STEP III, 2017, Q10 MS

Question 10

The first result is obtained by conserving energy for the rod and particle together (rotational kinetic energy and gravitational potential energy) and simplifying the algebra. Differentiating that result with respect to time and then simplifying gives $2(3a^2+l^2)\ddot{\theta} = g(3a+2l)\cos\theta$. Alternatively, the same result can be obtained by taking moments about an axis through P. Resolving perpendicular to the rod for the particle and rearranging the equation generated yields an expression for the normal reaction, $mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$, having used the previously obtained expression for $\ddot{\theta}$. This is demonstrably positive under the given conditions. Resolving along the rod towards P (i.e. radially inwards) yields an expression for the friction which is simplified using the first obtained result of the question, and then applying the conditions for limiting friction yields the given result. In the case $l > 2a$, the particle loses contact immediately as the rod falls away quicker than the particle accelerates downwards; this can be shown either by considering the equation of rotational motion for the rod alone about P and finding $l\ddot{\theta} = \frac{lg}{2a}$, or by observing from previous working that the normal reaction of the rod on the particle would need to be negative for the particle to stay in contact with the rod.

Moment of inertia of PQ about axis through P is $\frac{1}{3}m(3a)^2 = 3ma^2$ **B1**

Conserving energy, $0 = \frac{1}{2}3ma^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mg\frac{3}{2}a\sin\theta - mgl\sin\theta$ **M1 A1 A1 A1**

Thus $(3a^2 + l^2)\dot{\theta}^2 = g(3a + 2l)\sin\theta$ **A1* (6)**

Differentiating with respect to time,

$$2(3a^2 + l^2)\dot{\theta}\ddot{\theta} = g(3a + 2l)\cos\theta\dot{\theta}$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Alternatively, taking moments about axis through P

$$m(3a^2 + l^2)\ddot{\theta} = mg\left(\frac{3}{2}a + l\right)\cos\theta$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Resolving perpendicular to the rod for the particle,

$$mg\cos\theta - R = ml\ddot{\theta}$$

M1 A1

Thus

$$R = mg\cos\theta - ml\ddot{\theta} = mg\cos\theta\left(1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)}\right)$$

M1 A1

Thus

$$R = mg\cos\theta - ml\ddot{\theta} = mg\cos\theta\left(1 - \frac{l(3a+2l)}{2(3a^2+l^2)}\right)$$

M1 A1

$$1 - \frac{l(3a+2l)}{2(3a^2+l^2)} = \frac{6a^2+2l^2-3al-2l^2}{2(3a^2+l^2)} = \frac{3a(2a-l)}{2(3a^2+l^2)} > 0$$

because $l < 2a$ **A1 (5)**

Resolving along the rod towards P for the particle,

$$F - mg\sin\theta = ml\dot{\theta}^2$$

M1 A1

Thus

$$F = mg\sin\theta + ml\dot{\theta}^2 = mg\sin\theta\left(1 + \frac{l(3a+2l)}{(3a^2+l^2)}\right) = mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right)$$

M1

On the point of slipping $F = \mu R$, so **B1**

$$mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right) = \mu mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$$

Thus

$$\tan\theta = \frac{\mu a(2a-l)}{2(a^2+al+l^2)}$$

A1* (5)

At the instant of release, the equation of rotational motion for the rod ignoring the particle is

$$mg\frac{3a}{2} = 3ma^2\ddot{\theta}$$

and thus

$$\ddot{\theta} = \frac{g}{2a}$$

M1

Therefore the acceleration of the point on the rod where the particle rests equals $l\ddot{\theta} = \frac{lg}{2a} > g$ if $l > 2a$, and so the rod drops away from the particle faster than the particle accelerates and the particle immediately loses contact. **A1 (2)**

(Alternatively, for particle to accelerate with rod from previous working $R < 0$, **M1** meaning that it would have to be attached to so accelerate, and as it is only placed on the rod, this cannot happen.) **A1 (2)**