Classical and Quantum Algorithms for Variants of Subset-Sum via Dynamic Programming

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The Problems

Subset-Sum
Input: Multiset $\{a_1,\ldots,a_n\}$ and target m
Output: $S\subseteq\{1,\ldots,n\}$ such that $\sum_{i\in S}a_i=m$

Shifted-Sums
Input: Multiset $\{a_1,\ldots,a_n\}$ and shift s
Output: $S_1\neq S_2\subseteq\{1,\ldots,n\}$ s.t. $\sum_{i\in S_1}a_i+s=\sum_{i\in S_2}a_i$

Special cases of Shifted-Sums:

Equal-Sums
Input: Set $\{a_1,\ldots,a_n\}$
Output: $S_1\neq S_2\subseteq\{1,\ldots,n\}$ s.t. $\sum_{i\in S_1}a_i=\sum_{i\in S_2}a_i$

Pigeonhole Equal-Sums
Input: Set $\{a_1,\ldots,a_n\}$ s.t. $\sum_{i=1}^n a_i<2^n-1$
Output: $S_1\neq S_2\subseteq\{1,\ldots,n\}$ s.t. $\sum_{i\in S_1}a_i=\sum_{i\in S_2}a_i$

Known Results

Classical results:

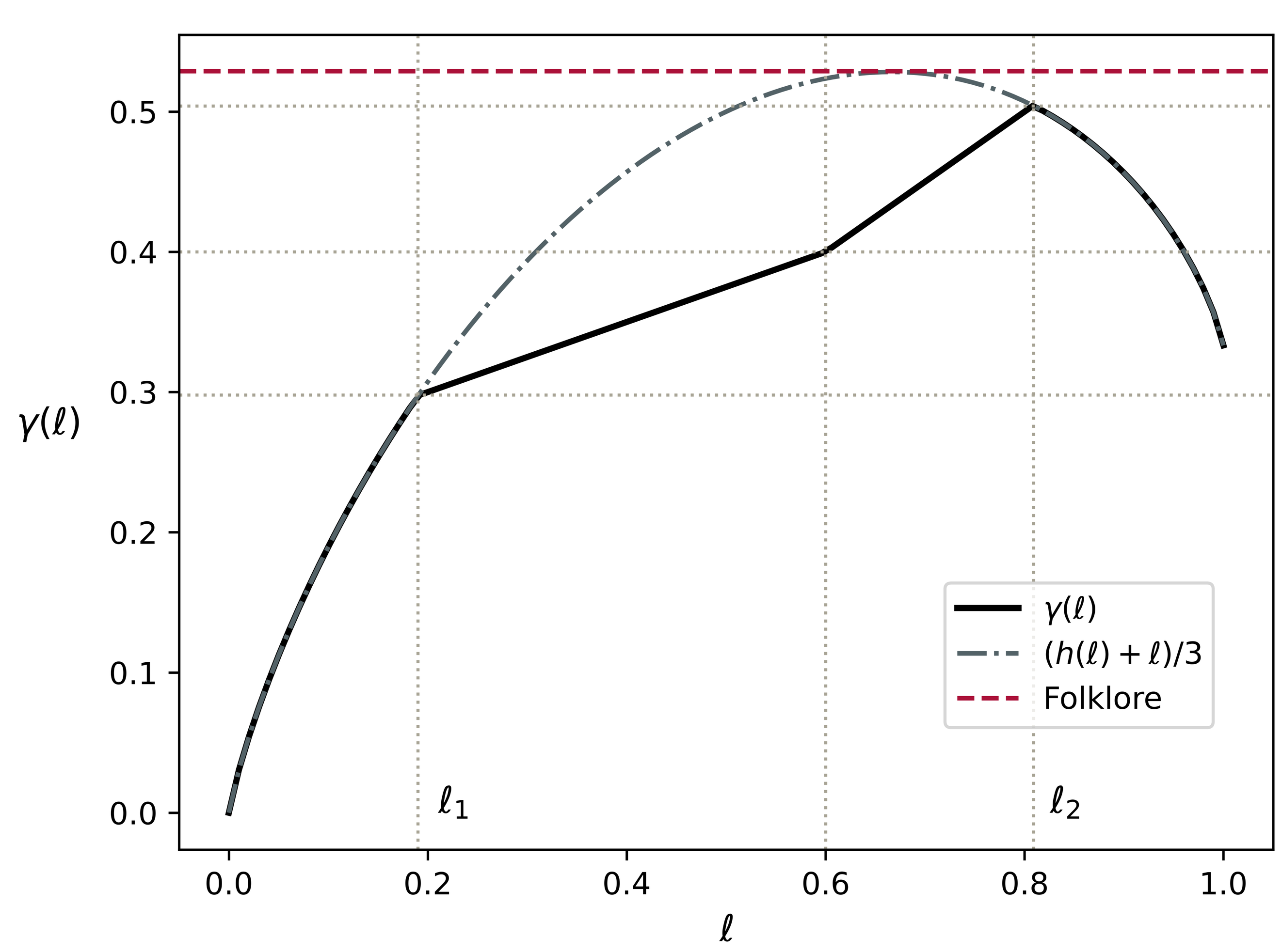
- Subset-Sum $\in\widetilde{O}(2^{n/2})$ [HS'74] Meet in the Middle
- Subset-Sum $\in\widetilde{O}(n+m)$ [Bri'17]
- SETH $\Rightarrow\forall\epsilon>0,\exists\delta>0,$ Subset-Sum $\notin O(m^{1-\epsilon}2^{n\delta})$ [ABHS'19]
- Equal-Sums $\in\widetilde{O}(3^{n/2})\leq O(2^{0.793n})$ [Woe'08] MiM
- Equal-Sums $\in O(2^{0.773n})$ [MNPW'19]

Quantum results:

- Subset-Sum $\in\widetilde{O}(2^{n/3})$ [BJLM'13] MiM
- Equal-Sums $\in\widetilde{O}(3^{n/3})\leq O(2^{0.529n})$ MiM

Our Results

	Classical	Quantum
Subset-Sum (not MiM)	$\widetilde{O}(2^{n/2})$	$\widetilde{O}(2^{n/3})$
Shifted-Sums	$O(2^{0.773n})$	$O(2^{0.504n})$
Pigeonhole Equal-Sums	$O(2^{n/2})$	$O(2^{0.4n})$



Dynamic Programming Datastructure

$$T_{p,k}=\left\{S\subseteq\{1,\ldots,n\}:\sum_{i\in S}a_i\equiv k\pmod p\right\}$$

1 - Compute the size $|T_{p,k}|$ by dynamic programming:

Let $t_p[i,j]=\left|\left\{S\subseteq\{1,\ldots,i\}:\sum_{i\in S}a_i\equiv j\pmod p\right\}\right|$.
The table $t_p[i,j]$ can be constructed in time $O(np)$:
$t_p[i,j]=t_p[i-1,j]+t_p[i-1,(j-a_i)\bmod p]$.

2 - Given random access to the table t_p, we construct an oracle that gives access to the elements of $T_{p,k}$ in time $O(n)$ (for a certain total order $\prec$ over $T_{p,k}$).

Main Ideas for Quantum Shifted-Sums

Maximum solution size, $\ell\in(0,1)$:

$$\ell n=\max\{|S_1|+|S_2|:S_1,S_2\text{ disjoint solutions}\}$$

Suppose that ℓ is known. Then:

– If $\ell<0.190$ or $\ell>0.809$, use quantum MiM.

– If $0.190\leq\ell\leq0.809$, use representation technique.

Example when $\ell=3/5$:

1 - Choose a random prime $p\in\{2^{2n/5},\ldots,2^{2n/5+1}\}$.

2 - Choose a random index $k\in[0,p-1]$.

3 - Search for a solution in $T_{p,k}$ using the dynamic programming oracle and Ambainis' Collision Finding.

$\to$ Expected bin size $|T_{p,k}|\approx2^{3n/5}$,

– Sufficiently large to contain a solution.

– Sufficiently small to keep the cost of search low: $\widetilde{O}(2^{2n/5})$.

Additional ideas when $\ell<3/5$:

– Choose $p\in\{2^{(1+\ell)n/4},\ldots,2^{(1+\ell)n/4+1}\}$.

– Multiple solutions in a random $T_{p,k}$.

– New faster Collision Finding for multiple collisions.

Additional ideas when $\ell>3/5$:

– Choose $p\in\{2^{2n/5},\ldots,2^{2n/5+1}\}$.

– Solution in a random $T_{p,k}$ with exp. small probability.

– Use Variable-Time Amplitude Amplification to boost the success probability of sampling a good $T_{p,k}$.