
# Algorithms Analysis and Design

## Chapter 3

## Brute Force and Exhaustive Search

# Introduction

After introducing the framework and methods for algorithm analysis in the preceding chapter, **we are ready to embark on a discussion of algorithm design strategies.**

- The subject of this chapter is **brute force** and its important special case, **exhaustive search**

- **Brute force** 
- Divide and conquer
- Decrease and conquer
- Transform and conquer
- Space and time tradeoffs

- Greedy approach
- Dynamic programming
- Iterative improvement
- Backtracking
- Branch and bound

# Brute Force Strategy

# Brute Force

- ***Brute force*** is a straightforward technique of problem-solving, usually directly based on the problem statement and definitions of the concepts involved.
- "Just do it!" would be another way to describe the prescription of the brute-force approach.
- This method relies more on compromising the power (**force**) of a computer system for solving a problem than on a good algorithm design.
- And **often**, the brute-force strategy is indeed the one that is easiest to apply.
- Rarely the most efficient but can be applied to wide range of problems.
- Rarely a source of clever algorithms.
- For some elementary problems, almost as good as most efficient. May not be worth cost.
- Sometimes the first one that comes to mind.

A first application of the brute-force approach often results in an algorithm that can be improved with a modest amount of effort.

# Examples

- Computing $a^n$ ($a > 0$, $n$ a nonnegative integer)
- Computing $n!$
- Multiplying two matrices
- Searching for a key of a given value in a list.
- Sorting algorithms (Selection sort , Bubble sort)
- Brute force string matching
- Closest-pair problem
- Brute force search (**Exhaustive search**)
  - Travelling salesman problem
  - 0/1 Knapsack problem
  - Assignment problem

# Importance of Brute Force approach

The brute-force approach should not be overlooked as an important algorithm design strategy

- Unlike some of the other strategies, brute force is applicable to a very wide variety of problems.
- The expense of designing a more efficient algorithm may be unjustifiable if only a few instances of a problem need to be solved and a brute-force algorithm can solve those instances with acceptable speed.
- Even if too inefficient in general, a brute-force algorithm can still be useful for solving small-size instances of a problem.
- A brute-force algorithm can serve an important theoretical or educational purpose as a criterion with which to judge more efficient alternatives for solving a problem.

# Calculating powers of a number

- Problem: exponentiation problem [ Computing $a^n$ ( *a* > 0, *n* a nonnegative integer ) ]

$$a^n = \underbrace{a * a * a \ldots\ldots\ldots * a}_{\text{n times}}$$

- Time complexity: $\theta\ (n)$

- **Can we find a better (faster) algorithm?**

Naive algorithm

```
ALGORITHM pow ( a , n )
{
    result = 1
    for i = 1 to n
        result = result * a
    return result
}
```

# Exercise

- Problem: exponentiation problem [ Computing $a^n$ ( $a > 0$, $n$ a nonnegative integer ) ]

$$a^n = \underbrace{a * a * a \ldots\ldots\ldots * a}_{\text{n times}}$$

1. Design a recursive algorithm for computing $a^n$
2. Derive a recurrence relation and solve it.
3. Find the time complexity.
4. Is it a good algorithm for solving this problem?

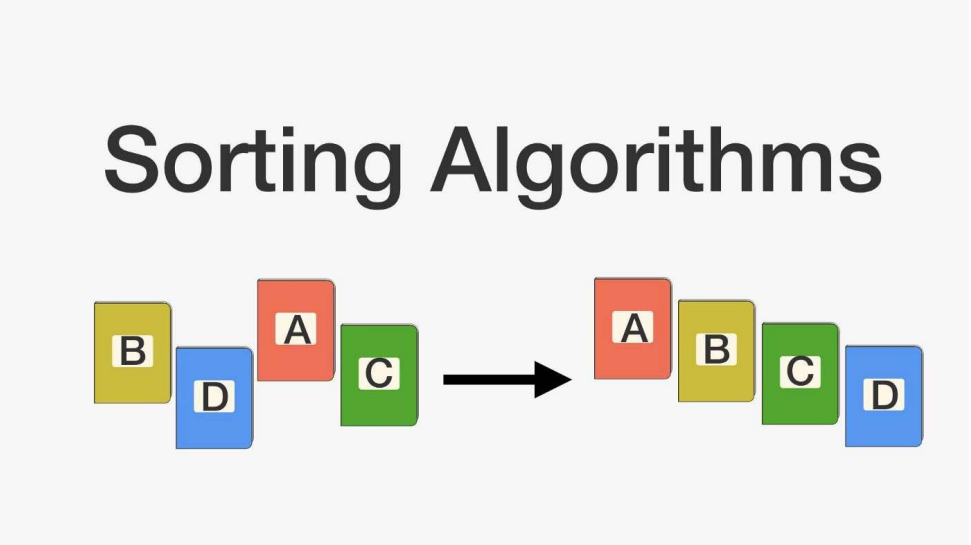
Recursive algorithm

```
ALGORITHM Rec_pow ( a , n )
{




}
```

The application of the brute-force approach to the problem of **sorting**

# Sorting Problem

- Sorting Problem: given a list of *n* orderable items (e.g., numbers, characters from some alphabet, character strings), rearrange them in nondecreasing order.
- "What would be the most straightforward method for solving the sorting problem?"
- The two prime candidates are:
  - Selection sort
  - Bubble sort



# Sorting Problem

Sorting algorithms can be categorized into several categories based on their approach to sorting the elements of an array or a list. Here are the main categories of sorting algorithms:

1. **Comparison-based sorting algorithms**:
   - These algorithms compare elements of the array to be sorted using a comparison operator, such as less than or greater than.
   - Examples of comparison-based sorting algorithms include bubble sort, selection sort, insertion sort, merge sort, quicksort, and heap sort.
2. **Non-comparison-based sorting algorithms**:
   - These algorithms do not rely on comparison operators to sort the elements of an array. Instead, they use specialized techniques to sort the elements more efficiently.
   - Examples of non-comparison-based sorting algorithms include counting sort, radix sort, and bucket sort.
3. **In-place sorting algorithms**:
   - These algorithms do not require any additional memory to sort the elements of an array. They rearrange the elements within the original array itself.
   - Examples of in-place sorting algorithms include selection sort, bubble sort, insertion sort, quicksort, and heap sort.
4. **"Out-of-place" or "External" sorting algorithms**
   - These algorithms work by dividing the input data into smaller parts that can fit into memory, sorting each part separately, and then merging the sorted parts into a single sorted output.
   - Out-of-place sorting algorithms are typically used when the data to be sorted is too large to fit into memory all at once, or when the input is stored on external storage such as a hard disk or tape drive.

# Sorting Problem

Sorting algorithms can be categorized into several categories based on their approach to sorting the elements of an array or a list. Here are the main categories of sorting algorithms:

4. **Stable sorting algorithms**:

- These algorithms maintain the relative order of elements with equal keys. This means that if two elements have the same value, they will appear in the same order in the sorted array as they did in the original unsorted array.
- Examples of stable sorting algorithms include insertion sort, merge sort, and counting sort.

5. **Unstable sorting algorithms**:

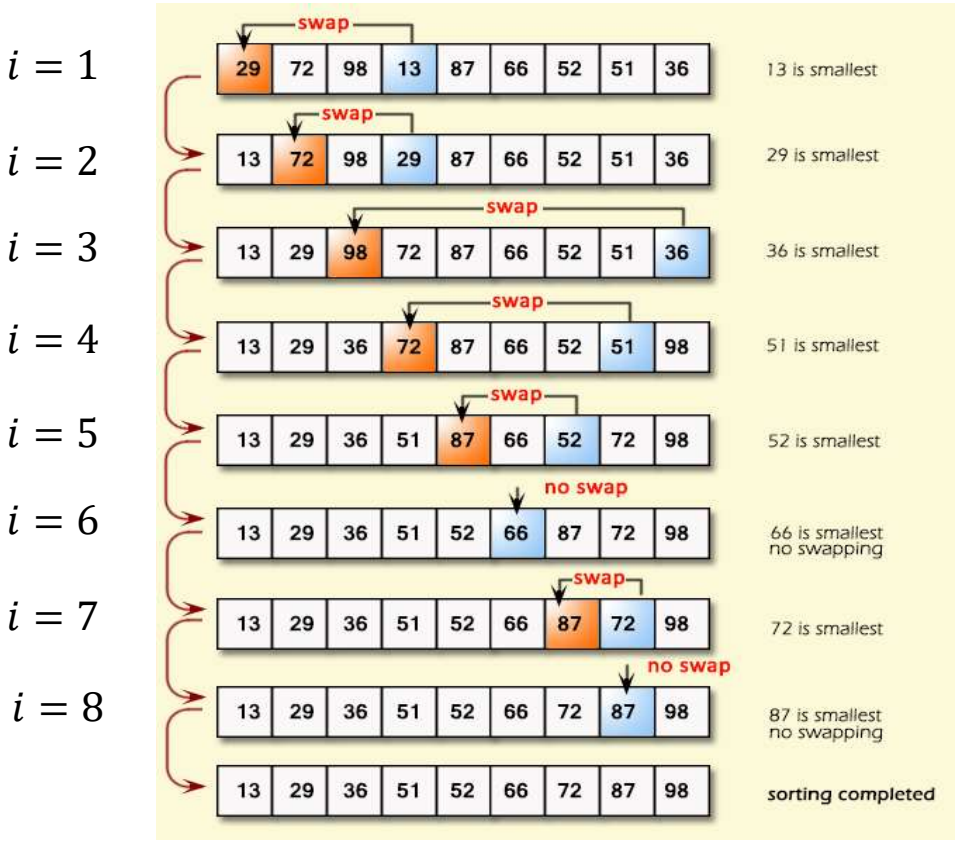
- These algorithms do not guarantee that the relative order of elements with equal keys will be maintained.
- Examples of unstable sorting algorithms include quicksort and heap sort.

# Selection Sort

- Scan the entire list to find its smallest element.
- swap it with the first element. First element now in its final sorted position.
- Scan remaining n-1 items, starting with second element, to find the smallest among them.
- Swap it with the second element. Second element now in its final sorted position.
- Repeat for the remaining elements.

# Example of selection sort

- Sort the list **29, 72, 98, 13, 87, 66, 52, 51, 36** in ascending order by **selection sort**



# selection sort

# Exact Analysis of selection sort

Sorts a given array by selection sort
**Input**: An array *A[1 … n]* of orderable elements
**Output**: Array *A[1 … n ]* sorted in ascending order

**ALGORITHM** $SelectionSort$ (A [1 …. n]

**for** $i \leftarrow 1$ **to** $n - 1$ **do**

min $\leftarrow i$

**for** $j \leftarrow i + 1$ **to** $n$ **do**

**if** A[ $j$ ] < A[min]

min $\leftarrow j$

swap A[$i$] and A[min]

1) What is the worst and best cases?
2) Do a line by line analysis for the worst and best cases and derive the time complexity (use Big O notation)

# Approximate Analysis of selection sort

**ALGORITHM** *SelectionSort* (A [1 …. n]

```
for i ← 1 to n − 1 do
    min ← i
    for j ← i + 1 to n do
        if A[ j ] < A[min]
            min ← j
    swap A[i] and A[min]
```

Thus, selection sort is a $\boldsymbol{\theta(n^2)}$

Note, however, that the number of key swaps is only $\boldsymbol{\theta(n)}$, or, more precisely, $n$ - 1(one for each repetition of the $i$ loop)

$$\sum_{i=m}^{n} c = c\,(n-m+1) \qquad \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

The basic operation is the key comparison $A[j] < A[min]$

The number of times it is executed depends on the array size and is given by the following sum:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1}[n-(i+1)+1]$$

$$= \sum_{i=1}^{n-1} n - i$$

$$= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i$$

$$= n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$$

# Exercise

Rewrite the ***SelectionSort*** algorithm to sort the elements of A in descending order.

**ALGORITHM** $SelectionSort$ (A [1 …. n]

    **for** $i \leftarrow 1$ **to** $n - 1$ **do**

        min $\leftarrow i$

        **for** $j \leftarrow i + 1$ **to** $n$ **do**

            **if** A[ $j$ ] < A[min]

                min $\leftarrow j$

        swap A[$i$] and A[min]

# Bubble sort

- The main concept of bubble sort is to compare adjacent elements of the list and exchange them if they are out of order. (**Compare and Swap**)
- By doing it repeatedly, we end up “bubbling up” the largest element to the last position on the list.
- The next pass bubbles up the second largest element, and so on.
- The same process goes for the remaining iterations (until after n − 1 passes the list is sorted.)

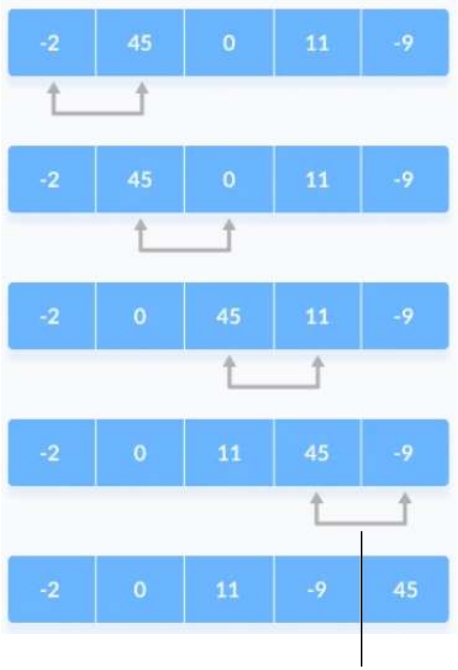
After each iteration, the largest element among the unsorted elements is placed at the end.

# Bubble sort

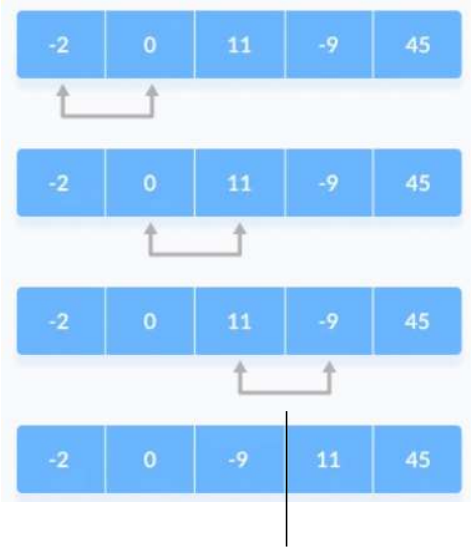
# Example of bubble sort

- Apply bubble sort on the list **-2, 45, 0, 11, -9**

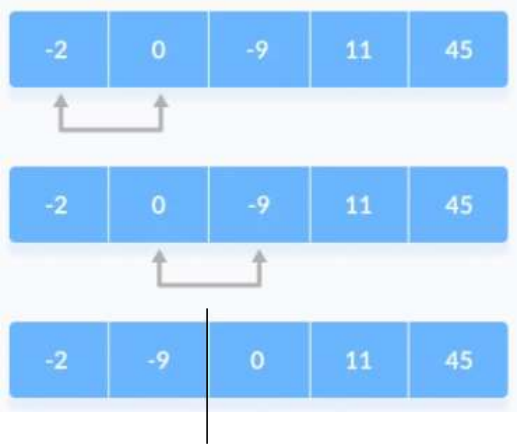
$first\ pass\ (i = 1)$



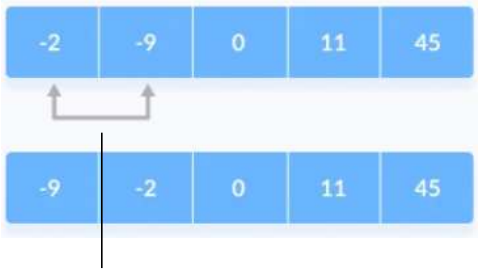

$second\ pass\ (i = 2)$




$third\ pass\ (i = 3)$




$forth\ pass\ (i = 4)$




After each iteration, the largest element among the unsorted elements is placed at the end.

The elements to the right of the vertical bar are in their final positions and are not considered in subsequent iterations of the algorithm.

# Exact Analysis of Bubble sort

Sorts a given array by bubble sort
**Input**: An array *A[1 … n]* of orderable elements
**Output**: Array *A[1 … n ]* sorted in ascending order

**ALGORITHM** $BubbleSort$ (A [1 …. n]

**for** $i \leftarrow 1$ **to** $n-1$ **do**

**for** $j \leftarrow 1$ **to** $n-i$ **do**

**if** A[ $j+1$ ] < A[$j$]

swap A[$j$] and A[$j+1$]

1) What is the worst and best cases?
2) Do a line by line analysis for the worst and best cases and derive the time complexity (use Big O notation)

# Approximate Analysis of Bubble sort

**ALGORITHM** *BubbleSort* (A [1 …. n]

**for** $i \leftarrow 1$ **to** $n-1$ **do**

**for** $j \leftarrow 1$ **to** $n-i$ **do**

**if** A[ $j+1$ ] < A[$j$]

swap A[$j$] and A[$j+1$]

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} 1 = \sum_{i=1}^{n-1} [n - i]$$

Identical to the sum for selection sort

Thus, Bubble sort is a $\boldsymbol{\theta(n^2)}$

Can we improve the crude version of bubble sort??

**Note**: A first application of the brute-force approach often results in an algorithm that can be improved with a modest amount of effort.

**Hint**: What if a pass through the list makes no changes?

# Selection sort vs Bubble sort

| Algorithm | Time complexity | | |
|---|---|---|---|
| | Best | Worst | Average |
| Selection sort | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ |
| Bubble sort | $O(n)$ | $O(n^2)$ | $O(n^2)$ |
| | | | |

We will explore and analyze other sorting algorithms later in this course

# Matrix Multiplication

- If A is an $m \times n$ matrix and B is an $n \times p$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

- The *matrix product* **C** = **AB** is defined to be the $m \times p$ matrix

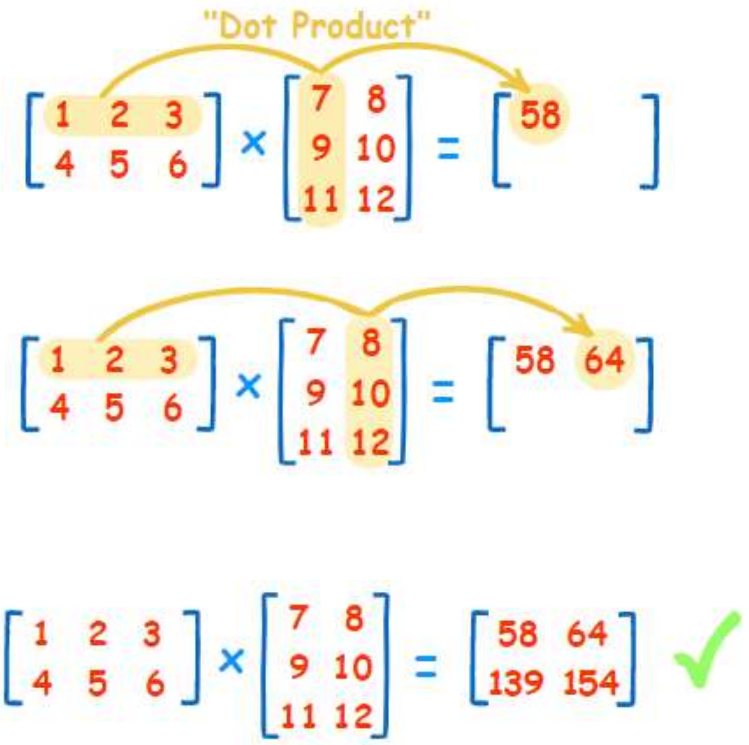
$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

- Such that:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$ for i = 1 …. m and j = 1 …… p

That is $C_{ij}$ is the dot product of the i$^{th}$ row of A and the j$^{th}$ column of B.

# Example



# Analysis of matrix multiplication

```
SQUARE-MATRIX-MULTIPLY(A, B)
1  n = A.rows
2  let C be a new n × n matrix
3  for i = 1 to n
4      for j = 1 to n
5          c_ij = 0
6          for k = 1 to n
7              c_ij = c_ij + a_ik · b_kj
8  return C
```

- Recall: We performed an exact analysis of matrix multiplication algorithm in chapter 2

the SQUARE-MATRIX-MULTIPLY procedure takes $\boldsymbol{\theta(n^3)}$time

- **Approximate analysis**

$$T(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} 1 = \sum_{i=1}^{n}\sum_{j=1}^{n} n = \sum_{i=1}^{n} n^2 = n^3$$

- **Can we find a better algorithm? (Later in chapter 4)**

# Sequential/Linear Search

- Searches for a given item (some search key X) in a list of $n$ elements by checking successive elements of the list until either a match with the search key is found (successful search) or the list is exhausted without finding a match (unsuccessful search)
- **Pseudocode** of Linear sequential search algorithm:

ALGORITHM $SequantialSearch\ (A, n\ , x)$

// Input: An array of n elements and a search key X
// Output: The index of the first element in A [1..n ] whose value is
// equal to K or −1 if no such element is found

$i \leftarrow 1$

**while** $(i <= n$ **and** A$[i] \neq X)$

$i \leftarrow i + 1$

**if** $i <= n$ **return** $i$

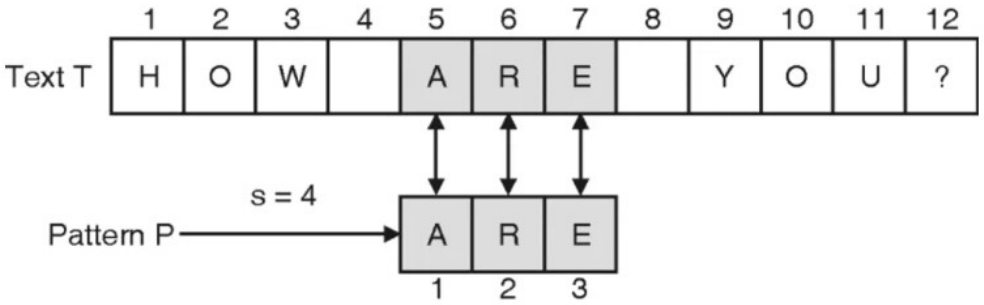
**else return** $-1$

We analyzed the efficiency of sequential search in the worst, best, and average cases in chapter 2

We will discuss later in this course other searching algorithms with a better time efficiency

# String Matching

- You probably often use your text editor to find some text in a file
  - e.g, Searching keywords in a file
- String Matching Problem: given a string of $n$ characters called the ***text*** and a string of $m$ characters ($m \leq n$) called the ***pattern***, find a substring of the text that matches the pattern.




- Commonly used algorithms:
  - Brute Force string matching
  - Moyer-Moore string matching
  - Knuth-Morris-Pratt (KMP)

# Brute Force String Marching

- String Matching Problem: given a string of $n$ characters called the ***text*** and a string of $m$ characters ($m \leq n$) called the ***pattern***, find a substring of the text that matches the pattern.

- **Brute-force algorithm**
  - Step 1 Align the pattern at the beginning of the text
  - Step 2 Moving from left to right, compare each character of the pattern to the corresponding character in text until
    - all characters are found to match (successful search); or
    - a mismatch is detected
  - Step 3 While pattern is not found and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

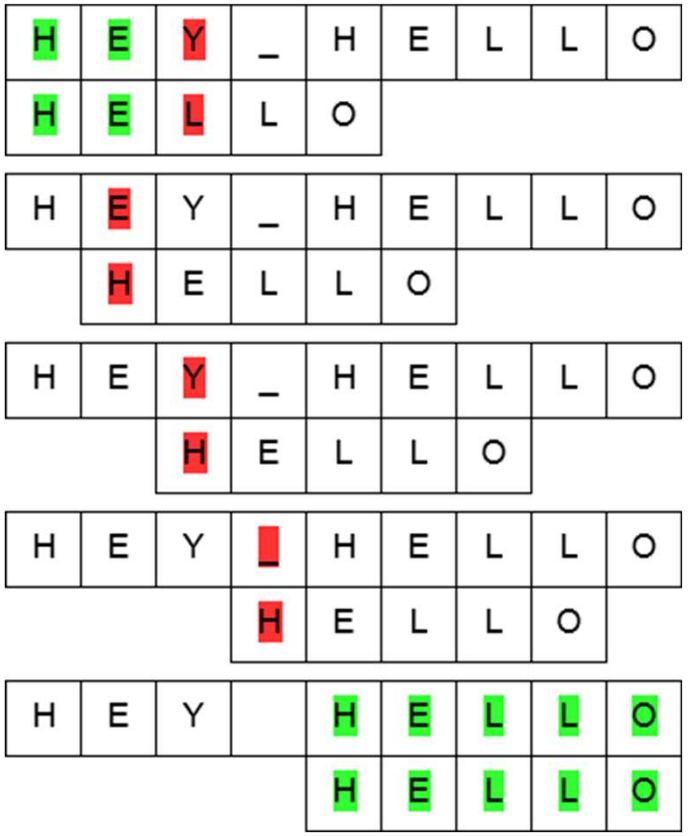
# Brute Force String Marching

- Example: Text = HEY_HELLO

  Pattern = HELLO

mismatch

match



# Brute Force String Marching

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  - Step 1 Align the pattern at the beginning of the text
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  - Step 3 While pattern is not found and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

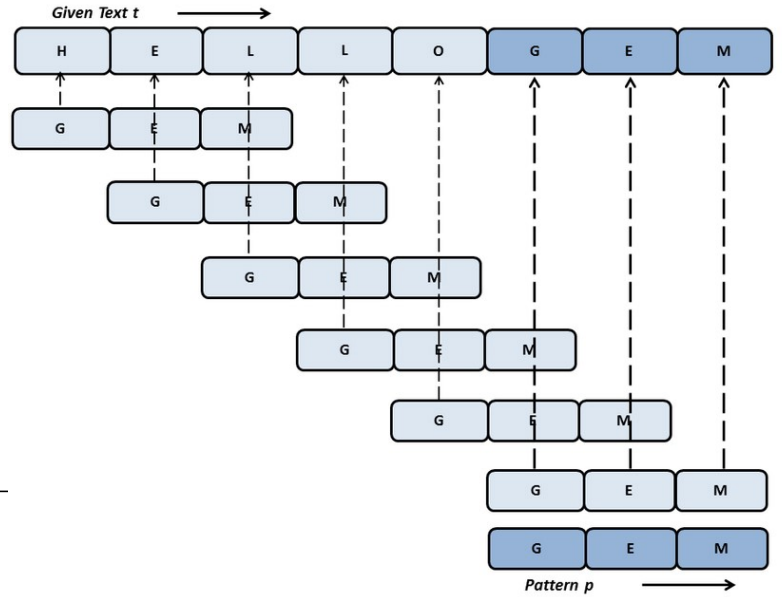
# Brute Force String Marching

- **Pseudocode** (provided the text positions are indexed from **1**)

ALGORITHM $BruteForceStringMatch\ (T[1 \ldots n], P[1 \ldots m])$

// Input: An array T[1…n] of **n** characters representing a **text**
An array P[1…m] of **m** characters representing a **pattern**
// Output: The index of the first character in the text that starts a matching substring or −1 if the search is unsuccessful

**for** $i = 1$ **to** $n - m + 1$ **do**
  $j = 1$
  **while** $j <= m$ **and** P[$j$] = $T\ [i + j - 1]$) **do**
    $j \leftarrow j + 1$
  **if** $j = m + 1$ **return** $i$
**return** $-1$



# Brute Force String Marching

# Analysis of Brute Force String Marching

- **Worst Case**:
  - The algorithm may have to make all m comparisons before shifting the pattern, and this can happen for each of the n − m + 1 tries.
  - The algorithm makes $m(n - m + 1)$ character comparisons ⟶ $O(n.m)$
- **Best Case**
  - The algorithm may have to make all $m$ comparisons for just one try (the first comparison with the first pattern succeeds immediately)
  - The algorithm makes $m$ character comparisons ⟶ $O(m)$

Text: **a a b a** h n f g

Pateern: **a a b a**

There are several more sophisticated and more efficient algorithms for string searching. The most widely known of them—by R. Boyer and J. Moore—(Boyer-Moore) is outlined in Section 7.2

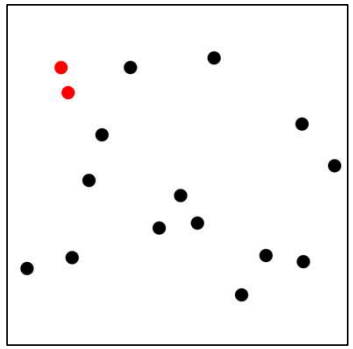
# Exercise

- Find the total number of character comparisons that will be made by the brute force algorithm in searching for a pattern of 4 characters in a text of 15 characters.

1) In the best case.
2) In the worst case.

# Closest-pair problem

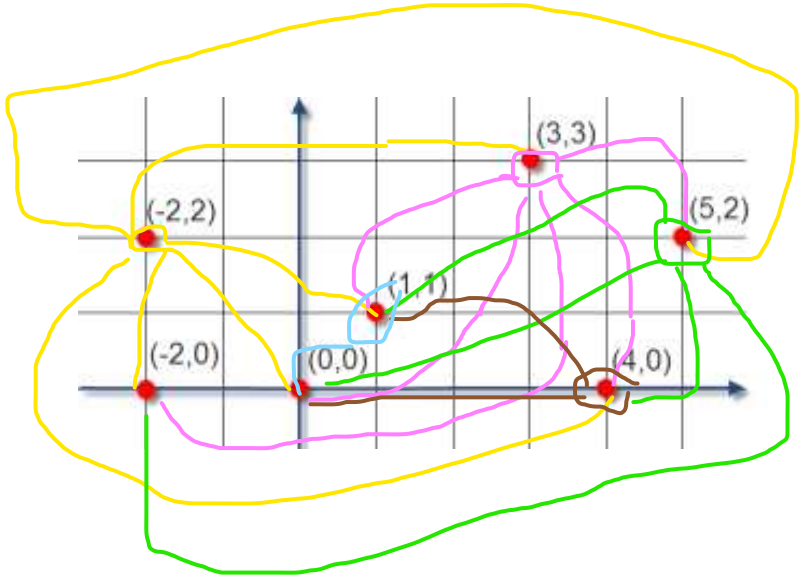
- Closest-pair problem: Find the two closest points in a set of $n$ points (in the two-dimensional Cartesian plane).
- arise in two important applied areas: computational geometry and operations research



- **Brute-force algorithm**
  - Compute the distance between every pair of distinct points.
  - and return the indexes of the points for which the distance is the smallest.

# Closest-pair problem

- **Brute-force algorithm**
  - Compute the distance between every pair of distinct points.
  - and return the indexes of the points for which the distance is the smallest.



# Closest-pair Brute Force Algorithm

**ALGORITHM** *BruteForceClosestPoints*$(P)$

//Input: A list $P$ of $n$ $(n \geq 2)$ points $P_1 = (x_1, y_1), \ldots, P_n = (x_n, y_n)$

//Output: Indices *index*1 and *index*2 of the closest pair of points

$dmin \leftarrow \infty$

**for** $i \leftarrow 1$ **to** $n - 1$ **do**

  **for** $j \leftarrow i + 1$ **to** $n$ **do**

    $d \leftarrow sqrt((x_i - x_j)^2 + (y_i - y_j)^2)$ //*sqrt* is the square root function

    **if** $d < dmin$

      $dmin \leftarrow d$; $index1 \leftarrow i$; $index2 \leftarrow j$

**return** *index*1, *index*2

Note: the distance between two points $p_i(x_i, y_i)$ and $p_j(x_j, y_j)$ is the standard **Euclidean distance**

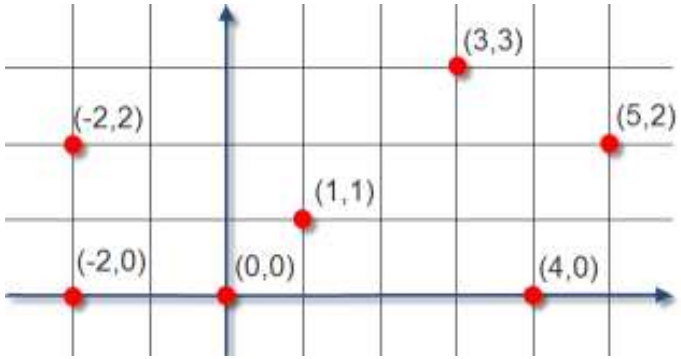
$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.$$

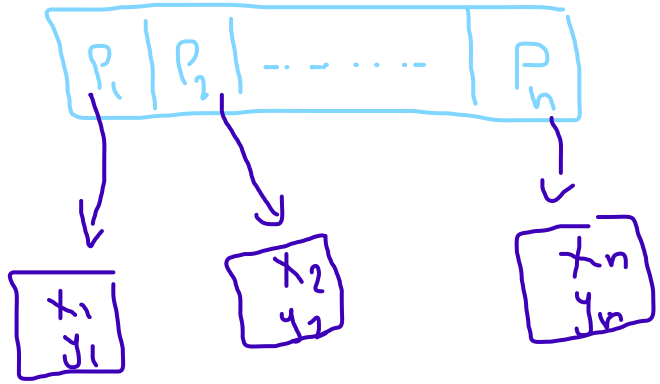
The basic operation of the algorithm will be **squaring a number**. The number of times it will be executed can be computed as follows:

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - i] \qquad \theta(n^2)$$

# Closest-pair problem

- **Brute-force algorithm**



# Closest-pair Brute Force Algorithm

For ***n*** number of points, we would need to measure **n(n-1)/2** distances and the cost is **$\Theta(n^2)$**

The basic operation of the algorithm will be **squaring a number**. The number of times it will be executed can be computed as follows:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \sum_{i=1}^{n-1} [n - i]$$

Search: What is the time complexity of getting the square root (sqrt) ???

# Exhaustive search

# Exhaustive search

- **Exhaustive search is a brute-force approach used to solve optimization problems.**
  - It is known as **brute-force search**.
  - It is also known as **generate and test**.
  - **It involves searching through all possible solutions in a given search space in order to find the best solution that optimizes a particular objective function.**

consists of systematically enumerating all possible candidates for the solution and checking whether each candidate satisfies the problem's statement.

- **It is guaranteed to find the global optimum solution.**
- **it is usually impractical for large search spaces due to its high computational complexity.**

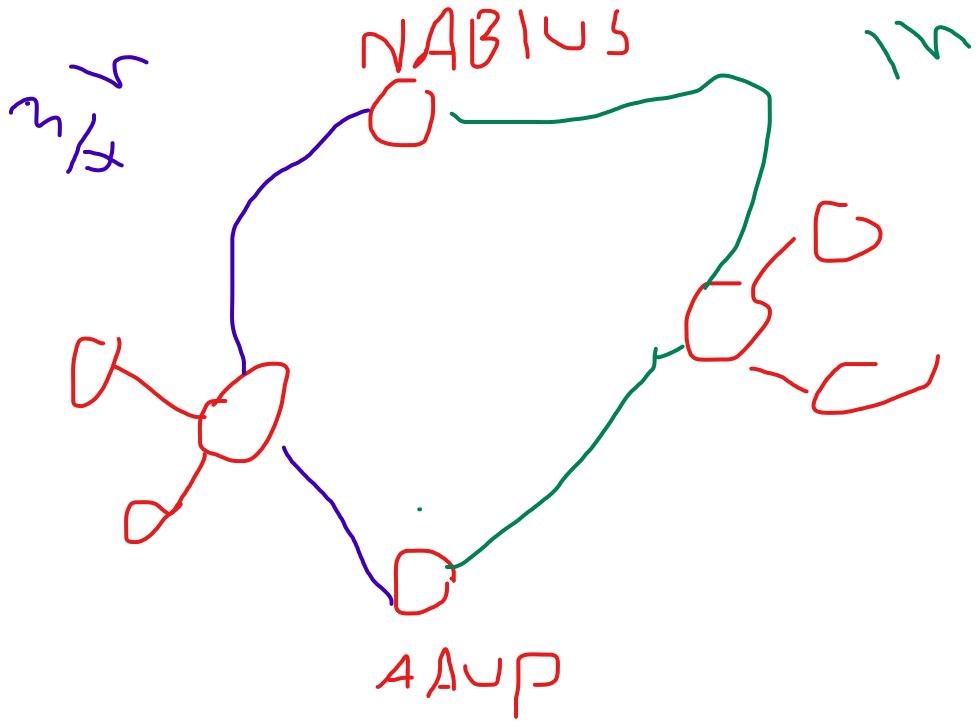
- While a brute-force search is simple to implement and will always find a solution if it exists, implementation costs are proportional to the number of candidate solutions – which in many practical problems tends to **grow very quickly as the size of the problem increases.**

# Introduction to optimization problems

# Optimization problem

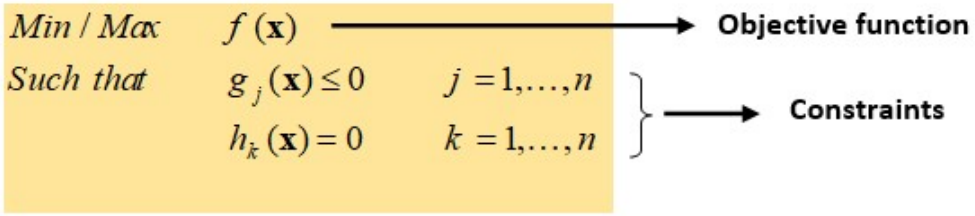
Here's a list of terminology that students should be familiar with regarding optimization:

- Optimization problem
- Objective function
- constraints
- Local and global optimum solutions
- Feasible and infeasible solutions
- Search Space



# Optimization problem

- An optimization problems can be formulated as follows:

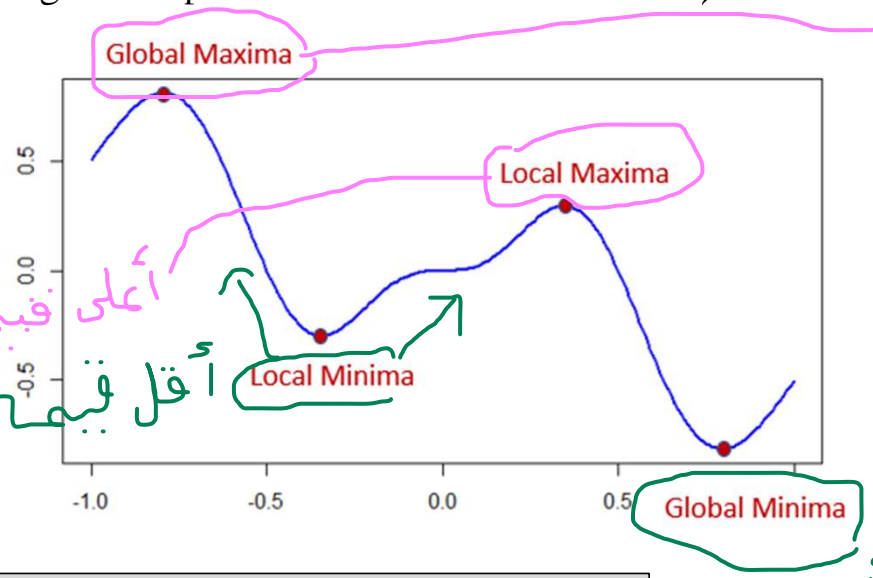



Find $x = (x_1, x_2, \ldots\ldots\ldots x_n)$ that maximize / minimize $f(x)$ considering the given constraints

- Objective function (Fitness): A function used to evaluate every solution of the search space (or assigns score for every solution)
- Feasible solutions: that satisfies all constraints.
- Optimal solution: A feasible solution that maximizing profit / minimizing cost

# Local vs Global optima

- Local optimum: is the best solution to a problem within a small neighborhood of possible solutions
- Global optimum: A solution $s^* \in S$ is a global optimum if it has a better objective function than all solutions of the search space.




The main goal of solving optimization problems is to find the global optimum ($s^*$)

# Example: 0/1 knapsack problem

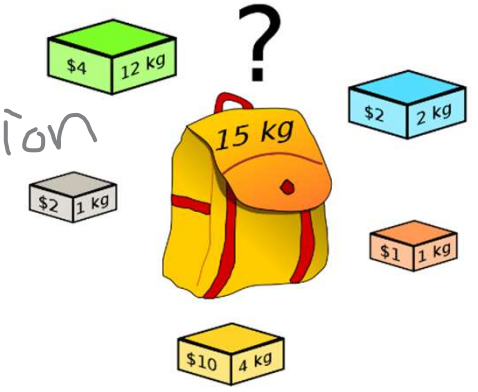
- Solution : The set of objects are encoded as a vector of zeros (not selected) and ones (selected)

Solution code or solution Presentation

| 0 | 1 | 1 | 0 | 1 |
|---|---|---|---|---|

يعني خزال الـ item الـ 2 والـ 3 والـ 5

يلي بحدده هو شكل المسألة




- Objective function (profit) : $\sum_{i=1}^{n} X_i \; p_i$
- Feasible solution: $\sum_{i=1}^{n} X_i \; W_i \; \leq$ M (capacity)

- Our goal is to find X that maximize $\sum_{i=1}^{n} X_i \; p_i$

subject to : $\sum_{i=1}^{n} X_i \; W_i \; \leq$ M and $X_i \in \{ 0 , 1 \}$

# Exhaustive search method

**Method:**

- **Generate** a list of all potential solutions to the problem in a systematic manner.
- **Evaluate** potential solutions one by one, **disqualifying infeasible ones** and, for an optimization problem, keeping track of the best one found so far
- when search ends, **announce** the solution(s) found

# Three important problems

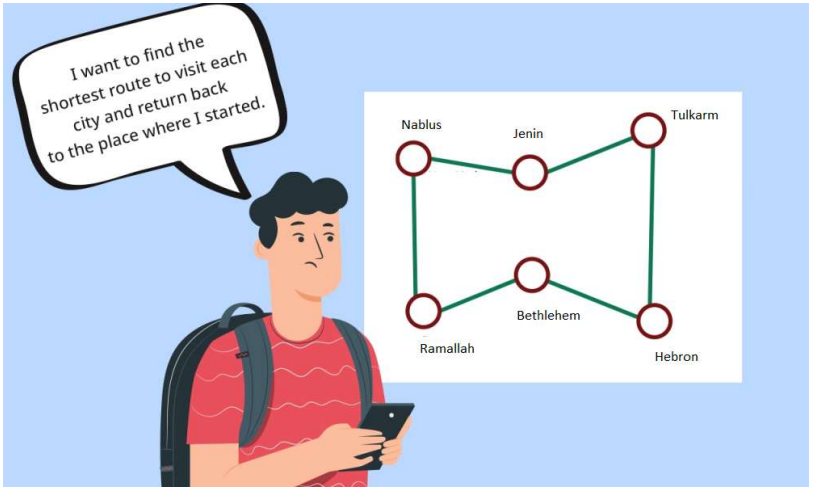
**To illustrate exhaustive search, we will apply it to three important problems:**

- Travelling salesman problem.
- Knapsack problem.
- Assignment problem.

# Travelling Salesman problem (TSP)

- **TSP problem**: Given $n$ cities with known distances between each pair, find the shortest tour that passes through all the cities exactly once before returning to the starting city
- TSP as **a graph problem**:
- The problem can be conveniently modeled by a **weighted graph**, with the graph's vertices representing the cities, graph's edges representing the paths, and the edge weights specifying the distances.

| TSP problem | TSP as graph problem |
|---|---|
| Cities | Graph's vertices |
| paths | Graph's edges |
| distances | Edge weights |



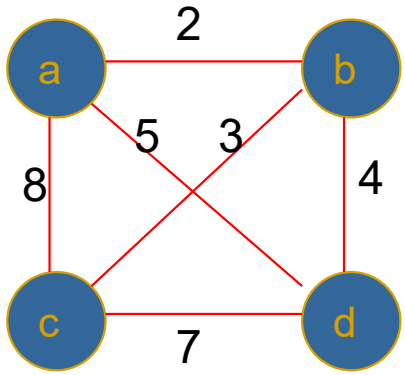

- Then the problem can be stated as the problem of finding the shortest **Hamiltonian circuit** of the graph

# TSP by Exhaustive Search

(n-1)!

- **Exhaustive search to solve the TSP:**
  - Get all tours by generating all the permutations of $n-1$ intermediate cities.
  - Evaluate each tour length.
  - Find the shortest among them

The total number of permutations needed is **(n-1)!**

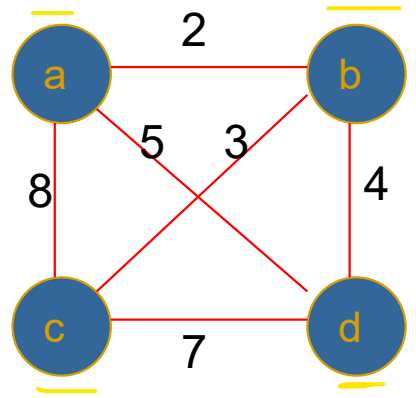



- **Example:**
  - Consider city (a) as the starting and ending point.
  - Generate all (n-1)! Permutations of cities.
  - Calculate cost of every permutation and keep track of minimum cost permutation
  - Return the permutation with minimum cost.

# TSP by Exhaustive Search

3!

| Tour | Length (Cost) | |
|---|---|---|
| a→b→c→d→a | 2+3+7+5 = 17 | optimal |
| a→b→d→c→a | 2+4+7+8 = 21 | |
| a→c→b→d→a | 8+3+4+5 = 20 | |
| a→c→d→b→a | 8+7+4+2 = 21 | |
| a→d→b→c→a | 5+4+3+8 = 20 | |
| a→d→c→b→a | 5+7+3+2 = 17 | optimal |




(4−1)! = 3!

4 nodes

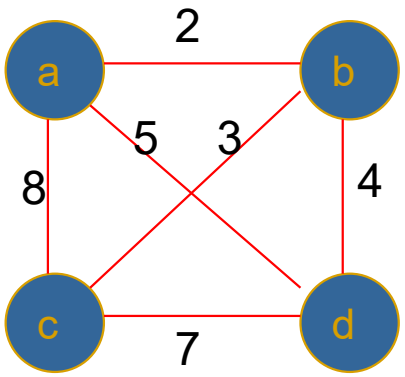
More tours????

Less tours????

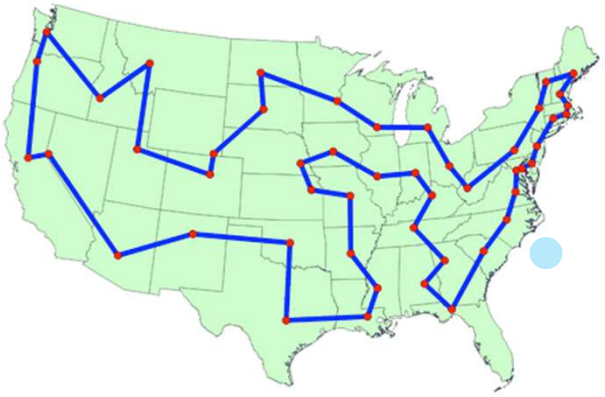
# TSP by Exhaustive Search

- **What is the time efficiency of the exhaustive search**? $O(n!)$ where n is the number of cities (vertices)
- The exhaustive-search approach is practical for very small values of *n*.

This solution becomes impractical even for only 20 cities



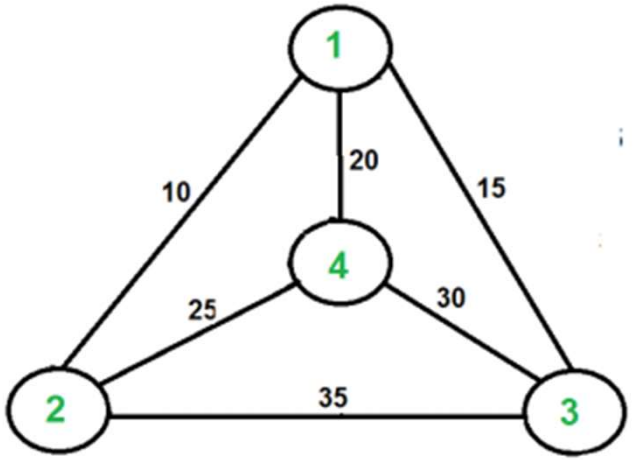

Small instance of TSP



Large instance of TSP

# Exercise

- Consider the graph shown. Find the shortest possible route that visits every city exactly once before returning to the starting city. What is the cost of the tour?
- Note: Consider city 1 as the starting and ending point



# Useful resources

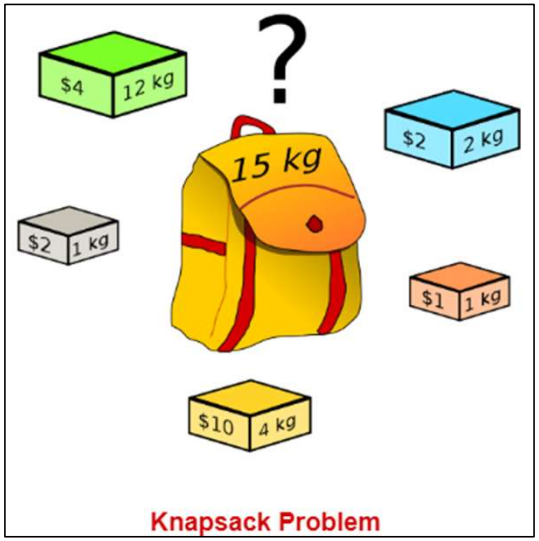
# Travelling salesman problem solver

- https://tspvis.com/

# Knapsack Problem

- **Knapsack problem**: Given $n$ items $I_1, I_2, \ldots\ldots, In$ of known weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$ and a knapsack of capacity $W$

  find **the most valuable subset of the items** that fit into the knapsack.



Knapsack Problem

- Which items should be placed into the knapsack such that:
  - The value or profit obtained by putting the items into the knapsack is maximum.
  - And the weight limit of the knapsack **does not exceed**.

- Application example:
  - A transport plane that has to deliver the most valuable set of items to a remote location without exceeding the plane's capacity

# Knapsack by Exhaustive Search

- **Exhaustive search to solve the 0/1 Knapsack problem:**
  - Generate **all the subsets** of the set of n items given.
  - **Compute the total weight** of each subset in order to identify feasible subsets (i.e., the ones with the total weight not exceeding the knapsack capacity)
  - Find a subset of the largest value among them Example

Since the number of subsets of an n-element set is $2^n$,

The exhaustive search leads to a $\Omega(2^n)$ algorithm, **no matter how efficiently individual subsets are generated**

# Knapsack by Exhaustive Search

- **Example on small instance of Knapsack problem:**

| Subset | Total weight | Total value |
|---|---|---|
| ∅ | 0 | $0 |
| {1} | 2 | $20 |
| {2} | 5 | $30 |
| {3} | 10 | $50 |
| {4} | 5 | $10 |
| {1,2} | 7 | $50 |
| {1,3} | 12 | $70 |
| {1,4} | 7 | $30 |
| **{2,3}** | **15** | **$80** |
| {2,4} | 10 | $40 |
| {3,4} | 15 | $60 |
| {1,2,3} | 17 | not feasible |
| {1,2,4} | 12 | $60 |
| {1,3,4} | 17 | not feasible |
| {2,3,4} | 20 | not feasible |
| {1,2,3,4} | 22 | not feasible |

← **optimal**

Knapsack capacity W=16

| item | weight | value |
|---|---|---|
| 1 | 2 | $20 |
| 2 | 5 | $30 |
| 3 | 10 | $50 |
| 4 | 5 | $10 |

Since number of items (n) is small, so it is feasible to find the optimal solution by brute force (using a computer)

But if n is much bigger, the brute force would take too long.

# Exercise

- You are given 5 elements needed to be put inside a knapsack with limited weight capacity W = 50. The elements weights with their values are given below:

| item | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Weight | 21 | 11 | 51 | 26 | 30 |
| value | 37 | 12 | 500 | 50 | 41 |

Apply brute force technique to find which subset of items should be put inside the knapsack such that the obtained profit is maximized and the weight limit of the knapsack does not exceed.

# Exercise

- You want to travel to someplace for a few days and have a single bag to carry which could fit 5 kgs. You have a list of items to take which could all not fit into that single bag. Assuming you cant take fractions of items, which items should be picked up to maximize the value of items according to your needs.

| item | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Weight | 2 | 3 | 4 | 5 |
| value | 3 | 4 | 5 | 6 |

# Discussion

How to generate all possible solutions for 0/1 knapsack or TSP problem?

Generating all possible solutions can be computationally expensive ????

The number of possible solutions grows exponentially with the size of the input???

# Inefficient exhaustive search

For both the traveling salesman and knapsack problems considered before, exhaustive search leads to algorithms that are extremely inefficient on every input.

- In fact, these two problems are the best-known examples of so called **NP-hard** problems.

- We will study other techniques to solve **some but not all instances** of these problems in less than exponential and factorial times.

# Search Topics


- NP-complete and NP-hard problems.
- Approximate algorithms.
  - Heuristic and Metaheuristic methods
  - Genetic algorithms
- Other techniques for solving 0/1 Knapsack problem
  - Greedy
  - Dynamic programming

# Useful resources

# 0/1 Knapsack solver

- https://augustineaykara.github.io/Knapsack-Calculator/
- https://monicagranbois.com/knapsack-algorithm-visualization/

# Travelling salesman problem solver

- https://tspvis.com/