 \cdot 2 = 2\,.$$

Alternatively,

$$\det(A) = 1 \cdot 4 - 1 \cdot 2 = 4 - 2 = 2\,.$$

Since $\det(A) \neq 0$, it follows that A is invertible. This is equivalent to the statement that the columns of A form a basis of $\mathbb{R}^2$.

Moving on to the matrix B, we use row operations to compute:

$$B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ 1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix} \overset{r_1 \mapsto \frac{1}{2} r_1}{\longmapsto} B_1 = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 & -1 \\ 1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix} \overset{r_3 \mapsto r_3 - r_1}{\longmapsto} B_2 = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Now, we review the operations we have carried out and note which effect they had on the determinant: in the first step, we produced a new matrix B_1 from B by multiplying one row by the factor $\frac{1}{2}$. Thus,

$$\det(B_1) = \frac{1}{2}\det(B)\,, \qquad \text{or, equivalently} \qquad \det(B) = 2 \cdot \det(B_1)\,.$$

B_2 is now of upper-triangular form. In the second operation we added a multiple of one row of B_1 to another row of B_1, which does not change the determinant; thus,

$$\det(B_2) = \det(B_1)\,.$$

Putting everything together, we have

$$\det(B) = 2 \cdot \det(B_1) = 2 \cdot \det(B_2) = 2 \cdot 1 \cdot 2 \cdot 3 = 12\,.$$

Alternatively, we can use the Leibniz Rule to compute

$$\begin{aligned}
\det(B) &= 2 \cdot 2 \cdot \frac{5}{2} + 1 \cdot (-1) \cdot 1 + 0 \cdot \frac{1}{2} \cdot (-1) - 1 \cdot 2 \cdot (-1) - \frac{1}{2} \cdot (-1) \cdot 2 - 0 \cdot 1 \cdot \frac{5}{2} \\
&= 10 - 1 + 0 + 2 + 1 - 0 \\
&= 12\,.
\end{aligned}$$

Since $\det(B) \neq 0$, it follows that B is invertible. This is equivalent to the statement that the columns of B form a basis of $\mathbb{R}^3$.

Bonus: B' arises from B by swapping row one and row two. Therefore,

$$\det(B') = -\det(B) = -12\,.$$

(P4) (a) We compute

$$A - \lambda I_2 = \begin{pmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix}.$$

(b) We compute

$$\begin{aligned}
&\det \begin{pmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{pmatrix} \\
&= (1-\lambda)(2-\lambda) - 2 \\
&= 2 - 3\lambda + \lambda^2 - 2 \\
&= \lambda^2 - 3\lambda \\
&= \lambda(\lambda - 3)\,.
\end{aligned}$$

This is the characteristic polynomial of A. The zeros of this polynomial, and thus the eigenvalues of A are

$$\lambda_1 = 0\,, \qquad \lambda_2 = 3\,.$$

each with algebraic multiplicity one, i.e.

$$\mu_{\text{alg}}(0) = 1 \qquad \text{and} \qquad \mu_{\text{alg}}(3) = 1\,.$$

(c) An eigenvector of A for eigenvalue λ is a non-zero vector $v \in \mathbb{R}^2$ satisfying the equation

$$Av = \lambda v\,, \qquad \text{or, equivalently,} \qquad (A - \lambda I_2)v = 0\,,$$

where I_2 is the $2{\times}2$-unit matrix. Thus, we have to find all non-zero solutions v to this equation, for $\lambda_1 = 0$ and $\lambda_2 = 3$.

For the eigenvalue $\lambda_1 = 0$, we have to solve the system

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \overset{r_2 \mapsto r_2 - r_1}{\longmapsto} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

The solutions to this system are the vectors

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{with} \qquad x + 2y = 0\,.$$

Thus, the solutions to our system are given by

$$\left\{ \begin{pmatrix} -2t \\ t \end{pmatrix} , \middle|\, t \in \mathbb{R} \right\} = \left\{ t \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} .$$

Since there is **one** parameter in this set of solutions, the maximal number of eigenvectors of A with eigenvalue $\lambda_0 = 0$ is **one**. We can take as our first eigenvector the vector

$$v_0 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} ,$$

or any non-zero multiple thereof (any such multiple would also be a correct answer). Hence,

$$\mu_{\text{geo}}(0) = 1 .$$

We move on to the eigenvalue $\lambda_1 = 3$: here we have to solve the system

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array} \right) .$$

We carry out row reduction steps:

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array} \right) \xmapsto[]{\substack{r_1 \mapsto -\frac{1}{2} r_1 \\ r_2 \mapsto r_2 + \frac{1}{2} r_1}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) .$$

Thus, the solutions are

$$\left\{ t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \middle|\, t \in \mathbb{R} \right\} ,$$

and since there is **one** parameter which parameterises this set of solutions, we can choose our second eigenvector as

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

or any non-zero multiple thereof. Hence,

$$\mu_{\text{geo}}(3) = 1 .$$

(d) (Bonus problem) Since $A \in \mathrm{Mat}(2{\times}2, \mathbb{R})$ and

$$\mu_{\text{geo}}(0) + \mu_{\text{geo}}(3) = 1 + 1 = 2 ,$$

the matrix A is diagonalisable.

The matrix S is the matrix whose columns consist of the eigenvectors of A; that is, here we obtain

$$S = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} .$$

Note that if you have used different eigenvectors in part (a), your matrix S will look different, but may still be a correct solution.

To compute $S\,D\,S^{-1}$, we need the inverse of S. Note that for $2{\times}2$-matrices we have a direct formula for the inverse,

$$S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix} = -\frac{1}{3} \cdot \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} .$$

Alternatively, we can use row reduction to find the inverse. We include this explicitly so that you can do it as an exercise and check yourself: we augment S by the unit matrix and compute

$$\left(\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \overset{r_1 \mapsto -\frac{1}{2} r_1}{\longmapsto} \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \overset{r_2 \mapsto r_2 - r_1}{\longmapsto} \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right)$$
$$\overset{r_2 \mapsto -\frac{2}{3} r_2}{\longmapsto} \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right) \overset{r_1 \mapsto r_1 + \frac{1}{2} r_2}{\longmapsto} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \end{array}\right)$$

So, indeed, we obtain the same expression for the inverse matrix S^{-1} as we got from our direct formula.

Finally, we compute

$$\begin{aligned} S\,D\,S^{-1} &= \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}}_{=S} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}}_{=D} \cdot \underbrace{\frac{1}{3} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}}_{=S^{-1}} \\ &= \frac{1}{3} \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= A\,. \end{aligned}$$