

# Classical and quantum dynamic programming for Subset-Sum and variants

Jonathan Allcock<sup>\*</sup>   Yassine Hamoudi<sup>†</sup>   Antoine Joux<sup>‡</sup>   Felix Klingelhöfer<sup>§</sup>  
Miklos Santha<sup>¶</sup>

## Abstract

SUBSET-SUM is an NP-complete problem where one must decide if a multiset of  $n$  integers contains a subset whose elements sum to a target value  $m$ . The best known classical and quantum algorithms run in time  $\tilde{O}(2^{n/2})$  and  $\tilde{O}(2^{n/3})$ , respectively, based on the well-known meet-in-the-middle technique. Here we introduce a novel dynamic programming data structure with applications to SUBSET-SUM and a number of variants, including EQUAL-SUMS (where one seeks two disjoint subsets with the same sum), 2-SUBSET-SUM (a relaxed version of SUBSET-SUM where each item in the input set can be used twice in the summation), and SHIFTED-SUMS, a generalization of both of these variants, where one seeks two disjoint subsets whose sums differ by some specified value.

Given any modulus  $p$ , our data structure can be constructed in time  $O(np)$ , after which queries can be made in time  $O(n)$  to the lists of subsets summing to a same value modulo  $p$ . We use this data structure to give new  $\tilde{O}(2^{n/2})$  and  $\tilde{O}(2^{n/3})$  classical and quantum algorithms for SUBSET-SUM, not based on the meet-in-the-middle method. We then use the data structure in combination with variable time amplitude amplification and a quantum pair finding algorithm, extending quantum element distinctness and claw finding algorithms to the multiple solutions case, to give an  $O(2^{0.504n})$  quantum algorithm for SHIFTED-SUMS, an improvement on the best known  $O(2^{0.773n})$  classical running time. We also study PIGEONHOLE EQUAL-SUMS and PIGEONHOLE MODULAR EQUAL-SUMS, variants of EQUAL-SUMS where the existence of a solution is guaranteed by the pigeonhole principle. For the former problem we give classical and quantum algorithms with running time  $\tilde{O}(2^{n/2})$  and  $\tilde{O}(2^{2n/5})$ , respectively. For the more general modular problem we give a classical algorithm which also runs in time  $\tilde{O}(2^{n/2})$ .

## 1 Introduction

SUBSET-SUM is the problem of deciding whether a given multiset of  $n$  integers has a subset whose elements sum to a target integer  $m$ .

**Problem 1** (SUBSET-SUM). Given a multiset  $\{a_1, \dots, a_n\}$  of positive integers and a target integer  $m$ , find a subset  $S \subseteq \{1, \dots, n\}$  such that  $\sum_{i \in S} a_i = m$ .

It is often useful to express SUBSET-SUM using inner product notation. We set  $\bar{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ , where the elements are taken in arbitrary order, and the task is to find  $\bar{e} \in \{0, 1\}^n$  such that

$$\bar{a} \cdot \bar{e} = \sum_{i=1}^n a_i e_i = m.$$

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<sup>\*</sup>Tencent Quantum Laboratory, Hong Kong.

<sup>†</sup>Simons Institute for the Theory of Computing, University of California, Berkeley.

<sup>‡</sup>CISPA Helmholtz Center for Information Security.

<sup>§</sup>G-SCOP, Université Grenoble Alpes.

<sup>¶</sup>Centre for Quantum Technologies, National University of Singapore, Singapore 117543.

The problem is famously NP-complete, and featured on Karp's list of 21 NP-complete problems [Kar72] in 1972 (under the name of knapsack). It can be solved classically in time  $\tilde{O}(2^{n/2})$  via the *meet-in-the-middle* technique [HS74]. Whether the problem can be solved in time  $\tilde{O}(2^{(1/2-\delta)n})$ , for some  $\delta > 0$ , is an important open problem, but we know that the Exponential Time Hypothesis implies that SUBSET-SUM cannot be computed in time  $m^{o(1)}2^{o(n)}$  [BLT15; JLL16]. SUBSET-SUM can also be solved in pseudopolynomial time, for instance in  $O(nm)$  by a textbook dynamic programming approach, which was improved to a highly elegant  $\tilde{O}(n+m)$  randomized algorithm by Bringmann [Bri17]. However, assuming the Strong Exponential Time Hypothesis (SETH), it can be shown that for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that SUBSET-SUM cannot be computed in time  $O(m^{1-\epsilon}2^{n\delta})$  [ABHS19]. On a quantum computer, the meet-in-the-middle approach can be combined with Grover search to solve SUBSET-SUM in time  $\tilde{O}(2^{n/3})$ .

A modular version of SUBSET-SUM can be similarly defined:

**Problem 2** (MODULAR SUBSET-SUM). Given a multiset  $\{a_1, \dots, a_n\}$  of positive integers, a target integer  $m$  and a modulus  $q$ , find a subset  $S \subseteq \{1, \dots, n\}$  such that  $\sum_{i \in S} a_i \equiv m \pmod{q}$ .

The  $\tilde{O}(2^{n/2})$  classical and  $\tilde{O}(2^{n/3})$  quantum meet-in-the-middle algorithms, as well as the classical  $O(nm)$  dynamic programming algorithm can be used to solve MODULAR SUBSET-SUM with the same running times, by replacing regular addition with modular addition. While the  $\tilde{O}(n+m)$  algorithm of Bringmann does not immediately give rise to an  $\tilde{O}(q)$  algorithm for MODULAR SUBSET-SUM, several recent algorithms have achieved this complexity [ABJ+19; ABB+21; CI21]. Also, SETH implies that for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that MODULAR SUBSET-SUM cannot be computed in time  $O(q^{1-\epsilon}2^{n\delta})$  because an instance of SUBSET-SUM where each  $a_i < m$  is a special case of MODULAR SUBSET-SUM when we choose  $q = nm$ .

## 1.1 Some variants of SUBSET-SUM

SUBSET-SUM has several close relatives we will be concerned with in this paper. First among these is EQUAL-SUMS, introduced by Woeginger and Yu [WY92], where one must decide if a set of  $n$  positive integers contains two disjoint subsets whose elements sum to the same value:

**Problem 3** (EQUAL-SUMS). Given a set  $\{a_1, \dots, a_n\}$  of positive integers, find two distinct subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  such that  $\sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i$ . In inner product notation we are looking for a nonzero vector  $\bar{e} \in \{-1, 0, 1\}^n$  such that  $\bar{a} \cdot \bar{e} = 0$ .

The folklore classical algorithm [Woe08] for EQUAL-SUMS runs in time  $\tilde{O}(3^{n/2}) \leq O(2^{0.793n})$ , and is also based on a meet-in-the-middle approach. More precisely, in the classical case we arbitrarily partition the input into two sets of the same size, giving rise to vectors  $\bar{a}_1, \bar{a}_2 \in \mathbb{N}^{n/2}$ . Then we compute and sort the possible  $3^{n/2}$  values  $\bar{a}_1 \cdot \bar{e}$ , for  $\bar{e} \in \{-1, 0, 1\}^{n/2}$ . Finally we compute the possible  $3^{n/2}$  values of the form  $\bar{a}_2 \cdot \bar{e}$  and, for each value, check via binary search if it has a collision (i.e. an item of the same value) in the first set of values. In the quantum case we use a different balancing, dividing the input into a set of size  $n/3$  and a set of size  $2n/3$ , and then use Grover search over the larger set to find a collision. The folklore quantum algorithm has running time  $\tilde{O}(3^{n/3}) \leq O(2^{0.529n})$ .

The classical running time of EQUAL-SUMS was reduced in a recent work by Mucha et al. [MNPW19] to  $O(2^{0.773n})$ , and it is an open problem whether this can be further improved.

The modular version of EQUAL-SUMS is defined as:

**Problem 4** (MODULAR EQUAL-SUMS). Given a set  $\{a_1, \dots, a_n\}$  of positive integers and a modulus  $q$ , find two distinct subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  such that  $\sum_{i \in S_1} a_i \equiv \sum_{i \in S_2} a_i \pmod{q}$ .

Similarly to SUBSET-SUM, the  $O(2^{0.793n})$  time meet-in-the-middle algorithm for EQUAL-SUMS gives rise to an algorithm of the same time for MODULAR EQUAL-SUMS. Moreover, we can suppose that  $q \geq 2^n$ , because otherwise we can just consider  $a_1, \dots, a_k$  from the input, where  $k$

satisfies  $2^{k-1} \leq q < 2^k$ . By the pigeonhole principle such an instance has a solution which we will show (see Theorem 34) can be found in time  $\tilde{O}(2^{k/2})$ . Thus, MODULAR EQUAL-SUMS can always be solved in time  $O(q^{0.793})$  classically.

It is somewhat intriguing that, when expressed as a function of  $n$ , faster algorithms are known for both SUBSET-SUM and MODULAR SUBSET-SUM than for EQUAL-SUMS and MODULAR EQUAL-SUMS, respectively, whereas expressed as function of  $q$  (or as function of  $\sum_{i=1}^n a_i$  in the non modular cases), the situation is the opposite. This holds both classically and quantumly.

A natural generalization of SUBSET-SUM is to allow each item in the input set to be used more than once in the summation, where the maximum number of times each item can be used is specified, as part of the input to the problem, by a given bound. This is the analog of bounded knapsack, a well studied problem in the literature (see for example [KPP04]). In particular, we will study the case when there is a uniform bound of 2, i.e. when every item can be used at most twice.

**Problem 5** (2-SUBSET-SUM). Given a multiset  $\{a_1, \dots, a_n\}$  of positive integers and a target integer  $0 < m < 2 \sum_{i=1}^n a_i$ , find a vector  $\bar{e} \in \{0, 1, 2\}^n$ , such that  $\bar{a} \cdot \bar{e} = m$ .

It turns out that there is a natural variant of SUBSET-SUM that generalizes both EQUAL-SUMS and 2-SUBSET-SUM. We call this variant SHIFTED-SUMS, whose investigation is the main subject of this paper.

**Problem 6** (SHIFTED-SUMS). Given a multiset  $\{a_1, \dots, a_n\}$  of positive integers and an integer  $0 \leq s < \sum_{i=1}^n a_i$ , find two subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  such that  $S_1 \neq S_2$  and  $\sum_{i \in S_1} a_i = s + \sum_{i \in S_2} a_i$ .

The condition  $S_1 \neq S_2$  is obviously satisfied by any solution  $S_1, S_2$  when  $s \neq 0$ , but it is necessary to specify it in the case  $s = 0$  to exclude the trivial solutions  $S_1 = S_2$ .

The problem EQUAL-SUMS is a special case of SHIFTED-SUMS, and it is easy to show (see Proposition 7) that 2-SUBSET-SUM can be reduced to SHIFTED-SUMS without increasing the size of the input. This means that any algorithm for SHIFTED-SUMS automatically gives rise to an algorithm of the same complexity for EQUAL-SUMS and 2-SUBSET-SUM, and therefore we focus on constructing a quantum algorithm for SHIFTED-SUMS.

We also consider the modular version of SHIFTED-SUMS:

**Problem 7** (MODULAR SHIFTED-SUMS). Given a multiset  $\{a_1, \dots, a_n\}$  of positive integers, an integer  $0 \leq s < \sum_{i=1}^n a_i$  and a modulus  $q$ , find two subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  such that  $S_1 \neq S_2$  and  $\sum_{i \in S_1} a_i \equiv s + \sum_{i \in S_2} a_i \pmod{q}$ .

We additionally study the following variant of EQUAL-SUMS where, by the pigeonhole principle, a solution is guaranteed to exist. This search problem is total in the sense that its decision version is trivial because the answer is always ‘yes’. Such problems belong to the complexity class TFNP [MP91] consisting of NP-search problems with total relations. Problems in TFNP cannot be NP-hard unless  $\text{NP} = \text{co-NP}$ . More precisely, PIGEONHOLE EQUAL-SUMS belongs to the Polynomial Pigeonhole Principle complexity class PPP, defined by Papdimitriou [Pap90], where the totality of the problem is syntactically guaranteed by the pigeonhole principle.

**Problem 8** (PIGEONHOLE EQUAL-SUMS). Given a set  $\{a_1, \dots, a_n\}$  of positive integers such that  $\sum_{i=1}^n a_i < 2^n - 1$ , find two distinct subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  such that  $\sum_{i \in S_1} a_i = \sum_{i \in S_2} a_i$ .

There are  $2^n$  subsets  $S \subseteq \{1, \dots, n\}$ . Since they all verify  $0 \leq \sum_{i \in S} a_i \leq 2^n - 2$  there must exist two distinct subsets  $S_1, S_2$  that sum to the same value, according to the pigeonhole principle.

We can also define a modular version of PIGEONHOLE EQUAL-SUMS which similarly belongs to the class PPP.

**Problem 9** (PIGEONHOLE MODULAR EQUAL-SUMS). Given a set  $\{a_1, \dots, a_n\}$  of positive integers and a modulus  $q$  such that  $q \leq 2^n - 1$ , find two distinct subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  such that  $\sum_{i \in S_1} a_i \equiv \sum_{i \in S_2} a_i \pmod{q}$ .

Observe that PIGEONHOLE EQUAL-SUMS is a special case of PIGEONHOLE MODULAR EQUAL-SUMS when  $q = 2^n - 1$ .

## 1.2 Our contributions and techniques

Our main contribution is the design of new classical and quantum algorithms for three problems related to SUBSET-SUM. The results are succinctly stated below and are summarized in Table 1. The algorithms for SUBSET-SUM achieves the same complexity bound as the currently known best algorithms based on the meet-in-the-middle method. Our algorithm for SHIFTED-SUMS (and for its special cases of EQUAL-SUMS and 2-SUBSET-SUM) improves on the currently best known  $O(2^{0.529n})$  quantum algorithm for these problems, which is also based on meet-in-the-middle method. Our quantum algorithm for PIGEONHOLE EQUAL-SUMS further improves, in this special case, on our algorithm for general EQUAL-SUMS. We also initiate the study of the PIGEONHOLE EQUAL-SUMS problem (and its modular variant) in the classical setting, where we obtain a better complexity than what was known before for the general EQUAL-SUMS problem.

**Theorems 22, 23** (Restated). *There are representation technique based classical and quantum algorithms for SUBSET-SUM that run in time  $\tilde{O}(2^{n/2})$  and  $\tilde{O}(2^{n/3})$ , respectively.*

**Theorem 25, Appendix A** (Restated). *There are classical and quantum algorithms for SHIFTED-SUMS that run in time  $O(2^{0.773n})$  and  $O(2^{0.504n})$ , respectively.*

**Theorem 33** (Restated). *There is a quantum algorithm for PIGEONHOLE EQUAL-SUMS that runs in time  $\tilde{O}(2^{2n/5})$ .*

**Theorems 32, 34** (Restated). *There are classical deterministic algorithms for PIGEONHOLE EQUAL-SUMS and PIGEONHOLE MODULAR EQUAL-SUMS that run in time  $\tilde{O}(2^{n/2})$ .*

	Classical	Quantum
SUBSET-SUM	$2^{n/2}$ [HS74], [Thm. 23]	$2^{n/3}$ [BJLM13], [Thm. 22]
SHIFTED-SUMS	$2^{0.773n}$ [MNPW19], [App. A]	$2^{0.504n}$ [Thms. 25, 43]
PIGEONHOLE EQUAL-SUMS	$2^{n/2}$ [Thm. 32]	$2^{2n/5}$ [Thm. 33]
PIGEONHOLE MODULAR EQUAL-SUMS	$2^{n/2}$ [Thm. 34]	-

Table 1: Best known classical and quantum running times for variants of SUBSET-SUM. Our results are indicated by reference to the corresponding theorems in this paper. The  $\tilde{O}(\cdot)$  notation is implied for all running times. For SUBSET-SUM our dynamic programming based results have the same time complexity as the best previous algorithms which were based on meet-in-the-middle. The results of [MNPW19] are for EQUAL-SUMS, a special case of SHIFTED-SUMS. In Appendix A we give an extension of their algorithm to SHIFTED-SUMS.

At a high level, all of our algorithms use a representation technique approach. While this technique was originally designed to solve SUBSET-SUM when the instances are drawn from some specific distribution [HJ10], here we follow the path of Mucha et al. [MNPW19] and use it in a worst case analysis. Among our three main algorithms, the quantization of the representation technique approach for SHIFTED-SUMS is the most challenging. We will therefore explain first, via this algorithm, the difficulties we had to address and the methods we used to tackle them.

**SHIFTED-SUMS.** The representation technique approach for SHIFTED-SUMS first consists of selecting a random prime  $p \in \{2^{bn}, \dots, 2^{bn+1}\}$ , where  $b \in (0, 1)$  is some appropriate constant, and a random integer  $k \in \{0, \dots, p-1\}$ . Then we consider the random bin  $T_{p,k}$ , defined as

$$T_{p,k} = \left\{ S \subseteq \{1, \dots, n\} : \sum_{i \in S} a_i \equiv k \pmod{p} \right\},$$

and we search that bin and  $T_{p,(k-s) \bmod p}$  for a solution. The choice of the bin size (which, on average, is roughly  $2^{(1-b)n}$ ) should balance two opposing requirements: the bin(s) should be sufficiently large to contain a solution and also sufficiently small to keep the cost of search low.

In order to satisfy the above two requirements, our algorithm uses the concept of a *maximum solution*. By definition, this is the maximum of  $|S_1| + |S_2|$ , when  $S_1, S_2$  are disjoint and form a solution. Let this maximum solution size be  $\ell n$ , for some  $\ell \in (0, 1)$ . The algorithm itself consists of two different procedures, designed to handle different maximum solution sizes. For  $\ell$  close to 0 or close to 1, the quantization of the meet-in-the-middle method adapted to solutions of size  $\ell n$  is used because it performs better. In this case, the quantization does not present any particular difficulties: it is a straightforward application of Grover search with the appropriate balancing. We therefore focus the discussion on the representation technique procedure used for values of  $\ell$  away from 0 or 1. When  $S_1, S_2$  form a maximum solution of size  $\ell n$  then, for every  $X \subseteq \overline{S_1 \cup S_2}$ , the pairs  $S_1 \cup X, S_2 \cup X$  also form a solution, and all these solutions have different values (see Lemma 27). This makes it possible to bound from below, not only the number of solutions, but also the number of *solution values* by  $2^{(1-\ell)n}$ , which makes the use of the representation technique successful.

The most immediate way to quantize the procedure is to replace classical collision finding by the quantum element distinctness algorithm of Ambainis [Amb07]. However, in a straightforward application of this algorithm we face a difficulty. For the sake of concreteness, we explain this when  $\ell = 3/5$ . In that case, by the above, the total number of solutions with different values is at least  $2^{2n/5}$ . This is handy for applying quantum element distinctness: we can select a random prime  $p \in \{2^{2n/5}, \dots, 2^{2n/5+1}\}$  and expect to have a solution in the random bin  $T_{p,k}$  with reasonable probability. The expected size  $|T_{p,k}|$  of the bin is about  $2^{3n/5}$ , and therefore the running time of Ambainis' algorithm should be of the order of  $|T_{p,k}|^{2/3}$  which is also about  $2^{2n/5}$ . However, the quantum element distinctness algorithm requires us to perform queries to  $T_{p,k}$ . That is, for some indexing  $T_{p,k} = \{S_1, \dots, S_{|T_{p,k}|}\}$  of the elements of  $T_{p,k}$ , we need to implement the oracle

$$O_{T_{p,k}}|I\rangle|0\rangle = |I\rangle|S_I\rangle,$$

where  $1 \leq I \leq |T_{p,k}|$ . In other words, given  $1 \leq I \leq |T_{p,k}|$ , we have to be able to find the  $I$ th element in  $T_{p,k}$  (for some ordering of that set). In the usual description of the element distinctness algorithm there is a simple way to do that (for example, the set over which the algorithm is run is just a set of consecutive integers). However, finding a simple bijection among the first  $|T_{p,k}|$  integers and  $T_{p,k}$  is not a trivial task. Perhaps the most interesting feature of our quantum algorithm is the way it implements this bijection and, therefore, the above oracle call, at a low cost. Unlike in the classical case, explicitly enumerating  $T_{p,k}$  is not an option because this would take too long, requiring about  $2^{3n/5}$  time steps. Instead, we use dynamic programming to compute the table

$$t_p[i, j] = \left| \left\{ S \subseteq \{1, \dots, i\} : \sum_{s \in S} a_s \equiv j \pmod{p} \right\} \right|.$$

Computing the cardinality of the bins is much cheaper than computing their contents, and can be done in time  $O(np) = \tilde{O}(2^{2n/5})$ . The crucial point is that, once the table  $t_p$  is constructed, one can deduce the paths through it that led to  $t_p[n, k] = |T_{p,k}|$ , in order to find each element of  $T_{p,k}$  in time  $O(n)$ . To do this, we define a strict total order  $\prec$  over  $\mathcal{P}([n])$  by setting  $S_1 \prec S_2$  if  $\max\{i : i \in S_1 \Delta S_2\} \in S_2$ . We are then able to prove:

**Theorem 17** (Restated). *Let  $T_{p,k}$  be enumerated as  $T_{p,k} = \{S_1, \dots, S_{|T_{p,k}|}\}$  where  $S_1 \prec \dots \prec S_{|T_{p,k}|}$ . Given any integer  $I \in \{1, \dots, |T_{p,k}|\}$  and random access to the elements of the table  $t_p$ , the set  $S_I$  can be computed in time  $O(n)$ .*

This novel dynamic programming data structure will be used in our algorithms for SUBSET-SUM, SHIFTED-SUMS (and therefore also for EQUAL-SUMS and 2-SUBSET-SUM) and PIGEONHOLE EQUAL-SUMS.

We now describe the additional quantum tools we use for SHIFTED-SUMS. The algorithm randomly chooses a bin size of about  $2^{(1-b)n}$  where  $b$  is defined differently depending on whether  $\ell$  is above or below  $3/5$ . It turns out that the quantum tools required are different in these two regions. When  $\ell < 3/5$ , with high probability a random bin contains multiple solutions from which we can profit. As a technical contribution, we construct a quantum algorithm for finding a pair marked by a binary relation  $\mathcal{R}(x, y)$  that is determined by checking if two underlying values  $f(x)$  and  $g(y)$  are equal or not. Our algorithm essentially generalizes the quantum element distinctness [Amb07] and claw finding [Tan09] algorithms to the case of multiple marked pairs. Using an appropriate variant of the birthday paradox (see Lemma 11) we prove:

**Theorem 12** (Restated). *Consider two sets of  $N \leq M$  elements, respectively, and an evaluation function on each set. Suppose that there are  $K$  disjoint pairs in the product of the two sets such that in each pair the elements evaluate to the same value. There is a quantum algorithm that finds such a pair in time*

$$\begin{cases} \tilde{O}((NM/K)^{1/3}), & \text{if } N \leq M \leq KN^2, \\ \tilde{O}((M/K)^{1/2}), & \text{if } M \geq KN^2. \end{cases}$$

The best complexity is then obtained by choosing the bin size as a function of  $\ell$  which balances the cost of the construction of the dynamic programming table and the quantum pair finding.

When  $\ell > 3/5$ , choosing a bin size  $2^{(1-b)n}$ , for  $b \leq 1 - \ell$ , guarantees that a random bin contains at least one solution with high probability. However, a better running time at first seems to be achievable by the following argument: Choose  $b > 1 - \ell$ , for which there is exponentially small probability that a random bin contains a solution, and use amplitude amplification to boost the success probability. Balancing again the dynamic programming and quantum pair finding costs would then give an optimal bin size of  $2^{3n/5}$ , independent of  $\ell$ . However, this argument contains a fallacy. Using standard amplitude amplification requires that the random bins  $T_{p,k}$  simultaneously satisfy two conditions: beside containing a solution, they should also have sizes close to the expected size of about  $2^{(1-b)n}$ . But there is no guarantee that these two events coincide, and a priori it could be that the exponentially small fraction of  $T_{p,k}$  containing a solution also happen to have sizes that far exceed the expectation. Fortunately, by carefully bounding the expectation of the product of bin sizes, we can use the variable time amplitude amplification algorithm of Ambainis [Amb12], and achieve the same running time as given by the above argument. We believe that this is a nice and natural application of this method.

The running time of our algorithm for SHIFTED-SUMS, as a function of  $\ell$ , is shown in Fig. 1.

**PIGEONHOLE EQUAL-SUMS.** The PIGEONHOLE EQUAL-SUMS problem can be of course solved by any (classical or quantum) algorithm which solves the general EQUAL-SUMS (or SHIFTED-SUMS) problem. However, one can make use of the explicit promise of  $a_1 + \dots + a_n < 2^n - 1$  to design faster algorithms than provided for by the general case when  $\ell > 3/5$ . Indeed, by the pigeonhole principle (see Lemma 31), for *any* value of  $p$ , if a bin  $T_{p,k}$  has size larger than  $2^n/p$  then it must contain a solution. Moreover, there must exist at least one such oversized bin. The array  $t_p$  can now be constructed both for *locating the index  $k$*  of one oversized bin and searching for a solution in it. We thus obtain a classical algorithm running in time  $\tilde{O}(p + 2^n/p)$ , and a quantum algorithm running in time  $\tilde{O}(p + (2^n/p)^{2/3})$ . These two quantities are minimized by *deterministically* choosing  $p = 2^{n/2}$  and  $p = 2^{2n/5}$  respectively (see Section 6.1).



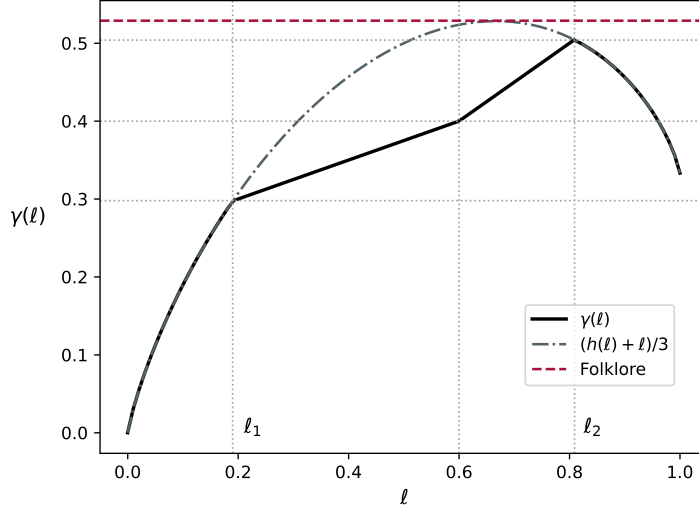


Figure 1: Running time exponent  $\gamma(\ell)$  of the quantum SHIFTED-SUMS algorithm, as a function of the maximum solution ratio  $\ell$  (see Theorem 25). The functional form of  $\gamma$  depends on the value of  $\ell$  (see Theorem 25), and  $\gamma(\ell)$  has a maximum value of 0.504 which occurs at  $\ell = \ell_2 \approx 0.809$ . For reference, the curve  $(h(\ell) + \ell)/3$  corresponding to Theorem 30 is plotted for all values of  $\ell$ , as is the value of 0.529 corresponding to the exponent of the folklore (quantized) meet-in-the-middle algorithm, as applied to SHIFTED-SUMS.

**PIGEONHOLE MODULAR EQUAL-SUMS.** In general, we do not know how to extend our algorithms to the modular variants of SUBSET-SUM and SHIFTED-SUMS with modulo  $q$ . A natural approach would be to consider the bins  $T_{p,k} = \{S \subseteq \{1, \dots, n\} : (\sum_{i \in S} a_i \bmod q) \equiv k \pmod{p}\}$ , but it becomes unclear how to compute the corresponding table  $t_p$  by dynamic programming. We describe a solution to this problem for PIGEONHOLE MODULAR EQUAL-SUMS that works in the classical setting only (see Section 6.2). Our approach consists of defining the bins based on the *quotient* in the division by  $p$  instead of the remainder, that is  $T_{p,k} = \{S \subseteq \{1, \dots, n\} : \lfloor (\sum_{i \in S} a_i \bmod q)/p \rfloor = k\}$ . We show that computing the cardinality of  $T_{p,k}$  and enumerating its elements can be done with the help of yet another table for the bins  $T'_{p,k} = \{S \subseteq \{1, \dots, n\} : (\sum_{i \in S} \lfloor a_i/p \rfloor) \equiv k \pmod{\lfloor q/p \rfloor}\}$ . This last table can be constructed with the same dynamic programming technique as before. We do not know how to extend this approach to the more general EQUAL-SUMS or SHIFTED-SUMS problems due to the lack of good statistics on how a random bin  $T_{p,k}$  behaves in this case. We also have no quantum speed-up for PIGEONHOLE MODULAR EQUAL-SUMS due to a bottleneck when going from  $T'_{p,k}$  to  $T_{p,k}$  that we cannot seemingly reduce with quantum techniques.

### 1.3 Related works

The closest previous work to our contribution is the paper of Mucha et al. [MNPW19] solving EQUAL-SUMS classically in time  $O(2^{0.773n})$ . Indeed, our work initiated partly as an investigation into the possibility of quantizing their algorithm. Their algorithm and ours both use the same two basic procedures, based respectively on the meet-in-the-middle method and on the representation technique. Let us point out some of the differences. Unlike our algorithm which is based around the concept of the size of a maximum solution, the classical algorithm is analyzed as function of a *minimum size* solution, defined as  $|S_1| + |S_2|$ , where this sum is minimized over all solutions. The use the classical algorithm makes from a minimum solution  $S_1, S_2$  of size  $\ell n$  is that when  $\ell > 1/2$  the number of *solution values* can be bounded from below by  $2^{(1-\ell)n}$ . The reason for our

change is that this argument does not hold for 2-SUBSET-SUM when  $\ell n$  is the size of a minimum solution, but *is* valid for both EQUAL-SUMS and 2-SUBSET-SUM when it is the size of a maximum solution. Another difference between our work and [MNPW19] is that the classical representation technique based algorithm always samples  $p$  from the same set  $\{2^{(1-\ell)n}, \dots, 2^{(1-\ell)n+1}\}$ , while we randomly choose  $p \in \{2^{bn}, \dots, 2^{bn+1}\}$  where  $b$  is defined differently depending on in which of two distinct regions  $\ell$  lies: if  $\ell \leq 3/5$  then  $b = (1 + \ell)/4$  and if  $\ell > 3/5$  then  $b = 2/5$ . This makes it possible to use different quantum techniques in these two regions.

The representation technique was designed by Howgrave-Graham and Joux [HJ10] to solve random SUBSET-SUM instances under some hypotheses (heuristics) about how such instances behave during the run of the algorithm. The basic idea is to decompose a single solution to the initial problem into many distinct decompositions of a sum of half-solutions. To compensate for this blow-up, an additional linear constraint is added to select approximately one of these decompositions. Quite surprisingly, this reduces the overall complexity under some reasonable looking hypotheses. The method can even be used recursively by using a well-parametrized binary tree of decompositions of depth  $d$ . This decomposes the initial solution into a sum of  $2^d$  partial sums satisfying various linear conditions. Under some rather strong assumptions, which are satisfied for a large fraction of randomly chosen instances, [HJ10] can solve SUBSET-SUM instances in time  $O(2^{0.337n})$ . Since then, several variants of this classical method have been proposed [BCJ11; EM20; BBSS20; CJS21], while others have investigated quantum algorithms based on the representation technique. Bernstein et al. [BJLM13] improved on [HJ10] using quantum walks, and their algorithm (again under some hypotheses) runs in time  $O(2^{0.242n})$ . Further quantum improvements were made in this context by [HM18] and [BBSS20]. However, we would like to emphasize that the algorithms in all these papers work for random inputs generated from some distributions. The paper [MNPW19] gave the first classical algorithm based on the representation technique that works for worst case inputs and with proven bounds. To our knowledge, for worst case inputs with provable guarantees the first quantum algorithm based on the representation technique is given in our work.

Dynamic programming, a fundamental algorithmic technique, is notoriously hard to quantize. Arguably, a key obstacle to doing so is the intrinsically sequential way in which the solution to a large problem is constructed from the solutions to smaller subproblems. In classical fine-grained complexity theory, several dynamic programming algorithms were shown to be optimal under the condition that SETH holds [ABW15; AW14; BI18; Bri14; BK15; KPS17]. Some attempts were also recently made to show the optimality of some quantum algorithms based on quantum analogues of SETH. For instance, an  $\Omega(n^{3/2})$  lower-bound was obtained under such a hypothesis in [BPS21] for the edit distance problem, but there is still a gap to close to  $O(n^2)$ , the best known quantum (and classical) upper bound. In another example [ABI+20], query lower bounds were proven for the connectivity problem on the two dimensional grid.

Certain basic dynamic programming algorithms can be trivially accelerated by Grover search or quantum minimum/maximum finding – see for example some discussion in [Abb19] – but beyond that few other quantum improvements are known. The notable exception to this is the work of Ambainis et al. [ABI+19] who succeeded in giving faster quantum algorithms for several NP-hard problems, including the travelling salesperson problem, for which the best classical algorithms use dynamic programming. Their algorithms precompute solutions for smaller instances via dynamic programming and then use non-trivial Grover search recursively on the rest of the problem. It is worth remarking that the dynamic programming used in these algorithms is essentially taking place on the hypercube representing all subsets of some universe. In this respect, one novelty of our results is that they speed up classical algorithms where the dynamic programming takes place in a constant dimensional grid graph.

The class PPP is arguably less studied than other syntactically definable subclasses of TFNP, such as PLS (Polynomial Local Search) and PPA (Polynomial Parity Argument), and it is not known whether PIGEONHOLE EQUAL-SUMS is complete in PPP. In fact, the first complete



problem for the class was only identified relatively recently [SZZ18]. Our results for PIGEONHOLE EQUAL-SUMS suggest that the problem is indeed simpler to solve than EQUAL-SUMS. In spirit, a similar result was obtained in [BST02] where it was shown that, for an optimization problem closely related to EQUAL-SUMS, better approximation schemes can be obtained for instances with guaranteed solutions.

## 1.4 Structure of the paper

The paper is organized as follows. In Section 2 we define some basic notations, and state (and, in some cases, prove) a number of facts, propositions and algorithmic tools used in subsequent sections. In Section 3 we introduce our dynamic programming data structure and show how it can be used to implement fast queries to the bins  $T_{p,k}$  used in our algorithms. In Section 4 we use this data structure to give classical and quantum algorithms for SUBSET-SUM which achieve the same complexity as the best known algorithms which are based on meet-in-the-middle. In Sections 5 and 6 we give our quantum algorithms for SHIFTED-SUMS and the pigeonhole variants of EQUAL-SUMS, respectively.

## 2 Preliminaries

### 2.1 Notations

Throughout the paper we use the  $\tilde{O}(x)$  and  $\tilde{\Omega}(x)$  notation to hide factors that are polylogarithmic in the argument  $x$ . For integers  $0 \leq m < n$ , we denote by  $[m..n]$  the set  $\{m, m+1, \dots, n\}$ , and by  $[n]$  the set  $[1..n]$ . For sets  $S, S' \subseteq [n]$  we denote by  $\bar{S}$  the set  $[n] \setminus S$ , and by  $S \Delta S'$  the symmetric difference of  $S$  and  $S'$ . Given a multiset  $A = \{a_1, \dots, a_n\}$  and subset  $S \subseteq [n]$  we denote  $\Sigma_A(S) := \sum_{i \in S} a_i$ . When the set  $A$  is clear from the context, we will omit the subscript and simply denote the subset sum by  $\Sigma(S)$ . The power set of  $[n]$  will be denoted by  $\mathcal{P}([n]) := \{S : S \subseteq [n]\}$ . For arbitrary integers  $a$  and  $b$  and a modulus  $p$  we say that  $a$  is congruent to  $b$  modulo  $p$ , and we write  $a \equiv b \pmod{p}$  or  $a \equiv_p b$  if  $a - b$  is divisible by  $p$ . By  $a \bmod p$  we denote the unique integer in  $\{0, \dots, p-1\}$  which is congruent to  $a$  modulo  $p$ . The binary entropy function will be denoted by  $h(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ .

### 2.2 Basic facts and propositions

We will make use of the following well known facts.

**Fact 1** ([Gal68], page 530). *For every constant  $\ell \in (0, 1)$  and for every large enough integer  $n$ , the following bounds hold:*

$$\frac{2^{nh(\ell)}}{\sqrt{8n\ell(1-\ell)}} \leq \binom{n}{\ell n} < \frac{2^{nh(\ell)}}{\sqrt{2\pi n\ell(1-\ell)}}.$$

**Fact 2** ([HW75], page 371). *For every constant  $b > 0$ , and for every large enough integer  $n$ , the number of primes in the interval  $[2^{bn}..2^{bn+1}]$  is at least  $2^{bn}/bn$ .*

**Corollary 3.** *Let  $b > 0$  be a constant,  $p$  be a random prime in the interval  $[2^{bn}..2^{bn+1}]$  and let  $a_1 \neq a_2 \in 2^{O(n)}$ . Then,  $\Pr_p[a_1 \equiv_p a_2] \in O\left(\frac{n}{2^{bn}}\right)$ .*

**Fact 4** (Paley-Zygmund inequality). *Let  $X \geq 0$  be a non identically zero random variable. For all  $0 < \theta < 1$ , we have*

$$\Pr[X \geq \theta \mathbb{E}[X]] \geq (1-\theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

**Fact 5** (Moment function is non-decreasing). *Let  $X$  be a random variable, and let  $0 < r \leq s$ . Then,*

$$(\mathbb{E}[|X|^r])^{1/r} \leq (\mathbb{E}[|X|^s])^{1/s}.$$

**Fact 6** (Enumerating solutions to SUBSET-SUM [BCJ11]). *Let  $A = \{a_1, \dots, a_n\}$  where  $a_i = 2^{O(n)}$  for all  $i$ . Let  $p = 2^{O(n)}$ ,  $0 \leq k \leq p - 1$ , and  $T_{p,k} = \{S \subseteq [n] : \Sigma_A(S) \equiv k \pmod{p}\}$ . Then, for any  $c \leq |T_{p,k}|$ , it is possible to find  $c$  elements of  $T_{p,k}$  in time  $\tilde{O}(2^{n/2} + c)$ .*

In Section 3.1 we will show how dynamic programming can be used to achieve the same task in time  $\tilde{O}(p + c)$  which is advantageous when  $p < 2^{n/2}$  (see Corollary 18).

We say that a polynomial time reduction between two problems is *size preserving* if it preserves the number of input items. The following propositions justify the assumption in our algorithms that the input items are bounded by  $2^{4n}$ .

**Proposition 7.** *There exist size preserving reductions from EQUAL-SUMS to SHIFTED-SUMS and from 2-SUBSET-SUM to SHIFTED-SUMS.*

*Proof.* In fact, the claimed reductions are not only size preserving, but also keep the input items  $\{a_1, \dots, a_n\}$  intact. EQUAL-SUMS is simply the restriction of SHIFTED-SUMS to the case  $s = 0$ . Consider an instance of 2-SUBSET-SUM, and set  $W = \sum_{i=1}^n a_i$ . We can suppose, without loss of generality, that  $W < m < 2W$  because  $\sum_{i=1}^n a_i \cdot e_i = m$  if and only if  $\sum_{i=1}^n a_i \cdot (2 - e_i) = 2W - m$ , and the  $m = W$  case is trivial.

We claim that  $\bar{a} = (a_1, \dots, a_n), m$  is a positive instance of 2-SUBSET-SUM if and only if  $\bar{a}, m - W$  is a positive instance of SHIFTED-SUMS. For a vector  $\bar{e} \in \{0, 1, 2\}^n$ , define  $e_i^1 = 1$  if  $e_i = 2$  and 0 otherwise, and similarly define  $e_i^2 = 1$  if  $e_i = 0$  and 0 otherwise. Then  $\bar{e}^1, \bar{e}^2 \in \{0, 1\}^n$  and  $e_i = e_i^1 - e_i^2 + 1$  implying  $\bar{a} \cdot \bar{e} = m$  if and only if  $\bar{a} \cdot \bar{e}^1 = \bar{a} \cdot \bar{e}^2 + m - W$ . Therefore  $\bar{e}$  is a solution of 2-SUBSET-SUM if and only if  $\bar{a}, m - W$  is a solution of SHIFTED-SUMS.  $\square$

Given a positive instance of 2-SUBSET-SUM, the above reduction produces a positive instance of SHIFTED-SUMS with  $s \neq 0$ . It is easy to check that the reduction also works in the reverse direction. Taken together, this implies that SHIFTED-SUMS with  $s \neq 0$  is just a reformulation of 2-SUBSET-SUM and, as we have already observed, with  $s = 0$  is exactly EQUAL-SUMS.

**Proposition 8.** *There exists probabilistic size preserving reduction from SHIFTED-SUMS to an instance of SHIFTED-SUMS where the input items satisfy  $\sum_{i=1}^n a_i < 2^{4n}$ . A similar statement holds also for SUBSET-SUM.*

*Proof.* We prove the statement for SHIFTED-SUMS, the proof for SUBSET-SUM is analogous. Let  $a_1, \dots, a_n, s$  be an instance of SHIFTED-SUMS. Without loss of generality we can suppose that  $\sum_{i=1}^n a_i < 2^{2n}$  because otherwise we can solve the instance in polynomial time of the input size. We choose a random prime  $p \in \{2^{4n-1}, \dots, 2^{4n}\}$  and define the reduced instance by  $s' = s \bmod p$  and  $a'_i = a_i \bmod p$ . It is obvious that if the original instance has a solution then so does the reduced instance. We now show that if the original instance does not have a solution, then the reduced instance has a solution only with exponentially small probability. We say that a prime is *bad* if there exist  $S_1 \neq S_2$  such that  $p$  divides  $\sum_{i \in S_1} a_i - \sum_{i \in S_2} a_i - s$ . For any  $S_1 \neq S_2$ , this sum has less than  $2^n$  prime divisors, therefore the number of bad primes is less than  $2^{3n}$ . Using Fact 2 we can conclude that the probability that the reduced instance has a solution is  $O(\frac{n}{2^n})$ .  $\square$

### 2.3 Algorithmic tools

We use the following generalization of Grover's search to the case of an unknown number of solutions.

**Fact 9** (QUANTUM SEARCH, Theorem 3 in [BBHT98]). *Consider a function  $f : [N] \rightarrow \{0, 1\}$  with an unknown number  $K = |f^{-1}(1)|$  of marked items. Suppose that  $f$  can be evaluated in time  $\tau$ . Then, the Quantum Search algorithm finds a marked item in  $f$  in expected time  $\tilde{O}(\sqrt{N/K} \cdot \tau)$ .*

Given a classical subroutine with stopping  $\tau$  that returns a marked item with probability  $\rho$ , we can convert it into a constant success probability algorithm with expected running time  $O(\mathbb{E}[\tau]/\rho)$  by repeating it  $O(1/\rho)$  times. Ambainis proved a similar result for the case of quantum subroutines, with a dependence on the second moment of the stopping time  $\tau$ , and a Grover-like speed-up for the dependence on  $\rho$ .

**Fact 10** (VARIABLE-TIME AMPLITUDE AMPLIFICATION, Theorem 2 in [Amb12]). *Let  $\mathcal{A}$  be a quantum algorithm which looks for a marked element in some set. Let  $\tau$  be the random variable corresponding to the stopping time of the algorithm, and let  $\rho$  be its success probability. Then the variable-time amplitude amplification algorithm finds a marked element in the above set with constant success probability in maximum time  $\tilde{O}(\sqrt{\mathbb{E}[\tau^2]/\rho})$ .*

The next result is a variant of the Birthday paradox over a product space  $[N] \times [M]$ , where at least  $K$  disjoint pairs are marked by some binary relation  $\mathcal{R}$ . Two pairs  $(x, y)$  and  $(x', y')$  are said to be disjoint if  $x \neq x'$  and  $y \neq y'$ . The disjointness assumption is made to simplify the analysis and will be satisfied in our applications.

**Lemma 11** (VARIANT OF THE BIRTHDAY PARADOX). *Consider three integers  $1 \leq K \leq N \leq M$ . Let  $\mathcal{R} : [N] \times [M] \rightarrow \{0, 1\}$  be a binary relation such that there exist at least  $K$  mutually disjoint pairs  $(x_1, y_1), \dots, (x_K, y_K) \in [N] \times [M]$  with  $\mathcal{R}(x_k, y_k) = 1$  for all  $k \in [K]$ . Given an integer  $r \leq O(\sqrt{NM/K})$ , define  $\epsilon(r)$  to be the probability of obtaining both elements from at least one marked pair when  $r$  numbers from  $[N]$  and  $r$  numbers from  $[M]$  are chosen independently and uniformly at random. Then,*

$$\epsilon(r) \geq \Omega\left(\frac{r^2 K}{NM}\right).$$

*Proof.* Fix any  $K$  disjoint marked pairs  $(x_1, y_1), \dots, (x_K, y_K)$ . Let  $X_1, \dots, X_r$  (resp.  $Y_1, \dots, Y_r$ ) be  $r$  independent and uniformly distributed random variables over  $[N]$  (resp.  $[M]$ ). For any indices  $i, j$ , let  $Z_{i,j}$  denote the binary random variable that takes value 1 if  $\{X_i, X_j\}$  is one of the  $K$  fixed pairs, and set  $Z = \sum_{i,j} Z_{i,j}$ . By definition, we have  $\epsilon(r) \geq \Pr[Z \neq 0]$ . We lower bound this quantity by using the inclusion-exclusion principle,

$$\epsilon(r) \geq \sum_{i,j} \Pr(Z_{i,j} = 1) - \frac{1}{2} \sum_{(i,j) \neq (k,\ell)} \Pr(Z_{i,j} = 1 \wedge Z_{k,\ell} = 1).$$

The first term on the right-hand side is equal to  $\sum_{i,j} \Pr(Z_{i,j} = 1) = \frac{r^2 K}{NM}$ . For the second term, the analysis depends on whether the indices  $i, k$  and  $j, \ell$  are distinct or not. If they are distinct then  $\Pr(Z_{i,j} = 1 \wedge Z_{k,\ell} = 1) = \frac{K^2}{(NM)^2}$  since  $Z_{i,j}$  and  $Z_{k,\ell}$  are independent. Otherwise, suppose for instance that  $i = k$ . Since the  $K$  fixed pairs are disjoint, we have  $\Pr(Z_{i,j} = 1 \wedge Z_{i,\ell} = 1) = \Pr(Z_{i,j} = 1 \wedge Y_j = Y_\ell) = \frac{K}{NM^2}$ . Finally, there are  $4\binom{r}{2}^2$  ways of choosing the indices  $i, j, k, \ell$  when  $i \neq k$  and  $j \neq \ell$ , and  $4r\binom{r}{2}$  ways when  $i = k$  or  $j = \ell$ . By putting everything together we obtain that,

$$\epsilon(r) \geq \frac{r^2 K}{NM} - 2\binom{r}{2}^2 \frac{K^2}{(NM)^2} - 2r\binom{r}{2} \frac{K}{NM^2} \geq \Omega\left(\frac{r^2 K}{NM}\right).$$

□

We use the above result to construct a quantum algorithm for finding a marked pair when the relation  $\mathcal{R}(x, y)$  is determined by checking if two underlying values  $f(x)$  and  $g(y)$  are equal or not. Our analysis essentially generalizes the quantum element distinctness [Amb07] and claw finding [Tan09] algorithms to the case of  $K > 1$ .

**Theorem 12** (QUANTUM PAIR FINDING). *There is a bounded-error quantum algorithm with the following properties. Consider four integers  $1 \leq K \leq N \leq M \leq R$  with  $R \leq N^{O(1)}$ . Let  $f : [N] \rightarrow [R]$  and  $g : [M] \rightarrow [R]$  be two functions that can be evaluated in time  $\tau$ . Define  $\mathcal{R} : [N] \times [M] \rightarrow \{0, 1\}$  to be any of the two following binary relations:*

1.  $\mathcal{R}(x, y) = 1$  if and only if  $f(x) = g(y)$ .
2.  $\mathcal{R}(x, y) = 1$  if and only if  $f(x) = g(y)$  and  $x \neq y$ .

Suppose that there exist at least  $K$  mutually disjoint pairs  $(x, y) \in [N] \times [M]$  such that  $\mathcal{R}(x, y) = 1$ . Then, the algorithm returns one such pair in time

$$\begin{cases} \tilde{O}((NM/K)^{1/3} \cdot \tau) & \text{if } N \leq M \leq KN^2, \\ \tilde{O}((M/K)^{1/2} \cdot \tau) & \text{if } M \geq KN^2. \end{cases}$$

*Proof.* If  $N \leq M \leq KN^2$  the algorithm consists of running a quantum walk over the product Johnson graph  $J(N, r) \times J(M, r)$  with  $r = (NM/K)^{1/3}$ . This walk has spectral gap  $\delta = \Omega(1/r)$  and the fraction  $\epsilon$  of vertices containing both elements from at least one marked pair satisfies  $\epsilon \geq \Omega\left(\frac{r^2 K}{NM}\right)$  by Lemma 11. Using the MNRS framework [MNRS11], the query complexity of finding one marked pair is then  $O(S + \frac{1}{\sqrt{\epsilon}}(\frac{U}{\sqrt{\delta}} + C))$ , where the setup cost is  $S = r$ , the update cost is  $U = O(1)$ , and the checking cost is  $C = 0$ . This leads to a query complexity of  $O(r + 1/\sqrt{\epsilon\delta}) = O((NM/K)^{1/3})$ . By a simple adaptation of the data structures described in [Amb07, Section 6.2] or [Jef14, Section 3.3.4], this can be converted to a similar upper bound on the time complexity with a multiplicative overhead of  $\tau$ .

If  $M \geq KN^2$ , the algorithm instead stores all pairs  $\{(x, f(x))\}_{x \in [N]}$  in a table – sorted according to the value of the first coordinate – and then runs the Quantum Search algorithm on the function  $h : [M] \rightarrow \{0, 1\}$  where  $h(x') = 1$  if there exists  $x \in [N]$  such that  $\mathcal{R}(x, y) = 1$ . There are at least  $K$  marked items and  $h$  can be evaluated in time  $O(\tau + \log N)$  using the sorted table. Thus, the running time is  $\tilde{O}(N \cdot \tau + (M/K)^{1/2} \cdot (\tau + \log N)) = \tilde{O}((M/K)^{1/2})$  by Proposition 9.  $\square$

### 3 Dynamic programming data structure

Here we introduce our dynamic programming data structure, and show how it can be used to implement low cost queries to the elements of the set  $T_{p,k}$  defined as follows.

**Definition 13.** Let  $A = \{a_1, \dots, a_n\}$  be a multiset of  $n$  integers. For integers  $p \geq 2$  and  $k \in \{0, 1, \dots, p-1\}$ , define the set  $T_{p,k}$  by

$$T_{p,k} = \{S \subseteq [n] : \Sigma_A(S) \equiv k \pmod{p}\},$$

and denote the cardinality of  $T_{p,k}$  by  $t_{p,k} := |T_{p,k}|$ .

Our main tool is the table  $t_p$ , defined below, constructed by dynamic programming. Note that  $t_{p,k} = t_p[n, k]$  and thus, once the table is constructed, the size  $t_{p,k}$  of  $T_{p,k}$  can be read off the last row.

**Lemma 14.** Let  $n, p$  be non-negative integers. In time  $O(np)$ , the  $(n+1) \times p$  table

$$t_p[i, j] = |\{S \subseteq \{1, \dots, i\} : \Sigma(S) \equiv j \pmod{p}\}|$$

can be constructed by dynamic programming, where  $i \in [0..n]$  and  $j \in [0..p-1]$ .

*Proof.* To compute the elements of the table, observe that  $t_p[0, 0] = 1$  and  $t_p[0, j] = 0$  for  $j > 0$ . The remaining elements can be deduced from the relation

$$t_p[i, j] = t_p[i-1, j] + t_p[i-1, (j - a_i) \bmod p].$$

The  $i^{\text{th}}$  row of  $t_p$  can thus be deduced from the  $(i-1)^{\text{th}}$  row and  $a_i$  in time  $O(p)$  and the computation of all rows can be completed in time  $O(np)$ .  $\square$

### 3.1 Fast SUBSET-SUM oracle

We now show how to use the table  $t_p$  to quickly query any element of  $T_{p,k}$ . To do so, we first define an ordering of the elements of  $T_{p,k}$ .

**Definition 15.** Let  $\prec$  be the relation over  $\mathcal{P}([n])$  defined as follows: for all  $S_1, S_2 \subseteq [n]$ ,  $S_1 \prec S_2$  if and only if  $\max\{i : i \in S_1 \Delta S_2\} \in S_2$ .

**Lemma 16.** *The relation  $\prec$  is a strict total order.*

*Proof.* For every subset  $S \subseteq [n]$ , we define  $\chi(S) = \sum_{i \in S} 2^i$ . Then,  $S_1 \prec S_2$  if and only if  $\chi(S_1) < \chi(S_2)$ . Since  $<$  is a total order over the integers, so is  $\prec$  over  $\mathcal{P}([n])$ .  $\square$

Using the above relation, we now show that the query function  $f : [1 \dots t_{p,k}] \rightarrow T_{p,k}$ , defined by  $f(I) = S_I$ , can be computed in time  $O(n)$ .

**Theorem 17.** *Let  $T_{p,k}$  be enumerated as  $T_{p,k} = \{S_1, \dots, S_{t_{p,k}}\}$  where  $S_1 \prec \dots \prec S_{t_{p,k}}$ . Given any integer  $I \in [1 \dots t_{p,k}]$  and random access to the elements of the table  $t_p$ , the set  $S_I$  can be computed in time  $O(n)$ .*

*Proof.* Algorithm 1 gives a process which starts from  $t_p[n, k]$  (i.e. the total number of subsets  $S \subseteq [n]$  that sum to  $k$  modulo  $p$ ) and an empty set  $Z$ , and constructs  $S_I$  by going backwards ( $i = n, \dots, 1$ ) through the rows of  $t_p$ . At the  $i$ -th step we examine  $t_p[i-1, j]$  and decide whether to include  $i$  in  $Z$  or not. If we do include  $i$  then we examine another element in that row to decide a new value of  $I$ , and we also reset  $j$ .

---

#### Algorithm 1: Fast Subset-Sum oracle

---

**Input:** Table  $t_p$ , integers  $k \in [0 \dots p-1]$  and  $I \in [1 \dots t_{p,k}]$ .

**Output:** The  $I$ -th subset  $Z \subseteq [n]$  (according to  $\prec$ ) such that  $\Sigma(Z) \equiv k \pmod{p}$ .

---

```

1  $j = k$ 
2  $Z = \emptyset$ 
3 for  $i = n, \dots, 1$  do
4   if  $I \leq t_p[i-1, j]$  then
5     Do nothing
6   else
7      $Z = Z \cup \{i\}$ 
8      $I = I - t_p[i-1, j]$ 
9      $j = j - a_i \pmod{p}$ 
10 Return  $Z$ 

```

---

The algorithm consists of  $n$  iterations, each of which can be performed in constant time assuming random access to the elements of  $t_p$ , and therefore the running time is  $O(n)$ . What is left to prove is that the output of the algorithm is indeed  $S_I$ .

First, we provide a high level explanation of why the algorithm works. The total ordering defined by  $\prec$  implies that  $T_{p,k}$  can be written as the disjoint union of two sets,

$$T_{p,k} = \{S_1, \dots, S_{t_p[n-1,k]}\} \cup \{S_{t_p[n-1,k]+1}, \dots, S_{t_p[n,k]}\},$$

where  $n$  is not contained in any  $S_i$  in the first (left) set, and is contained in every  $S_i$  of the second (right) set. Thus, we add  $n$  to the working set  $Z$  only if  $S_I > t_p[n-1, k]$ . If this is the case,  $S_I$  is the  $I - t_p[n-1, k]$ -th element of the right set. We note that removing  $n$  from each  $S_i$  in the right set gives the next bin defined over a smaller universe of size  $n-1$ ,

$$\{S \subseteq [n-1] : \Sigma(S) \equiv k - a_n \pmod{p}\}$$

which has  $t_p[n-1, (k-a_n) \bmod p]$  elements. Therefore, by updating  $n \leftarrow n-1$ ,  $I \leftarrow I - t_p[n-1, k]$  and  $k \leftarrow (k - a_n) \bmod p$  we can repeat the process to determine whether to add  $n-1$  to the working set, and so on, until we reach the value 1.

More formally, denote by  $I_i, j_i$  and  $Z_i$  the values of the respective variables at the beginning of the  $i$ th iteration. With this notation we initially have  $I_n = I, j_n = k$  and  $Z_n = \emptyset$ , and the final output corresponds to  $Z_0$ . We prove by backwards induction for  $i = n, \dots, 1$  the following two statements, which clearly hold for  $i = n$ :

1.  $Z_i = S_I \cap [i+1 \dots n]$ ,
2.  $I_i \leq t_p[i, j_i]$  and in the enumeration of  $T_p[i, j_i] := \{S_1, \dots, S_{t_p[i, j_i]}\}$  according to  $\prec$  we have  $S_{I_i} = S_I \cap [1 \dots i]$ .

Note that the first statement implies that the final output is  $Z_0 = S_I$ .

To prove the inductive step for the first statement, we must show that  $Z_{i-1} = S_I \cap [i \dots n]$ . The inductive hypothesis implies that this holds exactly when  $i \in Z_{i-1} \Leftrightarrow i \in S_I$ . Observe that  $T_p[i-1, j_i] = \{S \in T_p[i, j_i] : i \notin S\}$ . Therefore the set  $T_p[i, j_i]$  is the distinct union of  $T_p[i-1, j_i]$  and  $\mathcal{F}$  where  $\mathcal{F} = \{S \in T_p[i, j_i] : i \in S\}$ . By the definition of  $\prec$  over  $T_p[i, j_i]$ , for every  $X \in T_p[i-1, j_i]$  and for every  $Y \in \mathcal{F}$ , we have  $X \prec Y$ . This implies that  $i \in S_\ell \Leftrightarrow t_p[i-1, j_i] < \ell$ , for every  $\ell \in [1 \dots t_p[i, j_i]]$ . Therefore we have the following equivalences:

$$\begin{aligned} i \in Z_{i-1} &\Leftrightarrow t_p[i-1, j_i] < I_i \\ &\Leftrightarrow i \in S_{I_i} \\ &\Leftrightarrow i \in S_I \cap [1 \dots i] \\ &\Leftrightarrow i \in S_I, \end{aligned}$$

where the first equivalence follows from the definition of  $Z_{i-1}$  and the third equivalence from the second statement of the inductive hypothesis.

We now prove the inductive step for the second statement. Let the enumeration of  $T_p[i-1, j_{i-1}]$  according to  $\prec$  be  $T_p[i-1, j_{i-1}] = \{S'_1, \dots, S'_{t_p[i-1, j_{i-1}]}\}$ . We analyze separately the case  $I_i \leq t_p[i-1, j_i]$  and the case  $t_p[i-1, j_i] < I_i$ .

When  $I_i \leq t_p[i-1, j_i]$  then  $I_{i-1} \leq t_p[i-1, j_{i-1}]$  because  $I_{i-1} = I_i$  and  $j_{i-1} = j_i$ . Also,  $S'_\ell = S_\ell$ , for every  $\ell \in [1 \dots t_p[i-1, j_{i-1}]]$ . Therefore we have the following equalities

$$S'_{I_{i-1}} = S_{I_i} = S_I \cap [1 \dots i] = S_I \cap [1 \dots i-1],$$

where the second equality is the inductive hypothesis, and the third equality holds because  $i \notin S_{I_i}$  when  $I_i \leq t_p[i-1, j_i]$ .

When  $t_p[i-1, j_i] < I_i \leq t_p[i, j_i]$  then  $I_{i-1} \leq t_p[i-1, j_{i-1}]$  because  $I_{i-1} = I_i - t_p[i-1, j_i]$  and  $t_p[i-1, j_{i-1}] = t_p[i, j_i] - t_p[i-1, j_i]$  (here we used the definition of  $j_{i-1}$ ). Also,  $S'_\ell = S_{\ell+t_p[i-1, j_i]} \setminus \{i\}$ , for every  $\ell \in [1 \dots t_p[i-1, j_{i-1}]]$ . Therefore we have the following equalities:

$$S'_{I_{i-1}} = S_{I_i} \setminus \{i\} = (S_I \cap [1 \dots i]) \setminus \{i\} = S_I \cap [1 \dots i-1],$$

where again the second equality is the inductive hypothesis. □

**Corollary 18** (Enumerating solutions to SUBSET-SUM via dynamic programming). *Let  $A = \{a_1, \dots, a_n\}$  and  $T_{p,k} = \{S \subseteq [n] : \Sigma_A(S) \equiv k \pmod{p}\}$ . For any  $c \leq |T_{p,k}|$ , it is possible to find  $c$  elements of  $T_{p,k}$  in time  $\tilde{O}(p+c)$ .*

*Proof.* By Lemma 14 the table  $t_p[i, j]$  can be constructed in time  $O(np)$ . Thereafter, by Theorem 17 each set  $S_I$  for  $I \in [1 \dots t_{p,k}]$  can be computed in time  $O(n)$ . □

In comparison with Fact 6, where the same task is completed in time  $\tilde{O}(2^{n/2} + c)$ , enumerating solutions to SUBSET-SUM via dynamic programming can be advantageous when  $p < 2^{n/2}$ .



### 3.2 Statistics about random bins

In this section  $b \in (0, 1)$  is a constant,  $p$  is a random integer in  $[2^{bn} \dots 2^{bn+1}]$ , and  $k$  is a random integer in  $[0 \dots p - 1]$ . Therefore,  $T_{p,k}$  is a random bin and its cardinality  $t_{p,k}$  is a random integer. To upper bound  $t_{p,k}$  we can use Markov's inequality once we bound its expectation.

**Lemma 19.** *The expected bin size can be upper bounded as  $\mathbb{E}_{p,k}[t_{p,k}] \leq 2^{(1-b)n}$ .*

*Proof.* The expected size of  $T_{p,k}$  is at most  $\mathbb{E}_{p,k}[t_{p,k}] \leq 2^{(1-b)n}$  since  $\{T_{p,k} : 0 \leq k < p\}$  is a partition of  $\mathcal{P}([n])$  with  $p \geq 2^{bn}$ .  $\square$

The above result can be extended to a similar upper-bound on the second moment of  $t_{p,k}$ , under the assumption that the input does not contain too many solution pairs. This result is needed to analyze the complexity of the variable time quantum amplitude amplification algorithm later on.

**Lemma 20.** *Fix any integer  $s \geq 0$  and any real  $b \in [0, 1]$ . If there are at most  $2^{(2-b)n}$  pairs  $(S_1, S_2) \in \mathcal{P}([n])^2$  such that  $\Sigma(S_1) = \Sigma(S_2) + s$ , then the expected product of the sizes of two bins at distance  $s \bmod p$  from each other is at most  $\mathbb{E}_{p,k}[t_{p,k}t_{p,(k-s) \bmod p}] \leq \tilde{O}(2^{2(1-b)n})$ .*

*Proof.* The expectation of  $t_{p,k}t_{p,(k-s) \bmod p}$  is equal to the expected number of pairs  $(S_1, S_2)$  such that both  $\Sigma(S_1)$  and  $\Sigma(S_2) + s$  are congruent to  $k$  modulo  $p$ , that is

$$\mathbb{E}_{p,k}[t_{p,k}t_{p,(k-s) \bmod p}] = \mathbb{E}_{p,k} \left[ \sum_{S_1, S_2} \mathbb{1}_{\Sigma(S_1) \equiv_p \Sigma(S_2) + s \equiv_p k} \right].$$

Since  $k$  is uniformly distributed in  $[0 \dots p - 1]$ , this is equal to  $\sum_{S_1, S_2} \mathbb{E}_p \left[ \frac{1}{p} \mathbb{1}_{\Sigma(S_1) \equiv_p \Sigma(S_2) + s} \right] \leq 2^{-bn} \sum_{S_1, S_2} \Pr_p[\Sigma(S_1) \equiv_p \Sigma(S_2) + s]$ , where we used that  $p \geq 2^{bn}$ . This sum can be decomposed as

$$2^{-bn} \left( \sum_{\Sigma(S_1) = s + \Sigma(S_2)} 1 + \sum_{\Sigma(S_1) \neq s + \Sigma(S_2)} \frac{\Pr[\Sigma(S_1) \equiv_p \Sigma(S_2) + s]}{p} \right),$$

where the first inner term is at most  $2^{(2-b)n}$  by assumption, and the second term is at most  $2^{2n}n2^{-bn}$  by Corollary 3. We conclude that  $\mathbb{E}[t_{p,k}t_{p,(k-s) \bmod p}] \leq O(2^{-bn}(2^{(2-b)n} + 2^{2n}n2^{-bn})) \leq \tilde{O}(2^{2(1-b)n})$ .  $\square$

Finally, we provide a lower bound on the number of *distinct* subset sum values that get hashed to the random bin  $T_{p,k}$ .

**Lemma 21.** *Let  $V$  be any subset of the image set  $\{v \in \mathbb{N} : \exists S \subseteq [n], \Sigma(S) = v\}$ . Let  $v_{p,k}$  denote the number of values  $v \in V$  such that  $v \equiv k \pmod{p}$ . Suppose that  $|V| \geq 2^{(1-\ell)n}$  for some  $\ell \in [0, 1]$ . Then,*

$$\begin{cases} \Pr_{p,k}[v_{p,k} \geq 2^{(1-\ell-b)n-2}] = \Omega(1/n), & \text{when } \ell \leq 1-b, \\ \Pr_{p,k}[v_{p,k} \geq 1] = \Omega(2^{(1-\ell-b)n}), & \text{when } \ell > 1-b. \end{cases}$$

*Proof.* The expected size of  $V_{p,k}$  is at least  $\mathbb{E}_{p,k}[v_{p,k}] \geq |V|/p \geq |V|2^{-bn-1}$  since  $\{V_{p,k} : 0 \leq k < p\}$  is a partition of  $V$ . Similarly to Lemma 20, the second moment satisfies that  $\mathbb{E}_{p,k}[v_{p,k}^2] \leq O(2^{-bn}(|V| + |V|^2n2^{-bn}))$  by using Corollary 3. If  $\ell \leq 1-b$  then the  $|V|^2nb2^{-bn}$  term dominates the  $|V|$  term since  $|V| \geq 2^{(1-\ell)n}$  by assumption (we cannot say which term dominates when  $\ell > 1-b$ ). The Paley-Zygmund inequality (Fact 4, with  $\theta = 1/2$ ) completes the proof, noting that  $\Pr[v_{p,k} \geq 1] = \Pr[v_{p,k} > 0]$  because  $v_{p,k}$  is an integer.  $\square$

## 4 SUBSET-SUM

As an illustration of the utility of the data structure introduced in the previous section, here we show how it can be used to give quantum and classical algorithms for *worst-case* instances of SUBSET-SUM based on the representation technique, with running times  $\tilde{O}(2^{n/3})$  and  $\tilde{O}(2^{n/2})$  respectively. These algorithms therefore achieve the same complexity as the best known algorithms for worst case complexity based on the meet-in-the-middle principle. Both algorithms use, as a first step, a simple search procedure to handle the case where many solutions exist. Note that the tables constructed by the algorithms do not depend on the target value  $m$ .

---

### Algorithm 2: Quantum representation technique for SUBSET-SUM

---

**Input:** Instance of SUBSET-SUM of size  $n$  and target  $m \leq 2^{4n}$ .

**Output:** A subset  $S \subseteq [n]$  satisfying  $\Sigma(S) = m$  if one exists, otherwise output None.

- 1 Run the Quantum Search algorithm (Proposition 9) over the sets  $S \in \mathcal{P}([n])$ , where a set is marked if  $\Sigma(S) = m$ . Stop it and proceed to step 2 if the running time exceeds  $\tilde{O}(2^{n/3})$ , otherwise output the set it found within the allotted time.
  - 2 Choose a random prime  $p \in [2^{n/3} \dots 2^{n/3+1}]$ .
  - 3 Construct the table  $t_p[i, j]$  for  $i = 0, \dots, n$  and  $j = 0, \dots, p-1$  (see Section 3).
  - 4 Run Quantum Search on the set  $T_{p, m \bmod p}$  marking elements  $S \in T_{p, m \bmod p}$  that satisfy  $\Sigma(S) = m$ .
- 

**Theorem 22** (SUBSET-SUM, quantum). *Algorithm 2 solves SUBSET-SUM in time  $\tilde{O}(2^{n/3})$  with high probability.*

*Proof.* If the number of solutions is at least  $|\{S : \Sigma(S) = m\}| \geq 2^{n/3}$  then step 1 suffices to solve the problem with high probability since Quantum Search needs time  $\tilde{O}(\sqrt{2^n/2^{n/3}}) = \tilde{O}(2^{n/3})$ . Hence, let us assume that the number of solutions is at most  $2^{n/3}$ . The table  $t_p$  can be constructed in time  $\tilde{O}(2^{n/3})$  according to Lemma 14. The expected size of  $T_{p, m \bmod p}$  can be bounded by  $\mathbb{E}_p[t_{p, m \bmod p}] = |\{S : \Sigma(S) = m\}| + \sum_{S: \Sigma(S) \neq m} \Pr[\Sigma(S) \equiv_p m] = \tilde{O}(2^{2n/3})$  since  $\Pr[\Sigma(S) \equiv_p m] \in O(n/2^{n/3})$  by Corollary 3. Finally, by Markov's inequality, with high probability  $t_{p, m \bmod p}$  is no more than a small multiple of this expectation, and Quantum Search over a bin of size  $\tilde{O}(2^{2n/3})$  takes time  $\tilde{O}(2^{n/3})$  (using the fast oracle of Theorem 17).  $\square$

The classical algorithm is similar to the quantum one, except that it constructs a bigger table  $t_p$  to balance the cost of this construction and the cost of the classical search.

---

### Algorithm 3: Classical representation technique for SUBSET-SUM

---

**Input:** Instance of SUBSET-SUM of size  $n$  and target  $m \leq 2^{4n}$ .

**Output:** A subset  $S \subseteq [n]$  satisfying  $\Sigma(S) = m$  if one exists, otherwise output None.

- 1 Sample  $2^{n/2}$  subsets  $S \subseteq [n]$  uniformly at random and check if  $\Sigma(S) = m$ . If no solution is found, proceed to step 2.
  - 2 Choose a random prime  $p \in [2^{n/2} \dots 2^{n/2+1}]$ .
  - 3 Construct the table  $t_p[i, j]$  for  $i = 0, \dots, n$  and  $j = 0, \dots, p-1$  (see Section 3).
  - 4 Enumerate the elements of the set  $T_{p, m \bmod p}$  until finding  $S \in T_{p, m \bmod p}$  that satisfies  $\Sigma(S) = m$ .
- 

**Theorem 23** (SUBSET-SUM, classical). *Algorithm 3 solves SUBSET-SUM in time  $\tilde{O}(2^{n/2})$  with high probability.*

*Proof.* Step 1 does not suffice to solve the problem when the number of solutions is smaller than  $O(2^{n/2})$ . Let us suppose that we are in this case. From Lemma 14, the  $(n+1) \times p$  table  $t_p$  can be constructed in time  $\tilde{O}(2^{n/2})$ , after which each query to  $T_{p,m \bmod p} = \{S \subseteq [n] : \Sigma(S) \equiv_p m\}$  can be made in time  $O(n)$ . By linearity of expectation, the expected size of  $T_{p,m \bmod p}$  can be bounded by  $\mathbb{E}_p[t_{p,m \bmod p}] = |\{S : \Sigma(S) = m\}| + \sum_{S: \Sigma(S) \neq m} \Pr[\Sigma(S) \equiv_p m] = \tilde{O}(2^{n/2})$  since, by Corollary 3,  $\Pr[\Sigma(S) \equiv_p m] \in O(n/2^{n/2})$  for each of the (at most  $2^n$ ) sets  $S$  for which  $\Sigma(S) \neq m$ . By Markov's inequality, with high probability  $t_{p,m \bmod p}$  is no more than a small multiple of this expectation and thus can be enumerated in time  $\tilde{O}(2^{n/2})$  using Theorem 17.  $\square$

## 5 SHIFTED-SUMS

In this section we present the two quantum algorithms for solving the SHIFTED-SUMS problem. The running time of both algorithms – expressed in Theorems 28 and 30 – are functions of the size of a *maximum solution* of the input. This notion plays a central role in our algorithms and is defined next.

**Definition 24** (Maximum solution). We say that two disjoint subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  that form a solution to an instance of SHIFTED-SUMS are a *maximum solution* if their size  $|S_1| + |S_2| = \ell n$  is largest among all such solutions. We call  $\ell \in (0, 1)$  the *maximum solution ratio*.

By choosing the faster of these two algorithms for each  $\ell \in \{1/n, 2/n, \dots, (n-1)/n\}$  until a solution has been found (or it can be concluded that no solution exists), we obtain an overall quantum algorithm for SHIFTED-SUMS with performance captured by the following theorem.

**Theorem 25** (SHIFTED-SUMS, quantum). *There is a quantum algorithm which, given an instance of SHIFTED-SUMS with maximum solution ratio  $\ell \in (0, 1)$ , outputs a solution with at least inverse polynomial probability in time  $\tilde{O}(2^{\gamma(\ell)n})$  where*

$$\gamma(\ell) = \begin{cases} (1 + \ell)/4 & \text{if } \ell_1 \leq \ell < 3/5, & \text{(Theorem 28)} \\ \ell/2 + 1/10 & \text{if } 3/5 \leq \ell < \ell_2, & \text{(Theorem 28)} \\ (h(\ell) + \ell)/3 & \text{otherwise} & \text{(Theorem 30)} \end{cases}$$

and  $\ell_1 \approx 0.190$  and  $\ell_2 \approx 0.809$  are solutions to the equations  $(h(\ell) + \ell)/3 = (1 + \ell)/4$  and  $(h(\ell) + \ell)/3 = \ell/2 + 1/10$  respectively. In particular, the worst case complexity of the algorithm is  $O(2^{0.504n})$ .

Since a potential solution can be verified in polynomial time in  $n$ , in what follows we describe the two algorithms on yes instances with maximum solution ratio  $\ell$ . As presented, the algorithms find a solution with inverse polynomial probability in  $n$ , which can be amplified to constant probability in polynomial time.

Note that the classical algorithm of Mucha et. al [MNPW19] for EQUAL-SUMS is based on the concept of a *minimum* solution (Definition 42), rather than a *maximum* solution. In Appendix B, we present an analogous quantum algorithm for EQUAL-SUMS whose complexity is expressed in terms of the minimum solution ratio  $\ell'$ . While this does not change the algorithmic complexity in the worse case, for a given instance of EQUAL-SUMS the quantity  $\ell'$  may be smaller than  $\ell$ .

### 5.1 Representation technique algorithm

Our first algorithm uses a representation technique approach, based on the dynamic programming table of Section 3. Similarly to the algorithms for SUBSET-SUM presented in Section 4, before constructing the table we first ensure that the input does not contain too many solution pairs.

Depending on the value of the maximum solution ratio  $\ell$ , we may need to apply variable time amplitude amplification on top of quantum pair finding. The overall process is described in Algorithm 4.

---

**Algorithm 4:** Quantum representation technique for SHIFTED-SUMS

---

**Input:** Instance  $(a, s)$  of SHIFTED-SUMS with  $\sum_{i=1}^n a_i < 2^{4n}$  and maximum solution ratio  $\ell$ .

**Output:** Two subsets  $S_1, S_2 \subseteq [n]$ .

- 1 Set  $b = (1 + \ell)/4$  if  $\ell \leq 3/5$  and  $b = 2/5$  if  $\ell > 3/5$ .
  - 2 Run the Quantum Search algorithm (Proposition 9) over the set of pairs  $(S_1, S_2) \in \mathcal{P}([n])^2$ , where a pair is marked if  $\Sigma(S_1) = \Sigma(S_2) + s$  and  $S_1 \neq S_2$ . Stop it and proceed to step 3 if the running time exceeds  $\tilde{O}(2^{bn/2})$ , otherwise output the pair it found within the allotted time.
  - 3 If  $\ell > 3/5$  then run variable time amplitude amplification on steps 2 - 4, otherwise run them once:
  - 4 Choose a random prime  $p \in [2^{bn} \dots 2^{bn+1}]$  and a random integer  $k \in [0 \dots p - 1]$ .
  - 5 Construct the table  $t_p[i, j]$  for  $i = 0, \dots, n$  and  $j = 0, \dots, p - 1$  (see Section 3).
  - 6 Run the quantum Pair Finding algorithm (Theorem 12) to find if there exists two sets  $S_1 \in T_{p,k}$  and  $S_2 \in T_{p,(k-s) \bmod p}$  such that  $\Sigma(S_1) = \Sigma(S_2) + s$  and  $S_1 \neq S_2$ . If so, output the pair  $(S_1, S_2)$  it found.
- 

The analysis of the above algorithm relies on the random bins statistics presented in Section 3.2. We first define the *collision values set* which contains the values of all the possible solution pairs.

**Definition 26** (COLLISION VALUES SET). Given an instance  $(a, s)$  to the SHIFTED-SUMS problem, the *collision values set* is the set  $V = \{v \in \mathbb{N} : \exists S_1 \neq S_2, v = \Sigma(S_1) = \Sigma(S_2) + s\}$ .

We show that the collision values set  $V$  is of size at least  $2^{(1-\ell)n}$  when the maximum solution ratio is  $\ell$ . Thus, by Lemma 21, we can lower bound the number of values in  $V$  that get hashed to a random bin  $T_{p,k}$ .

**Lemma 27.** *If the maximum solution ratio is  $\ell$  then  $|V| \geq 2^{(1-\ell)n}$ .*

*Proof.* Let  $S_1, S_2 \subseteq \{1, \dots, n\}$  be a maximum solution of size  $|S_1| + |S_2| = \ell n$ . Then for any  $S \subseteq [n] \setminus (S_1 \cup S_2)$  the sets  $S_1 \cup S$  and  $S_2 \cup S$  form a solution, and for  $S \neq S'$ , the values  $\Sigma(S_1 \cup S)$  and  $\Sigma(S_1 \cup S')$  must be distinct. Indeed, if this were not the case then  $S_1 \cup (S \setminus S')$  and  $S_2 \cup (S' \setminus S)$  would form a disjoint solution of size larger than  $\ell$ . Therefore there are at least  $|V| \geq 2^{(1-\ell)n}$  distinct collision values.  $\square$

We now turn to the analysis of Algorithm 4.

**Theorem 28** (SHIFTED-SUMS, representation). *Given an instance of SHIFTED-SUMS with  $\sum_{i=1}^n a_i < 2^{4n}$  and maximum solution ratio  $\ell \in (0, 1)$ , Algorithm 4 finds a solution with inverse polynomial probability in time  $\tilde{O}(2^{(1+\ell)n/4})$  if  $\ell \leq 3/5$ , and  $\tilde{O}(2^{(\ell/2+1/10)n})$  if  $\ell > 3/5$ .*

*Proof.* Step 2 of Algorithm 4 handles the (easy) case where the total number of solution pairs exceeds  $2^{(2-b)n}$ . Indeed, in this situation the standard Quantum Search algorithm can find a solution pair in time  $\tilde{O}(\sqrt{2^{2n}/2^{(2-b)n}}) = \tilde{O}(2^{bn/2})$ , which is smaller than the time complexity given in Theorem 28.

**Analysis when  $\ell \leq 3/5$ .** In this case the algorithm executes steps 4 - 6 only once. From Lemma 14, the table  $t_p$  can be constructed in time  $O(n2^{bn})$ , after which each query to the elements of  $T_{p,k}$  can be performed in time  $O(n)$  (Theorem 17). By Lemma 21, the number of disjoint solution pairs contained in  $T_{p,k} \times T_{p,(k-s) \bmod p}$  is at least  $v_{p,k} \geq 2^{(1-\ell-b)n-2}$  with

probability  $\Omega(1/n)$ . By Lemma 19 and Markov's inequality, the sizes of  $T_{p,k}$  and  $T_{p,(k-s) \bmod p}$  are at most  $t_{p,k}, t_{p,(k-s) \bmod p} \leq n^2 2^{(1-b)n}$  with probability at least  $1 - 1/n^2$ . Thus, with probability  $\Omega(1/n)$  we can assume that both of these events occur. If this is the case, then the time to execute step 6 of the algorithm is  $\tilde{O}\left((t_{p,k} t_{p,(k-s) \bmod p} / v_{p,k})^{1/3}\right) = \tilde{O}(2^{(1+\ell-b)n/3})$  since the first complexity given in Theorem 12 is the largest one for our choice of parameters. This is at most  $\tilde{O}(2^{(1+\ell)n/4})$  when  $b = (1 + \ell)/4$ .

**Analysis when  $\ell > 3/5$ .** We assume that the total number of solution pairs is at most  $2^{(2-b)n}$  (otherwise we would have found a collision at step 2 with high probability). Given  $p$  and  $k$ , the base algorithm (steps 4 - 6) succeeds if there is a solution in  $T_{p,k} \times T_{p,(k-s) \bmod p}$ , i.e.  $v_{p,k} \geq 1$ . Therefore by Lemma 21, we have for its success probability  $\rho = \Omega(2^{(1-\ell-b)n})$ . We claim that  $\mathbb{E}[\tau^2] = \tilde{O}(2^{2bn})$  where  $\tau$  is the stopping time of the base algorithm. Constructing the table  $t_p$  takes time  $\tilde{O}(p)$ , and by summing the two complexities given in Theorem 12 the quantum pair finding algorithm takes time at most  $\tilde{O}((t_{p,k} t_{p,(k-s) \bmod p})^{1/3} + \sqrt{\max(t_{p,k}, t_{p,(k-s) \bmod p})})$ . Therefore we have

$$\begin{aligned} \mathbb{E}[\tau^2] &= \tilde{O}\left(\mathbb{E}_{p,k}\left[\left(p + (t_{p,k} t_{p,(k-s) \bmod p})^{1/3} + \sqrt{\max(t_{p,k}, t_{p,(k-s) \bmod p})}\right)^2\right]\right) \\ &\leq \tilde{O}\left(\mathbb{E}_{p,k}[p^2] + \mathbb{E}_{p,k}\left[t_{p,k}^{2/3} t_{p,(k-s) \bmod p}^{2/3}\right] + \mathbb{E}_{p,k}[t_{p,k}]\right) \\ &\leq \tilde{O}\left(\mathbb{E}_{p,k}[p^2] + \mathbb{E}_{p,k}\left[t_{p,k} t_{p,(k-s) \bmod p}\right]^{2/3} + \mathbb{E}_{p,k}[t_{p,k}]\right) \\ &\leq \tilde{O}(2^{2bn}) + \tilde{O}(2^{4(1-b)n/3}) + 2^{(1-b)n}, \end{aligned}$$

where the second inequality uses that the moment function is non-decreasing (Fact 5) and the last inequality uses Lemmas 19 and 20. Since  $b = 2/5$  we obtain that  $\mathbb{E}[\tau^2] \leq \tilde{O}(2^{2bn})$ . Finally, by Proposition 10, the overall time of steps 3 - 6 is  $\tilde{O}(\sqrt{\mathbb{E}[\tau^2]}/\rho) = \tilde{O}(2^{bn}/2^{(1-\ell-b)n/2}) = \tilde{O}(2^{(\frac{\ell}{2} + \frac{1}{10})n})$ .  $\square$

## 5.2 Meet-in-the-middle algorithm

Our second algorithm uses the standard meet-in-the-middle technique combined with Quantum Search to solve the SHIFTED-SUMS problem. We first state a lemma that if we randomly partition the input into two sets of relative sizes 1:2, then with at least inverse polynomial probability a maximum solution will be distributed in the same proportion between the two sets.

**Lemma 29.** *Let  $S_1, S_2$  be a maximum solution of ratio  $\ell$ . Then with at least inverse polynomial probability the random partition  $X_1 \cup X_2$  satisfies  $|(S_1 \cup S_2) \cap X_1| = \ell n/3$ ,  $|(S_1 \cup S_2) \cap X_2| = 2\ell n/3$ .*

*Proof.* There are  $\binom{n}{n/3}$  ways to partition  $[n]$  into two subsets  $X_1$  and  $X_2$  of respective sizes  $n/3$  and  $2n/3$ . Of these, there are  $\binom{\ell n}{\ell n/3} \cdot \binom{n-\ell n}{\frac{n-\ell n}{3}}$  partitions such that  $|(S_1 \cup S_2) \cap X_1| = \ell n/3$ ,  $|(S_1 \cup S_2) \cap X_2| = 2\ell n/3$ . The probability that  $|(S_1 \cup S_2) \cap X_1| = \ell n/3$ ,  $|(S_1 \cup S_2) \cap X_2| = 2\ell n/3$  is thus  $\frac{\binom{\ell n}{\ell n/3} \cdot \binom{n-\ell n}{\frac{n-\ell n}{3}}}{\binom{n}{n/3}}$ . Fact 1 gives that this quantity is at least  $\Omega(n^{-1/2})$ .  $\square$

We use the above result in the design of Algorithm 5, which is analyzed in the next theorem. We observe that the obtained time complexity is at most  $\tilde{O}(3^{n/3})$  and it is maximized at  $\ell = 2/3$ .

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**Algorithm 5:** Quantum meet-in-the-middle technique for SHIFTED-SUMS

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**Input:** Instance  $(a, s)$  of SHIFTED-SUMS with maximum solution ratio  $\ell$ .

**Output:** Two subsets  $S_1, S_2 \subseteq [n]$ .

- 1 Randomly split  $[n]$  into disjoint subsets  $X_1 \cup X_2$  such that  $|X_1| = n/3, |X_2| = 2n/3$ .
  - 2 Classically compute and sort  $V_1 = \{\Sigma(S_{11}) - \Sigma(S_{21}) : S_{11}, S_{21} \subseteq X_1 \text{ and } S_{11} \cap S_{21} = \emptyset \text{ and } |S_{11}| + |S_{21}| = \ell n/3\}$ .
  - 3 Apply Quantum Search over the set  $V_2 = \{\Sigma(S_{12}) - \Sigma(S_{22}) : S_{12}, S_{22} \subseteq X_2 \text{ and } S_{12} \cap S_{22} = \emptyset \text{ and } |S_{12}| + |S_{22}| = 2\ell n/3\}$ , where an element  $v_2 \in V_2$  is marked if there exists  $v_1 \in V_1$  such that  $v_1 + v_2 = s$ . For a marked  $v_2$ , output  $S_1 = S_{11} \cup S_{12}$  and  $S_2 = S_{21} \cup S_{22}$ .
- 

**Theorem 30** (SHIFTED-SUMS, meet-in-the-middle). *Given an instance of SHIFTED-SUMS with maximum solution ratio  $\ell \in (0, 1)$ , Algorithm 5 finds a solution with at least inverse polynomial probability in time  $\tilde{O}(2^{n(h(\ell)+\ell)/3})$ .*

*Proof.* There are  $\binom{n/3}{\ell n/3} 2^{\ell n/3}$  different ways to select two sets  $S_{11}, S_{21} \subseteq X_1$  such that  $S_{11} \cap S_{21} = \emptyset$ ,  $|S_{11}| + |S_{21}| = \ell n/3$ . Computing and sorting  $V_1$  thus takes time  $\tilde{O}(\binom{n/3}{\ell n/3} 2^{\ell n/3})$ . In the next step of the algorithm, Quantum Search is performed over all  $\binom{2n/3}{2\ell n/3} 2^{2\ell n/3}$  sets  $S_{12}, S_{22} \subseteq X_2$  such that  $S_{12} \cap S_{22} = \emptyset$ ,  $|S_{12}| + |S_{22}| = 2\ell n/3$ . We mark an element  $v_2 \in V_2$  if there exists  $v_1 \in V_1$  such that  $v_1 + v_2 = s$ . Since  $V_1$  is sorted this check can be done in time  $\text{polylog}(|V_1|)$ . The total time required is therefore  $\tilde{O}\left(\binom{n/3}{\ell n/3} 2^{\ell n/3} + \sqrt{\binom{2n/3}{2\ell n/3} 2^{2\ell n/3}}\right) = \tilde{O}\left(2^{\ell n/3} \left(\binom{n/3}{\ell n/3} + \sqrt{\binom{2n/3}{2\ell n/3}}\right)\right) = \tilde{O}(2^{\frac{n}{3}(h(\ell)+\ell)})$ . By Lemma 29, when the instance has a maximum solution of size  $\ell n$ , the set  $V_2$  has a marked element with at least inverse polynomial probability, and in that case a solution is found.  $\square$

## 6 Pigeonhole variants of EQUAL-SUMS

### 6.1 PIGEONHOLE EQUAL-SUMS

We now turn to the PIGEONHOLE EQUAL-SUMS problem, and give classical and quantum algorithms based on dynamic programming which run in time  $\tilde{O}(2^{n/2})$  and  $\tilde{O}(2^{2n/5})$ , respectively. In contrast with our quantum algorithm for SHIFTED-SUMS which made use of a random prime modulus, in the case of PIGEONHOLE EQUAL-SUMS we can deterministically choose a modulus  $p$ , and the pigeonhole principle guarantees a collision in at least one bin.

**Lemma 31.** *There is a classical deterministic algorithm such that, given an instance of PIGEONHOLE EQUAL-SUMS and a modulus  $p$  that divides  $2^n$ , it finds in time  $\tilde{O}(p)$  an integer  $k$  such that there exists two distinct subsets  $S_1, S_2$  with  $\Sigma(S_1) \equiv \Sigma(S_2) \equiv k \pmod{p}$ .*

*Proof.* Denote by  $\bar{0}, \bar{1}, \dots, \overline{p-1}$  the congruence classes modulo  $p$ . Each of these classes contains exactly  $2^n/p$  numbers between 0 and  $2^n - 2$ , except the last class  $\overline{p-1}$  that has only  $2^n/p - 1$  numbers. Since all  $2^n$  subsets  $S \subseteq [n]$  have a sum  $\Sigma(S)$  between 0 and  $2^n - 2$  there are two possible cases:

- either there is some class  $\bar{k}$  such that  $\Sigma(S) \in \bar{k}$  for strictly more than  $2^n/p$  subsets  $S$ ,
- or there are  $2^n/p$  subsets  $S$  such that  $\Sigma(S) \in \overline{p-1}$ .

Denote by  $\bar{k}$  a class that verifies one of these two points. By definition, there are *strictly more* subsets  $S$  such that  $\Sigma(S) \in \bar{k}$  than the number of elements between 0 and  $2^n - 2$  that belong to  $\bar{k}$ . However, for all  $S \subseteq [n]$ , we have  $\Sigma(S) \leq 2^n - 2$ . Thus, there must be two subsets  $S_1 \neq S_2$  such that  $\Sigma(S_1), \Sigma(S_2) \in \bar{k}$  and  $\Sigma(S_1) = \Sigma(S_2)$ .



From Lemma 14, the table  $t_p[i, j] = |\{S \subseteq \{1, \dots, i\} : \Sigma(S) \equiv j \pmod{p}\}|$  can be constructed in time  $\tilde{O}(p)$ . From the table, we can read off a value  $k$  that satisfies the above condition.  $\square$

**Theorem 32** (PIGEONHOLE EQUAL-SUMS, classical). *There is a classical deterministic algorithm for the PIGEONHOLE EQUAL-SUMS problem that runs in time  $\tilde{O}(2^{n/2})$ .*

*Proof.* Choose  $p = 2^{n/2}$ . By Lemma 31, in time  $\tilde{O}(2^{n/2})$  we can find  $k$  such that there exists  $S_1 \neq S_2$  satisfying  $\Sigma(S_1) \equiv \Sigma(S_2) \equiv k \pmod{2^{n/2}}$ . Once we know a bin that contains a collision, by Corollary 18, we can enumerate in time  $\tilde{O}(2^{n/2})$  a sufficient number of subsets in that bin to locate a collision.  $\square$

**Theorem 33** (PIGEONHOLE EQUAL-SUMS, quantum). *There is a quantum algorithm for the PIGEONHOLE EQUAL-SUMS problem that runs in time  $\tilde{O}(2^{2n/5})$ .*

*Proof.* We set  $p = 2^{2n/5}$  and, by Lemma 31, in time  $\tilde{O}(2^{2n/5})$  we can identify  $k$  such that there exists  $S_1 \neq S_2$  satisfying  $\Sigma(S_1) \equiv \Sigma(S_2) \equiv k \pmod{2^{2n/5}}$ . By Theorem 17, each query to  $T_{p,k} = \{S \subseteq [n] : \Sigma(S) \equiv k \pmod{p}\}$  can be made in time  $O(n)$ . We use Ambainis' element distinctness algorithm [Amb07] on these elements to find a collision. We do not want to run it on an unnecessarily large set. Therefore, if  $t_{p,k} > 2^{3n/5+1}$  then we run it only on the first  $2^{3n/5+1}$  elements of  $T_{p,k}$ , according to the ordering defined by  $\prec$ . A collision is then found in time  $\tilde{O}((2^{3n/5})^{2/3}) = \tilde{O}(2^{2n/5})$ . The overall running time of the algorithm is thus  $\tilde{O}(2^{2n/5})$ .  $\square$

## 6.2 PIGEONHOLE MODULAR EQUAL-SUMS

**Theorem 34** (PIGEONHOLE MODULAR EQUAL-SUMS, classical). *There is a classical deterministic algorithm for the PIGEONHOLE MODULAR EQUAL-SUMS problem that runs in time  $\tilde{O}(2^{n/2})$ .*

*Proof.* Without loss of generality we suppose that for every input  $a_i$ , the inequality  $a_i < q$  holds, where  $q \leq 2^n - 1$  is the modulus in the input. For such a modulus there exists a unique couple  $(q_1, q_2)$  with  $0 \leq q_1, q_2 < 2^{n/2}$  such that  $q = q_1 2^{n/2} + q_2$ . We define the one dimensional array  $B[j]$  for  $0 \leq j \leq q_1 - 1$  by

$$B[j] = \left\{ S \subseteq \{1, \dots, n\} : \left\lfloor \frac{\Sigma(S) \bmod q}{2^{n/2}} \right\rfloor = j \right\}.$$

We denote the cardinality of  $B[j]$  by  $b[j]$ , and we define  $\beta[j] = \left| \left\{ 0 \leq k \leq q - 1 : \left\lfloor \frac{k}{2^{n/2}} \right\rfloor = j \right\} \right|$ . Observe that  $\beta[j] \leq 2^{n/2}$ , for all  $j$ , and that

$$\sum_{j=0}^{q_1-1} \beta_j = q < 2^n = \sum_{j=0}^{q_1-1} b[j].$$

Therefore there exists  $j$  such that  $\beta[j] < b[j]$ , and we will call such an index *marked*. The algorithm will identify a marked  $j$ , and then will find  $\beta[j] + 1$  different sets in  $B[j]$ . We will show that this can be done in time  $\tilde{O}(2^{n/2})$ , and by the pigeonhole principle there are two sets  $S_1, S_2 \subseteq \{1, \dots, n\}$  among them such that  $\Sigma(S_1) \equiv \Sigma(S_2) \pmod{q}$ .

Computing directly the values in the array  $b$  is not easy, we will do that with the help of another one dimensional array  $C$ . For  $1 \leq k \leq n$ , we set  $a'_k = \left\lfloor \frac{a_k}{2^{n/2}} \right\rfloor$  and  $A' = \{a'_1, \dots, a'_k\}$ . Then we define

$$C[j] = \{S \subseteq \{1, \dots, n\} : \Sigma_{A'}(S) \equiv j \pmod{q_1}\},$$

for  $0 \leq j \leq q_1 - 1$ , and we set  $c[j] = |C[j]|$ . By Lemma 14 the full array  $c$  can be computed in time  $\tilde{O}(2^{n/2})$ , and by Theorem 17, for every  $0 \leq j \leq q_1 - 1$ , the entry  $C[j]$  can be enumerated in  $O(n)$  time per set.

The arrays  $B$  and  $C$  are of course not identical, but the following lemma shows that any set in  $B[j]$  must be contained in  $C$  at an index close to  $j$ . For any  $0 \leq i, j \leq q_1 - 1$ , we define

$$B[i \dots j] = \begin{cases} \cup_{k=i}^j B[k] & \text{if } i \leq j \\ \cup_{k=i}^{q_1-1} B[k] \cup \cup_{k=0}^j B[k] & \text{otherwise,} \end{cases}$$

and  $C[i \dots j]$  is defined analogously. We denote their respective cardinalities as  $b[i \dots j] = |B[i \dots j]|$  and  $c[i \dots j] = |C[i \dots j]|$ . Finally, for  $0 \leq i \leq j \leq q_1$ , we set  $\beta[i \dots j] = \sum_{k=i}^j \beta[k]$ .

**Lemma 35.** *For every  $0 \leq j \leq q_1 - 1$ , the inclusion*

$$B[j] \subseteq C[(j - n + 1) \bmod q_1 \dots (j + n - 1) \bmod q_1]$$

*holds.*

*Proof.* Let  $S \in B[j]$ . Then by definition  $\Sigma(S) \bmod q = j \cdot 2^{n/2} + j'$ , for some  $0 \leq j' \leq 2^{n/2} - 1$ . Consequently  $\Sigma(S) = k_1 q + j \cdot 2^{n/2} + j'$ , where  $0 \leq k_1 \leq n - 1$  because  $a_i < q$  for every  $i$ , and therefore  $\Sigma(S) < nq$ . This implies that

$$\begin{aligned} \left\lfloor \frac{\Sigma(S)}{2^{n/2}} \right\rfloor &= j + \left\lfloor \frac{k_1 q + j'}{2^{n/2}} \right\rfloor \\ &= j + \left\lfloor \frac{k_1(q_1 2^{n/2} + q_2) + j'}{2^{n/2}} \right\rfloor \\ &= j + k_1 q_1 + \left\lfloor \frac{k_1 q_2 + j'}{2^{n/2}} \right\rfloor \\ &= j + k_1 q_1 + k_2, \end{aligned}$$

where  $0 \leq k_2 \leq n - 1$ . Therefore

$$\Sigma_{A'}(S) = j + k_1 q_1 + k_2 - k_3,$$

where  $0 \leq k_3 \leq n - 1$  since the set  $S$  contains at most  $n$  elements. We can thus conclude that

$$\Sigma_{A'}(S) \equiv j + k_4 \pmod{q_1},$$

where for  $k_4 = k_2 - k_3$  we have  $-n + 1 \leq k_4 \leq n - 1$ . □

**Corollary 36.** *Let  $0 \leq i, j \leq q_1 - 1$  with  $2n \leq j - i \leq (q_1 - 1)/2$ . Then*

$$C[i + n - 1 \dots j - n + 1] \subseteq B[i \dots j] \subseteq C[(i - n + 1) \bmod q_1 \dots (j + n - 1) \bmod q_1].$$

*Proof.* The second inclusion follows immediately from Lemma 35. The lemma also implies

$$B[j + 1 \bmod q_1 \dots i - 1 \bmod q_1] \subseteq C[(j - n + 2) \dots (i + n - 2)].$$

Taking the complement of the set on each side gives the first inclusion. □

**Corollary 37.** *Let  $0 \leq i \leq q_1 - 1$ . Then there exists  $k \in \{i - n, i + n\}$  such that*

$$C[i] \subseteq B[(k - 2n + 1) \dots (k + 2n - 1)]$$

.

*Proof.* We have either  $i \geq 3n - 1$  or  $i \leq q_1 - 3n$ . If only the first case is true choose  $k = i - n$ , if only the second case is true choose  $k = i + n$ , if both cases are true choose arbitrarily. Obviously  $C[i] \subseteq C[(k - n) \dots (k + n)]$ , therefore Corollary 36 implies the statement. □

**Corollary 38.** *Let  $0 \leq i \leq q_1 - 1$ . In time  $\tilde{O}(2^{n/2})$  we can either enumerate  $C[i]$  or we can find a marked index.*

*Proof.* If  $c[i] \leq (4n - 1)2^{n/2}$  then by Theorem 17 we can enumerate  $C[i]$ . Otherwise, by Corollary 37, we test each  $S \in C[i]$  to see which set  $B[j]$ , for  $k - 2n + 1 \leq j \leq k + 2n - 1$ , it belongs to until an index  $j$  satisfying  $b[j] > 2^{n/2}$  is identified.  $\square$

We now describe the procedure to find a marked index. It is essentially a dichotomic search over shorter and shorter intervals  $[i \dots j] = \{i, \dots, j\}$ , with the invariant property  $\beta[i \dots j] < b[i \dots j]$ , and where in every step we halve the size of  $j - i$ . Initially we set  $i = 0$  and  $j = q_1 - 1$ , and we stop the process when  $2n \leq j - i < 4n$ . Clearly the number of iterations is less than  $n/2$ . We now describe one iteration. Let us suppose that our current interval is  $[i \dots j]$ , and let  $m = (j - i)/2$ , rounding it arbitrarily, if necessary. We will compute  $b[i \dots m]$  and  $b[m + 1 \dots j]$  and keep one of the two intervals for which the invariant property holds.

We claim that for some fixed indices  $i, j$ , in time  $\tilde{O}(2^{n/2})$  we can either compute  $b[i \dots j]$  or we find a marked index. From Corollary 36 it follows that

$$b[i \dots j] = c[i + n - 1 \dots j - n + 1] + \left| B[i \dots j] \cap \left( C[(i - n + 1) \bmod q_1 \dots i + n - 2] \cup C[(j - n + 2) \dots (j + n - 1) \bmod q_1] \right) \right|.$$

The first term  $c[i + n - 1 \dots j - n + 1]$  is computed in time  $\tilde{O}(q_1)$  by adding the corresponding entries in the array  $c$ . For the second term, since there at most  $4n$  entries of  $C$  involved in it, we can either enumerate all the elements they contain in time  $\tilde{O}(2^{n/2})$  by Corollary 38 and check for each element if it belongs to  $B[i \dots j]$  (hence we can compute  $b[i \dots j]$ ), or we can find a marked index.

Unless we already found a marked index during the process, the dichotomic search stops with less than  $4n$  candidate indices out of which at least one is marked. Therefore the last thing to show is that given a marked index  $j$ , how do we find a solution in time  $\tilde{O}(2^{n/2})$ . From Lemma 35 and Corollary 36 we know that

$$B[j] \subseteq C[(j - n + 1) \bmod q_1 \dots (j + n - 1) \bmod q_1] \subseteq B[j - 2n + 2 \bmod q_1 \dots j + 2n + 2 \bmod q_1].$$

We start enumerating  $\mathcal{C} = C[(j - n + 1) \bmod q_1 \dots (j + n - 1) \bmod q_1]$  until one of the following two things happens. If  $|\mathcal{C}| \leq (4n - 3)2^{n/2}$  then we fully enumerate  $\mathcal{C}$  and therefore can also fully enumerate  $B[j]$ , and find a solution there. Otherwise we stop after having enumerated  $(4n - 3)2^{n/2} + 1$  elements of  $\mathcal{C}$ , and for each of them we determine the index where they belong in  $B[j - 2n + 2 \bmod q_1 \dots j + 2n + 2 \bmod q_1]$ . There will be an index where we have found more than  $2^{n/2}$  subsets and therefore also a solution.  $\square$

### 6.3 Open problems

We suggest two directions for future work on the modular versions of the problems studied in this paper:

1. Quantization of PIGEONHOLE MODULAR EQUAL-SUMS. That is, can we construct a quantum algorithm improving on the classical running time given in Theorem 34?
2. Can we prove a modular version of the algorithm of [MNPW19]? That is, can MODULAR EQUAL-SUMS be solved classically in time  $O(q^{0.773})$ ?

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## A Classical algorithm for SHIFTED-SUMS

Here we adapt the classical  $O(2^{0.773n})$  algorithm for EQUAL-SUMS of Mucha et al. [MNPW19] to apply to SHIFTED-SUMS, with the same worst-case running time.



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**Algorithm 6:** Classical meet-in-the-middle technique for SHIFTED-SUMS

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**Input:** Instance  $(a, s)$  of SHIFTED-SUMS with maximum solution ratio  $\ell$ .

**Output:** Two subsets  $S_1, S_2 \subseteq [n]$ .

- 1 Randomly split  $[n]$  into disjoint subsets  $X_1 \cup X_2$  such that  $|X_1| = |X_2| = n/2$ .
  - 2 Classically compute and sort  $V_1 = \{\Sigma(S_{11}) - \Sigma(S_{21}) : S_{11}, S_{21} \subseteq X_1 \text{ and } S_{11} \cap S_{21} = \emptyset \text{ and } |S_{11}| + |S_{21}| = \ell n/2\}$ .
  - 3 Classically compute  $V_2 = \{\Sigma(S_{12}) - \Sigma(S_{22}) : S_{12}, S_{22} \subseteq X_2 \text{ and } S_{12} \cap S_{22} = \emptyset \text{ and } |S_{12}| + |S_{22}| = \ell n/2\}$ .
  - 4 For each  $v_2 \in V_2$ , binary search for  $v_1 \in V_1$  such that  $v_1 + v_2 = s$ . If such a  $v_2$  is found, output  $S_1 = S_{11} \cup S_{12}$  and  $S_2 = S_{21} \cup S_{22}$ , where  $S_{11}, S_{21} \subseteq X_1$  are such that  $v_1 = \Sigma(S_{11}) - \Sigma(S_{21})$  and  $S_{12}, S_{22}$  are such that  $v_2 = \Sigma(S_{12}) - \Sigma(S_{22})$ .
- 

**Theorem 39** (SHIFTED-SUMS, classical meet-in-the-middle). *Given an instance of SHIFTED-SUMS with maximum solution ratio  $\ell \in (0, 1)$ , Algorithm 6 finds a solution with at least inverse polynomial probability in time  $\tilde{O}(2^{n(h(\ell)+\ell)/2})$ .*

*Proof.* By the the same proof technique as Lemma 29, it follows that with at least inverse polynomial probability the random partition  $X_1 \cup X_2$  satisfies  $|(S_1 \cup S_2) \cap X_1| = |(S_1 \cup S_2) \cap X_2| = \ell n/2$ . If this is the case, the algorithm will succeed.

The sets  $V_1, V_2$  have cardinality  $|V_1| = |V_2| = \binom{n/2}{\ell n/2} 2^{\ell n/2}$ . Computing and sorting  $V_1, V_2$  thus takes time  $\tilde{O}(\binom{n/2}{\ell n/2} 2^{\ell n/2})$ . For each element  $v_2 \in V_2$ , binary search over  $V_1$  takes times logarithmic in  $|V_1|$ . The result follows from Fact 1.  $\square$

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**Algorithm 7:** Classical representation technique for SHIFTED-SUMS

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**Input:** Instance  $(a, s)$  of SHIFTED-SUMS with  $\sum_{i=1}^n a_i < 2^{4n}$  and maximum solution ratio  $\ell$ .

**Output:** Two subsets  $S_1, S_2 \subseteq [n]$ .

- 1 Set  $b = 1 - \ell$  if  $\ell > 1/2$  and  $b = 1/2$  otherwise.
  - 2 Choose a random prime  $p \in [2^{bn} \dots 2^{b(n+1)}]$  and a random integer  $k \in [0 \dots p - 1]$ .
  - 3 Construct the table  $t_p[i, j]$  for  $i = 0, \dots, n$  and  $j = 0, \dots, p - 1$  (see Section 3).
  - 4 Enumerate  $T_{p,k}$  and  $T_{p,(k-s) \bmod p}$ , and sort  $T_{p,(k-s) \bmod p}$ . For each  $S_1 \in T_{p,k}$ , binary search for  $S_2 \in T_{p,(k-s) \bmod p}$  such that  $\Sigma(S_1) = \Sigma(S_2) + s$  and  $S_1 \neq S_2$ . If found, output the pair  $(S_1, S_2)$ .
- 

**Theorem 40** (SHIFTED-SUMS, classical representation). *Given an instance of SHIFTED-SUMS with  $\sum_{i=1}^n a_i < 2^{4n}$  and maximum solution ratio  $\ell \in (0, 1)$ , Algorithm 7 finds a solution with inverse polynomial probability in time  $\tilde{O}(2^{bn} + 2^{(1-b)n})$ , where  $b = 1 - \ell$  if  $\ell > 1/2$  and  $b = 1/2$  otherwise.*

*Proof.* The choice of  $b$  satisfies  $b \leq 1 - \ell$ . By Lemmas 21 and 27, with probability  $\Omega(1/n)$  there is at least one solution pair contained in  $T_{p,k} \times T_{p,(k-s) \bmod p}$ . By Lemma 19 and Markov's inequality, the sizes of  $T_{p,k}$  and  $T_{p,(k-s) \bmod p}$  are at most  $t_{p,k}, t_{p,(k-s) \bmod p} \leq n^2 2^{(1-b)n}$  with probability at least  $1 - 1/n^2$ . Thus, with probability  $\Omega(1/n)$  we can assume that both of these events occur. If this is the case, then enumeration and sorting of  $T_{p,k}, T_{p,(k-s) \bmod p}$  can be completed in time  $\tilde{O}(2^{(1-b)n})$  (Theorem 17) after constructing the table  $t_p$  in time  $O(n2^{bn})$  (Lemma 14).  $\square$

For each value of  $\ell$ , choosing the better of Algorithms 6 and 7 gives the following result.

**Theorem 41** (SHIFTED-SUMS, classical). *There is a classical algorithm which, given an instance of SHIFTED-SUMS with maximum solution ratio  $\ell \in (0, 1)$ , outputs a solution with at least inverse polynomial probability in time  $\tilde{O}(2^{\gamma(\ell)n})$  where*

$$\gamma(\ell) = \begin{cases} 1/2 & \text{if } \ell_1 \leq \ell < 1/2, \\ \ell & \text{if } 1/2 \leq \ell < \ell_2, \\ (h(\ell) + \ell)/2 & \text{otherwise} \end{cases}$$

and  $\ell_1 \approx 0.227$  and  $\ell_2 \approx 0.773$  are solutions to the equations  $(h(\ell) + \ell)/2 = 1/2$  and  $(h(\ell) + \ell)/2 = \ell$  respectively. In particular, the worst case complexity of the algorithm is  $O(2^{0.773n})$ .

In comparison with the above result, the algorithm of [MNPW19] for EQUAL-SUMS has running time  $\tilde{O}(2^{\gamma(\ell')n})$  where  $\ell'$  is the minimum solution ratio (rather than maximum), and  $\gamma(\ell') = \ell'$  for  $1/2 \leq \ell' < \ell_2$  and  $\gamma(\ell') = (h(\ell') + \ell')/2$  otherwise. We do not know if a similar algorithm exists for SHIFTED-SUMS based on the minimum solution ratio.

## B Quantum EQUAL-SUMS in terms of minimum solution ratio

We first recall the concept of a *minimum solution*, introduced in [MNPW19].

**Definition 42** (Minimum solution). Two disjoint subsets  $S_1, S_2 \subseteq \{1, \dots, n\}$  that form a solution to an instance of EQUAL-SUMS are a *minimum solution* if their size  $|S_1| + |S_2| = \ell'$  is smallest among all such solutions. We call  $\ell' \in (0, 1)$  the minimum solution ratio.

We prove that, for the special case of EQUAL-SUMS, we can reformulate the results of Section 5 to make use of the minimum solution ratio  $\ell'$  instead of the maximum one.

**Theorem 43** (EQUAL-SUMS, quantum). *There is a quantum algorithm which, given an instance of EQUAL-SUMS with minimum solution ratio  $\ell' \in (0, 1)$ , outputs a solution with at least inverse polynomial probability in time  $\tilde{O}(2^{\gamma'(\ell')n})$  where*

$$\gamma'(\ell') = \begin{cases} \frac{1}{2} - \frac{1-\ell'}{4} h\left(\frac{\ell'}{2(1-\ell')}\right) & \text{if } \ell'_1 \leq \ell' < 1/2 \\ (1 + \ell')/4 & \text{if } 1/2 \leq \ell' < 3/5, \\ \ell'/2 + 1/10 & \text{if } 3/5 \leq \ell' < \ell'_2, \\ (h(\ell') + \ell')/3 & \text{otherwise} \end{cases}$$

and  $\ell'_1 \approx 0.273$  and  $\ell'_2 \approx 0.809$  are solutions to the equations  $(h(\ell') + \ell')/3 = 1/2 - (1 - \ell')h(\frac{\ell'}{2(1-\ell')})/4$  and  $(h(\ell') + \ell')/3 = \ell'/2 + 1/10$  respectively. In particular, the worst case complexity of the algorithm is  $O(2^{0.504n})$ .

The proof follows closely that of the quantum algorithm for SHIFTED-SUMS (Theorem 25), with the main difference coming from a bound on the size of the collision values set  $V = \{v \in \mathbb{N} : \exists S_1 \neq S_2, v = \Sigma(S_1) = \Sigma(S_2)\}$  for EQUAL-SUMS. For SHIFTED-SUMS, Lemma 27 gives the bound  $|V| \geq 2^{(1-\ell)n}$  in terms of the *maximum solution ratio*  $\ell$ . For EQUAL-SUMS we can obtain a similar statement in terms of the *minimum solution ratio*.

**Lemma 44.** *If an instance of EQUAL-SUMS has minimum solution ratio  $\ell'$  then the collision values set  $V = \{v \in \mathbb{N} : \exists S_1 \neq S_2, v = \Sigma(S_1) = \Sigma(S_2)\}$  satisfies*

$$|V| \geq \begin{cases} 2^{(1-\ell')n} & \text{if } \ell' > 1/2, \\ 2^{(1-\ell')h(\frac{\ell'}{2(1-\ell')})n} & \text{otherwise.} \end{cases}$$

*Proof.* The case  $\ell' > 1/2$  is dealt with in [MNPW19], therefore consider  $\ell' \leq 1/2$ . Let  $S_1, S_2 \subseteq \{1, \dots, n\}$  be a minimum solution of size  $\ell'n$ . Then for any  $S \subseteq \overline{S_1 \cup S_2}$ , with  $|S| = \frac{\ell'n}{2} - 1$ , the sets  $S \cup S_1$  and  $S \cup S_2$  form a solution, and for  $S \neq S'$ , the values  $\Sigma(S_1 \cup S)$  and  $\Sigma(S_1 \cup S')$  are distinct. Indeed, if this were not the case then  $S \setminus S'$  and  $S' \setminus S$  would form a disjoint solution of size less than  $\ell'n$ . Therefore  $|V| \geq \binom{n(1-\ell')}{\frac{\ell'n}{2}-1}$ , and the statement follows from Fact 1.  $\square$

We can now prove Theorem 43.

*Proof.* For each minimum solution ratio  $\ell' \in (0, 1)$ , we use the better of Algorithms 4 and 5 with two modifications: (i) the minimum solution ratio  $\ell'$  is used in place of the maximum solution ratio  $\ell$  in the input of the algorithms, and (ii) we choose the value of  $b$  in step 1 of Algorithm 4 to be

$$b(\ell') = \begin{cases} \frac{1}{2} - \frac{1-\ell'}{4} h\left(\frac{\ell'}{2(1-\ell')}\right) & \text{if } \ell' \leq 1/2, \\ (1 + \ell')/4 & \text{if } 1/2 < \ell' \leq 3/5, \\ 2/5 & \text{if } 3/5 < \ell'. \end{cases}$$

The analysis of Algorithm 5 is unaffected by the change to minimum solution ratio, as is the analysis of Algorithm 4 for  $\ell' > 1/2$ . For Algorithm 4 and  $\ell' \leq 1/2$ , repeating the analysis of Lemma 21 using  $|V| \geq 2^{(1-\ell')h\left(\frac{\ell'}{2(1-\ell')}\right)n}$  gives

$$\Pr_{p,k}[v_{p,k} \geq 2^{(z-b)n-2}] = \Omega(1/n)$$

where  $z = (1 - \ell')h\left(\frac{\ell'}{2(1-\ell')}\right)$ . Recalling proof of Theorem 28, construction of the dynamic programming table takes time  $\tilde{O}(2^{bn})$ , and a collision can be found in time  $\tilde{O}(t_{p,k}^{2/3}/v_{p,k}^{1/3}) = \tilde{O}(2^{(2-z-b)/3})$ . The running time for  $\ell' < 1/2$  follows from balancing these two costs, i.e. by setting  $b = (2 - z - b)/3$ .  $\square$