Mixed-Integer Programming: MILP and MIQP

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Introduction

Optimization

- Optimization is a process of finding the best solution to a problem based on a set of objectives, subject to constraints.

Importance of Optimization in Real-World Applications

- Optimization is a widely used technique in various fields, such as finance, engineering, logistics, healthcare, and many others.
- It helps in making better decisions, improving efficiency, reducing costs, and achieving optimal outcomes.

Linear Programming

$$\text{minimize } q^T x \tag{1}$$

$$Ax \preceq b \tag{2}$$

To find decision variables that minimize (or maximize) a linear objective function, subject to a set of linear constraints.
The x is the vector of decision variables, q and b are vectors of coefficients, and A is a matrix of coefficients with number of rows and columns equal to the number of linear constraints and decision variable, respectively.

Quadratic Programming

$$\text{minimize } \frac{1}{2}x^T Qx + q^T x \tag{3}$$

$$Ax \preceq b \tag{4}$$

To find decision variables that minimize (or maximize) a quadratic objective function, subject to a set of linear constraints.
The x is the vector of decision variable, q and b are vectors of coefficients, A is a matrix of coefficients with number of rows and columns equal to the number of linear constraints and decision variable, respectively, and Q is a diagonal matrix of coefficients with number of rows and columns equal to the number of decision variable.

Formulation and Components

Formulation

- The formulation involves defining decision variables to find, an objective function to minimize or maximize, and linear constraints on the decision variables that must be satisfied.
- It needs to be noted that linear programming and quadratic programming is a problem formulation, not an algorithm.

Components

- Decision variables represent the quantities of the items to be produced, purchased, or allocated.
- Objective function is an expression that defines the value to be optimized, such as profit, revenue, or cost.
- Constraints are inequalities that restrict the values of the decision variables, such as production capacities, material availability, and demand constraints, and must be in the form of linear expression.

Decision Variables

- Decision variable can be strictly integer or continuous.
- If an optimization problem consist only strictly integer decision variables, it is called integer programming.
- If an optimization problem consist of both integer and continuous of decision variables, it is called mixed-integer programming.

Objective Functions

- Quadratic programming is linear programming with quadratic objective function.
- mixed-integer linear programming is mixed-integer programming with linear objective function
- mixed-integer quadratic programming is mixed-integer programming with quadratic objective function

Linear Constraints

- Each constraint in linear constraints must be expressed in linear expressions terms. It means that each constraint must be expressed as a summation of constants that is multiplied with decision variables raised to the power of 1.
- The constants itself can be 0 if a decision variable is not used in that constraint.
- The use of $\preceq$ in constraints means that each row of Ax must be less than the value of b at that row.

Example (problem)

Example: Clothing Manufacturing

A clothing manufacturer wants to produce summer and winter shirts, both made from cotton. The summer shirts require 2 yards of blue cotton and 3 yards of red cotton to produce. The winter shirts require 3 yards of blue cotton and 1 yard of red cotton to produce.

The manufacturer has a total of 30 yards of blue cotton and 20 yards of red cotton available to use for production. The profit from selling one summer shirt is $20, and the profit from selling one winter shirt is $25. The manufacturer wants to maximize their profit from selling the summer and winter shirts, while using no more than the available cotton. How many summer and winter shirts should they produce to achieve this goal?

Example (break downs)

- Decision variables: number of summer and winter shirts.
- Objective function: maximize profit.
- Constraints: available cotton.

Example (decision variables)

To solve this problem, the manufacturer can use linear programming. Let's define the decision variables as follows:

- Let x be the number of summer shirts produced.
- Let y be the number of winter shirts produced.

The manufacturer can only produce non-negative quantities of shirts. Therefore, the first constraint is decision variables bounds:

$$0 \leq x, y \leq \infty \tag{5}$$

Example (linear constraints)

The amount of blue cotton used for summer shirts is $2x$, and for winter shirts is $3y$. The total amount of blue cotton available is 30 yards. Therefore, the second constraint is:

$$2x + 3y \leq 30 \tag{6}$$

The amount of red cotton used for summer shirts is $3x$, and for winter shirts is $1y$. The total amount of red cotton available is 20 yards. Therefore, the third constraint is:

$$3x + y \leq 20 \tag{7}$$

Example (objective function)

The objective function that the manufacturer wants to maximize is the total profit:

$$20x + 25y \tag{8}$$

Example (linear programming formulation)

By combining the objective function and the constraints, we can formulate the following linear programming problem:
Maximize:

$$20x + 25y \tag{9}$$

Subject to:

$$\begin{align}
x, y &\geq 0 \tag{10}\\
x, y &\leq \infty \tag{11}\\
2x + 3y &\leq 30 \tag{12}\\
3x + y &\leq 20 \tag{13}\\
\tag{14}
\end{align}$$

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Thank You

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