Proof sketches

1 Lecture 1

Lemma 1.1. $\||\psi^0_{\vec{0}}\rangle - |\psi^0_{\vec{i}}\rangle\| = 0$

Proof. $|\psi^0_{\vec{0}}\rangle = |\psi^0_{\vec{i}}\rangle = U_0|0,0\rangle$ $\square$

Lemma 1.2. $\||\psi^T_{\vec{0}}\rangle - |\psi^T_{\vec{i}}\rangle\| \geq 1/3$ *if the algorithm succeeds wp* $\geq 2/3$ *after* T *queries*

Proof. Success conditions: $\|(\mathrm{Id} \otimes |0\rangle\langle 0|)|\psi^T_{\vec{0}}\rangle\|^2 \geq 2/3$ and $\|(\mathrm{Id} \otimes |1\rangle\langle 1|)|\psi^T_{\vec{i}}\rangle\|^2 \geq 2/3$

$$
\begin{aligned}
\||\psi^T_{\vec{0}}\rangle - |\psi^T_{\vec{i}}\rangle\|^2 &= 2(1 - \mathrm{Re}(\langle\psi^T_{\vec{0}}|\psi^T_{\vec{i}}\rangle)) \\
&= 2(1 - \mathrm{Re}(\langle\psi^T_{\vec{0}}|\,(\mathrm{Id} \otimes |0\rangle\langle 0|)\,|\psi^T_{\vec{i}}\rangle) - \mathrm{Re}(\langle\psi^T_{\vec{0}}|\,(\mathrm{Id} \otimes |1\rangle\langle 1|)\,|\psi^T_{\vec{i}}\rangle)) \\
&\geq 2(1 - \|(\mathrm{Id} \otimes |0\rangle\langle 0|)\psi^T_{\vec{0}}\| \cdot \|(\mathrm{Id} \otimes |0\rangle\langle 0|)\psi^T_{\vec{i}}\| - \|(\mathrm{Id} \otimes |1\rangle\langle 1|)\psi^T_{\vec{0}}\| \cdot \|(\mathrm{Id} \otimes |1\rangle\langle 1|)\psi^T_{\vec{i}}\|) \\
&\qquad \text{by Cauchy-Schwarz inequality} \\
&\geq 2(1 - 2\sqrt{2}/3) \qquad \text{by success conditions} \\
&\geq 1/9
\end{aligned}
$$

$\square$

Lemma 1.3. $\||\psi^{t+1}_{\vec{0}}\rangle - |\psi^{t+1}_{\vec{i}}\rangle\| \leq \||\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle\| + \sqrt{q^t_i}$

Proof.

$$
\begin{aligned}
\||\psi^{t+1}_{\vec{0}}\rangle - |\psi^{t+1}_{\vec{i}}\rangle\| &= \|U_{t+1}|\psi^t_{\vec{0}}\rangle - U_{t+1}O_{\vec{i}}|\psi^t_{\vec{i}}\rangle\| && \text{by definition and } O_{\vec{0}} = \mathrm{Id} \\
&= \||\psi^t_{\vec{0}}\rangle - O_{\vec{i}}|\psi^t_{\vec{i}}\rangle\| && \text{unitary preserves norm} \\
&= \|O_{\vec{i}}(|\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle) + (\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\| \\
&\leq \|O_{\vec{i}}(|\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle)\| + \|(\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\| && \text{by triangle inequality} \\
&= \||\psi^t_{\vec{0}}\rangle - |\psi^t_{\vec{i}}\rangle\| + \|(\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\|
\end{aligned}
$$

We have $\mathrm{Id} - O_{\vec{i}} = |i\rangle\langle i| \otimes (\mathrm{Id} - X)$ where $X = |1\rangle\langle 0| + |0\rangle\langle 1|$. Hence, $\|(\mathrm{Id} - O_{\vec{i}})|\psi^t_{\vec{0}}\rangle\| = \|(|i\rangle\langle i| \otimes (\mathrm{Id} - X))|\psi^t_{\vec{0}}\rangle\|\| \leq 2\|(|i\rangle\langle i| \otimes \mathrm{Id})|\psi^t_{\vec{0}}\rangle\|\| = \sqrt{q^t_i}$, where we used that $\|\mathrm{Id} \otimes (\mathrm{Id} - X)\| \leq 2$. $\square$

Theorem 1.4. $Q(\mathrm{OR}) \geq \sqrt{n}/3$

Proof. $n/3 \leq \sum_{i=1}^n \sum_{t=0}^T \sqrt{q^t_i} \leq \sqrt{nT \sum_{i=1}^n \sum_{t=0}^T q^t_i} = \sqrt{n}T \Rightarrow T \geq \sqrt{n}/3.$ $\square$