

Proof sketches

1 Lecture 1

Lemma 1.1. $\| |\psi_0^0\rangle - |\psi_i^0\rangle \| = 0$

Proof. $|\psi_0^0\rangle = |\psi_i^0\rangle = U_0|0, 0\rangle$ □

Lemma 1.2. $\| |\psi_0^T\rangle - |\psi_i^T\rangle \| \geq 1/3$ if the algorithm succeeds w.p. $\geq 2/3$ after T queries

Proof. Success conditions: $\|(\text{Id} \otimes |0\rangle\langle 0|)|\psi_0^T\rangle\|^2 \geq 2/3$ and $\|(\text{Id} \otimes |1\rangle\langle 1|)|\psi_i^T\rangle\|^2 \geq 2/3$

$$\begin{aligned}
 \| |\psi_0^T\rangle - |\psi_i^T\rangle \|^2 &= 2(1 - \text{Re}(\langle \psi_0^T | \psi_i^T \rangle)) \\
 &= 2(1 - \text{Re}(\langle \psi_0^T | (\text{Id} \otimes |0\rangle\langle 0|) |\psi_i^T\rangle) - \text{Re}(\langle \psi_0^T | (\text{Id} \otimes |1\rangle\langle 1|) |\psi_i^T\rangle)) \\
 &\geq 2(1 - \|(\text{Id} \otimes |0\rangle\langle 0|)\psi_0^T\| \cdot \|(\text{Id} \otimes |0\rangle\langle 0|)\psi_i^T\| - \|(\text{Id} \otimes |1\rangle\langle 1|)\psi_0^T\| \cdot \|(\text{Id} \otimes |1\rangle\langle 1|)\psi_i^T\|) \\
 &\hspace{15em} \text{by Cauchy-Schwarz inequality} \\
 &\geq 2(1 - 2\sqrt{2}/3) \hspace{15em} \text{by success conditions} \\
 &\geq 1/9
 \end{aligned}$$

□

Lemma 1.3. $\| |\psi_0^{t+1}\rangle - |\psi_i^{t+1}\rangle \| \leq \| |\psi_0^t\rangle - |\psi_i^t\rangle \| + \sqrt{q_i^t}$

Proof.

$$\begin{aligned}
 \| |\psi_0^{t+1}\rangle - |\psi_i^{t+1}\rangle \| &= \| U_{t+1}|\psi_0^t\rangle - U_{t+1}O_{\vec{i}}|\psi_i^t\rangle \| && \text{by definition and } O_{\vec{0}} = \text{Id} \\
 &= \| |\psi_0^t\rangle - O_{\vec{i}}|\psi_i^t\rangle \| && \text{unitary preserves norm} \\
 &= \| O_{\vec{i}}(|\psi_0^t\rangle - |\psi_i^t\rangle) + (\text{Id} - O_{\vec{i}})|\psi_0^t\rangle \| \\
 &\leq \| O_{\vec{i}}(|\psi_0^t\rangle - |\psi_i^t\rangle) \| + \| (\text{Id} - O_{\vec{i}})|\psi_0^t\rangle \| && \text{by triangle inequality} \\
 &= \| |\psi_0^t\rangle - |\psi_i^t\rangle \| + \| (\text{Id} - O_{\vec{i}})|\psi_0^t\rangle \|
 \end{aligned}$$

We have $\text{Id} - O_{\vec{i}} = |i\rangle\langle i| \otimes X$ where $X = |1\rangle\langle 0| + |0\rangle\langle 1|$. Hence, $\|(\text{Id} - O_{\vec{i}})|\psi_0^t\rangle\| = \|(|i\rangle\langle i| \otimes \text{Id})|\psi_0^t\rangle\| = \sqrt{q_i^t}$. □

Theorem 1.4. $Q(\text{OR}) \geq \sqrt{n}/3$

Proof. $n/3 \leq \sum_{i=1}^n \sum_{t=0}^T \sqrt{q_i^t} \leq \sqrt{nT \sum_{i=1}^n \sum_{t=0}^T q_i^t} = \sqrt{nT} \Rightarrow T \geq \sqrt{n}/3$. □