Near-optimal Quantum Algorithms for Multivariate Mean Estimation

arXiv:2111.09787

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June 20th, 2022

Research Center for Quantum Software

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Applications:

1. Physics/chemistry simulations
2. Computer graphics
3. Finance
4. Shadow tomography

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Calls to these routines are *samples*.

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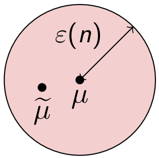
$$\mathbb{P}\left[\|\mu - \widetilde{\mu}\|_2 \leq \varepsilon(n)\right] \geq \frac{2}{3}.$$



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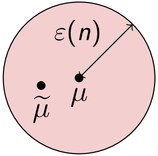

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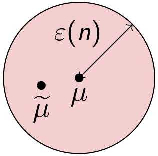

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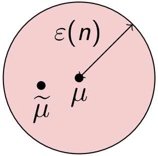

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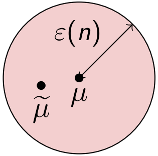

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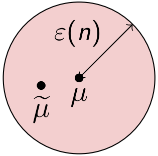

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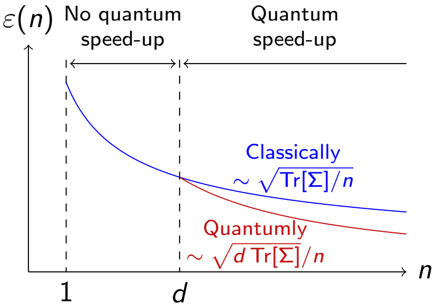

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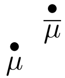

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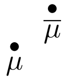

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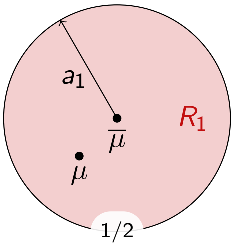

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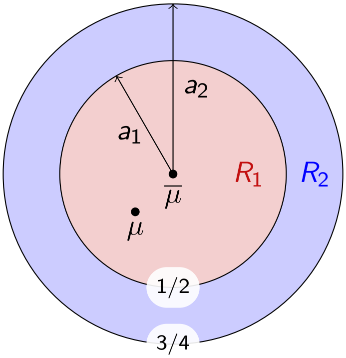

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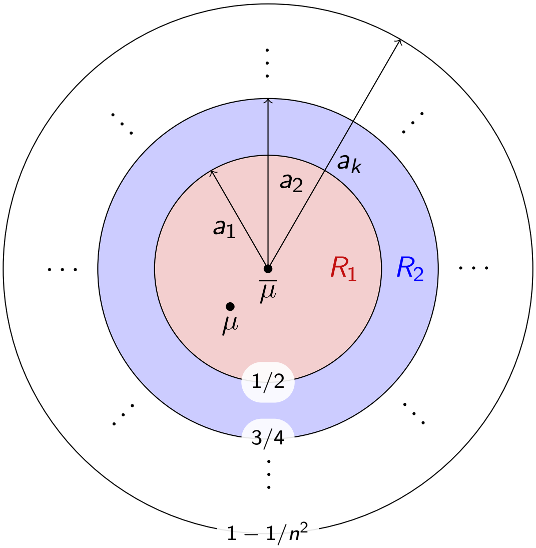

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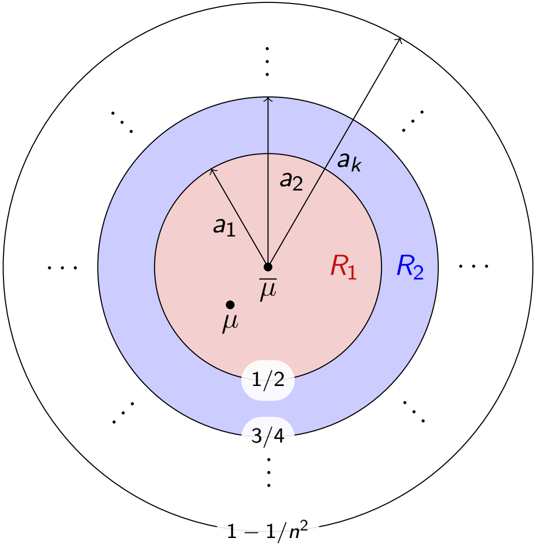

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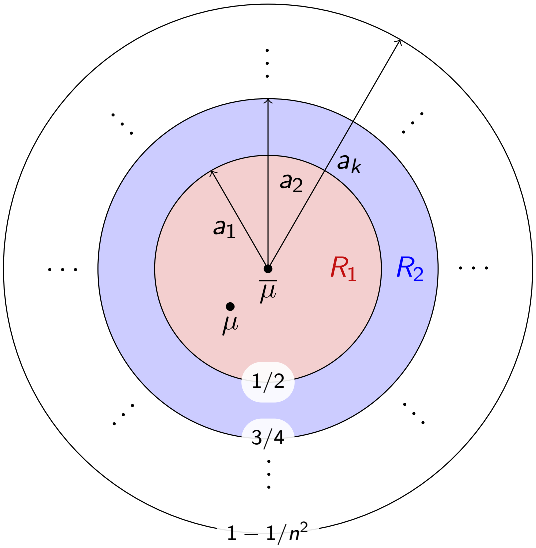
   $$\mathbb{E}[X] \approx \sum_{\ell=1}^{k} \mathbb{E}[X \cdot \mathbb{1}_{X \in R_\ell}].$$



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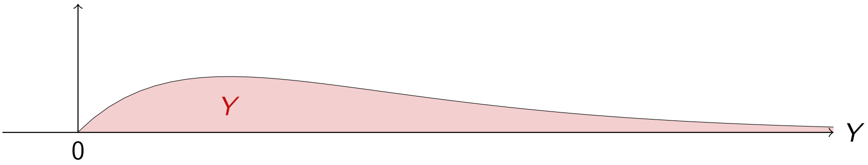

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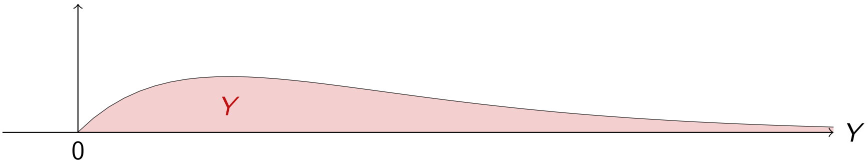

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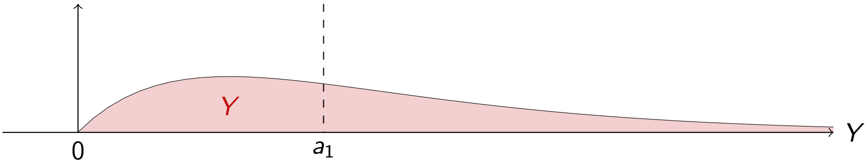

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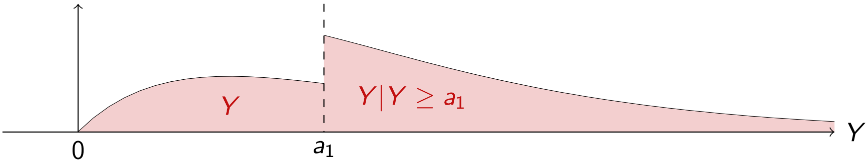

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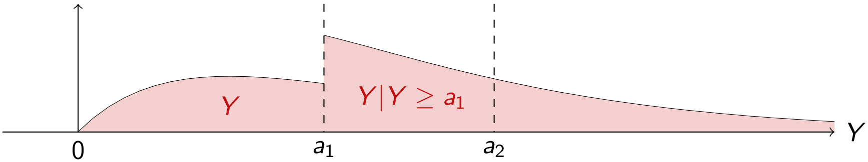

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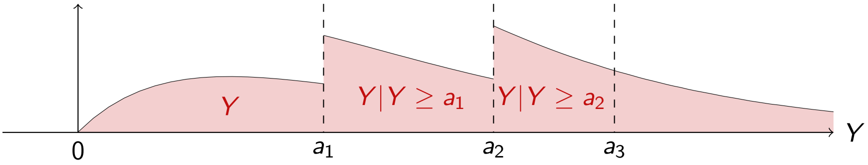

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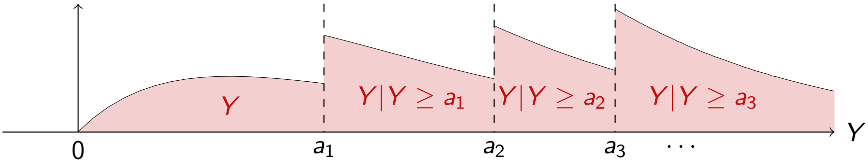

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ℓ. $\widetilde{O}(1)$ samples from $Y | Y \geq a_{\ell-1}$
Requires $\widetilde{O}(\sqrt{2^{\ell-1}})$ samples from Y
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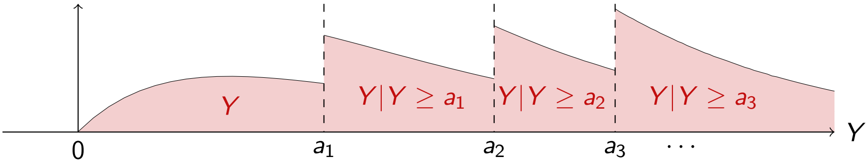

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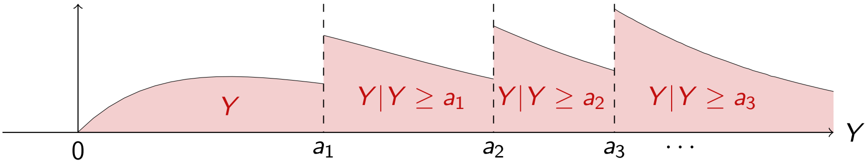
$\vdots$

ℓ. $\widetilde{O}(1)$ samples from $Y | Y \geq a_{\ell-1}$ $\hookrightarrow$ Let a_ℓ be the median $\mathbb{P}[Y \geq a_\ell] \approx \frac{1}{2^\ell}$
   Requires $\widetilde{O}(\sqrt{2^{\ell-1}})$ samples from Y by *amplitude amplification*

$\vdots$

$2\log(n)$. $\widetilde{O}(1)$ samples from $Y | Y \geq a_{2\log(n)}$ $\hookrightarrow$ Let $a_{2\log(n)}$ be the median $\mathbb{P}[Y \geq a_{2\log(n)}] \approx \frac{1}{n^2}$
   Requires $\widetilde{O}(n)$ samples from Y by *a.a.*



Mean estimation on the rings

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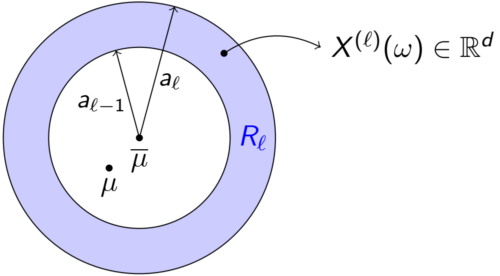
We consider $X^{(\ell)} = X \cdot \mathbb{1}_{X \in R_\ell} - \overline{\mu}$.
Goal: Estimate $\mathbb{E}[X^{(\ell)}]$.



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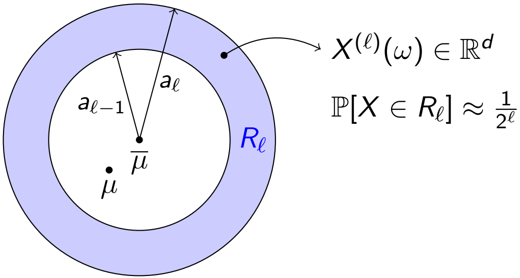

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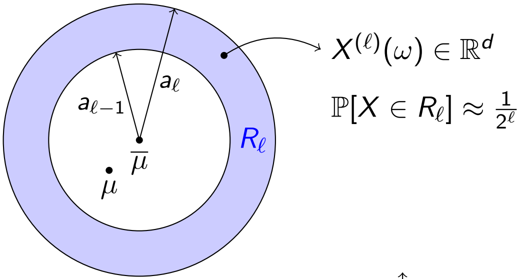

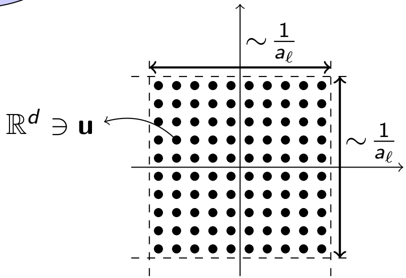

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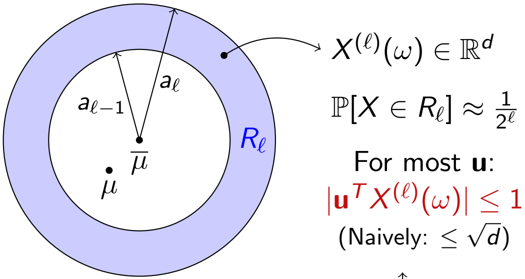

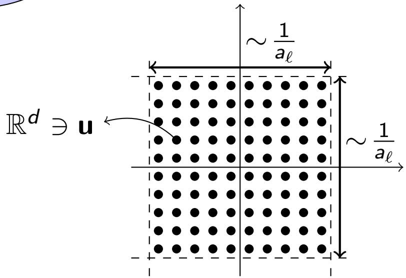

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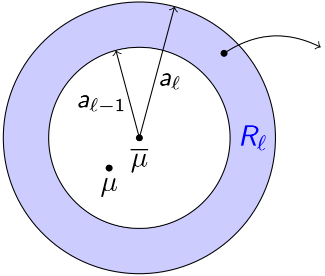

$\mathbb{P}[X \in R_\ell] \approx \frac{1}{2^\ell}$

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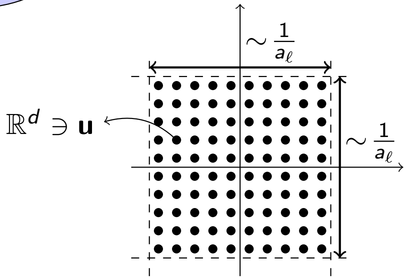

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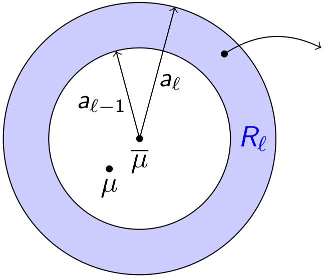

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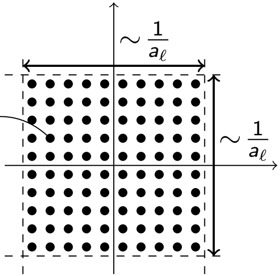

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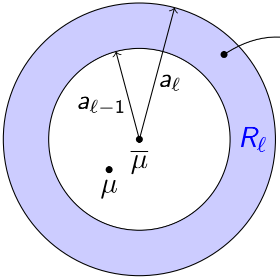

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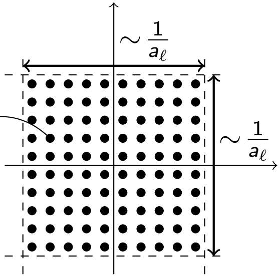

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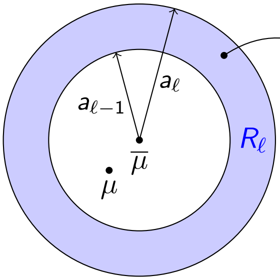

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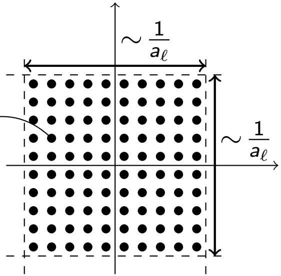

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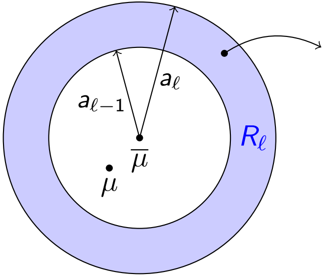

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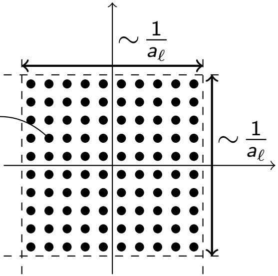

Concluding remarks

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Main result:
Optimal estimator $\widetilde{\mu}$ with n samples, s.t.

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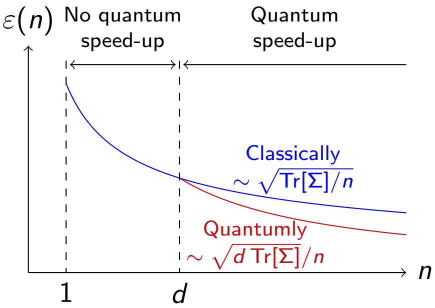

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Thanks for your attention!

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