# Incentive policies to reduce Carbon emissions

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# 1 Abstract

Carbon emission is amongst one of the most serious issues of our time. One of the main contributors for this externality is the Industry. In this project, we investigate using a rather simple model the efficiency of an incentive policy between a state and one representative of the companies. This falls under a Principal-Agent problem, with three parameters being the action taken by the company, the penalty imposed by the government and the salary given by the latter, with both the penalty and the salaries being incentives.

The characteristic of this problem is that the action taken is not observable, which puts us in the moral hazard context. This approach also takes the form of a Stackelberg game because the contract is optimized in two steps: first we find the action taken by the company, then the government applies penalties accordingly.

The goal of this project is to derive numerically or analytically the optimal action of the agent and evaluate the impact of the incentive policy with respect to this optimal action.

After testing and comparing the results [1], we concluded that in most cases, the companies can be incited to take considerable actions, yet the choice of the optimal contract needs to be discussed according to the needs and constraints of each party.



**Keywords** : Carbon emissions, incentive policy, Principal-Agent problem, Moral hazard, optimization.


# 2 Introduction

Climate change is one of the most controversial and serious issues of our time. One of its most infamous causes is pollution due to Carbon emissions whose origins are multiple and create polemics. The Industry contributes to these emissions. Thus to control pollution, actions ought to be taken : either by regulating Carbon emissions or by creating incentive policies with salaries. The latter intervenes on a fiscal level, where the government can impose penalties on companies, such as Carbon taxes. This is better known as the **Principal-Agent Problem**, where one party encourages the other to act in the former's best interests rather than the latter's. [2]

The aim of this contract is to encourage the agent to lower its Carbon emissions, by giving it a salary as an incentive. According to the action it decides to take, the principal can apply penalties. In our case however, there is asymmetric information, in the sense that the way the agent decides to perform under the terms of the contract is not observed by the principal, even though it does affect it. We are in the case of **Moral Hazard**.

Subsequently, the optimization of the contract is done in two steps, not all at once. We first optimize the company's problem, which will give us the action that was taken. Then, we optimize the principal's problem using the action previously set. This is how a Stackelberg leadership model is created. [3]

Another notion to be introduced is the utility : a utility function is a representation to define individual preferences for goods or services beyond the explicit monetary value of those goods or services. The economic problem of the principal is to construct a contract that can be accepted by the agent, and maximizes its utility. This is known as the agent participation constraint. The problem of this method of penalizing the agent after the action has been taken is the fact that other factors beyond the agent's control may get involved.

Mathematically, the agent's problem can be translated into the following general equations :

$$a^*(p) = \max_a \quad \mathbb{E}[U_A(p,a)].$$

Assuming here we find only one optimal action $a^*(p)$ , we then solve the principal's problem :

$$p^* = \max_p \quad \mathbb{E}[U_P(p,a^*(p))],$$

with respect to the constraint:

$$\mathbb{E}\Big[U_A\Big(p,a^*(p)\Big)\Big] \geq U_A(y).$$

Pragmatically speaking, we know that the choice of the agent's action is just arbitrary, as there are external factors it cannot control, such as the worldwide trend of Carbon emissions. This why we choose to optimize the action $a$, the penalty $p$ , then come back to optimizing $a$ according to the $p$ found.

## 3 Principal-Agent Problem under moral hazard

In our study, the government cannot control the company's actions towards pollution. The principal does not know the company's internal action. This puts us under the assumptions of moral hazard. This problem can be dealt with in two steps : the first is where we estimate the minimal action $a$ taken by the company, the second where we optimize the penalty $p$ and salary $s$ given this minimal $a$.

We are interested in a company whose Carbon emissions at time t are given by the discrete process :

$$E_t^a = \exp(X_t^a),$$

where $X$ is a stochastic process defined by :

$$\begin{cases} X_0^a &= x_0 \\ X_1^a &= x_0(\omega - a), \end{cases}$$

where $\omega \sim \mathcal{N}(0,1)$ and $a \in \mathbb{R}^+$ the action of the company.

In this rather simple model, the emission are represented by an exponential function as we considered that all company create pollution. Furthermore, this simulation is only one step, so the final time is 1. The company can only work on the initial emission $x_0$ whose can be reduced through the action $a$. Indeed, we can see that when $a$ increases $X_1^a$ decreases. We choose $a$ to be non negative. Otherwise, this can be interpreted as the company polluting on purpose.

The real random variable $\omega$ represents the proportion of pollution in the world that the principal (here the government) cannot control.

We aim to find a contract that optimizes the relationship between the two parties : the government and the company. Given a wage $s$ issued by the government at time $t = 0$ , the action $a$ taken by the company will have an effect on its level of emissions $X_1^a$ (at time $t = 1$) , and this will determine if it will pay a penalty $p$ as a tax, according to a threshold $B$ fixed beforehand.

We take into consideration four types of taxes :

$$P(x) = \begin{cases} p\,(x - B) & , \quad p \in (0,1) \\ p\,1_{x \geq B} & , \quad p \in \mathbb{R}_+^* \\ p\,x\,1_{x \geq B} & , \quad p \in (0,1) \\ p\,(x - B)_+ & , \quad p \in (0,1) \end{cases}$$

Our goal is to find the optimal parameters ($a$, $s$, $p$). Because of the hidden information, the principal does not know the action of the agent.

First, we have to solve the company's problem :

$$a^*(s,p) = \underset{a \in \mathbb{R}^+}{\arg\sup} \quad \mathbb{E}\Big[U_A\Big(s - K(a) - P(X_1^a)\Big)\Big],$$

where $U_A$ is the utility function defined as $U_A(x) = -\exp(-\gamma_A x)$ with $\gamma_A > 0$. This gives an optimal action denoted $a^*(s,p)$ that depends on the initial wage $s$ and the value of $p$ in the penalty function $P$.

Secondly, we turn to the government's problem :

$$(p^*, s^*) = \underset{p,s}{\arg\sup} \quad \mathbb{E}\Big[-E_1^a - (s - P(X_1^a))\Big],$$

such that :

$$\mathbb{E}\Big[U_A\Big(s - K(a) - P(X_1^a)\Big)\Big] \geq U_A(y).$$

The end result is an optimal incentive policy $\Big(a^*(s^*,p^*), s^*, p^*\Big)$.

Finally, we compute the optimal incentive policy for each penalty function before analyzing and comparing them.

## 3.1 Optimization with $P(x) = p(x-B)$ , $p \in (0,1)$

### 3.1.1 Optimization of the company's action $a$

The action depend on the wage and the penalty so the variable action is written $a(s,p)$. The optimization of the action for the first penalty is :

$$a^*(s,p) = \underset{a\in\mathbb{R}^+}{\arg\sup} \quad \mathbb{E}\Big[U_A\Big(s - K(a(s,p)) - P(X_1^{a(s,p)})\Big)\Big].$$

The problem can be written in the standard form: maximizing a function is equivalent to minimizing its opposite.[4]

So the global minimum of -f and the global maximum of f are identical, therefore :

$$a^*(s,p) = \underset{a\in\mathbb{R}^+}{\arg\inf} \quad -\mathbb{E}\Big[U_A\Big(s - K(a(s,p)) - P(X_1^{a(s,p)})\Big)\Big].$$

Developing expressions of each function :

$$a^*(s,p) = \underset{a\in\mathbb{R}^+}{\arg\inf} \quad -\mathbb{E}\Big[-e^{-\gamma_A\Big(s-\frac{k(a(s,p))^2}{2}-p\Big(x_0(\omega-a(s,p))-B\Big)\Big)}\Big],$$

where $\omega$ is the only random variable, all the others are deterministic and therefore can be taken out of the expected utility:

$$a^*(s,p) = \underset{a\in\mathbb{R}^+}{\arg\inf} \quad e^{-\gamma_A\big(s-\frac{k(a^*(s,p))^2}{2}+px_0a(s,p)+pB\big)}\ \mathbb{E}\Big[e^{-\gamma_A(-px_0\omega)}\Big]$$

$$a^*(s,p) = \underset{a\in\mathbb{R}^+}{\arg\inf} \quad e^{-\gamma_A\big(s-\frac{k(a(s,p))^2}{2}+px_0a(s,p)+pB\big)}\ e^{\frac{(\gamma_A px_0)^2}{2}}$$

$$a^*(s,p) = \underset{a\in\mathbb{R}^+}{\arg\inf} \quad f(a(s,p)).$$

$f$ is the function denoted as : $f(a(s,p)) = e^{-\gamma_A\big(s-\frac{k(a(s,p))^2}{2}+px_0a(s,p)+pB\big)}\ e^{\frac{(\gamma_A px_0)^2}{2}}$.

The problem is without constraint so it is equivalent to deriving the function with respect to $a$ and computing the stationary point,

$$\frac{\partial f(a(s,p))}{\partial a} = \gamma_A(ka(s,p) - px_0)\ e^{-\gamma_A(s-\frac{k(a(s,p))^2}{2}+px_0a(s,p)+pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}}.$$

The exponential functions and $\gamma_A$ are positive which means that :

$$\frac{\partial f(a^*(s,p))}{\partial a} = 0 \qquad \Leftrightarrow \qquad ka^*(s,p) - px_0 = 0 \qquad \Leftrightarrow \qquad a^*(s,p) = \frac{px_0}{k},$$

so :

$$a^*(s,p) = \frac{px_0}{k}.$$

The $a^*(s,p)$ does not depend on $s$ so, for more clarity, we only denote $a^*(p)$, and this notation will be use in all the computation of $p$ for the same reason, such that :

$$a^*(p) = \frac{px_0}{k}.$$

We compute the second order derivative and verify the nature of the point found above. We have that,

$$\frac{\partial^2 f(a^*(p))}{\partial a^2} = \gamma_A^2\ (ka^*(p) - px_0)^2\ e^{-\gamma_A\ (s-\frac{ka^*(p)^2}{2}+px_0a^*(p)+pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}} + \gamma_A\ k\ e^{-\gamma_A(s-\frac{ka^*(p)^2}{2}+px_0a^*(p)-pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}}.$$

We plug the expression of a*(p) :

$$\frac{\partial^2 f(\frac{px_0}{k})}{\partial a^2} = \gamma_A^2 \underbrace{(k\ (\frac{px_0}{k}) - px_0)}_{=0}^2\ e^{-\gamma_A\ (s - \frac{k\ (\frac{px_0}{k})^2}{2} + px_0\ (\frac{px_0}{k}) + pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}} + \gamma_A\ k\ e^{-\gamma_A (s - \frac{k\ (\frac{px_0}{k})^2}{2} + px_0\ (\frac{px_0}{k}) + pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}}$$

$$\frac{\partial^2 f(\frac{px_0}{k})}{\partial a^2} = \ \gamma_A\ k\ e^{-\gamma_A (s + \frac{(px_0)^2}{2k} + pB - \gamma_A \frac{(px_0)^2}{2})}.$$

The second derivative is a positive exponential function, therefore it is positive. Thus, $a^*(p) = \frac{px_0}{k}$ is the point of minimum global of $f(a^*(p))$. This is the solution of the company's problem.

### 3.1.2 Resolving the government's problem : optimization of the penalty $p$ and the salary $s$

The government's problem is an optimization problem with a constraint defined by :

$$\sup_{p,s} \quad \mathbb{E}\Big[ - E_1^a - (s - P(X_1^a))\Big],$$

such that

$$\mathbb{E}\Big[U_A\big(s - K(a) - P(X_1^a)\big)\Big] \geq U_A(y).$$

The problem can written in standard form :

$$\inf_{p,s} f(s,p) = \inf_{p,s} \ - \ \mathbb{E}\Big[ - E_1^a - (s - P(X_1^a))\Big],$$

with f the objective function defined by :

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(s,p) = \mathbb{E}\Big[E_1^a + (s - P(X_1^a))\Big],$$

under condition that:

$$g(s,p) = U_A(y) - \mathbb{E}\Big[\ U_A\big(s - K(a) - P(X_1^a)\big)\ \Big] \leq 0,$$

with the constraint $g : \mathbb{R}^2 \to \mathbb{R}$.
First, we have to make sure that the problem is convex ie the functions $f$ and $g$ are convex.

- Convexity of $f$ :

$$f(s,p) = \mathbb{E}\Big[E_1^a + (s - P(X_1^a))\Big],$$
$$= \mathbb{E}\Big[e^{x_0(\omega - a^*)} + s - p\big(x_0(\omega - a^*) - B\big)\Big]$$

by the linearity of the expectation, and assume $\omega \sim \mathcal{N}(0,1)$ and the Laplace transform, and knowing that $a^*(p) = \frac{px_0}{k}$ we obtain :

$$f(s,p) = e^{-a^* x_0 + \frac{x_0^2}{2}} + s + px_0 a^* + pB$$

The function $f$ is a sum of convex and differenciable functions with positive coefficients, $f$ is also convex and differentiable.

- convexity of $g$ :

$$g(s,p) = U_A(y) - \mathbb{E}\Big[U_A\big(s - K(a^*) - P(X_1^{a^*})\big)\Big]$$

by putting the expression of $a^*$ we have :

$$g(s,p) = -e^{-\gamma_A y} + \ e^{-\gamma_A (s - \frac{(px_0)^2}{2k} + \frac{(px_0)^2}{k} + pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}}$$

The function $g$ is a exponential function with positive coefficient, it is convex and differentiable.

Hence, the government's problem is convex and differentiable. We can resolve the optimization problem through the sufficient Karush–Kuhn–Tucker (KKT) conditions.

We define the Lagrangian $L(s,p)$ as :

$$L(s,p) = f(s,p) + \mu\, g(s,p),$$

with $\mu$ a positive KKT multiplier. To simplify the writing, we denote $a^*(s,p)$ as $a^*$ :

$$L(s,p) = \mathbb{E}\Big[E_1^{a*} + (s - P(X_1^{a*}))\Big] + \mu\Bigg(U_A(y) - \mathbb{E}\Big[U_A\big(s - K(a^*) - P(X_1^{a*})\big)\Big]\Bigg),$$

$$L(s,p) = e^{-\frac{px_0^2}{k}+\frac{x_0^2}{2}} + s + \frac{(px_0)^2}{k} + pB + \mu\Big(-e^{-\gamma_A y} + \ e^{-\gamma_A(s-\frac{(px_0)^2}{2k}+\frac{(px_0)^2}{k}+pB)}\ e^{\frac{(\gamma_A px_0)^2}{2}}\Big).$$

The KKT conditions are cited below, there is a pair $(p^*, s^*)$ such that,

$$\begin{cases} \frac{\partial L(p^*,s^*)}{\partial p} = 0 \\ \frac{\partial L(p^*,s^*)}{\partial s} = 0 \\ \mu\left(-e^{-\gamma_A y} + e^{-\gamma_A\left(s^*+\frac{(p^*x_0)^2}{2k}+p^*B\right)} e^{\frac{(\gamma_A p^* x_0)^2}{2}}\right) = 0. \end{cases}$$

It is equivalent to :

$$\begin{cases} -\frac{x_0^2}{k}e^{-\frac{p^*x_0^2}{k}+\frac{x_0^2}{2}} + \frac{2p^*x_0^2}{k} + B + \mu\left((\gamma_A x_0)^2 p^* + \gamma_A(\frac{-p^*x_0^2}{k} + \frac{x_0^2}{k} + B))e^{-\gamma_A(s^*-\frac{(p^*x_0)^2}{2k}+\frac{(p^*x_0)^2}{k}+p^*B)}\ e^{\frac{(\gamma_A p^* x_0)^2}{2}}\right) = 0 \\ 1 - \mu\gamma_A e^{-\gamma_A(s^*-\frac{(p^*x_0)^2}{2k}+\frac{(p^*x_0)^2}{k}+p^*B)}\ e^{\frac{(\gamma_A p^* x_0)^2}{2}} = 0 \\ \mu\Big(-e^{-\gamma_A y} + \ e^{-\gamma_A(s-\frac{(p^*x_0)^2}{2k}+\frac{(p^*x_0)^2}{k}+p^*B)}\ e^{\frac{(\gamma_A p^* x_0)^2}{2}}\Big) = 0. \end{cases}$$

- **First case : the constraint is inactive, ie $\mu = 0$.**

  Consequently, the KKT conditions are summarized as follows :

  $$\begin{cases} -\frac{x_0^2}{k}e^{-\frac{p^*x_0^2}{k}+\frac{x_0^2}{2}} + \frac{2p^*x_0^2}{k} + B = 0 \\ 1 = 0 \end{cases}$$

  However, we have a false equation so the point of minimum global does not exist with $\mu = 0$

- **Second case : the constraint is active, ie $\mu \neq 0$.**

  Considering that the problem is convex, we know that there is at least one solution $(p^*, s^*)$. Yet, we have just proven that there is no solution when the constraint is active. Consequently, the solution $(p*, s*)$ exists with active constraint.

  The procedure to resolve the system computes $p^*$ then $s^*$. The minimization of the Lagrangian $L$ on $p$ is independent of the value of $s$, so we can process in two steps.

  We notice that the constraint of participation is always saturated. The calculation of $s$ is based on the constraint of participation. The value of $s$ is the one which saturates the inequation ie
  $\mathbb{E}[U_A(s^* - K(a^*) - P(X_1^{a*}))] = U_A(y)$.

**Optimization of the penalty $p$**

This computation tries to find the optimal penalty $p^*$ with respect to the action $a^*(p)$ (The action is written $a^*(p)$ to simplify computation) previously found :

$$p^* = \underset{p\in(0,1)}{\arg\sup} \quad \mathbb{E}[-E_1^{a^*(p)} - (s - P(X_1^{a^*(p)}))].$$

As seen above, finding $p^*$ is equivalent to resolving this equation :

$$\frac{\partial f(s,p)}{\partial p} = 0 \qquad \Leftrightarrow \qquad -\frac{x_0^2}{k} e^{\left(-\frac{p^* x_0^2}{k} + \frac{x_0^2}{2}\right)} + \frac{2p^* x_0^2}{k} + B = 0$$

The form of this gradient is too complex for an explicit resolution. We turn to a different method : a gradient descent algorithm applied to the derivative of the objective function : for this $B$ and $K$ are chosen arbitrarily. The numerical application finds a minimum for $p = 0.5$ (Figure 1)

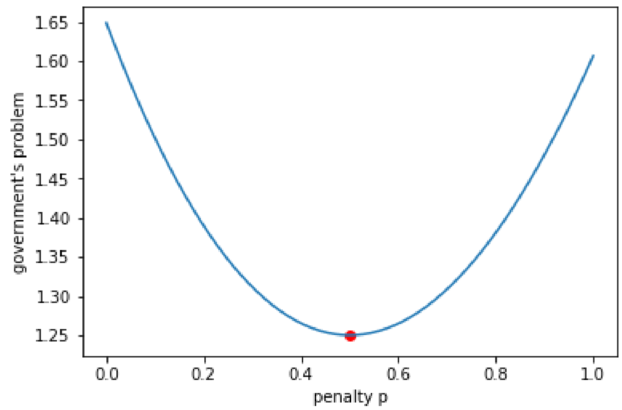


Figure 1: Representation of the company's problem depending on the penalty $p$

Do the values chosen for $B$ have an influence on the minimum $p^*$ ? To answer this question, we compute the minimum of $p^*$ according to $B$, with $B$ a non negative real number close to zero.

With Figure 2, the first conclusion is that the correlation between the two variables is linear : so the choice of $B$ does not change the order of magnitude of $p^*$.

But what if $B$ can be negative and large enough, but on a small interval ? Figure 2 shows that the correlation between the two variables looks like a logarithm function. This is coherent with the expression of $p^*$. This new result shows that we need to look further than the conditions set on the parameters.

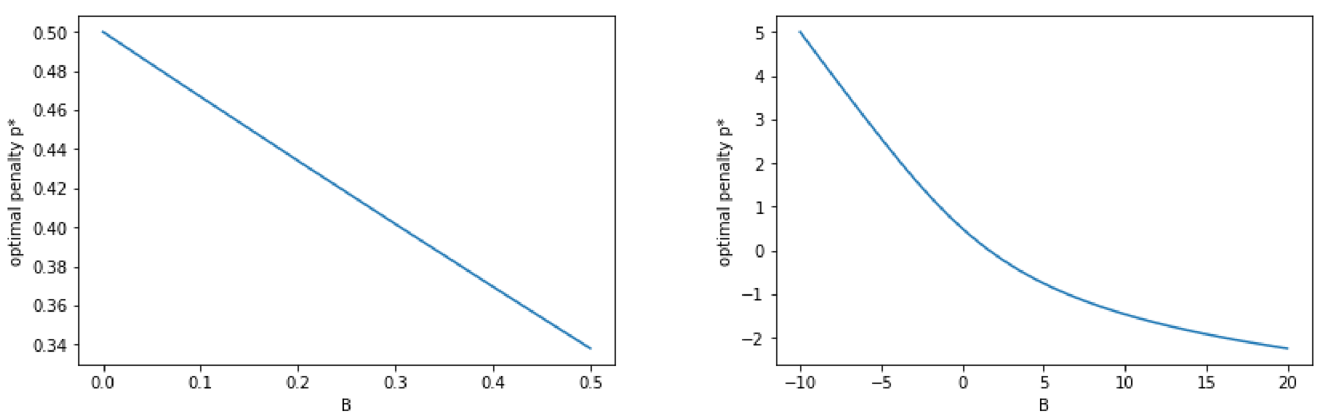


Figure 2: Optimization of p in function of B with fixed parameters s=0, k=1 and $x_0 = 1$

In order to verify that $p$ is the minimum, the second derivative is computed directly :

$$\frac{\partial^2 f(s,p)}{\partial p^2} = \frac{x_0^4}{k^2} e^{-\frac{p x_0^2}{k} + \frac{x_0^2}{2}} + \frac{2x_0^2}{k}.$$

This derivative is a sum of positive terms, hence :

$$\forall p \in (0,1) \quad , \frac{\partial^2 f(s,p)}{\partial p^2} \geq 0.$$

The point $p^*$ found with the gradient descent method is the minimum global. So $p^*$ is the approximation of the solution of the government's problem.

**Optimization of the salary s :** In all the computation for the salary, the action and penalty will be note $a^*$ and $p^*$. To recall, $s^*$ has to saturate the participation constraint :

$$\mathbb{E}\Big[U_A\Big(s^* - K(a^*) - P(X_1^{a*})\Big)\Big] = U_A(y).$$

Thus, it makes the problem easier, because it is just an equation with only one unknown parameter.

$$\mathbb{E}\Big[-e^{-\gamma_A\left(s^* - \frac{ka^{*2}}{2} - p^*(x_0(\omega - a^*) - B)\right)}\Big] = -e^{-\gamma_A y}$$

$$e^{-\gamma_A\left(s^* - \frac{ka^{*2}}{2} + p^* x_0 a^* + p^* B\right)}\mathbb{E}\Big[e^{-\gamma_A p^* x_0 \omega}\Big] = e^{-\gamma_A y}$$

$$e^{-\gamma_A\left(s^* - \frac{ka^{*2}}{2} + p^* x_0 a^* + p^* B\right)} e^{\frac{(\gamma_A p^* x_0)^2}{2}} = e^{-\gamma_A y}$$

$$e^{-\gamma_A\left(s^* - \frac{ka^{*2}}{2} + p^* x_0 a^* + p^* B - \gamma_A \frac{(p^* x_0)^2}{2}\right)} = e^{-\gamma_A y}.$$

As the exponential function is increasing, the order does not change, so :

$$s^* - \frac{ka^{*2}}{2} + p^* x_0 a^* + p^* B - \gamma_A \frac{(p^* x_0)^2}{2} = y$$

$$s^* = \frac{ka^{*2}}{2} - p^* x_0 a^* - p^* B + \gamma_A \frac{(p^* x_0)^2}{2} + y.$$

From now on, the optimization method of $p$ and $s$ for the three other penalties will be similar to this first case.

**To summarize the results of this section :** we have numerically found that $a^* = 0.18507, p^* = 0.18507$, and $s^*$ negative but as it as no sens on this problem we took $s^* = 0$, for a choice of parameters $k = x_0 = B = 1$ and $y_0 = 0$. We will compute the optimal contract for the three other policies before discussing the choice to be made.

## 3.2 Optimization with $P(x) = p1_{x \geq B}$ , $p \in \mathbb{R}_+^*$

### 3.2.1 Optimization of a

The optimization of $a(s,p)$ for the second penalty is :

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\sup}\ \mathbb{E}\Big[U_A\big(s - K(a(s,p)) - P(X_1^a(s,p))\big)\Big].$$

First the problem is written in the standard form, as seen in the first penalty :

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\ -\mathbb{E}\Big[U_A\big(s - K(a(s,p)) - P(X_1^a(s,p))\big)\Big].$$

By developing the expression of each function, we obtain the following :

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\ -\mathbb{E}\Big[-e^{-\gamma_A(s - \frac{ka(s,p)^2}{2} - p1_{x_0(\omega - a(s,p))\geq B})}\Big],$$

where $\omega$ is the only random variable, while all the others are deterministic ; they can therefore be taken out of the expected utility:

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\; e^{-\gamma_A(s-\frac{ka(s,p)^2}{2})}\;\mathbb{E}\Big[e^{\gamma_A p 1_{x_0(\omega-a(s,p))\geq B}}\Big].$$

Let us denote Z a Bernoulli variable with parameter $\mathbb{P}(x_0(\omega - a(s,p)) \geq B)$ such that :

$$\mathbb{E}[e^z] = e^1\,\mathbb{P}\big(x_0(\omega - a(s,p)) \geq B\big) + e^0\,\Big(1 - \mathbb{P}\big(x_0(\omega - a(s,p)) \geq B\big)\Big).$$

We plug it in the previous expression of $a^*(s,p)$ :

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\; e^{-\gamma_A(s-\frac{ka(s,p)^2}{2})}\Big[e^{\gamma_A p}\mathbb{P}(\omega \geq \frac{B}{x_0} + a(s,p)) + 1 - \mathbb{P}(\omega \geq \frac{B}{x_0} + a(s,p))\Big].$$

Since $\omega$ is a normal variable, we recognize its cumulative distribution function $\phi$ , hence,

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\; e^{-\gamma_A(s-\frac{ka(s,p)^2}{2})}\Big[e^{\gamma_A p}\big(1 - \phi(\frac{B}{x_0} + a(s,p))\big) + \phi(\frac{B}{x_0} + a(s,p))\Big].$$

Using this expression of $a^*(s,p)$, its value, for fixed parameters, can be numerically computed to find the coordinates of the minimum.

First, we fix the values of the parameters $k, B$ and $p$. This allows us to have a first overview of the function. Furthermore, the parameters can be changed manually to see the behaviour of $a^*(s,p)$. As $a^*(s,p)$ depends on $p$, we can see on Figure 3, the value of the minimum in function of $p$ for fixed parameters.

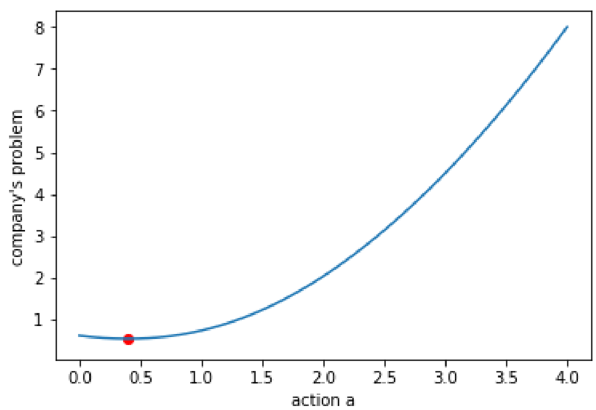


Figure 3: Optimization of the agent problem, with fixed parameters $B = 0$, $k = 1$ and $p = 1$

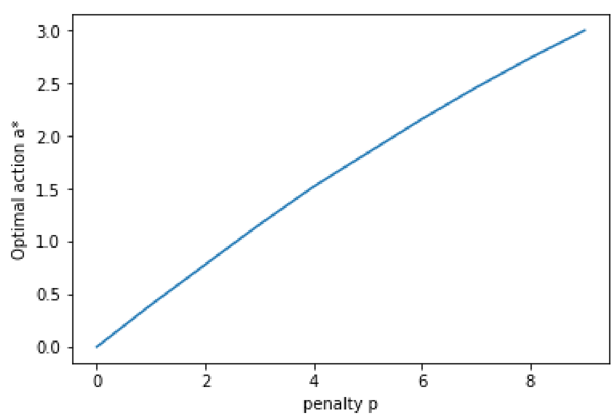


Figure 4: Optimization of $a^*(s,p)$ in function of $p$, with $B = 0$ and $k = 1$

On Figure 4, the minimum of $a$ according to $p$ has the same shape as a logarithm function. A logarithm expression is coherent with the expression of $a^*(s,p)$. At first glance, the function seems concave, so the minimum of $a$ should be for $p = 0$.

### 3.2.2 Numerical optimization $p$

To simplify the lecture, we denote the action $a^*(p)$ so the expression of $p^*$ for this penalty is:

$$p^* = \underset{p\in\mathbb{R}^+}{\arg\inf} \quad \mathbb{E}\Big[E_1^{a^*(p)} + (s - P(X_1^{a^*(p)}))\Big]$$

$$p^* = \underset{p\in\mathbb{R}^+}{\arg\inf} \quad \mathbb{E}\Big[e^{x_0(\omega - a^*(p))} + s - p\,1_{x_0(\omega - a^*(p))\geq B}\Big]$$

$$p^* = \underset{p\in\mathbb{R}^+}{\arg\inf} \quad e^{-x_0 a^*(p)}\mathbb{E}\big[e^{x_0\omega}\big] + s - p\,\mathbb{E}\big[1_{\omega\geq \frac{B}{x_0}+a^*(p)}\big]$$

$$p^* = \underset{p\in\mathbb{R}^+}{\arg\inf} \quad e^{-x_0 a^*(p) + \frac{x_0^2}{2}} + s - p\,\mathbb{P}\Big(\omega \geq \frac{B}{x_0} + a^*(p)\Big)$$

$$p^* = \underset{p\in\mathbb{R}^+}{\arg\inf} \quad e^{-x_0 a^*(p) + \frac{x_0^2}{2}} + s - p\,\Big(1 - \Phi(\frac{B}{x_0} + a^*(p))\Big).$$

To compute numerically the minimum of $p^*$, a value for $a$ is needed. So for each value of $p$ we first find the value of $a^*(p)$ and then compute $p^*$ using the optimal action $a^*(s,p)$. Finally, Figure 6 shows the value of $p^*$.

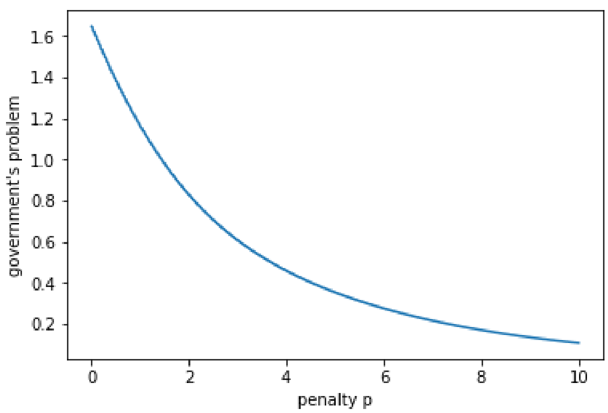


Figure 5: Optimization of the principal problem, with parameters $B = 1$ and $k = 1$

Figure 5 shows the value of the principal problem decreasing with $p$, the function look concave. So the minimum is the upper bound of the axis p : in this example $p = 10$.

### 3.2.3 Numerical optimization of s

$s^*$ satisfies the constraint saturation such as :

$$\mathbb{E}\Big[U_A(s^* - K(a^*) - P(X_1^{a^*}))\Big] = U_A(y).$$

Then by developing the terms, we have :

$$\mathbb{E}\left[-e^{-\gamma_A(s^* - \frac{ka^{*2}}{2} - p^* 1_{x_0(\omega - a^*)\geq B})}\right] = -e^{-\gamma_A y}$$

$$\mathbb{E}\left[e^{\gamma_A p^* 1_{\omega\geq \frac{B}{x_0}+a^*}}\right] = e^{-\gamma_A(y - s^* + \frac{ka^{*2}}{2})}.$$

However, by the definition of the expectation:

$$\mathbb{E}\left[e^{\gamma_A p^* 1_{\omega\geq \frac{B}{x_0}+a^*}}\right] = \int_{-\infty}^{+\infty} e^{\gamma_A p^* 1_{u\geq \frac{B}{x_0}+a^*}} \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u,$$

with the Chasles relation, we have :

$$\mathbb{E}\left[e^{\gamma_A p^* 1_{\omega \geq \frac{B}{x_0}+a^*}}\right] = \int_{-\infty}^{\frac{B}{x_0}+a^*} \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u + \int_{\frac{B}{x_0}+a^*}^{+\infty} e^{\gamma_A p^*}\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u.$$

In the first integral, we recognize the cumulative distribution function evaluated in the point $\frac{B}{x_0}+a^*$ , $\Phi\left(\frac{B}{x_0}+a^*\right)$. For the second integral, we have to rewrite it such that we identify a usual density function so :

$$\int_{\frac{B}{x_0}+a^*}^{+\infty} e^{\gamma_A p^*}\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u = e^{\gamma_A p^*}\underbrace{\int_{\frac{B}{x_0}+a^*}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u}_{1-\Phi\left(\frac{B}{x_0}+a^*\right)}.$$

Hence, back to the constraint saturation (initial problem):

$$\mathbb{E}\left[e^{\gamma_A p^* 1_{\omega \geq \frac{B}{x_0}+a^*}}\right] = \Phi\left(\frac{B}{x_0}+a^*\right) + \left(1-\Phi\left(\frac{B}{x_0}+a^*\right)\right)e^{\gamma_A p^*} = e^{-\gamma_A(y-s^*+\frac{ka^{*2}}{2})}.$$

By using the logarithm function, we obtain :

$$\ln\left[\Phi\left(\frac{B}{x_0}+^* a\right) + \left(1-\Phi\left(\frac{B}{x_0}+a^*\right)\right)e^{\gamma_A p^*}\right] = -\gamma_A(y-s^*+\frac{ka^{*2}}{2}),$$

so finally :

$$s^* = y + \frac{ka^{*2}}{2} + \frac{1}{\gamma_A}\ln\left[\Phi\left(\frac{B}{x_0}+a^*\right) + \left(1-\Phi\left(\frac{B}{x_0}+a^*\right)\right)e^{\gamma_A p^*}\right].$$

## 3.3 Optimization with $P(x) = px1_{x\geq B}$ , $p \in (0,1)$

### 3.3.1 Numerical optimization of $a$

The expression of $a^*(s,p)$ for the third penalty is:

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\sup}\ \mathbb{E}[U_A(s-K(a(s,p))-P(X_1^{a(s,p)}))].$$

We substitute the functions with their expressions. Hence we have :

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\sup}\ \mathbb{E}\left[-e^{-\gamma_A\left(s-\frac{ka^2}{2}-px_0(\omega-a(s,p))1_{x_0(\omega-a(s,p))\geq B}\right)}\right]$$

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\ e^{-\gamma_A\left(s-\frac{ka(s,p)^2}{2}\right)}\mathbb{E}\left[e^{\gamma_A p x_0(\omega-a(s,p))1_{\omega\geq\frac{B}{x_0}+a(s,p)}}\right].$$

Yet, by the definition of the expectation :

$$\mathbb{E}\left[e^{\gamma_A p x_0(\omega-a(s,p))1_{\omega\geq\frac{B}{x_0}+a(s,p)}}\right] = \int_{-\infty}^{\frac{B}{x_0}+a(s,p)} \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u + \int_{\frac{B}{x_0}+a(s,p)}^{+\infty} e^{\gamma_A p x_0(u-a(s,p))}\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u$$

$$\mathbb{E}\left[e^{\gamma_A p x_0(\omega - a(s,p))1_{\omega \geq \frac{B}{x_0}+a(s,p)}}\right] = \Phi\left(\frac{B}{x_0}+a(s,p)\right) + e^{-\gamma_A p x_0 a(s,p)} \int_{\frac{B}{x_0}+a(s,p)}^{+\infty} e^{\gamma_A p x_0 u} \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}} \mathrm{d}u$$

$$\mathbb{E}\left[e^{\gamma_A p x_0(\omega - a(s,p))1_{\omega \geq \frac{B}{x_0}+a(s,p)}}\right] = \Phi\left(\frac{B}{x_0}+a(s,p)\right) + e^{-\gamma_A p x_0 a(s,p)} \int_{\frac{B}{x_0}+a(s,p)}^{+\infty} \frac{e^{-\frac{1}{2}(u^2 - 2\gamma_A p x_0 u)}}{\sqrt{2\pi}} \mathrm{d}u$$

$$\mathbb{E}\left[e^{\gamma_A p x_0(\omega - a(s,p))1_{\omega \geq \frac{B}{x_0}+a(s,p)}}\right] = \Phi\left(\frac{B}{x_0}+a(s,p)\right) + e^{-\gamma_A p x_0 a(s,p)} \int_{\frac{B}{x_0}+a(s,p)}^{+\infty} \frac{e^{-\frac{1}{2}(u - \gamma_A p x_0)^2 + \frac{1}{2}\gamma_A p x_0}}{\sqrt{2\pi}} \mathrm{d}u$$

$$\mathbb{E}\left[e^{\gamma_A p_0(\omega - a(s,p))1_{\omega \geq \frac{B}{x_0}+a(s,p)}}\right] = \Phi\left(\frac{B}{x_0}+a(s,p)\right) + e^{-\gamma_A p x_0 a(s,p) + \frac{1}{2}(\gamma_A p x_0)^2} \int_{\frac{B}{x_0}+a(s,p)}^{+\infty} \frac{e^{-\frac{1}{2}(u - \gamma_A p x_0)^2}}{\sqrt{2\pi}} \mathrm{d}u.$$

We changed variables to identify the standard normal distribution : $z = u - \gamma_A p x_0$,

$$\mathbb{E}\left[e^{\gamma_A p x_0(\omega - a(s,p))1_{\omega \geq \frac{B}{x_0}+a}}\right] = \Phi\left(\frac{B}{x_0}+a(s,p)\right) + e^{-\gamma_A p x_0 a(s,p) + \frac{1}{2}(\gamma_A p x_0)^2} \int_{\frac{B}{x_0}+a(s,p)-\gamma_A p x_0}^{+\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \mathrm{d}z$$

.

So :

$$a^*(s,p) = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\; e^{-\gamma_A (s - \frac{k a(s,p)^2}{2})}[\Phi\left(\frac{B}{x_0}+a(s,p)\right) + e^{-\gamma_A p x_0 (a(s,p) - \frac{\gamma_A p x_0}{2})}(1 - \Phi\left(\frac{B}{x_0}+a(s,p) - \gamma_A p x_0\right)).$$

With the expression of $a^*(s,p)$, its value can be numerically computed.

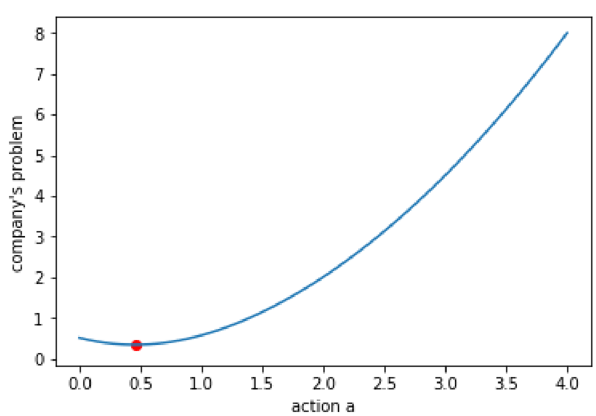


Figure 6: Optimization of the agent problem with fixed parameters $p = 1$ $B = 1$ and $k = 1$

Figure 6 shows the agent problem for fixed parameters. We see that we obtain a minimum near zero.

### 3.3.2 Numerical optimization of p

The expression of $p^*$ for this penalty is :

$$p^* = \underset{p\in(0,1)}{\arg\inf}\quad \mathbb{E}[E_1^{a^*(p)} + (s - P(X_1^{a^*(p)}))]$$

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad \mathbb{E}\Big[e^{x_0(\omega - a^*(p))} + s - px_0(\omega - a)1_{x_0(\omega-a^*(p))\geq B}\Big]$$

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad e^{-x_0a^*(p)}\mathbb{E}[e^{x_0\omega}] + s + px_0a^*(p)\mathbb{E}[1_{\omega\geq \frac{B}{x_0}+a^*(p)}] - px_0\mathbb{E}[\omega 1_{\omega\geq \frac{B}{x_0}+a^*(p)}]$$

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad e^{-x_0a^*(p)}e^{\frac{x_0^2}{2}} + s + px_0a^*(p)(1-\Phi\left(\frac{B}{x_0}+a^*(p)\right)) - px_0\mathbb{E}[\omega 1_{\omega\geq \frac{B}{x_0}+a^*(p)}].$$

Yet, we know that :

$$\mathbb{E}[\omega 1_{\omega\geq \frac{B}{x_0}+a^*(p)}] = \int_{\frac{B}{x_0}+a^*(p)-\gamma_A px_0}^{+\infty} u\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u,$$

we recognize the derivative function of $\frac{-e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}$. Consequently :

$$\int_{\frac{B}{x_0}+a^*(p)-\gamma_A px_0}^{+\infty} u\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u = \frac{1}{\sqrt{2\pi}}\left[-e^{\frac{-u^2}{2}}\right]_{\frac{B}{x_0}+a^*(p)}^{+\infty}$$

$$\mathbb{E}[\omega 1_{\omega\geq \frac{B}{x_0}+a^*(p)}] = \frac{1}{\sqrt{2\pi}}\left[-e^{\frac{-u^2}{2}}\right]_{\frac{B}{x_0}+a^*(p)}^{+\infty} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{B}{x_0}+a^*(p)\right)^2}.$$

Finally, we have that :

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad e^{-x_0a^*(p)+\frac{x_0^2}{2}} + s + px_0a^*(1-\Phi\left(\frac{B}{x_0}+a^*(p)\right)) - px_0\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{B}{x_0}+a^*(p)\right)^2}.$$

As before, we can numerically compute $p^*$ using this expression the value $a^*(p)$ is found for each $p$ from the previous section : the optimization of $a$.

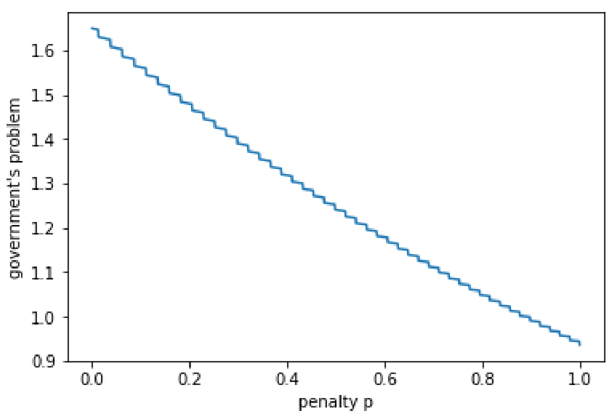


Figure 7: Optimization of the government's problem $p$ with $B = 1$ and $k = 1$

Figure 7 shows the value of the principal problem for fixed parameters. Like the previous penalty, the minimum is on the bond of the interval of p as the problem look like concave.

### 3.3.3 Numerical optimization of s

Since it is the same expression of company's problem, the optimization of s is :

$$\mathbb{E}\left[-e^{-\gamma_A\left(s^*-\frac{ka^{*2}}{2}-p^*x_0(\omega-a^*)1_{x_0(\omega-a^*)\geq B}\right)}\right]=-e^{-\gamma_A y}$$

$$e^{-\gamma_A(s^*-\frac{ka^{*2}}{2})}\left[\Phi(\frac{B}{x_0}+a^*)+e^{-\gamma_A p^* x_0(a^*-\frac{\gamma_A p^* x_0}{2})}(1-\Phi\left(\frac{B}{x_0}+a^*-\gamma_A p^* x_0\right)\right]=-e^{-\gamma_A y}.$$

We compute the logarithm of this expression :

$$-\gamma_A(s^*-\frac{ka^{*2}}{2})+ln\left[\Phi(\frac{B}{x_0}+a^*)+e^{-\gamma_A p^* x_0(a^*-\frac{\gamma_A p^* x_0}{2})}(1-\Phi(\frac{B}{x_0}+a^*-\gamma_A p^* x_0)\right]=-\gamma_A y.$$

Hence :

$$s^*=y+\frac{ka^{*2}}{2}+\frac{1}{\gamma_A}ln\left[\Phi(\frac{B}{x_0}+a^*)+e^{-\gamma_A p^* x_0(a^*-\frac{\gamma_A p^* x_0}{2})}(1-\Phi(\frac{B}{x_0}+a^*-\gamma_A p^* x_0)\right].$$

## 3.4 Optimization with $P(x)=p(x-B)_+$ , $p\in(0,1)$

### 3.4.1 Numerical optimization of a

The expression of $a^*(s,p)$ for the last penalty is :

$$a^*(s,p)=\underset{a(s,p)\in\mathbb{R}^+}{\arg\sup}\ \mathbb{E}[U_A(s-k(a(s,p))-P(X_1^{a(s,p)}))]$$

$$a^*(s,p)=\underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\ \mathbb{E}[e^{-\gamma_A(s-\frac{ka(s,p)^2}{2}-p(x_0(\omega-a(s,p))-B)_+)}]$$

$$a^*(s,p)=\underset{a(s,p)\in\mathbb{R}^+}{\arg\inf}\ e^{-\gamma_A(s-\frac{ka(s,p)^2}{2})}\mathbb{E}[e^{\gamma_A p(x_0(\omega-a(s,p))-B)_+}].$$

Yet, $(x_0(\omega-a(s,p))-B)_+$ is equivalent to the function $(x_0(\omega-a(s,p))-B)1_{x_0(\omega-a(s,p))\geq B}$ or $(x_0(\omega-a(s,p))-B)1_{\omega\geq\frac{B}{x_0}+a(s,p)}$.

We compute the expectation :

$$\mathbb{E}[e^{\gamma_A p(x_0(\omega-a(s,p))-B)_+}]=\int_{-\infty}^{\frac{B}{x_0}+a(s,p)}\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u+\int_{\frac{B}{x_0}+a(s,p)}^{+\infty}e^{\gamma_A p(x_0(u-a(s,p))-B)}\ \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u$$

$$\mathbb{E}[e^{\gamma_A p(x_0(\omega-a(s,p))-B)_+}]=\Phi\left(\frac{B}{x_0}+a(s,p)\right)+e^{-\gamma_A p(x_0 a(s,p)+B)}\int_{\frac{B}{x_0}+a(s,p)}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\gamma_A p x_0 u-\frac{u^2}{2}}\,\mathrm{d}u.$$

With the same process of above, we compute the integral such as to recover a density function :

$$\mathbb{E}[e^{\gamma_A p(x_0(\omega-a(s,p))-B)_+}]=\Phi\left(\frac{B}{x_0}+a(s,p)\right)+e^{-\gamma_A p(x_0 a(s,p)+B-\frac{(\gamma_A p x_0)^2}{2})}\int_{\frac{B}{x_0}+a(s,p)}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(u-\gamma_A p x_0)^2}{2}}\,\mathrm{d}u,$$

with the same change of variable as seen in the previous part, we obtain :

$$\mathbb{E}[e^{\gamma_A p(x_0(\omega-a(s,p))-B)_+}]=\Phi\left(\frac{B}{x_0}+a(s,p)\right)+e^{-\gamma_A p(x_0 a(s,p)+B-\frac{(\gamma_A p x_0)^2}{2})}\int_{\frac{B}{x_0}+a(s,p)-\gamma_A p x_0}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,\mathrm{d}y$$

$$a(s,p)^* = \underset{a(s,p)\in\mathbb{R}^+}{\arg\inf} \quad e^{-\gamma_A(s-\frac{ka^2}{2})}\left[\Phi\left(\frac{B}{x_0}+a(s,p)\right)+e^{-\gamma_A p(x_0 a+B-\frac{(\gamma_A p x_0)^2}{2})}(1-\Phi\left(\frac{B}{x_0}+a(s,p)-\gamma_A p x_0\right))\right].$$

The expression of $a^*(s,p)$, for fixed parameters, can be numerically computed.

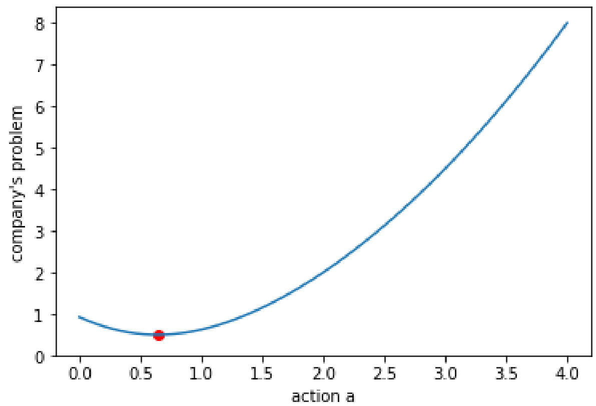


Figure 8: Optimization of a with $B=1$ and $k=1$

Figure 8 shows the agent problem for the fourth penalty for fixed parameters. We obtain an minimum near zero.

### 3.4.2 Numerical optimization of p

The expression of $p^*$ in that case is :

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad \mathbb{E}[E_1^{a^*(p)}+(s-P(X_1^{a^*(p)}))]$$

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad \mathbb{E}[e^{x_0(\omega-a^*(p))}+s-p(x_0(\omega-a^*(p))-B)_+]$$

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad e^{-x_0 a}\mathbb{E}[e^{x_0\omega}]+s-p\mathbb{E}[(x_0(\omega-a^*(p))-B)1_{\omega\geq\frac{B}{x_0}+a}].$$

The computation of this expectation is made separately :

$$\mathbb{E}[p(x_0(w-a^*(p)-B)1_{\omega\geq\frac{B}{x_0}+a}]=\frac{1}{\sqrt{2\pi}}\int_{\frac{B}{x_0}+a^*(p)}^{+\infty}(x_0(u-a^*(p))-B)\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\mathrm{d}u.$$

Using the Laplace transform, we obtain : $\mathbb{E}[e^{x_0\omega}]{=}e^{\frac{x_0^2}{2}}$.
So :

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad e^{-x_0 a^*(p)+\frac{x_0^2}{2}}+s-p\left[\int_{\frac{B}{x_0}+a^*(p)}^{+\infty}(x_0(u-a^*(p))-B)\ \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u\right]$$

$$p^* = \underset{p\in(0,1)}{\arg\inf} \quad e^{-x_0 a^*(p)+\frac{x_0^2}{2}}+s-p\left[\int_{\frac{B}{x_0}+a^*(p)}^{+\infty}x_0 u\ \frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u-(x_0a^*(p)+B)\int_{\frac{B}{x_0}+a^*(p)}^{+\infty}\frac{e^{\frac{-u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}u\right]$$

$$p^* = \arg\inf_{p\in(0,1)} \quad e^{-x_0 a^*(p)+\frac{x_0^2}{2}} + s - p\left[\frac{x_0}{\sqrt{2\pi}}\int_{\frac{B}{x_0}+a^*(p)}^{+\infty}[-\,e^{\frac{-u^2}{2}}]_{\frac{B}{x_0}+a^*(p)}^{+\infty} - (x_0 a^*(p)+B)(1-\Phi\left(\frac{B}{x_0}+a^*(p)\right))\right]$$

$$p^* = \arg\inf_{p\in(0,1)} \quad e^{-x_0 a^*(p)+\frac{x_0^2}{2}} + s - p\left(\frac{x_0}{\sqrt{2\pi}}e^{\frac{-(\frac{B}{x_0}+a^*(p))^2}{2}} - \left(x_0+B)(1-\Phi\left(\frac{B}{x_0}+a^*(p)\right)\right)\right).$$

This expression allows to numerically plots $p^*$. As before the value $a$ is computed from the first part of this penalty with the expression of $a^*(p)$ for every p.

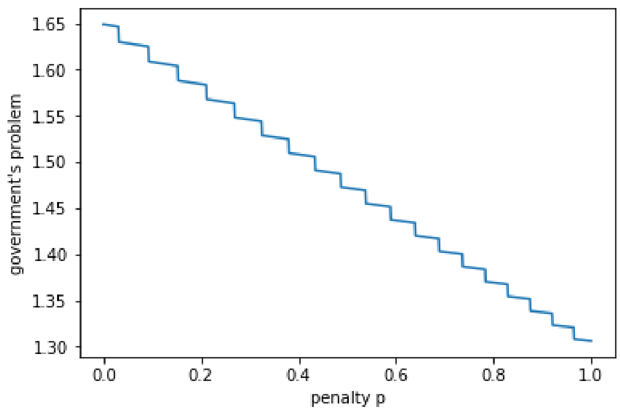


Figure 9: Representation of the government's problem depending on the penalty $p$ (with $B = 1$ and $k = 1$).

Figure 9 shows the principal problem of the fourth penalty for fixed parameters. In this case the function is also concave on the interval and so the minimum is on the bound.

### 3.4.3 Numerical optimization of s

We can compute $s^*$ in the same way as in previous penalties :

$$s^* = \frac{ka^{*2}}{2} + y + \frac{1}{\gamma_A}\ln\left[(1-\Phi(\frac{B}{x_0}+a^*-\gamma_A p^* x_0))e^{-\gamma_A p^*(x_0 a^*+B-\gamma_A p^*\frac{x_0^2}{2})} + \Phi(a^*+\frac{B}{x_0})\right].$$

# 4 Effects of the parameters

In this section we aim to find the effect of the parameters we fixed arbitrarily on the previous part

## 4.1 The threshold : B

B is the threshold defined in three out of the four penalty functions. $B$ is part of either the threshold in the company's obligation to pay the penalty, or the value of the penalty.

We compared the last three penalties, as the first one does not require a threshold. To do so, the value of the penalty $p$ is computed for each function and for different values of $B$ starting from 0.

In Figure 10, we see a threshold on the values of $p$ for $B$ in $[7, 9]$, after which the value $p$ for each penalties go to zero. This can be easily explained by the definition of those penalties. As $B$ increases, the level of emission which the companies have to reach in order to get penalized increases as well. It means that the more B increases, the less the companies are penalized. Consequently, when B is very large, the penalty goes to zero, and does not have any purpose anymore.

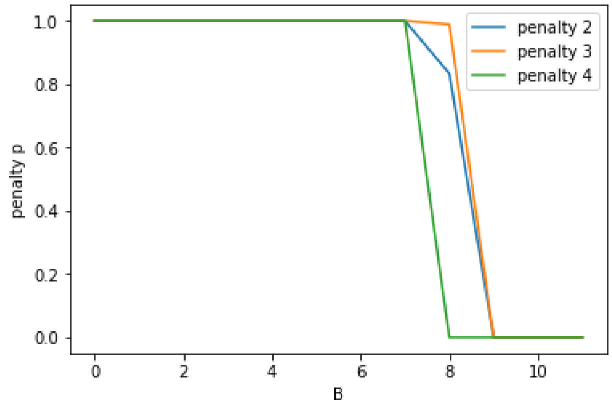


Figure 10: Representation of the penalty $p$ in function of $B$ for different penalties

In conclusion, we need to choose a value of $B$ near zero so as to keep the significance of the penalty. So the value $B = 1$ chosen previously can be maintained.

### 4.2 The parameter of the agent's utility : $\gamma_A$

The value $\gamma_A$ is a positive parameter of the agent's utility.To determine its effect on the action, we plot the $a$ in function of $\gamma_A$ for the last three penalties, as the first one does not depend on $\gamma_A$.

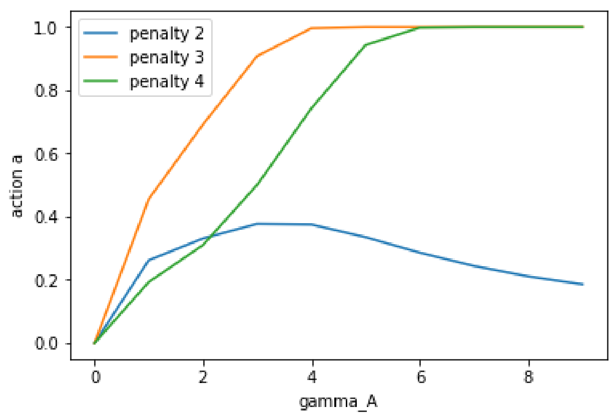


Figure 11: Representation of the action $a$ depending on $\gamma_A$

Figure 11 shows the effect of $\gamma_A$. We can see that the action of the second penalty has a concave curve, with a peak at $\gamma_A = 3$. The actions of the third and fourth penalties increase with $\gamma_A$ to reach a maximum at 1 where the companies have provided their best action to reduce the emissions.

The action of the second penalty has a concave curve, with a peak at $\gamma_A = 3$.

We can understand from this penalty that as $\gamma_A$ increases , the utility increases, the effect of the penalty on the agent decreases, which is coherent with the definition of the utility.

# 5 Optimization of the government's problem using its utility

With the last three penalty functions, we obtain a penalty $p^*$ on the maximum bond of the interval. The government's problem is rewritten as the agent problem with a utility. Hence, the new expression of the government's problem is :

$$p^* = \arg\sup_p \quad \mathbb{E}\Big[U_P\big( - E_1^a - (s - P(X_1^a))\big)\Big],$$

where $U_P$ the utility function defined by : $U_P(x) = -exp(-\gamma_P x)$, with $\gamma_P \geq 0$ is fixed.

This expression is computed in the same manner as the previous government's problem (search of a minimum).

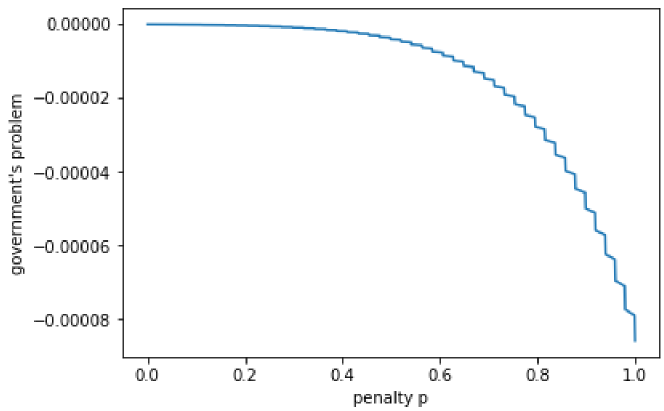


Figure 12: Optimization of the government's problem with $\gamma_P = 10$ on the third penalty function

Figure 12 represents the government's problem for the third penalty function with fixed parameters. The minimum of the government's problem is reached at the upper bond of the interval : 1, and this is the same result we have previously found with the first expression of the government's problem. Hence, rewritting the function does not change the value $p$ with this penalty function.

Nonetheless, the study of this new government's problem could be push further, and can be studied in more detail.

# 6 Comparison of the policies

For the four penalties, we have an expression for the quantities $a^*$, $p^*$ and $s^*$. With fixed parameters, these expressions can be used numerically to compute a value. The comparison of these values will allow us to find the best penalty for the government. In this case the parameters are : $B = 1$, $K = 1$, $y = 0$ and $x_0 = 1$

Table 1: Considered scenarios

| Taxes | $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|---|---|---|---|---|
| Action $a^*$ | 0.18507 | 0.260260 | 0.50016 | 0.19346 |
| Penalty $p^*$ | 0.18507 | 1.0 | 1.0 | 1.0 |
| Salary $s^*$ | 0 | 0.197971 | 0.34151 | 0.28604 |
| Carbon emissions $E_1^{a*}$ | 1.3701 | 1.2709 | 1.0403 | 1.3631 |
| Value of the agent's problem | -0.831040 | -1.0 | -1.0 | -1.0 |
| Value of the principal's problem | -1.589484 | -1.36510 | -1.24526 | -0.98543 |
| a=p=s=0 | | | | |
| Emissions de Carbone $E_1^0$ | 2.718 | 2.718 | 2.718 | 2.718 |
| Valeur du problème de l'agent | -1.0 | -1.0 | -1.0 | -1.0 |
| Valeur du problème du principal | -1.649 | -1.649 | -1.649 | -1.649 |

We first study each row separately :

- **The government's problem** : The government would prioritize the fourth penalty as it maximizes its value. The order would be : P4-P3-P2-P1.
- **The company's problem** : We can only deduce that the agent would prefer the first penalty because it matches the highest agent value. The other penalties provide the same agent value, so another factor needs to be taken into consideration.
- **The action** : The third policy leads to the highest action. It is followed respectively by the second, fourth and finally the first policies, but the difference between these last three is negligible.
- **The penalty** : the agent will naturally choose to pay the smallest penalty, and therefore would choose the first policy.

We can see that the optimal contract changes according to the goal we try to derive from it. Thus, to reach a unanimous decision, we need to consider a combination of all these cases, and study the pros and cons of each policy :

- **P1** : The government cannot accept this contract as it generates the lowest value of the principal, and has the lowest action, even if there is no salary to be paid by the government.
- **P2** : This policy can be considered by the government as the salary is the lowest, and the action generated is acceptable compared to others.
- **P3** : This contract generates the greatest action, but requires the highest salary.
- **P4** : With this contract, we obtain the highest value of the government's problem, but the action generated almost equals the one obtained with the first policy, which is the smallest.

All in all, choosing the best incentive policy depends on how we perceive what "best" is. If we are looking for the policy that costs the government the least, it would be the first penalty, as the salary given is 0. Yet, we can see that the action created by this contract is the lowest, which is understandable : the company is not incited at all, so it will only make the least effort possible in order not to pay a large penalty. By looking at the penalty, we see that it is the lowest that we have found, which is consistent with the company wanting to avoid paying a large penalty.

If the goal is to have the policy that holds the highest incentive, we would prefer the third policy, as it is the one that corresponds to the highest $a$. It would be natural to assume that this contract would be the most expensive for the government, and our results verify this : we can see that the second policy requires the highest salary to be paid to the company.

Furthermore, there are some policies than can be ruled out when compared to one another: if we consider the second and third, we can see that the action induced from P3 is double the action of P2, while the salaries do not differ in the same order (P3's salary is not double P2's). It would be logical in this case to assume that the government would rather accept to pay this salary to obtain a great action.

So to sum it all up, we can only reach a final optimal contract if we choose to compromise. And since the logical goal would be to lower the Carbon emissions, which is translated by having the highest action, we would recommend the third policy. It is the perfect example of a compromise, because on one side, the government commits itself to paying a high salary, and on the other, the agent acts accordingly and generates the largest action.

However, the parameters we chose in this model are only arbitrary : we have chosen them to get a general overview of the behavior of all the parameters put together. Consequently the conclusion we drew is not final : the discussion can still be pushed further.

Finally, we can take from these results that an optimal contract is achievable, and is a first step towards implementing solutions that could bring a difference in the Industry sector.

## 7 Conclusion

In the first part of this paper, we understood the form of the policies and the values of the principal and agent. The next step was the computation. We developed each equation to find direct expression of the extrema, and simplified the expressions that were to be computed numerically. Finally, we implemented our expressions and compared them. We discussed the results to find the best incentive policy, for some fixed parameters.

After testing and comparing the results, we concluded that in most cases, companies can be incited to take considerable action, yet the choice of the optimal contract needs to be discussed according to the needs and constraints of each party, and only achievable if both parties are willing to compromise. This study proves nonetheless the existence of an optimal contract,and is only the first step towards implementing efficient incentive policies to reduce Carbon emissions.

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