Quantum Query Complexity
PCMI Graduate Summer School 2023
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Course page: `https://yassine-hamoudi.github.io/pcmi2023/`

Problem Session 4

The adversary method and its dual

Problem 1 (Simplifying the dual adversary)

Recall the dual formulation of the adversary method:

$$\begin{array}{rl} \mathrm{Adv}(f) = \min_{\{w^{(x,i)}\}} & \max_x \sum_i \|w^{(x,i)}\|^2 \\ \text{s.t.} & \sum_{i:x_i \neq y_i} \langle w^{(x,i)} | w^{(y,i)} \rangle = \mathbf{1}_{f(x)\neq f(y)} \quad \forall x, y \end{array}$$

The goal of this problem is to show that the above program is equivalent to the next one (which may come in handy in Problems 3 and 4):

$$\begin{array}{rl} \mathrm{Adv}^\star(f) = \min_{\{w^{(x,i)}\}} & \sqrt{C_0 C_1} \\ \text{s.t.} & C_0 = \max_{x:f(x)=0} \sum_i \|w^{(x,i)}\|^2 \\ & C_1 = \max_{x:f(x)=1} \sum_i \|w^{(x,i)}\|^2 \\ & \sum_{i:x_i \neq y_i} \langle w^{(x,i)} | w^{(y,i)} \rangle = 1 \qquad \forall x, y,\ f(x) \neq f(y) \end{array}$$

Question 1. Show that $\mathrm{Adv}^\star(f) \leq \mathrm{Adv}(f)$.

Question 2. Let $\{w^{(x,i)}\}$ be a feasible solution to the first program. Define $C_0 = \max_{x:f(x)=0} \sum_i \|w^{(x,i)}\|^2$ and $C_1 = \max_{x:f(x)=1} \sum_i \|w^{(x,i)}\|^2$ (the solution has value $\max\{C_0, C_1\}$). Show that there exists another feasible solution to the same program of value $\sqrt{C_0 C_1}$.

Question 3. Let $\{w^{(x,i)}\}$ be a feasible solution to the second program. Define $|v^{(x,i)}\rangle = |w^{(x,i)}\rangle |x_i \oplus f(x)\rangle$. Show that it satisfies $\sum_{i:x_i \neq y_i} \langle v^{(x,i)} | v^{(y,i)} \rangle = \mathbf{1}_{f(x)\neq f(y)}$ for all x, y.

Question 4. Conclude that $\mathrm{Adv}^\star(f) = \mathrm{Adv}(f)$.

Problem 2 (Connectivity)

Consider the function $\textsc{Connectivity} : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ whose quantum query complexity was shown to be $\Omega(n^{3/2})$ in the last problem session. The goal of this problem is to give a matching upper bound by constructing a feasible solution to the dual adversary program.

We start with the easier-to-analyze st-$\textsc{Connectivity}$ problem where the goal is to decide if there exists a path between two given vertices s and t. Without loss of generality, we fix $s = 1$ and $t \in \{2, \ldots, n\}$.

Let $V_x(v) \subseteq \{1,\dots,n\}$ denote the set of vertices that belong to the same connected component as vertex $v \in \{1,\dots,n\}$ in graph $x \in \{0,1\}^{\binom{n}{2}}$. The st-Connectivity problem asks to decide if $t \in V_x(s)$. Define $\mathcal{G}_0$ as the set of graphs that are not st-connected, and $\mathcal{G}_1$ as the graphs that are st-connected. For each edge query $\{i,j\} \in \binom{n}{2}$, a vector $|w^{(x,\{i,j\})}\rangle \in \mathrm{span}\{|k\rangle : 1 \le k \le n\}$ for the dual adversary program is chosen as follows.
If $x \in \mathcal{G}_0$ then:

$$|w^{(x,\{i,j\})}\rangle = \begin{cases} |i\rangle - |j\rangle & \text{if } i \in V_x(1) \text{ and } j \notin V_x(1) \\ 0 & \text{otherwise} \end{cases}$$

If $x \in \mathcal{G}_1$ then fix any shortest length st-path in x and define:

$$|w^{(x,\{i,j\})}\rangle = \begin{cases} 0 & \text{if } \{i,j\} \text{ is not an edge on that path} \\ |i\rangle & \text{if } \{i,j\} \text{ is an edge and its orientation on the } st\text{-path is } i \to j \\ |j\rangle & \text{if } \{i,j\} \text{ is an edge and its orientation on the } st\text{-path is } j \to i \end{cases}$$

Question 1. Show that for all $x \in \mathcal{G}_0$, $y \in \mathcal{G}_1$ we have $\sum_{\{i,j\}: x_{\{i,j\}} \neq y_{\{i,j\}}} \langle w^{(x,\{i,j\})} | w^{(y,\{i,j\})} \rangle = 1$.

Question 2. Modify the above construction to show that $Q(\text{Connectivity}) = O(n^{3/2})$. As a hint, observe that a graph is connected if and only if it is st-connected for $s = 1$ and *all* $t \in \{2,\dots,n\}$.

> The above algorithm uses only $O(\log n)$ qubits of memory[1]. This is in contrast to an older quantum algorithm[2] that required $O(n \log n)$ space.

Problem 3 (Composition)

Given two functions $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$, define their composition $f \bullet g : \{0,1\}^{n \times m} \to \{0,1\}$ as $f \bullet g(X) = f(g(X_{1,1},\dots,X_{1,m}),\dots,g(X_{n,1},\dots,X_{n,m}))$. A striking property of the adversary method is that $\mathrm{Adv}(f \bullet g) = \mathrm{Adv}(f)\mathrm{Adv}(g)$. This problem studies some parts of the proof of this result.

Question 1. Show that $\mathrm{Adv}(f \bullet g) \le \mathrm{Adv}(f)\mathrm{Adv}(g)$.

Hint: Take any dual adversary solutions $\{w_f^{(x,i)}\}$ and $\{w_g^{(x,j)}\}$ for f and g respectively, and consider $|w_{f\bullet g}^{(X,(i,j))}\rangle = |w_f^{(((g(X_1),\dots,g(X_n)),i)}\rangle |w_g^{(X_i,j)}\rangle$.

Question 2. Suppose that f is the OR function. Show that $\mathrm{Adv}(f \bullet g) \ge \sqrt{n} \cdot \mathrm{Adv}(g)$.

Hint: Start from a primal adversary solution Γ for g and construct a primal adversary solution for $f \bullet g$.

[1] "Span Programs and Quantum Algorithms for st-Connectivity and Claw Detection". A. Belovs and B. Reichardt. *Proc. of ESA*, 2012. "Span-Program-Based Quantum Algorithms for Graph Bipartiteness and Connectivity". A. Āriņš. *Proc. of MEMICS*, 2015.

[2] "Quantum Query Complexity of Some Graph Problems". C. Dürr, M. Heiligman, P. Høyer, M. Mhalla. *SICOMP*, 2006.