# Quantum Sub-Gaussian Mean Estimator

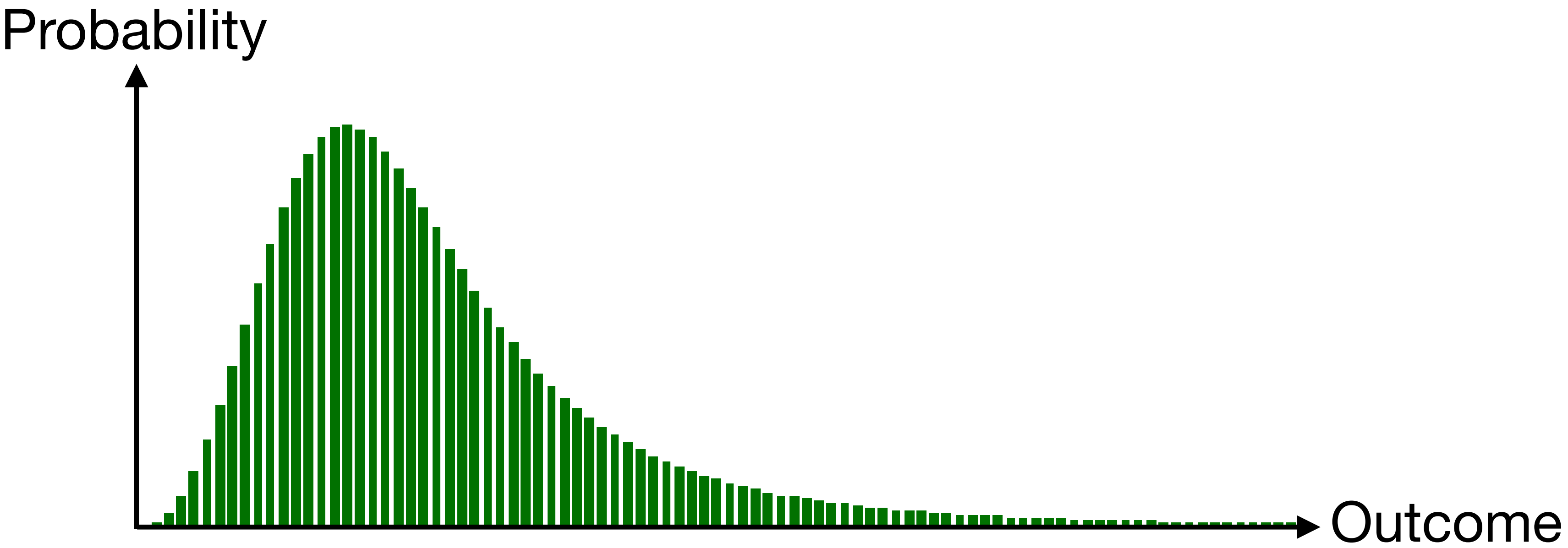
Yassine Hamoudi

ESA 2021

# Mean Estimation problem (over $\mathbb{R}$)



Experiment with unknown outcome distribution $D$



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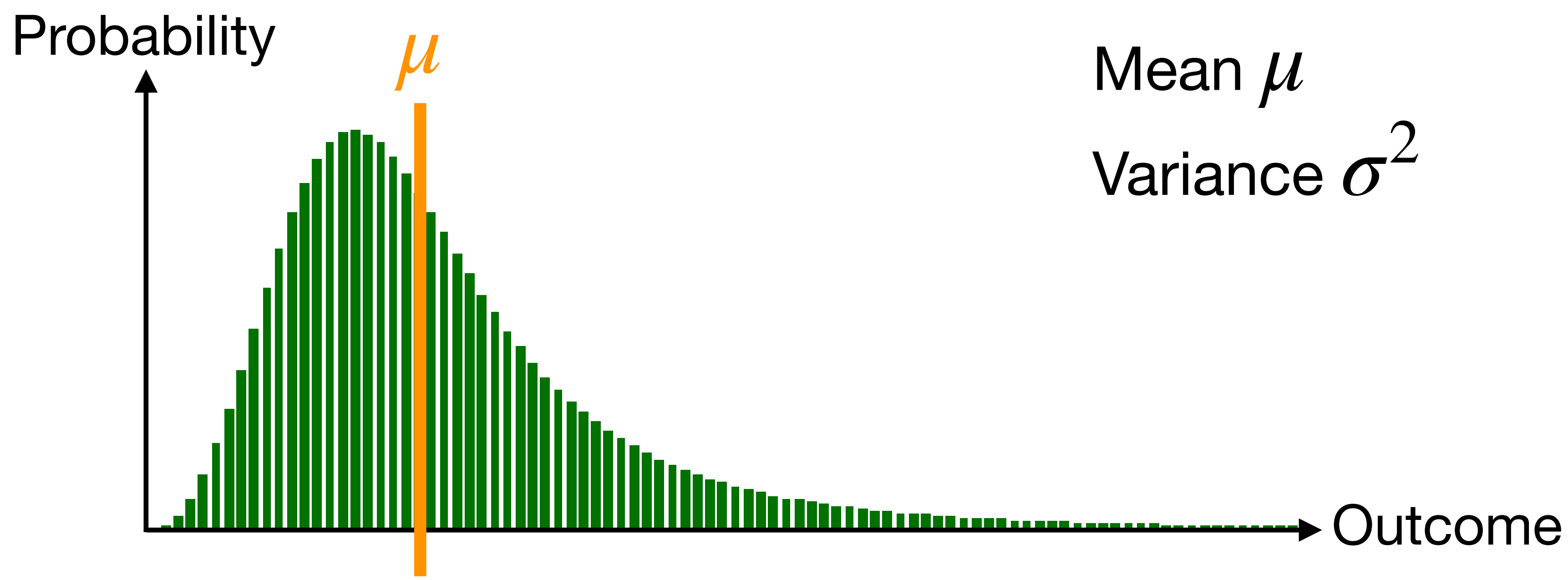


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Variance $\sigma^2$

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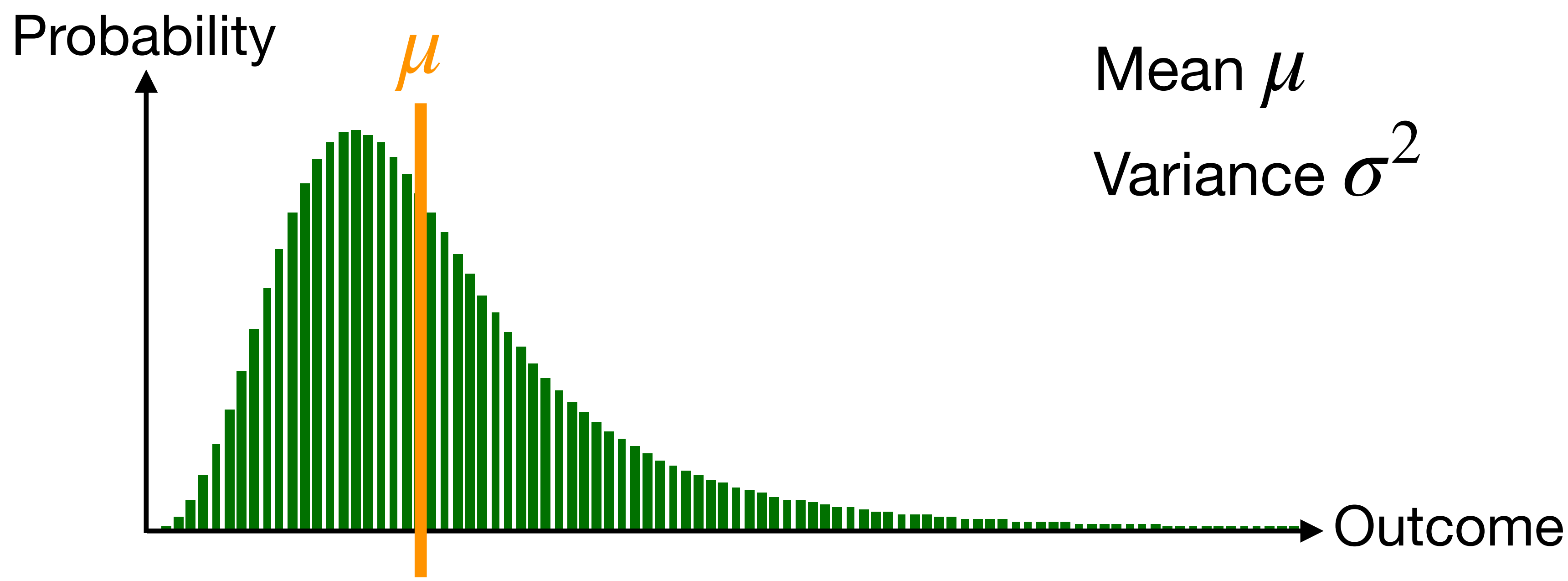


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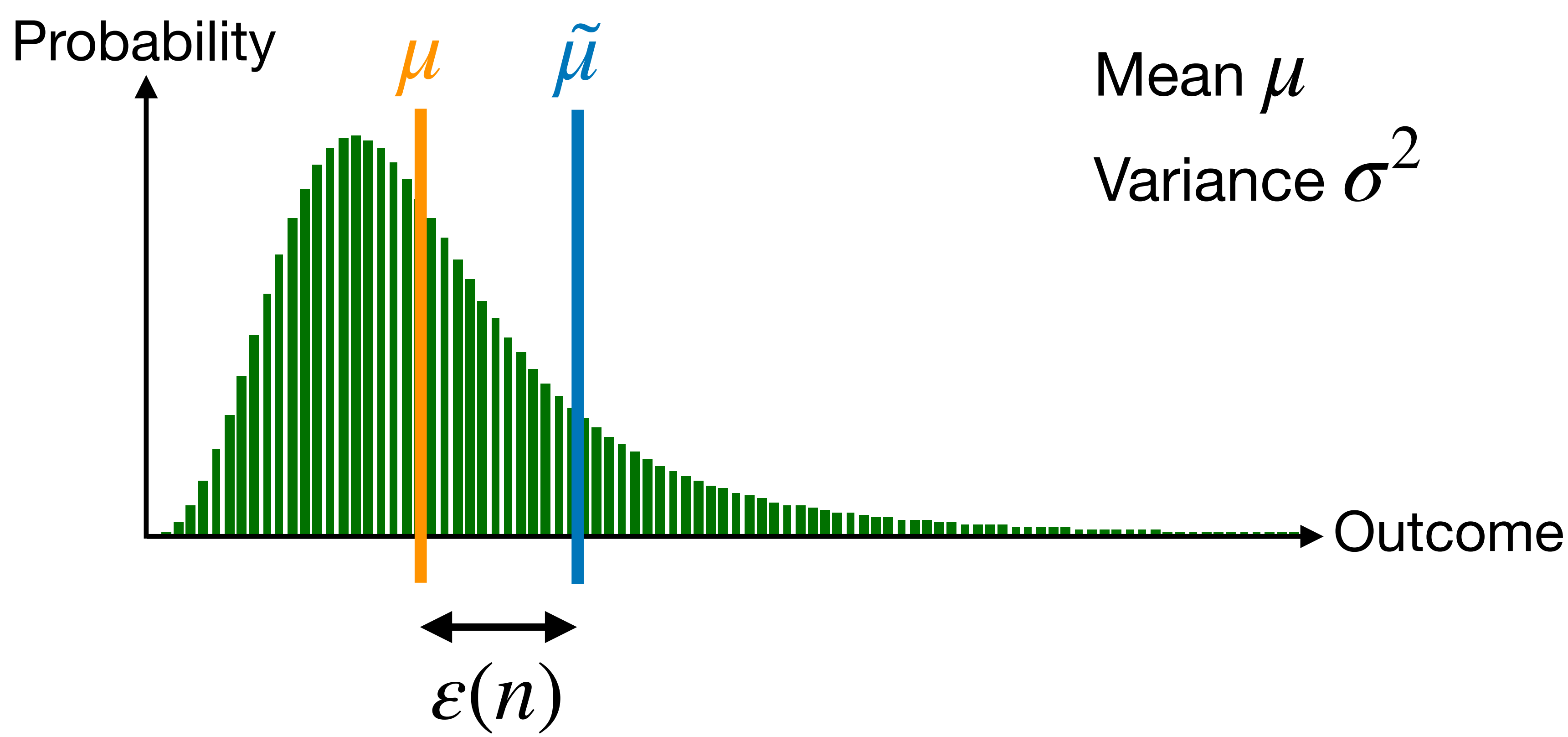


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**<u>Goal:</u>** compute an estimate $\tilde{\mu}$ that minimizes the error $\varepsilon(n)$ such that

$$\Pr\left[\,|\mu - \tilde{\mu}| > \varepsilon(n)\right] < \delta \qquad \text{given } \delta \in (0,1)$$

# Optimal classical estimators



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optimal $\sqrt{2} + o(1)$ factor in $O(\,.\,)$

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[H.,Magniez'19]

# Our result



Quantum "sub-Gaussian" estimator:

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✓ Doesn't require any prior knowledge about $D$

✓ Subsumes all previous quantum estimators

✓ Better than any classical sub-Gaussian estimator (recall: $\sqrt{\frac{\sigma^2 \log(1/\delta)}{n}}$)

✓ Optimal (lower bound by reduction from Quantum Search)

**up to log factors*

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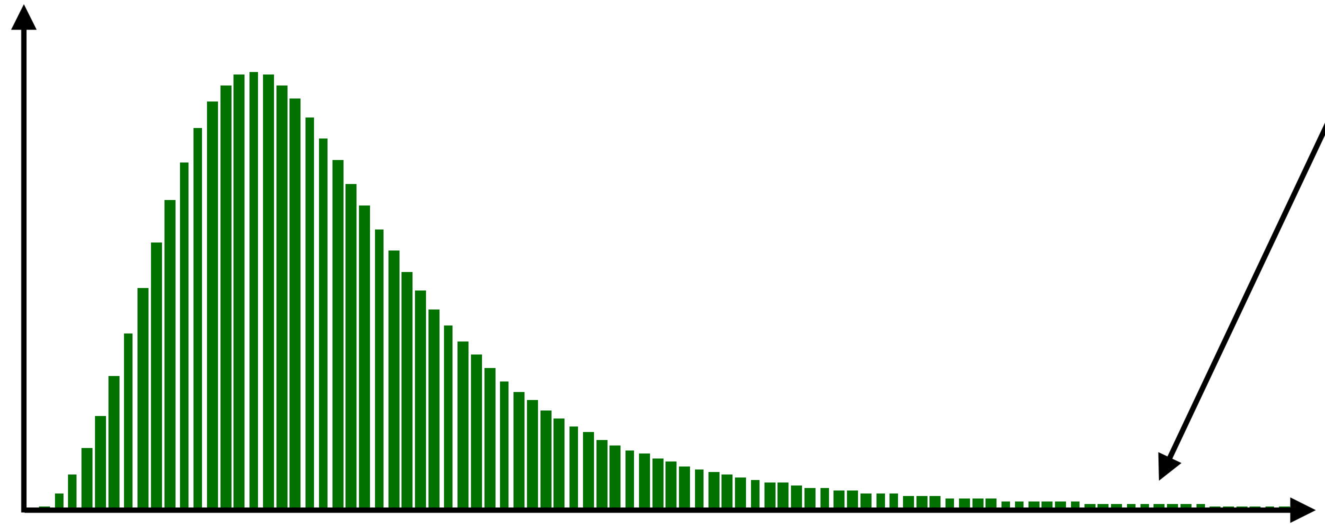
→ Similar models in other works on estimating/testing statistics [Nayak,Wu'99] [Bravyi,Harrow,Hassidim'11][Montanaro'15]…

# Technique: Outliers and truncation



Empirical Mean and Amplitude Estimation are too sensitive to **outliers**



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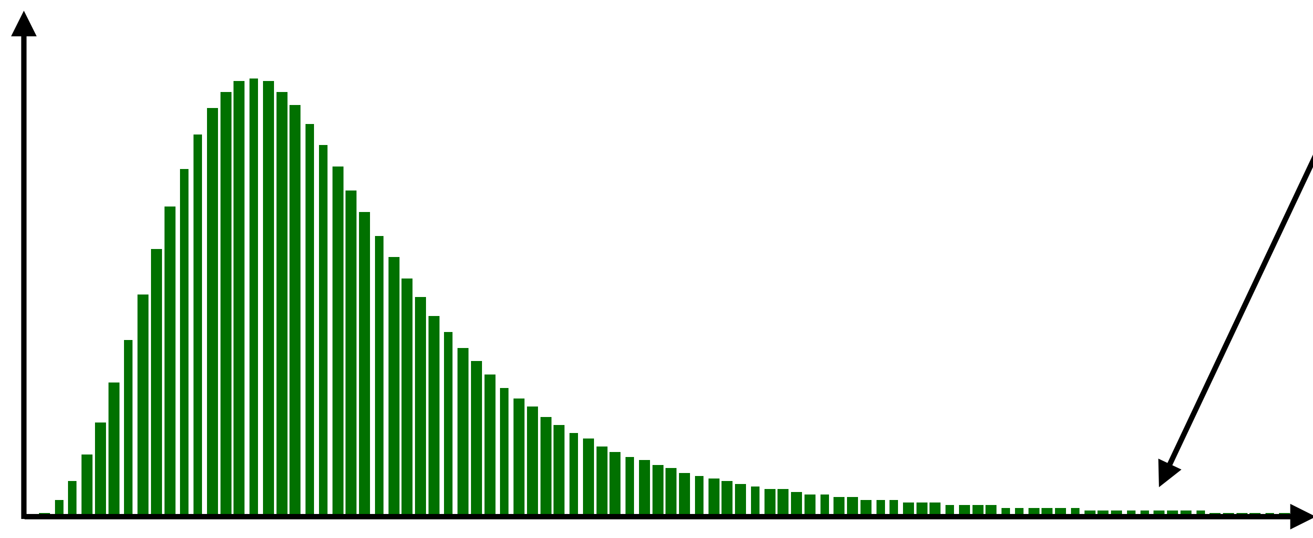

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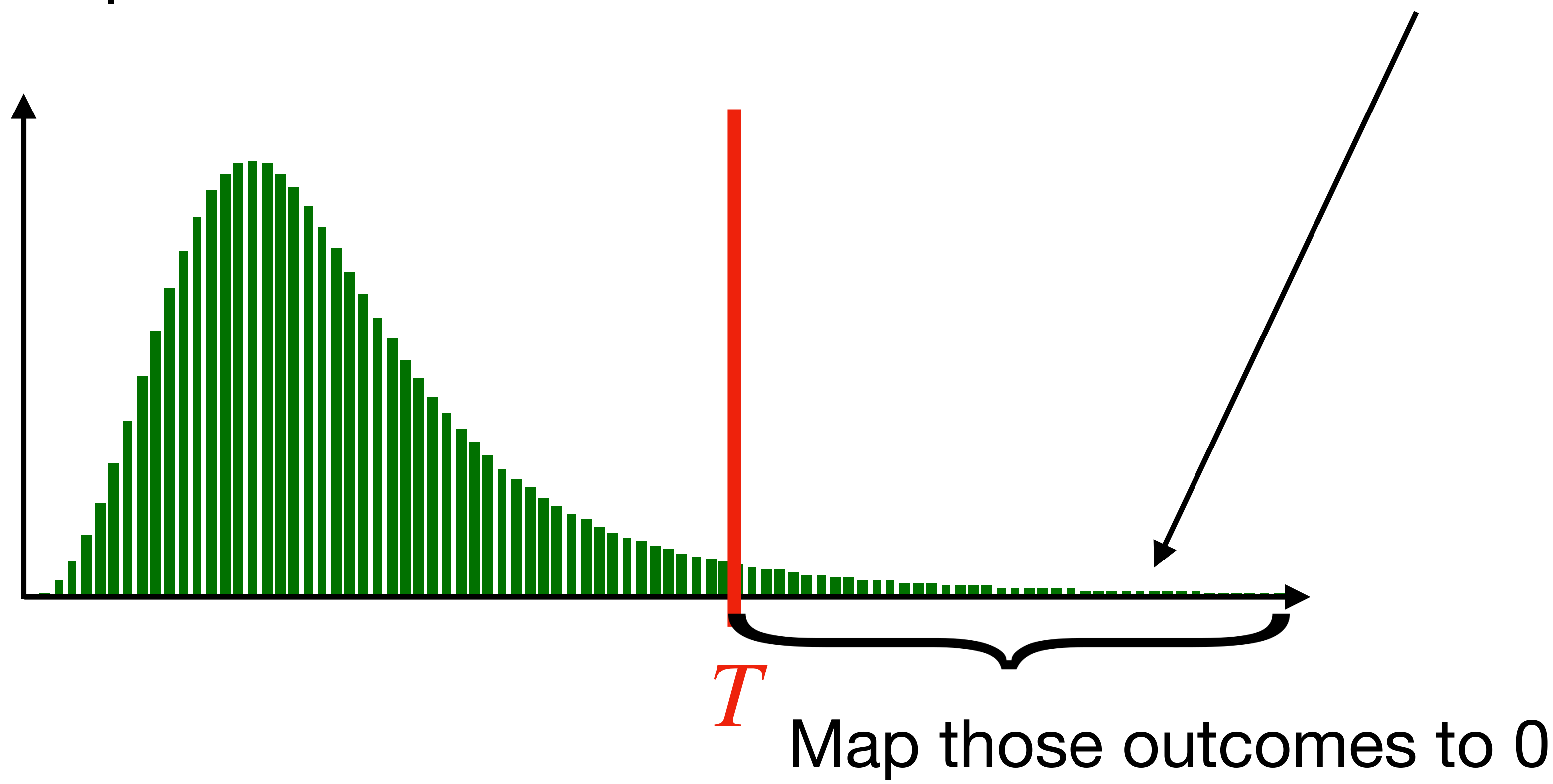


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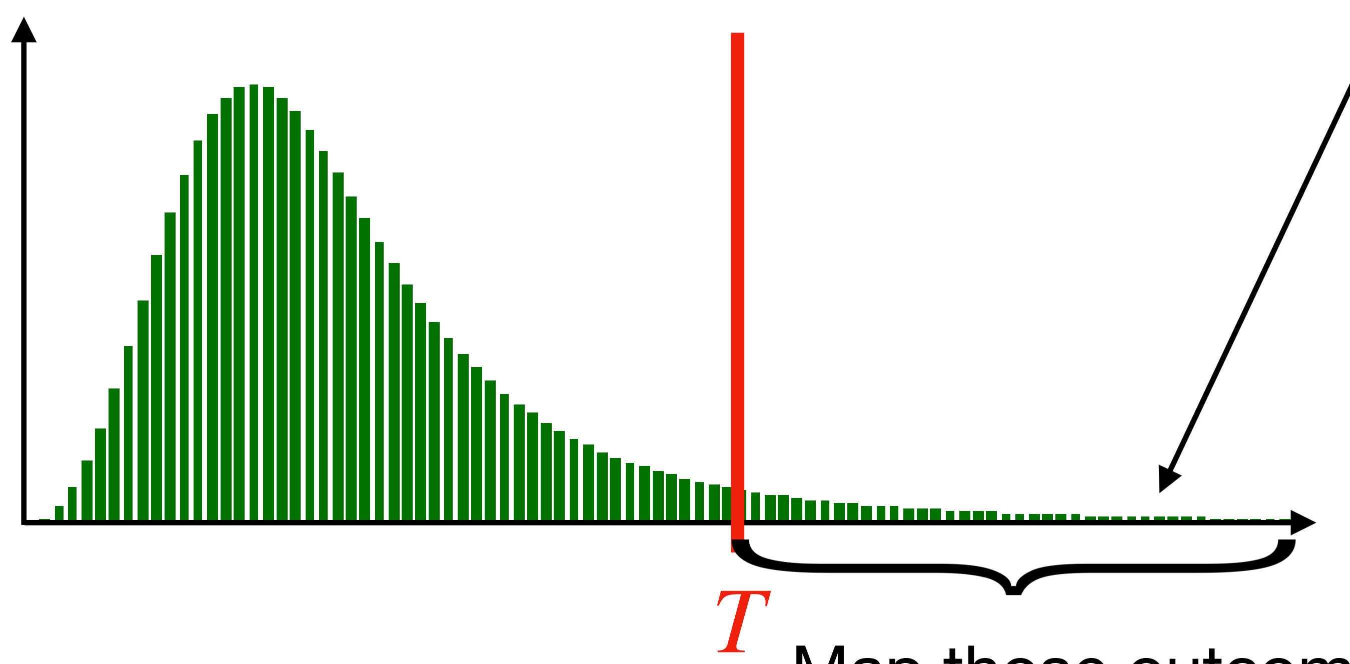


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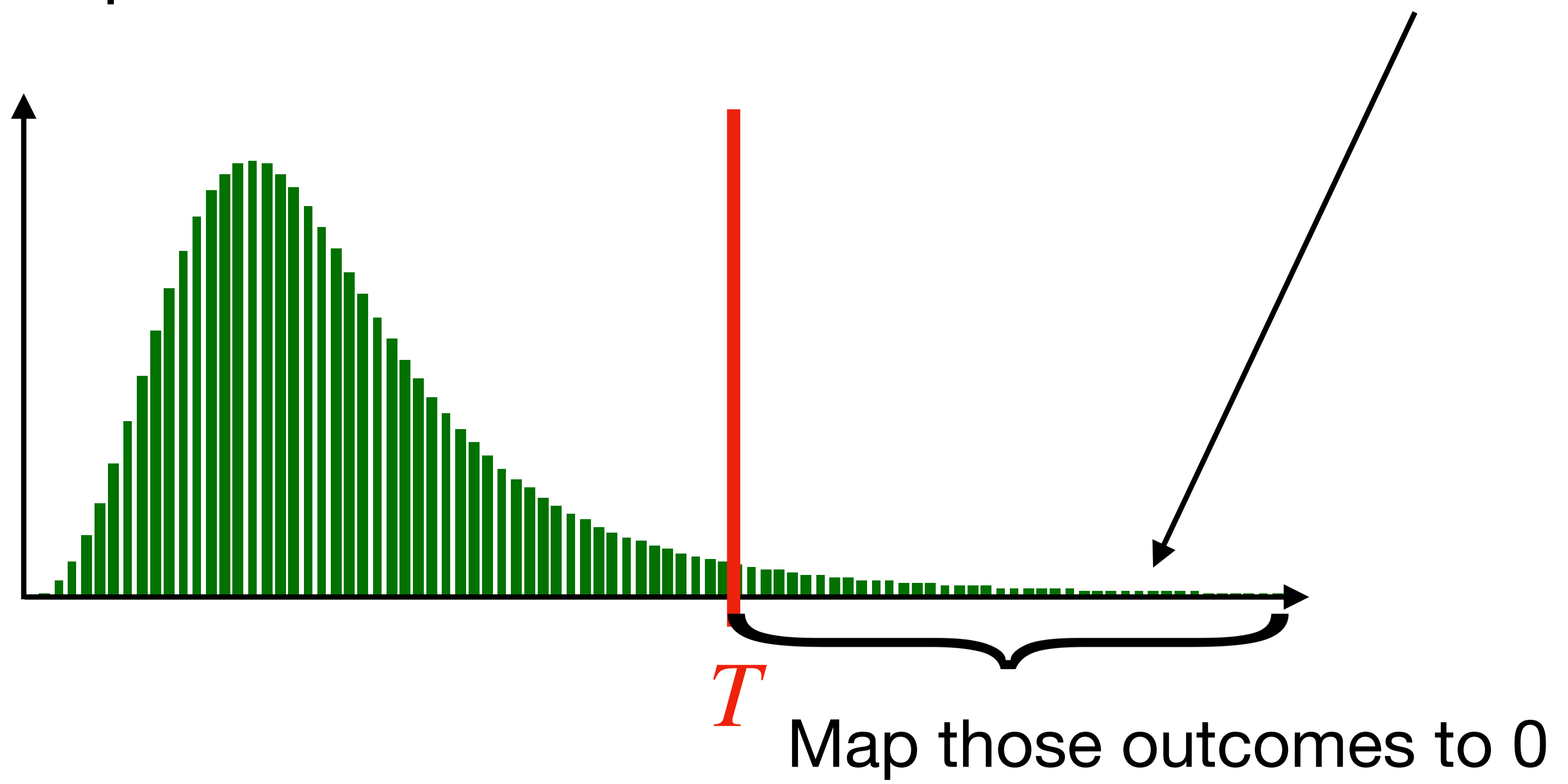


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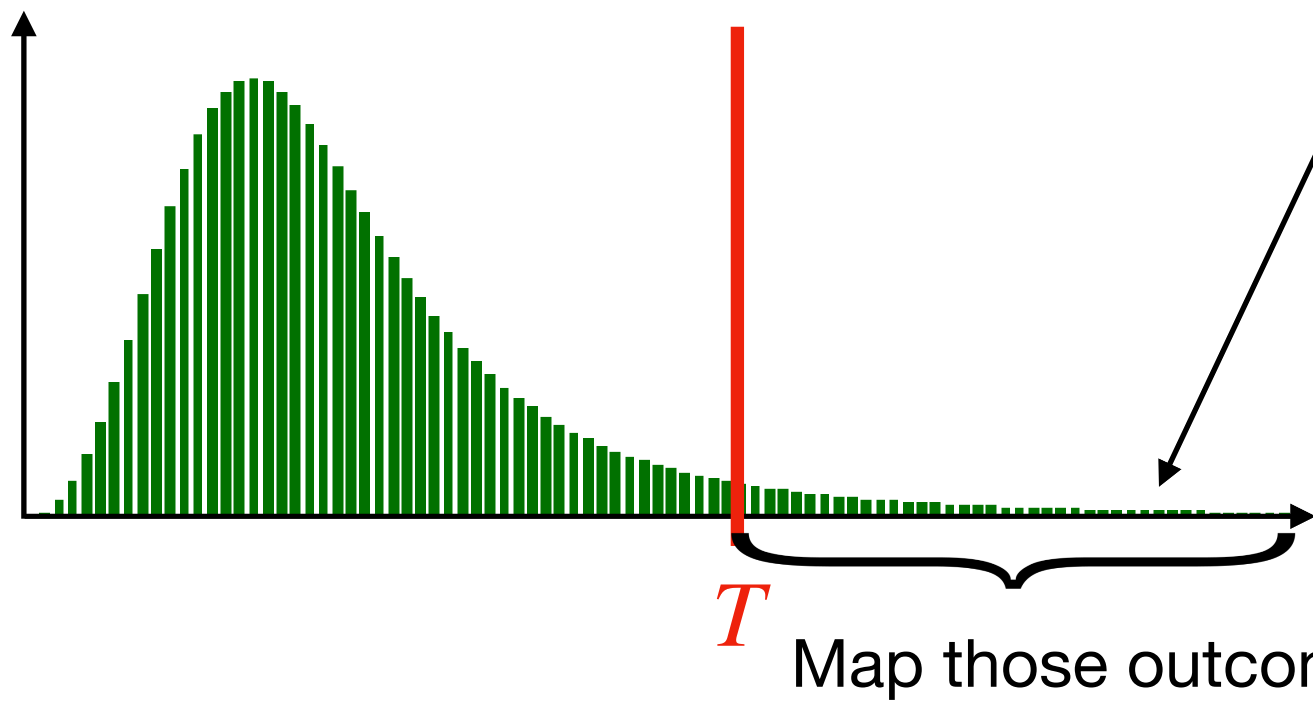


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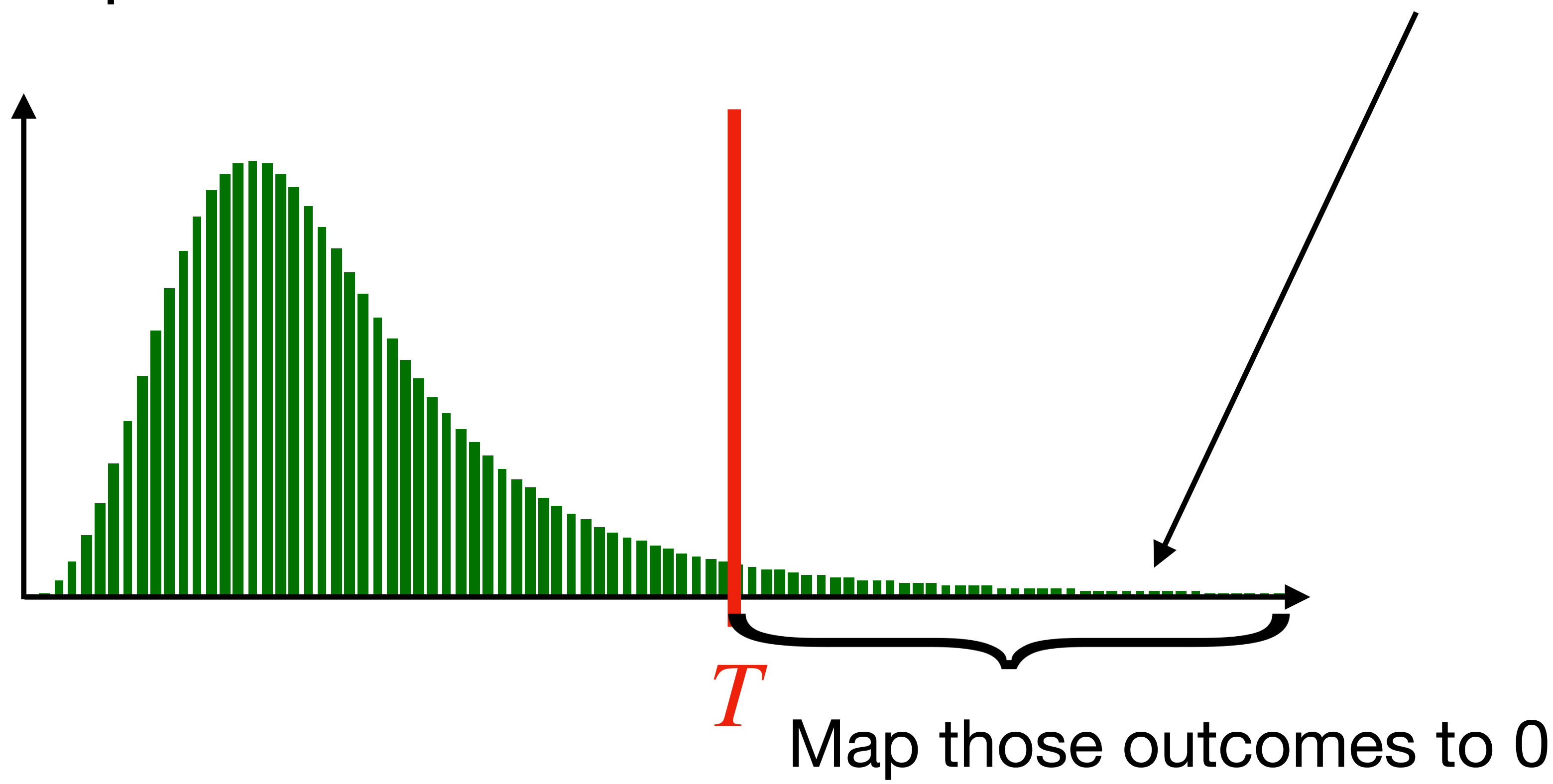


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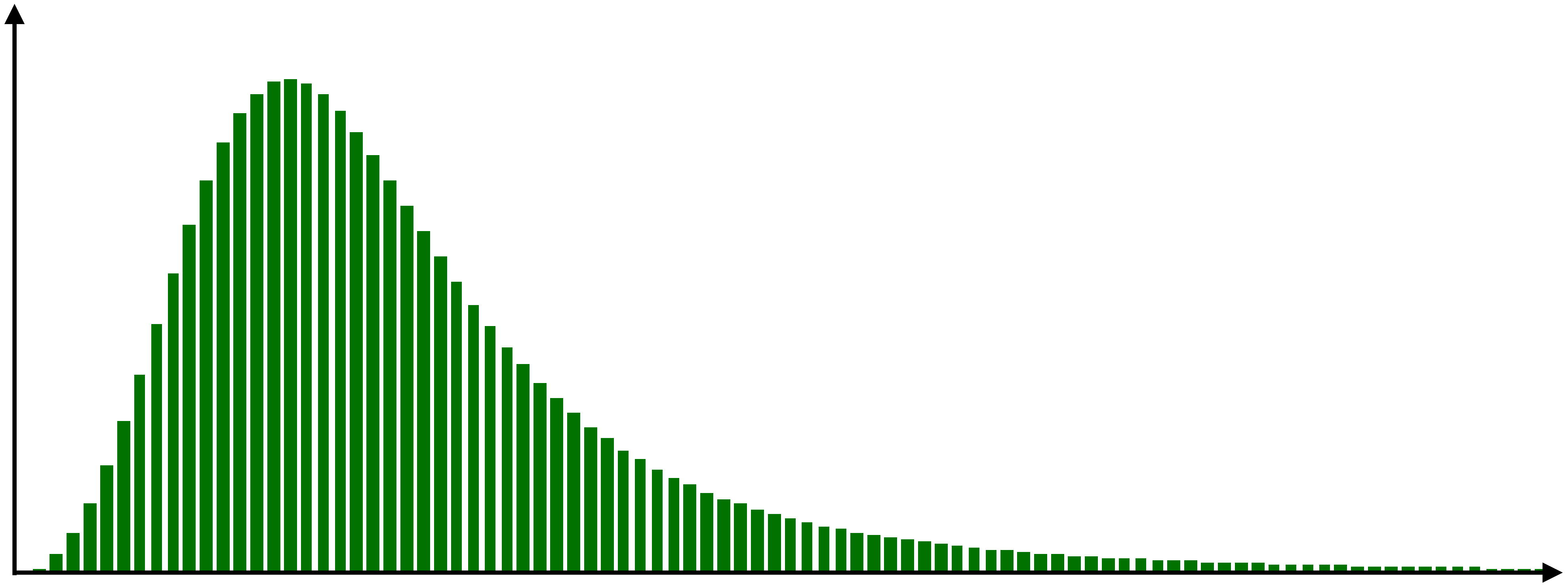

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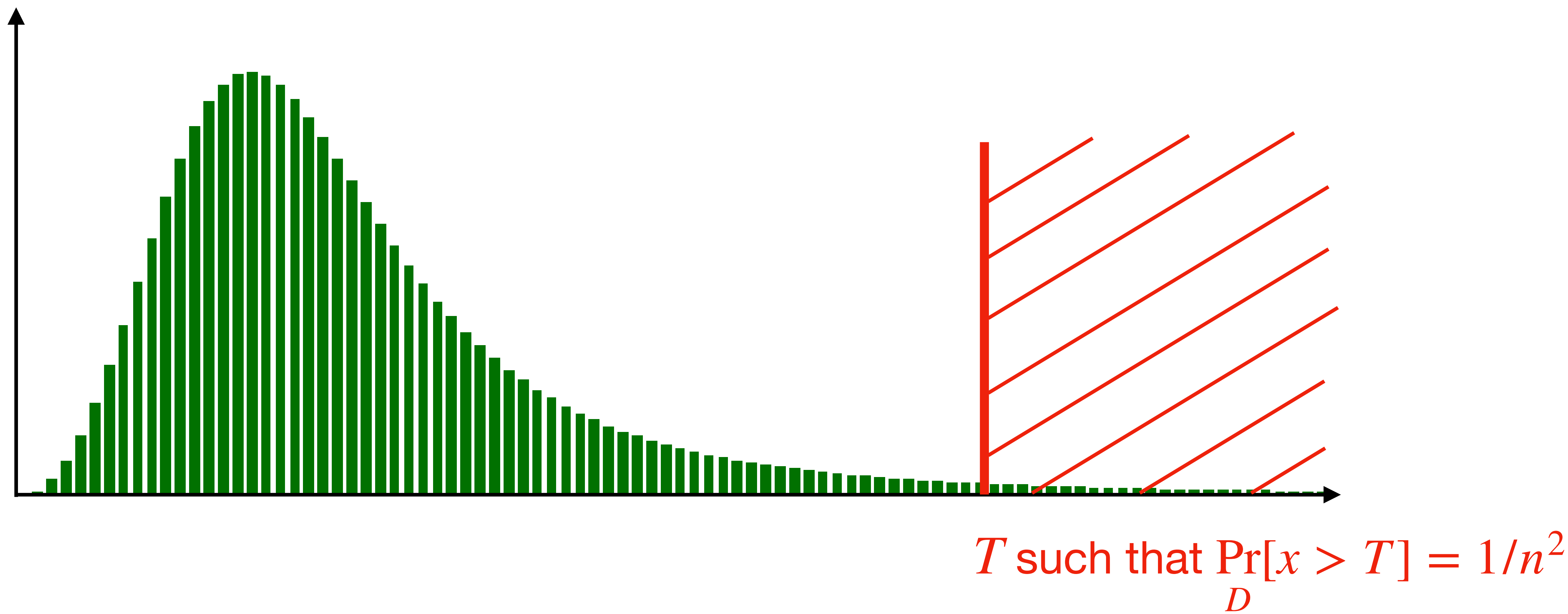

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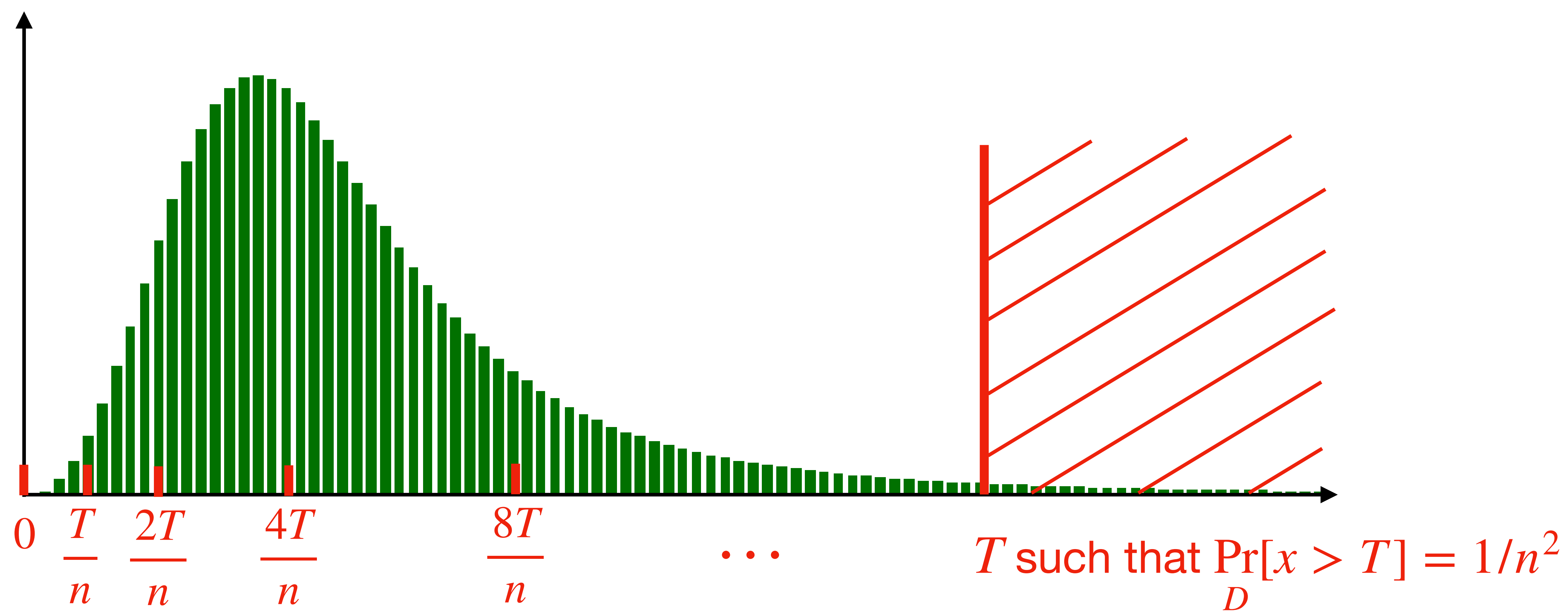

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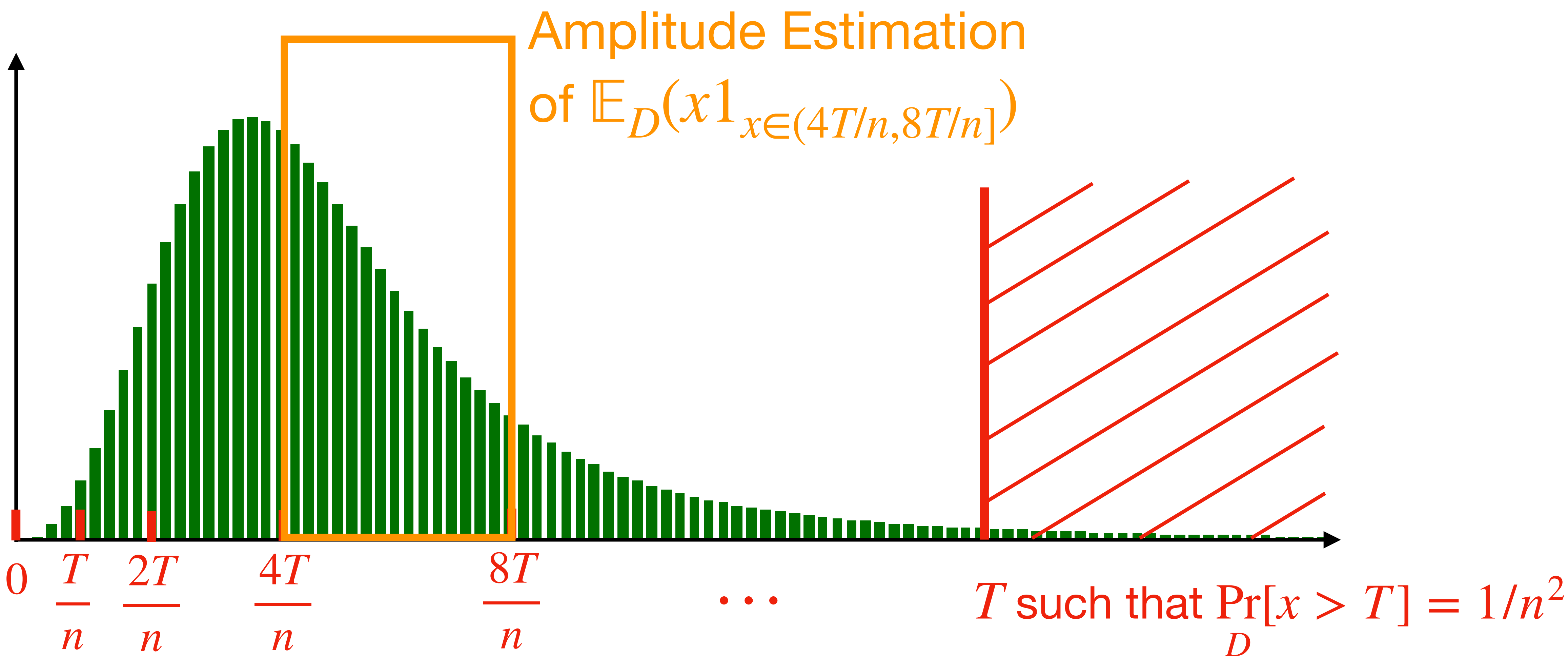

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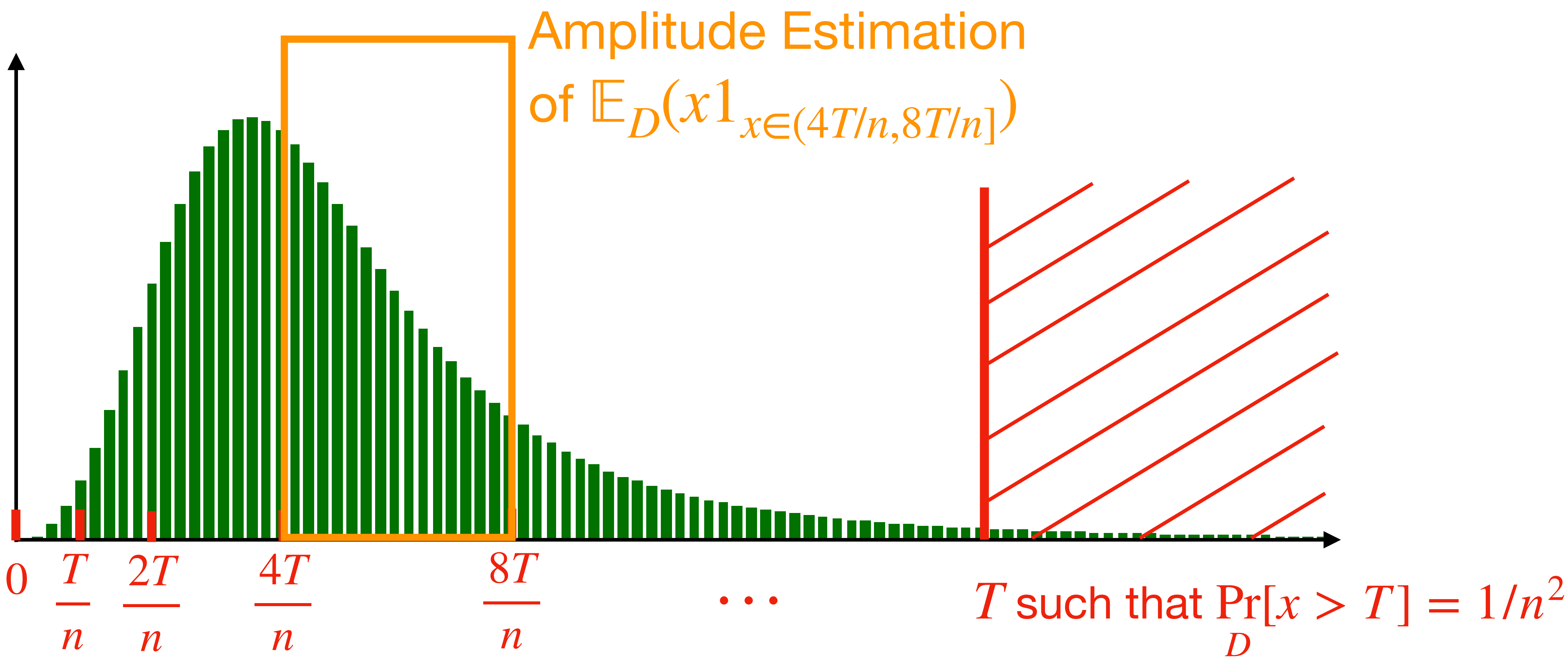


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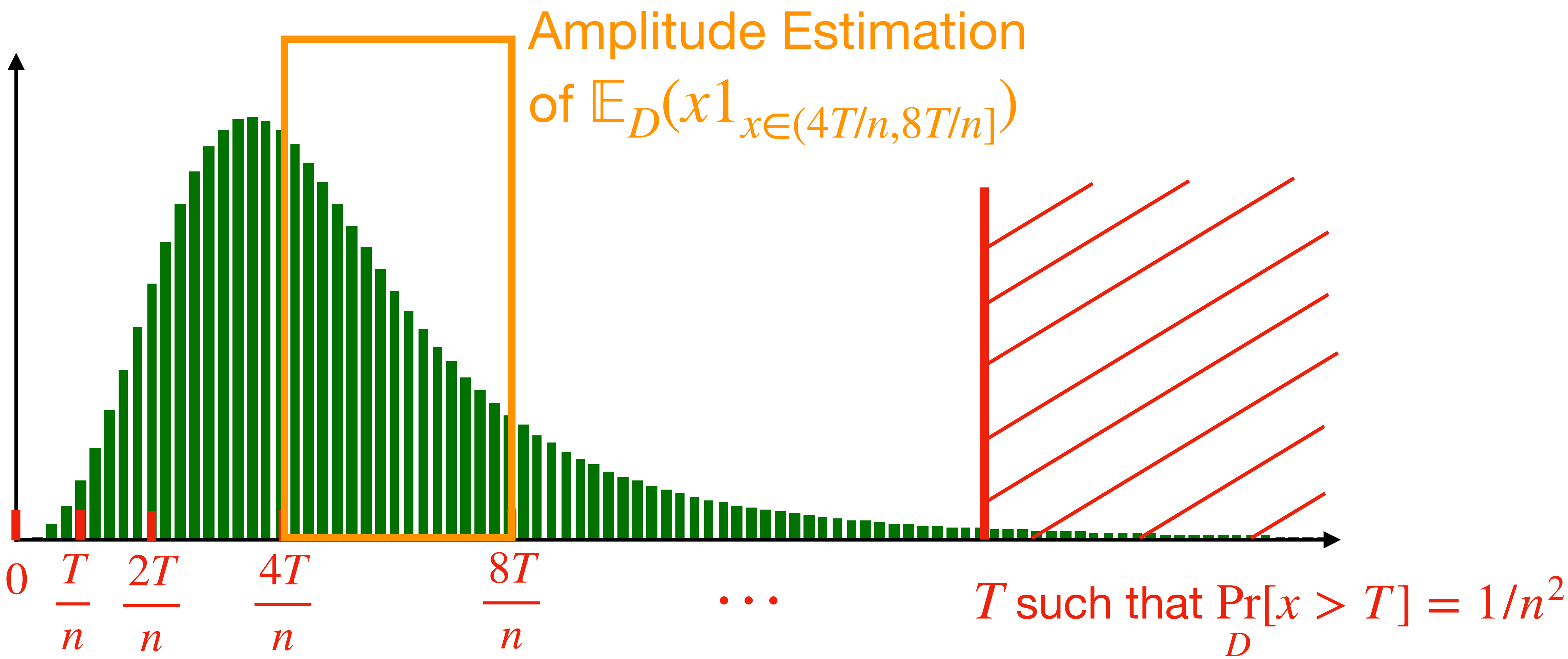


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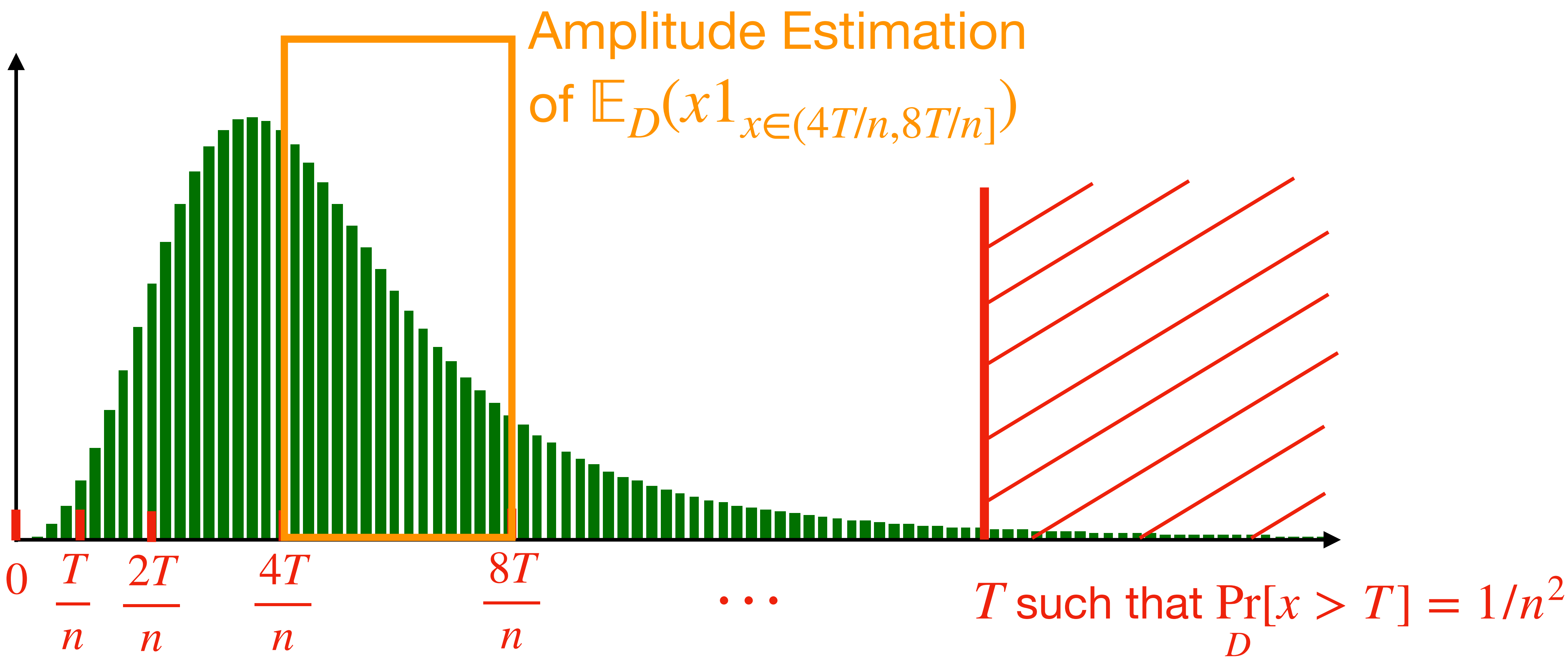


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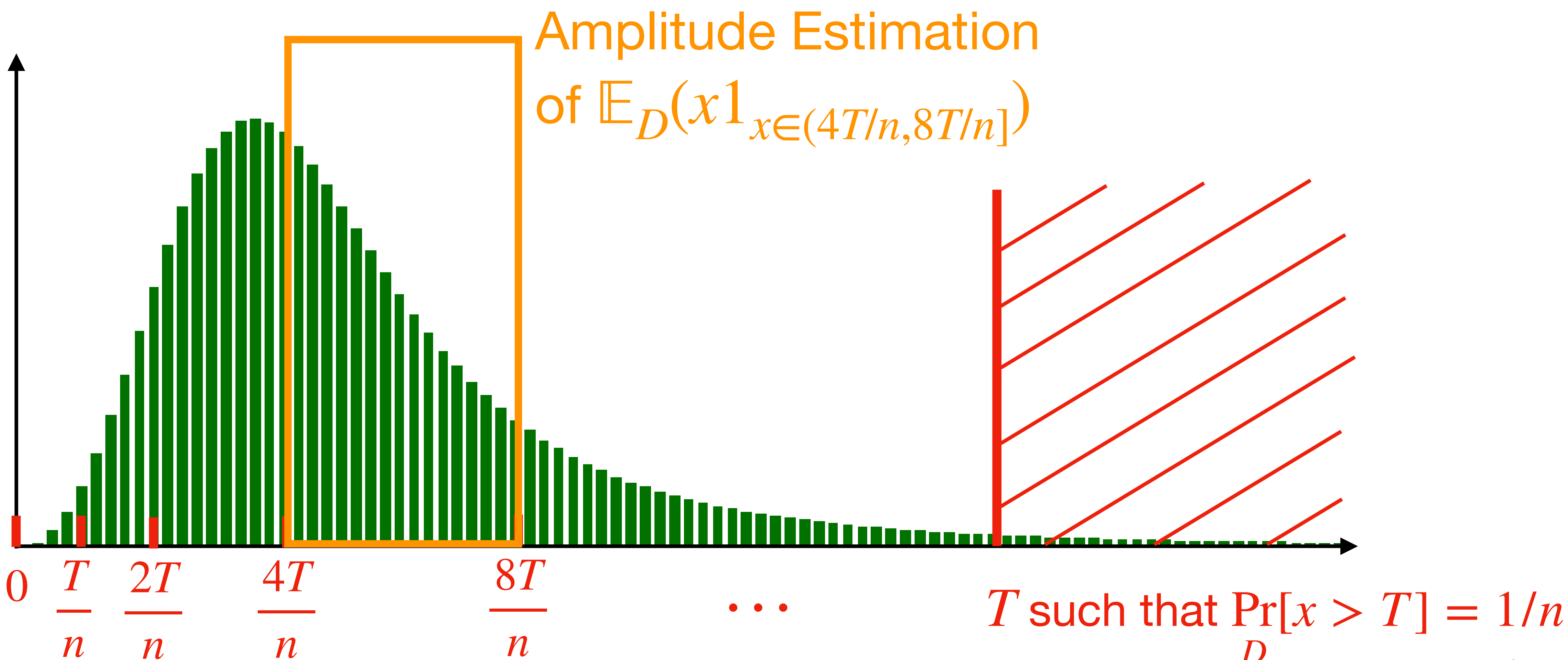


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# Conclusion



- **Sub-Gaussian estimators**

$$\Theta\left(\sqrt{\frac{\sigma^2 \log(1/\delta)}{n}}\right) \quad \text{vs} \quad \tilde{\Theta}\left(\frac{\sigma \log(1/\delta)}{n}\right)$$

Classical Quantum

- **Open questions**

  - Improve the log-factors

  - Extend to the **multivariate** setting (recent work: [Cornelissen,Jerbi'21])

  - Find other applications