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Quantum Query Complexity Problem Session 4

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Problem 1

Question 1

The statement that $\mathrm{Adv}^*(f) \le \mathrm{Adv}(f)$ will follow from the following two observations.

1. The feasible region of the second program is a superset of the feasible region of the first program. Since both programs are minimizing, having a larger feasible region means that a smaller quantity may appear and decrease the optimum of the second program.

   We can see that the feasible region is a superset, since the first program has the constraint that for all x, y

   $$\sum_{i:\ x_i \neq y_i} \left\langle v^{(x,i)} \middle| v^{(y,i)} \right\rangle = \begin{cases} 1 & f(x) \neq f(y) \\ 0 & f(x) = f(y) \end{cases} .$$

   The second program only constraints the sum of the inner products to be equal to 1 if $f(x) \neq f(y)$, but otherwise the sum of the inner products can be any arbitrary value.

2. The objective function of the second program is at most as large as the objective function of the first program at every point.

   This is because if we use the variables of the second program, the first objective equals $\max\{C_0, C_1\}$. Then it holds that

   $$\max\{C_0, C_1\} = \sqrt{\max\{C_0, C_1\}^2} \ge \sqrt{C_0 C_1}.$$

Question 2

Given a feasible solution $\{v^{(x,i)}\}$ to the first program, we will construct another feasible solution with value $\sqrt{C_0C_1}$. This solution will satisfy

$$u^{(x,i)} = \left(\frac{C_1}{C_0}\right)^{1/4} \cdot v^{(x,i)} \text{ for } f(x) = 0,$$

$$u^{(x,i)} = \left(\frac{C_0}{C_1}\right)^{1/4} \cdot v^{(x,i)} \text{ for } f(x) = 1.$$

We can verify that

$$\begin{aligned}\sum_{i:x_i\neq y_i} \left\langle u^{(x,i)} | u^{(y,i)} \right\rangle &= \sum_{i:x_i\neq y_i} \left(\frac{C_0}{C_1}\right)^{1/4} \cdot \left(\frac{C_1}{C_0}\right)^{1/4} \cdot \left\langle v^{(x,i)} | v^{(y,i)} \right\rangle \\ &= \sum_{i:x_i\neq y_i} \cdot \left\langle v^{(x,i)} | v^{(y,i)} \right\rangle \\ &= \mathbf{1}_{f(x)\neq f(y)}.\end{aligned}$$

Additionally,

$$\max_{x:f(x)=0} \sum_i \|u^{(x,i)}\|^2 = \sqrt{\frac{C_1}{C_0}} \cdot \max_{x:f(x)=0} \sum_i \|v^{(x,i)}\|^2 = \sqrt{C_0C_1},$$

$$\max_{x:f(x)=1} \sum_i \|u^{(x,i)}\|^2 = \sqrt{\frac{C_0}{C_1}} \cdot \max_{x:f(x)=1} \sum_i \|v^{(x,i)}\|^2 = \sqrt{C_0C_1}.$$

Thus the value of the program for $u^{(x,i)}$ is $\sqrt{C_0C_1}$.

Question 3

$$\sum_{i:x_i\neq y_i} \left\langle w^{(x,i)} | w^{(y,i)} \right\rangle = \sum_{i:x_i\neq y_i} \left\langle v^{(x,i)}, x_i \oplus f(x) | v^{(y,i)}, y_i \oplus f(y) \right\rangle$$

If $f(x) = f(y)$, then $x_i \oplus f(x) \neq y_i \oplus f(y)$, and thus the inner product is zero for all i. If on the other hand, $f(x) \neq f(y)$, then $x_i \oplus f(x) = y_i \oplus f(y)$ and thus the summation becomes

$$= \sum_{i:x_i\neq y_i} \left\langle v^{(x,i)} | v^{(y,i)} \right\rangle,$$

which is equal to 1 since $\{v^{(x,i)}\}$ is a feasible solution to the second program.

Question 4

Let $v^{(x,i)}$ be such that achieves the minimum value of $\sqrt{C_0 C_1}$ in the second program. Then we define $w^{(x,i)}$ as in Question 3, which implies that $w^{(x,i)}$ is a feasible solution to the first program. Additionally, note that $\|w^{(x,i)}\| = \|v^{(x,i)}\|$. Thus the values of C_0 and C_1 in the first program for the w variables are the same as the C_0, C_1 in the second program for the v variables. Now we use Question 2 and obtain that there exists a feasible solution to the first program that obtains the value $\sqrt{C_0 C_1}$.

Thus the minimum value of the first program is at least as small as the minimum value of the second program $\implies \mathrm{Adv}^*(f) \geq \mathrm{Adv}(f)$. Finally, from Question 1 we conclude the equality.

Problem 2

Question 1

WLOG, say that $y \in G_1$ is the cycle that maps $1 \to 2 \to \ldots \to n \to 1$. This is because we can relabel the vertices such that our cycle has the form above. Then $v^{(y,\{j,j+1\})} = |j\rangle$ for $j \in \{1, \ldots, n-1\}$ and $v^{(y,\{1,n\})} = |n\rangle$.

Thus we only need to consider the sum over the edges of y, which are not edges of x, since the rest of the $v^{(y,\{u,v\})}$ values are equal to $\emptyset$. Thus

$$\sum_{i: x_i \neq y_i} \langle v^{(x,i)} | v^{(y,i)} \rangle = \sum_{\{j,j+1\} \notin x} \langle v^{(x,\{j,j+1\})} | v^{(y,\{j,j+1\})} \rangle$$

$$= \sum_{j=1, \{j,j+1\} \notin x}^{n-1} \langle v^{(x,\{j,j+1\})} | j \rangle + \mathbf{1}[\{1,n\} \notin x] \cdot \langle v^{(x,\{1,n\})} | n \rangle.$$

Now note that

$$\langle v^{(x,\{j,j+1\})} | j \rangle = \begin{cases} 1, & 1 \text{ and } j \text{ are in the same c.c., but not } j \text{ and } j+1 \\ -1, & 1 \text{ and } j+1 \text{ are in the same c.c., but not } j \text{ and } j+1 \\ 0, & j \text{ and } j+1 \text{ are in the same c.c. (which includes } \{j,j+1\} \in x). \end{cases}$$

Thus consider the cut between the two connected components of x and draw a cycle $1 \to 2 \to \ldots \to n \to 1$. Let the 'left' c.c. be the c.c. with 1, and the other c.c. be the 'right' one. Then every time we cross from left to right, the sum increases by one, whereas every time we cross from right to left, the sum decreases by one. If following the cycle doesn't change the connected component, the sum does not change.

Since the cycle returns to one, the number of times we cross left to right must equal the number of times we cross right to left. Thus the summation is equal to 0 as desired.

Question 2

We will use the alternative characterization that we derived in Problem 1. We will keep $v^{(y,i)}$ for $y \in G_1$ as-is. We will also extend $v^{(y,i)}$ to all connected y that are not only cycles. Since y is connected, we can fix any cycle y' and orientation and define $v^{(y,i)}$ to be the same as $v^{(y',i)}$ (note that the cycle does not have to go ever every node). We will also extend $v^{(x,i)}$ to all graphs x that are disconnected but may have more than 2 connected components. The definition is the same, and the argument for orthogonality still holds in the same way.

This will make $C_0 = O(n^2)$ and $C_1 = O(n)$. The maximum value of C_0 is achieved via the union of two disjoint cliques of sizes $\Theta(n)$ each, whereas any graph achieves the maximum value of C_1 in G_1. Thus $\sqrt{C_0 C_1}$ for our construction of the v variables is equal to $O(n^{3/2})$.

Note, however, that v is not a feasible point; we want the sum of the inner products to be 1 for $f(x) \neq f(y)$. To fix this, we will modify our graphs slightly by adding a fake node f to all graphs in $G_0 \cup G_1$ and edge $\{f, 1\}$ to all graphs in G_1. This does not affect the connectivity of our graph, but it introduces another variable $v^{(x,\{f,i\})}$, which we will set to $|\sigma\rangle$, where $|\sigma\rangle$ is a new special symbol. Then the inner product will become

$$\begin{aligned}
\sum_{i:x_i \neq y_i} \langle v^{(x,i)} | v^{(y,i)} \rangle &= \langle v^{(x,\{f,1\})} | v^{(y,\{f,1\})} \rangle + \sum_{i=\{u,v\}:x_i \neq y_i} \langle v^{(x,i)} | v^{(y,i)} \rangle \\
&= 1 + \sum_{i=\{u,v\}:x_i \neq y_i} \langle v^{(x,i)} | v^{(y,i)} \rangle \\
&= 1.
\end{aligned}$$

This small transformation only changes the norm of the $|v\rangle$'s by at most 1, hence the value of C_0, C_1 does not change asymptotically.

Problem 3

Question 1

We will use the candidate dual adversary solution provided in the hint. Then we can compute for $f \circ g(X) = 0$ and $f \circ g(Y) = 1$:

$$\begin{aligned}
&\sum_{\substack{(i,j)\\ X_{(i,j)} \neq Y_{(i,j)}}} \langle v_{f\circ g}^{(X,(i,j))} | v_{f\circ g}^{(Y,(i,j))} \rangle \\
&= \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \sum_{\substack{j\\ X_{(i,j)} \neq Y_{(i,j)}}} \langle v_g^{(X_i,j)} | v_g^{(Y_i,j)} \rangle \\
&= \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \mathbf{1}[g(X_i) \neq g(Y_i)] \\
&= \sum_{\substack{i\\ g(X_i) \neq g(Y_i)}} \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \\
&= \mathbf{1}[f\circ g(X) \neq f \circ g(Y)]
\end{aligned}$$

Thus the candidate solution is a feasible point. Let us now compute its value:

$$\begin{aligned}
\sum_{(i,j)} \left\| v_{f\circ g}^{(X,(i,j))} \right\|^2 &= \max_X \sum_{(i,j)} \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \langle v_g^{(X_j,j)} | v_g^{(Y_i,j)} \rangle \\
&\leq \max_X \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \max_X \sum_j \langle v_g^{(X_i,j)} | v_g^{(Y_i,j)} \rangle \\
&\leq \max_X \sum_i \langle v_f^{(g(X_1),\dots,g(X_n)),i)} | v_f^{(g(Y_1),\dots,g(Y_n)),i)} \rangle \cdot \mathrm{Adv}(g) \\
&= \mathrm{Adv}(f) \cdot \mathrm{Adv}(g)
\end{aligned}$$

Where for the last two equalities we used the fact that v_f, v_g are the dual adversary solutions for f and g. Thus we have constructed a feasible solution for the $f \circ g$ function that has a value at most $\mathrm{Adv}(f) \cdot \mathrm{Adv}(g)$. Since the adversary program is minimizing, its optimal value may be even lower, and thus the statement holds.

Question 2

As per the hint, let Γ be the primal adversary solution for g. We will construct a primal adversary solution for $f \circ g$ as a block matrix. For simplicity, define $G_0 = \{X \mid g(X) = 0\}$ and $G_1 = \{Y \mid g(Y) = 1\}$. Then our new adversary matrix Γ' will have blocks that are indexed as n-tuples of G_0, G_1. In particular,

$$\Gamma'[(i_1, \ldots, i_n), (j_1, \ldots, j_n)] = \begin{cases} \Gamma & \text{if } \{\sum_k i_k, \sum_k j_k\} = \{0, 1\} \\ \mathbf{0} & \text{o.w.} \end{cases}$$

This is just the adversary matrix we saw for the OR function, but each block now includes the adversary matrix Γ for g.

One can verify that Γ' is a valid adversary matrix since it is symmetric and is equal to zero for inputs that map to the same value.

It remains for us to bound

$$\frac{\|\Gamma'\|}{\max_i \|\Gamma'_i\|}.$$

It is easy to see that $\|\Gamma'\| = \sqrt{n} \cdot \|\Gamma\|$ since it is a block matrix with n copies of Γ in each block.

Let us now investigate the form of Γ'_i. Here i is a coordinate from 1 to mn. Coordinate i lies in the r^{th} block and c^{th} coordinate of the block, where $r = \lceil i/m \rceil$ and $c = i \mod m + 1$.

Then Γ'_i will have all blocks equal to $\mathbf{0}$, except possibly block $(0, \ldots, 0)$ and $(0, \ldots, 1, \ldots, 0)$, where the 1 is in the r^{th} position. This block will be equal to Γ_c. Thus $\|\Gamma'_i\| = \|\Gamma_c\|$, due to the block format of the big matrix. Thus

$$\mathrm{Adv}(f \circ g) = \frac{\|\Gamma'\|}{\max_i \|\Gamma'_i\|} \geq \frac{\sqrt{n} \cdot \|\Gamma\|}{\max_{c=1}^m \|\Gamma_c\|} = \sqrt{n} \cdot \mathrm{Adv}(g).$$