Computer Vision Sheet01

19th October 2024

Proof of Associativity of Convolution

Let f, g, and h be functions defined on R. The convolution of two functions f and g is defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\,du.$$

We want to show that

$$(f * g) * h = f * (g * h).$$

First, we compute the convolution $(f * g)(x)$:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\,du.$$

Then, we compute $((f * g) * h)(x)$:

$$((f * g) * h)(x) = \int_{-\infty}^{\infty} (f * g)(x-v)h(v)\,dv.$$

Substituting the expression for $(f * g)(x-v)$:

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f((x-v)-u)g(u)\,du \right) h(v)\,dv.$$

Using Fubini's theorem, we can interchange the order of integration:

$$= \int_{-\infty}^{\infty} g(u) \left(\int_{-\infty}^{\infty} f((x-v)-u)h(v)\,dv \right) du.$$

We simplify the inner integral:

$$\int_{-\infty}^{\infty} f((x-v)-u)h(v)\,dv.$$

Let's set $w = x - v$ (implying $v = x - w$ and $dv = -dw$). Then, we have:

$$= \int_{-\infty}^{\infty} f(w-u)h(x-w)(-dw).$$

Rearranging gives:

$$= \int_{-\infty}^{\infty} f(w-u)h(x-w)\,dw.$$

We can rewrite $((f*g)*h)(x)$ as:

$$((f*g)*h)(x) = \int_{-\infty}^{\infty} g(u)\left(\int_{-\infty}^{\infty} f(w-u)h(x-w)\,dw\right)\,du.$$

We compute the right-hand side $f*(g*h)$:
First, compute $(g*h)(x)$:

$$(g*h)(x) = \int_{-\infty}^{\infty} g(x-u)h(u)\,du.$$

Now we compute $f*(g*h)(x)$:

$$(f*(g*h))(x) = \int_{-\infty}^{\infty} f(x-v)(g*h)(v)\,dv.$$

Substituting the expression for $(g*h)(v)$:

$$= \int_{-\infty}^{\infty} f(x-v)\left(\int_{-\infty}^{\infty} g(v-u)h(u)\,du\right)\,dv.$$

Using Fubini's theorem again, we change the order of integration:

$$= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} f(x-v)g(v-u)\,dv\right)\,du.$$

Now, let's make another change of variables in the inner integral. Let's set $w = x - v$ (thus $v = x - w$ and $dv = -dw$):

$$= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} f(w)g(x-u-w)(-dw)\right)\,du.$$

Rearranging gives:

$$= \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} f(w)g(x-u-w)\,dw\right)\,du.$$

The inner integral can again be recognized as the convolution of f and g:

$$= \int_{-\infty}^{\infty} (f*g)(x-u)h(u)\,du.$$

Proof: Convolution of Gaussian Kernels

The Gaussian kernel with standard deviation σ is defined as:

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

The convolution of two functions f and g is defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x - t) \, dt$$

Now, we want to convolve the Gaussian kernel with itself:

$$(G_\sigma * G_\sigma)(x) = \int_{-\infty}^{\infty} G_\sigma(t) G_\sigma(x - t) \, dt$$

Substituting the expression for G_σ:

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} \right) \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-t)^2}{2\sigma^2}} \right) dt$$

Combining the exponential terms:

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-\frac{(x-t)^2}{2\sigma^2}} \, dt$$

The exponent simplifies to:

$$-\frac{t^2}{2\sigma^2} - \frac{(x-t)^2}{2\sigma^2} = -\frac{1}{2\sigma^2} \left(2t^2 - 2tx + x^2 \right)$$

This results in:

$$(G_\sigma * G_\sigma)(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2}{4\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(t-\frac{x}{2})^2}{2\sigma^2}} \, dt$$

The integral evaluates to $\sqrt{2\pi}\sigma$:

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{x^2}{4\sigma^2}} \cdot \sqrt{2\pi}\sigma$$

Combining all results gives:

$$(G_\sigma * G_\sigma)(x) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{x^2}{4\sigma^2}} = G_{\sqrt{2}\sigma}(x)$$

Thus, we conclude that convolving a Gaussian with itself results in a Gaussian with standard deviation $\sqrt{2}\sigma$:

$$G_\sigma * G_\sigma = G_{\sqrt{2}\sigma}$$